INEQUALITY SUMMARY

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Famous Theorems

Chapter 1

Jensen's Inequality

1.1 Statement

Definition 1.1.1. *Suppose f is a function of one real variable defined on an interval X. f is called* convex *if:*

$$\forall x_1, x_2 \in X, \forall t \in [0,1] : \qquad f(tx_1 + (1-t)x_2) \leqslant tf(x_1) + (1-t)f(x_2).$$

f is called strictly convex *if:*

$$\forall x_1, x_2 \in X (x_1 \neq x_2), \forall t \in [0,1] : \qquad f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$$

A function f is said to be (strictly) concave *if -f is (strictly) convex.*

Theorem 1.1.1. *For a real convex function f, numbers $x_1, x_2, \cdots, x_n$ in its domain, and positive weights p_i, Jensen's inequality can be stated as:*

$$f\left(\frac{\sum\limits_{i=1}^{n} p_i x_i}{\sum\limits_{i=1}^{n} p_i}\right) \leqslant \frac{\sum\limits_{i=1}^{n} p_i f(x_i)}{\sum\limits_{i=1}^{n} p_i} \tag{1.1}$$

and the inequality is reversed if f is concave, which is

$$f\left(\frac{\sum\limits_{i=1}^{n} p_i x_i}{\sum\limits_{i=1}^{n} p_i}\right) \geqslant \frac{\sum\limits_{i=1}^{n} p_i f(x_i)}{\sum\limits_{i=1}^{n} p_i}. \tag{1.2}$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$ or f is linear [1].

As a particular case, if the weights p_i are all equal, then (1.1) and (1.2) become

$$f\left(\frac{\sum x_i}{n}\right) \leqslant \frac{\sum f(x_i)}{n} \tag{1.3}$$

and

$$f\left(\frac{\sum x_i}{n}\right) \geqslant \frac{\sum f(x_i)}{n}. \tag{1.4}$$

For instance, if λ_1 and λ_2 are two arbitrary nonnegative real numbers such that $\lambda_1 + \lambda_2 = 1$ then convexity of φ implies

$$\forall x_1, x_2 : \qquad \varphi\left(\lambda_1 x_1 + \lambda_2 x_2\right) \leqslant \lambda_1 \varphi(x_1) + \lambda_2 \varphi(x_2).$$

This can be generalized: if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are nonnegative real numbers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, then

$$\varphi\left(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n\right) \leqslant \lambda_1 \varphi(x_1) + \lambda_2 \varphi(x_2) + \cdots + \lambda_n \varphi(x_n) \tag{1.5}$$

for any $x_1, x_2, \cdots, x_n$. We also notice that if let $\lambda_i = \frac{p_i}{\sum_{k=1}^{n} p_k} (i = 1, 2, \cdots, n)$, then formule 1.5 is equivalent to formule 1.1.

Example 1.1.1. *Let $a, b, c > 0$ subject to $a + b + c = 1$, prove that*

$$\left(a + \frac{1}{a}\right)^3 + \left(b + \frac{1}{b}\right)^3 + \left(c + \frac{1}{c}\right)^3 \geqslant \frac{1000}{3}.$$

Proof. Consider a function $f(x) = \left(x + \frac{1}{x}\right)^3 (x > 0)$. Since its second derivative $f''(x) = \frac{6(x^6 + x^2 + 2)}{x^5} > 0$, f is convex on the interval $(0, +\infty)$. Using the formule 1.3 we have

$$f\left(\frac{a + b + c}{3}\right) \leqslant \frac{f(a) + f(b) + f(c)}{3}.$$

Applyng $a + b + c = 1$ to the above formula, what to be proved turns out.

□

1.2 Proof of the inequality

Proof. This finite form of the Jensen's inequality can be proved by induction: by convexity hypotheses, the statement is true for $n = 2$. Suppose it is true also for some n, one needs to prove it for $n + 1$. At least one of the λ_i is strictly positive, say λ_1; therefore by convexity inequality:

$$\varphi\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = \varphi\left(\lambda_1 x_1 + (1 - \lambda_1) \sum_{i=2}^{n+1} \frac{\lambda_i}{1 - \lambda_1} x_i\right)$$

$$\leqslant \lambda_1\varphi(x_1) + (1-\lambda_1)\varphi\left(\sum_{i=2}^{n+1}\frac{\lambda_i}{1-\lambda_1}x_i\right).$$

Since

$$\sum_{i=2}^{n+1}\frac{\lambda_i}{1-\lambda_1} = 1,$$

one can apply the induction hypotheses to the last term in the previous formula to obtain

$$\varphi\left(\sum_{i=2}^{n+1}\frac{\lambda_i}{1-\lambda_1}x_i\right) \leqslant \sum_{i=2}^{n+1}\frac{\lambda_i}{1-\lambda_1}\varphi\left(x_i\right).$$

Now we can see

$$\varphi\left(\sum_{i=1}^{n+1}\lambda_i x_i\right) \leqslant \lambda_1\varphi(x_1) + (1-\lambda_1)\sum_{i=2}^{n+1}\frac{\lambda_i}{1-\lambda_1}\varphi\left(x_i\right) = \sum_{i=1}^{n+1}\lambda_i\varphi\left(x_i\right),$$

namely the finite form of the Jensen's inequality, thereby showing it is also true for $n+1$. By mathematical induction, formule 1.5 is true for all $n \in \mathbb{N}^*$ [2]. $\square$

1.3 Application

Example 1.3.1. *Let positive numbers b_i such that $\sum\limits_{i=1}^{n} b_i = 1 (i = 1, 2, \cdots, n)$. Prove that*

$$\frac{1}{n} \leqslant b_1^{b_1} b_2^{b_2} \cdots b_n^{b_n} \leqslant \sum_{i=1}^{n} b_i^2.$$

Proof. Consider a concave function $f(x) = \ln x$ $(x \in \mathbb{R}^+)$.Since $\sum\limits_{i=1}^{n} b_i = 1$, we have

$$\sum_{i=1}^{n} b_i f(b_i) \leqslant f\left(\sum_{i=1}^{n} b_i \cdot b_i\right), \quad \sum_{i=1}^{n} b_i f(\frac{1}{b_i}) \leqslant f\left(\sum_{i=1}^{n} b_i \cdot \frac{1}{b_i}\right),$$

so

$$\ln\left(\prod_{i=1}^{n} b_i^{b_i}\right) \leqslant \ln\left(\sum_{i=1}^{n} b_i^2\right), \quad -\ln\left(\prod_{i=1}^{n} b_i^{b_i}\right) \leqslant \ln n.$$

We note that $f(x)$ is increasing, thus it follows that

$$\frac{1}{n} \leqslant \prod_{i=1}^{n} b_i^{b_i} \leqslant \sum_{i=1}^{n} b_i^2.$$

$\square$

Example 1.3.2. *Let $x_i \in \mathbb{R}^+ (i = 1, 2, \cdots, n)$. Show that*

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\cdots\left(x_n + \frac{1}{x_n}\right) \geqslant \left(\sqrt[n]{x_1 x_2 \cdots x_n} + \frac{1}{\sqrt[n]{x_1 x_2 \cdots x_n}}\right)^n.$$

Proof. We can assume $x_i = \mathrm{e}^{y_i}$ and so $y_i = \ln x_i (i = 1, 2, \cdots, n)$, thus

$$\prod_{i=1}^{n}\left(x_i + \frac{1}{x_i}\right) \geqslant \left(G_n + G_n^{-1}\right)^n$$

is equivalent to

$$\prod_{i=1}^{n}\left(\mathrm{e}^{y_i} + \mathrm{e}^{-y_i}\right) \geqslant \left(\mathrm{e}^{\frac{\sum y_i}{n}} + \mathrm{e}^{-\frac{\sum y_i}{n}}\right)^n.$$

Since logarithmic function $y = \ln x$ is increasing, we only need to prove

$$\frac{1}{n}\sum_{i=1}^{n}\ln\left(\mathrm{e}^{y_i} + \mathrm{e}^{-y_i}\right) \geqslant \ln\left(\mathrm{e}^{\frac{\sum y_i}{n}} + \mathrm{e}^{-\frac{\sum y_i}{n}}\right).$$

Let $f(x) = \ln\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)$. $f''(x) = \dfrac{4\mathrm{e}^{2x}}{\left(\mathrm{e}^{2x} + 1\right)^2} > 0$, and therfore f is strictly convex. By Jensen's Inequality, it follows that

$$\frac{1}{n}\sum_{i=1}^{n} f(y_i) \geqslant f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right).$$

This completes the proof.

□

Example 1.3.3. *Let $a_i \in \mathbb{R}^+, 0 \leqslant x_i \leqslant 1 (i = 1, 2, \cdots, n)$ and $\sum\limits_{i=1}^{n} a_i = 1$. Prove that*

$$\sum_{i=1}^{n}\frac{a_i}{1 + x_i} \leqslant \frac{1}{1 + x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. If some x_j equals 0, we obtain

$$\sum_{i=1}^{n}\frac{a_i}{1 + x_i} \leqslant \sum_{i=1}^{n} a_i = 1 = \frac{1}{1 + x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}.$$

If $0 < x_i \leqslant 1$, consider a function $f(t) = \dfrac{1}{1+e^t}(t \leqslant 0)$. Since $f''(t) = \dfrac{e^t(e^t-1)}{(e^t+1)^3} < 0$, that is to say, $f(t)$ is concave for all $x \in (-\infty, 0]$, we have

$$\sum_{i=1}^{n} a_i f(\ln x_i) \leqslant f\left(\sum_{i=1}^{n} a_i \ln x_i\right).$$

Simplify it and we get

$$\sum_{i=1}^{n} \frac{a_i}{1+x_i} \leqslant (1 + x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n})^{-1}.$$

□

Example 1.3.4. *Assume a matrix consisting of $m \times n$ numbers*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

such that $a_{ij} > 1$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$. If we mark the arithmetic mean of the i^{th} row as $A_i = \frac{1}{n}\sum\limits_{j=1}^{n} a_{ij}$, show that

$$\prod_{j=1}^{n} \left(\frac{a_{1j}a_{2j}\cdots a_{mj}+1}{a_{1j}a_{2j}\cdots a_{mj}-1}\right) \geqslant \left(\frac{A_1A_2\cdots A_m+1}{A_1A_2\cdots A_m-1}\right)^n.$$

Proof. Assume $f(x) = \ln\dfrac{e^x+1}{e^x-1}\,(x \in \mathbb{R}^+)$. We claim f(x) is convex since $f''(x) = \dfrac{2e^x(e^{2x}+1)}{(e^{2x}-1)^2} > 0$. Set the geometrical mean of the i^{th} row $G_i = (a_{i1}a_{i2}\cdots a_{in})^{\frac{1}{n}}$ and $x_j = \ln(a_{1j}a_{2j}\cdots a_{mj})$. We can easily check that

$$\frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}\sum_{j=1}^{n}\ln\prod_{i=1}^{m} a_{ij} = \frac{1}{n}\sum_{i=1}^{m}\ln\prod_{j=1}^{n} a_{ij} = \sum_{i=1}^{m}\ln G_i.$$

From Jensen's Inequality it follows that

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geqslant f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right),$$

i.e.

$$\frac{1}{n}\sum_{j=1}^{n}\ln\left(\frac{a_{1j}a_{2j}\cdots a_{mj}+1}{a_{1j}a_{2j}\cdots a_{mj}-1}\right) \geqslant \ln\left(\frac{G_1G_2\cdots G_m+1}{G_1G_2\cdots G_m-1}\right).$$

It is well known that $G_i \leqslant A_i$ (See also Chapter 2). Since $f'(x) = -\dfrac{2e^x}{e^{2x}-1}$ and so $f(x)$ is decreasing, we get

$$\sum_{j=1}^{n} \ln\left(\frac{a_{1j}a_{2j}\cdots a_{mj}+1}{a_{1j}a_{2j}\cdots a_{mj}-1}\right) \geqslant \ln\left(\frac{G_1G_2\cdots G_m+1}{G_1G_2\cdots G_m-1}\right)^n \geqslant \ln\left(\frac{A_1A_2\cdots A_m+1}{A_1A_2\cdots A_m-1}\right)^n.$$

Rewrite it as we used to do and we can obtain the inequality to be proved. □

Example 1.3.5. *Maximize* $f(x,y,z) = \dfrac{x(2y-z)}{1+x+3y} + \dfrac{y(2z-x)}{1+y+3z} + \dfrac{z(2x-y)}{1+z+3x}$ *given* $x,y,z > 0$ *such that* $x+y+z=1$.

Solution. Notice that

$$f(x,y,z) = \sum_{cyc} \frac{x(2y-z)}{1+x+3y} = \sum_{cyc} \frac{x(2y-z)}{2+(2y-z)}.$$

Consider a function $p(t) = \dfrac{t}{t+2}(t>0)$. We can see $p(t)$ is strictly concave as $p''(x) = -\dfrac{4}{(t+2)^3} < 0$. By Jensen's Inequality we have

$$f(x,y,z) = \sum_{cyc} xp(2y-z) \leqslant p\left(\sum_{cyc} x(2y-z)\right) = p\left(xy+yz+xz\right).$$

$p(t) = 1 - \dfrac{2}{t+2}(t>0)$ is increasing and

$$(x-y)^2+(y-z)^2+(x-z)^2 \geqslant 0 \Rightarrow xy+yz+xz \leqslant \frac{1}{3}(x+y+z)^2 = \frac{1}{3}.$$

Therefore

$$f(x,y,z) \leqslant p\left(xy+yz+xz\right) \leqslant p\left(\frac{1}{3}\right) = \frac{1}{7},$$

where equality holds if and only if $x=y=z=\dfrac{1}{3}$. Thus we assert $f(x,y,z)$ has a maximum $f\left(\dfrac{1}{3},\dfrac{1}{3},\dfrac{1}{3}\right) = \dfrac{1}{7}$.

□

Example 1.3.6. *Let* $a,b,c \in \mathbb{R}^+$. *Prove*

$$f(a,b,c) = \frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ac}} + \frac{c}{\sqrt{c^2+8ab}} \geqslant 1.$$

Proof. Without loss of generality, we can suppose $a+b+c=1$ .Otherwise $a'+b'+c'=\lambda$, then

$$f(a',b',c')=\sum_{cyc}\frac{a'}{\sqrt{a'^{\,2}+8b'c'}}=\sum_{cyc}\frac{a'/\lambda}{\sqrt{(a'/\lambda)^2+8(b'/\lambda)(c'/\lambda)}}=f\left(\frac{a'}{\lambda},\frac{b'}{\lambda},\frac{c'}{\lambda}\right)\geqslant 1.$$

Let $p(t)=\dfrac{1}{\sqrt{t}}(t>0)$.The first derivative $p'(t)=-\dfrac{1}{2}t^{-\frac{3}{2}}<0$ and the second derivative $p''(t)=\dfrac{3}{4}t^{-\frac{5}{2}}>0$. So p is convex which implies

$$f(a,b,c)=\sum_{cyc}ap(a^2+8bc)\geqslant p\left(\sum_{cyc}a\left(a^2+8bc\right)\right).$$

Notice $p(t)$ is decreasing and $p(1)=1$, thus we only need to prove that

$$\sum_{cyc}a(a^2+8bc)\leqslant 1=(a+b+c)^3.$$

Actually it clearly follows that

$$\sum_{cyc}a^3+8abc\leqslant(a+b+c)^3 \iff 3\sum_{cyc}a(b-c)^2\geqslant 0,$$

where equality holds if and only if $a=b=c$. □

Example 1.3.7. *For all $x,y,z\in\mathbb{R}^+$, prove that*

$$\sum_{cyc}\sqrt{3x(x+y)(x+z)}\leqslant\sqrt{4(x+y+z)^3}.$$

Proof. Without loss of generality, we can suppose $x+y+z=1$. It is easy to show function $p(t)=\sqrt{t}$ is concave for $x>0$. Accordingly we have

$$\begin{aligned}\sum_{cyc}\sqrt{3x(x+y)(x+z)}=\sum_{cyc}x\sqrt{\frac{3(x+y)(x+z)}{x}}&\leqslant\sqrt{\sum_{cyc}x\frac{3(x+y)(x+z)}{x}}\\&=\sqrt{\sum_{cyc}3(1-z)(1-y)}\\&=\sqrt{3\left(1+xy+yz+xz\right)}.\end{aligned}$$

In Example 1.3.5 we have shown that $xy+yz+xz\leqslant\dfrac{1}{3}(x+y+z)^2=\dfrac{1}{3}$. Hence

$$\sum_{cyc}\sqrt{3x(x+y)(x+z)}\leqslant\sqrt{3\left(1+\frac{1}{3}\right)}=2=\sqrt{4(x+y+z)^3}.$$

□

1.4 Generalization

Definition 1.4.1. *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ *where* $x_i > 0$, $p_i > 0 (i = 1, 2, \cdots, n)$. *We define*

$$M_n^r(\mathbf{x}, \mathbf{p}) = \begin{cases} \left(\dfrac{\sum\limits_{i=1}^{n} p_i x_i^r}{\sum\limits_{i=1}^{n} p_i} \right)^{\frac{1}{r}} & , 0 < |r| < +\infty, \\ \left(\prod\limits_{i=1}^{n} x_i^{p_i} \right)^{1 \big/ \sum\limits_{i=1}^{n} p_i} & , r = 0. \end{cases}$$

Definition 1.4.2. *Assume positive continuous function f(x) on interval* $I \subset \mathbb{R}^+$. *For all* $x_1, x_2 \in I$ *and* $p_1, p_2 \in \mathbb{R}^+$. $f(x)$ *is* generally convex *if and only if*

$$f\left(M_2^r(\mathbf{x}, \mathbf{p})\right) \leqslant M_2^r(\mathbf{f}(\mathbf{x}), \mathbf{p}), \tag{1.6}$$

where $\mathbf{f}(\mathbf{x}) = (f(x_1), f(x_2))$. $f(x)$ *is* generally strictly convex *when equality in formule 1.6 holds if and only if* $x_1 = x_2$. *Also formule 1.6 can be writen as*

$$\left(f\left(\frac{p_1}{p_1 + p_2} x_1^r + \frac{p_2}{p_1 + p_2} x_2^r \right) \right)^{\frac{1}{r}} \leqslant \left(\frac{p_1}{p_1 + p_2} (f(x_1))^r + \frac{p_2}{p_1 + p_2} (f(x_2))^r \right)^{\frac{1}{r}} \qquad if\ r \neq 0,$$

$$f\left((x_1^{p_1} \cdot x_2^{p_2})^{\frac{1}{p_1 + p_2}} \right) \leqslant ((f(x_1))^{p_1} \cdot (f(x_2))^{p_2})^{\frac{1}{p_1 + p_2}} \qquad if\ r = 0.$$

If the inequality in the formule 1.6 is reversed, then $f(x)$ *is* (strictly) generally concave *on the interval* I.

If we set $p_1 = p_2 = \frac{1}{2}$, then different particular values of r give different results:

$$\text{(i)} r = 2, f\left(\sqrt{\frac{x_1^2 + x_2^2}{2}} \right) \leqslant \sqrt{\frac{(f(x_1))^2 + (f(x_2))^2}{2}} \tag{1.7}$$

$$\text{(ii)} r = 1, f\left(\frac{x_1 + x_2}{2} \right) \leqslant \frac{f(x_1) + f(x_2)}{2} \tag{1.8}$$

$$\text{(iii)} r = 0, f\left(\sqrt{x_1 \cdot x_2} \right) \leqslant \sqrt{f(x_1) \cdot f(x_2)} \tag{1.9}$$

$$\text{(iv)} r = -1, f\left(\frac{2x_1 x_2}{x_1 + x_2} \right) \leqslant \frac{2f(x_1)f(x_2)}{f(x_1) + f(x_2)} \tag{1.10}$$

Theorem 1.4.1. *Assume* $f(x)$ *is generally convex on interval* $I \subset \mathbb{R}^+$.*For all* $x_i \in I$ *and* $p_i \in \mathbb{R}^+ (i = 1, 2, \cdots, n)$,*we have*

$$f\left(M_n^r(\mathbf{x}, \mathbf{p})\right) \leqslant M_n^r(\mathbf{f}(\mathbf{x}), \mathbf{p}) \tag{1.11}$$

or

$$f\left(\left(\frac{\sum_{i=1}^{n} p_i x_i^r}{\sum_{i=1}^{n} p_i}\right)^{\frac{1}{r}}\right) \leqslant \left(\frac{\sum_{i=1}^{n} p_i (f(x_i))^r}{\sum_{i=1}^{n} p_i}\right)^{\frac{1}{r}} \qquad if\ r \neq 0,$$

$$f\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/\sum_{i=1}^{n} p_i} \leqslant \left(\prod_{i=1}^{n} (f(x_i))^{p_i}\right)^{1/\sum_{i=1}^{n} p_i} \qquad if\ r = 0.$$

Proof. We can prove it by induction. If $r \neq 0$, the trick is similar to what we use in the proof of Jensen's inequality. Then let's consider the case $r = 0$.

1.Basis: According to the definition of generally convex function, we have

$$f\left(M_2^r(\mathbf{x}, \mathbf{p})\right) \leqslant M_2^r(\mathbf{f}(\mathbf{x}), \mathbf{p})$$

or

$$f\left((x_1^{p_1} \cdot x_2^{p_2})^{\frac{1}{p_1+p_2}}\right) \leqslant ((f(x_1))^{p_1} \cdot (f(x_2))^{p_2})^{\frac{1}{p_1+p_2}}.$$

2.Inductive step: Assume the inequality holds for some unspecified value of k. It must then be shown that it holds that :

$$f\left(\prod_{i=1}^{k+1} x_i^{p_i}\right)^{1/\sum_{i=1}^{k+1} p_i} \leqslant \left(\prod_{i=1}^{k+1} (f(x_i))^{p_i}\right)^{1/\sum_{i=1}^{k+1} p_i}$$

Using the basis we can show that

$$f\left(\prod_{i=1}^{k+1} x_i^{p_i}\right)^{1/\sum_{i=1}^{k+1} p_i} = f\left(x_1^{p_1} \cdot \prod_{i=2}^{k+1} x_i^{p_i}\right)^{1/\sum_{i=1}^{k+1} p_i}$$

□

Chapter 2

Average Value Inequality

2.1 The basic form of the inequality of average value

Theorem 2.1.1. *Let $a_i > 0$, $i = 1, 2, \ldots, n$.*

$$\sqrt[n]{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geqslant \frac{a_1 + a_2 + \cdots + a_n}{n} \geqslant \sqrt{a_1 a_2 \ldots a_n} \geqslant \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}, \tag{2.1}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n = 0$ [3]. The above inequality is often written in this form

$$\boldsymbol{Q_n} \geqslant \boldsymbol{A_n} \geqslant \boldsymbol{G_n} \geqslant \boldsymbol{H_n} \ .$$

2.1.1 AM-GM inequality

Definition 2.1.1. *Assume $a = (a_1, a_2, \ldots, a_n)$, $a_k \geqslant 0$,$1 \leqslant k \leqslant n$. $\boldsymbol{A_n} = \frac{1}{n}\sum_{k=1}^{n} a_k$, called the* Arithmetic Mean *of $a_1, a_2, \ldots, a_n$. $\boldsymbol{G_n} = (\prod_{k=1}^{n} a_k)^{\frac{1}{n}}$, called the* Geometric Mean *of $a_1, a_2, \ldots, a_n$ [4].*

Theorem 2.1.2.

$$\boldsymbol{A_n} \geqslant \boldsymbol{G_n} \tag{2.2}$$

or

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^n \geqslant \prod_{k=1}^{n} a_k \ . \tag{2.3}$$

Proof. Note that geometrical mean $\boldsymbol{G_n}$ is actually equal to

$$\exp\left(\frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n}\right),$$

Since the natural logarithm is strictly increasing, AM-GM inequality is equivalent to

$$\ln\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n}.$$

Since the natural logarithmic function is strictly concave, according to Jensen's inequality we can imply that the above inequality holds. We have thus proved the theorem.

□

Example 2.1.1. *Assume $n \in \mathbb{N}^+, x \in \mathbb{R}^+$.Prove that*

$$x + \frac{1}{nx^n} \geqslant \frac{n+1}{n} .$$

Proof. From $\boldsymbol{A_n} \geqslant \boldsymbol{G_n}$, we have

$$\frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} \geqslant \sqrt[n+1]{a_1 a_2 \ldots a_{n+1}} ,$$

$$x + \frac{1}{nx^n} = \underbrace{\frac{x}{n} + \cdots + \frac{x}{n}}_{n\ numbers} + \frac{1}{nx^n} \geqslant (n+1)\sqrt[n+1]{\frac{1}{n^{n+1}}} = \frac{n+1}{n} .$$

□

Theorem 2.1.3.

$$\boldsymbol{G_n}(\mathbf{a}, \mathbf{q}) \leqslant \boldsymbol{A_n}(\mathbf{a}, \mathbf{q}) , \tag{2.4}$$

where

$$\boldsymbol{G_n}(\mathbf{a}, \mathbf{q}) = \prod_{k=1}^{n} a_k^{q_k} , \ \boldsymbol{A_n}(\mathbf{a}, \mathbf{q}) = \sum_{k=1}^{n} q_k a_k , \ \sum_{k=1}^{n} q_k = 1, \ q_k > 0 \ (k = 1, 2, \cdots, n),$$

in other words,

$$a_1^{q_1} a_2^{q_2} \cdots a_n^{q_n} \leqslant a_1 q_1 + a_2 q_2 + \cdots + a_n q_n , \ q_1 + q_2 + \cdots + q_n = 1 .$$

$\boldsymbol{G_n}$ and $\boldsymbol{A_n}$ can be connected by logarithmic transformation . Suppose $\ln \mathbf{a} = (\ln a_1, \ln a_2, \ldots, \ln a_n)$, thus

$$\ln \boldsymbol{G_n}(\mathbf{a}, \mathbf{q}) = \ln a_1^{q_1} a_2^{q_2} \cdots a_n^{q_n} = \sum_{k=1}^{n} q_k \ln a_k = \boldsymbol{A_n}(\ln \mathbf{a}, \mathbf{q}) .$$

Formula (3) can be generalized. Let $a_{jk} > 0, \ q_k > 0, \ and \sum_{k=1}^{n} q_k = 1$, then

$$\sum_{j=1}^{m}\left(\prod_{k=1}^{m} a_{jk}^{q_k}\right) \leqslant \prod_{k=1}^{m}\left(\sum_{j=1}^{m} a_{jk}\right)^{q_k} \tag{2.5}$$

Example 2.1.2. *Let $a, b, c, d > 0$. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geqslant 2.$$

Proof.

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geqslant 2$$

$$\iff \frac{2a+b+c}{b+c}+\frac{2b+c+d}{c+d}+\frac{2c+d+a}{d+a}+\frac{2d+a+b}{a+b} \geqslant 8$$

$$\iff \left(\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}\right)+\left(\frac{a+c}{b+c}+\frac{c+a}{d+a}\right)+\left(\frac{b+d}{c+d}+\frac{d+b}{a+b}\right) \geqslant 8. \tag{a}$$

From $\boldsymbol{A_n} \geqslant \boldsymbol{G_n}$, we obtain

$$\left(\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}\right) \geqslant 4\sqrt[4]{\frac{a+b}{b+c}\cdot\frac{b+c}{c+d}\cdot\frac{c+d}{d+a}\cdot\frac{d+a}{a+b}} = 4 \tag{b}$$

and

$$(a+b+c+d)\left(\frac{1}{b+c}+\frac{1}{d+a}\right) = \frac{a+d}{b+c}+\frac{b+c}{a+d}+2 \geqslant 2\sqrt{\frac{a+d}{b+c}\cdot\frac{b+c}{a+d}}+2 = 4$$

$$\iff \frac{a+c}{b+c}+\frac{c+a}{d+a} \geqslant \frac{4(a+c)}{a+b+c+d}. \tag{c}$$

In a similar way, we have

$$\frac{b+d}{c+d}+\frac{d+b}{a+b} \geqslant \frac{4(b+d)}{a+b+c+d}. \tag{d}$$

Add (b), (c), (d) up and we get (a). This completes the proof. □

2.1.2 RMS-AM Inequality

Theorem 2.1.4. *Root mean square-arithmetic mean inequality states that for positive numbers $x_1, x_2, \cdots, x_n$,*

$$\sqrt[n]{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geqslant \frac{a_1+a_2+\cdots+a_n}{n}. \tag{2.6}$$

2.2 Power Mean Inequality

Definition 2.2.1. *If p is a non-zero real number, we can define the* generalized mean *or* power mean *with exponent p of the positive real numbers $x_1, x_2, \cdots, x_n$ as:*

$$M_p(x_1, x_2, \cdots, x_n) = \left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}}.$$

Furthermore, for a sequence of positive weights w_i with sum $\sum a_i = 1$, we define the weighted power mean *as:*

$$M_p(x_1, x_2, \cdots, x_n) = \left(w_1 x_1^p + w_2 x_2^p + \cdots + w_n x_n^p\right)^{\frac{1}{p}}.$$

The unweighted means correspond to setting all $w_i = \frac{1}{n}$.

Theorem 2.2.1. *In general, if $p \leq q$, then*

$$M_p(x_1, x_2, \cdots, x_n) \leqslant M_q(x_1, x_2, \cdots, x_n), \tag{2.7}$$

and the two means are equal if and only if $x_1 = x_2 = \cdots = x_n$.

If $p \geq 1$, then

$$\frac{x_1^p + x_2^p + \cdots + x_n^p}{n} \geqslant \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^p.$$

Proof. It follows from the fact that, for all real p,

$$\frac{\partial}{\partial p} M_p(x_1, x_2, \cdots, x_n) \geqslant 0,$$

which can be proved using Jensen's inequality. □

Chapter 3

Bernoulli's inequality

3.1 Statement

Theorem 3.1.1. *If $x > -1$, then*

$$(1+x)^r \geqslant 1+rx \tag{3.1}$$

for $r \geqslant 1$ or $r \leqslant 0$, and

$$(1+x)^r \leqslant 1+rx \tag{3.2}$$

for $0 \leqslant r \leqslant 1$, with equality if and only if $x = 0$ or $r = 0, 1$. [8]

3.2 Proof of the inequality

Proof. For $r = 0$,

$$(1+x)^0 \geq 1+0x$$

is equivalent to 11 which is true as required. Now suppose the statement is true for $r = k$:

$$(1+x)^k \geq 1+kx.$$

Then it follows that

$$(1+x)(1+x)^k \geq (1+x)(1+kx) \quad \text{(by hypothesis, since } (1+x) \geq 0) \tag{3.3}$$

$$\iff (1+x)^{k+1} \geq 1+kx+x+kx^2, \tag{3.4}$$

$$\iff (1+x)^{k+1} \geq 1+(k+1)x+kx^2. \tag{3.5}$$

However, as

$$1+(k+1)x+kx^2 \geq 1+(k+1)x(sincekx^2 \geq 0),$$

it follows that

$$(1+x)^{k+1}1+(k+1)x,$$

which means the statement is true for $r = k+1$ as required. By induction we conclude the statement is true for all $r \geq 0$. $\square$

Chapter 4

Rearrangement Inequality

4.1 Statement

Theorem 4.1.1.

$$x_n y_1 + \cdots + x_1 y_n \leqslant x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \leqslant x_1 y_1 + \cdots + x_n y_n \tag{4.1}$$

for every choice of real numbers

$$x_1 \leqslant \cdots \leqslant x_n \quad and \quad y_1 \leqslant \cdots \leqslant y_n$$

and every permutation

$$x_{\sigma(1)}, \ldots, x_{\sigma(n)}$$

of

$$x_1, \ldots, x_n \ .$$

If the numbers are different, meaning that

$$x_1 < \cdots < x_n \quad and \quad y_1 < \cdots < y_n \ ,$$

then the lower bound is attained only for the permutation which reverses the order, i.e., $\sigma(i) = n - i + 1$ for all $i = 1, \ldots, n$, and the upper bound is attained only for the identity, i.e., $\sigma(i) = i$ for all $i = 1, \ldots, n$.

Note that the rearrangement inequality makes no assumptions on the signs of the real numbers. [7]

4.2 Generalization

Theorem 4.2.1.

$$\sum_{j=1}^{n} \prod_{i=1}^{m} a'_{ij} \leqslant \sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij}, \tag{4.2}$$

$$\prod_{j=1}^{n}\sum_{i=1}^{m} a'_{ij} \geqslant \prod_{j=1}^{n}\sum_{i=1}^{m} a_{ij} \tag{4.3}$$

for every choice of real numbers $0 \leqslant a_{i1} \leqslant a_{i2} \cdots \leqslant a_{in}$ *and every permutation* $a'_{i1} \leqslant a'_{i2} \cdots \leqslant a'_{in}$ *of* $a_{i1}, a_{i2}, \cdots, a_{in}$ $(i = 1, 2, \cdots, m)$.

(4.2) and (4.3) are called *S inequality* and *T inequality* in short. To help understand and apply the two inequalities, we consider two matrixes arranged by $m \times n$ numbers ($a'_{i1} \leqslant a'_{i2} \cdots \leqslant a'_{in}$, $i = 1, 2, \cdots, m$):

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

$$A' = \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{pmatrix},$$

Chapter 5

Schur's Inequality

5.1 Statement

Theorem 5.1.1. *For all non-negative real numbers x, y, z and a positive number t,*

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \geqslant 0 \tag{5.1}$$

with equality if and only if $x = y = z$ or two of them are equal and the other is zero.

When $t = 1$, the following well-known special case can be derived:

$$x^3 + y^3 + z^3 + 3xyz \geqslant xy(x+y) + xz(x+z) + yz(y+z). \tag{5.2}$$

5.2 Proof of the inequality

Proof. Since the inequality is symmetric in x, y, z. we may assume without loss of generality that $x \geqslant y \geqslant z$. Then the inequality

$$(x-y)[x^t(x-z) - y^t(y-z)] + z^t(x-z)(y-z) \geqslant 0$$

clearly holds, since every term on the left-hand side of the equation is non-negative. This rearranges to Schur's inequality. □

Chapter 6

Mahler's Inequality

6.1 Statement

Theorem 6.1.1. *Mahler's inequality, named after Kurt Mahler, states that the geometric mean of the term-by-term sum of two finite sequences of positive numbers is greater than or equal to the sum of their two separate geometric means:*

$$\prod_{i=1}^{n}(x_i + y_i)^{\frac{1}{n}} \geqslant \prod_{i=1}^{n} x_i^{\frac{1}{n}} + \prod_{i=1}^{n} y_i^{\frac{1}{n}}, \tag{6.1}$$

when $x_i, y_i > 0$ for all k [5].

6.2 Proof of the inequality

Proof. By the inequality of arithmetic and geometric means, we have:

$$\prod_{i=1}^{n}\left(\frac{x_i}{x_i + y_i}\right)^{\frac{1}{n}} \leqslant \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{x_i + y_i}\right)$$

and

$$\prod_{i=1}^{n}\left(\frac{y_i}{x_i + y_i}\right)^{\frac{1}{n}} \leqslant \frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i}{x_i + y_i}\right).$$

Hence,

$$\prod_{i=1}^{n}\left(\frac{x_i}{x_i + y_i}\right)^{\frac{1}{n}} + \prod_{i=1}^{n}\left(\frac{y_i}{x_i + y_i}\right)^{\frac{1}{n}} \leqslant \frac{1}{n}n = 1.$$

Clearing denominators then gives the desired result. □

Chapter 7

Cauchy-Schwartz Inequality

7.1 Statement

Theorem 7.1.1. *Let* $a_i,\ b_i \in \mathbb{R},\ i = 1, 2, ..., n.$

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geqslant \left(\sum_{i=1}^{n} a_i b_i\right)^2, \tag{7.1}$$

Equality holds if and only if $a_i = \lambda b_i$ *(*λ *is a constant).* [6]

7.2 Proof of the inequality

Proof. We can check that

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1\leqslant i<j\leqslant n} (a_i b_j - a_j b_i)^2 \geqslant 0.$$

It is now obvious that the theorem holds. □

7.3 Application

Example 7.3.1. *Prove that for all positive* x, y *and* z*, we have*

$$x + y + z \leqslant 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right).$$

Proof. Indeed,

$$(x+y+z)^2 = \left(x\sqrt{\frac{y+z}{y+z}} + y\sqrt{\frac{x+z}{x+z}} + z\sqrt{\frac{x+y}{x+y}}\right)^2$$

$$\leqslant \left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right)(y+z+x+z+x+y)$$

$$\iff x+y+z \leqslant 2\left(\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\right).$$

An alternative proof is simply obtained with the Jensen inequality.
The inequality is

$$f\left(\frac{x+y+z}{3}\right) \leqslant \frac{f(x)+f(y)+f(z)}{3}$$

for all positive x,y,z with sum A, where $f(u)=\dfrac{u^2}{A-u}$, and $A=x+y+z$. This is equivalent to $f(u)$ being convex in the interval $(0,A)$.The second derivative is

$$f''(x)=\frac{2A^2}{(A-u)^3} \geqslant 0.$$

□

Example 7.3.2. *Let $a_i > 0 (i=1,2,\ldots,n)$, and $\sum\limits_{i=1}^{n} a_i = 1$.Then*

$$\sum_{i=1}^{n}\left(a_i+\frac{1}{a_i}\right)^2 \geqslant \frac{(1+n^2)^2}{n}.$$

Proof.

$$(1^2+1^2+\cdots+1^2)\sum_{i=1}^{n}\left(a_i+\frac{1}{a_i}\right)^2$$

$$\geqslant \left[\left(a_1+\frac{1}{a_1}\right)+\left(a_2+\frac{1}{a_2}\right)+\cdots+\left(a_n+\frac{1}{a_n}\right)\right]^2$$

$$=\left(\sum_{i=1}^{n} a_i+\sum_{i=1}^{n}\frac{1}{a_i}\right)^2.$$

Note that

$$\sum_{i=1}^{n} a_i = 1, \quad \left(\sum_{i=1}^{n} a_i\right)\cdot\left(\sum_{i=1}^{n}\frac{1}{a_i}\right) \geqslant n^2.$$

Thus, we can infer that

$$\sum_{i=1}^{n}\left(a_i+\frac{1}{a_i}\right)^2 \geqslant \frac{(1+n^2)^2}{n}.$$

□

Example 7.3.3. *Let $a_1,a_2,\ldots,a_n \in \mathbb{R}^+$,$n \in \mathbb{N}^+$,$n \geqslant 2$.Then*

$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n} \geqslant \frac{n^2}{n-1},$$

where $s=a_1+a_2+\cdots+a_n$.

Proof. Notice that

$$(n-1)s = ns - (s = a_1 + a_2 + \cdots + a_n) = (s - a_1) + (s - a_2) + \cdots + (s - a_n).$$

According to CauchyâĂŞSchwarz inequality, it follows

$$[(s-a_1) + (s-a_2) + \cdots + (s-a_n)]\left(\frac{1}{s-a_1} + \frac{1}{s-a_2} + \cdots + \frac{1}{s-a_n}\right)$$

$$\geqslant \left(\sqrt{s-a_1}\cdot\frac{1}{\sqrt{s-a_1}} + \sqrt{s-a_2}\cdot\frac{1}{\sqrt{s-a_2}} + \cdots + \sqrt{s-a_n}\cdot\frac{1}{\sqrt{s-a_n}}\right)^2 = n^2.$$

Hence,

$$s(n-1)\left(\frac{1}{s-a_1} + \frac{1}{s-a_2} + \cdots + \frac{1}{s-a_n}\right) \geqslant n^2.$$

Thus we have derived that

$$\frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_n} \geqslant \frac{n^2}{n-1}.$$

□

7.4 Corollary

Theorem 7.4.1. Triangle inequality *states that*

$$\left|\sqrt{\sum_{i=1}^n a_i^2} - \sqrt{\sum_{i=1}^n b_i^2}\right| \leqslant \sqrt{\sum_{i=1}^n (a_i \pm b_i)^2} \leqslant \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n a_i^2}. \tag{7.2}$$

Proof. According to Cauchy inequality, it follows that

$$\left|\sum_{i=1}^n a_i b_i\right| \leqslant \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}.$$

Then we have

$$\begin{aligned}
\sum_{i=1}^n (a_i \pm b_i)^2 &\leqslant \sum_{i=1}^n a_i^2 + 2\left|\sum_{i=1}^n a_i b_i\right| + \sum_{i=1}^n b_i^2 \\
&\leqslant \sum_{i=1}^n a_i^2 + 2\sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2} + \sum_{i=1}^n b_i^2 \\
&= \left(\sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n a_i^2}\right)^2.
\end{aligned}$$

and similarly

$$\begin{aligned}
\sum_{i=1}^{n}(a_i \pm b_i)^2 &\geqslant \sum_{i=1}^{n} a_i^2 - 2\left|\sum_{i=1}^{n} a_i b_i\right| + \sum_{i=1}^{n} b_i^2 \\
&\geqslant \sum_{i=1}^{n} a_i^2 - 2\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + \sum_{i=1}^{n} b_i^2 \\
&= \left(\sqrt{\sum_{i=1}^{n} a_i^2} - \sqrt{\sum_{i=1}^{n} a_i^2}\right)^2 .
\end{aligned}$$

By the above two we get

$$\left(\sqrt{\sum_{i=1}^{n} a_i^2} - \sqrt{\sum_{i=1}^{n} b_i^2}\right)^2 \leqslant \sum_{i=1}^{n}(a_i \pm b_i)^2 \leqslant \left(\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} a_i^2}\right)^2 ,$$

which is exactly equivalent to what to be proved. □

Theorem 7.4.2.

$$\sqrt{\sum_{i=1}^{n}(a_i + b_i)^2} \leqslant \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} a_i^2}. \tag{7.3}$$

Chapter 8

Chebyshev's Sum Inequality

8.1 Statement

Theorem 8.1.1. *If*

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$$

and

$$b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n,$$

then

$$\frac{1}{n}\sum_{k=1}^{n} a_k \cdot b_k \geqslant \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \ . \tag{8.1}$$

Similarly, if

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$$

and

$$b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n,$$

then

$$\frac{1}{n}\sum_{k=1}^{n} a_k \cdot b_k \leqslant \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \ . \tag{8.2}$$

8.2 Proof of the inequality

Proof. Consider the sum

$$S = \sum_{j=1}^{n}\sum_{k=1}^{n}(a_j - a_k)(b_j - b_k).$$

The two sequences are non-increasing, therefore $a_j - a_k$ and $b_j - b_k$ have the same sign for any j, k. Hence $S \geqslant 0$. Opening the brackets,we reduce:

$$0 \leqslant 2n\sum_{j=1}^{n} a_j b_j - 2\sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k,$$

whence

$$\frac{1}{n}\sum_{k=1}^{n} a_k \cdot b_k \geqslant \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) .$$

$\square$

Chapter 9

Young's Inequality

9.1 Statement

Theorem 9.1.1. *If a and b are nonnegative real numbers and p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}. \tag{9.1}$$

Equality holds if and only if $a^p = b^q$.

9.2 Proof of the inequality

Proof. The claim is certainly true if $a = 0$ or $b = 0$. Therefore, assume $a > 0$ and $b > 0$ in the following. Put $t = \frac{1}{p}$, and $(1 - t) = \frac{1}{q}$. Then since the logarithmfunction is strictly concave

$$\log\left(ta^p + (1-t)b^q\right) \geqslant t\log\left(a^p\right) + (1-t)\log\left(b^q\right) = \log(a) + \log(b) + w$$

with equality if and only if $a^p = b^q$. Young's inequality follows by exponentiating. $\square$

This form of Young's inequality is a special case of the inequality of weighted arithmetic and geometric means and can be used to prove Hölder's inequality.

Chapter 10

Hölder's Inequality

10.1 Statement

Theorem 10.1.1. *Let real numbers $a_i, b_i \geqslant 0 (i = 1, 2, \cdots, n)$, $p, q \in \mathbb{R}\backslash\{0\}$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $p > 1$, then*

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}, \tag{10.1}$$

and if $p < 1$ and $p \neq 0$, then

$$\sum_{i=1}^{n} a_i b_i \geqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}, \tag{10.2}$$

with equality if and only if $\alpha a_i^p = \beta b_i^q$, where $i = 1, 2, \ldots, n$, $\alpha^2 + \beta^2 \neq 0$.

10.2 Application

Example 10.2.1. *Let $a_i, b_i, m > 0$ .We have*

$$\sum_{i=1}^{n} \frac{a_i^{m+1}}{b_i^m} \geqslant \frac{\left(\sum\limits_{i=1}^{n} a_i\right)^{m+1}}{\left(\sum\limits_{i=1}^{n} b_i\right)^{m}}, \tag{10.3}$$

with equality if and only if $a_i = \lambda b_i$.

Proof. In formula(10.1), assume $p = m + 1$ with $m > 0$, whence we get

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^{m+1}\right)^{\frac{1}{m+1}} \left(\sum_{i=1}^{n} b_i^{\frac{m+1}{m}}\right)^{\frac{m}{m+1}}.$$

Then let $a_i = \dfrac{a_i}{b_i^{\frac{m}{m+1}}}$, $b_i = b_i^{\frac{m}{m+1}}$, so the above Inquality becomes

$$\sum_{i=1}^{n} a_i \leqslant \left(\sum_{i=1}^{n} \frac{a_i^{m+1}}{b_i^m}\right)^{\frac{1}{m+1}} \left(\sum_{i=1}^{n} b_i\right)^{\frac{m}{m+1}}.$$

Rewriting it we obtain

$$\sum_{i=1}^{n} \frac{a_i^{m+1}}{b_i^m} \geqslant \frac{\left(\sum_{i=1}^{n} a_i\right)^{m+1}}{\left(\sum_{i=1}^{n} b_i\right)^{m}}.$$

□

Chapter 11

Minkowski Inequality

11.1 Statement

Theorem 11.1.1. *If $a_k \geqslant 0$, $b_k \geqslant 0, k = 1, 2, \cdots, n$, and $p \geq 1$, then*

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}, \tag{11.1}$$

with equality if and only if $a_k = \lambda b_k$.

11.2 Proof of the inequality

Proof.

$$\sum_{k=1}^{n} (a_k + b_k)^p = \sum_{k=1}^{n} a_k (a_k + b_k)^{p-1} + \sum_{k=1}^{n} b_k (a_k + b_k)^{p-1}.$$

According to Hölder's inequality, since $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, we have

$$\sum_{k=1}^{n} a_k (a_k + b_k)^{p-1} \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (a_k + b_k)^{q(p-1)}\right)^{\frac{1}{q}}, \tag{a}$$

$$\sum_{k=1}^{n} b_k (a_k + b_k)^{p-1} \leqslant \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (a_k + b_k)^{q(p-1)}\right)^{\frac{1}{q}}. \tag{b}$$

Add (a) to (b) and we have

$$\sum_{k=1}^{n} (a_k + b_k)^p \leqslant \left(\left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}\right) \left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{q}}.$$

Divide the inequality by $\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{\frac{1}{q}}$, thus we obtain (11.1). □

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