- For two integers a and b, we say a **divides** b (and use the notation $a|b$) if $b = ka$ for some integer k.
- An integer n is said to be **even** if 2 divides n.
- An integer is said to be **odd** if it is not even.

Problem 1. For integers $a, b,$ and c, if a divides b then a divides bc.

Proof. Since $a|b$, $b = ka$ for some integer k. Then

$$bc \;=\; (ka)c \;=\; (kc)a$$

which is an integral multiple of a. Therefore bc is divisible by a. □

Problem 2. The sum of two even integer is also even.

Problem 3. The product of two even integer is also even.

Problem 4. For any integer n, $n^2 - n$ is even.

Note. An observation is that $n^2 - n = n(n-1)$, so either n or $(n-1)$ is even. Therefore $n(n-1)$ must be divisible by 2.

Proof. (Case 1) If n is even, then $n = 2k$ for some integer k, and hence

$$n^2 - n \;=\; (2k)^2 - 2k \;=\; 4k^2 - 2k \;=\; 2(2k^2 - k)$$

which is an integral multiple of 2. Therefore $n^2 - n$ is even.

(Case 2) If n is odd, then $n = 2k + 1$ for some integer k, and hence

$$n^2 - n \;=\; n(n-1) \;=\; (2k+1)(2k+1-1) \;=\; 2(2k+1)k$$

which is an integral multiple of 2. Therefore $n^2 - n$ is even. □

Problem 5. If the integers a and b are both even or both odd, then $a^2 - b^2$ must be even.

Proof. (Case 1) Suppose a and b are both even, then $a = 2m$ and $b = 2n$ for some integers m and n respectively. Then

$$a^2 - b^2 \;=\; (2m)^2 - (2n)^2 \;=\; 4m^2 - 4n^2 \;=\; 2(2m^2 - 2n^2)$$

which is an integral multiple of 2. Therefore $a^2 - b^2$ is even.

(Case 2) Suppose a and b are both odd, then $a = 2m + 1$ and $b = 2n + 1$ for some integers m and n respectively. Then

$$a^2 - b^2 \;=\; (2m+1)^2 - (2n+1)^2 \;=\; (4m^2 + 4m + 1) - (4n^2 + 4n + 1) \;=\; 2(2m^2 - 2n^2 + 2m - 2n)$$

which is an integral multiple of 2. Therefore $a^2 - b^2$ is even.

In both cases, $a^2 - b^2$ is even, so if the integers a and b are both even or both odd, then $a^2 - b^2$ must be even. □