A "Brunnermeier-Sannikov" Model with Jumps

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1 Introduction

2 Model

2.1 Set Up

Time is infinite and continuous. The economy is populated by two unit masses of agents, "households" and "experts". I will denote households and experts, respectively, by a subscript of $j \in \{h, e\}$.

2.1.1 Technology and Financial Markets

There is a single factor of production called "capital". If an agent i holds k_t units of capital, then the stock of capital evolves according to the law of motion[1]

$$\frac{dk_{i,t}}{k_{i,t}} = (\Phi(\iota_{i,t}) - \delta)\, dt + \sigma\, dW_t - \kappa_{i,t}\, dJ_t, \tag{1}$$

[1] I omit t^- subscripts in the statement of the law of motion to reduce notation. The more formal statement of the law of motion is

$$\frac{dk_t}{k_{t^-}} = (\Phi(\iota_{i,t^-}) - \delta)\, dt + \sigma\, dW_t - \kappa_{i,t}\, dJ_t,$$

which makes it clear that the process for k_t is cadlag.

where $\Phi(\cdot)$ is a standard investment technology with rate of internal investment $\iota_{i,t}$; δ is the rate of depreciation; σ is the volatility of the exogenous standard Brownian motion W_t; and $\kappa_{i,t}$ is an idiosyncratic Bernoulli random variable equaling $\tilde{\kappa}$ with probability θ and 0 with probability $1-\theta$, conditional on a realization of the Poisson process J_t, which has intensity λ. Conditional on $dJ_t = 0$, the growth rate of capital is controlled by internal investment subject to Gaussian shocks. When $dJ_t = 1$, however, an individual agent has a probability θ of exposure to the crisis shock. Agents exposed to the shock lose a fraction $\tilde{\kappa}$ of their capital stock. Absent redistributive insurance schemes, I assume $\tilde{\kappa} < 1$ so that no agent is ever wiped out by the crisis shock, or else they would obtain infinitely negative utility.

A final consumption good is produced from capital linearly, but the productivity depends on the agent using the capital. If an expert uses k_t units of capital to produce consumption, then they produce output at the rate $a_e k_t$. In contrast, a household produces output at the rate $a_h k_t$, where $a_e \geq a_h > 0$.

Capital is traded in a perfectly competitive market at a price q_t. Because capital is the only real asset in the economy, I will refer to q_t interchangeably as the asset price. I conjecture that the price of capital evolves endogenously according to

$$\frac{dq_t}{q_t} = \mu_{q,t}\, dt + \sigma_{q,t}\, dW_t - \kappa_{q,t}\, dJ_t. \tag{2}$$

There is also a market for risk-free debt in zero net supply. The risk-free interest rate is denoted by $dr_{f,t}$.[2]

The rate of return on capital for agent i of type j is the sum of the dividend yield

[2]It is assumed that the risk-free rate process has paths of finite variation (or absolutely continuous paths).

and capital gains, hence

$$dr_{kij,t} = \left(\frac{a_j - \iota_{j,t}}{q_t} + \mu_{q,t} + \Phi(\iota_{j,t}) - \delta + \sigma\sigma_{q,t}\right) dt \\ + (\sigma + \sigma_{q,t})\, dW_t - (\kappa_{i,t} + \kappa_{q,t} - \kappa_{i,t}\kappa_{q,t})\, dJ_t. \tag{3}$$

After aggregating across agents of type j, the rate of return on capital becomes

$$dr_{kj,t} = \left(\frac{a_j - \iota_{j,t}}{q_t} + \mu_{q,t} + \Phi(\iota_{j,t}) - \delta + \sigma\sigma_{q,t}\right) dt \\ + (\sigma + \sigma_{q,t})\, dW_t - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})\, dJ_t, \tag{4}$$

where $\kappa = \theta\tilde{\kappa}$ is the expected destruction of capital.

2.1.2 Preferences

Households and experts both have recursive preferences. For an agent of type $j \in \{h, e\}$, their objective function is their lifetime expected utility

$$V_{j,0} = \mathbb{E}_0\left[\int_0^\infty f(c_{j,t}, V_{j,t})\, dt\right], \tag{5}$$

where

$$f(c_j, V_j) = \left(\frac{1-\gamma_j}{1 - 1/\psi_j}\right) V_j \left[\left(\frac{c_j}{((1-\gamma_j)V_j)^{1/(1-\gamma_j)}}\right)^{1-1/\psi_j} - (\rho_j + \nu_j)\right]. \tag{6}$$

when $\psi_j \neq 1$ and

$$f(c_j, V_j) = (\rho_j + \nu_j)(1-\gamma_j)V_j \left(\log(c_j) - \frac{1}{1-\gamma_j}\log((1-\gamma_j)V_j)\right) \tag{7}$$

when $\psi_j = 1$.

For generality of the model, we allow experts' and households' preferences to be generically different. Agents may differ in their risk aversion γ_j, elasticity of intertemporal substituion ψ_j, time preference (or discount) rate ρ_j, and death rate ν_j.[3]

2.1.3 Portfolio Choice

The portfolio choice problem for both agents are virtually identical, so I only state the problem for experts. To simplify the statement, I present the representative expert's problem directly:

$$V_{e,0} = \max_{C_{e,t},\iota_{e,t},\varphi_{e,t}} \mathbb{E}_0 \left[\int_0^\infty f(C_{j,t}, V_{j,t})\, dt \right],$$

subject to the law of motion for net worth

$$dN_{e,t} = N_{e,t}(dr_{f,t} + \varphi_{e,t}(dr_{ke,t} - dr_{f,t})) - C_{e,t}. \tag{8}$$

The expert's controls are its consumption rate $C_{e,t}$, its rate of internal investment $\iota_{e,t}$, and the share of its net worth invested in the risky asset $\varphi_{e,t}$.

2.2 Solving Equilibrium

I look for a recursive stationary Markov equilibrium.

2.2.1 Portfolio Choice with Recursive Preferences

To solve experts' and households' portfolio choice problems, I conjecture that their value functions are given by

$$V_{j,t} = \frac{(\xi_{j,t} N_{j,t})^{1-\gamma_j}}{1-\gamma_j}, \tag{9}$$

[3] I assume that at every instance a fraction ν_j of agents of type j die and are immediately replaced by new agents, who are endowed with the wealth of dying agents. More elaborate redistribution schemes can implement wealth redistribution due to death dynamics.

where $\xi_{j,t}$ is the marginal value of additional net worth, is independent of $N_{j,t}$, and follows the jump diffusion

$$\frac{d\xi_{j,t}}{\xi_{j,t}} = \mu_{\xi j,t}\,dt + \sigma_{\xi j,t}\,dW_t + \kappa_{\xi j,t}\,dJ_t. \tag{10}$$

I may thus represent the value function as $V_{j,t} \equiv V_j(N_{j,t})$.

The Hamilton-Jacobi-Bellman equation is

$$0 = \max_{C_{e,t},\iota_{e,t},\varphi_{e,t}} f(C_{e,t}, V_e(N_{e,t})) + \mathbb{E}\left[d\left(\frac{(\xi_{e,t}N_{e,t})^{1-\gamma_e}}{1-\gamma_e}\right)\right] \tag{11}$$

To write the expectation, I apply Ito's lemma for jump diffusions.

Write the law of motion for aggregate expert net worth as

$$\frac{dN_{e,t}}{N_{e,t}} = \mu_{Ne,t}\,dt + \sigma_{Ne,t}\,dW_t - \kappa_{Ne,t}\,dJ_t, \tag{12}$$

where

$$\mu_{Ne,t} = dr_{f,t} - \frac{C_{e,t}}{N_{e,t}} + \varphi_{e,t}\left(\frac{a_e - \iota_{e,t}}{q_t} + \mu_{q,t} + \Phi(\iota_{e,t}) - \delta + \sigma\sigma_{q,t} - dr_{f,t}\right) \tag{13}$$

$$\sigma_{Ne,t} = \varphi_{e,t}(\sigma + \sigma_{q,t}) \tag{14}$$

$$\kappa_{Ne,t} = \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}). \tag{15}$$

By Ito's product rule,

$$\frac{d(\xi_{e,t}N_{e,t})}{\xi_{e,t}N_{e,t}} = (\mu_{\xi e,t} + \mu_{Ne,t} + \sigma_{\xi e,t}\sigma_{Ne,t})\,dt + (\sigma_{\xi e,t} + \sigma_{Ne,t})\,dW_t + (\kappa_{\xi e,t} - \kappa_{Ne,t} - \kappa_{\xi e,t}\kappa_{Ne,t})\,dJ_t.$$

By Ito's lemma, if x_t follows a jump diffusion, then

$$
\begin{aligned}
dx_t^{1-\gamma_e} &= \left((1-\gamma_e)x_t^{-\gamma_e}\mu_{x,t}x_t - \frac{(1-\gamma_e)\gamma_e}{2}x_t^{-\gamma_e-1}(\sigma_{x,t}x_t)^2\right)\,dt \\
&\quad + (1-\gamma_e)x_t^{-\gamma_e}\sigma_{x,t}x_t\,dW_t + (((1+\kappa_{x,t})x_t)^{1-\gamma_e} - x_t^{1-\gamma_e})\,dJ_t \\
\frac{1}{1-\gamma_e}\frac{dx_t^{1-\gamma_e}}{x_t^{1-\gamma_e}} &= (\mu_{x,t} - \frac{\gamma_e}{2}\sigma_{x,t}^2)\,dt + \sigma_{x,t}\,dW_t + \frac{(1+\kappa_{x,t})^{1-\gamma_e}-1}{1-\gamma_e}\,dJ_t,
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{1}{1-\gamma_e}\frac{d(\xi_{e,t}N_{e,t})^{1-\gamma_e}}{(\xi_{e,t}N_{e,t})^{1-\gamma_e}} &= \left(\mu_{\xi e,t} + \mu_{Ne,t} + \sigma_{\xi e,t}\sigma_{Ne,t} - \frac{\gamma_e}{2}(\sigma_{\xi e,t}+\sigma_{Ne,t})^2\right)\,dt \\
&\quad + (\sigma_{\xi e,t}+\sigma_{Ne,t})\,dW_t + \frac{(1+\kappa_{\xi e,t}-\kappa_{Ne,t}-\kappa_{\xi e,t}\kappa_{Ne,t})^{1-\gamma_e}-1}{1-\gamma_e}\,dJ_t
\end{aligned}
$$

The jump size further simplifies to

$$
(1+\kappa_{\xi e,t}-\kappa_{Ne,t}-\kappa_{\xi e,t}\kappa_{Ne,t})^{1-\gamma_e} = ((1+\kappa_{\xi e,t}) - \kappa_{Ne,t}(1+\kappa_{\xi e,t}))^{1-\gamma_e} = ((1+\kappa_{\xi e,t})(1-\kappa_{Ne,t}))^{1-\gamma_e}
$$

Additionally, the aggregator simplifies to

$$
f(C_{e,t}, V_e(N_{e,t})) = \begin{cases} \frac{(\xi_{e,t}N_{e,t})^{1-\gamma_e}}{1-1/\psi_e}\left[\left(\frac{C_{e,t}}{\xi_{e,t}N_{e,t}}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right] & \psi_e \neq 1 \\ (\rho_e+\nu_e)(\xi_{e,t}N_{e,t})^{1-\gamma_e}\log\left(\frac{C_{e,t}}{\xi_{e,t}N_{e,t}}\right) & \psi_e = 1 \end{cases}
$$

I plug these simplified expressions into (11) and divide by $(\xi_{e,t}N_{e,t})^{1-\gamma_e}$ to obtain

$$
\begin{aligned}
0 &= \frac{1}{1-1/\psi_e}\left[\left(\frac{C_{e,t}}{\xi_{e,t}N_{e,t}}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right]\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\log\left(\frac{C_{e,t}}{\xi_{e,t}N_{e,t}}\right)\mathbb{1}_{\psi_e=1} \\
&\quad + \mu_{\xi e,t} + \mu_{Ne,t} - \frac{\gamma_e}{2}(\sigma_{\xi e,t}^2 + \sigma_{Ne,t}^2) + (1-\gamma_e)\sigma_{\xi e,t}\sigma_{Ne,t} \\
&\quad + \frac{\lambda}{1-\gamma_e}(((1+\kappa_{\xi e,t})(1-\kappa_{Ne,t}))^{1-\gamma_e} - 1).
\end{aligned} \tag{16}
$$

The first-order conditions are

$$
\begin{aligned}
(C_{e,t}): \quad 0 &= \frac{C_{e,t}^{-1/\psi_e}}{(\xi_{e,t}N_{e,t})^{1-1/\psi_e}}\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\frac{1}{C_{e,t}}\mathbb{1}_{\psi_e=1} - \frac{1}{N_{e,t}} \\
C_{e,t}^{1-1/\psi_e} &= (\xi_{e,t}^{1-1/\psi_e}N_{e,t}^{-1/\psi_e})^{1-\psi_e} = \xi_{e,t}^{2-\psi_e-1/\psi_e}N_{e,t}^{1-1/\psi_e} \qquad \text{if } \psi_e \neq 1 \\
C_{e,t} &= (\rho_e+\nu_e)N_{e,t} \qquad \text{if } \psi_e = 1 \\
(\iota_{e,t}): \quad 0 &= -\frac{1}{q_t} + \Phi'(\iota_{e,t}) \\
(\varphi_{e,t}): \quad 0 &= \frac{a_e-\iota_{e,t}}{q_t} + \mu_{q,t} + \Phi(\iota_{e,t}) - \delta + \sigma\sigma_{q,t} - dr_{f,t} - \gamma_e\varphi_{e,t}(\sigma+\sigma_{q,t})^2 + (1-\gamma_e)\sigma_{\xi e,t}(\sigma+\sigma_{q,t}) \\
&\quad - \lambda(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\kappa_{Ne,t})^{-\gamma_e}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})
\end{aligned}
$$

Observe that

$$
\mathbb{E}[dr_{ke,t}] = \frac{a_e-\iota_{e,t}}{q_t} + \mu_{q,t} + \Phi(\iota_{e,t}) - \delta + \sigma\sigma_{q,t} - \lambda(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}),
$$

hence the first-order condition for leverage can be written as

$$
\begin{aligned}
\mathbb{E}[dr_{ke,t}] - dr_{f,t} = {} & \gamma_e\varphi_{e,t}(\sigma+\sigma_{q,t})^2 + (\gamma_e-1)\sigma_{\xi e,t}(\sigma+\sigma_{q,t}) \\
& + \lambda(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})((1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e}-1).
\end{aligned}
$$

To summarize, the FOCs are

$$
C_{e,t} = \xi_{e,t}^{1-\psi_e}N_{e,t}\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)N_{e,t}\mathbb{1}_{\psi_e=1} \tag{17}
$$

$$
\Phi'(\iota_{e,t}) = \frac{1}{q_t} \tag{18}
$$

$$
\begin{aligned}
&\frac{a_e-\iota_{e,t}}{q_t} + \mu_{q,t} + \Phi(\iota_{e,t}) - \delta + \sigma\sigma_{q,t} - dr_{f,t} \\
&= \gamma_e\varphi_{e,t}(\sigma+\sigma_{q,t})^2 + (\gamma_e-1)\sigma_{\xi e,t}(\sigma+\sigma_{q,t}) \\
&\quad + \lambda(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e}
\end{aligned} \tag{19}
$$

After plugging these quantities back into the HJB, I obtain

$$
\begin{aligned}
0 =& \frac{1}{1-1/\psi_e}\left[\left(\frac{\xi_{e,t}^{1-\psi_e}N_{e,t}}{\xi_{e,t}N_{e,t}}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right]\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\log\left(\frac{(\rho_e+\nu_e)N_{e,t}}{\xi_{e,t}N_{e,t}}\right)\mathbb{1}_{\psi_e=1} \\
&+ dr_{f,t} - \xi_{e,t}^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} - (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1} + \mu_{\xi e,t} - \frac{\gamma_e}{2}\sigma_{\xi e,t}^2 \\
&+ \varphi_{e,t}(\gamma_e\varphi_{e,t}(\sigma+\sigma_{q,t}) + (\gamma_e-1)\sigma_{\xi e,t}(\sigma+\sigma_{q,t})) \\
&+ \varphi_{e,t}\lambda(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e} \\
&- \frac{\gamma_e}{2}(\varphi_{e,t}(\sigma+\sigma_{q,t}))^2 + (1-\gamma_e)\varphi_{e,t}\sigma_{\xi e,t}(\sigma+\sigma_{q,t}) \\
&+ \frac{\lambda}{1-\gamma_e}(((1+\kappa_{\xi e,t})(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})))^{1-\gamma_e} - 1) \\
=& \left(\xi_{e,t}^{1-\psi_e}\frac{1/\psi_e}{1-1/\psi_e} - \frac{\rho_e+\nu_e}{1-1/\psi_e}\right)\mathbb{1}_{\psi\neq 1} + (\rho_e+\nu_e)\left(\log\left(\frac{\rho_e+\nu_e}{\xi_{e,t}}\right) - 1\right)\mathbb{1}_{\psi_e=1} \\
&+ dr_{f,t} - \frac{\lambda}{1-\gamma_e} + \mu_{\xi e,t} - \frac{\gamma_e}{2}\sigma_{\xi e,t}^2 + \frac{\gamma_e}{2}(\varphi_{e,t}(\sigma+\sigma_{q,t}))^2 \\
&+ \lambda(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e} \\
&\times\left(\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}) + \frac{1}{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))\right) \\
=& \left(\xi_{e,t}^{1-\psi_e}\frac{1/\psi_e}{1-1/\psi_e} - \frac{\rho_e+\nu_e}{1-1/\psi_e}\right)\mathbb{1}_{\psi\neq 1} + (\rho_e+\nu_e)\left(\log\left(\frac{\rho_e+\nu_e}{\xi_{e,t}}\right) - 1\right)\mathbb{1}_{\psi_e=1} \\
&+ dr_{f,t} - \frac{\lambda}{1-\gamma_e} + \mu_{\xi e,t} - \frac{\gamma_e}{2}\sigma_{\xi e,t}^2 + \frac{\gamma_e}{2}((\varphi_{e,t}(\sigma+\sigma_{q,t}))^2 \\
&+ \frac{\lambda}{1-\gamma_e}(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e}\left(1-\gamma_e\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})\right)
\end{aligned}
$$

2.2.2 Portfolio Choice with Log Utility

With log utility ($\psi_e = \gamma_e = 1$), the value function is

$$V_e(N_{e,t}) = \frac{\log(N_{e,t})}{\rho_e+\nu_e} + \xi_{e,t}, \tag{20}$$

where $\xi_{e,t}$ is still assumed to follow a jump diffusion. Then

$$\mathbb{E}[dV_e(N_{e,t})] = \frac{1}{\rho_e + \nu_e}\mathbb{E}[d\log(N_{e,t})] + \mathbb{E}[d\xi_{e,t}],$$

and by Ito's lemma,

$$\begin{aligned} d\log(N_{e,t}) &= \left(\frac{1}{N_{e,t}}\mu_{Ne,t}N_{e,t} - \frac{1}{2}\frac{1}{N_{e,t}^2}(\sigma_{Ne,t}N_{e,t})^2\right)\,dt \\ &\quad + \frac{1}{N_{e,t}}\sigma_{Ne,t}N_{e,t}\,dW_t + (\log((1-\kappa_{Ne,t})N_{e,t}) - \log(N_{e,t}))\,dJ_t \\ &= \left(\mu_{Ne,t} - \frac{1}{2}\sigma_{Ne,t}^2\right)\,dt + \sigma_{Ne,t}\,dW_t + (\log((1-\kappa_{Ne,t})N_{e,t}) - \log(N_{e,t}))\,dJ_t. \end{aligned}$$

The HJB in the log utility case becomes

$$\begin{aligned} \log(N_{e,t}) + (\rho_e + \nu_e)\xi_{e,t} = \log(C_{e,t}) &+ \frac{1}{(\rho_e+\nu_e)}\left(\mu_{Ne,t} - \frac{1}{2}\sigma_{Ne,t}^2\right) \\ &+ \lambda\frac{\log((1-\kappa_{Ne,t})N_{e,t}) - \log(N_{e,t})}{\rho_e+\nu_e} + \xi_{e,t}(\mu_{\xi e,t} + \lambda\kappa_{\xi e,t}) \end{aligned}$$

and simplifies to

$$\begin{aligned} 0 = \max_{C_{e,t},\iota_{e,t},\varphi_{e,t}} &\log\left(\frac{C_{e,t}}{N_{e,t}}\right) + \xi_{e,t}(\mu_{\xi e,t} + \lambda\kappa_{\xi e,t} - (\rho_e+\nu_e)) \\ &+ \frac{1}{\rho_e+\nu_e}\left(\mu_{Ne,t} - \frac{1}{2}\sigma_{Ne,t}^2 + \lambda\log(1-\kappa_{Ne,t})\right). \end{aligned} \tag{21}$$

The first-order conditions are

$$\begin{aligned} (C_{e,t}): &\quad 0 = \frac{1}{C_{e,t}} + \frac{1}{(\rho_e+\nu_e)N_{e,t}} \\ (\iota_{e,t}): &\quad 0 = \Phi'(\iota_{e,t}) - \frac{1}{q_t} \\ (\varphi_{e,t}): &\quad 0 = \frac{a_e - \iota_{e,t}}{q_t} + \mu_{q,t} + \Phi(\iota_{e,t}) - \delta + \sigma\sigma_{q,t} - dr_{f,t} - \varphi_{e,t}(\sigma+\sigma_{q,t})^2 - \lambda\frac{\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}}{1 - \varphi_{e,t}(\kappa+\kappa_{q,t} - \kappa\kappa_{q,t})}, \end{aligned}$$

hence

$$
\begin{aligned}
C_{e,t} &= (\rho_e + \nu_e) N_{e,t} && (22)\\
\Phi'(\iota_{e,t}) &= \frac{1}{q_t} && (23)\\
&\frac{a_e - \iota_{e,t}}{q_t} + \mu_{q,t} + \Phi(\iota_{e,t}) - \delta + \sigma\sigma_{q,t} - dr_{f,t} && (24)\\
&\quad = \varphi_{e,t}(\sigma + \sigma_{q,t})^2 + \lambda \frac{\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}}{1 - \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}. && (25)
\end{aligned}
$$

After plugging the first-order conditions into the HJB, I obtain

$$
\begin{aligned}
0 &= \log(\rho_e + \nu_e) + \xi_{e,t}(\mu_{\xi e,t} + \lambda\kappa_{\xi e,t} - (\rho_e + \nu_e))\\
&\quad + \frac{1}{\rho_e + \nu_e}\left(dr_{f,t} + \varphi_{e,t}^2(\sigma + \sigma_{q,t})^2 + \lambda \frac{\kappa_{Ne,t}}{1 - \kappa_{Ne,t}} - (\rho_e + \nu_e) - \frac{1}{2}\varphi_{e,t}(\sigma + \sigma_{q,t})^2 + \lambda \log(1 - \kappa_{Ne,t}) \right)\\
0 &= (\rho_e + \nu_e)\log(\rho_e + \nu_e) + (\rho_e + \nu_e)\xi_{e,t}(\mu_{\xi e,t} + \lambda\kappa_{\xi e,t} - (\rho_e + \nu_e))\\
&\quad + dr_{f,t} - (\rho_e + \nu_e) + \frac{\varphi_{e,t}^2}{2}(\sigma + \sigma_{q,t})^2\\
&\quad + \lambda \left(\frac{\varphi_{e,t}}{1 - \varphi_{e,t}(\kappa + \kappa_{q,t} + \kappa\kappa_{q,t})} + \log(1 - \varphi_{e,t}(\kappa + \kappa_{q,t} + \kappa\kappa_{q,t})) \right).
\end{aligned}
$$

2.2.3 State Variables

The two state variables are $\eta_t \equiv N_{e,t}/(q_t K_t)$ and K_t. Because of the linear homogeneity in agents' decision rules, the economy evolves on a balanced growth path, so all equilibrium quantities can be solved as functions of η_t alone. The additional state K_t is required only to determine the actual levels of quantities, such as output.

Recursive Preferences The aggregate law of motion for experts' net worth is

$$\begin{aligned}\frac{dN_{e,t}}{N_{e,t}} &= dr_{f,t} + \left(\gamma_e \varphi_{e,t}^2(\sigma+\sigma_{q,t})^2 + (\gamma_e - 1)\varphi_{e,t}\sigma_{\xi e,t}(\sigma+\sigma_{q,t}) - \xi_{e,t}^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} - (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1}\right) dt \\ &\quad + (\lambda\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e})\, dt \\ &\quad + \varphi_{e,t}(\sigma+\sigma_{q,t})\, dW_t - \varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})\, dJ_t.\end{aligned}$$

By Ito's product rule,

$$\frac{d(q_tK_t)}{q_tK_t} = (\mu_{q,t} + \Phi(\iota_t) - \delta + \sigma\sigma_{q,t})\, dt + (\sigma+\sigma_{q,t})\, dW_t - (\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})\, dJ_t.$$

Note that I use $\Phi(\iota_t)$ for the growth term rather than $\varphi_{e,t}\Phi(\iota_{e,t})N_{e,t}/q_t + \varphi_{h,t}\Phi(\iota_{h,t})N_{h,t}/q_t$. Because experts' FOC for $\iota_{e,t}$ depends only on the market price of capital q_t, experts will invest the same quantity as households. To reduce notation, I denote the rate of internal investment by the common quantity ι_t.

By Ito's lemma, for a process of the form $dx_t/x_t = \mu_{x,t}\, dt + \sigma_{x,t}\, dW_t - \kappa_{x,t}\, dJ_t$, the reciprocal follows

$$\begin{aligned}d(1/x_t) &= -\frac{1}{x_t^2}\mu_{x,t}x_t\, dt + \frac{1}{2}\frac{2}{x_t^3}(\sigma_{x,t}x_t)^2\, dt - \frac{1}{x_t^2}\sigma_{x,t}x_t\, dW_t + \left(\frac{1}{(1-\kappa_{x,t})x_t} - \frac{1}{x_t}\right) dJ_t \\ &= \left(-\frac{1}{x_t}\mu_{x,t} + \frac{1}{x_t}\sigma_{x,t}^2\right) dt - \frac{1}{x_t}\sigma_{x,t}\, dW_t + \frac{1}{x_t}\left(\frac{1}{1-\kappa_{x,t}} - 1\right) dJ_t \\ \frac{d(1/x_t)}{1/x_t} &= \left(\sigma_{x,t}^2 - \mu_{x,t}\right)\, dt - \sigma_{x,t}\, dW_t + \frac{\kappa_{x,t}}{1-\kappa_{x,t}}\, dJ_t.\end{aligned}$$

Thus, the reciprocal of aggregate wealth evolves according to

$$\begin{aligned}\frac{d(1/(q_tK_t))}{1/(q_tK_t)} &= ((\sigma+\sigma_{q,t})^2 - (\mu_{q,t} + \Phi(\iota_t) - \delta + \sigma\sigma_{q,t}))\, dt \\ &\quad - (\sigma+\sigma_{q,t})\, dW_t + \frac{\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}}{1-(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})}\, dJ_t.\end{aligned}$$

By (19),

$$
\begin{aligned}
-(\mu_{q,t} + \Phi(\iota_t) - \delta + \sigma\sigma_{q,t}) = \frac{a_e - \iota_{e,t}}{q_t} &- dr_{f,t} - \gamma_e\varphi_{e,t}(\sigma + \sigma_{q,t}) - (\gamma_e - 1)\sigma_{\xi e,t}(\sigma + \sigma_{q,t}) \\
&- \lambda(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})(1 + \kappa_{\xi e,t})^{1-\gamma_e}(1 - \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}))^{-\gamma_e}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{d(1/(q_t K_t))}{1/(q_t K_t)} = &\left((\sigma + \sigma_{q,t})^2 + \frac{a_e - \iota_{e,t}}{q_t} - dr_{f,t} - \gamma_e\varphi_{e,t}(\sigma + \sigma_{q,t})^2 - (\gamma_e - 1)\sigma_{\xi e,t}(\sigma + \sigma_{q,t})\right) dt \\
&- \lambda(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})(1 + \kappa_{\xi e,t})^{1-\gamma_e}(1 - \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}))^{-\gamma_e}\, dt \\
&- (\sigma + \sigma_{q,t})\, dW_t + \frac{\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}}{1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}\, dJ_t.
\end{aligned}
$$

By Ito's product rule, η_t can be expressed as a jump diffusion of the form

$$
\frac{d\eta_t}{\eta_t} = \mu_{\eta,t}\, dt + \sigma_{\eta,t}\, dW_t - \kappa_{\eta,t}\, dJ_t.
$$

First, I consider the volatility and jump size terms.

$$
\begin{aligned}
\sigma_{\eta,t} &= (\varphi_{e,t} - 1)(\sigma + \sigma_{q,t}) \\
-\kappa_{\eta,t} &= -\varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}) + \frac{\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}}{1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})} - \varphi_{e,t}\frac{(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})^2}{1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})} \\
&= \frac{-\varphi_{e,t}(1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}))(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}) + (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}) - \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})^2}{1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})} \\
&= \frac{(1 - \varphi_{e,t})(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}{1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})} \\
&= -\frac{(\varphi_{e,t} - 1)(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}{1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}.
\end{aligned}
$$

The drift term becomes

$$
\begin{aligned}
\mu_{\eta,t} &= dr_{f,t} + \Big(\gamma_e \varphi_{e,t}^2(\sigma+\sigma_{q,t})^2 + (\gamma_e-1)\varphi_{e,t}\sigma_{\xi e,t}(\sigma+\sigma_{q,t}) - \xi_{e,t}^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} - (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1}\Big) \\
&\quad + (\lambda\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e}) \\
&\quad + \left((\sigma+\sigma_{q,t})^2 + \frac{a_e-\iota_{e,t}}{q_t} - dr_{f,t} - \gamma_e\varphi_{e,t}(\sigma+\sigma_{q,t})^2 - (\gamma_e-1)\sigma_{\xi e,t}(\sigma+\sigma_{q,t})\right) \\
&\quad - (\lambda(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e} + \varphi_{e,t}(\sigma+\sigma_{q,t})^2) \\
&= \gamma_e\varphi_{e,t}(\varphi_{e,t}-1)(\sigma+\sigma_{q,t})^2 + (\gamma_e-1)(\varphi_{e,t}-1)\sigma_{\xi e,t}(\sigma+\sigma_{q,t}) - (\varphi_{e,t}-1)(\sigma+\sigma_{q,t})^2 \\
&\quad + \frac{a_e-\iota_{e,t}}{q_t} - \xi_{e,t}^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} - (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1} \\
&\quad + \lambda(\varphi_{e,t}-1)(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e} \\
&= \frac{a_e-\iota_{e,t}}{q_t} - \xi_{e,t}^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} - (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1} \\
&\quad + (\varphi_{e,t}-1)((\gamma_e\varphi_{e,t}-1)(\sigma+\sigma_{q,t})^2 + (\gamma_e-1)\sigma_{\xi e,t}(\sigma+\sigma_{q,t})) \\
&\quad + \lambda(\varphi_{e,t}-1)(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e}
\end{aligned}
$$

To summarize, the law of motion for the state variable η_t is

$$
\frac{d\eta_t}{\eta_t} = \mu_{\eta,t}\,dt + \sigma_{\eta,t}\,dW_t - \kappa_{\eta,t}\,dJ_t, \tag{26}
$$

where

$$
\begin{aligned}
\mu_{\eta,t} = \frac{a_e-\iota_{e,t}}{q_t} &- \xi_{e,t}^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} - (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1} \\
&+ (\varphi_{e,t}-1)((\gamma_e\varphi_{e,t}-1)(\sigma+\sigma_{q,t})^2 + (\gamma_e-1)\sigma_{\xi e,t}(\sigma+\sigma_{q,t})) \\
&+ \lambda(\varphi_{e,t}-1)(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})(1+\kappa_{\xi e,t})^{1-\gamma_e}(1-\varphi_{e,t}(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_e},
\end{aligned} \tag{27}
$$

$$
\sigma_{\eta,t} = (\varphi_{e,t}-1)(\sigma+\sigma_{q,t}), \tag{28}
$$

$$
\kappa_{\eta,t} = (\varphi_{e,t}-1)\frac{(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})}{1-(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})}. \tag{29}
$$

Log Utility In the case of log utility, the processes are simpler. The law of motion for $N_{e,t}$ becomes

$$\frac{dN_{e,t}}{N_{e,t}} = dr_{f,t} + \left(\varphi_{e,t}^2(\sigma + \sigma_{q,t})^2 - (\rho_e + \nu_e) + \lambda \frac{\varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}{1 - \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}\right) dt$$
$$+ \varphi_{e,t}(\sigma + \sigma_{q,t})\, dW_t - \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})\, dJ_t.$$

The law of motion for the recpirocal of aggregate wealth becomes

$$\frac{d(1/(q_t K_t))}{1/(q_t K_t)} = \left((\sigma + \sigma_{q,t})^2 + \frac{a_e - \iota_{e,t}}{q_t} - dr_{f,t} - \varphi_{e,t}(\sigma + \sigma_{q,t})^2 - \lambda \frac{\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}}{1 - \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}\right) dt$$
$$- (\sigma + \sigma_{q,t})\, dW_t + \frac{\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}}{1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}\, dJ_t.$$

By Ito's product rule, η_t follows a jump diffusion with the same form as (26), but where

$$\mu_{\eta,t} = \frac{a_e - \iota_{e,t}}{q_t} - (\rho_e + \nu_e) + (\varphi_{e,t} - 1)^2(\sigma + \sigma_{q,t})^2 + \lambda \frac{(\varphi_{e,t} - 1)(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}{1 - \varphi_{e,t}(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}, \tag{30}$$

$$\sigma_{\eta,t} = (\varphi_{e,t} - 1)(\sigma + \sigma_{q,t}), \tag{31}$$

$$\kappa_{\eta,t} = (\varphi_{e,t} - 1)\frac{(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}{1 - (\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})}. \tag{32}$$

3 Numerical Algorithm

For this section, I suppress all time subscripts unless explicitly required.

3.1 Equilibrium Conditions

Equilibrium is characterized by a system of functional equations and algebraic constraints. In the case of no jumps, these equations reduce to a system of differential algebraic equations.

Algebraic Equations Market-clearing for consumption and capital are two equilibrium conditions that can be expressed as algebraic constraints. To write the market-clearing condition for consumption, I need to specify a functional form for $\Phi(\cdot)$. Following the literature, I use the form

$$\Phi(\iota) = \frac{\chi_1}{\chi_2}\log(\chi_2\iota + 1), \tag{33}$$

This functional form yields the first-order condition

$$\frac{\chi_1}{\chi_2\iota + 1} = \frac{1}{q},$$

hence internal investment satisfies

$$\iota(q) = \frac{\chi_1 q - 1}{\chi_2}, \tag{34}$$

$$\Phi(q) = \frac{\chi_1}{\chi_2}\log(\chi_1 q). \tag{35}$$

Using (33), the market-clearing condition for consumption is

$$\begin{aligned}
&N_e(\xi_e^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1}) + N_h(\xi_h^{1-\psi_h}\mathbb{1}_{\psi_h\neq 1} + (\rho_h+\nu_h)\mathbb{1}_{\psi_h=1}) \\
&= a_e\varphi_e\frac{N_e}{q} + a_h\varphi_h\frac{N_h}{q} - \iota K \\
&q(\eta(\xi_e^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1}) + (1-\eta)(\xi_h^{1-\psi_h}\mathbb{1}_{\psi_h\neq 1} + (\rho_h+\nu_h)\mathbb{1}_{\psi_h=1})) \\
&= a_e\varphi_e\eta + a_h\varphi_h(1-\eta) - \frac{\chi_1}{\chi_2}q + \frac{1}{\chi_2}.
\end{aligned}$$

This equation re-arranges to

$$q = \begin{cases} \frac{\chi_2(a_e\varphi_e\eta + a_h\varphi_h(1-\eta))+1}{\chi_2((\rho_e+\nu_e)\eta + (\rho_h+\nu_h)(1-\eta))+\chi_1} & \text{if } \psi = 1, \\ \frac{\chi_2(a_e\varphi_e\eta + a_h\varphi_h(1-\eta))+1}{\chi_2(\xi_e^{1-\psi_e}\eta + \xi_h^{1-\psi_h}(1-\eta))+\chi_1} & \text{if } \psi \neq 1. \end{cases} \tag{36}$$

The market-clearing condition for capital is

$$\varphi_e \frac{N_e}{q} + \varphi_h \frac{N_h}{q} = K,$$

so dividing by K yields

$$\varphi_e \eta + \varphi_h (1-\eta) = 1.$$

Rearranging yields

$$\varphi_h = \frac{1-\varphi_e \eta}{1-\eta}. \tag{37}$$

Functional Equations The three functional equations are an asset pricing condition and the HJBs for experts and households. To write these conditions, I first verify my conjecture that q, ξ_e, and ξ_h are jump diffusions. I accomplish this by conjecturing these objects are functions of η and applying Ito's lemma.

$$\begin{aligned}
\frac{dq(\eta)}{q(\eta)} &= \left(\frac{q'(\eta)}{q(\eta)}\mu_\eta \eta + \frac{1}{2}\frac{q''(\eta)}{q(\eta)}(\sigma_\eta \eta)^2\right) dt + \frac{q'(\eta)}{q(\eta)}\sigma_\eta \eta \, dW_t + \left(\frac{q((1-\kappa_\eta)\eta)}{q(\eta)} - 1\right) dJ_t, \\
\frac{d\xi_e(\eta)}{\xi_e(\eta)} &= \left(\frac{\xi_e'(\eta)}{\xi_e(\eta)}\mu_\eta \eta + \frac{1}{2}\frac{\xi_e''(\eta)}{\xi_e(\eta)}(\sigma_\eta \eta)^2\right) dt + \frac{\xi_e'(\eta)}{\xi_e(\eta)}\sigma_\eta \eta \, dW_t + \left(\frac{\xi_e((1-\kappa_\eta)\eta)}{\xi_e(\eta)} - 1\right) dJ_t, \\
\frac{d\xi_h(\eta)}{\xi_h(\eta)} &= \left(\frac{\xi_h'(\eta)}{\xi_h(\eta)}\mu_\eta \eta + \frac{1}{2}\frac{\xi_h''(\eta)}{\xi_h(\eta)}(\sigma_\eta \eta)^2\right) dt + \frac{\xi_h'(\eta)}{\xi_h(\eta)}\sigma_\eta \eta \, dW_t + \left(\frac{\xi_h((1-\kappa_\eta)\eta)}{\xi_h(\eta)} - 1\right) dJ_t.
\end{aligned}$$

The unknown σ_q satisfies the fixed point

$$\sigma_q = \frac{q'}{q}\eta(\varphi_e - 1)(\sigma + \sigma_q),$$

which simplifies to

$$\sigma_q = \frac{q'\eta(\varphi_e - 1)\sigma}{q - q'\eta(\varphi_e - 1)}, \tag{38}$$

and the unknown κ_q satisfies the fixed point

$$\kappa_q = \frac{q\left(\left(\eta - \eta(\varphi_e - 1)\frac{\kappa+\kappa_q-\kappa\kappa_q}{1-(\kappa+\kappa_q-\kappa\kappa_q)}\right)\right)}{q(\eta)} - 1. \tag{39}$$

The terms $\sigma_{\xi e}$, $\sigma_{\xi h}$, $\kappa_{\xi e}$, and $\kappa_{\xi h}$ are given by

$$\sigma_{\xi e} = \frac{\xi_e'}{\xi_e}\sigma_\eta \eta \tag{40}$$

$$\sigma_{\xi h} = \frac{\xi_h'}{\xi_h}\sigma_\eta \eta \tag{41}$$

$$\kappa_{\xi e} = \frac{\xi_e\left(\eta - \eta(\varphi_e - 1)\frac{\kappa+\kappa_q-\kappa\kappa_q}{1-(\kappa+\kappa_q-\kappa\kappa_q)}\right)}{\xi_e(\eta)} - 1, \tag{42}$$

$$\kappa_{\xi h} = \frac{\xi_h\left(\eta - \eta(\varphi_e - 1)\frac{\kappa+\kappa_q-\kappa\kappa_q}{1-(\kappa+\kappa_q-\kappa\kappa_q)}\right)}{\xi_h(\eta)} - 1. \tag{43}$$

The equilibrium asset pricing condition is obtained by differencing experts' and households' FOCs for leverage, which yields

$$\begin{aligned} \frac{a_e - a_h}{q} \geq\ & (\gamma_e\varphi_e - \gamma_h\varphi_h)(\sigma + \sigma_q)^2 + ((\gamma_e - 1)\sigma_{\xi e} - (\gamma_h - 1)\sigma_{\xi h})(\sigma + \sigma_q) \\ & + \lambda(\kappa + \kappa_q - \kappa\kappa_q)(1 + \kappa_{\xi e})^{1-\gamma_e}(1 - \varphi_e(\kappa + \kappa_q - \kappa\kappa_q)) \\ & - \lambda(\kappa + \kappa_q - \kappa\kappa_q)(1 + \kappa_{\xi h})^{1-\gamma_h}(1 - \varphi_h(\kappa + \kappa_q - \kappa\kappa_q)) \end{aligned} \tag{44}$$

The inequality binds when $\varphi_h > 0$ is slack when $\varphi_e = 1$.

Experts' HJB yields the equilibrium condition

$$\begin{aligned} 0 =\ & \left(\xi_e^{1-\psi_e}\frac{1/\psi_e}{1 - 1/\psi_e} - \frac{\rho_e + \nu_e}{1 - 1/\psi_e}\right)\mathbb{1}_{\psi\neq 1} + (\rho_e + \nu_e)\left(\log\left(\frac{\rho_e + \nu_e}{\xi_e}\right) - 1\right)\mathbb{1}_{\psi_e = 1} \\ & + dr_{f,t} - \frac{\lambda}{1-\gamma_e} + \mu_{\xi e} - \frac{\gamma_e}{2}\sigma_{\xi e}^2 + \frac{\gamma_e}{2}((\varphi_e(\sigma + \sigma_{q,t}))^2 \\ & + \frac{\lambda}{1-\gamma_e}(1 + \kappa_{\xi e})^{1-\gamma_e}(1 - \varphi_e(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t}))^{-\gamma_e}\left(1 - \gamma_e\varphi_e(\kappa + \kappa_{q,t} - \kappa\kappa_{q,t})\right). \end{aligned} \tag{45}$$

From experts' FOC for leverage,

$$
\begin{aligned}
dr_f = {} & \frac{a_e - \iota_e}{q} + \mu_q + \Phi(\iota_e) - \delta + \sigma\sigma_q \\
& - \gamma_e\varphi_e(\sigma+\sigma_q)^2 - (\gamma_e - 1)\sigma_{\xi e}(\sigma+\sigma_q) \\
& - \lambda(\kappa+\kappa_q-\kappa\kappa_q)(1+\kappa_{\xi e})^{1-\gamma_e}(1-\varphi_e(\kappa+\kappa_q-\kappa\kappa_q))^{-\gamma_e}.
\end{aligned}
\tag{46}
$$

The remaining quantities in experts' HJB are known, given (46). Households' HJB is similarly given by

$$
\begin{aligned}
0 = {} & \left(\xi_h^{1-\psi_h}\frac{1/\psi_h}{1-1/\psi_h} - \frac{\rho_h+\nu_h}{1-1/\psi_h}\right)\mathbb{1}_{\psi\neq 1} + (\rho_h+\nu_h)\left(\log\left(\frac{\rho_h+\nu_h}{\xi_h}\right)-1\right)\mathbb{1}_{\psi_h=1} \\
& + dr_f - \frac{\lambda}{1-\gamma_h} + \mu_{\xi h} - \frac{\gamma_h}{2}\sigma_{\xi h}^2 + \frac{\gamma_h}{2}((\varphi_h(\sigma+\sigma_{q,t}))^2 \\
& + \frac{\lambda}{1-\gamma_h}(1+\kappa_{\xi h})^{1-\gamma_h}(1-\varphi_h(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t}))^{-\gamma_h}\left(1-\gamma_h\varphi_h(\kappa+\kappa_{q,t}-\kappa\kappa_{q,t})\right).
\end{aligned}
\tag{47}
$$

As a check, during computation, I can also verify that these HJBs agree with the HJBs when FOCs for leverage and consumption are not substituted. For experts, the HJB is

$$
\begin{aligned}
0 = {} & \left[\frac{\left(\frac{1}{\xi_e}\frac{C_e}{N_e}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)}{1-1/\psi_e} - \frac{C_e}{N_e}\right]\mathbb{1}_{\psi_e\neq 1} + \left[(\rho_e+\nu_e)\log\left(\frac{1}{\xi_e}\frac{C_e}{N_e}\right) - \frac{C_e}{N_e}\right]\mathbb{1}_{\psi_e=1} \\
& + \mu_{\xi e,t} + dr_f + \varphi_e\left(\frac{a_e-\iota_e}{q} + \mu_q + \Phi(\iota_e) - \delta + \sigma\sigma_q - dr_f\right) \\
& - \frac{\gamma_e}{2}(\sigma_{\xi e}^2 + (\varphi_e(\sigma+\sigma_q))^2) + (1-\gamma_e)\sigma_{\xi e}\varphi_e(\sigma+\sigma_q) \\
& - \frac{\lambda}{1-\gamma_e}(((1+\kappa_{\xi e})(1-\varphi_e(\kappa+\kappa_q-\kappa\kappa_q)))^{1-\gamma_e} - 1),
\end{aligned}
\tag{48}
$$

where the consumption FOC (17) pins down the consumption to wealth ratio C_e/N_e, and the leverage FOC (19) pins down the expected excess returns from capital, conditional on no jumps. Similarly, households have the HJB

$$
\begin{aligned}
0 = {} & \left[\frac{\left(\frac{1}{\xi_h}\frac{C_h}{N_h}\right)^{1-1/\psi_h} - (\rho_h + \nu_h)}{1 - 1/\psi_h} - \frac{C_h}{N_h}\right] \mathbb{1}_{\psi_h \neq 1} + \left[(\rho_h + \nu_h)\log\left(\frac{1}{\xi_h}\frac{C_h}{N_h}\right) - \frac{C_h}{N_h}\right] \mathbb{1}_{\psi_h = 1} \\
& + \mu_{\xi h,t} + dr_f + \varphi_h\left(\frac{a_h - \iota_h}{q} + \mu_q + \Phi(\iota_h) - \delta + \sigma\sigma_q - dr_f\right) \\
& - \frac{\gamma_h}{2}(\sigma_{\xi e}^2 + (\varphi_h(\sigma + \sigma_q))^2) + (1 - \gamma_h)\sigma_{\xi e}\varphi_h(\sigma + \sigma_q) \\
& - \frac{\lambda}{1 - \gamma_h}(((1 + \kappa_{\xi e})(1 - \varphi_h(\kappa + \kappa_q - \kappa\kappa_q)))^{1-\gamma_h} - 1),
\end{aligned} \tag{49}
$$

where households have a similar consumption FOC pinning down C_h/N_h and φ_h is zero whenever households' leverage FOC is slack.

3.2 Transformed Equilibrium Conditions

3.2.1 Transformation 1

The previous section shows that solving equilibrium reduces to solving four unknowns q, φ_e, ξ_e, and ξ_h. The latter two quantities are required to characterize positive dynamics if recursive preferences are used, but with log utility, ξ_e and ξ_h are only used to calculate welfare. The four equilibrium conditions pinning these quantities down are (36), (44), (45), and (47). For computational reasons, it is useful to transform ξ_e and ξ_h when $\psi_e \neq 1$ and/or $\psi_h \neq 1$. Rather than solve directly for ξ_e and ξ_h, I solve for

$$v_e = \xi_e^{\psi_e - 1}\mathbb{1}_{\psi_e \neq 1} + \xi_e \mathbb{1}_{\psi_e = 1}, \tag{50}$$

$$v_h = \xi_h^{\psi_e - 1}\mathbb{1}_{\psi_h \neq 1} + \xi_h \mathbb{1}_{\psi_h = 1}, \tag{51}$$

so that $\xi_e^{1-\gamma_e} = (v_e^{1/(\psi_e-1)})^{1-\gamma_e} = v_e^{(1-\gamma_e)/(\psi_e-1)}$

$$V_e = \frac{v_e^{(1-\gamma_e)/(\psi_e-1)} N_e^{1-\gamma_e}}{1-\gamma_e} \tag{52}$$
$$V_h = \frac{v_h^{(1-\gamma_h)/(\psi_h-1)} N_h^{1-\gamma_h}}{1-\gamma_h}. \tag{53}$$

The derivatives of v_e are

$$\begin{aligned}
v_e' &= (\psi_e - 1)\xi_e^{\psi_e-2}\xi_e' \\
&= (\psi_e - 1)\xi_e^{\psi_e-1}\frac{\xi_e'}{\xi_e} \\
v_e'' &= (\psi_e - 1)(\psi_e - 2)\xi_e^{\psi_e-3}\xi_e' + (\psi_e - 1)\xi_e^{\psi>e-2}\xi_e'' \\
&= (\psi_e - 1)(\psi_e - 2)\xi_e^{\psi_e-2}\frac{\xi_e'}{\xi_e} + (\psi_e - 1)\xi_e^{\psi_e-1}\frac{\xi_e''}{\xi_e} \\
\frac{v_e''}{v_e} &= (\psi_e - 1)(\psi_e - 2)\frac{\xi_e'}{\xi_e^2} + (\psi_e - 1)\frac{\xi_e''}{\xi_e} \\
\frac{\xi_e''}{\xi_e} &= \frac{1}{\psi_e - 1}\frac{v_e''}{v_e} - (\psi_e - 2)\frac{\xi_e'}{\xi_e^2} \\
&= \frac{1}{\psi_e - 1}\frac{v_e''}{v_e} - \xi_e^{-1}\frac{(\psi_e - 2)}{(\psi_e - 1)}\frac{v_e'}{v_e},
\end{aligned}$$

hence

$$\frac{\xi_e'}{\xi_e} = \frac{1}{\psi_e - 1}\frac{v_e'}{v_e}, \tag{54}$$
$$\frac{\xi_e''}{\xi_e} = \frac{1}{\psi_e - 1}\frac{v_e''}{v_e} - v_e^{-1/(\psi_e-1)}\frac{\psi_e - 2}{\psi_e - 1}\frac{v_e'}{v_e}. \tag{55}$$

Alternatively, I can re-derive the HJB. The aggregator simplifies to

$$f(C_e, V_e(N_e)) = \frac{v_e^{(1-\gamma_e)/(\psi_e-1)} N_e^{1-\gamma_e}}{1 - 1/\psi_e}\left[\left(\frac{C_e}{v_e^{1/(\psi_e-1)} N_e}\right)^{1-1/\psi_e} - (\rho_e + \nu_e)\right].$$

By Ito's lemma, if v_e follows

$$\frac{dv_e}{v_e} = \mu_{ve}\, dt + \sigma_{ve}\, dW_t - \kappa_{ve}\, dJ_t,$$

then $v_e^{1/(\psi_e-1)}$ follows

$$\begin{aligned}
dv_e^{1/(\psi_e-1)} &= \frac{1}{\psi_e-1} v_e^{1/(\psi_e-1)-1}\mu_{ve} v_e\, dt + \frac{1}{\psi_e-1} v_e^{1/(\psi_e-1)-1}\sigma_{ve} v_e\, dW_t \\
&\quad + \frac{1}{2}\frac{1}{\psi_e-1}\left(\frac{1}{\psi_e-1}-1\right) v_e^{1/(\psi_e-1)-2}(\sigma_{ve}v_e)^2\, dt + \left(((1+\kappa_{ve})v_e)^{1/(\psi_e-1)} - v_e^{1/(\psi_e-1)}\right) dJ_t \\
\frac{dv_e^{1/(\psi_e-1)}}{v_e^{1/(\psi_e-1)}} &= \left(\frac{1}{\psi_e-1}\mu_{ve} + \frac{1}{2}\frac{1}{\psi_e-1}\left(\frac{1}{\psi_e-1}-1\right)\sigma_{ve}^2\right) dt + \frac{1}{\psi_e-1}\sigma_{ve}\, dW_t + \left((1+\kappa_{ve})^{1/(\psi_e-1)}-1\right) dJ_t \\
&= \left(\frac{1}{\psi_e-1}\mu_{ve} + \frac{1}{2}\frac{\psi_e}{(\psi_e-1)^2}\sigma_{ve}^2\right) dt + \frac{1}{\psi_e-1}\sigma_{ve}\, dW_t + \left((1+\kappa_{ve})^{1/(\psi_e-1)}-1\right) dJ_t.
\end{aligned}$$

By Ito's product rule,

$$\begin{aligned}
\frac{d(v_e^{1/(\psi_e-1)}N_e)}{v_e^{1/(\psi_e-1)}N_e} &= \left(\mu_{Ne} + \frac{1}{\psi_e-1}\mu_{ve} + \frac{1}{2}\frac{\psi_e}{(\psi_e-1)^2}\sigma_{ve}^2 + \frac{1}{\psi_e-1}\sigma_{ve}\sigma_{Ne}\right) dt + \left(\frac{1}{\psi_e-1}\sigma_{ve} + \sigma_{Ne}\right) dW_t \\
&\quad + \left(((1+\kappa_{ve})^{1/(\psi_e-1)}-1) - \kappa_{Ne} - ((1+\kappa_{ve})^{1/(\psi_e-1)}-1)\kappa_{Ne}\right) dJ_t.
\end{aligned}$$

Then experts' transformed value function follows

$$\begin{aligned}
&\frac{1}{1-\gamma_e}\frac{dv_e^{(1-\gamma_e)/(\psi_e-1)}N_e^{1-\gamma_e}}{v_e^{(1-\gamma_e)/(\psi_e-1)}N_e^{1-\gamma_e}} \\
&= \left(\mu_{Ne} + \frac{1}{\psi_e-1}\mu_{ve} + \frac{1}{2}\frac{\psi_e}{(\psi_e-1)^2}\sigma_{ve}^2 + \frac{1}{\psi_e-1}\sigma_{ve}\sigma_{Ne} - \frac{\gamma_e}{2}\left(\frac{1}{\psi_e-1}\sigma_{ve}+\sigma_{Ne}\right)^2\right) dt \\
&\quad + \left(\frac{1}{\psi_e-1}\sigma_{ve} + \sigma_{Ne}\right) dW_t \\
&\quad + \frac{\left(1 + ((1+\kappa_{ve})^{1/(\psi_e-1)}-1) - \kappa_{Ne} - ((1+\kappa_{ve})^{1/(\psi_e-1)}-1)\kappa_{Ne}\right)^{1-\gamma_e} - 1}{1-\gamma_e}\, dJ_t.
\end{aligned}$$

The HJB simplifies to

$$
\begin{aligned}
0 &= f(C_e, V_e(N_e)) + \mathbb{E}\left[d\left(\frac{v_e^{(1-\gamma_e)/(\psi_e-1)}N_e^{1-\gamma_e}}{1-\gamma_e}\right)\right] \\
0 &= \max_{\varphi_e,\iota_e,C_e} \frac{v_e^{(1-\gamma_e)/(\psi_e-1)}N_e^{1-\gamma_e}}{1-1/\psi_e}\left[\left(\frac{C_e}{v_e^{1/(\psi_e-1)}N_e}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right] \\
&\quad + \frac{v_e^{(1-\gamma_e)/(\psi_e-1)}N_e^{1-\gamma_e}}{1-\gamma_e}\mathbb{E}\left[\frac{dv_e^{(1-\gamma_e)/(\psi_e-1)}N_e^{1-\gamma_e}}{v_e^{(1-\gamma_e)/(\psi_e-1)}N_e^{1-\gamma_e}}\right] \\
&= \max_{\varphi_e,\iota_e,C_e} \frac{1}{1-1/\psi_e}\left[\left(\frac{C_e}{v_e^{1/(\psi_e-1)}N_e}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right] \\
&\quad + \mu_{Ne} + \frac{1}{\psi_e-1}\mu_{ve} + \frac{1}{2}\frac{\psi_e}{(\psi_e-1)^2}\sigma_{ve}^2 + \frac{1}{\psi_e-1}\sigma_{ve}\sigma_{Ne} - \frac{\gamma_e}{2}\left(\frac{1}{\psi_e-1}\sigma_{ve}+\sigma_{Ne}\right)^2 \\
&\quad + \frac{\lambda}{1-\gamma_e}((1+((1+\kappa_{ve})^{1/(\psi_e-1)}-1)-\kappa_{Ne}-((1+\kappa_{ve})^{1/(\psi_e-1)}-1)\kappa_{Ne})^{1-\gamma_e}-1) \\
&= \max_{\varphi_e,\iota_e,C_e} \frac{1}{1-1/\psi_e}\left[\left(\frac{C_e}{v_e^{1/(\psi_e-1)}N_e}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right] + \mu_{Ne} + \frac{1}{\psi_e-1}\mu_{ve} \\
&\quad + \frac{1}{2}\frac{\psi_e}{(\psi_e-1)^2}\sigma_{ve}^2 + \frac{1}{\psi_e-1}\sigma_{ve}\sigma_{Ne} - \frac{\gamma_e}{2}\left(\left(\frac{1}{\psi_e-1}\sigma_{ve}\right)^2 + \sigma_{Ne}^2\right) - \frac{\gamma_e}{\psi_e-1}\sigma_{ve}\sigma_{Ne} \\
&\quad + \frac{\lambda}{1-\gamma_e}((1+((1+\kappa_{ve})^{1/(\psi_e-1)}-1)-\kappa_{Ne}(1+((1+\kappa_{ve})^{1/(\psi_e-1)}-1)))^{1-\gamma_e}-1) \\
&= \max_{\varphi_e,\iota_e,C_e} \frac{1}{1-1/\psi_e}\left[\left(\frac{C_e}{v_e^{1/(\psi_e-1)}N_e}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right] + \mu_{Ne} + \frac{1}{\psi_e-1}\mu_{ve} \\
&\quad + \frac{1}{2}(\psi_e-\gamma_e)\left(\frac{\sigma_{ve}}{\psi_e-1}\right)^2 + \frac{1-\gamma_e}{\psi_e-1}\sigma_{ve}\sigma_{Ne} - \frac{\gamma_e}{2}\sigma_{Ne}^2 \\
&\quad + \frac{\lambda}{1-\gamma_e}(((1+((1+\kappa_{ve})^{1/(\psi_e-1)}-1))(1-\kappa_{Ne}))^{1-\gamma_e}-1) \\
&= \max_{\varphi_e,\iota_e,C_e} \psi_e\left[\left(\frac{C_e}{v_e^{1/(\psi_e-1)}N_e}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right] + (\psi_e-1)\mu_{Ne} + \mu_{ve} + \frac{1}{2}\frac{\psi_e-\gamma_e}{\psi_e-1}\sigma_{ve}^2 \\
&\quad + (1-\gamma_e)\sigma_{ve}\sigma_{Ne} - \frac{\gamma_e}{2}(\psi_e-1)\sigma_{Ne}^2 + \lambda\frac{\psi_e-1}{1-\gamma_e}(((1+((1+\kappa_{ve})^{1/(\psi_e-1)}-1))(1-\kappa_{Ne}))^{1-\gamma_e}-1)
\end{aligned}
$$

The FOC for internal investment remains the same, but the FOC for consumption and leverage differ. For C_e, I obtain

$$
\begin{aligned}
0 &= v_e^{(1-1/\psi_e)/(\psi_e-1)} N_e^{1/\psi_e - 1} C_e^{-1/\psi_e} - N_e^{-1} \\
\left(\frac{C_e}{N_e}\right)^{-1/\psi_e} &= v_e^{(\psi_e-1)/((\psi_e-1)/\psi_e)} \\
\left(\frac{C_e}{N_e}\right)^{-1/\psi_e} &= v_e^{1/\psi_e} \\
\frac{C_e}{N_e} &= v_e^{-1}.
\end{aligned}
$$

For φ_e, I obtain

$$
\begin{aligned}
0 = {} & (\psi_e - 1)\left(\frac{a_e - \iota_e}{q} + \mu_q + \Phi(\iota_e) - \delta + \sigma\sigma_q - dr_f\right) + (1-\gamma_e)\sigma_{ve}(\sigma + \sigma_q) - \gamma_e(\psi_e - 1)\varphi_e(\sigma + \sigma_q)^2 \\
& - \lambda(\psi_e - 1)(1 + ((1+\kappa_{ve})^{1/(\psi_e-1)} - 1))^{1-\gamma_e}(1 - \varphi_e(\kappa + \kappa_q - \kappa\kappa_q))^{-\gamma_e}(\kappa + \kappa_q - \kappa\kappa_q),
\end{aligned}
$$

which re-arranges to

$$
\begin{aligned}
& (\psi_e - 1)\left(\frac{a_e - \iota_e}{q} + \mu_q + \Phi(\iota_e) - \delta + \sigma\sigma_q - dr_f\right) \\
& = \gamma_e(\psi_e - 1)\varphi_e(\sigma + \sigma_q)^2 + (\gamma_e - 1)\varphi_e(\sigma + \sigma_q) \\
& \quad + \lambda(\psi_e - 1)(1 + ((1+\kappa_{ve})^{1/(\psi_e-1)} - 1))^{1-\gamma_e}(1 - \varphi_e(\kappa + \kappa_q - \kappa\kappa_q))^{-\gamma_e}(\kappa + \kappa_q - \kappa\kappa_q) \\
& \frac{a_e - \iota_e}{q} + \mu_q + \Phi(\iota_e) - \delta + \sigma\sigma_q - dr_f \\
& = \gamma_e\varphi_e(\sigma + \sigma_q)^2 + \frac{\gamma_e - 1}{\psi_e - 1}\sigma_{ve}(\sigma + \sigma_q) \\
& \quad + \lambda(1 + ((1+\kappa_{ve})^{1/(\psi_e-1)} - 1))^{1-\gamma_e}(1 - \varphi_e(\kappa + \kappa_q - \kappa\kappa_q))^{-\gamma_e}(\kappa + \kappa_q - \kappa\kappa_q).
\end{aligned}
$$

Then the equilibrium conditions become

$$q = \frac{\chi_2(a_e\varphi_e\eta + a_h\varphi_h(1-\eta)) + 1}{\chi_2(\eta/v_e + (1-\eta)/v_h) + \chi_1} \tag{56}$$

$$\begin{aligned} dr_f = {} & \frac{a_e - \iota}{q} + \mu_q + \Phi(\iota) - \delta + \sigma\sigma_q - \gamma_e\varphi_e(\sigma+\sigma_q)^2 - \frac{\gamma_e - 1}{\psi_e - 1}\sigma_{ve}(\sigma+\sigma_q) \\ & - \lambda(1 + ((1+\kappa_{ve})^{1/(\psi_e-1)} - 1))^{1-\gamma_e}\frac{\kappa + \kappa_q - \kappa\kappa_q}{(1-\varphi_e(\kappa+\kappa_q-\kappa\kappa_q))^{\gamma_e}} \end{aligned} \tag{57}$$

$$\begin{aligned} \frac{a_e - a_h}{q} \geq {} & (\gamma_e\varphi_e - \gamma_h\varphi_h)(\sigma+\sigma_q)^2 + \left(\frac{\gamma_e - 1}{\psi_e - 1}\sigma_{ve} - \frac{\gamma_h - 1}{\psi_h - 1}\sigma_{vh}\right)(\sigma+\sigma_q) \\ & + \lambda(\kappa+\kappa_q-\kappa\kappa_q)(1 + ((1+\kappa_{ve})^{1/(\psi_e-1)} - 1))^{1-\gamma_e}(1-\varphi_e(\kappa+\kappa_q-\kappa\kappa_q)) \\ & - \lambda(\kappa+\kappa_q-\kappa\kappa_q)(1 + ((1+\kappa_{vh})^{1/(\psi_h-1)} - 1))^{1-\gamma_h}(1-\varphi_h(\kappa+\kappa_q-\kappa\kappa_q)) \end{aligned} \tag{58}$$

$$\mu_{Ne} = (1-\varphi_e)dr_f + \varphi_e\left(\frac{a_e - \iota}{q} + \mu_q + \Phi(\iota) - \delta + \sigma\sigma_q\right) - \frac{C_e}{N_e} \tag{59}$$

$$\mu_{Nh} = (1-\varphi_h)dr_f + \varphi_h\left(\frac{a_h - \iota}{q} + \mu_q + \Phi(\iota) - \delta + \sigma\sigma_q\right) - \frac{C_h}{N_h} \tag{60}$$

$$\begin{aligned} 0 = {} & \psi_e\left[\left(\frac{C_e}{N_e}\frac{1}{v_e^{1/(\psi_e-1)}}\right)^{1-1/\psi_e} - (\rho_e + \nu_e)\right] + (\psi_e - 1)\mu_{Ne} \\ & + \mu_{ve} + \frac{1}{2}\frac{\psi_e - \gamma_e}{\psi_e - 1}\sigma_{ve}^2 + (1-\gamma_e)\sigma_{ve}\varphi_e(\sigma+\sigma_q) - \frac{\gamma_e}{2}(\psi_e - 1)(\varphi_e(\sigma+\sigma_q))^2 \\ & + \lambda\frac{\psi_e - 1}{1-\gamma_e}(((1 + ((1+\kappa_{ve})^{1/(\psi_e-1)} - 1))(1-\varphi_e(\kappa+\kappa_q-\kappa\kappa_q)))^{1-\gamma_e} - 1) \end{aligned} \tag{61}$$

$$\begin{aligned} 0 = {} & \psi_h\left[\left(\frac{C_h}{N_h}\frac{1}{v_h^{1/(\psi_h-1)}}\right)^{1-1/\psi_h} - (\rho_h + \nu_h)\right] + (\psi_h - 1)\mu_{Nh} \\ & + \mu_{vh} + \frac{1}{2}\frac{\psi_h - \gamma_h}{\psi_h - 1}\sigma_{vh}^2 + (1-\gamma_h)\sigma_{vh}\varphi_h(\sigma+\sigma_q) - \frac{\gamma_h}{2}(\psi_h - 1)(\varphi_h(\sigma+\sigma_q))^2 \\ & + \lambda\frac{\psi_h - 1}{1-\gamma_h}(((1 + ((1+\kappa_{vh})^{1/(\psi_h-1)} - 1))(1-\varphi_h(\kappa+\kappa_q-\kappa\kappa_q)))^{1-\gamma_h} - 1) \end{aligned} \tag{62}$$

3.2.2 Transformation 2

An alternative transformation, used by Brunnermeier and Sannikov (2016), is

$$v_e = (\xi_e\eta q)^{1-\gamma_e}, \tag{63}$$

which implies the value function

$$V_e = \frac{(\xi_e N_e)^{1-\gamma_e}}{1-\gamma_e} = \frac{(\xi_e \eta q K)^{1-\gamma_e}}{1-\gamma_e} = v_e \frac{K^{1-\gamma_e}}{1-\gamma_e}$$

and the identity

$$\xi_e = \frac{v_e^{1/(1-\gamma_e)}}{\eta q}.$$

By Ito's product rule,

$$\begin{aligned}\frac{d(v_{e,t}K_t^{1-\gamma_e})}{v_{e,t}K_t^{1-\gamma_e}} &= \left(\mu_{ve,t} + (1-\gamma_e)(\Phi(\iota_t)-\delta) - \frac{\gamma(1-\gamma)}{2}\sigma^2 + (1-\gamma)\sigma_{ve,t}\sigma\right) dt \\ &\quad + (\sigma_{ve,t} + (1-\gamma)\sigma)\, dW_t + \big(\kappa_{ve,t} + (1+\kappa_{ve,t})((1-\kappa)^{1-\gamma_e}-1)\big)\, dJ_t.\end{aligned}$$

The time subscripts have been temporarily added to be clear that differentiate time-varying terms from constant terms in the jump-diffusion.

By Ito's lemma, the HJB is

$$\begin{aligned}0 &= f(C_e, V_e) + \mathbb{E}[dV_e] \\ 0 &= f(C_e, V_e) + \frac{v_e K^{1-\gamma_e}}{1-\gamma_e}\mathbb{E}\left[\frac{dv_e K^{1-\gamma_e}}{v_e K^{1-\gamma_e}}\right] \\ &= f(C_e, V_e) + v_e\frac{K^{1-\gamma_e}}{1-\gamma_e}\left(\mu_{ve} + (1-\gamma_e)(\Phi(\iota)-\delta) - \frac{\gamma(1-\gamma_e)}{2}\sigma^2 + (1-\gamma)\sigma_{ve}\sigma\right) \\ &\quad + v_e\frac{K^{1-\gamma_e}}{1-\gamma_e}\lambda\left(\kappa_{ve,t} + (1+\kappa_{ve,t})((1-\kappa)^{1-\gamma_e}-1)\right).\end{aligned}$$

Dividing by the value function yields

$$
\begin{aligned}
0 = &\frac{1}{1-1/\psi_e}\left[\left(\frac{C_e}{\xi_e N_e}\right)^{1-1/\psi_e} - (\rho_e+\nu_e)\right]\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\log\left(\frac{C_e}{\xi_e N_e}\right)\mathbb{1}_{\psi=1} \\
&+ \frac{\mu_{ve}}{1-\gamma_e} + (\Phi(\iota)-\delta) - \frac{\gamma}{2}\sigma^2 + \sigma_{ve}\sigma + \frac{\lambda}{1-\gamma_e}\left(\kappa_{ve,t} + (1+\kappa_{ve,t})((1-\kappa)^{1-\gamma_e}-1)\right).
\end{aligned}
$$

The first-order condition for consumption is

$$
\frac{C_e}{N_e} = \xi_e^{1-\psi_e}\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\mathbb{1}_{\psi_e=1},
$$

hence the first line simplifies to

$$
\begin{aligned}
&\frac{1}{1-1/\psi_e}\left[\xi_e^{-\psi_e(1-1/\psi_e)} - (\rho_e+\nu_e)\right]\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)(\log(\rho_e+\nu_e) - \log(\xi_e))) \\
&= \frac{1}{1-1/\psi_e}(\xi_e^{1-\psi_e} - (\rho_e+\nu_e))\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\left(\log(\rho_e+\nu_e) - \log\left(\frac{v_e^{1/(1-\gamma_e)}}{\eta q}\right)\right) \\
&= \frac{1}{1-1/\psi_e}\left(\frac{v_e^{(1-\psi_e)/(1-\gamma_e)}}{(\eta q)^{1-\psi_e}} - (\rho_e+\nu_e)\right)\mathbb{1}_{\psi_e\neq 1} + (\rho_e+\nu_e)\left(\log(\rho_e+\nu_e) + \log(\eta) + \log(q) - \frac{\log(v_e)}{1-\gamma_e}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
0 = &\frac{1}{1-1/\psi_e}\left(\frac{v_e^{(1-\psi_e)/(1-\gamma_e)}}{(\eta q)^{1-\psi_e}} - (\rho_e+\nu_e)\right)\mathbb{1}_{\psi_e\neq 1} \\
&+ (\rho_e+\nu_e)\left(\log(\rho_e+\nu_e) + \log(\eta) + \log(q) - \frac{\log(v_e)}{1-\gamma_e}\right) \\
&+ \frac{\mu_{ve}}{1-\gamma_e} + (\Phi(\iota)-\delta) - \frac{\gamma}{2}\sigma^2 + \sigma_{ve}\sigma + \frac{\lambda}{1-\gamma_e}\left(\kappa_{ve,t} + (1+\kappa_{ve,t})((1-\kappa)^{1-\gamma_e}-1)\right).
\end{aligned}
$$

The asset pricing condition can be rewritten by applying Ito's lemma to re-write

$\sigma_{\xi e}$, $\sigma_{\xi h}$, $\kappa_{\xi e}$ and $\kappa_{\xi h}$. First, note that

$$
\begin{aligned}
d(1-\eta_t) &= -\mu_{\eta,t}\eta_t\,dt - \sigma_{\eta,t}\eta_t\,dW_t + ((1-(1-\kappa_{\eta,t})\eta_t) - (1-\eta_t))\,dJ_t \\
\frac{d(1-\eta_t)}{1-\eta_t} &= -\frac{\eta_t}{1-\eta_t}\mu_{\eta,t}\,dt - \frac{\eta_t}{1-\eta_t}\sigma_{\eta,t}\,dW_t + \frac{1}{1-\eta_t}(\eta_t - (1-\kappa_{\eta,t})\eta_t)\,dJ_t \\
&= -\frac{\eta_t}{1-\eta_t}\mu_{\eta,t}\,dt - \frac{\eta_t}{1-\eta_t}\sigma_{\eta,t}\,dW_t + \frac{\eta_t}{1-\eta_t}\kappa_{\eta,t}\,dJ_t
\end{aligned}
$$

By Ito's product rule,

$$
\begin{aligned}
\frac{d(\eta_t q_t)}{\eta_t q_t} &= \mu_{\eta q,t}\,dt + (\sigma_{\eta,t} + \sigma_{q,t})\,dW_t + (-\kappa_{\eta,t} - \kappa_{q,t} + \kappa_{\eta,t}\kappa_{q,t})\,dJ_t \\
&= \mu_{\eta q,t}\,dt + (\sigma_{\eta,t} + \sigma_{q,t})\,dW_t + (-\kappa_{q,t} - (1-\kappa_{q,t})\kappa_{\eta,t})\,dJ_t \\
\frac{d((1-\eta_t)q_t)}{(1-\eta_t)q_t} &= \mu_{(1-\eta)q,t}\,dt + \left(-\frac{\eta_t}{1-\eta_t}\sigma_{\eta,t} + \sigma_{q,t}\right)\,dW_t + \left(\frac{\eta_t}{1-\eta_t}\kappa_{\eta,t} - \kappa_{q,t} - \frac{\eta_t}{1-\eta_t}\kappa_{\eta,t}\kappa_{q,t}\right)\,dJ_t \\
&= \mu_{(1-\eta)q,t}\,dt + \left(-\frac{\eta_t}{1-\eta_t}\sigma_{\eta,t} + \sigma_{q,t}\right)\,dW_t + \left(-\kappa_{q,t} + (1-\kappa_{q,t})\frac{\eta_t}{1-\eta_t}\kappa_{\eta,t}\right)\,dJ_t.
\end{aligned}
$$

The drift is omitted because it is not needed for the remainder of this transformation. After applying Ito's product rule again, I acquire

$$
\begin{aligned}
\frac{d(\xi_{e,t}\eta_t q_t)}{\xi_{e,t}\eta_t q_t} &= \mu_{\xi_e \eta q,t}\,dt + (\sigma_{\xi e,t} + \sigma_{\eta,t} + \sigma_{q,t})\,dW_t \\
&\quad + (\kappa_{\xi e,t} + (-\kappa_{q,t} - (1-\kappa_{q,t})\kappa_{\eta,t}) + \kappa_{\xi e,t}(-\kappa_{q,t} - (1-\kappa_{q,t})\kappa_{\eta,t}))\,dJ_t \\
\frac{d(\xi_{e,t}\eta_t q_t)}{\xi_{e,t}\eta_t q_t} &= \mu_{\xi_e \eta q,t}\,dt + (\sigma_{\xi e,t} + \sigma_{\eta,t} + \sigma_{q,t})\,dW_t + (\kappa_{\xi e,t} - (1+\kappa_{\xi e,t})(\kappa_{q,t} + (1-\kappa_{q,t})\kappa_{\eta,t}))\,dJ_t \\
\frac{d(\xi_{h,t}(1-\eta_t)q_t)}{\xi_{h,t}(1-\eta_t)q_t} &= \mu_{\xi_h(1-\eta)q,t}\,dt + \left(\sigma_{\xi h,t} - \frac{\eta_t}{1-\eta_t}\sigma_{\eta,t} + \sigma_{q,t}\right)\,dW_t \\
&\quad + \left(\kappa_{\xi h,t} + (1+\kappa_{\xi h,t})\left(-\kappa_{q,t} + (1-\kappa_{q,t})\frac{\eta_t}{1-\eta_t}\kappa_{\eta,t}\right)\right)\,dJ_t.
\end{aligned}
$$

Finally, I apply Ito's lemma to $x_t^{1-\gamma_e}$, which yields the law of motion

$$\frac{dx_t^{1-\gamma_e}}{x_t^{1-\gamma_e}} = (1-\gamma_e)(\mu_{x,t} - \frac{\gamma_e}{2}\sigma_{x,t}^2)\, dt + (1-\gamma_e)\sigma_{x,t}\, dW_t + ((1+\kappa_{x,t})^{1-\gamma_e} - 1)\, dJ_t.$$

Substitution implies

$$\begin{aligned}
\sigma_{ve} &= (1-\gamma_e)(\sigma_{\xi e} + \sigma_\eta + \sigma_q) \\
\sigma_{vh} &= (1-\gamma_h)\left(\sigma_{\xi h} - \frac{\eta}{1-\eta}\sigma_\eta + \sigma_q\right) \\
\kappa_{ve} &= (1 + (\kappa_{\xi e} - (1+\kappa_{\xi e})(\kappa_q + (1-\kappa_q)\kappa_\eta)))^{1-\gamma_e} - 1 \\
\kappa_{vh} &= \left[1 + \left(\kappa_{\xi h} + (1+\kappa_{\xi h})(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta)\right)\right]^{1-\gamma_e} - 1
\end{aligned}$$

After re-arranging the volatilities, I obtain

$$\begin{aligned}
\sigma_{\xi e} &= \frac{\sigma_{ve}}{1-\gamma_e} - \sigma_\eta - \sigma_q \\
\sigma_{\xi h} &= \frac{\sigma_{vh}}{1-\gamma_h} + \frac{\eta}{1-\eta}\sigma_\eta - \sigma_q.
\end{aligned}$$

After some algebraic manipulations, the jump sizes become

$$\begin{aligned}
(1+\kappa_{ve})^{1/(1-\gamma_e)} - 1 &= \kappa_{\xi e} - (1+\kappa_{\xi e})(\kappa_q + (1-\kappa_q)\kappa_\eta) \\
(1+\kappa_{ve})^{1/(1-\gamma_e)} + \kappa_q + (1-\kappa_q)\kappa_\eta - 1 &= \kappa_{\xi e}(1 - (\kappa_q + (1-\kappa_q)\kappa_\eta)) \\
\kappa_{\xi e} &= \frac{(1+\kappa_{ve})^{1/(1-\gamma_e)} + \kappa_q + (1-\kappa_q)\kappa_\eta - 1}{1 - (\kappa_q + (1-\kappa_q)\kappa_\eta)} \\
(1+\kappa_{vh})^{1/(1-\gamma_h)} - \left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right) - 1 &= \kappa_{\xi h}\left(1 + \left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right)\right) \\
\kappa_{\xi h} &= \frac{(1+\kappa_{vh})^{1/(1-\gamma_h)} - \left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right) - 1}{1 + \left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right)}
\end{aligned}$$

It follows that

$$
\begin{aligned}
1+\kappa_{\xi e} &= \frac{(1+\kappa_{ve})^{1/(1-\gamma_e)} + \kappa_q + (1-\kappa_q)\kappa_\eta - 1 + 1 - (\kappa_q + (1-\kappa_q)\kappa_\eta)}{1-(\kappa_q+(1-\kappa_q)\kappa_\eta)} \\
&= \frac{(1+\kappa_{ve})^{1/(1-\gamma_e)}}{1-(\kappa_q+(1-\kappa_q)\kappa_\eta)} \\
1+\kappa_{\xi h} &= \frac{(1+\kappa_{vh})^{1/(1-\gamma_h)} - \left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right) - 1 + 1 + \left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right)}{1+\left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right)} \\
&= \frac{(1+\kappa_{vh})^{1/(1-\gamma_h)}}{1+\left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right)}
\end{aligned}
$$

which implies that the jump sizes in the asset pricing condition are

$$
\begin{aligned}
(1+\kappa_{\xi e})^{1-\gamma_e} &= \frac{1+\kappa_{ve}}{(1-(\kappa_q+(1-\kappa_q)\kappa_\eta))^{1-\gamma_e}} \\
(1+\kappa_{\xi h})^{1-\gamma_h} &= \frac{1+\kappa_{vh}}{\left(1+\left(-\kappa_q + (1-\kappa_q)\frac{\eta}{1-\eta}\kappa_\eta\right)\right)^{1-\gamma_h}}
\end{aligned}
$$

Finally, to write the market-clearing condition for consumption when $\psi_e \neq 1$, notice that

$$
\begin{aligned}
\frac{C_e}{N_e} &= \xi_e^{1-\psi_e} = \frac{v_e^{(1-\psi_e)/(1-\gamma_e)}}{(\eta q)^{1-\psi_e}} \\
\frac{C_h}{N_h} &= \frac{v_h^{(1-\psi_h)/(1-\gamma_h)}}{((1-\eta)q)^{1-\psi_h}}.
\end{aligned}
$$

3.2.3 Transformation 3

A third transformation combines the previous transformations to avoid the calculation of negative numbers under the case of $\psi \neq 1$. Let $\tilde{v}_e$ be the second transformation. Recall that

$$
\xi_e = \frac{\tilde{v}_e^{1/(1-\gamma_e)}}{\eta q},
$$

so the first transformation implies

$$v_e = \xi_e^{\psi_e - 1} = \frac{\tilde{v}_e^{(\psi_e - 1)/(1-\gamma_e)}}{(\eta q)^{\psi_e - 1}}.$$

3.3 Functional Iteration Algorithm

In the case of log utility, we apply the algorithm developed by Li (2020). The algorithm proceeds in two main steps.

1. Initialize a guess $q^{(0)}$ over the state space.[4]
2. Do for i in $1:N$ or until $q^{(i)}$ is sufficiently close to $q^{(i-1)}$
   (a) Given $q^{(i-1)}$, calculate new proposals for $\varphi_e^{(i)}$ and $\kappa_q^{(i)}$ by differencing agents' first-order conditions for leverage and solving the jump size fixed point (39) for q.
   (b) Given the new proposal for $\varphi_e^{(i)}$, calculate a new proposal $\hat{q}^{(i)}$ using market-clearing for consumption (36).
   (c) Set $q^{(i)} = \alpha \hat{q}^{(i)} + (1-\alpha) q^{(i-1)}$.

The parameter α is the learning rate and faciliates "tempered updating". Without a sufficiently small α, the algorithm may not converge because solutions may not exist when $q^{(i)}$ changes too rapidly. It is also important to use monotonocity-preserving interpolants for q. Otherwise, the algorithm may not converge because of the difficulty in solving the endogenous jump size κ_q.

3.4 Pseudo-Transient Functional Iteration Algorithm

Can either directly solve the HJB problem, or we can transform the value functions further using the approach of Brunnermeier-Sannikov (2016). It is worthwhile trying both to see

[4] A decent guess is the equilibrium with no jumps.

which is faster for convergence or is more numerically stable. By default though, we will use the HJB approach since that seems to be Di Tella's preferred approach.