These notes loosely follow Fernández-Villaverde and Levintal (2018) "Solution Methods for Models with Rare Disasters" but omits several features, such as recursive preferences and disaster risk.

1 Model

1.1 Household

The model admits a representative agent, so I directly write households' problem as the representative agent's. The representative household solves, in the cashless limit,

$$\max_{C_t, L_t, B_{t+1}, X_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \exp(\eta_{L,t}) \frac{L_t^{1+\nu}}{1+\nu} \right) \tag{1}$$

subject to the budget constraint

$$C_t + \frac{B_{t+1}}{P_t} + X_t \leq W_t L_t + R_{K,t} K_t + R_{t-1} \frac{B_t}{P_t} + F_t + T_t, \tag{2}$$

where C_t is consumption, B_t nominal bonds, X_t investment, $W_{N,t}$ the real wage, L_t labor, $R_{KN,t}$ the gross real rental rate on capital, K_t capital, R_t the gross nominal interest rate on bonds, F_t real profits from firms, and T_t real lump-sum transfers from the government. The price of the final consumption good is P_t. Markets are assumed complete, but securities are in zero net supply. Because there is a representative agent, I may omit the Arrow securities from the budget constraint. Our notation treats B_t and K_t as pre-determined at time t. Investment for capital follows the law of motion

$$K_{t+1} = (1-\delta)K_t + \exp(\eta_{K,t}) \left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right] \right) X_t, \tag{3}$$

where the investment function takes the functional form

$$\Phi \left[\frac{X_t}{X_{t-1}} \right] = \frac{\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2.$$

The Lagrangian for the household is

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \exp(\eta_{L,t}) \frac{L_t^{1+\nu}}{1+\nu} \right]$$

$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \lambda_t \left[W_t L_t + R_{K,t} K_t + R_{t-1} \frac{B_t}{P_t} + F_t + T_t - C_t - \frac{B_{t+1}}{P_t} - X_t \right]$$

$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \lambda_t Q_t \left[(1-\delta)K_t + \exp(\eta_{K,t}) \left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right] \right) X_t - K_{t+1} \right],$$

which implies first-order conditions

$$0 = C_t^{-\gamma} - \lambda_t$$
$$0 = -\varphi \exp(\eta_{L,t}) L_t^{\nu} + \lambda_t W_t$$
$$0 = -\exp(\eta_{\beta,t})\lambda_t + \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1})\lambda_{t+1} R_t]$$
$$0 = -\exp(\eta_{\beta,t})\lambda_t + \exp(\eta_{\beta,t})\lambda_t Q_t \exp(\eta_{K,t})\left(1 - \Phi\left[\frac{X_t}{X_{t-1}}\right]\right)$$
$$- \exp(\eta_{\beta,t})\lambda_t Q_t \exp(\eta_{K,t})\Phi'\left[\frac{X_t}{X_{t-1}}\right]\frac{X_t}{X_{t-1}}$$
$$- \beta \mathbb{E}_t\left[\exp(\eta_{\beta,t+1})\lambda_{t+1} Q_{t+1} \exp(\eta_{K,t+1})\Phi'\left[\frac{X_{t+1}}{X_t}\right]\left(-\frac{X_{t+1}}{X_t^2}\right)X_{t+1}\right]$$
$$0 = \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1})\lambda_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))] - \exp(\eta_{\beta,t})\lambda_t Q_t$$

The first two equations can be combined by isolating λ_t, which obtains the intratemporal consumption-labor condition

$$C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^{\nu},$$

Using $\lambda_t = C_t^{-\gamma}$ and defining the gross inflation rate $\Pi_t \equiv P_t/P_{t-1}$, I can obtain the Euler equation for households

$$\exp(\eta_{\beta,t})\frac{C_t^{-\gamma}}{P_t} = \beta \mathbb{E}_t\left[\exp(\eta_{\beta,t+1})\frac{C_{t+1}^{-\gamma}}{P_{t+1}}R_t\right]$$
$$1 = \beta \mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})}\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}\frac{R_t}{\Pi_{t+1}}\right].$$

I can further simplify the Euler equation by defining the (real) stochastic discount factor

$$M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})}\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}$$

After dividing through by $\exp(\eta_{\beta,t})\lambda_t$ and re-arranging, the investment condition becomes

$$1 = \exp(\eta_{K,t})Q_t\left(1 - \Phi\left[\frac{X_t}{X_{t-1}}\right]\right) - \exp(\eta_{K,t})Q_t\Phi'\left[\frac{X_t}{X_{t-1}}\right]\frac{X_t}{X_{t-1}}$$
$$+ \mathbb{E}_t\left[M_{t+1}\exp(\eta_{K,t+1})Q_{t+1}\Phi'\left[\frac{X_{t+1}}{X_t}\right]\frac{X_{t+1}^2}{X_t^2}\right]$$
$$= \exp(\eta_{K,t})Q_t\left(\left(1 - \Phi\left[\frac{X_t}{X_{t-1}}\right]\right) - \Phi'\left[\frac{X_t}{X_{t-1}}\right]\frac{X_t}{X_{t-1}}\right)$$
$$+ \mathbb{E}_t\left[M_{t+1}\exp(\eta_{K,t+1})Q_{t+1}\Phi'\left[\frac{X_{t+1}}{X_t}\right]\frac{X_{t+1}^2}{X_t^2}\right].$$

Finally, after dividing through by $\exp(\eta_{\beta,t})\lambda_t$, the first-order condition for next-period capital is

$$Q_t = \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))]$$

In summary, households' optimality conditions are

$$C_t^{-\gamma}W_t = \varphi\exp(\eta_{L,t})L_t^\nu, \tag{4}$$

$$M_{t+1} = \beta\frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})}\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, \tag{5}$$

$$1 = \beta\mathbb{E}_t\left[M_{t+1}\frac{R_t}{\Pi_{t+1}}\right], \tag{6}$$

$$\begin{aligned}
1 = \exp(\eta_{K,t})Q_t\left(\left(1-\Phi\left[\frac{X_t}{X_{t-1}}\right]\right) - \Phi'\left[\frac{X_t}{X_{t-1}}\right]\frac{X_t}{X_{t-1}}\right) \\
+ \mathbb{E}_t\left[M_{t+1}\exp(\eta_{K,t+1})Q_{t+1}\Phi'\left[\frac{X_{t+1}}{X_t}\right]\frac{X_{t+1}^2}{X_t^2}\right],
\end{aligned} \tag{7}$$

$$Q_t = \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))]. \tag{8}$$

1.2   Production

Final Producers   There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 1$ so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t\left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j)Y_t(j)\, dj.$$

The FOC for $Y_t(j)$ is

$$0 = P_t\frac{\epsilon}{\epsilon-1}\left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}}\right)^{\frac{1}{\epsilon-1}}\frac{\epsilon-1}{\epsilon}Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j)$$

$$0 = \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}}\right)^{\frac{1}{\epsilon-1}}Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t}$$

$$0 = \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}}\right)^{-\frac{\epsilon}{\epsilon-1}}Y_t(j) - \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon}$$

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon}Y_t.$$

Plugging this quantity into the identity

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j) \, dj$$

and simplifying yields the price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} \, dj \right)^{\frac{1}{1-\epsilon}}.$$

Intermediate Producers Intermediate goods are producing according to the Cobb-Douglas technology

$$Y_t(j) = A_t K_t^\alpha(j) L_t^{1-\alpha}(j).$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\min_{K_t(j), L_t(j)} R_{K,t} K_t(j) + W_t L_t(j) \qquad \text{s.t.} \qquad A_t K_t^\alpha(j) L_t^{1-\alpha}(j) \geq \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

The RHS of the inequality constraint is the demand from final goods producers for intermediate j. The Lagrangian is

$$\mathcal{L} = R_{K,t} K_t(j) + W_t L_t(j) + MC_t(j) \left(\left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - A_t K_t^\alpha(j) L_t^{1-\alpha}(j) \right),$$

so the first-order conditions are

$$0 = R_{K,t} - MC_t(j) \alpha A_t \left(\frac{L_t(j)}{K_t(j)} \right)^{1-\alpha}$$

$$0 = W_t - MC_t(j)(1-\alpha) A_t \left(\frac{K_t(j)}{L_t(j)} \right)^\alpha,$$

hence the optimal capital-labor ratio satisfies

$$\frac{R_{K,t}}{\alpha A_t (K_t(j)/L_t(j))^{\alpha-1}} = \frac{W_t}{(1-\alpha) A_t (K_t(j)/L_t(j))^\alpha}$$

$$\frac{K_t(j)}{L_t(j)} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}}.$$

Since the RHS does not vary with j, all firms choose the same capital-labor ratio. Given this optimal ratio, the marginal cost satisfies

$$MC_t = \frac{R_{K,t}}{\alpha A_t} \left(\frac{K_t}{L_t} \right)^{1-\alpha}$$

$$= \frac{R_{K,t}}{\alpha A_t} \left(\frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}} \right)^{1-\alpha}$$

$$= \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^\alpha \frac{W_t^{1-\alpha} R_{K,t}^\alpha}{A_t}.$$

It follows that

$$R_{K,t}K_t + W_tL_t = \left(\frac{R_{K,t}}{A_t}\left(\frac{K_t}{L_t}\right)^{1-\alpha} + \frac{W_t}{A_t}\left(\frac{L_t}{K_t}\right)^{\alpha}\right)(A_tK_t^{\alpha}L_t^{1-\alpha})$$

$$= (\alpha MC_t + (1-\alpha)MC_t)\,Y_t(j) = MC_tY_t(j).$$

Therefore, (real) profits for an intermediate producer become

$$F_t(j) = \frac{P_t(j)}{P_t}Y_t(j) - MC_tY_t(j).$$

In addition to the capital-labor choice, firms also have the chance to reset prices in every period with probability $1 - \theta$. This problem can be written as

$$\max_{P_t(j)}\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})}\frac{u'(C_{t+s})}{u'(C_t)}\left(\frac{P_t(j)}{P_{t+s}}\left(\frac{P_t(j)}{P_{t+s}}\right)^{-\epsilon}Y_{t+s} - mc_{t+s}\left(\frac{P_t(j)}{P_{t+s}}\right)^{-\epsilon}Y_{t+s}\right),$$

where I have imposed that intermediate output equals demand. The first-order condition is

$$0 = (1-\epsilon)P_t(j)^{-\epsilon}\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})}\frac{u'(C_{t+s})}{u'(C_t)}(P_{t+s})^{-(1-\epsilon)}Y_{t+s}$$

$$+ \epsilon P_t(j)^{-\epsilon-1}\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})}\frac{u'(C_{t+s})}{u'(C_t)}mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}$$

Divide by $P_t(j)^{-\epsilon}/(\exp(\eta_{\beta,t})u'(C_t))$, apply the abuse of notation that $\prod_{u=1}^{0}\Pi_{t+u} = 1$, and re-arrange to obtain

$$P_t(j) = \frac{\epsilon}{\epsilon - 1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})P_{t+s}^{\epsilon-1}Y_{t+s}}$$

$$= \frac{\epsilon}{\epsilon - 1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}P_t^{\epsilon}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})P_t^{\epsilon-1}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon-1}Y_{t+s}}$$

$$\frac{P_t(j)}{P_t} = \frac{\epsilon}{\epsilon - 1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon-1}Y_{t+s}}.$$

This expression gives the optimal (real) reset price $P_t^* \equiv P_t(j)/P_t$ (note that the RHS does not depend on j). Define

$$S_{1,t} = \mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}Y_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+s}\right)^{\epsilon},$$

$$S_{2,t} = \mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})Y_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+s}\right)^{\epsilon-1}.$$

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Using these definitions, I may write the optimal reset price more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{S_{1,t}}{S_{2,t}}$$

where $S_{1,t}$ and $S_{2,t}$ satisfy the recursions

$$S_{1,t} = \exp(\eta_{\beta,t})u'(C_t)MC_tY_t + \theta\beta\mathbb{E}_t\Pi_{t+s}^{\epsilon}S_{1,t+1}$$
$$S_{2,t} = \exp(\eta_{\beta,t})u'(C_t)Y_t + \theta\beta\mathbb{E}_t\Pi_{t+s}^{\epsilon-1}S_{2,t+1}.$$

These recursions can be further rewritten as

$$\frac{S_{1,t}}{\exp(\eta_{\beta,t})u'(C_t)} = MC_tY_t + \theta\beta\mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)}\Pi_{t+s}^{\epsilon}\frac{S_{1,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})}\right]$$
$$\frac{S_{2,t}}{\exp(\eta_{\beta,t})u'(C_t)} = Y_t + \theta\beta\mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)}\Pi_{t+s}^{\epsilon-1}\frac{S_{2,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})}\right],$$

By defining $\tilde{S}_{1,t} \equiv S_{1,t}/(\exp(\eta_{\beta,t})u'(C_t))$ and $\tilde{S}_{2,t} \equiv S_{2,t}/(\exp(\eta_{\beta,t})u'(C_t))$, I can simplify these recursions into the form I use for the numerical solution.

From this section, we obtain the following five equilibrium conditions:

$$MC_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{W_t^{1-\alpha}R_{K,t}^{\alpha}}{A_t}, \tag{9}$$

$$\frac{K_t}{L_t} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_{K,t}}, \tag{10}$$

$$P_t^* = \frac{\epsilon}{\epsilon-1}\frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, \tag{11}$$

$$\tilde{S}_{1,t} = MC_tY_t + \theta\beta\mathbb{E}_t[M_{t+1}\Pi_{t+1}^{\epsilon}\tilde{S}_{1,t+1}], \tag{12}$$

$$\tilde{S}_{2,t} = Y_t + \theta\beta\mathbb{E}_t[M_{t+1}\Pi_{t+1}^{\epsilon-1}\tilde{S}_{2,t+1}]. \tag{13}$$

1.3 Monetary Policy

To close the model, I specify the monetary policy rule as the following Taylor rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\phi_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y}\right)^{1-\phi_R}\exp(\eta_{R,t}) \tag{14}$$

Any proceeds from monetary policy are distributed as lump sum to the representative household.

1.4 Aggregation

The price level is currently characterized as the integral

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon}\, dj.$$

To represent the model entirely in terms of aggregates, notice that, without loss of generality, we may re-order the fraction ϕ of firms which cannot reset prices to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon}\, dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \phi \int_0^1 P_{t-1}(j)^{1-\epsilon}\, dj = (1-\phi)(P_t^*)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}.$$

Dividing by $P_{t-1}^{1-\epsilon}$ implies

$$\Pi_t^{1-\epsilon} = (1-\phi)(P_t^* \Pi_t)^{1-\epsilon} + \phi. \tag{15}$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$
\begin{aligned}
V_t^p &= \int_0^{1-\phi} (P_t^*)^{-\epsilon}\, dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\epsilon}\, dj \\
&= \int_0^{1-\phi} (P_t^* \Pi_t)^{-\epsilon} \left(\frac{1}{\Pi_t}\right)^{-\epsilon}\, dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon}\, dj \\
&= (1-\phi)(P_t^* \Pi_t)^{-\epsilon} \Pi_t^\epsilon + \Pi_t^\epsilon \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon}\, dj.
\end{aligned}
$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon}\, dj = \phi \int_0^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon}\, dj = \phi v_{t-1}.$$

Thus, we acquire

$$V_t^p = \Pi_t^\epsilon \left((1-\phi)(P_t^* \Pi_t)^\epsilon + \phi v_{t-1}\right) \tag{16}$$

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1.5 Equilibrium

To close the model, I need to specify the functional form for investment, aggregate shocks, and market-clearing conditions.

Following the literature, I assume quadratic adjustment investment costs so that

$$\Phi\left[\frac{X_t}{X_{t-1}}\right] = \frac{\chi}{2}\left(\frac{X_t}{X_{t-1}}\right)^2, \tag{17}$$

which has first derivative w.r.t. X_t/X_{t-1}

$$\Phi'\left[\frac{X_t}{X_{t-1}}\right] = \chi\left(\frac{X_t}{X_{t-1}}\right). \tag{18}$$

There are five shocks in the model: $\eta_{A,t}$, $\eta_{\beta,t}$, $\eta_{L,t}$, $\eta_{K,t}$, and $\eta_{R,t}$. Without loss of generality, I assume all shocks follow AR(1) processes with persistence ρ_i and standard deviation σ_i.

Markets must clear for capital, labor, bonds, final goods, and intermediate goods, . The first three markets clear as a consequence of optimality conditions and the assumption that bonds have zero net supply. To clear the market for final goods, we set the sum of aggregate consumption demand C_t and investment demand X_t equal to aggregate supply Y_t, which satisfies

$$\int_0^1 A_t K_t^\alpha L_t^{1-\alpha}\, dj = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t\, dj$$

$$A_t K_t^\alpha L_t^{1-\alpha} = Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj = V_t^p Y_t.$$

Re-arranging yields the output market-clearing condition

$$C_t + X_t = Y_t, \tag{19}$$

$$Y_t = \frac{A_t K_t^\alpha L_t^{1-\alpha}}{V_t^p}. \tag{20}$$

It can be shown that $V_t^p \geq 1$ by applying Jensen's inequality. For our purposes, because the dimensionality of our model is not too large, we add the auxiliary Y_t variable, even though we could substitute it out of the system of equations.

All together, the full set of endogenous equilibrium conditions are

$$C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^\nu, \tag{21}$$

$$M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, \tag{22}$$

$$1 = \beta \mathbb{E}_t \left[M_{t+1} \frac{R_t}{\Pi_{t+1}} \right], \tag{23}$$

$$1 = \exp(\eta_{K,t}) Q_t \left(1 - \frac{3\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2 \right) + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \chi \left(\frac{X_{t+1}}{X_t} \right)^3 \right], \tag{24}$$

$$Q_t = \mathbb{E}_t [M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))], \tag{25}$$

$$MC_t = \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^\alpha \frac{W_t^{1-\alpha} R_{K,t}^\alpha}{A_t}, \tag{26}$$

$$\frac{K_t}{L_t} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}}, \tag{27}$$

$$P_t^* = \frac{\epsilon}{\epsilon-1} \frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, \tag{28}$$

$$\tilde{S}_{1,t} = MC_t Y_t + \theta\beta \mathbb{E}_t [M_{t+1} \Pi_{t+1}^\epsilon \tilde{S}_{1,t+1}], \tag{29}$$

$$\tilde{S}_{2,t} = Y_t + \theta\beta \mathbb{E}_t [M_{t+1} \Pi_{t+1}^{\epsilon-1} \tilde{S}_{2,t+1}], \tag{30}$$

$$\Pi_t^{1-\epsilon} = (1-\phi)(P_t^* \Pi_t)^{1-\epsilon} + \phi, \tag{31}$$

$$V_t^p = \Pi_t^\epsilon ((1-\phi)(P_t^* \Pi_t)^\epsilon + \phi v_{t-1}), \tag{32}$$

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\phi_R} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_{t-1}} \right)^{\phi_y} \right)^{1-\phi_R} \exp(\eta_{R,t}), \tag{33}$$

$$C_t + X_t = Y_t, \tag{34}$$

$$Y_t = \frac{A_t K_t^\alpha L_t^{1-\alpha}}{V_t^p}, \tag{35}$$

as well as the law of motion for capital

$$K_{t+1} = (1-\delta)K_t + \exp(\eta_{K,t}) \left(1 - \frac{\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2 \right) X_t \tag{36}$$

and the five exogenous processes

$$\eta_{\beta,t+1} = \rho_\beta \eta_{\beta,t} + \sigma_\beta \varepsilon_{\beta,t+1}, \tag{37}$$

$$\eta_{L,t+1} = \rho_L \eta_{L,t} + \sigma_L \varepsilon_{L,t+1}, \tag{38}$$

$$\eta_{A,t+1} = \rho_A \eta_{A,t} + \sigma_A \varepsilon_{A,t+1}, \tag{39}$$

$$\eta_{K,t+1} = \rho_K \eta_{K,t} + \sigma_K \varepsilon_{K,t+1}, \tag{40}$$

$$\eta_{R,t+1} = \rho_R \eta_{R,t} + \sigma_R \varepsilon_{R,t+1}. \tag{41}$$

2 Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$0 = \log \mathbb{E}_t \left[\exp \left(\xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1} \right) \right]$$

$$z_{t+1} = \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t)\varepsilon_{t+1},$$

where z_t are (predetermined) state variables and y_t are (nondetermined) jump variables. For the remainder of this section, lower case variables are the logs of previously upper case variables, and with a small abuse of notation, let $s_{1,t} = \log(\tilde{S}_{1,t})$ and $s_{2,t} = \log(\tilde{S}_{2,t})$. Additionally, let $r_{k,t} = \log(R_{K,t})$ and $v_t = \log(V_t^p)$.

Equation (21) becomes

$$1 = \varphi \exp(\eta_{L,t}) \frac{L_t^\nu}{C_t^{-\gamma} W_t}$$

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\varphi) + \eta_{L,t} + \nu l_t - (-\gamma c_t + w_t)}_{\xi} \right) \right].$$

Equation (22) will not be used in the system of equations for the risk-adjusted linearization, but it simplifies the other equations. Taking logs and re-arranging yields

$$0 = \log(\beta) + \eta_{\beta,t+1} + (-\gamma c_{t+1}) - \eta_{\beta,t} - (-\gamma c_t) - m_{t+1}$$

$$m_{t+1} = \underbrace{\log(\beta) - \eta_{\beta,t} + \gamma c_t}_{\xi} + \underbrace{\eta_{\beta,t+1} - \gamma c_{t+1}}_{\text{forward-looking}}.$$

Equation (23) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\beta) + r_t}_{\xi} + \underbrace{m_{t+1} - \pi_{t+1}}_{\text{forward-looking}} \right) \right].$$

Equation (24) becomes

$$1 - \exp(\eta_{K,t}) Q_t \left(1 - \frac{3\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2 \right)$$

$$= \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \chi \left(\frac{X_{t+1}}{X_t} \right)^3 \right]$$

$$\log \left(1 - \exp(\eta_{K,t} + q_t) \left(1 - \frac{3\chi}{2} \exp(2x_t - 2x_{t-1}) \right) \right)$$

$$= \log \mathbb{E}_t \left[\exp \left(m_{t+1} + \eta_{K,t+1} + q_{t+1} + \log(\chi) + 3x_{t+1} - 3x_t \right) \right]$$

$$0 = \log \mathbb{E}_t \left[\frac{\exp \left(m_{t+1} + \eta_{K,t+1} + q_{t+1} + \log(\chi) + 3x_{t+1} - 3x_t \right)}{\exp \left(\log \left(1 - \exp(\eta_{K,t} + q_t) \left(1 - \frac{3\chi}{2} \exp(2x_t - 2x_{t-1}) \right) \right) \right)} \right].$$

Re-arranging yields

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\chi) - 3x_t - \log \left(1 - \exp(\eta_{K,t} + q_t) \left(1 - \frac{3\chi}{2} \exp(2x_t - 2x_{t-1}) \right) \right)}_{\xi} \right) \right]$$

$$+ \log \mathbb{E}_t \left[\exp(\underbrace{m_{t+1} + \eta_{K,t+1} + q_{t+1} + 3x_{t+1}}_{\text{forward-looking}}) \right].$$

Equation (25) becomes

$$1 = \mathbb{E}_t \left[\frac{M_{t+1}(R_{K,t+1} + Q_{t+1}(1 - \delta))}{Q_t} \right]$$

$$0 = \log \mathbb{E}_t \left[\exp \left(m_{t+1} + \log(\exp(r_{k,t+1}) + (1 - \delta) \exp(q_{t+1})) - q_t \right) \right].$$

This equation is not linear in the forward-looking variables, we define a new variable

$$\Omega_t = R_{K,t} + Q_t(1 - \delta), \tag{42}$$

so that we may represent (25) in the required form for risk-adjusted linearizations as the following two equations:

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{-q_t}_{\xi} + \underbrace{m_{t+1} + \omega_{t+1}}_{\text{forward-looking}} \right) \right],$$

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\omega_t - \log(\exp(r_{k,t}) + (1 - \delta) \exp(q_t))}_{\xi} \right) \right],$$

Equation (26) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{(1 - \alpha)w_t + \alpha r_{k,t} - a_t - (1 - \alpha) \log(1 - \alpha) - \alpha \log(\alpha) - mc_t}_{\xi} \right) \right].$$

Equation (27) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{k_t - l_t - \log \left(\frac{\alpha}{1 - \alpha} \right) - (w_t - r_{k,t})}_{\xi} \right) \right].$$

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Like the stochastic discount factor, equation (28) will not be used in the system of equations, but it will be useful to simplify other equations. Taking logs yields

$$p_t^* = \log\left(\frac{\epsilon}{\epsilon - 1}\right) + s_{1,t} - s_{2,t}.$$

Equation (29) becomes

$$\tilde{S}_{1,t} - MC_t Y_t$$
$$= \mathbb{E}_t \left[\exp\left(\log(\theta) + \log(\beta) + m_{t+1} + \epsilon\pi_{t+1} + s_{1,t+1}\right)\right]$$

$$0 = \log \mathbb{E}_t \left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - \log(\exp(s_{1,t}) - \exp(mc_t)\exp(y_t))}_{\xi} + \underbrace{m_{t+1} + \epsilon\pi_{t+1} + s_{1,t+1}}_{\text{forward-looking}}\right)\right]$$

and equation (30) becomes

$$0 = \log \mathbb{E}_t \left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - \log(\exp(s_{2,t}) - \exp(y_t))}_{\xi} + \underbrace{m_{t+1} + (\epsilon - 1)\pi_{t+1} + s_{2,t+1}}_{\text{forward-looking}}\right)\right].$$

Equation (31) becomes

$$1 = \frac{\Pi_t^{1-\epsilon}}{(1 - \phi)(P_t^*\Pi_t)^{1-\epsilon} + \phi}$$
$$0 = \log \mathbb{E}_t \left[\exp\left((1 - \epsilon)\pi_t - \log((1 - \phi)\exp((1 - \epsilon)(p_t^* + \pi_t)) + \phi)\right)\right].$$

Equation (32) becomes

$$1 = \frac{V_t^p}{\Pi_t^\epsilon((1 - \phi)(P_t^*\Pi_t)^{1-\epsilon} + \phi V_{t-1}^p)}$$
$$0 = \log \mathbb{E}_t \left[\exp\left(v_t - \log((1 - \phi)\exp(\epsilon(p_t^* + \pi_t)) + \phi\exp(v_{t-1}))\right)\right].$$

Equation (33) becomes

$$0 = \log \mathbb{E}_t \left[\exp\left(\phi_R r_{t-1} + (1 - \phi_R)r + (1 - \phi_R)(\phi_\pi(\pi_t - \pi) + \phi_y(y_t - y_{t-1})) + \eta_{R,t} - r_t\right)\right].$$

Equation (34) becomes

$$0 = \log \mathbb{E}_t \left[\exp(y_t - \log(\exp(c_t) + \exp(x_t)))\right].$$

Equation (35) becomes

$$0 = \log \mathbb{E}_t \left[\exp(a_t + \alpha k_t + (1 - \alpha)l_t - v_t - y_t)\right].$$

Equation (36) becomes

$$k_{t+1} = \log\left((1-\delta)\exp(k_t) + \exp(\eta_{K,t})\left(1 - \frac{\chi}{2}\exp(2(x_t - x_{t-1}))\right)\exp(x_t)\right).$$

The autoregressive processes (37) to (41) remain as they are.

The jump variables are y_t, c_t, l_t, w_t, r_t, π_t, q_t, x_t, $r_{k,t}$, ω_t, mc_t, $s_{1,t}$, $s_{2,t}$, and v_t. The state variables are k_t, x_{t-1}, v_{t-1}, r_{t-1}, y_{t-1}, and the autoregressive processes. The equations defining the evolution of the lags x_{t-1}, v_{t-1}, r_{t-1}, and y_{t-1} are obtained by the formula $z_{(t-1)+1} = z_t$.

This system has three forward difference equations (25), (29), and (30). To ensure accuracy of the risk-adjusted linearization, I derive N-period ahead forward difference equations for all three.

We can write (25) recursively as

$$\begin{aligned}
Q_t &= \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))] \\
&= \mathbb{E}_t[M_{t+1}R_{K,t+1} + (1-\delta)M_{t+1}\mathbb{E}_{t+1}[M_{t+2}(R_{K,t+2} + Q_{t+2}(1-\delta))]] \\
&= \mathbb{E}_t[M_{t+1}R_{K,t+1}] + (1-\delta)\mathbb{E}_t\mathbb{E}_{t+1}[M_{t+1}M_{t+2}(R_{K,t+2} + Q_{t+2}(1-\delta))].
\end{aligned}$$

By the tower property,

$$\begin{aligned}
Q_t &= \mathbb{E}_t[M_{t+1}R_{K,t+1}] + (1-\delta)\mathbb{E}_t[M_{t+1}M_{t+2}(R_{K,t+2} + Q_{t+2}(1-\delta))] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{2}(1-\delta)^{s-1}\left(\prod_{u=1}^{s}M_{t+s}\right)R_{K,t+s}\right) + M_{t+1}M_{t+2}Q_{t+2}(1-\delta)^2\right] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{2}(1-\delta)^{s-1}\left(\prod_{u=1}^{s}M_{t+s}\right)R_{K,t+s}\right) + M_{t+1}M_{t+2}(1-\delta)^2\mathbb{E}_{t+2}[M_{t+3}(R_{K,t+3} + Q_{t+3}(1-\delta))]\right] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{3}(1-\delta)^{s-1}\left(\prod_{u=1}^{s}M_{t+s}\right)R_{K,t+s}\right) + \prod_{s=1}^{3}((1-\delta)M_{t+s})Q_{t+3}\right]
\end{aligned}$$

and so on. Given this recursive structure, define $D_{Q,t}^{(n)}$ and $P_{Q,t}^{(n)}$ as

$$D_{Q,t}^{(n)} = \mathbb{E}_t\left[(1-\delta)M_{t+1}D_{Q,t+1}^{(n-1)}\right]$$

$$P_{Q,t}^{(n)} = \mathbb{E}_t\left[(1-\delta)M_{t+1}P_{Q,t+1}^{(n-1)}\right]$$

with boundary conditions

$$D_{Q,t}^{(0)} = \frac{R_{K,t+1}}{1-\delta}$$

$$P_{Q,t}^{(0)} = Q_{t+1}.$$

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Then I may write the N-period ahead recursive form of equation (25) as

$$Q_t = \sum_{n=1}^{N} D_{Q,t}^{(n)} + P_{Q,t}^{(N)}.$$

To see why this recursion works, it is simpler to first verify that $P_{Q,t}^{(3)}$ is correct:

$$
\begin{aligned}
P_{Q,t}^{(1)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}Q_{t+1}\right] \\
P_{Q,t}^{(2)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}(\mathbb{E}_{t+1}[(1-\delta)M_{t+2}Q_{t+2}])\right] \\
&= \mathbb{E}_t\left[\mathbb{E}_{t+1}[(1-\delta)^2 M_{t+1}M_{t+2}Q_{t+2}]\right] \\
&= \mathbb{E}_t\left[\prod_{s=1}^{2}((1-\delta)M_{t+s})Q_{t+2}\right],
\end{aligned}
$$

where the second equality for $P_{Q,t}^{(2)}$ follows from the fact that M_{t+1} is measurable with respect to the information set at time $t+1$ and can therefore be moved inside the conditional expectation $\mathbb{E}_{t+1}[\cdot]$. Continuing for one more recursion, I have

$$
\begin{aligned}
P_{Q,t}^{(3)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}\mathbb{E}_{t+1}\left[\prod_{s=1}^{2}((1-\delta)M_{t+1+s})Q_{t+3}\right]\right] \\
&= \mathbb{E}_t\left[\prod_{s=1}^{3}((1-\delta)M_{t+s})Q_{t+3}\right].
\end{aligned}
$$

Similarly, for $D_{Q,t}$, I have

$$
\begin{aligned}
D_{Q,t}^{(1)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}\frac{R_{K,t+1}}{1-\delta}\right] = \mathbb{E}_t[M_{t+1}R_{K,t+1}] \\
D_{Q,t}^{(2)} &= \mathbb{E}_t[(1-\delta)M_{t+1}\mathbb{E}_{t+1}[M_{t+2}R_{K,t+2}]] \\
&= \mathbb{E}_t[(1-\delta)M_{t+1}M_{t+2}R_{K,t+2}] \\
D_{Q,t}^{(3)} &= \mathbb{E}_t[(1-\delta)M_{t+1}\mathbb{E}_{t+1}[(1-\delta)M_{t+2}M_{t+3}R_{K,t+3}]] \\
&= \mathbb{E}_t[(1-\delta)^2 M_{t+1}M_{t+2}M_{t+3}R_{K,t+3}].
\end{aligned}
$$

Since $P_{Q,t}^{(n)}$ and $D_{Q,t}^{(n)}$ are time-t conditional expectations, they are measurable at time t, so they are not forward-looking variables. Thus, to get this version of (25) in the appropriate

form, define $d_{q,n,t} = \log(D_{Q,t}^{(n)})$ and $p_{q,n,t} = \log(P_{Q,t}^{(n)})$, and use the following $2N+1$ equations:

$$0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{q_t - \log\left(\sum_{n=1}^{N} \exp(d_{q,n,t}) + \exp(p_{q,n,t})\right)}_{\xi}\right)\right] \tag{43}$$

$$0 = \begin{cases} \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(1-\delta) - d_{q,n,t}}_{\xi} + \underbrace{m_{t+1} + d_{q,n-1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n > 1 \\[2em] \log \mathbb{E}_t\left[\exp\left(\underbrace{-d_{q,1,t}}_{\xi} + \underbrace{m_{t+1} + r_{k,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n = 1, \end{cases} \tag{44}$$

$$0 = \begin{cases} \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(1-\delta) - p_{q,n,t}}_{\xi} + \underbrace{m_{t+1} + p_{q,n-1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n > 1 \\[2em] \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(1-\delta) - p_{q,1,t}}_{\xi} + \underbrace{m_{t+1} + q_{t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n = 1. \end{cases} \tag{45}$$

For (29), observe that

$$\tilde{S}_{1,t} = MC_t Y_t + \theta\beta\mathbb{E}_t[M_{t+1}\Pi_{t+1}^{\epsilon}(MC_{t+1}Y_{t+1} + \theta\beta\mathbb{E}_{t+1}[M_{t+2}\Pi_{t+2}^{\epsilon}\tilde{S}_{1,t+2}])]$$
$$= MC_t Y_t + \theta\beta\mathbb{E}_t[M_{t+1}\Pi_{t+1}^{\epsilon}MC_{t+1}Y_{t+1} + \theta\beta M_{t+1}\Pi_{t+1}^{\epsilon}M_{t+2}\Pi_{t+2}^{\epsilon}\tilde{S}_{1,t+2}]$$
$$= MC_t Y_t + \mathbb{E}_t\left[\sum_{s=1}^{1}((\theta\beta)^s\prod_{u=1}^{s}(M_{t+u}\Pi_{t+u}^{\epsilon}))MC_{t+s}Y_{t+s}\right] + \mathbb{E}_t\left[\prod_{s=1}^{2}(\theta\beta M_{t+s}\Pi_{t+s}^{\epsilon})\tilde{S}_{1,t+2}\right].$$

Thus, define $D_{S1,t}^{(n)}$ and $P_{S1,t}^{(n)}$ as the recursions

$$D_{S1,t}^{(n)} = \mathbb{E}_t[\theta\beta M_{t+1}\Pi_{t+1}^{\epsilon}D_{S1,t+1}^{(n-1)}],$$
$$P_{S1,t}^{(n)} = \mathbb{E}_t[\theta\beta M_{t+1}\Pi_{t+1}^{\epsilon}P_{S1,t+1}^{(n-1)}],$$

with boundary conditions

$$D_{S1,t}^{(0)} = MC_t Y_t$$
$$P_{S1,t}^{(0)} = \tilde{S}_{1,t}.$$

Given these definitions, it follows that

$$D_{S1,t}^{(1)} = \mathbb{E}_t[\theta\beta M_{t+1}\Pi_{t+1}^\epsilon MC_{t+1}Y_{t+1}]$$
$$P_{S1,t}^{(1)} = \mathbb{E}_t[\theta\beta M_{t+1}\Pi_{t+1}^\epsilon \tilde{S}_{1,t+1}]$$
$$P_{S1,t}^{(2)} = \mathbb{E}_t[\theta\beta M_{t+1}\Pi_{t+1}^\epsilon \mathbb{E}_{t+1}[\theta\beta M_{t+2}\Pi_{t+2}^\epsilon \tilde{S}_{1,t+2}]]$$
$$= \mathbb{E}_t[(\theta\beta)^2 M_{t+1}\Pi_{t+1}^\epsilon M_{t+2}\Pi_{t+2}^\epsilon \tilde{S}_{1,t+2}].$$

Thus, defining $d_{s1,t} = \log(D_{S1,t})$ and $p_{s1,t} = \log(P_{S1,t})$, the N-period ahead recursive form of (29) results in the $2N+1$ equations

$$0 = \log\mathbb{E}_t\left[\exp\left(\underbrace{s_{1,t} - \log\left(\sum_{n=0}^{N-1}\exp(d_{s1,n,t}) + \exp(p_{s1,n,t})\right)}_{\xi}\right)\right] \tag{46}$$

$$0 = \begin{cases} \log\mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - d_{s1,n,t}}_{\xi} + \underbrace{m_{t+1} + \epsilon\pi_{t+1} + d_{s1,n-1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n \geq 1 \\ \log\mathbb{E}_t\left[\exp\left(\underbrace{d_{s1,0,t} - mc_t - y_t}_{\xi}\right)\right] & \text{if } n = 0. \end{cases} \tag{47}$$

$$0 = \begin{cases} \log\mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - p_{s1,n,t}}_{\xi} + \underbrace{m_{t+1} + \epsilon\pi_{t+1} + p_{s1,n-1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n > 1 \\ \log\mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - p_{s1,1,t}}_{\xi} + \underbrace{m_{t+1} + s_{1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n = 1. \end{cases} \tag{48}$$

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It is straightforward to show that a similar recursive form applies to (30):

$$
0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{ s_{2,t} - \log \left(\sum_{n=0}^{N-1} \exp(d_{s2,n,t}) + \exp(p_{s2,n,t}) \right) }_{\xi} \right) \right] \tag{49}
$$

$$
0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{ \log(\theta) + \log(\beta) - d_{s2,n,t} }_{\xi} + \underbrace{ m_{t+1} + (\epsilon - 1)\pi_{t+1} + d_{s2,n-1,t+1} }_{\text{forward-looking}} \right) \right] & \text{if } n \geq 1 \\[2em] \log \mathbb{E}_t \left[\exp \left(\underbrace{ d_{s2,0,t} - y_t }_{\xi} \right) \right] & \text{if } n = 0. \end{cases}
$$

$$\tag{50}$$

$$
0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{ \log(\theta) + \log(\beta) - p_{s2,n,t} }_{\xi} + \underbrace{ m_{t+1} + (\epsilon - 1)\pi_{t+1} + p_{s2,n-1,t+1} }_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\[2em] \log \mathbb{E}_t \left[\exp \left(\underbrace{ \log(\theta) + \log(\beta) - p_{s2,1,t} }_{\xi} + \underbrace{ m_{t+1} + s_{2,t+1} }_{\text{forward-looking}} \right) \right] & \text{if } n = 1, \end{cases}
$$

$$\tag{51}$$

where terms and boundary conditions are analogously defined.