Discussion about several properties of inverse CDF

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We can use either sup or inf to define inverse CDF. The left/right continuity of CDF is kind of confusing which depend on how you define the CDF.

$$F(x_0) = P\{x : x < x_0\} \tag{1}$$

$$F(x_0) = P\{x : x \leq x_0\} \tag{2}$$

In the case of (1), the CDF is left continuous. In (2), the CDF is right continuous. Here, we use (2) as the definition of CDF.

The continunity for inverse CDF is even more tricky. For inf, we have the following definition:

$$F^{-1}(x) = inf\{x : F(x) > y\} \tag{3}$$

$$F^{-1}(x) = inf\{x : F(x) \geq y\} \tag{4}$$

Similarly, we can use sup to define inverse CDF.

$$F^{-1}(x) = sup\{x : F(x) < y\} \tag{5}$$

$$F^{-1}(x) = sup\{x : F(x) \leq y\} \tag{6}$$

The following discussion is based on the definition (3) i.e.$F^{-1}(x) = sup\{x : F(x) < y\}$. You can derivative properties in other cases.

$\boxed{\text{Prop1: } F^{-1}(y) \text{ is non-decreasing.}}$

Assume $0 < y_1 < y_2 < 1$, because $F(x)$ is nondecreasing. Then we have

$$\{x : F(x) < y_1\} \subseteq \{x : F(x) < y_2\}$$

$$\Longleftrightarrow sup\{x : F(x) < y_1\} \leq sup\{x : F(x) < y_2\}$$

So $F^{-1}(y)$ is nondecreasing function.

$\boxed{\text{Prop2: } F(F^{-1}(y)) \geq y \text{ and } " = " \text{ holds when } F(x) \text{ is continuous at } F^{-1}(y).}$

$\boxed{\textbf{But } F(F^{-1}(y)) = y \textbf{ doesn't mean } F(x) \textbf{ is continuous at } F^{-1}(y)\textbf{.}}$

$H \equiv \{x : \ F(x) < y\}$
$F^{-1}(y) = sup\{x : \ F(x) < y\} = supH$
Here $F(x) \equiv P(\xi(\omega) \leq x)$ (right continuous)

- First show: $F(F^{-1}(y)) \geq y$

  For $\forall \epsilon > 0$, $F^{-1}(y) + \epsilon \notin H$ then $F(F^{-1}(y) + \epsilon) \geq y$, by the right continuity,

$$\Longrightarrow lim_{\epsilon\to 0}F(F^{-1}(y)+\epsilon) = F(F^{-1}(y)) \geq y$$

- Then show " $=$ " holds when $F(x)$ is continuous at $F^{-1}(y)$ i.e. $F(F^{-1}(y)) = y$

$$\forall \epsilon > 0,\ \exists x^*\ s,t,\ F^{-1}(y) - \epsilon < x^*\ and\ x^* \in H$$

  Since $F^{-1}(y)$ is nondecreasing,
$$F(F^{-1}(y) - \epsilon) \leq F(x^*) < y$$

  When $F(x)$ is continuous at $F^{-1}(y)$,

$$F(F^{-1}(y) - \epsilon) \to F(F^{-1}(y))\ \ as\ \epsilon \to 0$$

$$lim_{\epsilon\to 0}F(F^{-1}(y) - \epsilon) = F(F^{-1}(y)) \leq y$$

  So " $=$ " holds when $F(x)$ is continuous at $F^{-1}(y)$ i.e. $F(F^{-1}(y)) = y$

$\boxed{\text{Prop3: } F^{-1}(F(x)) \leq x}$

For fixed x_0, assume $M = \{x:\ F(x) < F(x_0)\}$, because $F(x)$ is nondecreasing, $\forall x \in M,\ \ x \leq x_0$. Since $F^{-1}(F(x_0)) = supM$, then $F^{-1}(F(x_0)) \leq x_0$.

$\boxed{\text{Prop 4.1: } F^{-1}(y) \leq x \Longleftrightarrow y \leq F(x);}$

$\boxed{\text{Prop 4.2: } F^{-1}(y) > x \Longleftrightarrow y > F(x)}$

Only show 4.1

- " $\Longleftarrow$ "

$$F^{-1}(y) = sup\{x:\ F(x) < y\}$$

$$H = \{x:\ F(x) < y\}$$

  If $y \leq F(x_0)$, $x_0 \geq x \in H$
$$x_0 \geq supH$$

- " $\Longrightarrow$ "
  If $F^{-1}(y) \leq x$, $y \leq F(F^{-1}(y)) \leq F(x)$

$\boxed{\text{Prop5: } F^{-1}(y) \text{ is left continuous}}$

$\forall y_0$ only need to show $lim_{y\to y_0^-}F^{-1}(y) \leq F^{-1}(y_0)$

- First show that $lim_{y\to y_0^-}F^{-1}(y) \leq F^{-1}(y_0)$ (obvious)

- Then show the other direction.

  By Prop 4.1, only need to show $y_0 \leq F(lim_{y \to y_0^-} F^{-1}(y))$

$$F(lim_{y \to y_0^-} F^{-1}(y)) \geq lim_{y \to y_0^-} F(F^{-1}(y)) \geq lim_{y \to y_0^-} y = y_0$$