A Highly-Accurate Industrial Robot Calibrator with Multi-Planer Constraints Supplementary Material

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This is the supplementary file for the paper, where the convergence proof of the AMPC algorithm is provided.
Given nodes $i \in N$, and iteration t, the convergence is divided into two sequential steps:
Step 1. The difference between Γ_{t+1} and Γ_t is bounded by that between $(\gamma_{t+1}, W_{t+1}, a_{1,t+1})$ and $(\gamma_t, W_t, a_{1,t})$;
Step 2. The augmented Lagrangian function (23) is non-increasing and low-bounded.

A. Proof of Step 1

This work provides proof with Γ as an active variable. Thus, *Lemma* 1 is presented as:
Lemma 1: With (29), $(\Gamma_{t+1} - \Gamma_t)^2$ is bounded by:

$$\left(\Gamma_{t+1} - \Gamma_t\right)^2 \le 2\left(\rho(\eta-1)\right)^2 \left(\frac{1}{N}\sum_{i=1}^{N}\left(\Phi_i\left(a_{t+1}, W_{t+1}, \gamma_{t+1}\right) - \Phi_i\left(a_t, W_t, \gamma_t\right)\right)\right)^2 + 2\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1} + \kappa_{1,i,t+1}\right) - \frac{1}{\kappa_{2,i,t}}\left(a_{1,t} + \kappa_{1,i,t}\right)\right)\right)^2 = v_\Gamma,$$
(S1)

Proof: Consider that (23) is non-convex, implying that any equilibrium point having a zero-gradient (e.g., a saddle point or a global/local optimum) has the potential to be a feasible solution. Given the solution to a_1 by (29c) to be $a_{1,t+1}$, yielding:

$$\Gamma_t + \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1} + \kappa_{1,i,t+1}\right) + \rho \cdot \Phi\left(a_{t+1}, W_{t+1}, \gamma_{t+1}\right)\right) = 0,$$
(S2)

By replacing the values of Γ derived from equations (27d) and (29d) into expression (S2), we can derive:

$$\Gamma_{t+1} = (\eta-1)\frac{\rho}{N}\sum_{i=1}^{N}\Phi_i\left(a_{1,t+1}, W_{t+1}, \gamma_{t+1}\right) - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1} + \kappa_{1,i,t+1}\right) = 0,$$
(S3)

Hence, the difference between Γ_{t+1} and Γ_t is given as:

$$\Gamma_{t+1} - \Gamma_t = (\eta-1)\frac{\rho}{N}\sum_{i=1}^{N}\left(\Phi_i\left(a_{1,t+1}, W_{t+1}, \gamma_{t+1}\right) - \Phi_i\left(a_{1,t}, W_t, \gamma_t\right)\right) - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1} + \kappa_{1,i,t+1}\right) - \frac{1}{\kappa_{2,i,t}}\left(a_{1,t} + \kappa_{1,i,t}\right)\right),$$
(S4)

With the inequality $(x-y)^2 \le 2(x^2+y^2)$, (S1) is fulfilled. Thus, *Lemma* 1 stands to implement Step 1.

B. Proof of Step 2

To facilitate the execution of Step 2, we present *Lemma* 2:
Lemma 2: If the following criteria are fulfilled, it is beneficial to define intermediate variables.

$$\rho \le 0, \ \eta \ge 0, \ \gamma_{x,t} \ge 0, \ \gamma_{y,t} \ge 0, \ \gamma_{z,t} \ge 0,$$
(S5a)

$$\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1} + \kappa_{1,i,t+1}\right) \le 0, \ \eta \ge \frac{1}{2},$$
(S5b)

Hence the following inequality is valid:

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) - f\left(\gamma_t, W_t, a_{1,t}, \Gamma_t\right) \le 0,$$
(S6a)

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) \ge 0.$$
(S6b)

Proof: Considering the second-order Taylor expansion of f at the point of $(\gamma_{t+1}, W_t, a_{1,t}$ and $\Gamma_t)$, we have:

$$f\left(\gamma_{t+1}, W_t, a_{1,t}, \Gamma_t\right) - f\left(\gamma_t, W_t, a_{1,t}, \Gamma_t\right) \overset{(31a),(32a)}{=} \frac{\rho}{2N}\sum_{i=1}^{N}\left[\begin{array}{l}\kappa_{4,i,t}^2\left(\gamma_{x,t+1} - \gamma_{x,t}\right)^2 + \kappa_{5,i,t}^2\left(\gamma_{y,t+1} - \gamma_{y,t}\right)^2 \\ + \kappa_{6,i,t}^2\left(\gamma_{z,t+1} - \gamma_{z,t}\right)^2\end{array}\right].$$
(S7)

Note that based on optimal conditions in (27a) and (29a), the first-order term is zero and has been omitted for brevity. Likewise, the difference between $f(\gamma_{t+1}, W_{t+1}, a_{1,t}$ and $\Gamma_t)$ and $f(\gamma_{t+1}, W_t, a_{1,t}$ and $\Gamma_t)$, and $f(\gamma_{t+1}, W_{t+1}, a_{1,t+1}$ and $\Gamma_t)$ and $f(\gamma_{t+1}, W_{t+1}, a_{1,t}$ and $\Gamma_t)$ can be formulated by:

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t}, \Gamma_t\right) - f\left(\gamma_{t+1}, W_t, a_{1,t}, \Gamma_t\right) \overset{(31b),(32b)}{=} \frac{\rho}{2N}\sum_{t=1}^{N}\left(\gamma_{x,t+1}\left(w_{x,t+1} - w_{x,t}\right)^2 + \gamma_{y,t+1}\left(w_{y,t+1} - w_{y,t}\right)^2 + \gamma_{z,t+1}\left(w_{z,t+1} - w_{z,t}\right)^2\right)$$
(S8)

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t+1}, \Gamma_t\right) - f\left(\gamma_{t+1}, W_{t+1}, a_{1,t}, \Gamma_t\right) \overset{(31c),(32c)}{=} \frac{1}{2N}\sum_{i=1}^{N}\left(1+\rho\kappa_{2,i,t}^2\right)\left(a_{1,t+1}-a_{1,t}\right)^2. \tag{S9}$$

Additionally, $f(\gamma_{t+1}, W_{t+1}, a_{1,t+1}$ and $\Gamma_{t+1})$ and $f(\gamma_{t+1}, W_{t+1}, a_{1,t+1}$ and $\Gamma_t)$ are differed as:

$$f\left(\gamma_t, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) - f\left(\gamma_t, W_t, a_{1,t}, \Gamma_t\right) \overset{(31d),(32d)}{=} \frac{\left(\Gamma_{t+1}-\Gamma_t\right)^2}{\eta\rho} \overset{S1}{\leq} \frac{\nu_\Gamma}{\eta\rho}. \tag{S10}$$

With (S9), the equality can be deduced from the update rules (27d) and (29d), whereas the inequality is contingent upon the premises established in Lemma 1. Through a logical combination of equations (S7) to (S10), the following result can be inferred:

$$f\left(\gamma_t, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) - f\left(\gamma_t, W_t, a_{1,t}, \Gamma_t\right) \leq \frac{\rho}{2N}\sum_{i=1}^{N}\left(\kappa_{4,i,t}^2\left(\gamma_{x,t+1}-\gamma_{x,t}\right)^2 + \kappa_{5,i,t}^2\left(\gamma_{y,t+1}-\gamma_{y,t}\right)^2 + \kappa_{6,i,t}^2\left(\gamma_{z,t+1}-\gamma_{z,t}\right)^2\right)$$

$$+\frac{\rho}{2N}\sum_{i=1}^{N}\left(\gamma_{x,t}\left(w_{x,t+1}-w_{x,t}\right)^2 + \gamma_{y,t}\left(w_{y,t+1}-w_{y,t}\right)^2 + \gamma_{z,t}\left(w_{z,t+1}-w_{z,t}\right)^2\right)^2 + \frac{1}{2N}\sum_{i=1}^{N}\left(1+\rho\kappa_{2,i,t}^2\right)\left(a_{1,t+1}-a_{1,t}\right)^2$$

$$+\frac{2}{\eta}(\eta-1)^2\rho\frac{1}{N}\sum_{i=1}^{N}\left(\Phi_i\left(a_{t+1}, W_{t+1}, \gamma_{t+1}\right)-\Phi_i\left(a_t, W_t, \gamma_t\right)\right)^2 + 2\frac{1}{\rho\eta}\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right) - \frac{1}{\kappa_{2,i,t}}\left(a_{1,t}+\kappa_{1,i,t}\right)\right)\right)^2 \leq 0, \tag{S11}$$

$$\Rightarrow \rho \leq 0, \gamma_{x,t} \geq 0, \gamma_{y,t} \geq 0, \gamma_{z,t} \geq 0, \frac{2}{\eta}(\eta-1)^2\rho \leq 0, \frac{1}{\rho\eta} \leq 0,$$

$$\Rightarrow \rho \leq 0, \gamma_{x,t} \geq 0, \gamma_{y,t} \geq 0, \gamma_{z,t} \geq 0, \eta \geq 0.$$

Hence, (S5a) and (S6a) are fulfilled, which demonstrates that (23) is non-increasing in this case. After $(t+1)$-*th* iteration, (23) can be reformulated as:

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) = \frac{1}{2N}\sum_{i=1}^{N}\left\|B_i - \hat{P}\right\|_2^2 + \frac{1}{N}\sum_{i=1}^{N}\left\langle\Phi_i\left(a_{1,t+1}, W_{t+1}, \gamma_{t+1}\right), \Gamma_{t+1}\right\rangle + \frac{1}{2N}\sum_{i=1}^{N}\rho\left\|\Phi_i\left(a_{1,t+1}, W_{t+1}, \gamma_{t+1}\right)\right\|_2^2, \tag{S12}$$

By substituting (S3) into (S11), we can obtain:

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) = \frac{1}{2N}\sum_{i=1}^{N}\left\|B_i - \hat{P}\right\|_2^2 + \frac{(2\eta-1)\rho}{2}\frac{1}{N}\sum_{i=1}^{N}\left(\Phi_i\left(a_{1,t+1}, W_{t+1}, \gamma_{t+1}\right)\right)^2 - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right), \tag{S13}$$

with (S5b) and (S13), (S6b) is fulfilled, i.e., (23) is lower-bounded. Hence, *Lamma* 2 stands, making Step 2 complete. To sum up, as per the aforementioned deductions, with the implementation of Steps 1 through 2, AMPC's convergence can be established with certainty in theory.