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REACHABILITY PROBABILITIES IN MARKOVIAN TIMED AUTOMATA

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Reachability Probabilities in Markovian Timed Automata[⋆]

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Abstract. We propose a novel stochastic extension of timed automata, i.e. *Markovian Timed Automata*. We study the problem of optimizing the reachability probabilities in this model. Two variants are considered, namely, the time-bounded and the unbounded case. In each case, we propose Bellman equations to characterize the probability. For the former, we provide two approaches to solve the Bellman equations, namely, a discretization and a reduction to Hamilton-Jacobi-Bellman equations. For the latter, we show that in the single-clock case, the problem can be reduced to solving a system of linear equations, whose coefficients are optimal time-bounded reachability probabilities in Continuous-Time Markov Decision Processes.

1 Introduction

This paper introduces **Markovian Timed Automata** (MTA), a novel extension of timed automata [1] with exponentially distributed location residence times. To give some motivation, let's look at the following example: A robot moves on a 3×3-grid (Fig. 1), starting from A and trying to reach B in T_2 units of time. At each cell, it can move up (u), down (d), left (l) and right (r) (when applicable). Cells are associated with rates, which intuitively represent the speed of the robot. As soon as it moves to a cell, the robot decides a direction to move to the next cell, and then waits in the current cell for a certain amount of time, which is governed by an exponential distribution with a given rate λ, i.e., the probability of leaving the cell within time t is $1-e^{\lambda t}$.[1] The robot is allowed to stay in consecutive dark cells for at most T_1 units of time, while there is no time constraint for the bright cells. At each cell, the robot has a probability 0.1 to

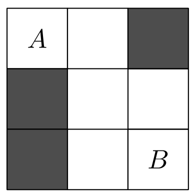

Fig. 1. Robot example

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[1] This seems to be restrictive. However, we note that in practice, one can approximate any distribution by phase-type distributions, resulting in series-parallel combinations of exponential distributions [12].

break down, apart from moving to the neighboring one. Since there are different ways to reach B, each of which is associated with some probability, one natural question is: What's the *maximum probability* to reach B from A within T_2 units of time? This problem can be readily formulated as a controller synthesis problem for MTA depicted in Fig. 2, where each location in $\{\ell_0, \ldots, \ell_8\}$ corresponds to a cell and x, y are *clocks* to specify the time constraints. (N.B. the failure location and all the transitions leading to it are omitted for the clearer illustration. For the same reason the representation of the MTA here is a bit different from the one in Fig. 3.)

In general, controller synthesis problems for MTA are to determine the sequence of actions that maximize the probability[2] to reach certain goal locations in MTA. Two variants of reachability properties are considered: what is the maximum likelihood to hit a set of goal locations within a given deadline (*time-bounded reachability*), and what is this probability in absence of such deadline (*time-unbounded reachability*)? To solve these issues, we first apply the standard region construction [1] to MTA. Then in the time-bounded case, we characterize maximum (time-bounded) reachability probabilities by a variant of the Bellman equation [3]. This provides the basis for two approaches to compute such probabilities. The first approach uses discretization, and shows that max-reachability probabilities can be reduced to maximum reachability probabilities in a finite state Markov decision process (MDP), for which various efficient algorithms, such as value iteration [3] exist. We show that the accuracy of our result is $(1-e^{-\lambda h})\cdot(1-e^{-\lambda T})$ where h is the discretization step, T is the deadline, and λ is the maximal rate of all exponential distributions in the MTA. The second approach is based on partial differential equations (PDEs), in particular Hamilton-Jacobi-Bellman equations [10]. Accordingly for the time-unbounded case, we provide a Bellman equation to characterize (time-unbounded) reachability probabilities, and show that for single-clock MTA, it boils down to solving a linear equation whose coefficients are reachability probabilities in (locally uniform) continuous-time Markov decision processes (CTMDPs) [13].

We point out that MTA are rather expressive: Zero-clock MTA correspond to a subclass of CTMDPs [2][5], whereas probabilistic timed automata (PTAs) [11] are obtained by basically ignoring the exit rates in any location in the MTA. In earlier work [9], we have used *deterministic* MTA as specification formalism for linear real-time properties over stochastic processes. Some more related works are in order: In [4] the authors consider stochastic timed games, which contain two types of locations: probabilistic ones and locations belonging to one of the players. The authors have addressed the reachability problem for this model: It was shown that the quantitative reachability problem is undecidable in general (for $2\frac{1}{2}$-player games), while the qualitative question "$= 0$" or "$= 1$" can be solved in PTIME for $1\frac{1}{2}$-player games with a single clock. MTA are essentially $1\frac{1}{2}$-player stochastic timed games. However, our focus is on *quantitative* analysis rather than on qualitative analysis or decidability issues. In [6], a game extension

[2] The minimal one can be treated in a completely dual fashion. We omitted it for the sake of simplicity.

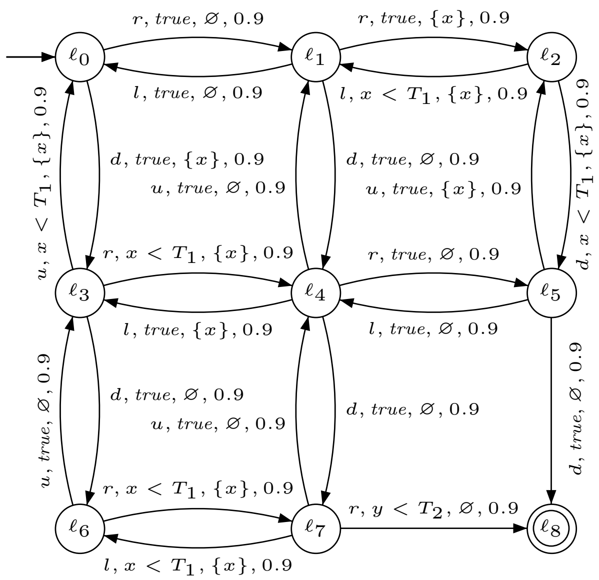

Fig. 2. MTA for the robot example

of SMDP was considered and the winning objective was specified by a deterministic timed automaton. Again, only the *qualitative* question were addressed.

As a by-product of our work we obtain two procedures to compute maximum time-bounded reachability probabilities in *locally uniform* CTMDPs. This problem has also been treated in [2][5][14], where [2][5] mainly addressed time-abstract schedulers which are not necessarily optimal, while here time-dependent schedulers are considered, as well as [14]. Comparing to [14], we discretize both the time and the state space. Moreover a system of PDEs is derived in order to characterize the maximum reachability probability. The error bound that we obtain improves the error bound given in [14] yielding a substantial reduction in the required number of iterations.

2 Markovian Timed Automata

Given a set H, let $\Pr : \mathcal{F}(H) \to [0, 1]$ be a probability measure on the measurable space $(H, \mathcal{F}(H))$, where $\mathcal{F}(H)$ is a σ-algebra over H. Let $Distr(H)$ denote the set of probability measures on this measurable space.

2.1 Markov Decision Processes

Definition 1 (MDP). *A (continuous-state)* Markov decision process *is a tuple* $\mathcal{D} = (Act, S, s_0, \mathcal{P})$ *where*

– *Act is a* denumerable *set of* actions*;*

– S *is a set of* states*;*
– $s_0 \in S$ *is the* initial state*;*
– $\mathcal{P} : S \times Act \times \mathcal{F}(S) \to [0,1]$ *is the* transition probability function*, where* $\mathcal{P}(s, \alpha, \cdot)$ *is a probability measure over* $\mathcal{F}(S)$ *for any* $s \in S$ *and* $\alpha \in Act$*, such that* $\mathcal{P}(\cdot, \cdot, A)$ *is measurable for any* $A \in \mathcal{F}(S)$*.*

The measure $\mathcal{P}(s, \alpha, A)$ is the one-step transition probability from state $s \in S$ to the set of states $A \in \mathcal{F}(S)$ by taking action $\alpha \in Act$. Notice that in general one can extend the MDP model to uncountably many actions (see [15]). In this paper we will consider only MDPs which have finitely many actions, i.e., finitely-branching MDPs.

2.2 Markovian Timed Automata

Let $\mathcal{X}$=$\{x_1, ..., x_n\}$ be a set of *nonnegative* variables in $\mathbb{R}$, called *clocks*. A clock-valuation is a function $\eta{:}\mathcal{X}{\to}\mathbb{R}_{\geqslant 0}$ assigning to each variable x a value $\eta(x)$. Let $\mathcal{V}(\mathcal{X})$ denote the set of all clock-valuations over $\mathcal{X}$. A *clock constraint* on $\mathcal{X}$, denoted by g, is a conjunction of expressions of the form $x \bowtie c$ for clock $x \in \mathcal{X}$, comparison operator $\bowtie \in \{<, \leqslant, >, \geqslant\}$ and $c \in \mathbb{N}$. Let $\mathcal{CC}(\mathcal{X})$ denote the set of clock constraints over $\mathcal{X}$. A clock valuation η *satisfies* constraint $x \bowtie c$, denoted $\eta \models x \bowtie c$, if and only if $\eta(x) \bowtie c$; it satisfies a conjunction of such expressions if and only if η satisfies all of them. Let $\vec{0}$ denote the valuation that assigns 0 to all clocks. For a subset $X \subseteq \mathcal{X}$, the reset of X, denoted $\eta[X := 0]$, is the valuation η' such that $\forall x \in X.\ \eta'(x) := 0$ and $\forall x \notin X.\ \eta'(x) := \eta(x)$. For $\delta \in \mathbb{R}_{\geqslant 0}$ and $\mathcal{X}$-valuation η, $\eta{+}\delta$ is the $\mathcal{X}$-valuation η'' such that $\forall x \in \mathcal{X}.\ \eta''(x) := \eta(x){+}\delta$, which implies that all clocks proceed at the same speed.

Definition 2 (MTA). *A* Markovian timed automaton *is a tuple* $\mathcal{M} = (Act, \mathcal{X}, Loc, \ell_0, E, \rightsquigarrow)$*, where*

– Act *is a finite set of* actions*;*
– $\mathcal{X}$ *is a finite set of* clocks*;*
– Loc *is a finite set of* locations*;*
– $\ell_0 \in Loc$ *is the* initial location*;*
– $E : Loc \to \mathbb{R}_{>0}$ *is the* exit rate function *and*
– $\rightsquigarrow \subseteq Loc \times Act \times \mathcal{CC}(\mathcal{X}) \times Distr(2^{\mathcal{X}} \times Loc)$ *is the* edge relation*.*

For simplicity we abbreviate $(\ell, \alpha, g, \zeta) \in \rightsquigarrow$ by $\ell \overset{\alpha,g}{\rightsquigarrow} \zeta$, where ζ is a probability distribution over $2^{\mathcal{X}} \times Loc$. Here we don't include location invariants, as in [1], and we don't require edge relation $\rightsquigarrow$ to be total, e.g., there might be some clock constraints g for which $\rightsquigarrow$ is not defined.

Example 1. An example MTA is shown in Fig. 3, where there are 6 locations with ℓ_0 the initial location. In ℓ_0 (resp. ℓ_2), there is a decision to be made between actions α_1 and α_2 (resp. γ_1 and γ_2) when the clock valuation is $\eta(x) \in [0,1)$ (resp. $\eta(x) \in (1,2)$). ℓ_3 is the *goal* location, which will be used later. For edge $\ell_0 \overset{\alpha_2, x\geqslant 0}{\rightsquigarrow} \zeta$, $\zeta(\varnothing, \ell_2) = 0.2$ and $\zeta(\varnothing, \ell_5) = 0.8$.

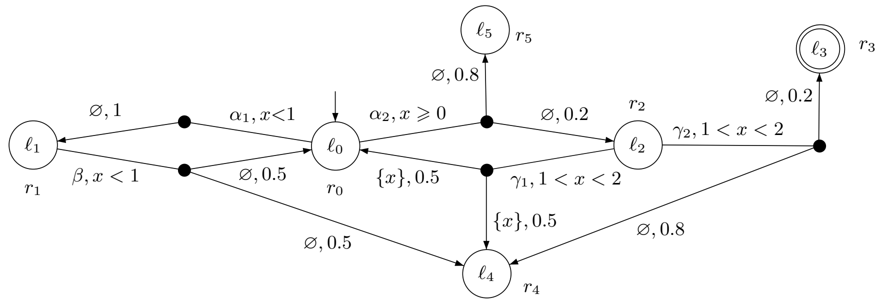

Fig. 3. An example MTA

Semantics. Intuitively, an MTA behaves as follows. Consider location ℓ_1 in Fig. 3. As soon as location ℓ_1 is entered with clock valuation η, action β is chosen and a waiting time τ in ℓ_1 is sampled from the probability distribution $E(\ell_1)e^{-E(\ell_1)\tau}$. If $\eta + \tau \models x < 1$, there will be a jump to location ℓ_0 with probability 0.5 or to location ℓ_4 with probability 0.5, otherwise no jump occurs and MTA remains in location ℓ_1. When for instance the next location ℓ_0 is entered with clock valuation η' ($= \eta+\tau$ in this example), action α_1 or α_2 is chosen and new the waiting time τ' in ℓ_0 is sampled from the probability distribution $E(\ell_0)e^{-E(\ell_0)\tau'}$. Suppose α_2 is picked and $\eta'+\tau' \models x \geqslant 0$ (the guard of action α_2), the MTA jumps to the next location according to the probability distribution associated with α_2. The MTA follows the same behavior continuously.

The semantics of an MTA with clock set $\mathcal{X}$ is given as a continuous-state MDP, where *states* are of the form (ℓ, η) where $\ell \in Loc$ and $\eta \in \mathcal{V}(\mathcal{X})$ is a clock valuation.

Definition 3 (Semantics). *Let $\mathcal{M} = (Act, \mathcal{X}, Loc, \ell_0, E, \rightsquigarrow)$ be an* MTA*. The* MDP *associated with $\mathcal{M}$ is $\mathcal{D}(\mathcal{M}) = (Act, S, s_0, \mathcal{P})$ where $S = Loc \times \mathcal{V}(\mathcal{X})$; $s_0 = (\ell_0, \vec{0})$; and*

– *for each edge $\ell \overset{\alpha,g}{\rightsquigarrow} \zeta$ in $\mathcal{M}$ with $\zeta(X, \ell') = p > 0$,*

$$\mathcal{P}((\ell, \eta), \alpha, A) := \int_0^\infty E(\ell)e^{-E(\ell)\tau} \cdot \mathbf{1}_g(\eta + \tau) \cdot p \; d\tau, \tag{1}$$

where $A = \{(\ell', \eta') \mid \exists \tau \in \mathbb{R}_{\geqslant 0}.\ \eta' = (\eta + \tau)[X := 0] \text{ and } \eta + \tau \models g\}$ and $\mathbf{1}_g(\cdot)$ is the characteristic function, i.e., $\mathbf{1}_g(\eta + \tau) = 1$ if $\eta + \tau \models g$; 0, otherwise.

We emphasize that MTA is a Markovian model with *decisions* instead of a pure stochastic model like *deterministic* MTA (DMTA) studied in [9], as in DMTA, there are no decision actions. To state it alternatively, any DMTA coincides with an MTA with $Act = \{\alpha\}$. Moreover, for DMTA, the edge relation is defined as $\rightsquigarrow \subseteq Loc \times \mathcal{CC}(\mathcal{X}) \times 2^{\mathcal{X}} \times Distr(Loc)$, while the MTA model allows for each set of transitions to reset their clocks differently. This has also been used in *probabilistic timed automata* (PTA, [11]). In this sense, our model can be considered as a continuous-time extension of PTA, due to the presence of exponential distributions. Any MTA for which the exit rate of any location is zero is a PTA. Locally uniform *continuous-time* MDP*s* (CTMDPs) [13] (the exit rate of a location doesn't depend on the action) with *finite* state space are zero-clock MTAs (i.e., $\mathcal{X} = \varnothing$). We note that as in MDPs, it is assumed that action labels from any location in MTA are pairwise different.

Finite paths in MTA $\mathcal{M}$ are of the form $\ell_0 \xrightarrow{\alpha_0,t_0} \ell_1 \xrightarrow{\alpha_1,t_1} \cdots \xrightarrow{\alpha_{n-1},t_{n-1}} \ell_n$, where for each edge $\ell_i \overset{\alpha_i,g_i}{\rightsquigarrow} \zeta_i$ of $\mathcal{M}$ with $\zeta_i(X_i, \ell_{i+1}) > 0$ ($\ell_i \in Loc$, $\alpha_i \in Act$, $t_i \in \mathbb{R}_{\geqslant 0}$, $X_i \subseteq \mathcal{X}$ and $0 \leqslant i < n$), we have that η_i is a valid clock valuation on *entering* location ℓ_i satisfying $\eta_0 = \vec{0}$, $(\eta_i + t_i) \models g_i$, and $\eta_{i+1} = (\eta_i + t_i)[X_i := 0]$. Let $Paths(\mathcal{M})$ (resp. $Paths_{\ell,\eta}(\mathcal{M})$) denote the set of finite paths (resp. starting in ℓ with initial clock valuation η) in $\mathcal{M}$. Given a set of locations $G \subseteq Loc$, we write $RPaths_{\ell,\eta}(G) \subseteq Paths_{\ell,\eta}(\mathcal{M})$ as the set of paths reaching G from location ℓ and clock-valuation η. We also define $Paths^n(\mathcal{M})$ (resp. $Paths^n_{\ell,\eta}(\mathcal{M})$) as the set of paths of length n. To simplify notation, we omit the references to $\mathcal{M}$ whenever possible. For $\rho \in Paths(\mathcal{M})$, let $\rho[n]{:=}\ell_n$ be the n-th location of ρ and $\rho\langle n\rangle{:=}t_n$ be the time spent in ℓ_n. We define the following sets: $\Omega = Act \times \mathbb{R}_{\geq 0} \times Loc$, $\mathcal{B}$ as the Borel σ-field over $\mathbb{R}_{\geq 0}$, $\mathcal{J}_{Loc} = 2^{Loc}$, $\mathcal{J}_{Act} = 2^{Act}$ and the σ-field $\mathcal{J} = \sigma(\mathcal{J}_{Act} \times \mathcal{B} \times \mathcal{J}_{Loc})$. The σ-field over the subsets of meassurable paths of length n of $Paths^n$ is defined as $\mathcal{J}_{Paths^n} = \sigma(\{Loc_0 \times M_0 \times \cdots \times M_{n-1} | Loc_0 \in \mathcal{J}_{Loc}, M_i \in \mathcal{J}\})$. A set $B \in \mathcal{J}_{Paths^n}$ is a base of a *cylinder set* C if $C = Cyl(B) = \{\rho \in Paths | \rho[0 \ldots n-1] \in B\}$, where $\rho[0 \ldots n-1]$ is the prefix of length n of path ρ. The σ-field $\mathcal{J}_{Paths}$ of measurable subsets of *Paths* is defined as $\mathcal{J}_{Paths} = \sigma(\cup_{n=0}^{\infty} \{Cyl(B) | B \in \mathcal{J}_{Paths^n}\})$.

Schedulers. The decision of which action to chose in an MTA is resolved by schedulers. [3] A scheduler must have enough "knowledge" to make such a decision which might be the current location and clock valuation (memoryless/positional schedulers), or the path from the initial to the current location (history dependent schedulers). We use $\mathcal{I}(\ell) \in Act$ to denote the set of actions enabled in location ℓ.

Definition 4 (Schedulers). *Let $\mathcal{M} = (Act, \mathcal{X}, Loc, \ell_0, E, \rightsquigarrow)$ be an* MTA*. A scheduler for $\mathcal{M}$ is a function $\theta : Paths(\mathcal{M}) \to Act$ such that for $n \in \mathbb{N}$,*

$$\theta(\ell_0 \xrightarrow{\alpha_0,t_0} \ell_1 \xrightarrow{\alpha_1,t_1} \cdots \xrightarrow{\alpha_{n-1},t_{n-1}} \ell_n) \in \mathcal{I}(\ell_n). \tag{2}$$

In the above definition we assume that the scheduler makes the decision as soon as a location is entered. These are called *early* schedulers [13]. In contrast, a *late* scheduler will decide which action to take upon leaving a location, i.e., besides the history it will consider also the waiting time. In this paper we will mainly consider early schedulers, although the theory can be adapted to late schedulers easily (see *Remark* 1).

[3] In control engineering, one tends to use another terminology, i.e., *controllers*, while in CS community, schedulers, adversaries, policies, strategies, *etc* are more common. We do *not* distinguish them in the current paper.

Probability measure. For an MTA $\mathcal{M}$ we define the following operators. Given a path $\rho \in \mathit{Paths}(\mathcal{M})$ and $m \in \Omega$ the *concatenation* of ρ and m is the path $\rho' = \rho \circ m$. Given a finite path ρ, $ll(\rho)$ return the last location of ρ. Given an initial clock valuation $\eta \in \mathcal{V}(\mathcal{X})$ and a finite path ρ, η_ρ is the clock valuation η' on entering location $ll(\rho)$ and for $m \in \Omega$, m_α returns the action associated to triple m. Given two locations ℓ, ℓ' and an action α, $g(\ell, \alpha, \ell')$ and $p(\ell, \alpha, \ell')$ returns the guard and probability associated to transition from ℓ to ℓ' under the action α, respectively. Now we can define the probability of measurable subsets of finite paths under a scheduler θ.

Definition 5 (Probability measure). *Let* $\mathcal{M} = (\mathit{Act}, \mathcal{X}, \mathit{Loc}, \ell_0, E, \rightsquigarrow)$ *be an* MTA*,* η *an initial clock valuation,* ν *an initial probability distribution,* $n \in \mathbb{N}$ *and* θ *a scheduler. Let* $\Pr^n_{\ell,\eta,\theta} : \mathcal{J}_{\mathit{Paths}^n} \to [0,1]$ *be the probability of all set of paths of length* $n > 0$ *starting in location* ℓ *with clock valuation* η*.* $\Pr^n_{\ell,\eta,\theta}$ *is defined inductively as follows*

$$\Pr^{n+1}_{\ell,\eta,\theta}(B) = \int_{\mathit{Paths}^n} \Pr^n_{\ell,\eta,\theta}(d\rho) \int_{\Omega} \mathbf{1}_B(\rho \circ m) \int_{\mathbb{R}_{\geq 0}} E(ll(\rho)) e^{-E(ll(\rho))\tau} \times \sum_{\ell' \in \mathit{Loc}} \mathbf{1}_m(\theta(\rho), \tau, \ell') \mathbf{1}_{g(ll(\rho),\theta(\rho),\ell')}(\eta_\rho + \tau) p(ll(\rho), \theta(\rho), \ell') dm d\tau,$$

where

- $B \in \mathit{Paths}^{n+1}$ *and for* $n = 0$ *we define* $\Pr^0_{\ell,\eta,\theta}(B) = 1$ *if* $\ell \in B$*, otherwise* 0*,*
- $\mathbf{1}_B(\rho \circ m) = 1$ *when* $\rho \circ m \in B$*, otherwise* 0*,*
- $\mathbf{1}_m(\theta(\rho), \tau, \ell') = 1$ *when* $m = (\theta(\rho), \tau, \ell')$*, otherwise* 0*,*
- $\mathbf{1}_{g(ll(\rho),\theta(\rho),\ell')}(\eta_\rho + \tau) = 1$ *when* $\eta_\rho + \tau \models g(ll(\rho), \theta(\rho), \ell')$*, otherwise* 0*.*

Intuitively $\Pr^{n+1}_{\ell,\eta,\theta}(B)$ is the probability of the set of paths ρ' of lenght $n+1$ defined as a product between the probability of the set of paths ρ of lenght n ($\rho' = \rho \circ m$) and the one-step transition probability to go from location $ll(\rho)$ to $ll(\rho')$.

For a measurable base $B \in \mathcal{J}_{\mathit{Paths}^n_{\ell,\eta}}$ and cylinder set $C = \mathit{Cyl}(B)$ we define $\Pr_{\ell,\eta,\theta}(C) = \Pr^n_{\ell,\eta,\theta}(B)$ as the probability of subsets of paths from $\mathit{Paths}_{\ell,\eta}$.

Maximum reachability. We are mainly interested in computing the maximum probability to reach a set of goal locations $G \subseteq \mathit{Loc}$ from the initial location ℓ_0.

Definition 6. *Let* $\mathcal{M} = (\mathit{Act}, \mathcal{X}, \mathit{Loc}, \ell_0, E, \rightsquigarrow)$ *be an* MTA *and Sched the set of all schedulers. The maximum probability to reach a set of goal locations* $G \subseteq \mathit{Loc}$ *from a location* ℓ *and clock valuation* η *is the function* $p^{\mathcal{M},G}_{\max} : \mathit{Loc} \times \mathcal{V}(\mathcal{X}) \to [0,1]$ *defined as*

$$p^{\mathcal{M},G}_{\max}(\ell, \eta) = \sup_{\theta \in \mathit{Sched}} \Pr_{\ell,\eta,\theta}(\mathit{RPaths}_{\ell,\eta}(G)).$$

In words, the maximum probability is the maximal one among all the schedulers.

The next theorem says that for MTA and maximum reachability probabilities, it suffices to consider *positional* schedulers instead of history dependent schedulers defined in Def. 4, i.e., the decision depends only on the current location and clock valuation.

Theorem 1 (Reachability in MTA). *Let $\mathcal{M} = (Act, \mathcal{X}, Loc, \ell_0, E, \rightsquigarrow)$ be an* MTA *and $G \subseteq Loc$ a set of goal locations. The maximum reachability probability $p^{\mathcal{M},G}_{\max}$ is the least fixpoint of the integral operator $\mathcal{F} : \big(Loc \times \mathcal{V}(\mathcal{X}) \to [0,1]\big) \to \big(Loc \times \mathcal{V}(\mathcal{X}) \to [0,1]\big)$, where for the given function $\Pr : Loc \times \mathcal{V}(\mathcal{X}) \to [0,1]$, location $\ell \in Loc$ and clock valuation η, $\mathcal{F}(\Pr)(\ell,\eta) = 1$ if $\ell \in G$, and $\mathcal{F}(\Pr)(\ell,\eta) = 0$ if $RPaths_{\ell,\eta}(G) = \varnothing$. For all other $\ell \notin G$ we have $\mathcal{F}(\Pr)(\ell,\eta) =$*

$$\max_{\alpha \in \mathcal{I}(\ell)} \left\{ \int_0^\infty E(\ell) e^{-E(\ell)\tau} \cdot \sum_{\ell \xmapsto[p,X]{\alpha,g} \ell'} \mathbf{1}_g(\eta+\tau) \cdot p \cdot \Pr(\ell',\eta') \; d\tau \right\}, \tag{3}$$

where transition $\ell \xmapsto[p,X]{\alpha,g} \ell'$ is defined by transition $\ell \overset{\alpha,g}{\rightsquigarrow} \zeta$, $\zeta(X,\ell') = p$ and $\eta' = (\eta+\tau)[X := 0]$.

Proof. In Eq.(3) the max function is outside the integral due to the fact that we consider only the early schedulers, i.e., the schedulers that don't chose an action depending on the waiting time in a location. First, we show that $p^{\mathcal{M},G}_{max}$ is a fixed point of $\mathcal{F}$.

- If $\ell \in G$ then $p^{\mathcal{M},G}_{max}(\ell,\eta) = 1 = \mathcal{F}(p^{\mathcal{M},G}_{max})(\ell,\eta)$
- If $\ell \notin G$ and $\rho \notin RPaths_{\ell,\eta}(G)$ then $p^{\mathcal{M},G}_{max}(\ell,\eta) = 0 = \mathcal{F}(p^{\mathcal{M},G}_{max})(\ell,\eta)$.
- If $\ell \notin G$ and $\rho \in RPaths_{\ell,\eta}(G)$ then we have to show that by using Eq.(3)

$$p^{\mathcal{M},G}_{max}(\ell,\eta) = \mathcal{F}(p^{\mathcal{M},G}_{max})(\ell,\eta).$$

We have that

$$\mathcal{F}(p^{\mathcal{M},G}_{max})(\ell,\eta) = \max_{\alpha \in \mathcal{I}(\ell)} \left\{ \int_0^\infty E(\ell) e^{-E(\ell)\tau} \right.$$
$$\left. \times \sum_{\ell \xmapsto[p,X]{\alpha,g} \ell'} \mathbf{1}_g(\eta+\tau) \cdot p \cdot \sup_{\theta \in Sched} \Pr_{\ell',\eta',\theta}(\Diamond G) \; d\tau \right\}.$$

Given the fact *Sched* is the class of early schedulers the choice of the action doesn't depending on the waiting time τ in a location. Therefore, we can take the "sup" term out of the integral.

$$\mathcal{F}(p^{\mathcal{M},G}_{max})(\ell,\eta) = \max_{\alpha \in \mathcal{I}(\ell)} \sup_{\theta \in Sched} \left\{ \int_0^\infty E(\ell) e^{-E(\ell)\tau} \right.$$
$$\left. \times \sum_{\ell \xmapsto[p,X]{\alpha,g} \ell'} \mathbf{1}_g(\eta+\tau) \cdot p \cdot \Pr_{\ell',\eta',\theta}(\Diamond G) \; d\tau \right\},$$

In the above equation, there are two decisions that are being made: over the set of action $\mathcal{I}(\ell)$ and the over the set of early schedulers *Sched*. We can combine both decisions into a single one given the fact that the set of early schedulers *Sched* are more general.

$$\mathcal{F}(p_{max}^{\mathcal{M},G})(\ell,\eta) = \sup_{\theta \in Sched} \left\{ \int_0^\infty E(\ell) e^{-E(\ell)\tau} \times \sum_{\ell \xrightarrow[p,X]{\alpha',g} \ell'} \mathbf{1}_g(\eta+\tau) \cdot p \cdot \Pr_{\ell',\eta',\theta}(\Diamond\, G)\, d\tau \right\},$$

where $\theta(\rho) = \alpha'$ for $\rho = \ell_0 \xrightarrow{\alpha_0,t_0} \ell_1 \xrightarrow{\alpha_1,t_1} \cdots \xrightarrow{\alpha_{n-1},t_{n-1}} \ell_n$, $\ell_n = \ell$ and $n \in \mathbb{N}$. Therefore, we obtain

$$\mathcal{F}(p_{max}^{\mathcal{M},G})(\ell,\eta) = \sup_{\theta \in Sched} \Pr_{\ell,\eta,\theta}(\Diamond\, G) = p_{max}^{\mathcal{M},G}(\ell,\eta).$$

Now we show that $p_{max}^{\mathcal{M},G}(\ell,\eta)$ is the least fixed point of $\mathcal{F}$. We define $p_{max}^{\mathcal{M},G,n}(\ell,\eta)$ as the probability to reach G in n steps. Using the same reasoning as from above one can show that

$$p_{max}^{\mathcal{M},G,n+1}(\ell,\eta) = \mathcal{F}(p_{max}^{\mathcal{M},G,n})(\ell,\eta).$$

By induction on n, we show $p_{max}^{\mathcal{M},G,n}(\ell,\eta) \leq \Pr(\ell,\eta)$ for another fixed point $\Pr(\ell,\eta)$ of $\mathcal{F}$.

– Base case: $p_{max}^{\mathcal{M},G,0}(\ell,\eta) = 1 = \Pr(\ell,\eta)$ for $\ell \in G$ and $p_{max}^{\mathcal{M},G,0}(\ell,\eta) = 0 = \Pr(\ell,\eta)$ if $\ell \notin G$ and $\rho \notin \mathit{RPaths}_{\ell,\eta}(G)$.
– Induction step:

$$p_{max}^{\mathcal{M},G,n+1}(\ell,\eta) = \max_{\alpha \in \mathcal{I}(\ell)} \left\{ \int_0^\infty E(\ell) e^{-E(\ell)\tau} \times \sum_{\ell \xrightarrow[p,X]{\alpha,g} \ell'} \mathbf{1}_g(\eta+\tau) \cdot p \cdot p_{max}^{\mathcal{M},G,n}(\ell',\eta')\, d\tau \right\}$$

$$p_{max}^{\mathcal{M},G,n+1}(\ell,\eta) \leq \max_{\alpha \in \mathcal{I}(\ell)} \left\{ \int_0^\infty E(\ell) e^{-E(\ell)\tau} \times \sum_{\ell \xrightarrow[p,X]{\alpha,g} \ell'} \mathbf{1}_g(\eta+\tau) \cdot p \cdot \Pr(\ell',\eta')\, d\tau \right\}$$

$$= \mathcal{F}(\Pr)(\ell,\eta) = \Pr(\ell,\eta).$$

It follows that $\Pr(\ell,\eta) \geq \lim_{n\to\infty} p_{max}^{\mathcal{M},G,n+1}(\ell,\eta) = p_{max}^{\mathcal{M},G}(\ell,\eta)$. □

Remark 1. One can transform Theorem 1 to deal with the class of *late* schedulers. This is obtained by moving the "max" term inside the integral of (3), i.e. $\Pr(\ell,\eta) =$

$$\int_0^\infty E(\ell) e^{-E(\ell)\tau} \cdot \max_{\alpha \in \mathcal{I}(\ell)} \left\{ \sum_{\ell \xrightarrow[p,X]{\alpha,g} \ell'} \mathbf{1}_g(\eta+\tau) \cdot p \cdot \Pr(\ell',\eta') \right\} d\tau.$$

2.3 Region construction for MTA

A main step in computing maximum reachability probability or the least fixpoint of the operator defined in Thm. 1 is to apply the *region construction* [1] in a similar way as for standard TA. Formally, a region is an equivalence class under $\cong$, an equivalence relation on clock valuations, which can be characterized by a specific form of a clock constraint. Let c_{x_i} be the largest constant with which $x_i \in \mathcal{X}$ is compared in some guard in the MTA. Clock evaluations $\eta, \eta' \in \mathcal{V}(\mathcal{X})$ are *clock-equivalent*, denoted $\eta \cong \eta'$, if and only if either

1. for any $x \in \mathcal{X}$ it holds: $\eta(x) > c_x$ and $\eta'(x) > c_x$, or
2. for any $x_i, x_j \in \mathcal{X}$ with $\eta(x_i), \eta'(x_i) \leqslant c_{x_i}$ and $\eta(x_j)$, $\eta'(x_j) \leqslant c_{x_j}$ it holds: $\eta(x_j) \leqslant \eta'(x_j)$ iff $\lfloor \eta(x_i) \rfloor = \lfloor \eta'(x_i) \rfloor$ and $\{\eta(x_i)\} \leqslant \{\eta'(x_i)\}$, where $\lfloor d \rfloor$ ($\{d\}$) is the integral (fractional) part of $d \in \mathbb{R}$.

This clock equivalence is coarser than the traditional definition [1] by merging the "non-delayable" regions (those with point constraints like "$x = 0$") into the "delayable" regions (those only with interval constraints like "$0 < y < 1$"). For instance, for $\mathcal{X} = \{x_1, x_2\}$, the non-delayable regions $(x_1 = 0, x_2 = 0)$, $(0 < x_1 < 1, x_2 = 0)$ and $(x_1 = 0, 0 < x_2 < 1)$ are merged with the delayable region $(0 < x_1 < 1, 0 < x_2 < 1)$ yielding $(0 \leqslant x_1 < 1, 0 \leqslant x_2 < 1)$. The reason for this slight change will become clear later. We define the boundary of a region Θ as $\partial\Theta = \overline{\Theta} \setminus \mathring{\Theta}$, where $\overline{\Theta}$ is the closure and $\mathring{\Theta}$ is the interior of Θ, respectively. For instance, a region $\Theta = (x_1 \leqslant 1, x_2 > 0)$ has its closure $\overline{\Theta} = (x_1 \leqslant 1, x_2 \geqslant 0)$, its interior $\mathring{\Theta} = (x_1 < 1, x_2 > 0)$ and its boundary $\partial\Theta = (x_1 = 1, x_2 = 2)$. Here Θ is viewed as a set of elements from $\mathcal{V}(\mathcal{X})$.

Let $\mathcal{R}e(\mathcal{X})$ be the set of regions over the set $\mathcal{X}$ of clocks. For $\Theta, \Theta' \in \mathcal{R}e(\mathcal{X})$, Θ' is the *successor region* of Θ if for all $\eta \models \Theta$ there exists $\delta \in \mathbb{R}_{>0}$ such that $\eta + \delta \models \Theta'$ and $\forall \delta' < \delta.\ \eta + \delta' \models \Theta \vee \Theta'$. The region Θ *satisfies* the guard g, denoted $\Theta \models g$, iff $\forall \eta \models \Theta.\ \eta \models g$. The *reset operation* on region Θ is defined as $\Theta[X := 0] := \big\{\eta[X := 0] \mid \eta \models \Theta\big\}$.

Notation: Given a tuple $v = (v_1, \cdots, v_n)$ with n components, by $v\lfloor_k$ we denote the k-th component of v. In particular, for a node $v = (\ell, \Theta)$, $v\lfloor_1$ returns a location, while $v\lfloor_2$ returns the associated region.

Definition 7 (Region graph of MTA). *The* region graph *of* MTA $\mathcal{M} = (Act, \mathcal{X}, Loc, \ell_0, E, \rightsquigarrow)$ *with the set of goal locations* $G \subseteq Loc$ *is* $\mathcal{G}(\mathcal{M}) = (Act, V, v_0, \Lambda, \hookrightarrow)$*, where*

- $V = Loc \times \mathcal{R}e(\mathcal{X})$ *is a finite set of* vertices *with* initial vertex $v_0 = (\ell_0, \Theta_0)$*, where* Θ_0 *is the initial region such that* $\vec{0} \in \Theta_0$*;*
- $\Lambda : V \to \mathbb{R}_{\geqslant 0}$ *is the* exit rate function *where:*

$$\Lambda(v) = \begin{cases} E(v\lfloor_1) & \text{if } v \overset{\alpha,p,X}{\hookrightarrow} v' \text{ for some } v' \in V \\ 0 & \text{otherwise.} \end{cases}$$

- $\hookrightarrow \subseteq V \times \Big(\big(Act \times [0,1] \times 2^{\mathcal{X}}\big) \cup \{\delta\}\Big) \times V$ *is the* transition (edge) relation*, such that:*

- ▶ $v \overset{\delta}{\hookrightarrow} v'$ *if* $v\lfloor_1 = v'\lfloor_1$, *and* $v'\lfloor_2$ *is the successor region of* $v\lfloor_2$*;*
- ▶ $v \overset{\alpha,p,X}{\hookrightarrow} v'$ *if* $v\lfloor_1 \xmapsto[p,X]{\alpha,g} v'\lfloor_1$ *with* $v\lfloor_2 \models g$*, and* $v\lfloor_2[X := 0] = v'\lfloor_2$*.*

We also define $\mathbb{G} = \{v \in V \mid v\lfloor_1 \in G\}$ to be the set of *goal vertices* in $\mathcal{G}(\mathcal{M})$.

Any vertex in the region graph is a pair consisting of a location and a region. For a vertex $v \in V$ and clock valuation $\eta \in \mathcal{V}(\mathcal{X})$ we define the boundary function $\flat(v, \eta) = \inf\{\delta \mid \eta + \delta \in \partial v\lfloor_2\}$, which is the minimum time (if it exists) to "hit" the boundary of the region corresponding to vertex v starting from a clock valuation η. Edges of the form $v \overset{\delta}{\hookrightarrow} v'$ are called *delay edges (jump)*, whereas those of the form $v \overset{\alpha,p,X}{\hookrightarrow} v'$ are called *Markovian edges (jump)*. Note that Markovian edges emanating from a vertex corresponding to a non-delayable region do *not* contribute to the reachability probability. The waiting time in such vertex is always zero. Therefore, we can safely remove all the Markovian edges emanating from vertices with non-delayable regions and combine each such non-delayable region with its unique delayable (direct) successor. In the sequel, by slight abuse of notation, we refer to this *simplified region graph* as $\mathcal{G}(\mathcal{M})$. Note that then $v\lfloor_2[X := 0] \subseteq v'\lfloor_2$ in the last item of Def. 7. An example region graph is shown in Fig. 4.

Example 2. For the MTA $\mathcal{M}$ in Fig. 3, the reachable part (forward reachable from the initial vertex and backward reachable from the accepting vertices) of the region graph $\mathcal{G}(\mathcal{M})$ is shown in Fig. 4.

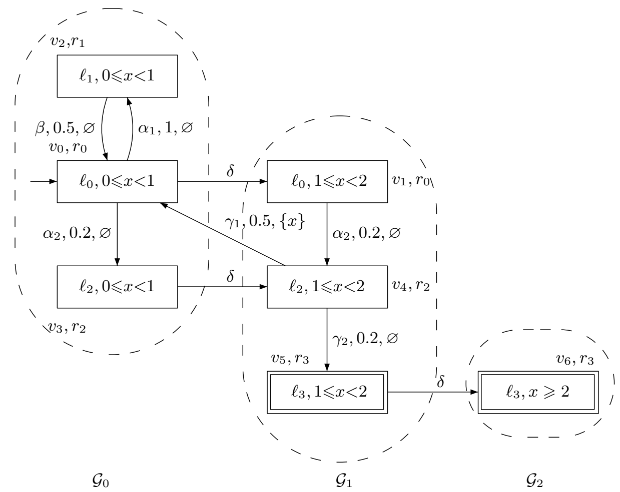

Fig. 4. Reachable region graph $\mathcal{G}(\mathcal{M})$

Now we define a system of integral equations on the region graph $\mathcal{G}(\mathcal{M})$ which will help to compute the maximum reachability probability from Thm. 1.

Definition 8. *Given the region graph $\mathcal{G}(\mathcal{M}) = (Act,\ V, v_0, \Lambda, \hookrightarrow)$ of the* MTA $\mathcal{M} = (Act, Loc,\ \mathcal{X}, \ell_0, E, \rightsquigarrow)$ *and the set of goal vertices* $\mathbb{G}$*, for the function* $Prob_v(\eta) : V \times \mathcal{V}(\mathcal{X}) \to [0,1]$ *let the operator* $\widetilde{\mathcal{F}} : (V \times \mathcal{V}(\mathcal{X}) \to [0,1]) \to (V \times \mathcal{V}(\mathcal{X}) \to [0,1])$ *be defined as* $\widetilde{\mathcal{F}}(Prob_v(\eta)) = 1$ *if* $v \in \mathbb{G}$*;* $\widetilde{\mathcal{F}}(Prob_v(\eta)) = 0$ *if* $v \notin \mathbb{G}$ *and* $\mathbb{G}$ *cannot be reached from* v*; and for* $v \notin \mathbb{G}$ *we have*

$$\begin{aligned}
\widetilde{\mathcal{F}}(Prob_v(\eta)) =& Prob_{v,\delta}(\eta) + \max_{\alpha \in \mathcal{I}(v\downharpoonright_1)} Prob_{v,\alpha}(\eta),\\
Prob_{v,\alpha}(\eta) =& \int_0^{\flat(v,\eta)} \Lambda(v)\cdot e^{-\Lambda(v)\tau} \cdot \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} p \cdot Prob_{v'}\big((\eta+\tau)[X{:=}0]\big)\ d\tau,\\
Prob_{v,\delta}(\eta) =& e^{-\Lambda(v)\flat(v,\eta)} \cdot Prob_{v'}\big(\eta + \flat(v,\eta)\big),
\end{aligned}$$

where $Prob_{v,\alpha}(\eta)$ *denotes the probability to reach* $\mathbb{G}$ *by taking a Markovian jump and* $Prob_{v,\delta}(\eta)$ *the probability to reach* $\mathbb{G}$ *through vertex* v' *by taking the delay jump* $v \overset{\delta}{\hookrightarrow} v'$*.*

Theorem 2. *Let* $\mathcal{M} = (Act, Loc, \mathcal{X}, \ell_0, E, \rightsquigarrow)$ *be an* MTA *with the set of goal locations* G *and* $Prob_v(\eta)$ *be the least fixpoint of the operator* $\widetilde{\mathcal{F}}$*, then for* $v\downharpoonright_1 = \ell$ *we have*

$$Prob_v(\eta) = p_{\max}^{\mathcal{M},G}(\ell,\eta).$$

Proof. Let $Prob_v(\eta) = \widetilde{\mathcal{F}}(Prob_v(\eta))$ and $\Pr(\ell,\eta) = \mathcal{F}(\Pr)(\ell,\eta))$ be two fixpoints from Definition 8 and Theorem 1, respectively. In order to prove the theorem we have to show that $Prob_v(\eta) = \Pr(\ell,\eta)$ for $v\downharpoonright_1 = \ell$.

From here on we will assume that the MTA has the clock constraints of the form $x \trianglelefteq c$, where $c \in \mathbb{N}_{\geqslant 0}$ and $\trianglelefteq \in \{\leq, <, \geq, >\}$. For a transition $\ell \xmapsto[p,X]{\alpha,g} \ell'$ with guard g and clock valuation η we get that

$$\mathcal{F}(\Pr)(\ell,\eta) = \max_{\alpha \in \mathcal{I}(\ell)} \left\{ \int_{t_1}^{t_2} \Lambda(\ell) e^{-\Lambda(\ell)\tau} \cdot \sum_{\ell \xmapsto[p,X]{\alpha,g} \ell'} p \cdot \Pr(\ell',\eta')\ d\tau \right\},$$

where $\eta + \tau \models g$ and $\tau \in]t_1, t_2[$, $t_1, t_2 \in \mathbb{Q}_{\geqslant 0} \cup \{\infty\}$. We define $\Pr^n(\ell,\eta)$ as the probability to reach the set of goal locations G in $n > 0$ steps

$$\mathcal{F}(\Pr^n)(\ell,\eta) = \max_{\alpha \in \mathcal{I}(\ell)} \left\{ \int_{t_1}^{t_2} \Lambda(\ell) e^{-\Lambda(\ell)\tau} \cdot \sum_{\ell \xmapsto[p,X]{\alpha,g} \ell'} p \cdot \Pr^{n-1}(\ell',\eta')\ d\tau \right\}.$$

For $n = 0$, $\Pr^0(\ell,\eta) = 1$ if $\ell \in G$ and $\Pr^0(\ell,\eta) = 0$ for $\ell \notin G$. In the same way we define $Prob_v^n(\eta)$ as the probability to reach the set of goal vertices $\mathbb{G}$ in $n > 0$

steps

$$\widetilde{\mathcal{F}}(Prob_v^n(\eta)) = Prob_{v,\delta}^n(\eta) + \max_{\alpha\in\mathcal{I}(v\downharpoonright_1)} Prob_{v,\alpha}^n(\eta),$$

$$Prob_{v,\alpha}^n(\eta) = \int_0^{\flat(v,\eta)} \Lambda(v)\cdot e^{-\Lambda(v)\tau}\cdot \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} p\cdot Prob_{v'}^{n-1}\big((\eta+\tau)[X{:=}0]\big)\ d\tau,$$

$$Prob_{v,\delta}^n(\eta) = e^{-\Lambda(v)\flat(v,\eta)}\cdot Prob_{v'}^n\big(\eta+\flat(v,\eta)\big).$$

For $n=0$, $Prob_v^0(\eta)=1$ if $v\in\mathbb{G}$ and $Prob_v^0(\eta)=0$ for $v\notin\mathbb{G}$. Now the task is to show that for all $n\in\mathbb{N}_{\geqslant 0}$ and $v\downharpoonright_1=\ell$,

$$\Pr^n(\ell,\eta) = Prob_v^n(\eta). \tag{4}$$

Notice that $\lim_{n\to\infty}\Pr^n(\ell,\eta)=\Pr(\ell,\eta)$ and $\lim_{n\to\infty} Prob_v^n(\eta)=Prob_v(\eta)$.

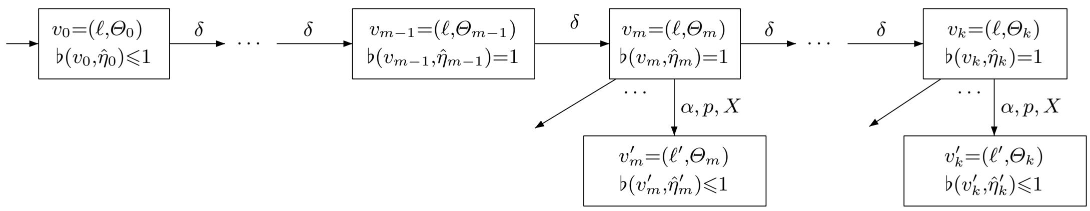

Fig. 5. The sub-region graph $\hat{\mathcal{G}}(\mathcal{M})$ for the transition from ℓ to ℓ'

We will show the validity of Eq.(4) by induction on n.

– Base case $n=0$: Eq.(4) trivially holds.
– Induction step: we have to show the validity of Eq.(4) for $n+1$. We consider the MTA transition $\ell \overset{\alpha,g}{\rightsquigarrow} \zeta$ and its corresponding region graph $\hat{\mathcal{G}}(\mathcal{M})$ shown in Fig. 5. For simplicity we consider that location ℓ induces the vertices $\{v_i=(\ell,\Theta_i)\mid 0\leqslant i\leqslant k\}$ with $v_0=v$. Note that for Markovian transitions, the regions stay the same. We denote $\hat\eta_i$ as the entering clock valuation in vertex v_i, for i the indices of the regions. Here $\hat\eta_0=\eta$ and $\hat\eta_i=\hat\eta_{i-1}+\flat(v_{i-1},\hat\eta_{i-1})$ for $1\le i\le k$. For any $\hat\eta\in\bigcup_{i=0}^{m-1}\Theta_i\cup\bigcup_{i>k}\Theta_i$, $\hat\eta\not\models g$; or more specifically,

$$t_1=\sum_{i=0}^{m-1}\flat(v_i,\hat\eta_i) \qquad\text{and}\qquad t_2=\sum_{i=0}^{k}\flat(v_i,\hat\eta_i).$$

For notation simplicity we define

$$\varXi_v^n(\eta) = Prob_{v,\delta}^n(\eta) + \max_{\alpha\in\mathcal{I}(v\downharpoonright_1)} Prob_{v,\alpha}^n(\eta).$$

Given the fact that from v_0 the process can only execute a delay transition before time t_1, it holds that

$$\varXi_{v_0}^{n+1}(\eta) = e^{-t_1\Lambda(v_0)}\cdot\varXi_{v_m}^{n+1}(\hat\eta_m),$$

$$\varXi_{v_m}^{n+1}(\hat\eta_m) = Prob_{v_m,\delta}^{n+1}(\hat\eta_m) + \max_{\alpha\in\mathcal{I}(v_m\downharpoonright_1)} Prob_{v_m,\alpha}^{n+1}(\hat\eta_m).$$

Notice that $\Lambda(v_0) = \Lambda(v_i)$ for all $i \leq k$. Therefore, by substitution we obtain:

$$
\begin{aligned}
&\Xi_{v_0}^{n+1}(\eta) \\
&= e^{-t_1\Lambda(v_0)} \cdot Prob_{v_m,\delta}^{n+1}(\hat{\eta}_m) + e^{-t_1\Lambda(v_0)} \cdot \max_{\alpha\in\mathcal{I}(v_m\lfloor_1)} Prob_{v_m,\alpha}^{n+1}(\hat{\eta}_m) \\
&= e^{-t_1\Lambda(v_0)} \cdot Prob_{v_m,\delta}^{n+1}(\hat{\eta}_m) + e^{-t_1\Lambda(v_0)} \cdot \max_{\alpha\in\mathcal{I}(v_m\lfloor_1)} \Bigg\{ \int_0^{\flat(v_m,\hat{\eta}_m)} \Lambda(v_m) \cdot e^{-\Lambda(v_m)\tau} \\
&\qquad \times \sum_{v_m \overset{\alpha,p,X}{\hookrightarrow} v'_m} p \cdot Prob_{v'_m}^{n}\big((\hat{\eta}_m+\tau)[X:=0]\big) \Bigg\} d\tau \\
&= e^{-t_1\Lambda(v_0)} \cdot Prob_{v_m,\delta}^{n+1}(\hat{\eta}_m) + \max_{\alpha\in\mathcal{I}(v_m\lfloor_1)} \Bigg\{ \int_{t_1}^{t_1+\flat(v_m,\hat{\eta}_m)} \Lambda(v_m) \cdot e^{-\Lambda(v_m)\tau} \\
&\qquad \times \sum_{v_m \overset{\alpha,p,X}{\hookrightarrow} v'_m} p \cdot Prob_{v'_m}^{n}\big((\hat{\eta}_m+\tau-t_1)[X:=0]\big) \Bigg\} d\tau.
\end{aligned}
$$

Evaluating each term $Prob_{v_m,\delta}^{n+1}(\hat{\eta}_m)$ we get the following sum of integrals:

$$
\begin{aligned}
&\Xi_{v_0}^{n+1}(\eta) \\
&= \max_{\alpha\in\mathcal{I}(v_m\lfloor_1)} \Bigg\{ \sum_{i=0}^{k-m} \int_{t_1+\sum_{j=0}^{i-1}\flat(v_{m+j},\hat{\eta}_{m+j})}^{t_1+\sum_{j=0}^{i}\flat(v_{m+j},\hat{\eta}_{m+j})} \Lambda(v_{m+i}) \cdot e^{-\Lambda(v_{m+i})\tau} \cdot \sum_{v_{m+i} \overset{\alpha,p,X}{\hookrightarrow} v'_{m+i}} p \\
&\qquad \times Prob_{v'_{m+i}}^{n}\Big((\hat{\eta}_{m+i}+\tau-t_1-\sum_{j=0}^{i-1}\flat(v_{m+j},\hat{\eta}_{m+j}))[X:=0]\Big) \Bigg\} d\tau.
\end{aligned}
$$

Notice that $\mathcal{I}(v_m\lfloor_1) = \mathcal{I}(v_{m+i}\lfloor_1)$ for all $i \leq k$. Now we will rewrite the function $\Xi_{v_0}^{n+1}(\eta)$ into an equivalent and simpler form by using the auxiliary function $F_\alpha^n(t)$. We define the function $F_\alpha^n(t) : [t_1, t_2] \to [0,1]$, such that when $t \in [t_1 + \sum_{j=0}^{i-1}\flat(v_{m+j},\hat{\eta}_{m+j}), t_1 + \sum_{j=0}^{i}\flat(v_{m+j},\hat{\eta}_{m+j})]$ for $i \leqslant k-m$ then

$$
F_\alpha^n(t) = \sum_{v_{m+i} \overset{\alpha,p,X}{\hookrightarrow} v'_{m+i}} p \cdot Prob_{v'_{m+i}}^{n}\Big((\hat{\eta}_{m+i}+t-t_1-\sum_{j=0}^{i-1}\flat(v_{m+j},\hat{\eta}_{m+j}))[X:=0]\Big)
$$

Using $F_\alpha^n(t)$ we can rewrite $\Xi_{v_0}^{n+1}(\eta)$ to an equivalent form as:

$$
\Xi_{v_0}^{n+1}(\eta) = \max_{\alpha\in\mathcal{I}(v_m\lfloor_1)} \int_{t_1}^{t_2} \Lambda(v_0) \cdot e^{-\Lambda(v_0)\tau} \cdot F_\alpha^n(\tau) d\tau. \tag{5}
$$

Here notice that

$$
\hat{\eta}_{m+i} = \eta + \sum_{j=0}^{m-1}\flat(v_j,\hat{\eta}_j) + \sum_{j=0}^{i-1}\flat(v_{m+j},\hat{\eta}_{m+j}).
$$

Therefore, for any $t \in [t_1 + \sum_{j=0}^{i-1} \flat(v_{m+j}, \hat{\eta}_{m+j}), t_1 + \sum_{j=0}^{i} \flat(v_{m+j}, \hat{\eta}_{m+j})]$, $i \leqslant k - m$ we obtain

$$\hat{\eta}_{m+i} + t - t_1 - \sum_{j=0}^{i-1} \flat(v_{m+j}, \hat{\eta}_{m+j}) = \eta + t.$$

From the I.H. we know that $\Pr^n(\ell, \eta) = Prob^n_{v_0}(\eta)$ such that $v_0\downharpoonright_1 = \ell$. Therefore, for any $t \in [t_1 + \sum_{j=0}^{i-1} \flat(v_{m+j}, \hat{\eta}_{m+j}), t_1 + \sum_{j=0}^{i} \flat(v_{m+j}, \hat{\eta}_{m+j})]$ and $v'_{m+i}\downharpoonright_1 = \ell'$, $i \leqslant k - m$, we get

$$\begin{aligned}
F^n_\alpha(t) = & \sum_{v_{m+i} \overset{\alpha,p,X}{\hookrightarrow} v'_{m+i}} p \cdot Prob^n_{v'_{m+i}}\big((\hat{\eta}_{m+i} + t - t_1 \\
& - \sum_{j=0}^{i-1} \flat(v_{m+j}, \hat{\eta}_{m+j}))[X := 0]\big) \\
= & \sum_{v_{m+i} \overset{\alpha,p,X}{\hookrightarrow} v'_{m+i}} p \cdot Prob^n_{v'_{m+i}}\big((\eta + t))[X := 0]\big) \\
= & \sum_{v_{m+i} \overset{\alpha,p,X}{\hookrightarrow} v'_{m+i}} p \cdot \Pr^n(\ell', (\eta + t))[X := 0]) \\
= & \sum_{\ell \xmapsto[p,X]{\alpha,g} \ell'} p \cdot \Pr^n(\ell', (\eta + t))[X := 0]).
\end{aligned}$$

Eq.(5) results in

$$\varXi^{n+1}_{v_0}(\eta) = \max_{\alpha \in \mathcal{I}(\ell)} \left\{ \int_{t_1}^{t_2} \varLambda(\ell) \cdot e^{-\varLambda(\ell)\tau} \cdot \sum_{\ell \xmapsto[p,X]{\alpha,g} \ell'} p \cdot \Pr^n(\ell', (\eta + \tau))[X := 0]) \right\} d\tau.$$

As $Prob^{n+1}_{v_0}(\eta) = \varXi^{n+1}_{v_0}(\eta)$ (for $v_0\downharpoonright_1 \notin \mathbb{G}$) we get that $Prob^{n+1}_{v_0}(\eta) = \Pr^{n+1}(\ell, \eta)$. $\square$

Based on this theorem, we will focus on the efficient computation of maximum reachability probabilities in region graphs (instead of in MTAs) in the following two sections. We will study two cases: time-bounded and unbounded reachability probabilities.

3 Time-Bounded Reachability

In this section, we concentrate on maximizing *time-bounded reachability probabilities* in a region graph $\mathcal{G}(\mathcal{M}) = (Act, V, v_0, \varLambda, \hookrightarrow)$, i.e., given a set $\mathbb{G}$ of goal locations and time bound $T \in \mathbb{N}$, we are interested in maximizing the probability to reach $\mathbb{G}$ *within* T *time units*. To this end, we introduce a *fresh* clock t, denoting the *global* time which is initialized to zero and never reset. To distinguish the role of t, we write the state of $\mathcal{G}(\mathcal{M})$ as (v, η, t). As in this case time bounds to reach $\mathbb{G}$ have to be considered, $t \leqslant T$ should be added to each vertex of $\mathcal{G}(\mathcal{M})$. Formally, by notation abuse we overload $\flat(v, \eta, t)$ to be $\min\{\flat(v, \eta), T - t\}$, i.e. the minimum time to hit the boundary $\partial v\lfloor_2$ *at time* $t \leqslant T$. For instance, let $\partial v\lfloor_2$ be $x = 1 \wedge y = 2$, $T = 100$, and suppose $\eta(x) = 0.5$, $\eta(y) = 1.7$. If $t = 2.5$, then $\flat(v, \eta, t) = \min\{1 - 0.5, 2 - 1.7, 100 - 2.5\} = 0.3$; if $t = 99.9$, then $\flat(v, \eta, t) = \min\{1 - 0.5, 2 - 1.7, 100 - 99.9\} = 0.1$;

The following Bellman (dynamic programming) equations derived from Def. 8 play an essential role in solving the time-bounded reachability problem. Let $P(v, \eta, t)$ be the maximum probability at time t for state (v, η) to reach $\mathbb{G}$ within time bound T. $P(v, \eta, t) = 1$ if $v \in \mathbb{G}$ and $t \leqslant T$; 0 if $t > T$ or $\mathbb{G}$ is *not* reachable from v; and otherwise

$$\begin{aligned} P(v,\eta,t) = \; & \underbrace{\max_{\alpha \in \mathcal{I}(v)} \left\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^{\flat(v,\eta,t)} \underbrace{\Lambda(v) e^{-\Lambda(v)\tau} p}_{(\star)} \cdot P(v', \eta', t+\tau)\, d\tau \right\}}_{\text{(I)}} \\ & \quad + \underbrace{\underbrace{e^{-\Lambda(v)\flat(v,\eta,t)}}_{(\star\star)} P\Big(v'',\; \eta + \flat(v,\eta,t),\; t + \flat(v,\eta,t)\Big)}_{\text{(II)}}, \end{aligned} \tag{6}$$

where $\mathcal{I}(v)$ is the set of actions enabled in v, $\eta' = (\eta + \tau)[X := 0]$ and $v \overset{\delta}{\hookrightarrow} v''$, where v'' is the time successor of v. Term (I) represents the maximum reachability probability (among all enabled actions) by taking a Markovian jump $v \overset{\alpha,p,X}{\hookrightarrow} v'$ and (II) represents the probability of taking the boundary jump $v \overset{\delta}{\hookrightarrow} v''$. Note that $(\star)$ is the density function of taking $v \overset{\alpha,p,X}{\hookrightarrow} v'$ at time τ and $(\star\star)$ is the probability not to leave location v within $\flat(v, \eta, t)$ time units.

We will now provide two ways to solve (6): one by discretization (6) and the other based on the Hamilton-Jacobi-Bellman equation [10].

3.1 Discretization

Our first approach is to *discretize the continuous variables* in the Bellman equation. Using a discretization step $h = \frac{1}{N}$ ($N \in \mathbb{N}_{>0}$), the aim is to obtain a *finite-state* MDP $\mathcal{D}(\mathcal{M})$ from $\mathcal{G}(\mathcal{M})$. For this MDP, a similar Bellman equation can be derived and solved efficiently e.g. by value iteration [3]. Intuitively, h is the length of time in which a *single* Markovian jump takes place from a given location. By h, each Markovian jump in $\mathcal{G}(\mathcal{M})$ can be approximated by a Markovian jump which only takes place at time points $\{0, h, \ldots, NT\}$. This gives rise to an MDP:

Definition 9. *Given $\mathcal{G}(\mathcal{M}) = (Act, V, v_0, \Lambda, \hookrightarrow)$ and discretization step $h = \frac{1}{N}$ ($N \in \mathbb{N}_{>0}$), the* MDP $\mathcal{D}_h(\mathcal{M}) = (Act \cup \{\perp\}, S, s_0, \mathcal{P})$ *is defined as follows:*

- $S = \{(v, \eta, t) \mid v \in V \wedge \eta \in v\lfloor_2 \wedge t \leqslant T\}$;
- $s_0 = (v_0, \vec{0}, 0)$;
- $\perp$ *is a* fresh *action encoding the "delay" in* $\mathcal{G}(\mathcal{M})$;

For each $(v, \eta, t) \in S$ we distinguish three cases:

(i) If $h < \flat(v, \eta, t)$ and $v \overset{\alpha,p,X}{\hookrightarrow} v'$ then

$$\mathcal{P}\big((v, \eta, t), \alpha, (v', \eta[X := 0], t)\big) = p \cdot (1 - e^{-\Lambda(v)h});$$
$$\mathcal{P}\big(\,(v, \eta, t), \alpha, (v, \eta + h, t + h)\,\big) = e^{-\Lambda(v)h};$$

(ii) If $h < \flat(v, \eta, t)$ and $Act(v) = \varnothing$ then

$$\mathcal{P}\big(\,(v, \eta, t), \perp, (v, \eta + h, t + h)\,\big) = e^{-\Lambda(v)h};$$

(iii) If $h \geqslant \flat(v, \eta, t)$ and $v \overset{\delta}{\hookrightarrow} v'$ then

$$\mathcal{P}\big(\,(v, \eta, t), \perp, (v', \eta + \flat(v, \eta, t), t + \flat(v, \eta, t))\,\big) = e^{-\Lambda(v)\flat(v,\eta,t)}.$$

Each state in the MDP $\mathcal{D}_h(\mathcal{M})$ has an outgoing transition of type (i), (ii) or (iii).

Example 3. Fig. 6 depicts the first half (till $2h$) of the reachable part of the MDP $\mathcal{D}_h(\mathcal{M})$ for the region graph $\mathcal{G}(\mathcal{M})$ in Fig. 4 with the step size $h = \frac{1}{2}$ and time bound $T = 2$. This means that in the MDP, the goal state(s) should be reached within 4 steps. Therefore, the second half ($3h$ and $4h$) is not necessary anyway. We add (i), (ii) and (iii) onto the edge labels to indicate which type of transition it is.

Let $Y(v, \eta, t)$ be the maximum reachability probability in the MDP $\mathcal{D}_h(\mathcal{M})$. Then we have that

$$Y(v, \eta, t) = \tag{7}$$
$$\begin{cases} e^{-\Lambda(v)\flat(v,\eta,t)} Y\big(v'',\ \eta{+}\flat(v, \eta, t),\ t{+}\flat(v, \eta, t)\big), \text{if } h \geqslant \flat(v, \eta, t) \\ \max\limits_{\alpha \in \mathcal{I}(v)} \Big\{ \sum\limits_{v \overset{\alpha,p,X}{\hookrightarrow} v'} (1{-}e^{-\Lambda(v)h}) p{\cdot}Y(v', \widetilde{\eta}', t) \Big\} {+} e^{-\Lambda(v)h} Y(v, \eta{+}h, t{+}h), o/w, \end{cases}$$

where $\widetilde{\eta}' = \eta[X := 0]$ and $v \overset{\delta}{\hookrightarrow} v''$.

By the discretization step h, the regions of $\mathcal{G}(\mathcal{M})$ contain finitely many points (one point is a state in the MDP). To be more exact, each region has maximally h points for each clock. Together with the fact that there are only finitely many regions of interest in $\mathcal{G}(\mathcal{M})$, the number of states in the MDP is finite.

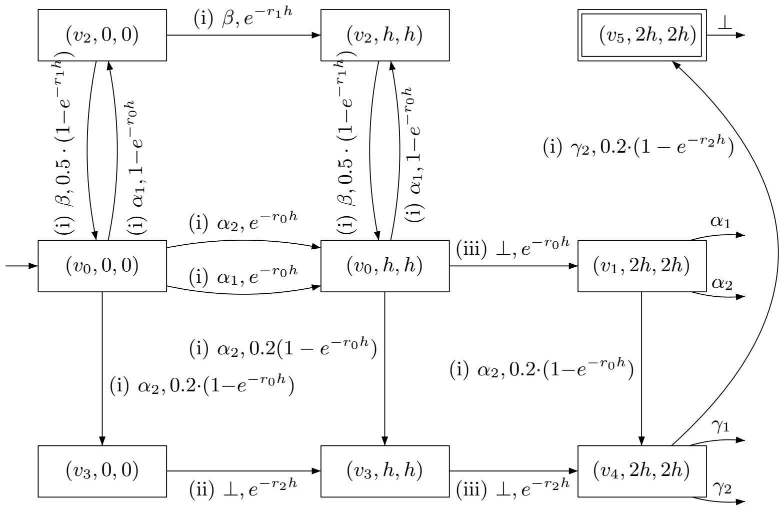

Fig. 6. MDP obtained from the region graph with discretization step h.

Based on Eq.(6) we define two integral operators

$$\mathcal{F} : (V \times \mathcal{V}(\mathcal{X}) \times \mathbb{R}_{\geqslant 0} \to [0,1]) \to (V \times \mathcal{V}(\mathcal{X}) \times \mathbb{R}_{\geqslant 0} \to [0,1]),$$
$$\widehat{\mathcal{F}} : (V \times \mathcal{V}(\mathcal{X}) \times \mathbb{R}_{\geqslant 0} \to [0,1]) \to (V \times \mathcal{V}(\mathcal{X}) \times \mathbb{R}_{\geqslant 0} \to [0,1]),$$

where

$$(\mathcal{F}H)(v,\eta,t) =$$
$$\begin{cases} \max_{\alpha\in\mathcal{I}(v)} \left\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} H(v',\eta',t+\tau)d\tau \right\} + \\ \qquad e^{-\Lambda(v)h} H(v,\eta+h,t+h), & \text{if } h < \flat(v,\eta,t) \\ e^{-\Lambda(v)\flat(v,\eta,t)} H(v'',\eta+\flat(v,\eta,t),t+\flat(v,\eta,t)), & \text{if } h \geq \flat(v,\eta,t) \end{cases}$$

and

$$(\widehat{\mathcal{F}}H)(v,\eta,t) =$$
$$\begin{cases} \max_{\alpha\in\mathcal{I}(v)} \left\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} H(v',\widetilde{\eta}',t)d\tau \right\} + \\ \qquad e^{-\Lambda(v)h} H(v,\eta+h,t+h), & \text{if } h < \flat(v,\eta,t) \\ e^{-\Lambda(v)\flat(v,\eta,t)} H(v'',\eta+\flat(v,\eta,t),t+\flat(v,\eta,t)), & \text{if } h \geq \flat(v,\eta,t) \end{cases}$$

such that $\eta' = (\eta+\tau)[X := 0]$ and $\widetilde{\eta}' = \eta[X := 0]$. The integral operators act on measurable functions $H : V \times \mathcal{V}(\mathcal{X}) \times \mathbb{R}_{\geqslant 0} \to [0,1]$ such that $(\mathcal{F}H)(v,\eta,t) = 1$ and $(\widehat{\mathcal{F}}H)(v,\eta,t) = 1$ if $v \in \mathbb{G}$ and $t \leq T$.

Lemma 1. *For any measurable function $H : V \times \mathcal{V}(\mathcal{X}) \times \mathbb{R}_{\geqslant 0} \to [0,1]$ and the least fixpoint $Y(v,\eta,t)$ for the equation $H(v,\eta,t) = (\widehat{\mathcal{F}}H)(v,\eta,t)$ we have $(\widehat{\mathcal{F}}H)(v,\eta,t) \geq Y(v,\eta,t)$, where $v \in V$, $\eta \in v\lfloor_2$ and $t \leq T$.*

Proof. Let $H(v,\eta,t)$ be a fixpoint for the equation $H(v,\eta,t) = (\widehat{\mathcal{F}}H)(v,\eta,t)$ in order to show that $(\widehat{\mathcal{F}}H)(v,\eta,t) \geq Y(v,\eta,t)$ we will show by induction on $n \in \mathbb{N}$ that $(\widehat{\mathcal{F}}H)(v,\eta,t) \geq Y^n(v,\eta,t)$, where $\lim_{n\to\infty} Y^n(v,\eta,t) = Y(v,\eta,t)$.

- Base case: $Y^0(v,\eta,t) = 1 = (\widehat{\mathcal{F}}H)(v,\eta,t)$ if $v \in \mathbb{G}$ and $t \leq T$ and $Y^0(v,\eta,t) = 0 \leq (\widehat{\mathcal{F}}H)(v,\eta,t)$, otherwise.
- Induction hypothesis: $Y^n(v,\eta,t) \leq (\widehat{\mathcal{F}}H)(v,\eta,t)$.
- Induction step: for $h < \flat(v,\eta,t)$ we get

$$\begin{aligned}
&Y^{n+1}(v,\eta,t) = (\widehat{\mathcal{F}}Y^n)(v,\eta,t) \\
&= \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} Y^n(v',\widetilde{\eta}',t)d\tau \Big\} + e^{-\Lambda(v)h} Y^n(v,\eta+h,t+h) \\
&\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} H(v',\widetilde{\eta}',t)d\tau \Big\} + e^{-\Lambda(v)h} H(v,\eta+h,t+h) \\
&\hspace{25em} \text{(By I.H.)} \\
&=(\widehat{\mathcal{F}}H)(v,\eta,t).
\end{aligned}$$

We obtain that $(\widehat{\mathcal{F}}H)(v,\eta,t) \geq Y^{n+1}(v,\eta,t)$ and $(\widehat{\mathcal{F}}H)(v,\eta,t) \geq \lim_{n\to\infty} Y^n(v,\eta,t) = Y(v,\eta,t)$. □

Lemma 2. *For the least fixpoints $P(v,\eta,t)$ and $Y(v,\eta,t)$ given by the equations $P(v,\eta,t) = (\mathcal{F}P)(v,\eta,t)$ and $Y(v,\eta,t) = (\widehat{\mathcal{F}}Y)(v,\eta,t)$ we get that $Y(v,\eta,t) \geq P(v,\eta,t)$, where $v \in V$, $\eta \in v\downharpoonright_2$ and $t \leq T$.*

Proof. We will prove that $P^n(v,\eta,t) - Y^n(v,\eta,t) \leq 0$, by induction on n, where $\lim_{n\to\infty} P^n(v,\eta,t) = P(v,\eta,t)$ and $\lim_{n\to\infty} Y^n(v,\eta,t) = Y(v,\eta,t)$.

- Base case: $P^0(v,\eta,t) = 1 = Y^0(v,\eta,t)$ if $v \in \mathbb{G}$, and $t \leq T$ and $P^0(v,\eta,t) = 0 = Y^0(v,\eta,t)$, otherwise.
- Induction hypothesis: $P^n(v,\eta,t) - Y^n(v,\eta,t) \leq 0$.
- Induction step: for $h < \flat(v,\eta,t)$ we get

$$\begin{aligned}
&P^{n+1}(v,\eta,t) - Y^{n+1}(v,\eta,t) \\
&= \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} P^n(v',\eta',t+\tau)d\tau \Big\} + e^{-\Lambda(v)h} P^n(v,\eta+h,t+h) \\
&\quad - \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} Y^n(v',\widetilde{\eta}',t)d\tau \Big\} - e^{-\Lambda(v)h} Y^n(v,\eta+h,t+h) \\
&\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} (P^n(v',\eta',t+\tau) - Y^n(v',\widetilde{\eta}',t))d\tau \Big\} \\
&\quad + e^{-\Lambda(v)h}(P^n(v,\eta+h,t+h) - Y^n(v,\eta+h,t+h)).
\end{aligned}$$

From the definition of the maximum time-bounded reachability we know that $P(v,\eta,t)$ is the maximum probability to reach a set of goal states in $T-t$ units of time. Therefore, the function $P(v,\eta,t)$ is monotonously decreasing in its third argument, i.e., by increasing t the value of $P(v,\eta,t)$ decreases. As a result we get that $P(v',\widetilde{\eta}',t) \geq P(v',\eta',t+\tau)$ for all $\tau \in [0,h]$. We obtain that $P^n(v',\widetilde{\eta}',t) \geq P^n(v',\eta',t+\tau)$ for all $\tau \in [0,h]$ and:

$$
\begin{aligned}
&P^{n+1}(v,\eta,t) - Y^{n+1}(v,\eta,t)\\
&\quad\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau}(P^n(v',\widetilde{\eta}',t) - Y^n(v',\widetilde{\eta}',t))d\tau \Big\}\\
&\quad+e^{-\Lambda(v)h}(P^n(v,\eta+h,t+h) - Y^n(v,\eta+h,t+h)).
\end{aligned}
$$

From induction hypothesis we know that $P^n(v',\widetilde{\eta}',t) - Y^n(v',\widetilde{\eta}',t) \leq 0$ and $P^n(v,\eta+h,t+h) - Y^n(v,\eta+h,t+h) \leq 0$ as result we obtain $P^{n+1}(v,\eta,t) - Y^{n+1}(v,\eta,t) \leq 0$, which proves the lemma.

□

Note that by using the integral operator $\widehat{\mathcal{F}}$ on function $Y(v,\eta,t)$ we obtain Eq. (7) for $h < \flat(v,\eta,t)$ as follows:

$$
\begin{aligned}
&Y(v,\eta,t)\\
&= \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} Y(v',\widetilde{\eta}',t)d\tau \Big\} + e^{-\Lambda(v)h}Y(v,\eta+h,t+h)\\
&= \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} d\tau \cdot Y(v',\widetilde{\eta}',t) \Big\} + e^{-\Lambda(v)h}Y(v,\eta+h,t+h)\\
&= \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} p(1-e^{-\Lambda(v)h}) Y(v',\widetilde{\eta}',t) \Big\} + e^{-\Lambda(v)h}Y(v,\eta+h,t+h).
\end{aligned}
$$

Theorem 3 (Error bound). *For any state (v,η), time bound T, discretization step $h = \frac{1}{N}$ and $\lambda = \max_{v\in V}\{\Lambda(v)\}$ which is the maximum rate of all exponential distributions appearing in the region graph:*

$$\sup_{t\in[0,T]} |P(v,\eta,t) - Y(v,\eta,t)| \leq (1-e^{-\lambda h})(1-e^{-\lambda T}).$$

Proof. By using the lemmas 1,2 we get that:

$$
\begin{aligned}
\sup_{t\in[0,T]} |P(v,\eta,t) - Y(v,\eta,t)| &\leq \sup_{t\in[0,T]} |P(v,\eta,t) - (\widehat{\mathcal{F}}P)(v,\eta,t)|\\
&\leq \sup_{t\in[0,T]} |(\mathcal{F}P)(v,\eta,t) - (\widehat{\mathcal{F}}P)(v,\eta,t)|,
\end{aligned}
$$

where $P(v,\eta,t)$ is the least fixpoint of the equation $P(v,\eta,t) = (\mathcal{F}P)(v,\eta,t)$. Therefore, we have to show that

$$\sup_{t\in[0,T]} \Big|(\mathcal{F}P)(v,\eta,t) - (\widehat{\mathcal{F}}P)(v,\eta,t)\Big| \leq (1-e^{-\lambda h})(1-e^{-\lambda T})$$

for every vertex v and clock valuation η.

For the case $h \geq \flat(v,\eta,t)$ the theorem trivially holds as $(\mathcal{F}P)(v,\eta,t) - (\widehat{\mathcal{F}}P)(v,\eta,t) = 0$. On the other hand for $h < \flat(v,\eta,t)$ we have the following

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left|(\mathcal{F}P)(v,\eta,t) - (\widehat{\mathcal{F}}P)(v,\eta,t)\right| \\
&\leq \sup_{t\in[0,T]} \left| \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau}(P(v',\eta',t+\tau) - P(v',\widetilde{\eta}',t))d\tau \Big\} \right| \\
&\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} \sup_{t\in[0,T]} \left|P(v',\eta',t+\tau) - P(v',\widetilde{\eta}',t)\right| d\tau \Big\} \\
&\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} \sup_{t\in[0,T]} \left|P(v',\widetilde{\eta}',t)\right| d\tau \Big\} \\
&\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^h p\Lambda(v)e^{-\Lambda(v)\tau} d\tau \sup_{t\in[0,T]} \left|P(v',\widetilde{\eta}',t)\right| \Big\} \\
&\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} p(1-e^{-\Lambda(v)h}) \sup_{t\in[0,T]} \left|P(v',\widetilde{\eta}',t)\right| \Big\} \\
&\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} p(1-e^{-\Lambda(v)h})P(v',\widetilde{\eta}',0) \Big\} \\
&\leq \max_{\alpha\in\mathcal{I}(v)} \Big\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} p(1-e^{-\Lambda(v)h})(1-e^{-\Lambda(v')T}) \Big\} \\
&\leq (1-e^{-\lambda h})(1-e^{-\lambda T}),
\end{aligned}
$$

where $\lambda = \max_{v\in V}\{\Lambda(v)\}$. Note that in the above derivations we have used an upper bound for the maximum time-bounded reachability, i.e., $P(v',\widetilde{\eta}',0) \leq (1-e^{-\Lambda(v')T})$. The upper bound represents the probability to reach a set of goal states in T units of time that are also all successor states of v'. □

Comparison with [14]. As said, CTMDPs are essentially zero-clock MTAs, hence the above error bound also applies to CTMDP. Recently in [14], the authors provided an error bound $\frac{\lambda^2 Th}{2}$. Our error bound actually improves this, yielding a substantial reduction in the required number of iterations. This is shown in Tab. 1).

3.2 Hamilton-Jacobi-Bellman Equations

As in traditional control theory [3], the dynamic programming principles lead to a first-order integro-differential equation, which is the *Hamilton-Jacobi-Bellman* (HJB) *partial differential equation* (PDE).

T	ϵ	N_1	N_2	$prob_1$	$prob_2$
1	10^{-2}	173	201	$3.07535 \cdot 10^{-6}$	$3.06691 \cdot 10^{-6}$
1	10^{-4}	17323	20055	$3.06783 \cdot 10^{-6}$	$3.06807 \cdot 10^{-6}$
1	10^{-6}	1732390	2005400	$3.06795 \cdot 10^{-6}$	$3.06795 \cdot 10^{-6}$
10	10^{-2}	200	20055	$7.1097 \cdot 10^{-5}$	$7.10174 \cdot 10^{-5}$
10	10^{-4}	20026	2005400	$7.10122 \cdot 10^{-5}$	$7.10156 \cdot 10^{-5}$
10	10^{-6}	2002700	200540366	$7.10156 \cdot 10^{-5}$	aborted
100	10^{-2}	200	2005400	0.000881425	0.000882816

Table 1. N_1 iterations corresponding to bound $(1-e^{-\lambda h})(1-e^{-\lambda T})$ and N_2 iterations corresponding to bound $\frac{\lambda^2 Th}{2}$ [14].

Given the region graph $\mathcal{G}(\mathcal{M}) = (Act, V, v_0, \Lambda, \hookrightarrow)$ let $f(v,\eta,t) := P(v,\eta,t)$ be the maximum time-bounded reachability probability at time t in $\mathcal{G}(\mathcal{M})$. For every $v \in V$ and $\eta \in \mathcal{V}(\mathcal{X})$ and $t \leqslant T$ let $f(v,\eta,t)$ be given as follows:

$$\frac{\partial f(v,\eta,t)}{\partial t} + \sum_{i=1}^{|\mathcal{X}|} \frac{\partial f(v,\eta,t)}{\partial \eta^{(i)}} =$$
$$\max_{\alpha \in \mathcal{I}(v)} \left\{ \Lambda(v) \cdot \sum_{v \overset{\alpha,X,p}{\hookrightarrow} v'} p\left(f(v,\eta,t) - f\left(v',\eta[X := 0],t\right)\right) \right\},$$

where $\eta^{(i)}$ is the i'th clock variable. The initial conditions of the above PDE are $f(v,\eta,T) = \mathbf{1}_{\mathbb{G}}(v,\eta)$ for any $v \in V$ and $\eta \in v\lfloor_2$. Moreover, for every $\eta \in \partial v\lfloor_2$ and transition $v \overset{\delta}{\hookrightarrow} v'$, the boundary conditions take the form $f(v,\eta,t) = f(v',\eta,t)$.

Example 4. For region graph $\mathcal{G}(\mathcal{M})$ in Fig. 4 and subgraph $\mathcal{G}_0$ we can define the following system of PDEs

$$\begin{aligned}
\frac{\partial f(v_0,x,t)}{\partial t} + \frac{\partial f(v_0,x,t)}{\partial x} &= \max\Big\{ r_0 \cdot (f(v_0,x,t) - f(v_2,x,t)), \\
&\qquad 0.2 \cdot r_0 \cdot (f(v_0,x,t) - f(v_3,x,t)) \Big\}, \\
\frac{\partial f(v_2,x,t)}{\partial t} + \frac{\partial f(v_2,x,t)}{\partial x} &= 0.5 \cdot r_1 \cdot (f(v_2,x,t) - f(v_0,x,t)),
\end{aligned}$$

with the boundary conditions $f(v_0,1,t) = f(v_1,1,t)$ and $f(v_3,1,t) = f(v_4,1,t)$. Here x is the clock variable.

Several methods can be used to solve the above HJB equation, e.g., the finite volume method [16] or the time and state space discretization technique [8].

3.3 Zero-Clock MTA

Following the same reasoning, for CTMDPs, i.e., zero-clock MTA, we can obtain similar results, except that instead of a system of PDEs we obtain a system of

ODEs. As CTMDPs have no clocks, the resulting state space is finite. In the above section we have defined $f(v, \eta, t)$ as the maximum reachability probability. Given a *finite* state space, $f(v, \eta, t)$ can be simplified to $P_{i,j}(t)$, which is the probability to reach state ℓ_j at time T starting from state ℓ_i at time t. For any two states ℓ_i and ℓ_j we obtain the ODE:

$$\frac{dP_{i,j}(t)}{dt} = \max_{\alpha \in \mathcal{I}(\ell_i)} \Big\{ E_i \sum_k p_{i,k}(\alpha) \left(P_{i,j}(t) - P_{k,j}(t) \right) \Big\},$$

where $E_i := E(\ell_i)$ and $p_{i,k}(\alpha) := p$ such that $\ell_i \overset{\alpha,p}{\rightsquigarrow} \zeta_i$ and $\zeta_i(\varnothing, \ell_i) = \ell_k$. The system of ODEs can be also rewritten in the following matrix form:

$$\frac{d\widehat{\mathbf{\Pi}}(t)}{dt} = - \max_{\alpha \in Act} \Big\{ \mathbf{Q}(\alpha) \widehat{\mathbf{\Pi}}(t) \Big\}, \quad t \leqslant T, \tag{8}$$

where $\widehat{\mathbf{\Pi}}(t)$ is the transition probability matrix at time t (the element (i, j) of $\widehat{\mathbf{\Pi}}(t)$ is $P_{i,j}(t)$), $\widehat{\mathbf{\Pi}}(T) = \mathbf{I}$, $\mathbf{Q}(\alpha) = \mathbf{R}(\alpha) - \mathbf{E}$ is the infinitesimal generator where $\mathbf{R}(\alpha)$ is the rate matrix (its element (i, j) is $E_i p_{i,j}(\alpha)$) and $\mathbf{E}$ is the exit rate matrix (all diagonal elements are the exit rates whereas the off-diagonal elements are zero). A recent work [7] reveals that the above system of ODEs can be solved more efficiently than the general system of PDEs by adopting adaptive uniformization.

4 Unbounded Reachability

In this section, we focus on maximizing *unbounded reachability probabilities*. In contrast to the *time-bounded* case, there is no constraint on the time to reach the goal states $\mathbb{G}$. Let $P(v, \eta)$ be the maximum probability to reach $\mathbb{G}$ starting from vertex v and clock valuation η. Directly from the characterization given in Def. 8, we obtain

$$P(v, \eta) = \max_{\alpha \in \mathcal{I}(v)} \Bigg\{ \sum_{v \overset{\alpha,p,X}{\hookrightarrow} v'} \int_0^{\flat(v,\eta)} \underbrace{\Lambda(v)\, e^{-\Lambda(v)\tau} p}_{(\star)} \cdot P(v', \eta') d\tau \Bigg\} + \underbrace{e^{-\Lambda(v)\flat(v,\eta)}}_{(\star\star)} P\big(v'', \eta + \flat(v, \eta)\big), \tag{9}$$

where $\eta' = (\eta + \tau)[X := 0]$, $v \overset{\delta}{\hookrightarrow} v''$ and $P(v, \eta) = 1$ for $v \in \mathbb{G}$. We can solve (9) by using the discretization approach from Def. 9, except that there is a minor difficulty, namely, the number of solutions of the system of integral equations is in general infinite, which means that the derived error bounds cannot be used directly. We proceed by considering some special cases, depending on the number of clocks in the model. For the *zero-clock* MTA (i.e. CTMDPs), it is trivial since one can use the *embedded* MDP of the CTMDP and solve the reachability probability optimization problem [15]. Note that in the time-bounded case the *embedded* MDP does not suffice. We now move to the *single-clock* case.

4.1 Single-Clock Case

For single-clock MTA, we will show that the Bellman equation (9) can be simplified to a system of *linear* equations where the coefficients are either maximum time-bounded reachability probabilities for CTMDPs, which serve as a special case and have been solved in Section 3; or maximum unbounded reachability probabilities of CTMDPs, which, by using the embedded MDP, can be calculated quite efficiently.

Given an MTA $\mathcal{M}$, we denote the set of constants appearing in the clock constraints of $\mathcal{M}$ as $\{c_0, \ldots, c_m\}$ with $c_0 = 0$. We assume the following order: $0 = c_0 < c_1 < \cdots < c_m$. Let $\Delta c_i = c_{i+1} - c_i$ for $0 \leqslant i < m$. Note that in this case, regions in the region graph $\mathcal{G}(\mathcal{M})$ (cf. Section 2.3) can be represented by the following intervals: $[c_0, c_1), \ldots, [c_m, \infty)$. We partition the region graph $\mathcal{G}(\mathcal{M}) = (Act, V, v_0, \Lambda, \hookrightarrow)$, or $\mathcal{G}$ for short, into a set of subgraphs $\{\mathcal{G}_i \mid 0 \leqslant i \leqslant m\}$ each with a set $\mathbb{G}_i$ of goal vertices:

$$\mathcal{G}_i = (Act, V_i, \Lambda_i, \{M_i^\alpha, B_i^\alpha, F_i\}_{\alpha \in Act}) \ ,$$

where $0 \leqslant i \leqslant m$ and $\Lambda_i(v) = \Lambda(v)$, if $v \in V_i$; 0 otherwise. These subgraphs are obtained by partitioning V, $\mathbb{G}$ and $\hookrightarrow$ as follows:

- $V = \bigcup_{0 \leqslant i \leqslant m}\{V_i\}$, where $V_i = \{(\ell, \Theta) \in V \mid \Theta \subseteq [c_i, c_{i+1})\}$;
- $\mathbb{G} = \bigcup_{0 \leqslant i \leqslant m}\{\mathbb{G}_i\}$, where $v \in \mathbb{G}_i$ iff $v \in V_i \cap \mathbb{G}$;
- $\hookrightarrow = \bigcup_{0 \leqslant i \leqslant m} \big(\bigcup_{\alpha \in Act}\{M_i^\alpha \cup B_i^\alpha\}\big) \cup F_i$, where for each $\alpha \in Act$, M_i^α is the set of *Markovian transitions (without reset)* between vertices inside $\mathcal{G}_i$ labeled by α; B_i^α is the set of *Markovian transitions (with reset)* from $\mathcal{G}_i$ to $\mathcal{G}_0$ (*B*ackward) labeled by α; and F_i is the set of *delay transitions* from the vertices in $\mathcal{G}_i$ to that in $\mathcal{G}_{i+1}$ (*F*orward). It is easy to see that M_i^α, F_i, and B_i^α are pairwise disjoint.

Example 5. We may partition the region graph $\mathcal{G}$ in Fig. 4 into $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2$ as in respective ovals. As an example for $\mathcal{G}_1$, the set of non-reset Markovian transitions is $M_1 = M_1^{\alpha_2} \cup M_1^{\gamma_2} = \{v_1 \overset{\alpha_2, 0.2, \varnothing}{\hookrightarrow} v_4, v_4 \overset{\gamma_2, 0.2, \varnothing}{\hookrightarrow} v_5\}$; the set of reset transitions is $B_1 = B_1^{\gamma_1} = \{v_4 \overset{\gamma_1, 0.5, \{x\}}{\hookrightarrow} v_0\}$ and the set of delay transitions is $F_1 = \{v_5 \overset{\delta}{\hookrightarrow} v_6\}$.

Given a subgraph $\mathcal{G}_i$ ($0 \leqslant i \leqslant m$) with k_i states, define the probability vector $\vec{U}_i(x) = [u_i^1(x), \ldots, u_i^{k_i}(x)]^\top \in \mathbb{R}(x)^{k_i \times 1}$, where $u_i^j(x)$ is the maximum probability to go from vertex $v_i^j \in V_i$ to some vertex in the goal set $\mathbb{G}$ (in $\mathcal{G}$) at time point x. We distinguish two cases:

\# **Case** $0 \leqslant i < m$. We first introduce some definitions.

- $\mathbf{P}_i^{\alpha,\mathsf{M}} \in [0,1]^{k_i \times k_i}$ and $\mathbf{P}_i^{\alpha,\mathsf{B}} \in [0,1]^{k_i \times k_0}$ are probability transition matrices for *Markovian* and *backward* transitions respectively, parameterized by action α. Namely, for each vertex v and action $\alpha \in \mathcal{I}(v)$, $\mathbf{P}_i^{\alpha,\mathsf{M}}[v, v'] = p$, if $v \overset{\alpha, p, \varnothing}{\hookrightarrow} v'$; 0 otherwise. Similarly $\mathbf{P}_i^{\alpha,\mathsf{B}}[v, v'] = p$ if $v \overset{\alpha, p, \{x\}}{\hookrightarrow} v'$; 0 otherwise. Note that $\sum_{v'} \mathbf{P}_i^{\alpha,\mathsf{M}}(v, v') + \sum_{v''} \mathbf{P}_i^{\alpha,\mathsf{B}}(v, v'') = 1$. Moreover, we write $\mathbf{P}_i^\alpha = \big(\mathbf{P}_i^{\alpha,\mathsf{M}} \big| \mathbf{P}_i^{\alpha,\mathsf{B}}\big)$, and note $\mathbf{P}_i^\alpha \in [0,1]^{k_i \times (k_i + k_0)}$; and each row of $\mathbf{P}_i^\alpha$ sums up to 1.

- $\mathbf{D}_i(x) \in \mathbb{R}^{k_i \times k_i}$ is the delay probability matrix, i.e. for any $1 \leqslant j \leqslant k_i$, $\mathbf{D}_i(x)[j,j] = e^{-E(v_i^j)x}$. The off diagonal elements are zero;
- $\mathbf{E}_i \in \mathbb{R}^{k_i \times k_i}$ is the exit rate matrix, i.e. for any $1 \leqslant j \leqslant k_i$, $\mathbf{E}_i[j,j] = E(v_i^j)$. The off-diagonal elements are zero;
- $\mathbf{M}_i^{\alpha}(x) = \mathbf{E}_i{\cdot}\mathbf{D}_i(x){\cdot}\mathbf{P}_i^{\alpha,\mathsf{M}} \in \mathbb{R}^{k_i \times k_i}$ is the probability density matrix for Markovian transitions inside $\mathcal{G}_i$ (i.e. for Markovian edges M_i^{α}); Namely, $\mathbf{M}_i^{\alpha}(x)[j,j']$ indicates the probability density function to take the Markovian jump *without* reset from the j-th vertex to the j'-th vertex in $\mathcal{G}_i$;
- $\mathbf{B}_i^{\alpha}(x) = \mathbf{E}_i{\cdot}\mathbf{D}_i(x){\cdot}\mathbf{P}_i^{\alpha,\mathsf{B}} \in \mathbb{R}^{k_i \times k_0}$ is the probability density matrix for the reset edges B_i^{a}. Namely, $\mathbf{B}_i^{\alpha}(x)[j,j']$ indicates the probability density function to take the Markovian jump *with* reset from the j-th vertex in $\mathcal{G}_i$ to the j'-th vertex in $\mathcal{G}_0$;
- $\mathbf{F}_i \in \mathbb{R}^{k_i \times k_{i+1}}$ is the incidence matrix for delay edges F_i. More specifically, $\mathbf{F}_i[j,j'] = 1$ indicates that there is a delay transition from the j-th vertex in $\mathcal{G}_i$ to the j'-th vertex in $\mathcal{G}_{i+1}$; 0 otherwise. Recall that for each action the delay edge is the same.

By instantiating (9), we obtain the following vector form for $x \in [0, \Delta c_i]$:

$$\vec{U}_i(x) = \max_{\alpha \in Act} \Big\{ \underbrace{\int_0^{\Delta c_i - x} \mathbf{M}_i^{\alpha}(\tau)\vec{U}_i(x+\tau)d\tau}_{(*)} \tag{10}$$

$$+ \underbrace{\int_0^{\Delta c_i - x} \mathbf{B}_i^{\alpha}(\tau)d\tau \cdot \vec{U}_0(0)}_{(**)} \Big\} + \mathbf{D}_i(\Delta c_i - x) \cdot \mathbf{F}_i \vec{U}_{i+1}(0), \tag{11}$$

Let us explain the above equation. First of all, $\flat(v,x) = \Delta c_i - x$ for each state $v \in V_i$. Note that this only holds for single clock case. Term $(*)$ (resp. $(**)$) reflects the case where clock x is not reset (resp. is reset and returning to $\mathcal{G}_0$). Note that $\mathbf{M}_i^{\alpha}(\tau)$ and $\mathbf{B}_i^{\alpha}(\tau)$ are the matrix forms of the density function $(\star)$ in (9). The matrix $\mathbf{D}_i(\Delta c_i - x)$ indicates the probability to delay until the "end" of region i, and $\mathbf{F}_i\vec{U}_{i+1}(0)$ denotes the probability to continue in $\mathcal{G}_{i+1}$ (at relative time point 0), and $\mathbf{D}_i(\Delta c_i - x)\mathbf{F}_i$ is the matrix form of the term $(\star\star)$ in (9).

Example 6 (Continuing Example 5). According to the definitions, we have the following matrices for $\mathcal{G}_1$, with which the vector $\vec{U}_1(x)$ can be obtained:

$$\mathbf{M}_1^{\alpha_2}(x) = \underbrace{\begin{pmatrix} r_0 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{E}_1} \underbrace{\begin{pmatrix} e^{-r_0 x} & 0 & 0 \\ 0 & e^{-r_2 x} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{D}_1(x)} \underbrace{\begin{pmatrix} 0 & 0.2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{P}_1^{\alpha_2,\mathsf{M}}(x)}$$

$$= \begin{pmatrix} 0 & 0.2r_0 e^{-r_0 x} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly,

$$\mathbf{M}_1^{\gamma_2}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.2r_2e^{-r_2x} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B}_1^{\gamma_1}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0.5r_2e^{-r_2x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{F}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

\# **Case** $i = m$. $\vec{U}_m(x)$ is simplified as follows:

$$\vec{U}_m(x)=\max_{\alpha\in Act}\Big\{\int_0^\infty \widehat{\mathbf{M}}_m^\alpha(\tau)\vec{U}_m(x+\tau)d\tau + \tilde{\mathbf{1}}_F + \int_0^\infty \mathbf{B}_m^\alpha(\tau)d\tau \cdot \vec{U}_0(0)\Big\}, \tag{12}$$

where $\widehat{\mathbf{M}}_m^\alpha(\tau)[v,\cdot] = \mathbf{M}_m^\alpha(\tau)[v,\cdot]$ for $v \notin \mathbb{G}$, 0 otherwise. $\tilde{\mathbf{1}}_F$ is an indicator vector such that $\tilde{\mathbf{1}}_F[v] = 1$ if $v \in \mathbb{G}$, 0 otherwise. Note that as the last subgraph $\mathcal{G}_m$ involves infinite regions, it has no delay transitions.

Our task is to solve the system of integral equations (10)-(12). We first observe that:

(i) Due to the fact that inside $\mathcal{G}_i$ there are only Markovian jumps with neither resets nor delay transitions, $\mathcal{G}_i$ with (V_i, Λ_i, M_i) forms a CTMDP $\mathcal{C}_i$. For each $\mathcal{G}_i$ we define an *augmented* CTMDP $\mathcal{C}_i^\star$ with state space $V_i \cup V_0$, such that all V_0-vertices are made absorbing (meaning no outgoing edges) in $\mathcal{C}_i^\star$. The edges connecting V_i to V_0 are kept and all the edges inside $\mathcal{C}_0$ are removed. The augmented CTMDP is used to calculate the probability to start from a vertex in $\mathcal{G}_i$ and take a reset edge in a certain time. The augmented CTMDP $\mathcal{C}_1^\star$ for $\mathcal{G}_1$ in Fig. 4 is shown in Fig. 7.

(ii) Given any (finite state) CTMDP, let $\mathbf{\Pi}(x)[\ell,\ell']$ be the probability to start in location ℓ at time 0 and reach ℓ' at time x. Note that the CTMDP coincides with its region graph, since the set of clocks is empty. By instantiating (9), we have the following equation (in the matrix form):

$$\mathbf{\Pi}(x) = \max_{\alpha\in Act}\left\{\int_0^x \widetilde{\mathbf{M}}^\alpha(\tau)\mathbf{\Pi}(x-\tau)d\tau\right\} + \mathbf{D}(x), \tag{13}$$

where $\widetilde{\mathbf{M}}^\alpha(\tau)[\ell,\ell'] = E(\ell)e^{-E(\ell)\tau}\cdot p$ if there is a transition $\ell \overset{\alpha,\varnothing}{\rightsquigarrow} \zeta$ and $p = \zeta(\varnothing,\ell')$; 0 otherwise. For augmented CTMDP $\mathcal{C}_i^\star$, $\widetilde{\mathbf{M}}^\alpha(\tau)$ links to $\mathbf{M}_i^\alpha(\tau)$ and $\mathbf{B}_i^\alpha(\tau)$ as $\widetilde{\mathbf{M}}^\alpha(\tau) = \left(\begin{array}{c|c} \mathbf{M}_i^\alpha(\tau) & \mathbf{B}_i^\alpha(\tau) \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right)$, where $\mathbf{0} \in \mathbb{R}^{k_0\times k_i}$ is the matrix with all 0's and $\mathbf{I} \in \mathbb{R}^{k_0\times k_0}$ is the identity matrix.

Now given any CTMDP $\mathcal{C}_i$ (resp. augmented CTMDP $\mathcal{C}_i^\star$) corresponding to $\mathcal{G}_i$, we obtain Eq. (13), and write its solution as $\Pi_i(x)$ (resp. $\Pi_i^\star(x)$). We then define $\bar{\mathbf{\Pi}}_i^\star \in \mathbb{R}^{k_i\times k_0}$ for an augmented CTMDP $\mathcal{C}_i^\star$ to be part of $\mathbf{\Pi}_i^\star$, where $\bar{\mathbf{\Pi}}_i^\star$ only keeps the probabilities starting from V_i and ending in V_0. As a matter of fact, $\mathbf{\Pi}_i^\star(x) = \left(\begin{array}{c|c} \mathbf{\Pi}_i(x) & \bar{\mathbf{\Pi}}_i^\star(x) \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right)$. The milestone of this section is the following theorem:

Theorem 4. *For subgraph $\mathcal{G}_i$ of $\mathcal{G}$ with k_i states, it holds:*

- *For $0 \leqslant i < m$,*

$$\vec{U}_i(0) = \mathbf{\Pi}_i(\Delta c_i) \cdot \mathbf{F}_i \vec{U}_{i+1}(0) + \bar{\mathbf{\Pi}}_i^\star(\Delta c_i) \cdot \vec{U}_0(0), (\dagger)$$

  where $\mathbf{\Pi}_i(\Delta c_i)$ and $\bar{\mathbf{\Pi}}_i^\star(\Delta c_i)$ are for (augmented) CTMDP *$\mathcal{C}_i$ and $\mathcal{C}_i^\star$, respectively.*
- *For $i = m$,*

$$\vec{U}_m(0) = \max_{\alpha \in Act} \left\{ \widehat{\mathbf{P}}_m^\alpha \cdot \vec{U}_m(0) + \vec{\mathbf{1}}_F + \widehat{\mathbf{B}}_m^\alpha \cdot \vec{U}_0(0) \right\}, (\ddagger)$$

  where $\widehat{\mathbf{P}}_m^\alpha(v, v') = \mathbf{P}_m^\alpha(v, v')$ if $v \notin \mathbb{G}$; 0 otherwise, and $\widehat{\mathbf{B}}_m^\alpha = \int_0^\infty \mathbf{B}_m^\alpha(\tau) d\tau$.

Proof. We first deal with the case $i < m$. If in $\mathcal{G}_i$, for some action α there exists some backward edge, namely, for some j, j', $\mathbf{B}_i^\alpha(x)[j, j'] \neq 0$, then we shall consider the *augmented* CTMDP $\mathcal{C}_i^\star$ with $k_i^\star = k_i + k_0$ states. In view of this, the augmented integral equation $\vec{V}_i(x)$ is defined as:

$$\vec{V}_i^\star(x) = \max_{\alpha \in Act} \left\{ \int_0^{\Delta c_i - x} \mathbf{M}_i^{\alpha,\star}(\tau) \vec{V}_i^\star(x+\tau) d\tau + \mathbf{D}_i^\star(\Delta c_i - x) \cdot \mathbf{F}_i^\star \cdot \vec{\widehat{V}}_i(0) \right\},$$

where

- $\vec{V}_i^\star(x) = \left(\begin{array}{c} \vec{V}_i(x) \\ \hline \vec{V}_i'(x) \end{array}\right) \in \mathbb{R}^{k_i^\star \times 1}$, where $\vec{V}_i'(x) \in \mathbb{R}^{k_0 \times 1}$ is the vector representing reachability probability for the augmented states in $\mathcal{G}_i$;
- $\mathbf{M}_i^{\alpha,\star}(\tau) = \left(\begin{array}{c|c} \mathbf{M}_i^\alpha(\tau) & \mathbf{B}_i^\alpha(\tau) \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \in \mathbb{R}^{k_i^\star \times k_i^\star}$. For the augmented states, we assume that their exit rates are 0.
- $\mathbf{D}_i^\star(\tau) = \left(\begin{array}{c|c} \mathbf{D}_i(\tau) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \in \mathbb{R}^{k_i^\star \times k_i^\star}$.
- $\mathbf{F}_i^\star = \left(\mathbf{F}_i' \middle| \mathbf{B}_i'\right) \in \mathbb{R}^{k_i^\star \times (k_{i+1}+k_0)}$ such that $\mathbf{F}_i' = \left(\begin{array}{c} \mathbf{F}_i \\ \hline \mathbf{0} \end{array}\right) \in \mathbb{R}^{k_i^\star \times k_{i+1}}$ is the incidence matrix for delay edges and $\mathbf{B}_i' = \left(\begin{array}{c} \mathbf{0} \\ \hline \mathbf{I} \end{array}\right) \in \mathbb{R}^{k_i^\star \times k_0}$, $\vec{\widehat{V}}_i(0) = \left(\begin{array}{c} \vec{U}_{i+1}(0) \\ \hline \vec{U}_0(0) \end{array}\right) \in \mathbb{R}^{(k_{i+1}+k_0) \times 1}$.

In the sequel, we shall prove two claims:

Claim 1. For each $0 \leqslant j \leqslant k_i$,

$$\vec{U}_i[j] = \vec{V}_i^\star[j] \ .$$

Proof of Claim 1. According to the definition, we have that

$$\vec{V}_i^\star(x) = \max_{\alpha\in Act}\left\{\int_0^{\Delta c_i - x}\left(\begin{array}{c|c}\mathbf{M}_i^\alpha(\tau) & \mathbf{B}_i^\alpha(\tau)\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right)\cdot\vec{V}_i^\star(x+\tau)d\tau\right.$$
$$\left. + \left(\begin{array}{c|c}\mathbf{D}_i(\Delta c_i - x) & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{I}\end{array}\right)\cdot\left(\begin{array}{c|c}\mathbf{F}_i & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{I}\end{array}\right)\cdot\left(\frac{\vec{U}_{i+1}(0)}{\vec{U}_0(0)}\right)\right\} \quad .$$

It follows immediately that $\vec{V}_i'(x) = \vec{U}_0(0)$. For $\vec{V}_i(x)$, we have that

$$\begin{aligned}
&\vec{V}_i(x)\\
&= \max_{\alpha\in Act}\Big\{\int_0^{\Delta c_i - x}\mathbf{M}_i^\alpha(\tau)\vec{V}_i(x+\tau)d\tau + \int_0^{\Delta c_i - x}\mathbf{B}_i^\alpha(\tau)\vec{V}_i'(x+\tau)d\tau\\
&\quad +\mathbf{D}_i(\Delta c_i - x)\cdot\mathbf{F}_i\cdot\vec{U}_{i+1}(0)\Big\}\\
&= \max_{\alpha\in Act}\Big\{\int_0^{\Delta c_i - x}\mathbf{M}_i^\alpha(\tau)\vec{V}_i(x+\tau)d\tau + \int_0^{\Delta c_i - x}\mathbf{B}_i^\alpha(\tau)d\tau\cdot\vec{U}_0(0)\\
&\quad +\mathbf{D}_i(\Delta c_i - x)\cdot\mathbf{F}_i\cdot\vec{U}_{i+1}(0)\Big\}\\
&= \vec{U}_i(x)
\end{aligned}$$

Claim 2.

$$\vec{V}_i^\star(x) = \mathbf{\Pi}_i^\star(\Delta c_i - x)\cdot\mathbf{F}_i^\star\vec{\hat{V}}_i(0) \quad ,$$

where

$$\mathbf{\Pi}_i^\star(x) = \max_{\alpha\in Act}\left\{\int_0^x \mathbf{M}_i^{\alpha,\star}(\tau)\mathbf{\Pi}_i^\star(x-\tau)d\tau + \mathbf{D}_i^\star(x)\right\} \quad . \tag{14}$$

Standard arguments yield that the optimal probability corresponds to the least fixpoint of a functional and can be computed iteratively from set $c_{i,x} = \Delta c_i - x$.

$$\begin{aligned}
\vec{V}_i^{\star,(0)}(x) &= \vec{0}\\
\vec{V}_i^{\star,(j+1)}(x) &= \max_{\alpha\in Act}\left\{\int_0^{c_{i,x}}\mathbf{M}_i^\alpha(\tau)\vec{V}_i^{\star,(j)}(x+\tau)d\tau + \mathbf{D}_i^\star(c_{i,x})\cdot\mathbf{F}_i^\star\vec{\hat{V}}_i(0)\right\}.
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{\Pi}_i^{\star,(0)}(c_{i,x}) &= \mathbf{0}\\
\mathbf{\Pi}_i^{\star,(j+1)}(c_{i,x}) &= \max_{\alpha\in Act}\left\{\int_0^{c_{i,x}}\mathbf{M}_i^\star(\tau)\mathbf{\Pi}_i^{\star,(j)}(c_{i,x}-\tau)d\tau + \mathbf{D}_i^\star(c_{i,x})\right\}.
\end{aligned}$$

By induction on j, we prove the following relation:

$$\vec{V}_i^{\star,(j)}(x) = \mathbf{\Pi}_i^{\star,(j)}(c_{i,x})\cdot\mathbf{F}_i\vec{\hat{V}}_i(0) \quad .$$

– Base case: $\vec{V}_i^{\star,(0)}(x) = \vec{0}$ and $\mathbf{\Pi}_i^{\star,(0)}(c_{i,x}) = \mathbf{0}$.

– Induction hypothesis:

$$\vec{V}_i^{\star,(j)}(x) = \mathbf{\Pi}_i^{\star,(j)}(c_{i,x}) \cdot \mathbf{F}_i^{\star}\vec{\widehat{U}}_i(0) \ .$$

– Induction step $j \to j+1$. We

$$\vec{V}_i^{\star,(j+1)}(x) = \max_{\alpha \in Act}\left\{\int_0^{c_{i,x}} \mathbf{M}_i^{\star,\alpha}(\tau)\vec{V}_i^{\star,(j)}(x+\tau)d\tau + \mathbf{D}_i^{\star}(c_{i,x}) \cdot \mathbf{F}_i^{\star}\vec{\widehat{U}}_i(0)\right\}. \tag{15}$$

By induction hypothesis it follows that

$$\begin{aligned}
\vec{V}_i^{\star,(j+1)}(x) &= \max_{\alpha \in Act}\left\{\int_0^{c_{i,x}} \mathbf{M}_i^{\star,\alpha}(\tau)\vec{V}_i^{\star,(j)}(x+\tau)d\tau + \mathbf{D}_i^{\star}(c_{i,x}) \cdot \mathbf{F}_i^{\star}\vec{\widehat{V}}_i(0)\right\} \\
&= \max_{\alpha \in Act}\left\{\int_0^{c_{i,x}} \mathbf{M}_i^{\star,\alpha}(\tau) \cdot \mathbf{\Pi}_i^{\star,(j)}(c_{i,x}-\tau) \cdot \mathbf{F}_i^{\star}\vec{\widehat{V}}_i(0)d\tau \right.\\
&\qquad \left. + \mathbf{D}_i^{\star}(c_{i,x}) \cdot \mathbf{F}_i^{\star}\vec{\widehat{V}}_i(0)\right\} \\
&= \max_{\alpha \in Act}\left\{\left(\int_0^{c_{i,x}} \mathbf{M}_i^{\star,\alpha}(\tau)\mathbf{\Pi}_i^{\star,(j)}(c_{i,x}-\tau)d\tau + \mathbf{D}_i^{\star}(c_{i,x})\right)\right.\\
&\qquad \left. \times \mathbf{F}_i^{\star}\vec{\widehat{V}}_i(0)\right\} \\
&= \max_{\alpha \in Act}\left\{\int_0^{c_{i,x}} \mathbf{M}_i^{\star,\alpha}(\tau)\mathbf{\Pi}_i^{\star,(j)}(c_{i,x}-\tau)d\tau + \mathbf{D}_i^{\star}(c_{i,x})\right\} \cdot \mathbf{F}_i^{\star}\vec{\widehat{V}}_i(0) \\
&= \mathbf{\Pi}_i^{\alpha,(j+1)}(c_{i,x}) \cdot \mathbf{F}_i^{\star}\vec{\widehat{V}}_i(0)
\end{aligned}$$

Clearly,

$$\mathbf{\Pi}_i^{\star}(c_{i,x}) = \lim_{j\to\infty} \mathbf{\Pi}_i^{\star,(j)}(c_{i,x})$$

and

$$\vec{V}_i^{\star}(x) = \lim_{j\to\infty} \vec{V}_i^{\star,(j)}(x).$$

Let $x = 0$ and we obtain

$$\vec{V}_i^{\star}(0) = \mathbf{\Pi}_i^{\star}(c_{i,0}) \cdot \mathbf{F}_i\vec{\widehat{V}}_i(0).$$

We can also write the above relation for $x = 0$ as:

$$\begin{aligned}
\begin{pmatrix} \vec{V}_i(0) \\ \hline \vec{V}_i'(0) \end{pmatrix} &= \mathbf{\Pi}_i^{\star}(\Delta c_i)\left(\mathbf{F}_i' \middle| \mathbf{B}_i'\right)\begin{pmatrix} \vec{U}_{i+1}(0) \\ \hline \vec{U}_0(0) \end{pmatrix} \\
&= \left(\begin{array}{c|c} \mathbf{\Pi}_i(\Delta c_i) & \bar{\mathbf{\Pi}}_i^{\star}(\Delta c_i) \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right)\left(\begin{array}{c|c} \mathbf{F}_i & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right)\begin{pmatrix} \vec{U}_{i+1}(0) \\ \hline \vec{U}_0(0) \end{pmatrix} \\
&= \left(\begin{array}{c|c} \mathbf{\Pi}_i(\Delta c_i)\mathbf{F}_i & \bar{\mathbf{\Pi}}_i^{\star}(\Delta c_i) \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right)\begin{pmatrix} \vec{U}_{i+1}(0) \\ \hline \vec{U}_0(0) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{\Pi}_i(\Delta c_i)\mathbf{F}_i\vec{U}_{i+1}(0) + \bar{\mathbf{\Pi}}_i^{\star}(\Delta c_i)\vec{U}_0(0) \\ \hline \vec{U}_0(0) \end{pmatrix}.
\end{aligned}$$

As a result we can represent $\vec{V}_i(0)$ in the following matrix form

$$\vec{V}_i(0) = \mathbf{\Pi}_i(\Delta c_i)\mathbf{F}_i\vec{U}_{i+1}(0) + \bar{\mathbf{\Pi}}_i^a(\Delta c_i)\vec{U}_0(0)$$

by noting that $\mathbf{\Pi}_i$ is formed by the first k_i rows and columns of matrix $\mathbf{\Pi}_i^\star$ and $\bar{\mathbf{\Pi}}_i^\star$ is formed by the first k_i rows and the last $k_i^\star - k_i = k_0$ columns of $\mathbf{\Pi}_i^\star$. The conclusion follows from Claim 1.

For $i = m$, i.e., the last graph $\mathcal{G}_m$, the region size is infinite, therefore delay transitions do not exist. Recall that

$$\vec{U}_m(x) = \max_{\alpha \in Act}\left\{\int_0^\infty \widehat{\mathbf{M}}_m^\alpha(\tau)\vec{U}_m(x+\tau)d\tau + \vec{1}_F + \int_0^\infty \mathbf{B}_m^\alpha(\tau)d\tau \cdot \vec{U}_0(0)\right\}$$

We first prove the following claim:

Claim. For any $x \in \mathbb{R}_{\geqslant 0}$, $\vec{U}_m(x)$ is a constant vector function.

Proof of the claim. We define

$$\vec{U}_m^{(0)}(x) = \vec{0}$$

$$\vec{U}_m^{(j+1)}(x) = \max_{\alpha \in Act}\left\{\int_0^\infty \widehat{\mathbf{M}}_m^\alpha(\tau)\vec{U}_m^{(j)}(x+\tau)d\tau + \vec{1}_F + \int_0^\infty \mathbf{B}_m^\alpha(\tau)d\tau \cdot \vec{U}_0(0)\right\}$$

It is not difficult to see that $\vec{U}_m(x) = \lim_{j\to\infty}\vec{U}_m^{(j)}(x)$. We shall show, by induction on j, that $\vec{U}_m^{(j)}(x)$ is a constant vector function.

- Base case: $\vec{U}_m^{(0)}(x) = \vec{0}$, which is clearly constant.
- I.H.: $\vec{U}_m^{(j)}(x)$ is a constant vector function.
- Induction step: $(j \to j+1)$

$$\begin{aligned}
\vec{U}_m^{(j+1)}(x) &= \max_{\alpha \in Act}\left\{\int_0^\infty \widehat{\mathbf{M}}_m^a(\tau)\vec{U}_m^{(j)}(x+\tau)d\tau + \vec{1}_F\right. \\
&\qquad \left. + \int_0^\infty \mathbf{B}_m^\alpha(\tau)d\tau \cdot \vec{U}_0(0)\right\} \\
&\overset{\text{I.H.}}{=} \max_{\alpha \in Act}\left\{\int_0^\infty \widehat{\mathbf{M}}_m^a(\tau)\cdot\vec{U}_m^{(j)}(x)d\tau + \vec{1}_F + \int_0^\infty \mathbf{B}_m^\alpha(\tau)d\tau \cdot \vec{U}_0(0)\right\} \\
&= \max_{\alpha \in Act}\left\{\int_0^\infty \widehat{\mathbf{M}}_m^a(\tau)d\tau\cdot\vec{U}_m^{(j)}(x) + \vec{1}_F + \int_0^\infty \mathbf{B}_m^\alpha(\tau)d\tau \cdot \vec{U}_0(0)\right\}
\end{aligned}$$

The conclusion follows trivially.

Since $\vec{U}_m(x)$ is constant vector function, we have that

$$\vec{U}_m(x) = \max_{\alpha \in Act}\left\{\int_0^\infty \widehat{\mathbf{M}}_m^\alpha(\tau)d\tau \cdot \vec{U}_m(x) + \vec{1}_F + \int_0^\infty \mathbf{B}_m^\alpha(\tau)d\tau \cdot \vec{U}_0(0)\right\}$$

More than that $\int_0^\infty \widehat{\mathbf{M}}_m^\alpha(\tau)d\tau$ boils down to $\widehat{\mathbf{P}}_m^\alpha$ and $\int_0^\infty \mathbf{B}_m^a(\tau)d\tau$ to $\widehat{\mathbf{B}}_m^\alpha$. Also we add the vector $\vec{1}_F$ to ensure that the probability to start from a state in G_F is one. □

Recall that we intend to solve the system of integral equations (10)-(12) to obtain vectors $\vec{U}_i(0)$ for $0 \leq i \leq m$. Thm. 4 entails that instead of accomplishing this directly, one could alternatively appeal to linear equations (†)-(‡), where $\vec{U}_i(0)$ $(0 \leq i \leq m)$ can be regarded as a family of variables and the coefficients of the linear equations can be obtained by computing the corresponding maximum time-bounded reachability probabilities of CTMDPs $\mathcal{C}_i^\star$. It is not difficult to see that one can use standard value iteration algorithms to solve (†)-(‡).

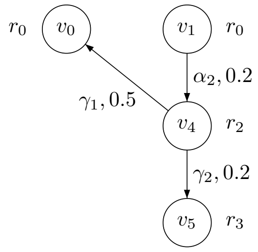

Fig. 7. Augmented CTDMP $\mathcal{C}_1^\star$

5 Conclusion

We have defined an extension of timed automata with exponentially distributed durations on locations. We constructed region graphs of such automata based on which two variants of reachability problems were investigated. We presented two approaches to determine maximum time-bounded reachability probabilities in MTA, by discritization and a reduction to HJB PDEs. For single-clock MTA, unbounded reachability probabilities were characterized as the solution of a system of linear equations whose coefficients are maximum reachability probabilities in CTMDPs, i.e., zero-clock MTA. We remark that in this paper only the locally uniform model (i.e., the exit rate of each location solely depends on the location instead of actions) was addressed, however the general case can be treated without any difficulty.

Many future works remain to be done. For example, we plan to extend the MTA model to rewards; also one can consider more general variables than clocks, resulting in Markovian *hybrid* automata. Implementation and case studies are in plan as well.

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