Change of Variables Formula in Measure Theory

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Let $(\Omega_i, \mathcal{F}_i), \quad i = 1, 2$ be two measureable spaces.
Let $f: \ \Omega_1 \longrightarrow \Omega_2 \ be \ (\mathcal{F}_1, \mathcal{F}_2) - measurable$
Let $h: \ \Omega_2 \longrightarrow \mathcal{R} \ be \ <\mathcal{F}_2, \mathcal{B}(\mathcal{R})> -measurable$
Let $I(x) \equiv x$ is identity function in $\mathcal{R}$
μ_1 is a measure on $(\Omega_1, \mathcal{F}_1)$, $\mu_2 = \mu_1 f^{-1}$ and $\mu_3 = \mu_2 f^{-1}$ then we have

$$g = h \circ f \in L^1(\Omega_1, \mathcal{F}_1, \mu_1) \Longleftrightarrow h(\cdot) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2) \Longleftrightarrow I(\cdot) \in L^1(\mathcal{R}, \mathcal{B}(\mathcal{R})) \tag{1}$$

$$\int_{\Omega_1} g d\mu_1 = \int_{\Omega_2} h d\mu_2 = \int_{\mathcal{R}} x d\mu_3 \tag{2}$$

Proof

1. $\boxed{\text{Claim } g = h \circ f \in L^1(\Omega_1, \mathcal{F}_1, \mu_1) \Longleftrightarrow h(\cdot) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2) \text{ and } \int_{\Omega_1} g d\mu_1 = \int_{\Omega_2} h d\mu_2}$

   (a) Let $h = I_A \quad A \in \mathcal{F}_2$

   $$\begin{aligned}\int_{\Omega_2} I_A \, d\mu_2 = \int_{\Omega_2} I_A \, d(\mu f^{-1}) = \mu_1 f^{-1}(A) = \mu_1[f^{-1}(A)] = \int_{\Omega_1} I_{f^{-1}(A)} \, d\mu_1 &= \int_{\Omega_1} I_A(f) \, d\mu_1 \\ &= \int_{\Omega_1} I_A \circ f \, d\mu_1 \\ &= \int_{\Omega_1} h \circ f \, d\mu_1\end{aligned}$$

   So for $h = I_A$, Claim 1 holds

   (b) By linearity, Claim 1 holds for all nonnegative simple function h in Ω_2

   (c) If h is nonnegative measurable function, then there exists $\{h_n\}_{n \geq 1}$, sequence of nonnegative simple functions, so that $0 \leq h_n \uparrow h$. Then $\{h_n \circ f\}_{n \geq 0}$ is a sequence of simple functions that increase to $h \circ f$. By MCT, Claim 1 holds.

   (d) If h is measurable function,

   $$\int_{\Omega_1} h^+ \circ f \, d\mu_1 = \int_{\Omega_2} h^+ d_{\mu_1 f^{-1}} \tag{3}$$

   $$\int_{\Omega_1} h^- \circ f \, d\mu_1 = \int_{\Omega_2} h^- d_{\mu_1 f^{-1}} \tag{4}$$

   and $h^{\pm} \circ f = (h \circ f)^{\pm}$. Then we have

   $$\int_{\Omega_1} h \circ f \, d\mu_1 = \int_{\Omega_1} h^+ \circ f \, d\mu_1 - \int_{\Omega_1} h^- \circ f \, d\mu_1 = \int_{\Omega_2} h^+ d\mu_1 f^{-1} - \int_{\Omega_2} h^- d\mu_1 f^{-1} = \int_{\Omega_2} h \, d\mu_1 f^{-1}$$

   So Claim 1 holds.

2. $\boxed{\text{Claim } \int_{\Omega_2} h \, d\mu_2 = \int_{\mathcal{R}} x \, d\mu_3}$

   $$\int_{\mathcal{R}} x \, d\mu_3 = \int_{\mathcal{R}} x \, d\mu_2 h^{-1} = \int_{\Omega_2} h I_{\mathcal{R}}(h) \, d\mu_2 = \int_{\Omega_2} h \, d\mu_2$$

   So Claim 2 holds.