Homework 2 [Due 9/15]

Problem 1. Rewrite the following sentences in the "if... then..." form. Then write down the *converse* and the *contrapositive* for each of them. (Notice that some form sounds more natural than others)

1. You will not have time to finish this homework unless you start right now.
2. Whenever a prime p divides $x \cdot y$, p divides x or y.
3. I will stay in my office, only if I'm busy.

Problem 2. Determine the truth values of the following statements and *explain your reasoning*.

1. If 3 is prime then so is 7. (The correct answer here is true. However, in what sense is it true?)
2. I will get a 4.0 in this course if and only if my course average is no less than 90%.
3. A positive integer p is prime if and only if p is not divisible by any positive integer no greater than $\sqrt{p}$.

Problem 3. Construct the truth tables for the expressions

- $P \Rightarrow (P \vee Q)$
- $(P \wedge (\sim Q)) \wedge (P \wedge Q)$
- $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
- $(P \Rightarrow Q) \Leftrightarrow ((\sim Q) \Rightarrow (\sim P))$

and identify the *tautologies and contradictions* appeared in the table.

Problem 4. 2.26 in the textbook (3rd edition)

Problem 5. 2.27 in the textbook (3rd edition)

Problem 6. 2.39 in the textbook (3rd edition)

Problem 7. 2.51 in the textbook (3rd edition)

Problem 8. List 5 (or 8 for extra credit) different tautologies. (You will have to show they are indeed tautologies by constructing truth tables)

Problem 9. Let $\mathbb{R}$ be the set of real numbers. Consider the following open sentences

$$\begin{aligned} P(x) &: x \text{ is not zero} \\ Q(x,y) &: x \cdot y = 1. \end{aligned}$$

Write the following expressions in words and then determine their truth values (with explanations, of course).

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, Q(x,y)$.
2. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \sim Q(x,y)$.
3. $\forall x \in \mathbb{R}, P(x) \Rightarrow (\exists y \in \mathbb{R}, Q(x,y))$.
4. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (P(y) \Rightarrow Q(x,y))$.
5. $\forall x \in \mathbb{R}, P(x) \Rightarrow (\exists y \in \mathbb{R}, P(y) \wedge Q(x,y))$.
6. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, Q(x,y)$. (Note the differences with statement number 1)
7. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, Q(x,y)$.
8. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \sim Q(x,y)$
9. $\exists x \in \mathbb{R}, P(x) \wedge (\forall y \in \mathbb{R}, \sim Q(x,y))$

Problem 10. For each of the expressions appeared in problem 7, state its negation.

Problem 11. Prove that for any positive integer k, $8^k - 1$ is divisible by 7. (Recall that in the previous homework you have proved $10^k - 1$ is divisible by 9. The same idea should, in principle, work here.)

Fake proofs

Mathematical fallacy refers to incorrect steps of deduction (either intentional or accidental) that are concealed in the presentation of the proof. They can be quite difficult to identify. Consider the following absurd "fake theorems". Their fake proofs may seem correct in the first glance. Identify the incorrect steps, and explain why they are incorrect. (The two examples below represent two very common mistakes related to calculus. In the future, you must be very careful to avoid this kind of mistakes in your own proofs.)

Theorem 1. $1 = 0$.

Proof. *Consider the indefinite integral*

$$\int \frac{dx}{x \ln x}.$$

Integrate by parts via

$$u = \ln x \quad \textit{and} \quad dv = \frac{1}{x}$$

and hence

$$du = -\frac{dx}{x \ln x} \quad \textit{and} \quad v = \ln x \,,$$

we get

$$\int \frac{dx}{x \ln x} = \left(\ln x \cdot \frac{1}{\ln x} \right) - \int -\frac{\ln x}{x\,(\ln x)^2}\, dx = 1 + \int \frac{dx}{x \ln x} \,.$$

After cancellation, we get $0 = 1$. □

Theorem 2. $1 = 2$.

Proof. (Another way) Since for any positive integer x we have

$$x^2 = \underbrace{x + x + \cdots + x}_{x\text{-times}}.$$

Differentiating both sides yields

$$2x = \underbrace{1 + 1 + \cdots + 1}_{x\text{-times}} = x \,.$$

Since x is positive and hence nonzero, we can cancel it from both sides, giving us

$$2 = 1.$$

□