Homework 4 [Due 9/29]

Problem 1. Fill in the addition and multiplication table for the "modulo 5 arithmetic": In both tables, for the a and b given, find $c \in \{0, 1, 2, 3, 4\}$ such that

$$\begin{aligned} a + b &\equiv c \pmod{m} \\ a \times b &\equiv c \pmod{m} \end{aligned}$$

respectively.

$+$	0	1	2	3	4
0					
1					
2					
3					
4					

$\times$	0	1	2	3	4
0					
1					
2					
3					
4					

Problem 2. Find an integer x such that

$$\begin{aligned} x &\equiv 1 \pmod{3} \\ x &\equiv 3 \pmod{5}. \end{aligned}$$

Problem 3. Prove that for two positive integers a and b, if $a|b$ and $b|a$ than $a = b$.

Problem 4. Let n be an integer. Prove that if $3|n^2$, then $3|n$.

Problem 5. For integers a, b, c, m with $m \geqslant 2$, prove that $a \equiv b \pmod{m}$ implies $ac \equiv bc \pmod{m}$.

Problem 6. Find integers a, b, c such that $ac \equiv bc \pmod{4}$ but $a \not\equiv b \pmod{4}$.

Problem 7. For integers a, b, c, m with $m \geqslant 2$, if $ac \equiv bc \pmod{m}$, can we "cancel" the c from both sides and conclude that $a \equiv b \pmod{m}$?

1. If you think we can do that, provide a proof.
2. If you think we cannot do that, explain why.

Problem 8. Prove that for any integer x not divisible by 3 there exists an integer y such that $xy \equiv 1 \pmod{3}$.

Problem 9. Prove that for any two integers x and y with y not divisible by 3, there exists an integer z such that $x \equiv yz \pmod{3}$.

Problem 10. Consider the statement "for any integer x not divisible by 4 there exists an integer y such that $xy \equiv 1 \pmod{4}$"

1. If you think it is true, then provide a proof.
2. If you think it is not true, provide an integer x for which there can be no number y such that $xy \equiv 1 \pmod{3}$.

Problem 11. Prove that for any integer x not divisible by 5 there exists an integer y such that $xy \equiv 1 \pmod{5}$.

Problem 12. Prove that for any two integers x and y with y not divisible by 5, there exists an integer z such that $x \equiv yz \pmod{5}$.

Problem 13. For integers a, b, m with $m \geqslant 2$, prove that $a \equiv b$ implies $a^2 \equiv b^2 \pmod{m}$.

Problem 14. For integers a, b, m with $m \geqslant 2$, does $a^2 \equiv b^2 \pmod{m}$ imply $a \equiv b$? Explain your answer.

Problem 15. Consider the statement "*for all $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ there exists a natural number n such that for any integers i and j both greater than n, $|x_i - x_j| < \epsilon$*".

1. Rewrite the statement using symbols such as $\forall$ and $\exists$.

2. Write down the negation of the statement using symbols.

3. Write down the negation of the statement in English.

Problem 16. Explain the following concepts:

1. Proper subset

2. Power set

3. Relative complement

4. Partition of a set

5. Cartesian product of two sets