ASYMPTOTIC SERIES SOLUTIONS

TO ONE-DIMENSIONAL HELMHOLTZ EQUATION

by

Yan Chen

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written by Yan Chen
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Harvey Segur

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Keith Julien

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Sujeet Bhat

Date__________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Chen, Yan (M.S., Applied Mathematics)

Asymptotic Series Solutions to One-Dimensional Helmholtz Equation

Thesis directed by Professor Harvey Segur

Abstract

Usually, a divergent asymptotic series can approximate a function, with fixed argument values, to some optimal accuracy, but no better. Dingle (1973) [12] began development of a method to create a follow-on series, which begins at the point of optimal accuracy of the original asymptotic series, and can be used to improve the accuracy further, but again with a minimal error that cannot be reduced by further use of the two series. Berry & Howls (1990) [7] developed Dingle's idea further, using a sequence of divergent series, with each series improving on the accuracy of the preceding one. Again, they eventually came to a (much smaller) minimal error that they could not reduce. They named this sequence of increasingly accurate asymptotic series *hyperasymptotics*. They demonstrated their approach by developing hyperasymptotic series, valid for large positive z, for the Airy function, $Ai(z)$.

We present a variation of the method of Berry & Howls, which eliminates some of the inherent error in their approach. Using Dingle's change of variables we transform the Airy differential equation into a new ODE that is exact, for all positive z. After reformulating the ODE as an integral equation, we solve the integral equation (exactly) with a recurrent series that converges absolutely for all positive z. Each term in our series can be expanded as an asymptotic series with the error term under our control because of the bound we develop for it. Comparing with other techniques, our solution maintains all the original function's information.

We discover a type of oscillation behavior hidden in the hyperasymptotic series. This behavior ought to be in the structure because of the way that the hyperseries is constructed. Each term in the hyperseries is in the form that consists of indexes and arguments. Each

index changes discretely, whereas each argument changes continuously. This is the fundamental reason that causes the oscillation phenomenon. The other researchers, like Berry [4]- [8], Howls [7] and Boyd [10], have not addressed this issue in their related work because they only approximate the solution when the argument is at a single value.

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Chapter 1

Introduction

1.1 Preliminary

A **series** is the sum of a sequence of numbers or functions with the form

$$\{C_0 + C_1 x + C_2 x^2 + \dots\} \tag{1.1}$$

where $\{C_n\}|_{n=0}^{\infty}$ is a set of known coefficients, and x is a parameter. The N-th partial sum of an infinite power series has the the form

$$S_N(x) = \sum_{n=0}^{N} C_n x^n. \tag{1.2}$$

The infinite series is **convergent** at a given x if $S_N(x) = \sum_{n=0}^{N} C_n x^n \to$ a finite-valued function of x as $N \to \infty$. The series is **asymptotic** [13] to a function $S(x)$ as $x \to x_0$ if for any non-negative integer N,

$$S(x) - \sum_{n=0}^{N} C_n (x - x_0)^n = o((x - x_0)^{(N)}) \tag{1.3}$$

as $x \to x_0$. In other words, if the series is truncated after the first N terms, and the sum of everything omitted is smaller than the last term that is kept in the N-th partial sum for any positive N as $x \to x_0$, then the series is asymptotic as $x \to x_0$.

Let f(x) and g(x) be two functions defined on some subset of the real numbers. We say f(x) is asymptotic to g(x) as $x \to x_0$ and write

$$f(x) = o(g(x)), \tag{1.4}$$

if and only if for any positive number ϵ, there exists a positive real number δ such that for all x with $|x - x_0| < \delta$ we have $|f(x)| \le \epsilon |g(x)|$. The two concepts, convergent series and asymptotic series, involve two different limits and are independent – a given series might have both properties, or one, or neither.

The Stirling approximation (1730) [16] for the Gamma function

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}}(\frac{z}{e})^z(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \dots) \tag{1.5}$$

where z is a large positive number, is a series that is asymptotic as z goes infinity, but it is not convergent for any finite z, as shown in Figure 1.1 [1]. The partial sums of the series first converge towards the exact value, and then start diverging. The partial sum that best approximates the function is at the optimal truncation term, when the error term reaches its minimum value. The optimal truncation term depends on the value of the argument z.

The Stirling series is an asymptotic series as $z \to \infty$. When we estimate the gamma function at value of 10, we can use any partial sums with the number of terms no larger than the optimal truncation term to do so, as shown in Figure 1.1. In general, if we approximate the gamma function at any large finite value, we can use those partial sums of the Stirling series with the number of terms no larger than the optimal truncation term to do so.

Berry [6] refers to an optimally-truncated asymptotic series as **superasymptotics**. The error is typically $\mathcal{O}(\exp(q/\epsilon))$ where $q > 0$ is a constant and ϵ is the small parameter of

[1] Cited from Stirling approximation Wikipedia page.

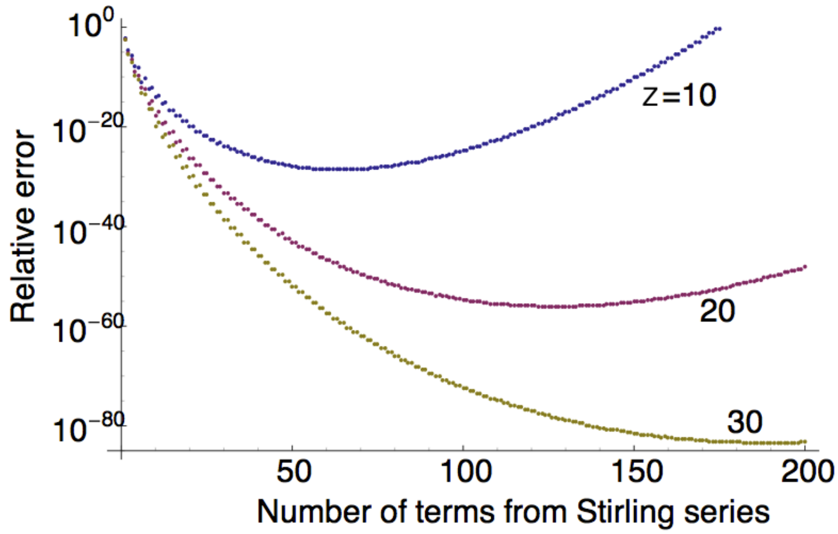

Figure 1.1: The relative error in a truncated Stirling series vs. the number of terms used.

the asymptotic series. The degree of the highest term retained in the optimal truncation, $N_{opt}(\epsilon)$, grows with $1/\epsilon$.

One can do no better than the optimal truncated series from the original asymptotic series if one wants to use only this series. If I may quote Niels Abel (1828) *"The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever"*. This issue leads to the main topic of my thesis work: to improve the approximations of some functions using asymptotic analysis.

In response to the high accuracy required by some problems, for example in quantum physics, people have developed methods that go beyond optimal truncations [10, 15]. Berry and Howls (1991) [7] demonstrated a technique that generates better approximation series, which they called **hyperasymptotic series**. These series achieve higher accuracy than a superasymptotic approximation by adding one or more terms of a second asymptotic series, with different scaling assumptions, to the optimal truncation of the original asymptotic expansion. With another rescaling, this process can be iterated by adding terms of a third asymptotic series, and so on.

With this new technique, Berry has unveiled one kind of structure hidden in the divergent part of the original asymptotic expansion. Also, Berry clarified the point that the variation of an asymptotic series in the neighbourhood of the Stokes lines was not discontinuous, which had been believed previously, but was rather smooth and extremely rapid in nature. [4,5] As such, researchers are able to apply hyperasymptotic technique to investigate the behaviour of asymptotic series in the neighbourhood surrounding Stokes lines.

1.2 One-dimensional Helmholtz equation

An example, which strongly involves the series expansion techniques mentioned above, is to approximate the solutions $y(z, \lambda)$ of the one-dimensional Helmholtz equation

$$d^2y(z, \lambda)/dz^2 = \lambda^2 Z(z)y(z, \lambda) \tag{1.6}$$

where $Z(z)$ is a known function, and λ is a constant. This is an eigenfunction problem where $y(z, \lambda)$ is the eigenfunction and λ is the eigenvalue, and the equation is also a type of one-dimensional Schrödinger equation. The goal of analysis is not only to obtain high accuracy of the solutions, but also to exhibit the detailed structure of a physical theory as an important parameter takes a limiting value.

1.2.1 Airy function

The simplest nontrivial model that has the form of a one-dimensional Helmholtz equation is the Airy equation

$$\frac{d^2}{dz^2}y(z) = zy(z) \tag{1.7}$$

We set $Z(z) = z$, and one can think of the constant λ as contained in the variable z in this case. The Airy function, $Ai(z)$, is a special function named after George Airy because of the paper [2] he published in 1838. The Airy function is a solution to the Airy equation (1.7).

1.3 Goal

The idea of my thesis work is inspired by Dingle [12], Berry & Howls [7]. (I refer to their method as the DBH method in my thesis.) They developed the hyperasymptotic series technique, which significantly increases the accuracy beyond that of the original series: the error is reduced to less than the square of the superasymptotic error. In our work, we are aiming to find an even better approximation to the solutions than the one generated by the DBH method.

1.4 Thesis structure

Chapter 2 of this thesis contains two main sections: the WKB method, developed in the 1920s, and the DBH method. The WKB method is one of the most common methods to approximate the solutions to the Schrödinger equation. The purpose of this part is because both DBH method and our method start with the WKB method. The DBH method has three main highlights: change variable, equation transformation, and Borel summation. The first two are constructed specially for the purpose of using Borel summation.

Chapter 3 of this thesis focusses on our method to approach the same approximation issue. The first step in our method is the same as the DBH method's first highlight: change variables. But the variation between the DBH and our method starts from the equation transformation. Our advantages are to transform the equation exactly without reducing its order, and to solve the new equation with a bounded, convergent series, with no approximation. Also, we find the boundaries for our error terms, as well as a mathematical model to approximate them.

Chapter 4 of this thesis focusses on the comparison among the WKB, the DBH, and our method to approximate the Airy function as an example. Both the numerical and analytical work demonstrate that all methods can achieve the same level of accuracy with their superasymptotic series. Then by making use of the error term, the DBH can achieve the approximation with the error less than the square of the superasymptotic error at $F = 16$. Our method currently achieves the accuracy between the WKB method's and the DBH method's. But more importantly, along the way of the approximation work, we found a type of oscillation behavior hidden in the errors, which we think is unavoid. We also think this is a phenomenon that happens in other hyperasymptotics methods.

Chapter 2

Asymptotic Series Approximation to 1D Helmholtz Equation

2.1 WKB method

The WKB method is a tool for obtaining a global approximation to the solution of a linear differential equation whose highest derivative is multiplied by a small parameter. It assumes the solutions have the form of asymptotic series, consisting of exponentials of elementary integrals of algebraic functions, and special functions. It is an approximation method that is suitable for linear differential equations of any order. The formal WKB expansion starts in a single exponential power series of the form

$$y(x, \lambda) \sim \exp\left[\lambda \sum_{n=0}^{\infty} \frac{S_n(x)}{\lambda^n}\right] \tag{2.1}$$

as $\lambda \to \infty$. The first step is to plug this form back into the linear differential equation for $y(x, \lambda)$. By grouping the terms in the new equation according to their powers of $(1/\lambda)$, the approximation series can be found. For 1D Helmholtz equation, which is a second order ODE, the WKB method provides the approximation series in the form

$$y(z,\lambda)_{\pm} = \frac{\exp\left[\pm\lambda\int_{z*}^{z} Z^{\frac{1}{2}}(t)dt\right]}{Z^{\frac{1}{4}}(z)}(1+\mathcal{O}(\frac{1}{\lambda})) \tag{2.2}$$

where $z*$ is an arbitrary number.

2.1.1 Approximate the Airy function by WKB method

The function now called the Airy function was introduced by Airy [2] in a form of an integral

$$Ai(z) = \frac{1}{\pi}\int_{0}^{\infty}\cos(\frac{t^3}{3}+zt)dt. \tag{2.3}$$

Expanding the integrand of the Airy function around $z = 0$, one can generate a series representation [11] for the Airy function

$$Ai(z) = \frac{1}{3^{\frac{2}{3}}\pi}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{3}n+\frac{1}{3})}{n!}\sin\{\frac{2}{3}(n+1)\pi\}(3^{\frac{1}{3}}z)^n. \tag{2.4}$$

The series is convergent for all finite $|z|$. Later on, people found that the Airy function is a solution to what we call the Airy equation

$$\frac{d^2}{dz^2}y(z) = zy(z). \tag{1.7}$$

The Airy function has the property that is bounded as $z \to +\infty$.

For large positive z, the convergent series in (2.4) is numerically inefficient. The WKB method is a way to compute the Airy function in this limit. Following the WKB method, one obtains the leading order approximation

$$Ai(z) = \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}z^{\frac{1}{4}}}(1+\mathcal{O}(z^{-\frac{3}{2}})) \tag{2.5}$$

for all finite positive z. Now we have introduced the WKB method to approximate the

Airy function. The series, implied by equation (2.5), is asymptotic as $z \to +\infty$. But it is divergent for any finite z, like the Stirling series in (1.5). The DBH method (in this chapter) improves the accuracy beyond that obtained by the optimal truncation. Our method (in chapter 3) is an alternative way to improve this accuracy.

2.2 The DBH method (Hyperasymptotic series)

As the method was first discovered by Robert Dingle in 1950s, and summarized in his 1973 book [12], this technique is for systematically reducing the exponentially small remainder of an asymptotic expansion truncated near its least term for the solutions to a one-dimensional Helmholtz equation. The two main ideas are: (1) changing variables and (2) repeatedly applying Borel summation. The improvements form a new series, known as a hyperasymptotic series, which sums up all the optimally truncated parts of asymptotic series generated by the method of Borel summation until the last series reaches the natural halt, which is a limitation of the Borel summation method. The hyperasymptotic series is a divergent series as well. However, it shows a kind of structure hidden in the divergent part of the original asymptotic series solution, as well as leads to an approximation whose error is less than the square of the original superaysmptotic error.

2.2.1 Change variable

To approximate the solutions to the same equation, 1D Helmholtz equation (1.6), the DBH method is developed based on the WKB method. Recall the WKB method asymptotic series

$$y(z,\lambda)_{\pm} = \frac{\exp\left[\pm\lambda\int_{z*}^{z} Z^{\frac{1}{2}}(t)dt\right]}{Z^{\frac{1}{4}}(z)}(1+\mathcal{O}(\frac{1}{\lambda})) \tag{2.2}$$

where $z*$ is an arbitrary number. The first key step used by Dingle [12] is introducing a new variable, which is the difference between two exponents in (2.2)

$$F(z) = 2\lambda \int_{z^*}^{z} Z^{\frac{1}{2}}(t)dt. \tag{2.6}$$

To analyze the subdominant solution of (1.6) when $\Re[F] > 0$, write equation (2.2) as

$$y(z,\lambda) = (\exp[-\frac{1}{2}F]/Z^{\frac{1}{4}}(z))Y(F) \tag{2.7}$$

where $Y(F)$ a new function. Berry and Howls [7] proposed a formal asymptotic series defined by

$$Y(F) \sim \sum_{r=0}^{\infty}(-1)^r Y_r(F) \tag{2.8}$$

where

$$Y_0(F) = 1. \tag{2.9}$$

Substituting the new form of the solution to the 1D Helmholtz equation (1.6) the DBH method leads a recurrence relation for the terms in the series

$$Y'_{r+1}(F) = -Y''_r(F) + G(F)Y_r(F) \tag{2.10}$$

where primes denote derivatives with respect to F and

$$G(F) = (Z^{\frac{1}{4}})''/Z^{\frac{1}{4}}. \tag{2.11}$$

2.2.2 Resurgence formula

Dingle discovered a "resurgence formula" for each term,

$$Y_r(F) \sim \frac{1}{2\pi F^r}\sum_{s=0}^{\infty}(r-s-1)!(-F)^s Y_s(F) \tag{2.12}$$

which satisfies the recurrence relation in (2.10). Numerically, this equation is meaningless because the factorials for $s > r - 1$ are infinite. But, the purpose of developing this formula is for the next step in the DBH method.

2.2.3 Borel summation

Borel approximation, introduced by Èmile Borel in 1899 [9], is a systematic method to assign a finite value to a divergent series. (It is not the only method, for example Euler's method [3] is an alternative.) It is particularly useful for summing divergent asymptotic series. As Dingle and Berry discovered, Borel summation can be continued analytically across a Stokes line, where some other summation methods fail [8] [12].

The way that the DBH method constructs its series solution is actually building a path specifically for the purpose of applying the Borel summation method. The new ODE that is built in equation (2.10) is a first-order ODE, so the DBH method reduces the order of the original 1D Helmholtz equation from second-order to first-order. *A* consequence of this reduction is that the formal series solution constructed in this way is again a divergent series.

With the least term at N_0, the series that approximates $Y(F)$ in the DBH method is divided into the sum of the first $N_0 - 1$ terms Y_r, and the replaced remaining series by equation (2.12) with interchanging the r and s labels.

$$Y(F) \sim S_0 + \sum_{r=0}^{\infty} Y_r(F)(-F)^r \sum_{s=N_0}^{\infty} \frac{(s-r-1)!}{2\pi F^s} \tag{2.13}$$

where

$$S_0 = \sum_{r=0}^{N_0-1} (-1)^r Y_r(F). \tag{2.14}$$

The truncated sum S_0 is the superasymptotics of the $Y(F)$ series, and it is also the zeroth level of hyperasymptotic series that the DBH method will eventually build up. The similar procedure can be applied to the new generated series. However, the repetition of this process stops when the new generated series reaches the natural halt. The reason of having the natural halt is, quote from Berry [6] (1991):

> Hyperasymptotics comes to a natural halt, because each hyperseries is shorter than its predecessor, and eventually contains only one term. The decreasing length is a consequence of the 'live now, pay later' philosophy, natural in asymptotics, that the terms must continue to decrease, not only within each hyperseries but from each hyperseries to the next.

Summing up all the convergent parts of the Borel method generated series, one can write

$$Y(F) = S_0 + S_1 + S_2 + \dots, \tag{2.15}$$

which is so called the hyperasymptotic series as $F \to \infty$.

2.3 Approximate the Airy function by the DBH method

Recall the Airy equation

$$\frac{d^2}{dz^2} y(z) = z y(z). \tag{1.7}$$

The first step in the DBH method is to change the independent variable,

$$F(z) = \frac{4}{3} z^{\frac{3}{2}} \tag{2.16}$$

and then to use this new variable to rewrite the WKB ansatz for the subdominant solution, given in (2.16):

$$y(z, \lambda) = (\exp[-\frac{1}{2}F]/Z^{\frac{1}{4}}(z))Y(F). \tag{2.7}$$

When one applies this procedure is applied to the Airy equation, (1.7), one finds

$$Ai(z) = \frac{\exp(-\frac{1}{2}F)}{2\pi^{\frac{1}{2}} z^{\frac{1}{4}}} Y(F) \tag{2.17}$$

or

$$Y(F) = 2\sqrt{\pi}(\frac{3}{4}F)^{\frac{1}{6}} \exp(\frac{1}{2}F) Ai\{(\frac{3}{4}F)^{\frac{2}{3}}\}. \tag{2.18}$$

To compute $Y(F)$ hyperasymptotically, the DBH method first uses the formal asymptotic expansion but with the new variable F

$$Y_r(F) = \frac{1}{(27F)^r} \frac{\Gamma(3r+\frac{1}{2})}{\Gamma(r+1)\Gamma(r+\frac{1}{2})}. \tag{2.19}$$

Then by using the resurgence formula, (2.12), and Borel summation, (2.13), the DBH method generates 5-stage hyperasymptotic series to the Airy function, for $F = 16$. The numerical work is shown below in Figure 2.1 and table (2.1) [1]

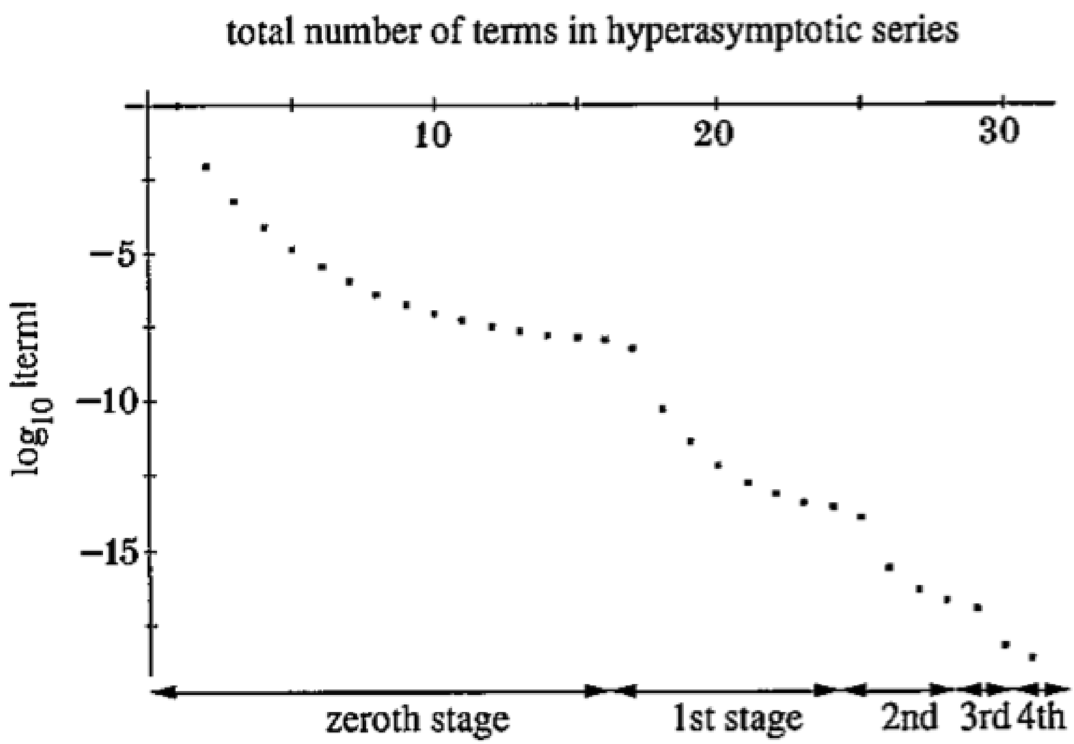

Figure 2.1: Decrease of the terms in the first five hyperseries of Y(16), for the Airy function.

The formal asymptotic series can achieve the accuracy of 9 decimal places for $F = 16$, as shown in the table. The DBH method achieves the accuracy of 18 decimal places for $F = 16$,

[1]Both Figure 2.1 and Table 2.1 are cited from [7].

level	approximation to $Y(F)$	approx. − exact
lowest	1	8.163×10^{-3}
S_0	0.991836793511323459 1100	-5.677×10^{-9}
S_0+S_1	0.991836799188251255 0983	-1.134×10^{-14}
$S_0+S_1+S_2$	0.991836799188262590 7500	-8.160×10^{-18}
$S_0+\ldots+S_3$	0.991836799188262599 8682	9.584×10^{-19}
$S_0+\ldots+S_4$	0.991836799188262600 6031	1.151×10^{-18}
exact	0.991836799188262598 9098	0

Table 2.1: Hyperasymptotic approximations to the Airy function (2.17) for $F = 16$.

which is twice as many decimals as the formal asymptotic series. As far as we know, this is the highest accuracy achieved, by any method, for the Airy function in the limit of $F \to \infty$. Berry & Howls [7] note that the same method applies as well for a subdominant solution of any 1D Helmholtz equation.

Chapter 3

Our Method

The technique that we present in this section was inspired by the work of Berry and Howls [7]. It differs from the DBH method because we sought to overcome some ambiguity at points in their analysis. Like them, we apply our method to the Airy function, not because the Airy function demands great precision, but because it is a simple example in which to demonstrate the method.

Using the DBH change of variables we transform the Airy differential equation into a new ODE that is exact, for all positive z. After reformulating the ODE as an integral equation, we solve the integral equation (exactly) with a series that converges absolutely for all positive z. Each term in our series can be expanded as an asymptotic series with the error term under our control because of the bound we develop for it. Comparing with the DBH hyperasymptotic series technique, our solution maintains all the original function's information that the DBH method has lost.

3.1 New ODE transformation for solving the Airy function

Using the same strategy as the DBH, we transform the original 1D Helmholtz equation into a new ODE which differs from what the DBH method did in equation (2.10). We first define a lemma to show how to obtain our ODE.

Lemma 1. For the Airy function, equation (2.6) and (2.7) become

$$F(z) = \frac{4}{3} z^{\frac{3}{2}} \tag{2.16}$$

and

$$y(z) = \frac{\exp(-\frac{1}{2}F)}{2\pi^{\frac{1}{2}} z^{\frac{1}{4}}} Y(F). \tag{3.1}$$

One simply changes variables in this way, and the Airy equation, (1.7), becomes

$$Y''(F) - Y'(F) - G(F)Y(F) = 0 \tag{3.2}$$

where

$$G(F) = -\frac{5}{36F^2}. \tag{3.3}$$

□

The variation between our method and the DBH method starts from equation (3.2). Recall that the DBH method transforms the Airy equation, (1.7), into the infinite sequence of first-order ODEs shown in (2.10). If the infinite series representation of $Y(F)$ in (2.8) were convergent, and if one could sum all of the ODEs presented in (2.10), then one would obtain the single ODE in (3.2). However, B&H [7] show that their series is divergent, so such infinite summations are ill-defined, and the ODE in (3.2) is not equivalent to the infinite sequence of ODEs in (2.10).

3.2 A bounded and convergent series solution for $Y(F)$

Theorem 1. The linear integral equation

$$Y(F) = A + \int_F^\infty (1 - e^{F-f})G(f)Y(f)df \tag{3.4}$$

where A is a constant and $F > 0$, has the following properties.

a) For any fixed, real-valued, nonzero A, (3.4) has a real-valued solution that can be represented in terms of a convergent series of the form

$$Y(F) = \sum_{m=0}^\infty Y_m(F) \tag{3.5}$$

where

$$Y_0(F) = A \tag{3.6}$$

where each subsequent term in the series is defined recursively by

$$Y_{m+1}(F) = \int_F^\infty (1 - e^{F-f})G(f)Y_m(f)df, m \geq 0, \tag{3.7}$$

where $G(F)$ is defined in (3.3). We emphasize that the series in (3.5) is not equivalent to that in (2.8).

b) Without loss of generality, we assume the constant $A > 0$. The series solution for $Y(F)$ in (3.7) can be bounded absolutely as shown in (3.8)

$$|Y(F)| \leq Ae^{\frac{5}{36F}}. \tag{3.8}$$

c) Except for $Y_0(F)$, every term in the sum in (3.5) goes to zero as $F \to +\infty$, so $Y(F) \to A$ as $F \to +\infty$.

d) The function defined by (3.4) is differentiable for $F > 0$, and the derivative, $Y'(F)$, is bounded as

$$|Y'(F)| \leq A(e^{\frac{5}{36F}} - 1). \tag{3.9}$$

e) Formally differentiating (3.4) twice leads to

$$Y''(F) = Y'(F) + G(F)Y(F). \tag{3.10}$$

It follows from (3.3), (3.8) and (3.9) that the right side of (3.10) is bounded for any $F > 0$, so $Y''(F)$ is bounded for $F > 0$ as well. In addition, (3.10) guarantees that the solution of (3.4) also satisfies (3.2). The proof of all the properties in Theorem 1 can be found in Appendix A. With these properties, we can conclude that we find a bounded, continuous and differentiable series solution to solve equation (3.2) for $F > 0$. Also, as $F \to +\infty$, this series approaches a finite constant. In addition, this series solution gives us an exact representation of the Airy function through equation (3.1). □

3.3 Analysis on the 1st term in $Y(F)$ series

In the previous section, we have shown that a convergent series solution can be found for $Y(F)$. The advantage we have over the DBH method at this level is that our series solution still contains all the information from the original equation. Meanwhile, without using Borel summation, we are not bringing any new information into our solution. In other words, the convergent series in (3.5) defines the exact solution of (3.2), without approximation. Therefore, with the correct choice of the constant A, substituting this $Y(F)$ into (3.1) gives the Airy function, also without approximation.

Let us take a close look at the 1st non-constant term in the $Y(F)$ series, $Y_1(F)$. Using the recurrent integral formula in equation (3.7), we can evaluate the $Y_1(F)$ term with asymptotic series expansion, shown in equation (3.11), by the method of integration by parts. N can be any positive integer in this case. But for the purpose of better approximation, one should choose the optimal N. Through the definition of the optimal truncation rule [10], N is the optimal index when its value makes the error term of an asymptotic series reach its minimum. In the definition, it says usually this is achieved by truncating the series so as to retain the smallest term in the series, discarding all terms of higher degree. However, in the asymptotic series we found for $Y_1(F)$, this is an unusual case. We found that for every positive F, after its value exceeds a little bit more than the mid value of two consecutive integers, the optimal term changes to the next integer. Therefore, we choose our optimal N by directly computing the error term itself, $N!Q_N(F)$. In addition, by controlling the error term, we do not lose any information from original equation.

$$
\begin{aligned}
Y_1(F;A) &= -\frac{5A}{36}\int_F^{\infty}(1-e^{F-f})f^{-2}df \\
&= \frac{5A}{36}[\sum_{n=1}^{N}(-1)^n(n-1)!F^{-n} + (-1)^{N-1}N!Q_N(F)],
\end{aligned} \tag{3.11}
$$

where

$$
N!Q_N(F) = \min_N \|N!\int_F^{\infty} e^{F-f}f^{-(N+1)}df\|. \tag{3.12}
$$

Since we want to deal with the minimum error, so we let $N!Q_N(F)$ denote the minimum of the quantity. N, in this case, means the specific value of index that achieves this minimization.

As discussed in Chapter 2, after one has truncated the (usual) asymptotic series at its point of minimal error (for that series), then Borel summation provides analytic formulae for the terms in the first "hyperasymptotic series". Without Borel summation, there are no such analytic formulae, so we must analyze the error term, given in (3.12), in more detail.

3.3.1 Bound the error term for $Y_1(F)$

We first study upper and lower bounds on the error term for $Y_1(F)$, (3.12). We focus on the case that the argument of the Airy function is on the positive axis, so the new variable F is also positive.

Theorem 2. For any finite positive F, the error term in (3.11), $N!Q_N(F)$, can be bounded as shown below

a) For the upper bound,

$$N! \int_F^\infty \frac{e^{F-f}}{f^{N+1}} df < \frac{e^{2-F}}{\sqrt{F}}. \tag{3.13}$$

b) For the lower bound,

$$\sqrt{(}2\pi)(1+\frac{\epsilon}{F})^{F+\epsilon+\frac{1}{2}} \frac{e^{-\epsilon}}{(1+\frac{1}{F})^{F+\epsilon+1}}(1-e^{-1}) \leq N!Q_N(F) \tag{3.14}$$

where $\epsilon \approx F\ln(1+\frac{1}{F})-1$, which is the leading order term in the Taylor expansion we found for ϵ. $\square$

Numerically, we plot the three functions: the upper bound, the error term, and the lower bound, over different ranges of F, shown in Figure 3.1, 3.2, and 3.3. (see Appendix B for the details of the derivation the bounds)

Theorem 2 provides strict upper and lower bounds on the dominant terms of the hyperasymptotic expansion. These bounds therefore establish part of the structure of the hyperasymptotic expansion.

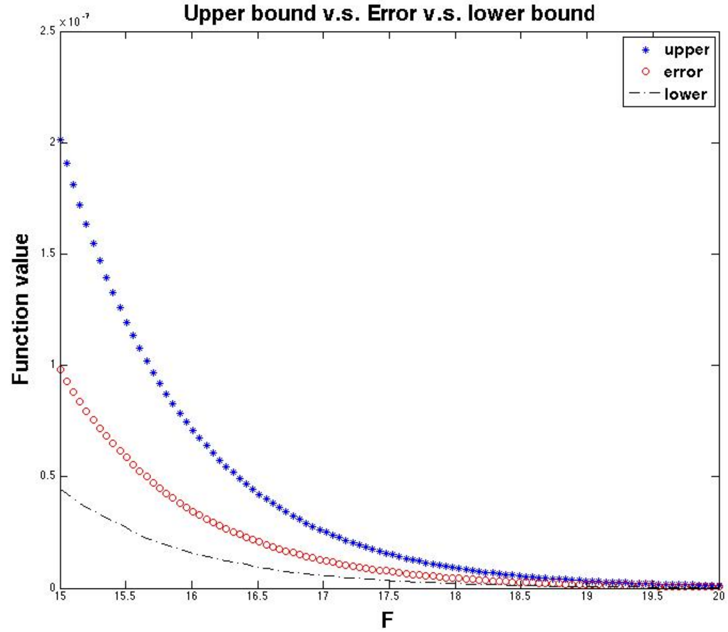

Figure 3.1: Three functions over F from 15 to 20

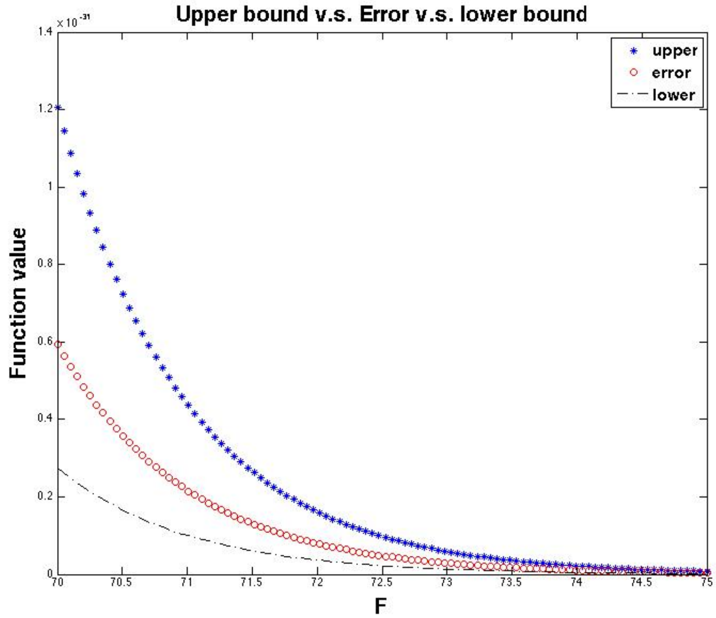

Figure 3.2: Three functions over F from 75 to 80

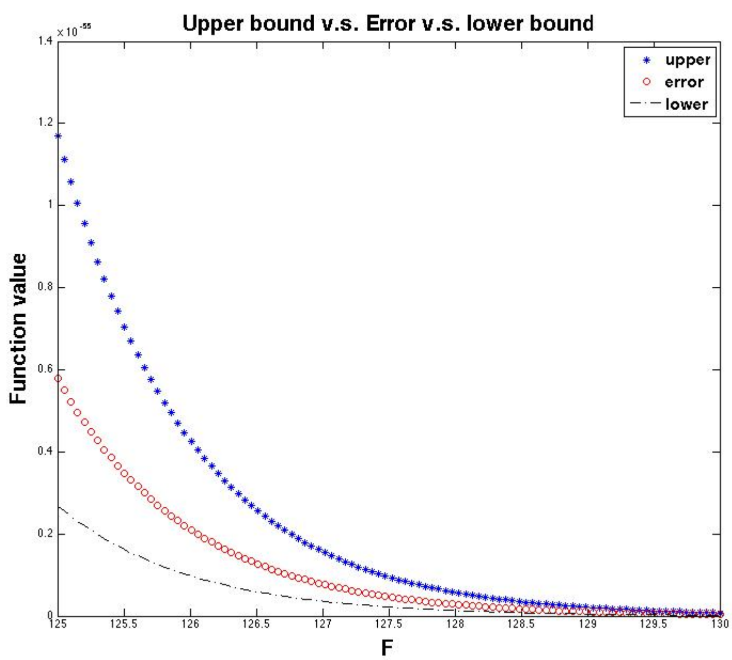

Figure 3.3: Three functions over F from 125 to 130

3.3.2 Curve fitting the leading order term for the error term

As for the error term itself, we are interested in its structure in great detail. Since we are able to compute the error term $N!Q_N(F)$ with a very high accuracy, the strategy we use to find out its behavior is curve fitting

$$N!Q_N(F) \approx \sqrt{\frac{\pi}{2}}\frac{e^{-F}}{\sqrt{F}} + \text{ Oscillation.} \tag{3.15}$$

With further manipulation among the terms, we find the error term has the oscillation behavior. We plot the ratio of the error term and the leading order term as we guess in (3.15) over F, and it shows as Figure 3.4

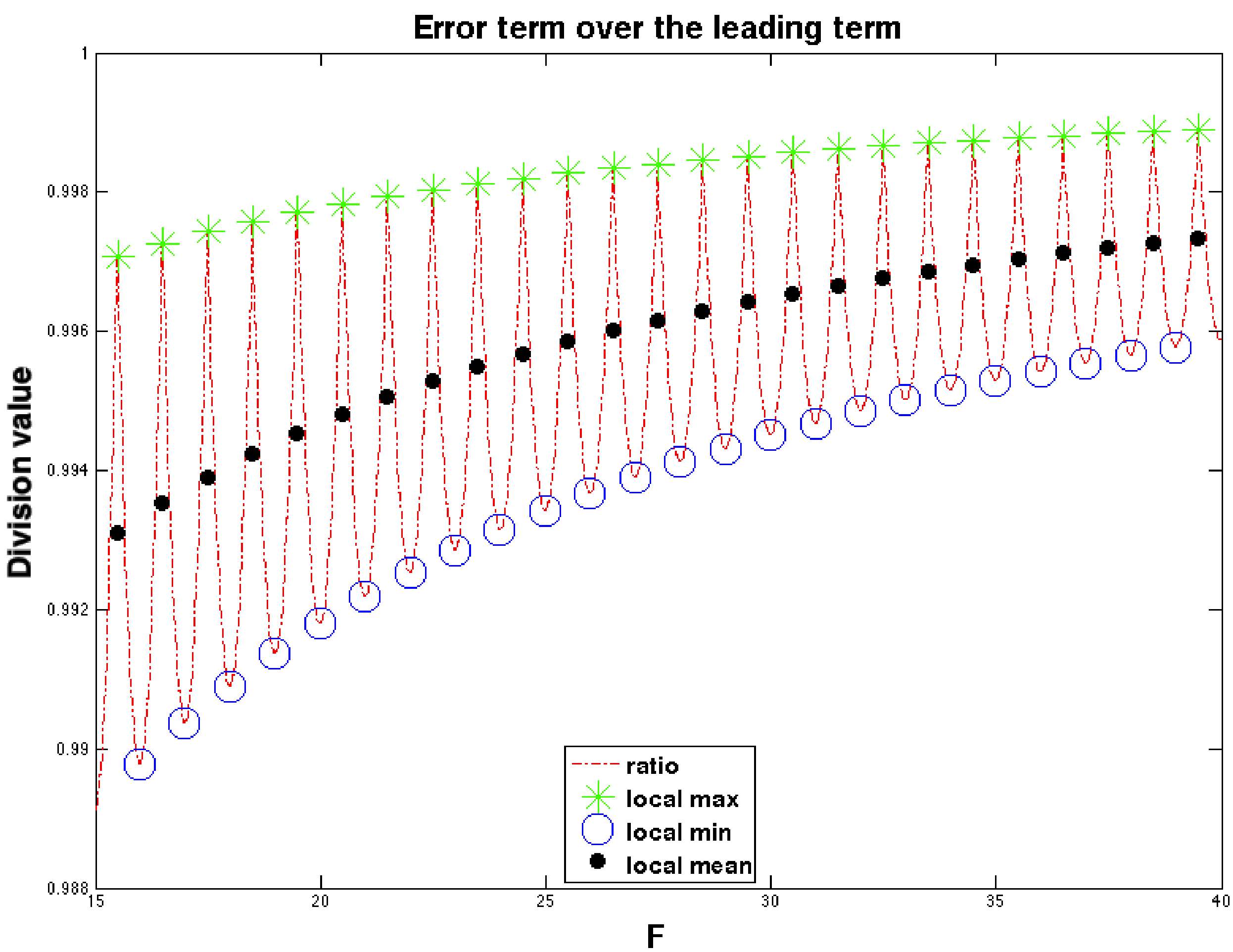

Figure 3.4: Ratio of error term and the leading order term

The second guess we have for this behavior is adding a modified sine wave into the expansion, shown in equation (3.16)

$$\frac{N!Q_N(F)}{\sqrt{\frac{\pi}{2}}\frac{\exp(-F)}{\sqrt{(F)}}} = 1 - \frac{0.12236|\sin(\pi F - \frac{\pi}{2})|}{F}. \tag{3.16}$$

The reason that we add a sine function is that the oscillation behavior shown in Figure 3.4 looks very like part of the sine function. If we compare the functions on both sides with plotting, we generate Figure 3.5

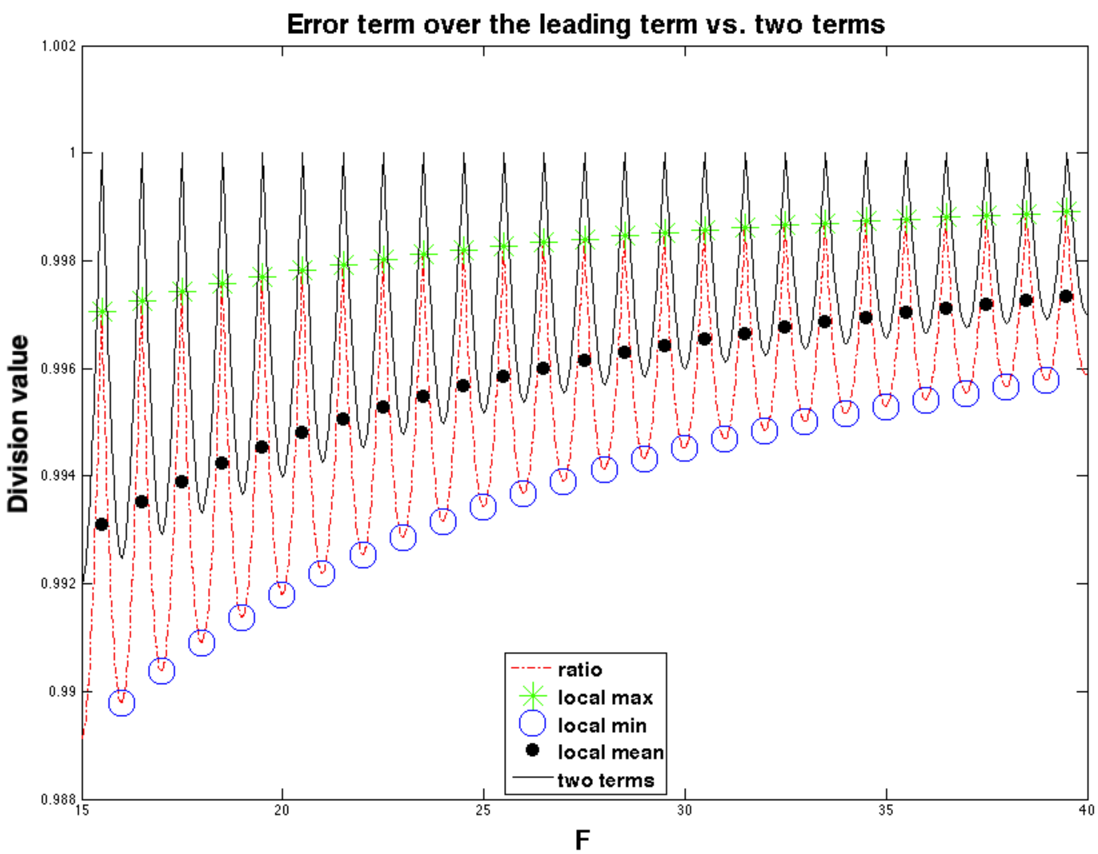

Figure 3.5: Comparison of two functions in equation (3.16).

The patterns and amplitudes between the black and red curves, shown in Figure 3.5, are very similar. Then we want shift the black curve to the red curve by finding a relation between the argument value and the local maximum, which is denoted by the green asterix in 3.5. The reason we choose the local maximum points is because the sinusoidal function we construct is a half-wave function which is not oscillating around a simple curve. But it goes back to 1 every period, so it is natural to shift this function by setting the local maximum points of the red curve as the moving target.

After some data mining, we find a linear relation, shown in Figure 3.6, between $\ln(F)$ and $\ln(\ln(R))$ where R denotes the local maximum values for each pattern of the error term over the leading term, $\frac{N!Q_N(F)}{\sqrt{\frac{\pi}{2}}\frac{\exp(-F)}{\sqrt{(F)}}}$.

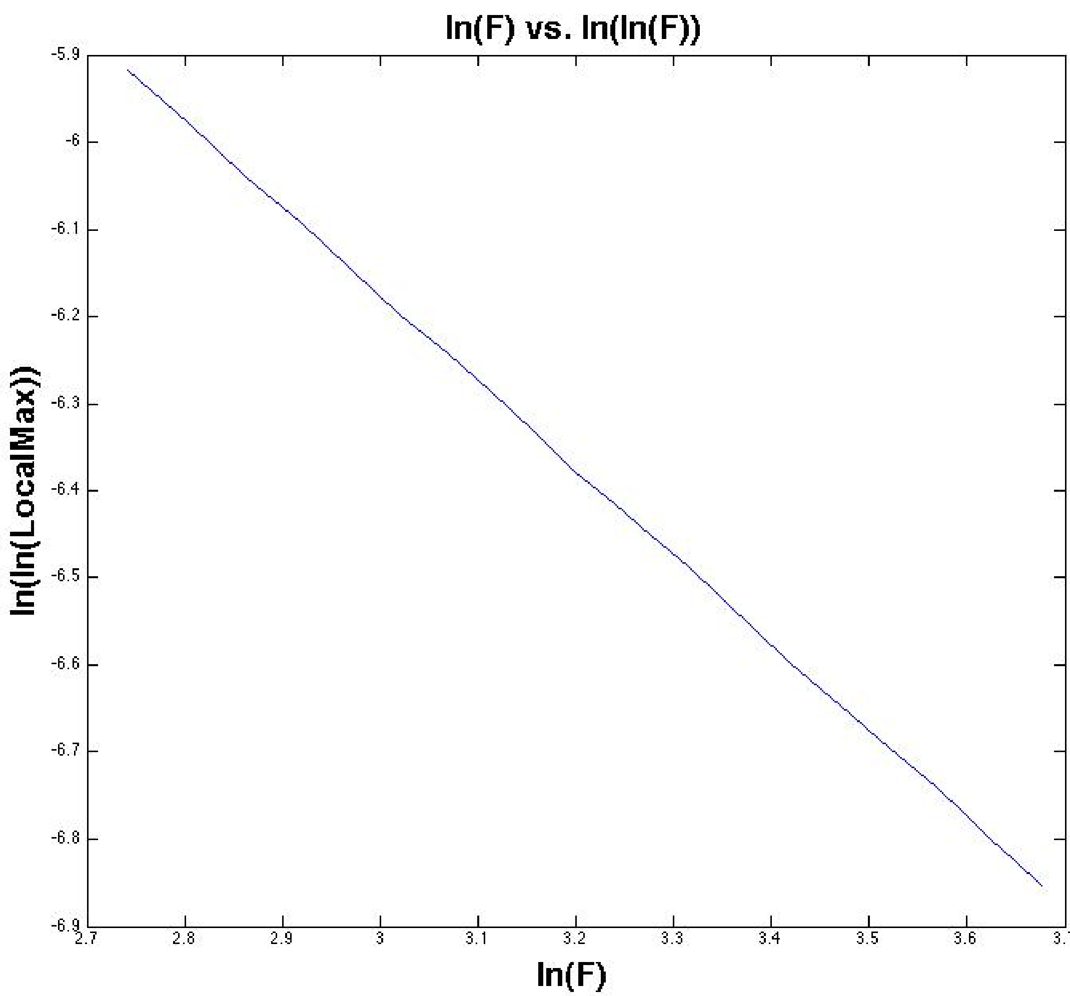

Figure 3.6: A linear relation between two functions.

We also notice that all the values of R are less than 1, which means the values of $\ln(\ln(R))$ are complex values. Also, Figure 3.6 only shows the real part values of $\ln(\ln(R))$ on the y axis. For convenience, let $\ln(F) = R_F$ and $\ln(\ln(R)) = I_{max} + R_{max}$, where I_{max} stands for the imaginary part, and R_{max} stands for the real part. Using least square fitting method, we find

$$R_{max} = -0.999 R_F - 3.178. \tag{3.17}$$

The imaginary part is very easy to find since $\ln(R)$ is only a negative real value. So I_{max} is $i\pi$ for all F. Now we can write

$$\begin{aligned}
\ln(\ln(R)) &= I_{max} + R_{max} \\
&= i\pi - 0.999 R_F - 3.178 \\
&= i\pi - 0.999 \ln(F) - 3.178.
\end{aligned} \tag{3.18}$$

Take exponential of both sides of the equation (3.18) we get

$$\ln(R) = -F^{-0.999} e^{-3.178}. \tag{3.19}$$

Do it again

$$R = e^{-F^{-0.999} e^{-3.178}}. \tag{3.20}$$

This is the line to approximate the local maximum curve, which is the green asterisk shown in Figure 3.5. Now we can adjust our model

$$\frac{N! Q_N(F)}{\sqrt{\frac{\pi}{2}} \frac{\exp(-F)}{\sqrt{(F)}}} = e^{-F^{-0.999} e^{-3.178}} - \frac{0.12236 |\sin(\pi F - \frac{\pi}{2})|}{F}. \tag{3.21}$$

Here is the final plot

The numerical data, shown in Table 3.1, indicates that we achieve 10^{-5} accuracy to approximate our error as F changes.

F	Error/Our Model
15	1.000032945
16	1.000018948
17	1.000007957
18	0.9999992537
19	0.9999923133
20	0.9999867486
21	0.9999822686
22	0.9999786518
23	0.9999757276
24	0.9999733634
25	0.9999714548
26	0.9999699191
27	0.9999686903
28	0.9999677152
29	0.9999669507
30	0.9999663616
31	0.999965919
32	0.9999655993
33	0.9999653827
34	0.9999652528
35	0.9999651959
36	0.9999652005
37	0.999965257
38	0.9999653572
39	0.9999654941
40	0.9999656619

Table 3.1: Numerical data between the error term and our model.

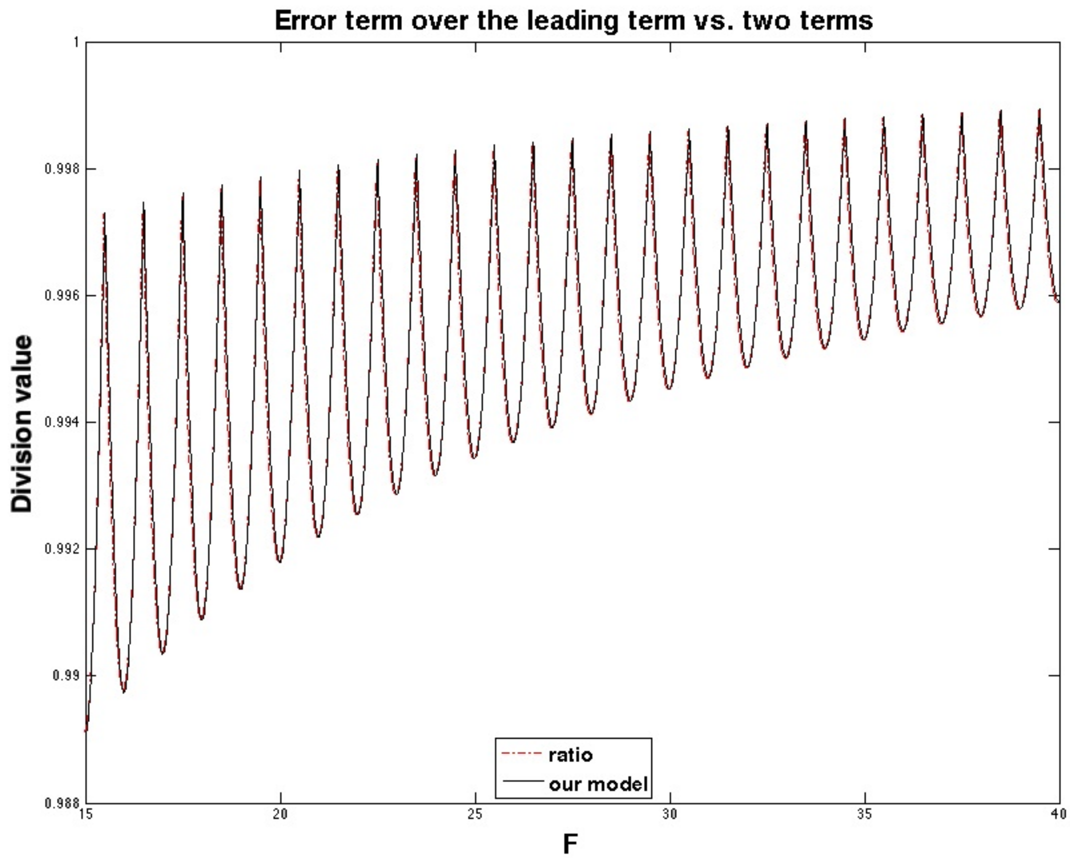

Figure 3.7: Final version of our model

In a conclusion, we construct a two-term mathematical expression as a leading representation to approximate the error term of $Y_1(F)$ in our series solution for the Airy function, $Ai(z)$. It can be written as

$$N!Q_N(F) \approx \sqrt{\frac{\pi}{2}}\frac{e^{-F}}{\sqrt{F}}\Big(e^{-F^{-0.999}e^{-3.178}} - \frac{0.12236|\sin(\pi F - \frac{\pi}{2})|}{F}\Big). \tag{3.22}$$

Chapter 4

Comparison

4.1 Algebra terms for three series are the same

For both the DBH and our method, the first step is changing variable based on the formal asymptotic series generated by the WKB method for computing the Airy function. If we compare the optimally truncated partial sum of the formal asymptotic series; the superasymptotic series of the DBH series; and the sum of the algebra terms in our series solution, they have the same format if one unifies the variable. Numerically, the three methods obtain the same accuracy when they approximate the Airy function only with these partial sums. As an example, please see the computational work of $Y_2(F)$ in Appendix C.

4.2 Later terms comparison

The comparison that we can make for the later terms is only between the DBH series and our method since the formal asymptotic series has only a divergent tail left. A main point that we discovered is the oscillation behavior hidden in our errors.

Theorem 3. The error term in $Y_1(F)$, $N!Q_N(F)$, appears in the later terms in $Y(F)$ series.

We find two properties about this error:

1) When we sum up all the terms that have the same form as $N!Q_N(F)$, they do not cancel with each other because they have the same sign and period,

2) the other smaller errors existing in the $Y(F)$ series do not affect the main error, $N!Q_N(F)$, since the summation of them is significantly smaller than $N!Q_N(F)$ as F goes large. $\square$

Combining with what we have shown numerically in Figure 3.4, 3.5 and 3.7 in Chapter 3 that the error in $Y_1(F)$, $N!Q_N(F)$, has the oscillation behavior, we conclude that there is a type of oscillation phenomenon in our series. Please see Appendix D for the proof in detail.

From the definition of hyperasymptotic series in Chapter 1, we know for the DBH method, our method, as well as many other approaches in the literature, to construct a hyperasymptotic series, the first step is to truncate the asymptotic series near the optimal truncated term. Then to analyze the error terms with some approximation methods. For example, for the DBH method, it uses Borel summation to make sense out of the divergent tail; also in the book [10], the example uses Euler acceleration method to deal with the error; for our method, it is curve fitting method to approximate the error.

For all of these methods, the optimal N is a new variable that depends on F, and that is introduced into the hyperseries. In other words, in the appended series, there are two variables,N and F. Since N takes only integer values, N changes discretely (changing by 1 each time), whereas F changes continuously. The result is small oscillations, with a period of approximately 1, in the hyperseries. These oscillations are not part of the original function (the Airy function, for the specific example under discussion), but are introduced by truncating the original asymptotic series at an integer N that is optimal for each F.

Chapter 5

Conclusion

We analyze the 1D Helmholtz equation asymptotically by constructing a new series solution. Comparing with the DBH method, we transform the original problem to the new ODE exactly without losing any information. Also, we show that the solution of our ODE is represented (exactly) by a bounded, convergent series. Certainly this series is different from the DBH hyperasymptotic series. In addition, each term in our convergent series can be represented by a hyperasymptotic series as $F \to \infty$, and the hyperasymptotic series obtained in this way is also different from the DBH hyperasymptotic series.

In addition, we discover that the hyperasymptotic series that we construct to approximate the Airy function contains small oscillations, even though $Ai(z)$ itself is not oscillatory for $z > 0$. We conclude that this phenomenon occurs commonly when constructing a hyperasymptotic series for a smooth, non-oscillatory function.

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Appendix A

Proof of Theorem 1

To prove all the properties listed in Theorem 1, we start with solving the ODE that we obtain

$$Y''(F) - Y'(F) - G(F)Y(F) = 0 \tag{A.1}$$

where $G = -\frac{5}{36F^2}$. Since $G(F)$ is not a constant, to reduce the difficulty of solving this ODE, as the argument F goes large, we approximate the ODE as:

$$Y''(F) - Y'(F) = 0. \tag{A.2}$$

We will bring the missing term back in later. Now to solve equation (A.2), we let

$$Y(F) = A + Be^F, \tag{A.3}$$

By the method of variation of parameter, we replace the equation (A.3) with

$$Y(F) = A(F) + B(F)e^F. \tag{A.4}$$

Here we actually introduced two unknown functions, $A(F)$ and $B(F)$. To solve $Y(F)$ function, we are allowed to impose one extra condition on these two unknown functions to help us. Set up the boundary condition as

$$B(\infty) = 0, \text{ and } Y(\infty) = A(\infty) = A, \tag{A.5}$$

where A is some constant. Then we differentiate equation (A.4) and get

$$Y'(F) = A'(F) + B'(F)e^F + B(F)e^F. \tag{A.6}$$

Without loss of generality, equation (A.6) also has to equal to the derivative of equation (A.3), so we set

$$A'(F) + B'(F)e^F = 0. \tag{A.7}$$

This is the extra condition we impose. Also we can obtain the second derivative of $Y(F)$

$$Y''(F) = B'(F)e^F + B(F)e^F.$$

Now if we plug all these back to equation (A.1), we have

$$0 = B'(F)e^F - G(F)Y(F) \tag{A.8}$$

$$= B'(F)e^F + \frac{5}{36F^2}(A(F) + B(F)e^F) \tag{A.9}$$

and now if we consider both equation (A.7) and (A.8), we are able to solve $A(F)$ and $B(F)$

$$A'(F) = -B'(F)e^F \tag{A.10}$$

$$= -G(F)Y(F) \tag{A.11}$$

and

$$B'(F) = G(F)Y(F)e^{-F}. \tag{A.12}$$

After we integrate both equation (A.11) and (A.12) from $F \to \infty$, we have

$$A(F) = A(\infty) + \int_F^\infty G(f)Y(f)df \tag{A.13}$$

$$B(F) = B(\infty) - \int_F^\infty G(f)Y(f)e^{-f}df. \tag{A.14}$$

Since $B(\infty)$ is zero and $A(\infty) = A$, we can plug equation (A.13),(A.14) to (A.4), and get the integral equation for $Y(F)$ as

$$Y(F) = A + \int_F^\infty (1 - e^{F-f})G(f)Y(f)df. \tag{A.15}$$

This is the same as equation (3.4). Now, we can construct a bounded series representation for $Y(F)$ from (3.4). We first let

$$Y(F) = Y_0(F) + \sum_{m=1}^{\infty} Y_r(F) \tag{A.16}$$

where $Y_0(F) = A$ and for $n \geq 0$:

$$Y_{m+1}(F) = \int_F^\infty (1 - e^{F-f})G(f)Y_m(f)df. \tag{A.17}$$

This is the recurrence equation (3.7) in property a) in Theorem 1 comes from. Now we can use induction to prove that the series is bounded

$$|Y_1(F)| = \Big|\frac{5A}{36}\int_F^\infty (1 - e^{F-f})f^{-2}df\Big| \tag{A.18}$$

$$\leq \frac{5A}{36}\int_F^\infty f^{-2}df \tag{A.19}$$

$$\leq \frac{5A}{36F}. \tag{A.20}$$

Let's do one more to find out the pattern,

$$|Y_2(F)| = \left| \frac{5}{36} \int_F^\infty (1 - e^{F-f}) f^{-2} Y_1(f) df \right| \tag{A.21}$$

$$\leq \frac{5^2 A}{36^2} \int_F^\infty f^{-3} df \tag{A.22}$$

$$\leq \frac{5^2 A}{36^2 2F^2}. \tag{A.23}$$

So now, we predict that the general formula to bound the nth term should be

$$|Y_n(F)| \leq A \frac{5^n}{(36F)^n n!}. \tag{A.24}$$

Let's test the (n+1)th term

$$|Y_{n+1}(F)| = \frac{5}{36} \int_F^\infty (1 - e^{F-f}) f^{-2} Y_n(f) df \tag{A.25}$$

$$\leq \frac{5^{n+1} A}{36^{n+1} n!} \int_F^\infty f^{-n-2} df \tag{A.26}$$

$$\leq \frac{5^{n+1} A}{(36F)^{n+1} (n+1)!}. \tag{A.27}$$

One can bound the entire series for $Y(F)$ by summing the bounds for each term, so we write

$$|Y(F)| \leq \sum_{n=0}^{\infty} \frac{5^n A}{(36F)^n (n)!}. \tag{A.28}$$

The right hand side is the Taylor series of exponential function, so we can rewrite it as

$$|Y(F)| \leq A e^{\frac{5}{36F}}. \tag{A.29}$$

That is the same inequality as in (3.8). So we have proved property b). As $F \to +\infty$, the bound of $|Y(F)| \to A$. That is the conclusion of property c). Further, we can find the first and second order derivative of $Y(F)$ are bounded as well. Now we can start from equation

(A.15), and take derivative of both sides and get:

$$
\begin{aligned}
|Y'(F)| &= \left| \frac{d}{dF} \frac{5}{36} \int_F^{\infty} (1 - e^{F-f}) f^{-2} Y(f) df \right| && \text{(A.30)} \\
&\leq \frac{5}{36} \int_F^{\infty} e^{F-f} f^{-2} |Y(f)| df && \text{(A.31)} \\
&\leq \frac{5}{36} \int_F^{\infty} f^{-2} |Y(f)| df && \text{(A.32)} \\
&\leq \frac{5A}{36} \int_F^{\infty} f^{-2} e^{\frac{5}{36f}} df. && \text{(A.33)}
\end{aligned}
$$

If we change variable, let $\frac{1}{f} = u$, so $du = -\frac{1}{f^2}$ then we have

$$
|Y'(F)| \leq \frac{5A}{36} \int_0^{1/F} e^{\frac{5u}{36}} du. \tag{A.34}
$$

Therefore,

$$
|Y'(F)| \leq A(e^{\frac{5}{36F}} - 1). \tag{A.35}
$$

This is the same inequality as in (3.9), so property d) is proved. Now we do it again. Differentiate the exact expression for $Y'(F)$, we will have

$$
Y''(F) = Y'(F) - \frac{5}{36F^2} Y(F) \tag{A.36}
$$

As we have proved before, the right hand side of this equation is absolutely bounded. Therefore, we are guaranteed that the second derivative of $Y(F)$ is also bounded. Property e) is proved here.

Appendix B

Proof of Theorem 2

The way to bound the error term is separate it into two parts first, and then find the bounds for each of them. Recall the error term:

$$N! \int_F^\infty \frac{e^{F-f}}{f^{N+1}} df. \tag{B.1}$$

It can be separated into the factorial term and the integral term. Let's first find the upper bound for the error term. For the factorial term, we use a modified version of the Stirling approximation inequality

$$N! < eN^{N+\frac{1}{2}} e^{-N}. \tag{B.2}$$

This inequality works when N is any positive integer which is what we want. The original Stirling approximation inequality [1] is

$$\Gamma(x) < \sqrt{\frac{2\pi}{x}} x^x e^{-x} e^{\frac{1}{12x}} \tag{B.3}$$

where $x > 0$. The proof from (B.3) to (B.2) is very easy. Assume x to be any positive integer, then $\Gamma(x) = (x-1)!$. Since we know the maximum value of $\sqrt{2\pi}e^{\frac{1}{12x}}$ is when $x = 1$. Therefore, one can find the inequality in (B.2).

As for the integral term, an upper bound is obtained by replacing $\frac{1}{f^{N+1}}$ by its maximum value in $F < f < \infty$

$$\int_F^\infty \frac{e^{F-f}}{f^{N+1}} df < \frac{1}{F^{N+1}} \int_F^\infty e^{F-f} df = \frac{1}{F^{N+1}}. \tag{B.4}$$

As it is mentioned in the paragraph below equation (3.12), the choice of N is either the floor or the ceil of F. Therefore, let's define a general case. Let

$$N = F + \epsilon \text{ , where } |\epsilon| < 1. \tag{B.5}$$

Putting the two separated terms (B.2) and (B.4) together now with their bounds, as well as the new variable, we can write

$$N! \int_F^\infty \frac{e^{F-f}}{f^{N+1}} df < e\left(\frac{F+\epsilon}{F}\right)^{F+\epsilon+\frac{1}{2}} \frac{e^{-F-\epsilon}}{F^{\frac{1}{2}}}. \tag{B.6}$$

As we look through the computation results of the error term with 20 decimal points accuracy, we guess the upper bound should have the main form of $C\frac{e^{-F}}{\sqrt{F}}$, where C is a constant. So the upper bound that we are looking for can be written as

$$e\left(\frac{F+\epsilon}{F}\right)^{F+\epsilon+\frac{1}{2}} \frac{e^{-F-\epsilon}}{F^{\frac{1}{2}}} \leq C\frac{e^{-F}}{\sqrt{F}}. \tag{B.7}$$

Now we can focus on finding the value of C that can satisfy the conditions of $|\epsilon| < 1$ and $F \geq 3$. The choice of 3 is because we want F to be away from 0, also be as general as possible. Cancel the main form on both sides, the inequality can be written as

$$\left(\frac{F+\epsilon}{F}\right)^{F+\epsilon+\frac{1}{2}} e^{1-\epsilon} \leq C \tag{B.8}$$

$$\left(1+\frac{\epsilon}{F}\right)^{F+\epsilon+\frac{1}{2}} \leq Ce^{\epsilon-1}. \tag{B.9}$$

Now we can discuss different cases in terms of the value of ϵ.

For $\epsilon = 0$,

$$1 \leq C * e^{-1} \quad \Rightarrow e \leq C \tag{B.10}$$

For $-1 < \epsilon < 0$,

$$(1 - \frac{|\epsilon|}{F})^{F+\epsilon+\frac{1}{2}} \leq Ce^{\epsilon-1}. \tag{B.11}$$

For the right hand side,

$$\inf\{Ce^{\epsilon-1}\} = Ce^{-2}. \tag{B.12}$$

For the left hand side

$$(1 - \frac{|\epsilon|}{F})^{F+\epsilon+\frac{1}{2}} \leq 1 \leq Ce^{-2} \tag{B.13}$$

$$\Rightarrow e^2 \leq C \tag{B.14}$$

For $0 < \epsilon < 1$, since $F \geq 3$, so let

$$\frac{\epsilon}{F} = \frac{1}{H}, \tag{B.15}$$

then we write

$$(1 + \frac{\epsilon}{F})^{F+\epsilon+\frac{1}{2}} = (1 + \frac{1}{H})^{H\epsilon+\epsilon+\frac{1}{2}} = \left[(1 + \frac{1}{H})^H\right]^\epsilon (1 + \frac{1}{H})^{\epsilon+\frac{1}{2}}. \tag{B.16}$$

For the expression in the square bracket in (B.16), since we set $F \geq 3$, using (B.15) we get $\min H = 3$. Also, recall a definition of the irrational number e

$$\lim_{H\to\infty} (1 + \frac{1}{H})^H = e. \tag{B.17}$$

Therefore,

$$\max_{H\geq 3}[(1 + \frac{1}{H})^H]^\epsilon = e. \tag{B.18}$$

For the expression outside the square bracket in (B.16), since $1 < (1 + \frac{1}{H}) \leq \frac{4}{3}$, so

$$\max(1 + \frac{1}{H})^{\epsilon + \frac{1}{2}} = (\frac{4}{3})^{\epsilon + \frac{1}{2}} < (\frac{4}{3})^{1.5} < e^2 \tag{B.19}$$

Overall, by comparing all the upper bounds across three different conditions for the value of ϵ, we can narrow down the cases to (B.10), (B.14), and (B.19). We find that the value of C that satisfies different values of ϵ and F is e^2. So we can write the equation (B.6) as

$$N! Q_N(F) < \frac{e^{2-F}}{\sqrt{F}}. \tag{B.20}$$

This is the same as (3.13).

For the lower bound, we still use the Stirling approximation to bound the factorial term. As Robbins remarked on the Stirling's formula [14], the lower bound can be written as

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} < n! \tag{B.21}$$

where n is a positive integer. To simplify the later computation, we modify the bound and get

$$\sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N} < N!. \tag{B.22}$$

For the integral, we change the upper limit of the integral to $F+1$ so that the value will be less than the it was when the limit is infinity.

$$\int_F^\infty \frac{e^{F-f}}{f^{N+1}} df > \int_F^{F+1} \frac{e^{F-f}}{f^{N+1}} df > \frac{1}{(F+1)^{N+1}} (1 - e^{-1}). \tag{B.23}$$

We still use the same change variable strategy as before, $N = F + \epsilon$. Combine the two lower bounds and obtain

$$N! \int_F^\infty \frac{e^{F-f}}{f^{N+1}} df > \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N} \frac{1}{(F+1)^{N+1}} (1-e^{-1}) \tag{B.24}$$

$$= \sqrt{2\pi}(1-e^{-1}) \Big[(1+\frac{\epsilon}{F})^{F+\epsilon+\frac{1}{2}} \frac{e^{-\epsilon}}{(1+\frac{1}{F})^{F+\epsilon+1}} \Big]. \tag{B.25}$$

To find the lower bound of the error term, we can find the minimum value of the function in equation (23) in terms of ϵ. To be convenient, let's remove the constant, and set the rest as a new function $P(\epsilon, F)$

$$P(\epsilon, F) = (1+\frac{\epsilon}{F})^{F+\epsilon+\frac{1}{2}} \frac{e^{-F-\epsilon}}{F^{\frac{1}{2}}(1+\frac{1}{F})^{F+\epsilon+1}}. \tag{B.26}$$

Take the first derivative with respect to ϵ, we obtain

$$\frac{\partial P}{\partial \epsilon} = (1+\frac{\epsilon}{F})^{F+\epsilon+\frac{1}{2}} \frac{e^{-F-\epsilon}}{F^{\frac{1}{2}}(1+\frac{1}{F})^{F+\epsilon+1}} \Big[\ln(1+\frac{\epsilon}{F}) + \frac{F+\epsilon+\frac{1}{2}}{F+\epsilon} - 1 - \ln(1+\frac{1}{F}) \Big] \tag{B.27}$$

$$= (1+\frac{\epsilon}{F})^{F+\epsilon+\frac{1}{2}} \frac{e^{-F-\epsilon}}{F^{\frac{1}{2}}(1+\frac{1}{F})^{F+\epsilon+1}} \Big[\ln(\frac{F+\epsilon}{1+F}) + \frac{1}{2(F+\epsilon)} \Big]. \tag{B.28}$$

Set $\frac{\partial P}{\partial \epsilon} = 0$ for $F \geq 3$, and $|\epsilon| < 1$. We first notice that if $\epsilon = 1$, then

$$\ln(\frac{F+\epsilon}{1+F}) = 0 \Rightarrow \frac{\partial P}{\partial \epsilon} > 0. \tag{B.29}$$

If $\epsilon = 0$, then

$$\ln(\frac{F}{1+F}) + \frac{1}{2F} < 0 \Rightarrow \frac{\partial P}{\partial \epsilon} > 0 \tag{B.30}$$

for $F > 3$. We also know that the function $P(\epsilon, F)$ is continuous with the restriction of $|\epsilon| < 1$ and $F > 3$. Therefore, we conclude that there must be at least one non-trivial value of ϵ that makes the equation $\frac{\partial P}{\partial \epsilon} = 0$ exist. We know the product outside of the square bracket is always positive under the same condition. Now let's focus on solving the function inside the bracket.

$$\ln(\frac{F+\epsilon}{1+F}) + \frac{1}{2(F+\epsilon)} = 0 \tag{B.31}$$

for $F \geq 3$ and $|\epsilon| < 1$, so $|\frac{\epsilon}{F}| < \frac{1}{3}$. Rewrite equation (29), we obtain

$$\ln(1+\frac{1}{F}) - \ln(1+\frac{\epsilon}{F}) = \frac{1}{2F(1+\frac{\epsilon}{F})}. \tag{B.32}$$

Expand $\ln(1+\frac{\epsilon}{F})$, we have

$$\ln(1+\frac{\epsilon}{F}) = \frac{\epsilon}{F} - \frac{\epsilon^2}{2F^2} + \frac{\epsilon^3}{3F^3} - \dots. \tag{B.33}$$

Expand $\frac{1}{2F(1+\frac{\epsilon}{F})}$, we have

$$\frac{1}{2F(1+\frac{\epsilon}{F})} = \frac{1}{2F}(1 - \frac{\epsilon}{F} - (\frac{\epsilon}{F})^2 + \dots) \tag{B.34}$$

$$= \frac{1}{2F} + \frac{\epsilon^2}{2F^2} + \frac{\epsilon^3}{2F^3} + \dots. \tag{B.35}$$

So equation (30) can be written as

$$\ln(1+\frac{1}{F}) - \frac{1}{2F} = (1-\frac{1}{2F})(\frac{\epsilon}{F}) - \frac{1}{2}(1-\frac{1}{F})(\frac{\epsilon}{F})^2 + (\frac{1}{3} - \frac{1}{2F})(\frac{\epsilon}{F})^3 + \dots. \tag{B.36}$$

For $F \to$ large, we use the first term in equation (34) to approximate $(\frac{\epsilon}{F})$ and get

$$\frac{\epsilon}{F} \sim \frac{(\ln(1+\frac{1}{F}) - \frac{1}{2F})}{1-\frac{1}{2F}} \tag{B.37}$$

$$= \frac{2F\ln(1+\frac{1}{F}) - 1}{2F-1}. \tag{B.38}$$

If we want higher accuracy for the location of the minimum value, we can take as many terms as we want. Let's try one more term,

$$(1-\frac{1}{2F})(\frac{\epsilon}{F}) \sim (\ln(1+\frac{1}{F})-\frac{1}{2F}) + \frac{\frac{1}{2}(1-\frac{1}{F})(2F\ln(1+\frac{1}{F})-1)^2}{(2F-1)^2} \tag{B.39}$$

which implies

$$\frac{\epsilon}{F} \sim \frac{(\ln(1+\frac{1}{F})-\frac{1}{2F})}{(1-\frac{1}{2F})} + \frac{(1-\frac{1}{F})(2F\ln(1+\frac{1}{F})-1)^2}{(2F-1)^3} \tag{B.40}$$

Therefore, we conclude that the lower bound for the error term is

$$\sqrt(2\pi)(1+\frac{\epsilon}{F})^{F+\epsilon+\frac{1}{2}}\frac{e^{-\epsilon}}{(1+\frac{1}{F})^{F+\epsilon+1}}(1-e^{-1}) \leq N!Q_N(F) \tag{B.41}$$

where

$$\epsilon \approx F\ln(1+\frac{1}{F})-1 \tag{B.42}$$

Appendix C

Computation of $Y_2(F)$

Using the recurrence formula, (3.7), we can compute the later terms in the $Y(F)$ that solves our ODE, (3.2). We show the computational work for $Y_2(F)$.

$$\begin{aligned} Y_2(F) &= -\frac{5}{36}\int_F^\infty (1-e^{F-f})f^{-2}Y_1(f)df \\ &= (\frac{5}{36})^2\int_F^\infty (1-e^{F-f})f^{-2}\Big[\sum_{n=0}^{N-1} n!(-1)^n f^{-(n+1)} \\ &+ (-1)^N N!\int_f^\infty e^{f-t}t^{-N-1}dt\Big]df \end{aligned} \tag{C.1}$$

where N is the same optimal N from the series of $Y_1(F)$. For convenience, let us write (C.1) as

$$Y_2(F) = (\frac{5}{36})^2(Y_{2a}(F) + Y_{2b}(F) + Y_{2c}(F) + Y_{2d}(F)). \tag{C.2}$$

Each part is addressed below.

$$\begin{aligned} Y_{2a}(F) &= \int_F^\infty f^{-2} \sum_{n=0}^{N-1} n!(-1)^n f^{-(n+1)} df \\ &= \sum_{n=0}^{N-1} n!(-1)^n \int_F^\infty f^{-(n+3)} df \\ &= \sum_{n=0}^{N-1} \frac{n!(-1)^n}{(n+2)F^{n+2}} \end{aligned} \tag{C.3}$$

$$\begin{aligned} Y_{2b}(F) &= \int_F^\infty f^{-2}(-1)^N N! \Big(\int_f^\infty e^{f-t} t^{-N-1} dt \Big) df \\ &= (-1)^N N! \int_F^\infty f^{-2} \int_f^\infty e^{f-t} t^{-N-1} dt df \\ &= (-1)^N N! \int_F^\infty f^{-2} Q_N(f) df. \end{aligned} \tag{C.4}$$

$$\begin{aligned} Y_{2c}(F) = &- \int_F^\infty e^{F-f} f^{-2} \sum_{n=0}^{N-1} n!(-1)^n f^{-(n+1)} df \\ &= \sum_{n=0}^{N-1} n!(-1)^{n+1} \int_F^\infty e^{F-f} f^{-(n+3)} df \\ &= \sum_{n=0}^{N-1} n!(-1)^{n+1} \Big[F^{-(n+3)} + \sum_{r=0}^{N_2-n-4} (-1)^{r+1} \prod_{i=0}^{r} (n+3+i) F^{-(n+4+r)} + \dots \\ &+ (-1)^{N_2-n-2} \prod_{i=0}^{N_2-n-3} (n+3+i) Q_{N_2}(F) \Big] \end{aligned} \tag{C.5}$$

where N_2 is the new index for the optimally truncated term in each series in (C.5). For the last term, we interchange two integrals so that the expression will be simplified.

$$
\begin{aligned}
Y_{2d}(F) &= -\int_F^\infty e^{F-f} f^{-2} (-1)^N N! \int_f^\infty e^{f-t} t^{-N-1} dt df \\
&= (-1)^{N+1} N! \int_F^\infty e^{F-t} t^{-N-1} \int_f^t f^{-2} df dt \\
&= (-1)^{N+1} N! \Big[\frac{-1}{(N+1)F^{N+1}} + \frac{N+1+F}{(N+1)F} \int_F^\infty e^{F-f} f^{-(N+1)} df \Big] \\
&= (-1)^{N+1} N! \Big[\frac{-1}{(N+1)F^{N+1}} + \frac{N+1+F}{(N+1)F} Q_N(f) \Big]
\end{aligned} \tag{C.6}
$$

Appendix D

Proof of Theorem 3

There are two things we want to show. The first one is to show when the integral involves the main error term, $N!Q_N(F)$, then the result will be significantly smaller than the main error term itself as F goes large. Also, when summing up all these small error terms, the result is still too small to affect the oscillation behavior from the main errors. I use the notation SE_i to denote these small errors show up in the i-th term in the $Y(F)$ series. Recall the recurrence formula

$$Y_{m+1}(F) = -\frac{5}{36}\int_F^\infty (1-e^{F-f})f^{-2}Y_m(f)df. \tag{D.1}$$

Now if we want to compute the small error in $Y_2(F)$, we have

$$\begin{aligned} SE_2 &= (\frac{5}{36})^2 \int_F^\infty (1-e^{F-f})f^{-2}\Big[(-1)^N N! \int_f^\infty e^{f-t}t^{-N-1}dtdf\Big] \\ &= (\frac{5}{36})^2 \int_F^\infty (1-e^{F-f})f^{-2}\Big[(-1)^N N! Q_N(f)df\Big]. \end{aligned} \tag{D.2}$$

Now we can bound SE_2 by using the bound we found for the main error term, $N!Q_N(F)$, in Theorem 2. Recall the upper bound of the main error term

$$N! \int_F^\infty \frac{e^{F-f}}{f^{N+1}} df = N!Q_N(F) < \frac{e^{2-F}}{\sqrt{F}}. \tag{D.3}$$

Then we have

$$\begin{aligned} |SE_2| &= |(\frac{5}{36})^2 \int_F^\infty (1-e^{F-f}) f^{-2} (-1)^N N! \int_f^\infty e^{f-t} t^{-N-1} dt df| \\ &< |(\frac{5}{36})^2 \int_F^\infty f^{-2} N! \int_f^\infty e^{f-t} t^{-N-1} dt df| \\ &< |(\frac{5}{36})^2 \int_F^\infty f^{-5/2} e^{2-f} df| \\ &< (\frac{5}{36})^2 e^{2-F} F^{-5/2} \\ &= (\frac{5}{36})^2 \frac{e^{2-F}}{F^{5/2}}. \end{aligned} \tag{D.4}$$

Similarly, we can bound the small error in the later terms. For $Y_3(F)$, we have

$$\begin{aligned} |SE_3| &= |(\frac{5}{36})^2 \int_F^\infty (1-e^{F-f}) f^{-2} SE_2 df| \\ &< |(\frac{5}{36})^3 \int_F^\infty f^{-2} f^{-5/2} e^{2-f} df| \\ &< (\frac{5}{36})^3 e^{2-F} F^{9/2} \\ &= (\frac{5}{36})^3 \frac{e^{2-F}}{F^{9/2}}. \end{aligned} \tag{D.5}$$

We found the pattern can be generalized as

$$|SE_n| < (\frac{5}{36})^n \frac{e^{2-F}}{F^{1/2+2(n-1)}}. \tag{D.6}$$

The sum can be bounded by using geometric sum formula

$$
\begin{aligned}
|\sum_{n=2}^{\infty} SE_n| &< (\frac{5}{36})^2 \frac{e^{2-F}}{F^{5/2}} \sum_{n=0}^{\infty} (\frac{5}{36})^n F^{-2n} \\
&= (\frac{5}{36})^2 \frac{e^{2-F}}{F^{5/2}} \frac{1}{1-(\frac{5}{36})F^{-2}} \\
&= \left[(\frac{5}{36})^2 \frac{e^2}{F^2 - \frac{5}{36}}\right] \frac{e^{-F}}{\sqrt{F}}.
\end{aligned}
\tag{D.7}
$$

As F goes large, the bound of the sum is much smaller than the leading order of the main error, $\sqrt{\frac{\pi}{2}}\frac{e^{-F}}{\sqrt{F}}$, which we developed in Chapter 3.

The second thing we want to show is the main errors in the later terms of $Y(F)$ series have the same phase and same sign. By showing this, we can conclude that when we sum up all the main errors, the oscillatory behavior remains, because there is no cancellation. We know that the main error term is generated when the algebra terms in the previous terms of $Y(F)$ series are involved in the integration. For the same phase, if we truncate the series when the index reaches the same value as the optimal N from $Y_1(F)$ series, then we will have the same form of the error term, $N!Q_N(F)$, so that the phase will be the same. For the same sign, recall the computation of $Y_2(F)$ in Appendix C, there are two parts for algebra term computation. The first part generates a new partial sum

$$
\begin{aligned}
Y_{2a}(F) &= \int_F^{\infty} f^{-2} \sum_{n=0}^{N-1} n!(-1)^n f^{-(n+1)} df \\
&= \sum_{n=0}^{N-1} n!(-1)^n \int_F^{\infty} f^{-(n+3)} df \\
&= \sum_{n=0}^{N-1} \frac{n!(-1)^n}{(n+2)F^{n+2}}
\end{aligned}
\tag{D.8}
$$

which has no effect on the oscillation behavior. The second part is

$$
\begin{aligned}
Y_{2c}(F) &= -\int_F^\infty e^{F-f} f^{-2} \sum_{n=0}^{N-1} n!(-1)^n f^{-(n+1)} df \\
&= \sum_{n=0}^{N-1} n!(-1)^{n+1} \int_F^\infty e^{F-f} f^{-(n+3)} df \\
&= \sum_{n=0}^{N-1} n!(-1)^{n+1} \Big[F^{-(n+3)} + \sum_{r=0}^{N_2-n-4} (-1)^{r+1} \prod_{i=0}^{r} (n+3+i) F^{-(n+4+r)} + \dots \\
&+ (-1)^{N_2-n-2} \prod_{i=0}^{N_2-n-3} (n+3+i) Q_{N_2}(F) \Big]
\end{aligned} \tag{D.9}
$$

where N_2 is the new index for the optimally truncated term in each series in (D.9). We notice that the algebra term in $Y_2(F)$ series has form that is similar with it is in $Y_1(F)$. The first difference is the power of F starts at $-(n+3)$ instead of $-(n+1)$. The other difference is that the first term in $Y_1(F)$ series, F^{-1}, starts with a minus sign; whereas in $Y_{2c}(F)$ series, the first term in the square bracket always has a plus sign. Therefore, if we let the partial sum be truncated at the same index value as the optimal N in $Y_1(F)$ series, then the number of terms in this series is one less than it is in the partial sum in $Y_1(F)$ series. Also, recall that the sign of the error term in $Y_1(F)$ is $(-1)^{N-1}$. In equation (D.9), let us first consider the case when $n = 0$. The sign outside the square bracket in equation (D.9) is a minus, and since the series starts at F^{-3}, so when we truncate the series, the number of terms will be $N-3$. The sign of the error term in the bracket will be -1 multipy $(-1)^{N-3}$. In addition to the reverse sign of the first term, the final sign of the error term is the same as $(-1)^{N-1}$. When $n = 1$, the sign outside the bracket is a plus, and F starts at F^{-4}, so there are $N-4$ terms in the optimal truncated series in this case. So the sign of the error term in the bracket is plus multiplied by $(-1)^{N-4}$. In addition to the reverse sign of the first term, the final sign of the error term is the same as $(-1)^{N-1}$. This is a same procedure for all the later terms in $Y_2(F)$. Eventually, when we sum up all the terms that have the form of $N!Q_N(F)$, they share the same sign which is $(-1)^{N-1}$. Fortunately, if we combine all the algebra terms in

$Y_{2a}(F)$ and $Y_{2c}(F)$, they have the same sign as well. It is easy to check since we know the sign of the algebra terms in $Y_{2c}(F)$ are the same if the power of F is the same. In this case, when $n = 1$, in $Y_{2a}(F)$ the term with the form of F^{-3} has the minus sign, so is it in $Y_{2c}(F)$ as we just worked it out. With this information, we can work out the sign of the errors in the later terms with a more general way. For the m-th term in $Y(F)$ series, m should be less than the optimal N, the partial sum from the previous term will have the form of F^{n+m} with sign of $(-1)^{n+m}$. But we care about the a portion of the entire partial sum, just like $Y_{2a}(F)$ will not affect the oscillation behavior. So the portion of the partial sum starts at F^{n+m+1} but with $(-1)^{n+m}$. For any n in the sum, the sign of the error is $(-1)^{n+m}$ multiplied by $(-1)^{N-m-n}$ and then one time reverse because of the sign of the first term. Therefore, the final sign of the error term is $(-1)^{n+m+N-m-n+1}$, which is the same as $(-1)^{N-1}$.

With this proof, we can sum up all the main error terms, which have the form of $N!Q_N(F)$, as our final main error in $Y(F)$ series. As we discussed before, our error term maintains all the information from the original equation since we can bound it from above and below. Therefore, we conclude that the hidden information in the errors in $Y(F)$ series, or in the Airy function, must have oscillation phenomena involved.