

Logical equivalence

P	Q	$P \Rightarrow Q$	$(\sim Q) \Rightarrow (\sim P)$
T	T		
T	F		
F	T		
F	F		

We use the notation

$$P \Rightarrow Q \equiv (\sim Q) \Rightarrow (\sim P)$$

and call the two sides **logically equivalent**.

Set notations

- $\{\dots\}$: set. E.g. $\{1, 2, 3\}$ is the set containing elements 1, 2, and 3.
- \emptyset : the empty set.
- \in : membership. E.g. $x \in S$ means x is an element of the set S . E.g. $x, y \in S$ means x and y are both in S .
- \subset : subset. E.g. $A \subset B$ means set A is a subset of set B .
- \mathbb{N} : the set of natural numbers.
- \mathbb{Z} : the set of integers.
- \mathbb{Q} : the set of rational numbers.
- \mathbb{R} : the set of real numbers.

Quantifiers

- **Universal** quantifier: “for all...”, “for any...”, “for every...” we use the notation “ \forall ” (an upside down “A”)
- **Existential** quantifier: “there exists...”, “there is...”, “for some...”, “for at least one...” we use the notation “ \exists ” (an “E” facing the wrong way)

Example 1 Let S be the set of all states in USA. Consider the following open sentences

$$P(x, y) : x \text{ and } y \text{ are neighbors (share land borders)}$$

Write the following expressions in words and then determine their truth values.

1. $\forall x \in S, \exists y \in S, P(x, y)$.
2. $\forall x \in S, \exists y \in S, \sim P(x, y)$.
3. $\exists x, y, z \in S, P(x, y) \wedge P(x, z) \wedge P(y, z)$.

$$4. \exists x \in S, \forall y \in S, P(x, y).$$

$$5. \exists x \in S, \forall y \in S, \sim P(x, y).$$

Negations of “for all...”

$$\begin{aligned} \sim(\forall x \in S, P(x)) &\equiv \exists x \in S, \sim P(x) \\ \sim(\text{“for all } x \text{ in } S, P(x) \text{ is true”}) &\equiv \text{“there exists an } x \text{ in } S \text{ such that } P(x) \text{ is false”} \end{aligned}$$

Negation of “there exists...”

$$\begin{aligned} \sim(\exists x \in S, P(x)) &\equiv \forall x \in S, \sim P(x) \\ \sim(\text{“there exists an } x \text{ in } S \text{ such that } P(x) \text{ is true”}) &\equiv \text{“for all } x \text{ in } S, P(x) \text{ is false”} \end{aligned}$$

Example 2 Consider the open sentence

$$P(x, y) : x < y.$$

Write the following expressions in words and then determine their truth values.

$$1. \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, P(x, y).$$

In words:

Its negation in symbols:

Its negation in words:

$$2. \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, P(y, x).$$

In words:

Its negation in symbols:

Its negation in words:

$$3. \forall x, y \in \mathbb{R}, P(x, y) \Rightarrow (\exists z \in \mathbb{R}, P(x, z) \wedge P(z, y)).$$

In words:

Its negation in symbols:

Its negation in words: