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Project Euler Questions 137/140

This short write-up will walk through the process that I used to solve Project Euler questions 137/140, and describe some of the benefit I found in using Haskell to code this solution.

First, let us recall the definition of the Fibbonacci numbers. We start with:

$$F_0 = 0, \quad F_1 = 1$$

and define further numbers by the relation:

$$F_n = F_{n-1} + F_{n-2}$$

This gives us the first few terms:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

(Please note that Project Euler omits the first index, but this will not change our calculations in any meaningful way, and I will try to be as explicit as possible about this.)

The natural generalization is to consider a different starting point. I will write

$$G(a, b, n) = G(a, b, n-1) + G(a, b, n-2)$$

given the starting point:

$$G(a, b, 0) = a, \quad G(a, b, 1) = b$$

and when we are working with a set a and b I will simply write G_n. You can see that $G(0, 1, n)$ gives the Fibbonacci numbers themselves.

These two Project Euler questions are concerned with the power series with coefficients given by the above sequences. That is, we are interested in the value of the summations:

$$\sum_{k=1}^{\infty} G(a,b,n)x^k = G(a,b,1)x + G(a,b,2)x^2 + G(a,b,3)x^3 + \dots$$

Again, if we take $a = 0$ and $b = 1$, we will have the Fibbonacci numbers:

$$\begin{aligned}\sum_{k=1}^{\infty} G(0,1,n)x^k &= F_1x + F_2x^2 + F_3x^3 + \dots \\ &= x + x^2 + 2x^3 + 3x^4 + 5x^5 \dots\end{aligned}$$

Interestingly, it is not too difficult to derive a closed form solution to these series. Let us fix $G_0 = a$ and $G_1 = b$. Then the power series is:

$$\sum_{k=0}^{\infty} G_k x^k = G_0 + G_1x + G_2x^2 + \dots$$

Please note that I have included the initial term G_0 that Project Euler omits. We can correct for this at the end by simply subtracting this term.

First, I will pull out the first two given terms from the series:

$$\sum_{k=0}^{\infty} G_k x^k = G_0 + G_1 x + \sum_{k=2}^{\infty} G_k x^k$$

Now I can Replace G_k with the relation that defines it, and split apart the summation further:

$$\begin{aligned}\sum_{k=0}^{\infty} G_k x^k &= G_0 + G_1 x + \sum_{k=2}^{\infty} (G_{k-1} + G_{k-2}) x^k \\ &= G_0 + G_1 x + \sum_{k=2}^{\infty} G_{k-1} x^k + \sum_{k=2}^{\infty} G_{k-2} x^k\end{aligned}$$

Next, I notice that multiplying my original series by x and x^2 gives:

$$\begin{aligned}x \sum_{k=0}^{\infty} G_k x^k &= G_0 x + G_1 x^2 + G_2 x^3 + \dots \\ x^2 \sum_{k=0}^{\infty} G_k x^k &= G_0 x^2 + G_1 x^3 + G_2 x^4 + \dots\end{aligned}$$

Substituting these into the above equation I have:

$$\sum_{k=0}^{\infty} G_k x^k = G_0 + G_1 x + x \sum_{k=0}^{\infty} G_k x^k - G_0 x + x^2 \sum_{k=0}^{\infty} G_k x^k$$

Denoting the original summation as $s(x)$ this is:

$$s(x) = G_0 + G_1 x + x s(x) - G_0 x + x^2 s(x)$$

and solving gives:

$$s(x) = \frac{G_0 - G_0 x + G_1 x}{1 - x - x^2}$$

Remember that this includes our first term. So we have[1]:

$$\sum_{k=0}^{\infty} G_k x^k = G_0 + G_1 x + G_2 x^2 + \cdots = \frac{G_0 - G_0 x + G_1 x}{1 - x - x^2}$$

For the Fibbonacci series we set $G_0 = 0$ and $G_1 = 1$ giving:

$$\sum_{k=0}^{\infty} F_k x^k = F_0 + F_1 x + F_2 x^2 + \cdots = \frac{x}{1 - x - x^2}$$

[1] Note Project Euler omits the first term, considering instead the series:

$$\sum_{k=1}^{\infty} G_k x^k = G_1 x + G_2 x^2 + \cdots = \frac{G_0 - G_0 x + G_1 x}{1 - x - x^2} - G_0$$

The next question under consideration is rational values of x give an integer value for our power series? I will write:

$$s(x) = \sum_{k=1}^{\infty} G_k x^k = G_0 + G_1 x + G_2 x^2$$

Above we had the equation:

$$s(x) = G_0 + G_1 x + x s(x) - G_0 x + x^2 s(x)$$

which solving for x using the quadratic formula[2] gives us:

$$x = \frac{-(G_1 - G_0 + s) + \sqrt{(G_1 - G_0 + s)^2 - 4s(G_0 - s)}}{2s}$$

Remember that we included the term G_0. If we would like to correct for this, we can make the transformation $s \to S - G_0$ where:

$$\begin{aligned} S(x) = \sum_{k=1}^{\infty} G_k x^k &= \\ &= \sum_{k=0}^{\infty} G_k x^k - G_0 \\ &= s(x) - G_0 \end{aligned}$$

which gives instead:

$$x = \frac{-(G_1 + S) + \sqrt{(G_1 + S)^2 - 4(S + G_0)(-S)}}{2(S + G_0)}$$

[2] We actually have two choices for roots, but one is outside the radius of convergence for the power series

Using the second equation and taking $G_0 = 0$ and $G_1 = 1$ to give the Fibbonacci power series, we have have the following values:

$$\begin{aligned} S\left(\sqrt{2}-1\right) &= 1 \\ S\left(\frac{1}{2}\right) &= 2 \\ S\left(\frac{\sqrt{13}-2}{3}\right) &= 3 \\ S\left(\frac{\sqrt{89}-5}{8}\right) &= 4 \\ &\vdots \end{aligned}$$

We can see for instance that $S = 2$ has the desired property that x is rational. What conditions must be met for this? First, look at our equation for x we we substitue the particuliar values for G_0 and G_1:

$$x = \frac{-(1+S) + \sqrt{(1+S)^2 - (2S)^2}}{2S}$$

This will be rational exactly when the discriminant $(1+S)^2 - (2S)^2$ is a perfect square [3]

[3] The following arguement follows from this article : Link

Since the discriminant is itself a sum of two squares, we can consider them as Pythagorean triples so that for integers m, n:

$$S + 1 = m^2 - n^2, \quad 2S = 2mn$$

This gives the equation:

$$m^2 - n^2 = mn + 1$$

which can be written as

$$(2m - n)^2 = 4 + 5n^2$$

We need $4 + 5n^2$ to be a perfect square to ensure a rational x. Unexpectedly, using $n = F_{2j}$ (where j is a positive integer) ensures this is satisfied:

$$\begin{aligned}
4 + 5n^2 = 4 + 5F_{2j} &= 4 + 5\left(\frac{(\varphi^{2j} - \psi^{2j})^2}{5}\right) \\
&= 4 + \varphi^{4j} - 2(\varphi\psi)^{2j} + \psi^{4j} \\
&= \varphi^{4j} + 2 + \psi^{4j} \\
&= \varphi^{4j} + 2(\varphi\psi)^{2j} + \psi^{4j} \\
&= (\varphi^{2j} + \psi^{2j})^2
\end{aligned}$$

Furthermore, substituing into the equation $(2m - n)^2 = 4 + 5n^2$ above gives:

$$\begin{aligned}
m &= \frac{1}{2}(\varphi^{2j} + \psi^{2j} + F_{2j}) \\
&= \frac{1}{2}\left(\varphi^{2j} + \psi^{2j} + \frac{\varphi^{2j} - \psi^{2j}}{\sqrt{5}}\right) \\
&= \frac{\varphi\varphi^{2j} - \psi\psi^{2j}}{\sqrt{5}} = \frac{\varphi^{2j+1} - \psi^{2j+1}}{\sqrt{5}} \\
&= F_{2j+1}
\end{aligned}$$

What an unexpected resutl! That we could consider successive coeffients in identify this rational points is certainly not intuitive. We see that these numbers also grow very quickly:

$$F_{2j}F_{2j+1}, n \in \mathbb{Z}^+ = \{2,15, 104, 714, 4895, 33552, 229970, 1576239, 10803704, 74049690, \\ 507544127, 3478759200, 23843770274, 163427632719, 1120149658760, \ldots\}$$

The last number in this list is the solution to problem 137. Problem 140 considers the initalization $G_0 = 3$ and $G_1 = 1$.