- For two integers a and b, we say a **divides** b (and use the notation $a|b$) if $b = ka$ for some integer k.
- An integer n is said to be **even** if 2 divides n.
- An integer is said to be **odd** if it is not even.
- Given two integers a and b with $b \neq 0$, there exists a unique pair of integers (q, r) with $0 \leqslant |r| < |b|$ such that

$$a = bq + r.$$

  This result is commonly known as *Euclidean Division* or *Division Algorithm*. In this context, q is known as the **quotient** and r is known as the **remainder**.

Problem 1. For any integer n, $n^2 - n$ is divisible by 2.

Note. An observation is that $n^2 - n = n(n-1)$, so either n or $(n-1)$ is even. Therefore $n(n-1)$ must be divisible by 2.

Proof. (Case 1) If n is even, then $n = 2k$ for some integer k, and hence

$$n^2 - n = n(n-1) = 2k(2k-1) = 2[k(2k-1)]$$

which is an integral multiple of 2. Therefore $n^2 - n$ is even.

(Case 2) If n is odd, then $n = 2k+1$ for some integer k, and hence

$$n^2 - n = n(n-1) = (2k+1)(2k+1-1) = 2(2k+1)k$$

which is an integral multiple of 2. Therefore $n^2 - n$ is even. □

Problem 2. For any integer n, $n^3 - n$ is divisible by 3.

Proof. (Case 1) If $n = 3k$ for some integer k, then

$$n^3 - n = n(n^2 - 1) = 3k\left((3k)^2 - 1\right)$$

is an integral multiple of 3. So 3 divides $n^3 - n$.

(Case 2) Alternatively, if $n = 3k + 1$ for some integer k, then

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1) = (3k+1)(3k+1-1)(3k+1+1) = (3k+1)3k(3k+2)$$

which is an integral multiple of 3. Therefore $n^3 - n$ is divisible by 3.

(Case 3) Finally, if $n = 3k + 2$ for some integer k, then

$$\begin{aligned} n^3 - n &= n(n-1)(n+1) \\ &= (3k+2)(3k+2-1)(3k+2+1) \\ &= (3k+2)(3k+1)(3k+3) \\ &= 3(3k+2)(3k+1)(k+1) \end{aligned}$$

which is, again, an integral multiple of 3. So in this case 3 also divides $n^3 - n$.

Since the three cases above cover all the possibilities, we can conclude that $n^3 - n$ is divisible by 3 for any integer n. $\square$

Problem 3. For an integer n, if n^2 is odd then n is also odd.

Proof. (Proof by contrapositive) Suppose n is even, then $n = 2k$ for some integer k. In that case

$$n^2 = (2k)^2 = 2(2k^2)$$

is an integral multiple of 2 and hence not odd. $\square$

Note. Here we used the fact that the implication $P \Rightarrow Q$ and its contrapositive $(\sim Q) \Rightarrow (\sim P)$ are logically equivalent.

Problem 4. For an integer n, n^2 is odd if and only if n is odd.

Proof. ($\Rightarrow$) Suppose n is even, then $n = 2k$ for some integer k, then

$$n^2 = (2k)^2 = 2(2k^2)$$

is an integral multiple of 2, so n^2 is not odd. This shows that n^2 is odd implies n is odd.

($\Leftarrow$) Conversely, suppose n is odd, then $n = 2k + 1$ for some integer k. In this case,

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

which is an odd number.

Combining the two parts, we can conclude that n^2 is odd if and only if n is odd. $\square$