Logical statements

A **statement** is a *declarative* sentence or assertion that is true or false (but not both).

Each statement has a **truth value**:

- **true** (denoted by T)
- **false** (denoted by F)

Similar (but slightly different) concepts in other context:

- Proposition
- Claim
- Assertion

These are not statements

- (Interrogative) What is the solution to $2x - 3 = 1$?
- (Imperative) Multiply x by 3.
- (Exclamatory) What a wonderful day!

Examples

Are these statements?

- 3 is an odd integer. (Yes, it is a statement)
- 3 is an even integer. (Yes, it is a statement)
- Is 3 even or odd? (No, it is not a statement)
- 3 is either odd or even. (Yes, it is a statement)
- Let x be 3. (No, it is not a statement)
- The sequence "666" appears in the decimal expansion of π infinitely many times. (Yes, it is a statement. But. . . is it true?)
- For two integers x and y with $x > y$, the greatest common divisor of x and y is also the greatest common divisor of $x - y$ and y. (Yes, it is a statement.)

Notation

We often use symbols to represent statements:

$$P: \ 3 \text{ is an even number.}$$

In this case, the truth value of P is F.

Statements with variables

A **open sentence** is a declarative sentence that contains variables.

The **domain** of an open sentence is the (prescribed) set of all possible values for the variables involved.

E.g.

$$P(x): \ x \text{ is an even number}$$

over the integers $\mathbb{Z}$ (its domain)

$$\begin{aligned} P(0) &: \ 0 \text{ is an even number} \\ P(1) &: \ 1 \text{ is an even number} \\ P(2) &: \ 2 \text{ is an even number} \\ P(999) &: \ 999 \text{ is an even number} \end{aligned}$$

It may take different truth value on different values for the variable(s)

Logical connectives

They takes statements as input and create new statements.

Another POV:

Truth functions

Yet another POV:

Operations on truth values

Yet another POV:

Binary arithmetics

Negation

For a statement P, the **negation** of P is the statement

not P

Notation:

$$\sim P$$

In (awkward) English, it can be expressed as

It is not the case that P

E.g.

$$P: \text{ 3 is an odd integer}$$

Then $\sim P$ can be expressed as:

- It is not the case that 3 is an odd integer
- 3 is not an odd integer

Negation

For a statement with variables, negation keeps the domain unchanged.

For a real number x, $P(x):\ x$ is nonnegative

$\sim P(x):\ x$ is not nonnegative

Note, however, x is a real number (the domain is $\mathbb{R}$), so we can also use

$\sim P(x):\ x$ is positive

Negation

$P(x, y)$: x and y are both positive

$\sim P(x, y)$: x and y are both not positive **WRONG**

$\sim P(x, y)$: x and y are **not** both positive

Negation: Truth table

P	$\sim P$
T	F
F	T

Can you see that $\sim(\sim P)$ is equivalent to P itself?

Disjunction

For two statements P and Q, the **disjunction** of the two is the statement

$$P \textbf{ or } Q$$

Notation:

$$P \vee Q$$

$P \vee Q$ is true when

1. one of P and Q is true; or
2. both P or Q are true

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction

For two statements P and Q, the **conjunction** of the two is the statement

$$P \textbf{ and } Q$$

Notation:

$$P \wedge Q$$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Try this

P	Q	$\sim (P \wedge Q)$	$(\sim P) \vee (\sim Q)$
T	T		
T	F		
F	T		
F	F		

Search for **XOR** and **NAND**

Implications (conditional)

Given two statements P and Q, the **implication** is the statement

If P **then** Q

or equivalently

P **implies** Q

Notation:

$$P \implies Q$$

In this setting, we often call P the **hypothesis** (premise) and Q the **conclusion**.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Ways to express $P \implies Q$

- If P then Q
- P implies Q
- P only if Q
- Q if P
- P is sufficient for Q
- Q is necessary for P

Why is [false implies anything] true?

If I have a billion dollars then I will give every student a 4.0.

If I have robbed a bank this morning then. . .

If pigs fly . . .

Rewrite this as if. . . then. . .

A real number is rational whenever it is an integer.

Try this

P	Q	$P \implies Q$	$(\sim P) \vee Q$
T	T		
T	F		
F	T		
F	F		