Process Mining - 02269

Lecture 1

Petri nets

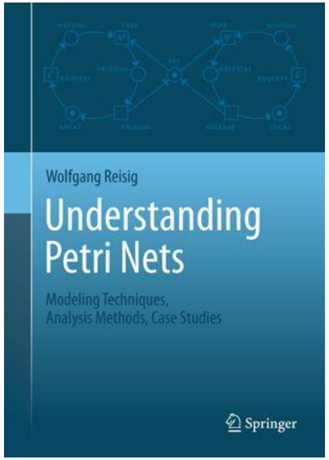
Andrea Burattin

Background: Petri nets

- Idea
  - Formal description of concurrent systems
  - Formal model and graphical representation are equivalent

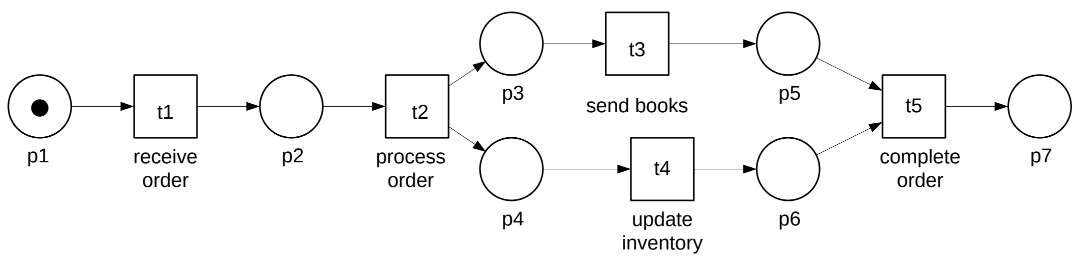
- Background
  - Foundations developed by Carl Adam Petri, 1962
  - Variety of variants and extensions
  - Here: modelling and analysis of business processes

- Reference book
  - https://link.springer.com/978-3-642-33278-4



The essence of Petri nets

- A Petri net is a directed graph consisting of places, transitions, and arcs between them
- Petri nets are often referred to as *bipartite* graphs
  - An arc is defined from a place to a transition, or from a transition to a place
- Notation by example:

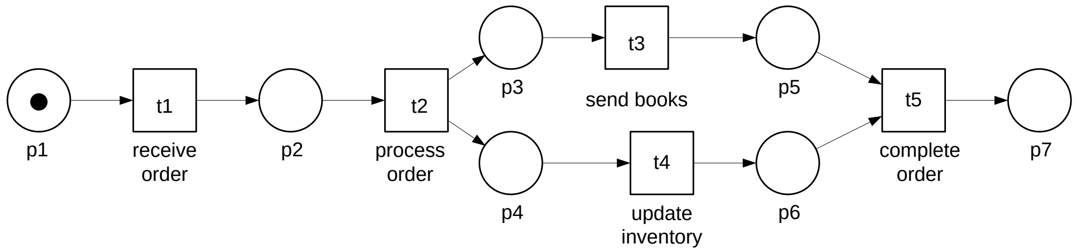


from M. Weske: Business Process Management, © Springer-Verlag Berlin Heidelberg 2007

Syntax

- A Petri net is a tuple (P, T, F) where
  - P is a finite set of places
  - T is a finite set of transitions such that $T \cap P = \emptyset$
  - $F \subseteq (P \times T) \cup (T \times P)$ is a flow relation
  - A place $p \in P$ is an input place of transition $t \in T$ iff $(p, t) \in F$. The set of input places for a transition t is denoted as $\bullet t$
  - A place $p \in P$ is an output place of transition $t \in T$ iff $(t, p) \in F$. The set of output places for a transition t is denoted as $t\bullet$
  - $p\bullet$ and $\bullet p$ denote the set of transition that share p as input and output places respectively

Example

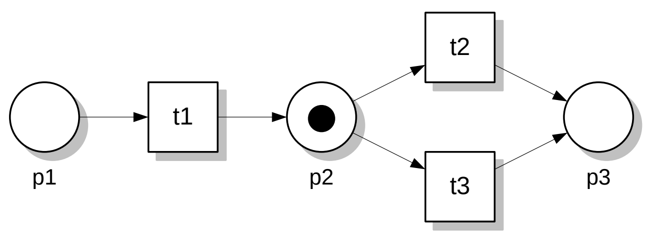


- $P = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$
- $T = \{t_1, t_2, t_3, t_4, t_5\}$
- $F = \{(p_1, t_1), (t_1, p_2), (p_2, t_2), (t_2, p_3), (t_2, p_4),$

Semantics

- Dynamic behaviour is represented by tokens in the Petri net
- State of a Petri net (the marking) is described as a distribution of tokens over places

- The marking (or state) of a Petri net (P, T, F) is defined by a function $M: P \to \mathbb{N}$ mapping the set of places onto the natural numbers, where $\mathbb{N}$ is the set of natural numbers including zero



Enabling, Firing, Reachability

- Let (P, T, F) be a Petri net and M a marking. Firing a transition is represented by a state change of the Petri net
  - A transition $t \in \mathrm{T}$ is *enabled* in a state M if $M(p) \geq 1$ for all $\mathrm{p} \in \bullet t$
  - *Firing* an enabled transition $t \in \mathrm{T}$ in state M results in state M' where

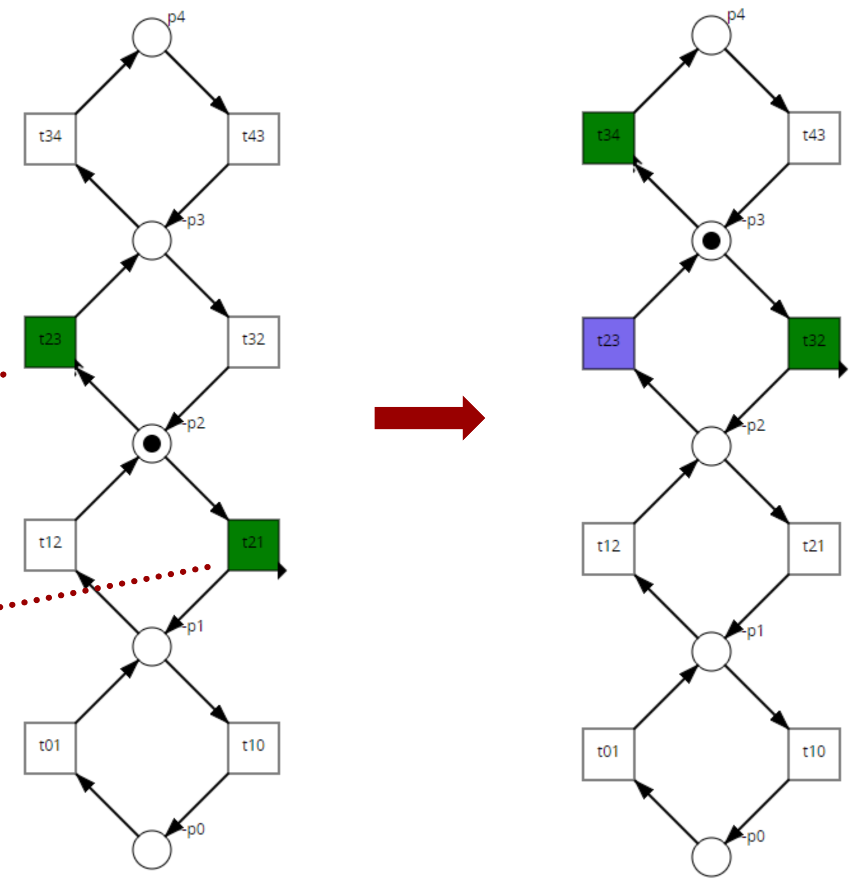
$$\forall p \in \bullet t \,.\, M'(p) = M(p) - 1 \wedge \forall p \in t\bullet .\, M'(p) = M(p) + 1$$

  - $M \xrightarrow{t} M'$ indicates that by firing t the state of the net changes from M to M'
  - $M_1 \xrightarrow{*} M_n$ means that there is a sequence of transitions $t_1, t_2, \ldots, t_{n-1}$ such that $M_i \xrightarrow{t_i} M_{i+1}$ for $1 \leq i < n$
  - A state M' is *reachable* from sate M iff $M \xrightarrow{*} M'$

Enabling, Firing, Reachability (cont.)

- Firing of a transition is atomic
- Multiple transitions may be enabled, but only one fires as part of a state change
- The number of tokens in a net may vary if there are transitions for which the number of input places is not equal to the number of output places
- The network, the net structure, is static

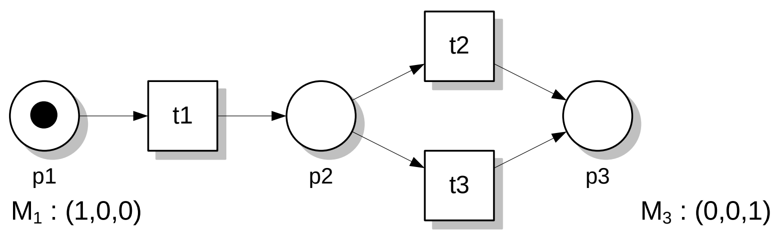
Non-Determinism



Two transitions are enabled but only one can fire

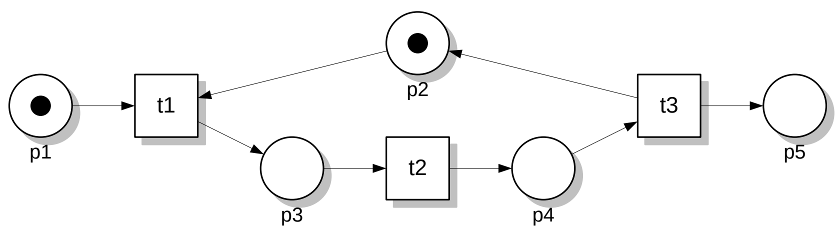
Petri Net System

- A Petri net system is a pair (N, M)
  - $N = (P, T, F)$ is a Petri net
  - M is the initial marking
- Consider this Petri net system:
  $\big((\{p_1, p_2, p_3\}, \{t_1, t_2, t_3\}, \{(p_1, t_1), (t_1, p_2), (p_2, t_2), (t_2, p_3), (p_2, t_3), (t_3, p_3)\}), (1,0,0)\big)$
- $M_1 \xrightarrow{o} M_3$ with $o = (t_1, t_2)$ changes the state of the system from marking M_1 to M_3



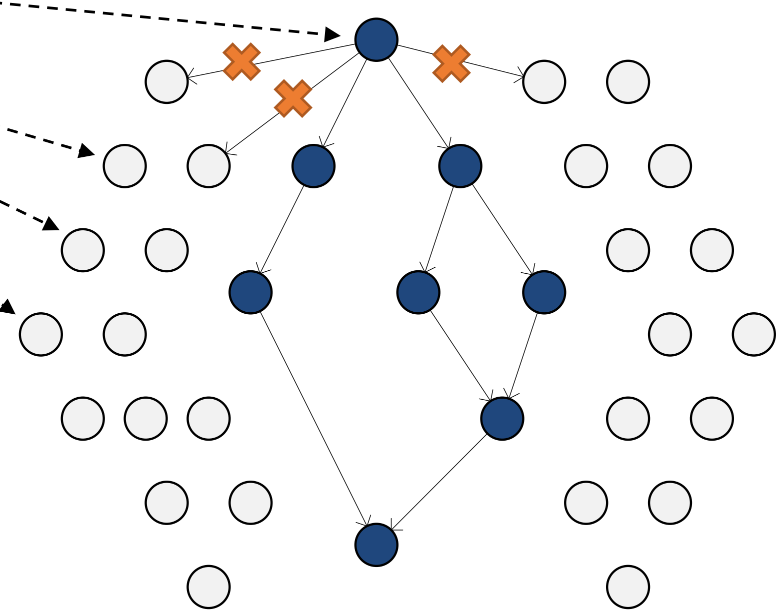
Reachability

- Examples
  - $(0,1,0,0,1)$ is reachable from $(1,1,0,0,0)$ by $o = (t_1, t_2, t_3)$
  - $(0,1,1,0,0)$ is NOT reachable from $(1,1,0,0,0)$ since there is no according firing sequence
- Remark
  - Multiset notation is often used for markings
  - In the example: $[p_1, p_2]$ instead of $(1,1,0,0,0)$
  - Multiple tokens in one place: $[p_1, p_1, p_2]$ or $[p_1^2, p_2]$ instead of $(2,1,0,0,0)$



Petri Nets into Transition Systems

- A Petri net system (N, M_0) defines the following transition system (S, TR, s_0)
- Intuitive idea
  - $s_0 = M_0$
  - $S = M: P \rightarrow \mathbb{N}_{\geq 0}$
  - Transitions only for allowed marking evolutions

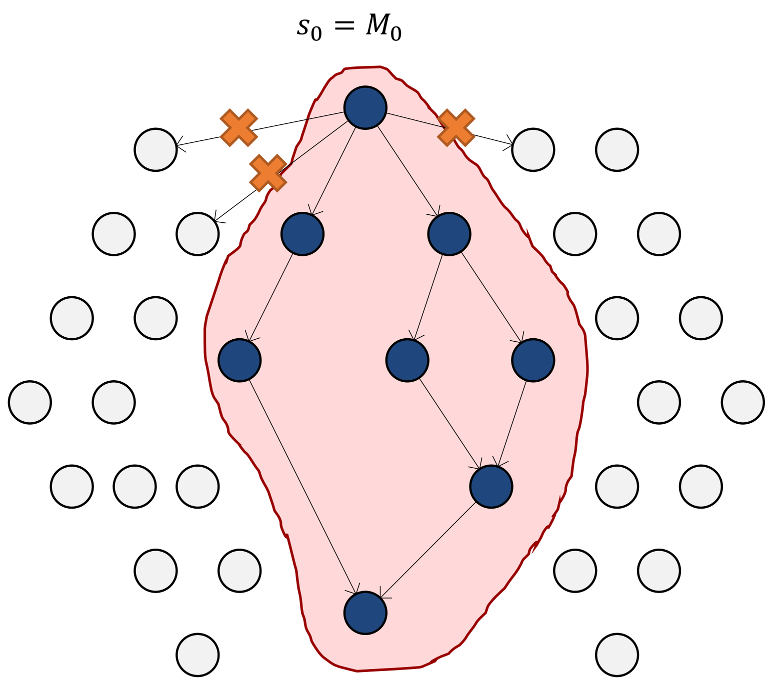


Petri Nets into Transition Systems

- Given a Petri net $N = (P, T, F)$
- A Petri net system (N, M_o) defines the following transition system (S, TR, s_0)
  - $S = M: P \rightarrow \mathbb{N}_{\geq 0}$ $(M \neq M_0)$
  - $s_0 = M_0$
  - $TR = \{(m, m') \in S \times S$ |
    $\exists t \in T: (\forall p \in \bullet t: m(p) > 0) \wedge$
    $\left(\forall p \in P: m'(p) = m(p) - w\big((p,t)\big) + w\big((t,p)\big)\right)\}$
  - Where
    - $w\big((x,y)\big) = 1$ if $(x, y) \in F$ and
    - $w\big((x,y)\big) = 0$ if $(x, y) \notin F$

Reachability Graph

- Reachability graph is the **reachable portion** of the transition system (reachable from initial marking)



Workflow Petri Nets

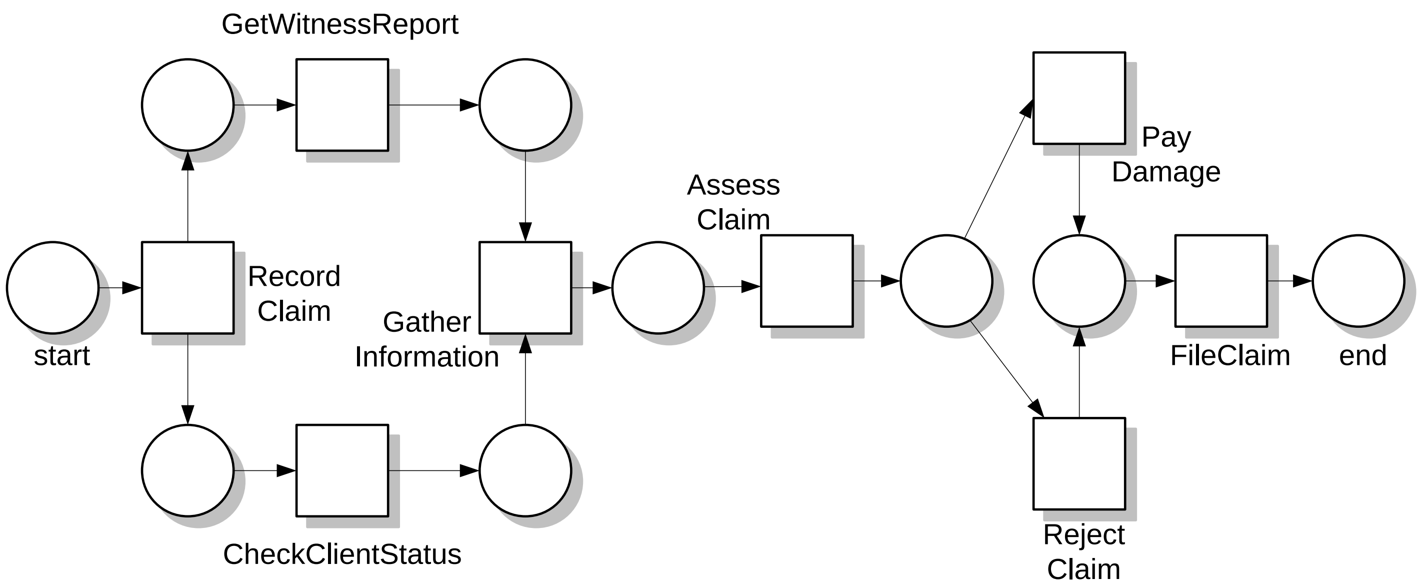
- Use Petri nets to model business processes
  - **Transitions** represent **activities**
  - **Places** represent **states** of the business process
  - **Arcs** represent **control flow dependencies**
  - Token represent instances (not always)
  - Behaviour of process instance is represented by firing rule
  - Behaviour of transitions is influenced by tokens in the preset

- In fact: many process modelling languages are rooted in Petri net systems

Definition

- A Petri net $N = (P, T, F)$ is called *workflow net* iff
  - There is a distinguished place $i \in P$ called *initial place* with no incoming edges, i.e.,$\bullet i = \emptyset$
  - There is a distinguished place $o \in P$ called *final place* with no outgoing edges, i.e., $o\bullet = \emptyset$
  - Every place and every transition are located on a path from the initial place to the final place

Example



from M. Weske: Business Process Management, © Springer-Verlag Berlin Heidelberg 2007

Properties

- i is the only initial place: for each $p \in P$: $\bullet p \neq \emptyset$ or $p = i$
- o is the only final place: for each $p \in P$: $p\bullet \neq \emptyset$ or $p = o$
- Let N be a WF-net, if it is extended with a transition t^*, which connects o with i (i.e. $\bullet t^* = \{o\}$ and $t^*\bullet = \{i\}$), the resulting net is strongly connected

- Strongly connected

  A graph is *strongly connected*, if and only if for each pair of nodes (x, y) there is a path from x to y