A Highly-Accurate Industrial Robot Calibrator with Multi-Planer Constraints Supplementary Material

Tinghui Chen, Shuai Li, *Senior Member*, *IEEE*, Weiyi Yang, and Xin Luo, *Senior Member*, *IEEE*

This is the supplementary file for the paper, where the convergence proof of the AMPC algorithm is provided. Given nodes $i \in N$, and iteration t, the convergence is divided into two sequential steps:

Step 1. The difference between Γ_{t+1} and Γ_t is bounded by that between (γ_{t+1}, W_{t+1}, $a_{1,t+1}$) and (γ_t, W_t, $a_{1,t}$);

Step 2. The augmented Lagrangian function (23) is non-increasing and low-bounded.

A. Proof of Step 1

This work provides proof with Γ as an active variable. Thus, *Lemma* 1 is presented as:

Lemma 1: With (29), $(\Gamma_{t+1}-\Gamma_t)^2$ is bounded by:

$$\left(\Gamma_{t+1}-\Gamma_t\right)^2 \le 2\left(\rho\left(\eta-1\right)\right)^2\left(\frac{1}{N}\sum_{i=1}^{N}\left(\Phi_i\left(a_{t+1},W_{t+1},\gamma_{t+1}\right)-\Phi_i\left(a_t,W_t,\gamma_t\right)\right)\right)^2+2\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right)-\frac{1}{\kappa_{2,i,t}}\left(a_{1,t}+\kappa_{1,i,t}\right)\right)\right)^2=v_\Gamma, \tag{S1}$$

Proof: Consider that (23) is non-convex, implying that any equilibrium point having a zero-gradient (e.g., a saddle point or a global/local optimum) has the potential to be a feasible solution. Given the solution to a_1 by (29c) to be $a_{1,t+1}$, yielding:

$$\Gamma_t+\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right)+\rho\cdot\Phi\left(a_{t+1},W_{t+1},\gamma_{t+1}\right)\right)=0, \tag{S2}$$

By replacing the values of Γ derived from equations (27d) and (29d) into expression (S2), we can derive:

$$\Gamma_{t+1}=\left(\eta-1\right)\frac{\rho}{N}\sum_{i=1}^{N}\Phi_i\left(a_{1,t+1},W_{t+1},\gamma_{t+1}\right)-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right)=0, \tag{S3}$$

Hence, the difference between Γ_{t+1} and Γ_t is given as:

$$\Gamma_{t+1}-\Gamma_t=\left(\eta-1\right)\frac{\rho}{N}\sum_{i=1}^{N}\left(\Phi_i\left(a_{1,t+1},W_{t+1},\gamma_{t+1}\right)-\Phi_i\left(a_{1,t},W_t,\gamma_t\right)\right)-\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right)-\frac{1}{\kappa_{2,i,t}}\left(a_{1,t}+\kappa_{1,i,t}\right)\right), \tag{S4}$$

With the inequality $(x-y)^2 \le 2(x^2+y^2)$, (S1) is fulfilled. Thus, *Lemma* 1 stands to implement Step 1.

B. Proof of Step 2

To facilitate the execution of Step 2, we present *Lemma* 2:

Lemma 2: If the following criteria are fulfilled, it is beneficial to define intermediate variables.

$$\rho\le 0,\ \eta\ge 0,\ \gamma_{x,t}\ge 0,\ \gamma_{y,t}\ge 0,\ \gamma_{z,t}\ge 0, \tag{S5a}$$

$$\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right)\le 0,\ \eta\ge\frac{1}{2}, \tag{S5b}$$

Hence the following inequality is valid:

$$f\left(\gamma_{t+1},W_{t+1},a_{1,t+1},\Gamma_{t+1}\right)-f\left(\gamma_t,W_t,a_{1,t},\Gamma_t\right)\le 0, \tag{S6a}$$

$$f\left(\gamma_{t+1},W_{t+1},a_{1,t+1},\Gamma_{t+1}\right)\ge 0. \tag{S6b}$$

Proof: Considering the second-order Taylor expansion of f at the point of (γ_{t+1}, W_t, $a_{1,t}$ and Γ_t), we have:

$$f\left(\gamma_{t+1},W_t,a_{1,t},\Gamma_t\right)-f\left(\gamma_t,W_t,a_{1,t},\Gamma_t\right)\overset{(31a),(32a)}{=}\frac{\rho}{2N}\sum_{i=1}^{N}\left[\begin{array}{l}\kappa_{4,i,t}^2\left(\gamma_{x,t+1}-\gamma_{x,t}\right)^2+\kappa_{5,i,t}^2\left(\gamma_{y,t+1}-\gamma_{y,t}\right)^2\\+\kappa_{6,i,t}^2\left(\gamma_{z,t+1}-\gamma_{z,t}\right)^2\end{array}\right]. \tag{S7}$$

Note that based on optimal conditions in (27a) and (29a), the first-order term is zero and has been omitted for brevity. Likewise, the difference between $f(\gamma_{t+1}$, W_{t+1}, $a_{1,t}$ and $\Gamma_t)$ and $f(\gamma_{t+1}$, W_t, $a_{1,t}$ and $\Gamma_t)$, and $f(\gamma_{t+1}$, W_{t+1}, $a_{1,t+1}$ and $\Gamma_t)$ and $f(\gamma_{t+1}$, W_{t+1}, $a_{1,t}$ and $\Gamma_t)$ can be formulated by:

$$f\left(\gamma_{t+1},W_{t+1},a_{1,t},\Gamma_t\right)-f\left(\gamma_{t+1},W_t,a_{1,t},\Gamma_t\right)\overset{(31b),(32b)}{=}\frac{\rho}{2N}\sum_{t=1}^{N}\left(\gamma_{x,t+1}\left(w_{x,t+1}-w_{x,t}\right)^2+\gamma_{y,t+1}\left(w_{y,t+1}-w_{y,t}\right)^2+\gamma_{z,t+1}\left(w_{z,t+1}-w_{z,t}\right)^2\right) \tag{S8}$$

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t+1}, \Gamma_t\right) - f\left(\gamma_{t+1}, W_{t+1}, a_{1,t}, \Gamma_t\right) \overset{(31c),(32c)}{=} \frac{1}{2N}\sum_{i=1}^{N}\left(1+\rho\kappa_{2,i,t}^2\right)\left(a_{1,t+1}-a_{1,t}\right)^2. \tag{S9}$$

Additionally, $f(\gamma_{t+1}, W_{t+1}, a_{1,t+1}$ and $\Gamma_{t+1})$ and $f(\gamma_{t+1}, W_{t+1}, a_{1,t+1}$ and $\Gamma_t)$ are differed as:

$$f\left(\gamma_t, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) - f\left(\gamma_t, W_t, a_{1,t}, \Gamma_t\right) \overset{(31d),(32d)}{=} \frac{\left(\Gamma_{t+1}-\Gamma_t\right)^2}{\eta\rho} \overset{S1}{\leq} \frac{v_\Gamma}{\eta\rho}. \tag{S10}$$

With (S9), the equality can be deduced from the update rules (27d) and (29d), whereas the inequality is contingent upon the premises established in Lemma 1. Through a logical combination of equations (S7) to (S10), the following result can be inferred:

$$\begin{aligned}
&f\left(\gamma_t, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) - f\left(\gamma_t, W_t, a_{1,t}, \Gamma_t\right) \leq \frac{\rho}{2N}\sum_{i=1}^{N}\left(\kappa_{4,i,t}^2\left(\gamma_{x,t+1}-\gamma_{x,t}\right)^2 + \kappa_{5,i,t}^2\left(\gamma_{y,t+1}-\gamma_{y,t}\right)^2 + \kappa_{6,i,t}^2\left(\gamma_{z,t+1}-\gamma_{z,t}\right)^2\right) \\
&+\frac{\rho}{2N}\sum_{i=1}^{N}\left(\gamma_{x,t}\left(w_{x,t+1}-w_{x,t}\right)^2 + \gamma_{y,t}\left(w_{y,t+1}-w_{y,t}\right)^2 + \gamma_{z,t}\left(w_{z,t+1}-w_{z,t}\right)^2\right)^2 + \frac{1}{2N}\sum_{i=1}^{N}\left(1+\rho\kappa_{2,i,t}^2\right)\left(a_{1,t+1}-a_{1,t}\right)^2 \\
&+\frac{2}{\eta}(\eta-1)^2\rho\frac{1}{N}\sum_{i=1}^{N}\left(\Phi_i\left(a_{t+1}, W_{t+1}, \gamma_{t+1}\right)-\Phi_i\left(a_t, W_t, \gamma_t\right)\right)^2 + 2\frac{1}{\rho\eta}\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right) - \frac{1}{\kappa_{2,i,t}}\left(a_{1,t}+\kappa_{1,i,t}\right)\right)\right)^2 \leq 0, \\
&\Rightarrow \rho \leq 0, \gamma_{x,t} \geq 0, \gamma_{y,t} \geq 0, \gamma_{z,t} \geq 0, \frac{2}{\eta}(\eta-1)^2\rho \leq 0, \frac{1}{\rho\eta} \leq 0, \\
&\Rightarrow \rho \leq 0, \gamma_{x,t} \geq 0, \gamma_{y,t} \geq 0, \gamma_{z,t} \geq 0, \eta \geq 0.
\end{aligned} \tag{S11}$$

Hence, (S5a) and (S6a) are fulfilled, which demonstrates that (23) is non-increasing in this case. After $(t+1)$-*th* iteration, (23) can be reformulated as:

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) = \frac{1}{2N}\sum_{i=1}^{N}\left\|B_i - \hat{P}\right\|_2^2 + \frac{1}{N}\sum_{i=1}^{N}\left\langle\Phi_i\left(a_{1,t+1}, W_{t+1}, \gamma_{t+1}\right), \Gamma_{t+1}\right\rangle + \frac{1}{2N}\sum_{i=1}^{N}\rho\left\|\Phi_i\left(a_{1,t+1}, W_{t+1}, \gamma_{t+1}\right)\right\|_2^2, \tag{S12}$$

By substituting (S3) into (S11), we can obtain:

$$f\left(\gamma_{t+1}, W_{t+1}, a_{1,t+1}, \Gamma_{t+1}\right) = \frac{1}{2N}\sum_{i=1}^{N}\left\|B_i - \hat{P}\right\|_2^2 + \frac{(2\eta-1)\rho}{2}\frac{1}{N}\sum_{i=1}^{N}\left(\Phi_i\left(a_{1,t+1}, W_{t+1}, \gamma_{t+1}\right)\right)^2 - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\kappa_{2,i,t+1}}\left(a_{1,t+1}+\kappa_{1,i,t+1}\right), \tag{S13}$$

with (S5b) and (S13), (S6b) is fulfilled, i.e., (23) is lower-bounded. Hence, *Lamma* 2 stands, making Step 2 complete. To sum up, as per the aforementioned deductions, with the implementation of Steps 1 through 2, AMPC's convergence can be established with certainty in theory.