These notes loosely follow Fernández-Villaverde and Levintal (2018) "Solution Methods for Models with Rare Disasters" but omits several features, such as recursive preferences and disaster risk.

1 Model

1.1 Household

The model admits a representative agent, so I directly write households' problem as the representative agent's. The representative household solves, in the cashless limit,

$$\max_{C_t, L_t, B_{t+1}, X_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \exp(\eta_{L,t}) \frac{L_t^{1+\nu}}{1+\nu} \right) \tag{1}$$

subject to the budget constraint

$$C_t + \frac{B_{t+1}}{P_t} + X_t \leq W_t L_t + R_{K,t} K_t + R_{t-1} \frac{B_t}{P_t} + F_t + T_t, \tag{2}$$

where C_t is consumption, B_t nominal bonds, X_t investment, $W_{N,t}$ the real wage, L_t labor, $R_{KN,t}$ the gross real rental rate on capital, K_t capital, R_t the gross nominal interest rate on bonds, F_t real profits from firms, and T_t real lump-sum transfers from the government. The price of the final consumption good is P_t. Markets are assumed complete, but securities are in zero net supply. Because there is a representative agent, I may omit the Arrow securities from the budget constraint. Our notation treats B_t and K_t as pre-determined at time t. Investment for capital follows the law of motion

$$K_{t+1} = (1-\delta) K_t + \exp(\eta_{K,t}) \left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right] \right) X_t, \tag{3}$$

where the investment function takes the functional form

$$\Phi \left[\frac{X_t}{X_{t-1}} \right] = \frac{\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2.$$

The Lagrangian for the household is

$$\begin{aligned}
\mathcal{L} = & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \exp(\eta_{L,t}) \frac{L_t^{1+\nu}}{1+\nu} \right] \\
& + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \lambda_t \left[W_t L_t + R_{K,t} K_t + R_{t-1} \frac{B_t}{P_t} + F_t + T_t - C_t - \frac{B_{t+1}}{P_t} - X_t \right] \\
& + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \lambda_t Q_t \left[(1-\delta) K_t + \exp(\eta_{K,t}) \left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right] \right) X_t - K_{t+1} \right],
\end{aligned}$$

which implies first-order conditions

$$
\begin{aligned}
0 &= C_t^{-\gamma} - \lambda_t \\
0 &= -\varphi \exp(\eta_{L,t}) L_t^{\nu} + \lambda_t W_t \\
0 &= -\exp(\eta_{\beta,t})\lambda_t + \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1})\lambda_{t+1} R_t] \\
0 &= -\exp(\eta_{\beta,t})\lambda_t + \exp(\eta_{\beta,t})\lambda_t Q_t \exp(\eta_{K,t}) \left(1 - \Phi\left[\frac{X_t}{X_{t-1}}\right]\right) \\
&\quad - \exp(\eta_{\beta,t})\lambda_t Q_t \exp(\eta_{K,t}) \Phi'\left[\frac{X_t}{X_{t-1}}\right] \frac{X_t}{X_{t-1}} \\
&\quad - \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1})\lambda_{t+1} Q_{t+1} \exp(\eta_{K,t+1}) \Phi'\left[\frac{X_{t+1}}{X_t}\right] \left(-\frac{X_{t+1}}{X_t^2}\right) X_{t+1} \\
0 &= \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1})\lambda_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))] - \exp(\eta_{\beta,t})\lambda_t Q_t
\end{aligned}
$$

The first two equations can be combined by isolating λ_t, which obtains the intratemporal consumption-labor condition

$$
C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^{\nu},
$$

Using $\lambda_t = C_t^{-\gamma}$ and defining the gross inflation rate $\Pi_t \equiv P_t / P_{t-1}$, I can obtain the Euler equation for households

$$
\begin{aligned}
\exp(\eta_{\beta,t}) \frac{C_t^{-\gamma}}{P_t} &= \beta \mathbb{E}_t \left[\exp(\eta_{\beta,t+1}) \frac{C_{t+1}^{-\gamma}}{P_{t+1}} R_t\right] \\
1 &= \beta \mathbb{E}_t \left[\frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \frac{R_t}{\Pi_{t+1}}\right].
\end{aligned}
$$

I can further simplify the Euler equation by defining the (real) stochastic discount factor

$$
M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}
$$

After dividing through by $\exp(\eta_{\beta,t})\lambda_t$ and re-arranging, the investment condition becomes

$$
\begin{aligned}
1 &= \exp(\eta_{K,t}) Q_t \left(1 - \Phi\left[\frac{X_t}{X_{t-1}}\right]\right) - \exp(\eta_{K,t}) Q_t \Phi'\left[\frac{X_t}{X_{t-1}}\right] \frac{X_t}{X_{t-1}} \\
&\quad + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \Phi'\left[\frac{X_{t+1}}{X_t}\right] \frac{X_{t+1}^2}{X_t^2}\right] \\
&= \exp(\eta_{K,t}) Q_t \left(\left(1 - \Phi\left[\frac{X_t}{X_{t-1}}\right]\right) - \Phi'\left[\frac{X_t}{X_{t-1}}\right] \frac{X_t}{X_{t-1}}\right) \\
&\quad + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \Phi'\left[\frac{X_{t+1}}{X_t}\right] \frac{X_{t+1}^2}{X_t^2}\right].
\end{aligned}
$$

Finally, after dividing through by $\exp(\eta_{\beta,t})\lambda_t$, the first-order condition for next-period capital is

$$Q_t = \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))]$$

In summary, households' optimality conditions are

$$C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^{\nu}, \tag{4}$$

$$M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, \tag{5}$$

$$1 = \beta \mathbb{E}_t \left[M_{t+1} \frac{R_t}{\Pi_{t+1}} \right], \tag{6}$$

$$\begin{aligned} 1 = \exp(\eta_{K,t}) Q_t & \left(\left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right] \right) - \Phi' \left[\frac{X_t}{X_{t-1}} \right] \frac{X_t}{X_{t-1}} \right) \\ & + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \Phi' \left[\frac{X_{t+1}}{X_t} \right] \frac{X_{t+1}^2}{X_t^2} \right], \end{aligned} \tag{7}$$

$$Q_t = \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))]. \tag{8}$$

1.2 Production

Final Producers There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 1$ so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j)\, dj.$$

The FOC for $Y_t(j)$ is

$$\begin{aligned} 0 &= P_t \frac{\epsilon}{\epsilon-1} \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} \frac{\epsilon-1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) \\ 0 &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t} \\ 0 &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} \\ Y_t(j) &= \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t. \end{aligned}$$

Plugging this quantity into the identity

$$P_tY_t = \int_0^1 P_t(j)Y_t(j)\,dj$$

and simplifying yields the price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon}\,dj\right)^{\frac{1}{1-\epsilon}}.$$

Intermediate Producers Intermediate goods are producing according to the Cobb-Douglas technology

$$Y_t(j) = A_tK_t^{\alpha}(j)L_t^{1-\alpha}(j).$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\min_{K_t(j),L_t(j)} R_{K,t}K_t(j) + W_tL_t(j) \qquad \text{s.t.} \qquad A_tK_t^{\alpha}(j)L_t^{1-\alpha}(j) \geq \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t.$$

The RHS of the inequality constraint is the demand from final goods producers for intermediate j. The Lagrangian is

$$\mathcal{L} = R_{K,t}K_t(j) + W_tL_t(j) + MC_t(j)\left(\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t - A_tK_t^{\alpha}(j)L_t^{1-\alpha}(j)\right),$$

so the first-order conditions are

$$\begin{aligned} 0 &= R_{K,t} - MC_t(j)\alpha A_t\left(\frac{L_t(j)}{K_t(j)}\right)^{1-\alpha} \\ 0 &= W_t - MC_t(j)(1-\alpha)A_t\left(\frac{K_t(j)}{L_t(j)}\right)^{\alpha}, \end{aligned}$$

hence the optimal capital-labor ratio satisfies

$$\begin{aligned} \frac{R_{K,t}}{\alpha A_t(K_t(j)/L_t(j))^{\alpha-1}} &= \frac{W_t}{(1-\alpha)A_t(K_t(j)/L_t(j))^{\alpha}} \\ \frac{K_t(j)}{L_t(j)} &= \frac{\alpha}{1-\alpha}\frac{W_t}{R_{K,t}}. \end{aligned}$$

Since the RHS does not vary with j, all firms choose the same capital-labor ratio. Given this optimal ratio, the marginal cost satisfies

$$\begin{aligned} MC_t &= \frac{R_{K,t}}{\alpha A_t}\left(\frac{K_t}{L_t}\right)^{1-\alpha} \\ &= \frac{R_{K,t}}{\alpha A_t}\left(\frac{\alpha}{1-\alpha}\frac{W_t}{R_{K,t}}\right)^{1-\alpha} \\ &= \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{W_t^{1-\alpha}R_{K,t}^{\alpha}}{A_t}. \end{aligned}$$

It follows that

$$
\begin{aligned}
R_{K,t}K_t + W_tL_t &= \left(\frac{R_{K,t}}{A_t}\left(\frac{K_t}{L_t}\right)^{1-\alpha} + \frac{W_t}{A_t}\left(\frac{L_t}{K_t}\right)^{\alpha}\right)\left(A_tK_t^{\alpha}L_t^{1-\alpha}\right) \\
&= \left(\alpha MC_t + (1-\alpha)MC_t\right)Y_t(j) = MC_tY_t(j).
\end{aligned}
$$

Therefore, (real) profits for an intermediate producer become

$$
F_t(j) = \frac{P_t(j)}{P_t}Y_t(j) - MC_tY_t(j).
$$

In addition to the capital-labor choice, firms also have the chance to reset prices in every period with probability $1-\theta$. This problem can be written as

$$
\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})}\frac{u'(C_{t+s})}{u'(C_t)}\left(\frac{P_t(j)}{P_{t+s}}\left(\frac{P_t(j)}{P_{t+s}}\right)^{-\epsilon}Y_{t+s} - mc_{t+s}\left(\frac{P_t(j)}{P_{t+s}}\right)^{-\epsilon}Y_{t+s}\right),
$$

where I have imposed that intermediate output equals demand. The first-order condition is

$$
\begin{aligned}
0 = &(1-\epsilon)P_t(j)^{-\epsilon}\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})}\frac{u'(C_{t+s})}{u'(C_t)}(P_{t+s})^{-(1-\epsilon)}Y_{t+s} \\
&+ \epsilon P_t(j)^{-\epsilon-1}\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})}\frac{u'(C_{t+s})}{u'(C_t)}mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}
\end{aligned}
$$

Divide by $P_t(j)^{-\epsilon}/(\exp(\eta_{\beta,t})u'(C_t))$, apply the abuse of notation that $\prod_{u=1}^{0}\Pi_{t+u} = 1$, and re-arrange to obtain

$$
\begin{aligned}
P_t(j) &= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})P_{t+s}^{\epsilon-1}Y_{t+s}} \\
&= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}P_t^{\epsilon}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})P_t^{\epsilon-1}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon-1}Y_{t+s}} \\
\frac{P_t(j)}{P_t} &= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon-1}Y_{t+s}}.
\end{aligned}
$$

This expression gives the optimal (real) reset price $P_t^* \equiv P_t(j)/P_t$ (note that the RHS does not depend on j). Define

$$
\begin{aligned}
S_{1,t} &= \mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}Y_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+s}\right)^{\epsilon}, \\
S_{2,t} &= \mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})Y_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+s}\right)^{\epsilon-1}.
\end{aligned}
$$

Using these definitions, I may write the optimal reset price more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon - 1}\frac{S_{1,t}}{S_{2,t}}$$

where $S_{1,t}$ and $S_{2,t}$ satisfy the recursions

$$\begin{aligned} S_{1,t} &= \exp(\eta_{\beta,t})u'(C_t)MC_tY_t + \theta\beta\mathbb{E}_t\Pi_{t+s}^{\epsilon}S_{1,t+1} \\ S_{2,t} &= \exp(\eta_{\beta,t})u'(C_t)Y_t + \theta\beta\mathbb{E}_t\Pi_{t+s}^{\epsilon-1}S_{2,t+1}. \end{aligned}$$

These recursions can be further rewritten as

$$\begin{aligned} \frac{S_{1,t}}{\exp(\eta_{\beta,t})u'(C_t)} &= MC_tY_t + \theta\beta\mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)}\Pi_{t+s}^{\epsilon}\frac{S_{1,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})}\right] \\ \frac{S_{2,t}}{\exp(\eta_{\beta,t})u'(C_t)} &= Y_t + \theta\beta\mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)}\Pi_{t+s}^{\epsilon-1}\frac{S_{2,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})}\right], \end{aligned}$$

By defining $\tilde{S}_{1,t} \equiv S_{1,t}/(\exp(\eta_{\beta,t})u'(C_t))$ and $\tilde{S}_{2,t} \equiv S_{2,t}/(\exp(\eta_{\beta,t})u'(C_t))$, I can simplify these recursions into the form I use for the numerical solution.

From this section, we obtain the following five equilibrium conditions:

$$\begin{aligned} MC_t &= \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{W_t^{1-\alpha}R_{K,t}^{\alpha}}{A_t}, && (9) \\ \frac{K_t}{L_t} &= \frac{\alpha}{1-\alpha}\frac{W_t}{R_{K,t}}, && (10) \\ P_t^* &= \frac{\epsilon}{\epsilon-1}\frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, && (11) \\ \tilde{S}_{1,t} &= MC_tY_t + \theta\beta\mathbb{E}_t[M_{t+1}\Pi_{t+s}^{\epsilon}\tilde{S}_{1,t+1}], && (12) \\ \tilde{S}_{2,t} &= Y_t + \theta\beta\mathbb{E}_t[M_{t+1}\Pi_{t+s}^{\epsilon-1}\tilde{S}_{2,t+1}]. && (13) \end{aligned}$$

1.3 Monetary Policy

To close the model, I specify the monetary policy rule as the following Taylor rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\phi_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\Pi}\left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_Y}\right)^{1-\phi_R}\exp(\eta_{R,t}) \tag{14}$$

Any proceeds from monetary policy are distributed as lump sum to the representative household.

1.4 Aggregation

The price level is currently characterized as the integral

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon}\, dj.$$

To represent the model entirely in terms of aggregates, notice that, without loss of generality, we may re-order the fraction ϕ of firms which cannot reset prices to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon}\, dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \phi \int_0^1 P_{t-1}(j)^{1-\epsilon}\, dj = (1-\phi)(P_t^*)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}.$$

Dividing by $P_{t-1}^{1-\epsilon}$ implies

$$\Pi_t^{1-\epsilon} = (1-\phi)(P_t^*\Pi_t)^{1-\epsilon} + \phi. \tag{15}$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$\begin{aligned}
v_t &= \int_0^{1-\phi} (P_t^*)^{-\epsilon}\, dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\epsilon} dj \\
&= \int_0^{1-\phi} (P_t^*\Pi_t)^{-\epsilon} \left(\frac{1}{\Pi_t}\right)^{-\epsilon} dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} dj \\
&= (1-\phi)(P_t^*\Pi_t)^{-\epsilon}\Pi_t^{\epsilon} + \Pi_t^{\epsilon} \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj.
\end{aligned}$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \phi \int_0^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \phi v_{t-1}.$$

Thus, we acquire

$$v_t = \Pi_t^{\epsilon}((1-\phi)(P_t^*\Pi_t)^{\epsilon} + \phi v_{t-1}) \tag{16}$$

1.5 Equilibrium

To close the model, I need to specify the functional form for investment, aggregate shocks, and market-clearing conditions.
Following the literature, I assume quadratic adjustment investment costs so that

$$\Phi\left[\frac{X_t}{X_{t-1}}\right] = \frac{\chi}{2}\left(\frac{X_t}{X_{t-1}}\right)^2, \tag{17}$$

which has first derivative w.r.t. X_t/X_{t-1}

$$\Phi'\left[\frac{X_t}{X_{t-1}}\right] = \chi\left(\frac{X_t}{X_{t-1}}\right). \tag{18}$$

There are five shocks in the model: $\eta_{A,t}$, $\eta_{\beta,t}$, $\eta_{L,t}$, $\eta_{K,t}$, and $\eta_{R,t}$. Without loss of generality, I assume all shocks follow AR(1) processes with persistence ρ_i and standard deviation σ_i.
Markets must clear for capital, labor, bonds, final goods, and intermediate goods, . The first three markets clear as a consequence of optimality conditions and the assumption that bonds have zero net supply. To clear the market for final goods, we set aggregate consumption demand C_t equal to aggregate supply Y_t, which satisfies

$$\begin{aligned}\int_0^1 A_t K_t^{\alpha} L_t^{1-\alpha}\, dj &= \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t\, dj \\ A_t K_t^{\alpha} L_t^{1-\alpha} &= Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj = v_t Y_t.\end{aligned}$$

Re-arranging yields the consumption market-clearing condition

$$C_t = \frac{A_t K_t^{\alpha} L_t^{1-\alpha}}{v_t}. \tag{19}$$

It can be shown that $v_t \geq 1$ by applying Jensen's inequality.

All together, the full set of endogenous equilibrium conditions are

$$
\begin{aligned}
C_t^{-\gamma} W_t &= \varphi \exp(\eta_{L,t}) L_t^{\nu}, && (20)\\
M_{t+1} &= \beta \frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, && (21)\\
1 &= \beta \mathbb{E}_t \left[M_{t+1} \frac{R_t}{\Pi_{t+1}} \right], && (22)\\
1 &= \exp(\eta_{K,t}) Q_t \left(1 - \frac{3\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2 \right) + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \chi \left(\frac{X_{t+1}}{X_t} \right)^3 \right], && (23)\\
Q_t &= \mathbb{E}_t [M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))], && (24)\\
MC_t &= \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^{\alpha} \frac{W_t^{1-\alpha} R_{K,t}^{\alpha}}{A_t}, && (25)\\
\frac{K_t}{L_t} &= \frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}}, && (26)\\
P_t^* &= \frac{\epsilon}{\epsilon - 1} \frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, && (27)\\
\tilde{S}_{1,t} &= MC_t Y_t + \theta \beta \mathbb{E}_t [M_{t+1} \Pi_{t+s}^{\epsilon} \tilde{S}_{1,t+1}], && (28)\\
\tilde{S}_{2,t} &= Y_t + \theta \beta \mathbb{E}_t [M_{t+1} \Pi_{t+s}^{\epsilon-1} \tilde{S}_{2,t+1}], && (29)\\
\Pi_t^{1-\epsilon} &= (1-\phi)(P_t^* \Pi_t)^{1-\epsilon} + \phi, && (30)\\
v_t &= \Pi_t^{\epsilon}((1-\phi)(P_t^* \Pi_t)^{\epsilon} + \phi v_{t-1}), && (31)\\
\frac{R_t}{R} &= \left(\frac{R_{t-1}}{R} \right)^{\phi_R} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\Pi} \left(\frac{Y_t}{Y_{t-1}} \right)^{\phi_Y} \right)^{1-\phi_R} \exp(\eta_{R,t}), && (32)\\
C_t &= \frac{A_t K_t^{\alpha} L_t^{1-\alpha}}{v_t}, && (33)
\end{aligned}
$$

as well as the law of motion for capital

$$
K_{t+1} = (1-\delta) K_t + \exp(\eta_{K,t}) \left(1 - \frac{\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2 \right) X_t \tag{34}
$$

and the five exogenous processes

$$
\begin{aligned}
\eta_{\beta,t+1} &= \rho_\beta \eta_{\beta,t} + \sigma_\beta \varepsilon_{\beta,t+1}, && (35)\\
\eta_{L,t+1} &= \rho_L \eta_{L,t} + \sigma_L \varepsilon_{L,t+1}, && (36)\\
\eta_{A,t+1} &= \rho_A \eta_{A,t} + \sigma_A \varepsilon_{A,t+1}, && (37)\\
\eta_{K,t+1} &= \rho_K \eta_{K,t} + \sigma_K \varepsilon_{K,t+1}, && (38)\\
\eta_{R,t+1} &= \rho_R \eta_{R,t} + \sigma_R \varepsilon_{R,t+1}. && (39)
\end{aligned}
$$

2 Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$
\begin{aligned}
0 &= \log \mathbb{E}_t \left[\exp\left(\xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1}\right)\right] \\
z_{t+1} &= \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t)\varepsilon_{t+1},
\end{aligned}
$$

where z_t are (predetermined) state variables and y_t are (nondetermined) jump variables. For the remainder of this section, lower case variables are the logs of previously upper case variables, and with a small abuse of notation, let $s_{1,t} = \log(\tilde{S}_{1,t})$ and $s_{2,t} = \log(\tilde{S}_{2,t})$. Additionally, let $r_{k,t} = \log(R_{K,t})$.

Equation (20) becomes

$$
\begin{aligned}
1 &= \varphi \exp(\eta_{L,t}) \frac{L_t^{\nu}}{C_t^{-\gamma} W_t} \\
0 &= \log \mathbb{E}_t \left[\exp\left(\underbrace{\log(\varphi) + \eta_{L,t} + \nu l_t - (-\gamma c_t + w_t)}_{\xi}\right)\right].
\end{aligned}
$$

Equation (21) will not be used in the system of equations for the risk-adjusted linearization, but it simplifies the other equations. Taking logs and re-arranging yields

$$
\begin{aligned}
0 &= \log(\beta) + \eta_{\beta,t+1} + (-\gamma c_{t+1}) - \eta_{\beta,t} - (-\gamma c_t) - m_{t+1} \\
&= \underbrace{\log(\beta) - \eta_{\beta,t} + \gamma c_t}_{\xi} + \underbrace{\eta_{\beta,t+1} - \gamma c_{t+1} - m_{t+1}}_{\text{forward-looking}}.
\end{aligned}
$$

Equation (22) becomes

$$
0 = \log \mathbb{E}_t \left[\exp\left(\underbrace{\log(\beta) + r_t}_{\xi} + \underbrace{m_{t+1} - \Pi_{t+1}}_{\text{forward-looking}}\right)\right].
$$

Equation (23) becomes

$$
\begin{aligned}
&1 - \exp(\eta_{K,t}) Q_t \left(1 - \frac{3\chi}{2}\left(\frac{X_t}{X_{t-1}}\right)^2\right) \\
&= \mathbb{E}_t\left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \chi \left(\frac{X_{t+1}}{X_t}\right)^3\right] \\
&\log\left(1 - \exp(\eta_{K,t} + q_t)\left(1 - \frac{3\chi}{2}\exp(2x_t - 2x_{t-1})\right)\right) \\
&= \log \mathbb{E}_t\left[\exp\left(m_{t+1} + \eta_{K,t+1} + q_{t+1} + \log(\chi) + 3x_{t+1} - 3x_t\right)\right] \\
0 &= \log \mathbb{E}_t\left[\frac{\exp\left(m_{t+1} + \eta_{K,t+1} + q_{t+1} + \log(\chi) + 3x_{t+1} - 3x_t\right)}{\exp\left(\log\left(1 - \exp(\eta_{K,t} + q_t)\left(1 - \frac{3\chi}{2}\exp(2x_t - 2x_{t-1})\right)\right)\right)}\right].
\end{aligned}
$$

Re-arranging yields

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\chi) - 3x_t - \log \left(1 - \exp(\eta_{K,t} + q_t) \left(1 - \frac{3\chi}{2} \exp(2x_t - 2x_{t-1}) \right) \right)}_{\xi} \right) \right]$$
$$+ \log \mathbb{E}_t \left[\exp(\underbrace{m_{t+1} + \eta_{K,t+1} + q_{t+1} + 3x_{t+1}}_{\text{forward-looking}}) \right].$$

Equation (24) becomes

$$\begin{aligned} 1 &= \mathbb{E}_t \left[\frac{M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))}{Q_t} \right] \\ 0 &= \log \mathbb{E}_t \left[\exp \left(m_{t+1} + \log(\exp(r_{k,t+1}) + (1-\delta)\exp(q_{t+1})) - q_t \right) \right]. \end{aligned}$$

This equation is not linear in the forward-looking variables, we define a new variable

$$N_t = R_{K,t} + Q_t(1-\delta), \tag{40}$$

so that we may represent (24) in the required form for risk-adjusted linearizations as the following two equations:

$$\begin{aligned} 0 &= \log \mathbb{E}_t \left[\exp \left(\underbrace{-q_t}_{\xi} + \underbrace{m_{t+1} + n_{t+1}}_{\text{forward-looking}} \right) \right], \\ 0 &= \log \mathbb{E}_t \left[\exp \left(\underbrace{n_t - \log(\exp(r_{k,t}) + (1-\delta)\exp(q_t))}_{\xi} \right) \right], \end{aligned}$$

Equation (25) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{(1-\alpha)w_t + \alpha r_{k,t} - a_t - (1-\alpha)\log(1-\alpha) - \alpha\log(\alpha) - mc_t}_{\xi} \right) \right].$$

Equation (26) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{k_t - l_t - \log \left(\frac{\alpha}{1-\alpha} \right) - (w_t - r_{k,t})}_{\xi} \right) \right].$$

Like the stochastic discount factor, equation (27) will not be used in the system of equations, but it will be useful to simplify other equations. Taking logs yields

$$p_t^* = \log\left(\frac{\epsilon}{\epsilon-1}\right) + s_{1,t} - s_{2,t}.$$

The first equation becomes

$$\begin{aligned}
1 &= \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{1+i_t}{1+\pi_{t+1}} \right] \\
0 &= \log \mathbb{E}_t \left[\exp\left(\log(\beta) - \sigma(c_{t+1} - c_t) + \tilde{i}_t - \tilde{\pi}_{t+1} \right) \right] \\
&= \log \mathbb{E}_t \left[\exp\left(\underbrace{\log(\beta) + \sigma c_t + \tilde{i}_t}_{\xi} - \underbrace{\sigma c_{t+1} - \tilde{\pi}_{t+1}}_{\text{Forward-Looking}} \right) \right],
\end{aligned}$$

where $c_t = \log(C_t)$, $\tilde{i}_t = \log(1+i_t)$, and $\tilde{\pi}_t = \log(1+\pi_{t+1})$.

The second equation becomes

The third equation becomes

$$\begin{aligned}
1 &= \frac{w_t}{A_t MC_t} \\
0 &= \log \mathbb{E}_t \left[\exp\left(\hat{w}_t - a_t - \tilde{mc}_t \right) \right].
\end{aligned}$$

The fourth and fifth equation become

$$0 = \log \mathbb{E}_t \left[\exp\left(c_t - a_t - n_t + \hat{v}_t \right) \right].$$

The sixth equation becomes

$$0 = \hat{v}_t - \epsilon\tilde{\pi}_t - \log((1-\phi)\exp(\tilde{\pi}_t^*)^{-\epsilon} + \phi\exp(\hat{v}_{t-1})),$$

where $\hat{v}_{t-1}$ will be treated as an additional state variable, i.e. if $a_t = \hat{v}_t$ is a jump variable and $b_t = \hat{v}_{t-1}$ is a state variable, then

$$b_{t+1} = a_t.$$

The seventh equation becomes

$$0 = (1-\epsilon)\tilde{\pi}_t - \log((1-\phi)\exp(\tilde{\pi}_t^*)^{1-\epsilon} + \phi).$$

The eighth equation becomes

$$0 = \tilde{\pi}_t^* - \log\left(\frac{\epsilon}{\epsilon-1}\right) - \tilde{\pi}_t - (\hat{x}_{1,t} - \hat{x}_{2,t})).$$

By plugging this expression for $\tilde{\pi}_t^*$ into the previous two equations, we can also remove one more variable from the system.

The ninth and tenth equation become

$$1 = \mathbb{E}_t \left[\phi\beta \frac{x_{1,t+1}(1+\pi_{t+1})}{x_{1,t} - C_t^{-\sigma} MC_t A_t N_t / v_t} \right]$$

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{1,t}) - \exp((1-\sigma)c_t + \tilde{mc}_t))}_{\xi} + \hat{x}_{1,t+1} + \epsilon\tilde{\pi}_{t+1} \right) \right]$$

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{2,t}) - \exp((1-\sigma)c_t))}_{\xi} + \hat{x}_{2,t+1} + (\epsilon - 1)\tilde{\pi}_{t+1} \right) \right],$$

where the fact that $x_{1,t}$ and $x_{2,t}$ must both be positive implies $x_{1,t} - C_t^{-\sigma} MC_t Y_t$ and $x_{2,t} - C_t^{-\sigma} Y_t$ are both positive, as the expectations on the RHS are also both positive.

For the monetary policy rule, we use $\tilde{i}_{t-1} \equiv \log(1 + i_{t-1})$ and $\varepsilon_{i,t}$ as states and treat i_t as a jump variable, hence

$$\tilde{i}_t = (1-\rho_i)\tilde{i} + \rho_i \tilde{i}_{t-1} + (1-\rho_i)\phi_\pi(\tilde{\pi}_t - \tilde{\pi}) + \varepsilon_{i,t}.$$

This formulation allows us to treat the policy rule as an expectational equation.

The above nine equations comprise the expectational equations. The following four equations comprise the states:

$$\begin{aligned} a_{t+1} &= \rho_a a_t + \varepsilon_{a,t+1} \\ \hat{v}_{(t-1)+1} &= \hat{v}_t \\ \tilde{i}_{(t-1)+1} &= \tilde{i}_t \\ \varepsilon_{i,t+1} &= \varepsilon_{i,t+1} \end{aligned}$$

To conclude, note that we have three forward difference equations. The first is the Euler equation, which can be expressed as