Something about Riemann and Lebesgue intergals

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0.1 "Limit"——footstone of mathematical analysis

1. In "Group Theory", the basic operation is " $\times$ ", we define the kind of mapping which keeps the operation as : homomorphic mapping. It is a key mapping in Group Theory.
2. In Mathematical Analysis, the foodstone is "limit" and the continuous functions are that keep "limit operation", i.e.

$$if\ x_n \longrightarrow x \quad \Rightarrow f(x_n) \longrightarrow f(x)$$

   So we can see the importance of continuous fuctions in Mathematical Analysis.

3. Different definition of Continuous Functions:

   Assume there are two metric space $(E,\ d)$, and $(S,\ \rho)$. $f:\ E \longrightarrow S$ then the following proposition are equivalent:

   (a) The definition of continuous functtion that most of us learnt in Mathematical Analysis:

$$\forall \epsilon > 0,\ \exists \delta > 0,\ such\ that\ f(B_d(x_0, \delta)) \subset B_\rho(f(x_0), \epsilon)$$

   Remark: Most of us are used to the definition in $(R,\ d)$. However, it is only a case of metric space. Though it helps to understand by realine R, or R^2, R^3......, metric space is much broader than that.

   (b) $\forall G \subset S$, G is open, we have $f^{-1}(G) = \{x \in E:\ f(x) \in G\}$ is open subset of E

   (c) $\forall F \subset S$, F is closed, we have $f^{-1}(F) = \{x \in E:\ f(x) \in F\}$ is closed subset of E

0.2 Riemann intergal and Lebesgue intergal

We learnt in "Mathematical Analysis" about Riemann intergal. In Measure Theory, there is another intergal called Lebesgue intergal. It is natural to ask : Why is Lebesgue intergal needed?

1. Riemann intergal is partitioned by the **range** of function f

Lebesgue intergal is prtitioned by the **domain** of function f

It is like two persons are counting their money. What Mr. Riemann does is: add the cashes one by one in order. What Mr. Lebesgue does is: sort the cash, put cashes with the same value together then count the number of $100, number of $50........ What Lebesgue does seems to be more effective ^_^.

2. Riemann intergable function space is not complete but the sapce for Lebesgue intergable function is complete. i.e, the limitation of a convergence sequence of **Riemann intergable functions** may not be **Riemann intergable**. But the limitation of a convergence sequence of **Lebesgue intergable functions** is **Lebesgue intergable**.

3. Riemann intergal requires stronger precondition to interchange the order of "limitaion" and "integration" that is: uniform convergence

   In Lebesgue, we only need "Lebesgue's dominated convergence theorem (DCT)"

4. Let f be a bounded function on a bounded interval [a, b]. Then

   (a) f is Riemann integrable on [a,b] iff f is continuous a.e. (m) on [a, b]

   You need to show that: upper-Riemann intergral=lower-Riemann intergral

   Proof (to be cont)

   (b) f is Lebesgue integrable on [a, b] and the Lebesgue integral $\int_{[a,b]} f dm$ equals the Riemann integral $\oint_{[a,b]} f$, i.e., the two integrals coincide.

5. If f is Riemann absolute integrabe on an unbounded area G and on every bounded subarea of G is Riemann integrable, then f is Lebesgue integrable on G and the two integrals are coincide.

   To show this, you need to use MCT and DCT

   For above $f(x_1, x_2, ..., x_n)$, Riemann absolute intergable is equivalent to Riemann intnergable when $n \geq 2$

0.3 Examples

$$f(x) = \begin{cases} 1 & x \in Q \\ 0 & x \in R/Q \end{cases}$$

$f(x)$ Legesbue integrable but not Riemann integrable