

These notes loosely follow Fernández-Villaverde and Levintal (2018) “Solution Methods for Models with Rare Disasters” but omits several features, such as recursive preferences and disaster risk.

1 Model

1.1 Household

The model admits a representative agent, so I directly write households’ problem as the representative agent’s. The representative household solves, in the cashless limit,

$$\max_{C_t, L_t, B_{t+1}, X_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \exp(\eta_{L,t}) \frac{L_t^{1+\nu}}{1+\nu} \right) \quad (1)$$

subject to the budget constraint

$$C_t + \frac{B_{t+1}}{P_t} + X_t \leq W_t L_t + R_{K,t} K_t + R_{t-1} \frac{B_t}{P_t} + F_t + T_t, \quad (2)$$

where C_t is consumption, B_t nominal bonds, X_t investment, $W_{N,t}$ the real wage, L_t labor, $R_{KN,t}$ the gross real rental rate on capital, K_t capital, R_t the gross nominal interest rate on bonds, F_t real profits from firms, and T_t real lump-sum transfers from the government. The price of the final consumption good is P_t . Markets are assumed complete, but securities are in zero net supply. Because there is a representative agent, I may omit the Arrow securities from the budget constraint. Our notation treats B_t and K_t as pre-determined at time t . Investment for capital follows the law of motion

$$K_{t+1} = (1 - \delta) K_t + \exp(\eta_{K,t}) \left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right] \right) X_t, \quad (3)$$

where the investment function takes the functional form

$$\Phi \left[\frac{X_t}{X_{t-1}} \right] = \frac{\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2.$$

The Lagrangian for the household is

$$\begin{aligned} \mathcal{L} = & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \exp(\eta_{L,t}) \frac{L_t^{1+\nu}}{1+\nu} \right] \\ & + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \lambda_t \left[W_t L_t + R_{K,t} K_t + R_{t-1} \frac{B_t}{P_t} + F_t + T_t - C_t - \frac{B_{t+1}}{P_t} - X_t \right] \\ & + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \lambda_t Q_t \left[(1 - \delta) K_t + \exp(\eta_{K,t}) \left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right] \right) X_t - K_{t+1} \right], \end{aligned}$$

which implies first-order conditions

$$\begin{aligned}
0 &= C_t^{-\gamma} - \lambda_t \\
0 &= -\varphi \exp(\eta_{L,t}) L_t^\nu + \lambda_t W_t \\
0 &= -\exp(\eta_{\beta,t}) \lambda_t + \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1}) \lambda_{t+1} R_t] \\
0 &= -\exp(\eta_{\beta,t}) \lambda_t + \exp(\eta_{\beta,t}) \lambda_t Q_t \exp(\eta_{K,t}) \left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right]\right) \\
&\quad - \exp(\eta_{\beta,t}) \lambda_t Q_t \exp(\eta_{K,t}) \Phi' \left[\frac{X_t}{X_{t-1}} \right] \frac{X_t}{X_{t-1}} \\
&\quad - \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1}) \lambda_{t+1} Q_{t+1} \exp(\eta_{K,t+1}) \Phi' \left[\frac{X_{t+1}}{X_t} \right] \left(-\frac{X_{t+1}}{X_t^2} \right) X_{t+1}] \\
0 &= \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1}) \lambda_{t+1} (R_{K,t+1} + Q_{t+1} (1 - \delta))] - \exp(\eta_{\beta,t}) \lambda_t Q_t
\end{aligned}$$

The first two equations can be combined by isolating λ_t , which obtains the intratemporal consumption-labor condition

$$C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^\nu,$$

Using $\lambda_t = C_t^{-\gamma}$ and defining the gross inflation rate $\Pi_t \equiv P_t/P_{t-1}$, I can obtain the Euler equation for households

$$\begin{aligned}
\exp(\eta_{\beta,t}) \frac{C_t^{-\gamma}}{P_t} &= \beta \mathbb{E}_t \left[\exp(\eta_{\beta,t+1}) \frac{C_{t+1}^{-\gamma}}{P_{t+1}} R_t \right] \\
1 &= \beta \mathbb{E}_t \left[\frac{\exp(\eta_{\beta,t+1}) \frac{C_{t+1}^{-\gamma}}{P_{t+1}} R_t}{\exp(\eta_{\beta,t}) \frac{C_t^{-\gamma}}{P_t}} \right].
\end{aligned}$$

I can further simplify the Euler equation by defining the (real) stochastic discount factor

$$M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1}) \frac{C_{t+1}^{-\gamma}}{P_{t+1}}}{\exp(\eta_{\beta,t}) \frac{C_t^{-\gamma}}{P_t}}$$

After dividing through by $\exp(\eta_{\beta,t}) \lambda_t$ and re-arranging, the investment condition becomes

$$\begin{aligned}
1 &= \exp(\eta_{K,t}) Q_t \left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right]\right) - \exp(\eta_{K,t}) Q_t \Phi' \left[\frac{X_t}{X_{t-1}} \right] \frac{X_t}{X_{t-1}} \\
&\quad + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \Phi' \left[\frac{X_{t+1}}{X_t} \right] \frac{X_{t+1}^2}{X_t^2} \right] \\
&= \exp(\eta_{K,t}) Q_t \left(\left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right]\right) - \Phi' \left[\frac{X_t}{X_{t-1}} \right] \frac{X_t}{X_{t-1}} \right) \\
&\quad + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \Phi' \left[\frac{X_{t+1}}{X_t} \right] \frac{X_{t+1}^2}{X_t^2} \right].
\end{aligned}$$

Finally, after dividing through by $\exp(\eta_{\beta,t})\lambda_t$, the first-order condition for next-period capital is

$$Q_t = \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1 - \delta))]$$

In summary, households' optimality conditions are

$$C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^\nu, \quad (4)$$

$$M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1}) C_{t+1}^{-\gamma}}{\exp(\eta_{\beta,t}) C_t^{-\gamma}}, \quad (5)$$

$$1 = \beta \mathbb{E}_t \left[M_{t+1} \frac{R_t}{\Pi_{t+1}} \right], \quad (6)$$

$$1 = \exp(\eta_{K,t}) Q_t \left(\left(1 - \Phi \left[\frac{X_t}{X_{t-1}} \right] \right) - \Phi' \left[\frac{X_t}{X_{t-1}} \right] \frac{X_t}{X_{t-1}} \right) \\ + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \Phi' \left[\frac{X_{t+1}}{X_t} \right] \frac{X_{t+1}^2}{X_t^2} \right], \quad (7)$$

$$Q_t = \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1 - \delta))]. \quad (8)$$

1.2 Production

Final Producers There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 1$ so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj.$$

The FOC for $Y_t(j)$ is

$$0 = P_t \frac{\epsilon}{\epsilon-1} \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{1}{\epsilon-1}} \frac{\epsilon-1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) \\ 0 = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t} \\ 0 = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} \\ Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

Plugging this quantity into the identity

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j) dj$$

and simplifying yields the price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}.$$

Intermediate Producers Intermediate goods are produced according to the Cobb-Douglas technology

$$Y_t(j) = A_t K_t^\alpha(j) L_t^{1-\alpha}(j).$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\min_{K_t(j), L_t(j)} R_{K,t} K_t(j) + W_t L_t(j) \quad \text{s.t.} \quad A_t K_t^\alpha(j) L_t^{1-\alpha}(j) \geq \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

The RHS of the inequality constraint is the demand from final goods producers for intermediate j . The Lagrangian is

$$\mathcal{L} = R_{K,t} K_t(j) + W_t L_t(j) + MC_t(j) \left(\left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - A_t K_t^\alpha(j) L_t^{1-\alpha}(j) \right),$$

so the first-order conditions are

$$\begin{aligned} 0 &= R_{K,t} - MC_t(j) \alpha A_t \left(\frac{L_t(j)}{K_t(j)} \right)^{1-\alpha} \\ 0 &= W_t - MC_t(j) (1-\alpha) A_t \left(\frac{K_t(j)}{L_t(j)} \right)^\alpha, \end{aligned}$$

hence the optimal capital-labor ratio satisfies

$$\begin{aligned} \frac{R_{K,t}}{\alpha A_t (K_t(j)/L_t(j))^{\alpha-1}} &= \frac{W_t}{(1-\alpha) A_t (K_t(j)/L_t(j))^\alpha} \\ \frac{K_t(j)}{L_t(j)} &= \frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}}. \end{aligned}$$

Since the RHS does not vary with j , all firms choose the same capital-labor ratio. Given this optimal ratio, the marginal cost satisfies

$$\begin{aligned} MC_t &= \frac{R_{K,t}}{\alpha A_t} \left(\frac{K_t}{L_t} \right)^{1-\alpha} \\ &= \frac{R_{K,t}}{\alpha A_t} \left(\frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}} \right)^{1-\alpha} \\ &= \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^\alpha \frac{W_t^{1-\alpha} R_{K,t}^\alpha}{A_t}. \end{aligned}$$

It follows that

$$\begin{aligned} R_{K,t}K_t + W_tL_t &= \left(\frac{R_{K,t}}{A_t} \left(\frac{K_t}{L_t} \right)^{1-\alpha} + \frac{W_t}{A_t} \left(\frac{L_t}{K_t} \right)^\alpha \right) (A_t K_t^\alpha L_t^{1-\alpha}) \\ &= (\alpha MC_t + (1-\alpha)MC_t) Y_t(j) = MC_t Y_t(j). \end{aligned}$$

Therefore, (real) profits for an intermediate producer become

$$F_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - MC_t Y_t(j).$$

In addition to the capital-labor choice, firms also have the chance to reset prices in every period with probability $1 - \theta$. This problem can be written as

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} \left(\frac{P_t(j)}{P_{t+s}} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} \right),$$

where I have imposed that intermediate output equals demand. The first-order condition is

$$\begin{aligned} 0 &= (1 - \epsilon) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} (P_{t+s})^{-(1-\epsilon)} Y_{t+s} \\ &\quad + \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} mc_{t+s} P_{t+s}^\epsilon Y_{t+s} \end{aligned}$$

Divide by $P_t(j)^{-\epsilon}/(\exp(\eta_{\beta,t})u'(C_t))$, apply the abuse of notation that $\prod_{u=1}^0 \Pi_{t+u} = 1$, and re-arrange to obtain

$$\begin{aligned} P_t(j) &= \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) mc_{t+s} P_{t+s}^\epsilon Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) P_{t+s}^{\epsilon-1} Y_{t+s}} \\ &= \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) mc_{t+s} P_t^\epsilon (\prod_{u=1}^s \Pi_{t+u})^\epsilon Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) P_t^{\epsilon-1} (\prod_{u=1}^s \Pi_{t+u})^{\epsilon-1} Y_{t+s}} \\ \frac{P_t(j)}{P_t} &= \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) mc_{t+s} (\prod_{u=1}^s \Pi_{t+u})^\epsilon Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) (\prod_{u=1}^s \Pi_{t+u})^{\epsilon-1} Y_{t+s}}. \end{aligned}$$

This expression gives the optimal (real) reset price $P_t^* \equiv P_t(j)/P_t$ (note that the RHS does not depend on j). Define

$$\begin{aligned} S_{1,t} &= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) mc_{t+s} Y_{t+s} \left(\prod_{u=1}^s \Pi_{t+u} \right)^\epsilon, \\ S_{2,t} &= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) Y_{t+s} \left(\prod_{u=1}^s \Pi_{t+u} \right)^{\epsilon-1}. \end{aligned}$$

Using these definitions, I may write the optimal reset price more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{S_{1,t}}{S_{2,t}}$$

where $S_{1,t}$ and $S_{2,t}$ satisfy the recursions

$$\begin{aligned} S_{1,t} &= \exp(\eta_{\beta,t})u'(C_t)MC_tY_t + \theta\beta\mathbb{E}_t\Pi_{t+s}^\epsilon S_{1,t+1} \\ S_{2,t} &= \exp(\eta_{\beta,t})u'(C_t)Y_t + \theta\beta\mathbb{E}_t\Pi_{t+s}^{\epsilon-1}S_{2,t+1}. \end{aligned}$$

These recursions can be further rewritten as

$$\begin{aligned} \frac{S_{1,t}}{\exp(\eta_{\beta,t})u'(C_t)} &= MC_tY_t + \theta\beta\mathbb{E}_t \left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)} \Pi_{t+s}^\epsilon \frac{S_{1,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})} \right] \\ \frac{S_{2,t}}{\exp(\eta_{\beta,t})u'(C_t)} &= Y_t + \theta\beta\mathbb{E}_t \left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)} \Pi_{t+s}^{\epsilon-1} \frac{S_{2,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})} \right], \end{aligned}$$

By defining $\tilde{S}_{1,t} \equiv S_{1,t}/(\exp(\eta_{\beta,t})u'(C_t))$ and $\tilde{S}_{2,t} \equiv S_{2,t}/(\exp(\eta_{\beta,t})u'(C_t))$, I can simplify these recursions into the form I use for the numerical solution.

From this section, we obtain the following five equilibrium conditions:

$$MC_t = \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^\alpha \frac{W_t^{1-\alpha} R_{K,t}^\alpha}{A_t}, \quad (9)$$

$$\frac{K_t}{L_t} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}}, \quad (10)$$

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, \quad (11)$$

$$\tilde{S}_{1,t} = MC_tY_t + \theta\mathbb{E}_t[M_{t+1}\Pi_{t+1}^\epsilon\tilde{S}_{1,t+1}], \quad (12)$$

$$\tilde{S}_{2,t} = Y_t + \theta\mathbb{E}_t[M_{t+1}\Pi_{t+1}^{\epsilon-1}\tilde{S}_{2,t+1}]. \quad (13)$$

1.3 Monetary Policy

To close the model, I specify the monetary policy rule as the following Taylor rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\phi_R} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_{t-1}} \right)^{\phi_y} \right)^{1-\phi_R} \exp(\eta_{R,t}) \quad (14)$$

Any proceeds from monetary policy are distributed as lump sum to the representative household.

1.4 Aggregation

The price level is currently characterized as the integral

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj.$$

To represent the model entirely in terms of aggregates, notice that, without loss of generality, we may re-order the fraction θ of firms which cannot reset prices to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\theta)(P_t^*)^{1-\epsilon} + \int_{1-\theta}^1 P_{t-1}(j)^{1-\epsilon} dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\theta)(P_t^*)^{1-\epsilon} + \theta \int_0^1 P_{t-1}(j)^{1-\epsilon} dj = (1-\theta)(P_t^*)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon}.$$

Dividing by $P_{t-1}^{1-\epsilon}$ implies

$$\Pi_t^{1-\epsilon} = (1-\theta)(P_t^* \Pi_t)^{1-\epsilon} + \theta. \quad (15)$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$\begin{aligned} V_t^p &= \int_0^{1-\theta} (P_t^*)^{-\epsilon} dj + \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_t} \right)^{-\epsilon} dj \\ &= \int_0^{1-\theta} (P_t^* \Pi_t)^{-\epsilon} \left(\frac{1}{\Pi_t} \right)^{-\epsilon} dj + \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t} \right)^{-\epsilon} dj \\ &= (1-\theta)(P_t^* \Pi_t)^{-\epsilon} \Pi_t^\epsilon + \Pi_t^\epsilon \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} dj. \end{aligned}$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} dj = \theta \int_0^1 \left(\frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} dj = \theta V_{t-1}^p.$$

Thus, we acquire

$$V_t^p = \Pi_t^\epsilon ((1-\theta)(P_t^* \Pi_t)^{-\epsilon} + \theta V_{t-1}^p) \quad (16)$$

1.5 Equilibrium

To close the model, I need to specify the functional form for investment, aggregate shocks, and market-clearing conditions.

Following the literature, I assume quadratic adjustment investment costs so that

$$\Phi \left[\frac{X_t}{X_{t-1}} \right] = \frac{\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2, \quad (17)$$

which has first derivative w.r.t. X_t/X_{t-1}

$$\Phi' \left[\frac{X_t}{X_{t-1}} \right] = \chi \left(\frac{X_t}{X_{t-1}} \right). \quad (18)$$

There are five shocks in the model: $\eta_{A,t}$, $\eta_{\beta,t}$, $\eta_{L,t}$, $\eta_{K,t}$, and $\eta_{R,t}$. Without loss of generality, I assume all shocks follow AR(1) processes with persistence ρ_i and standard deviation σ_i .

Markets must clear for capital, labor, bonds, final goods, and intermediate goods, . The first three markets clear as a consequence of optimality conditions and the assumption that bonds have zero net supply. To clear the market for final goods, we set the sum of aggregate consumption demand C_t and investment demand X_t equal to aggregate supply Y_t , which satisfies

$$\begin{aligned} \int_0^1 A_t K_t^\alpha L_t^{1-\alpha} dj &= \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t dj \\ A_t K_t^\alpha L_t^{1-\alpha} &= Y_t \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} dj = V_t^p Y_t. \end{aligned}$$

Re-arranging yields the output market-clearing condition

$$C_t + X_t = Y_t, \quad (19)$$

$$Y_t = \frac{A_t K_t^\alpha L_t^{1-\alpha}}{V_t^p}. \quad (20)$$

It can be shown that $V_t^p \geq 1$ by applying Jensen's inequality. For our purposes, because the dimensionality of our model is not too large, we add the auxiliary Y_t variable, even though we could substitute it out of the system of equations.

All together, the full set of endogenous equilibrium conditions are

$$C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^\nu, \quad (21)$$

$$M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1}) C_{t+1}^{-\gamma}}{\exp(\eta_{\beta,t}) C_t^{-\gamma}}, \quad (22)$$

$$1 = \mathbb{E}_t \left[M_{t+1} \frac{R_t}{\Pi_{t+1}} \right], \quad (23)$$

$$1 = \exp(\eta_{K,t}) Q_t \left(1 - \frac{3\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2 \right) + \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \chi \left(\frac{X_{t+1}}{X_t} \right)^3 \right], \quad (24)$$

$$Q_t = \mathbb{E}_t [M_{t+1} (R_{K,t+1} + Q_{t+1} (1 - \delta))], \quad (25)$$

$$MC_t = \left(\frac{1}{1 - \alpha} \right)^{1 - \alpha} \left(\frac{1}{\alpha} \right)^\alpha \frac{W_t^{1 - \alpha} R_{K,t}^\alpha}{A_t}, \quad (26)$$

$$\frac{K_t}{L_t} = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_{K,t}}, \quad (27)$$

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, \quad (28)$$

$$\tilde{S}_{1,t} = MC_t Y_t + \theta \mathbb{E}_t [M_{t+1} \Pi_{t+1}^\epsilon \tilde{S}_{1,t+1}], \quad (29)$$

$$\tilde{S}_{2,t} = Y_t + \theta \mathbb{E}_t [M_{t+1} \Pi_{t+1}^{\epsilon-1} \tilde{S}_{2,t+1}], \quad (30)$$

$$\Pi_t^{1-\epsilon} = (1 - \theta) (P_t^* \Pi_t)^{1-\epsilon} + \theta, \quad (31)$$

$$V_t^p = \Pi_t^\epsilon ((1 - \theta) (P_t^* \Pi_t)^{-\epsilon} + \theta V_{t-1}^p), \quad (32)$$

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\phi_R} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_{t-1}} \right)^{\phi_y} \right)^{1 - \phi_R} \exp(\eta_{R,t}), \quad (33)$$

$$C_t + X_t = Y_t, \quad (34)$$

$$Y_t = \frac{A_t K_t^\alpha L_t^{1-\alpha}}{V_t^p}, \quad (35)$$

as well as the law of motion for capital

$$K_{t+1} = (1 - \delta) K_t + \exp(\eta_{K,t}) \left(1 - \frac{\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2 \right) X_t \quad (36)$$

and the five exogenous processes

$$\eta_{\beta,t+1} = \rho_\beta \eta_{\beta,t} + \sigma_\beta \varepsilon_{\beta,t+1}, \quad (37)$$

$$\eta_{L,t+1} = \rho_L \eta_{L,t} + \sigma_L \varepsilon_{L,t+1}, \quad (38)$$

$$\eta_{A,t+1} = \rho_A \eta_{A,t} + \sigma_A \varepsilon_{A,t+1}, \quad (39)$$

$$\eta_{K,t+1} = \rho_K \eta_{K,t} + \sigma_K \varepsilon_{K,t+1}, \quad (40)$$

$$\eta_{R,t+1} = \rho_R \eta_{R,t} + \sigma_R \varepsilon_{R,t+1}. \quad (41)$$

1.6 Deterministic Steady State

To provide an initial guess for the risk-adjusted linearization and to provide a verification that the model is coded correctly, I determine some reasonable guesses for the deterministic steady state.

Within this subsection, I denote the deterministic steady state values by an absence of a time subscript. The exogenous processes, by construction, have steady states of 0, i.e. $\eta_\beta = \eta_L = \eta_A = \eta_K = \eta_R = 0$. Further, $A = 1$.

Focusing now on the endogenous equilibrium conditions, from (21),

$$W = \frac{\varphi L^\nu}{C^{-\gamma}}.$$

From (24),

$$1 = \left(1 - \frac{3\chi}{2}\right) Q + \chi Q \Rightarrow Q = \frac{1}{1 - \frac{\chi}{2}}.$$

From (25),

$$R_K = \delta Q.$$

Equation (26) remains as it is but with time subscripts removed. From (29),

$$\tilde{S}_1 = MC \cdot Y + \theta \Pi^\epsilon \tilde{S}_1 \Rightarrow \tilde{S}_1 = \frac{MC \cdot Y}{1 - \theta \Pi^\epsilon}.$$

From (30),

$$\tilde{S}_2 = Y + \theta \Pi^{\epsilon-1} \tilde{S}_2 \Rightarrow \tilde{S}_2 = \frac{Y}{1 - \theta \Pi^{\epsilon-1}}.$$

Thus,

$$P^* = \frac{\epsilon}{\epsilon - 1} MC \frac{1 - \theta \Pi^{\epsilon-1}}{1 - \theta \Pi^\epsilon}.$$

From (31),

$$\Pi^{1-\epsilon} = (1 - \theta) (P^* \Pi)^{1-\epsilon} + \theta$$

Note that P^* depends on MC and fundamental parameters, hence the above equation pins down MC , which then pins down the ratio of K to L . From (32),

$$\begin{aligned} V^p &= \Pi^\epsilon \left((1 - \theta) (P^* \Pi)^{-\epsilon} + \theta V^p \right) \\ V^p &= \frac{(1 - \theta) (P^* \Pi)^{-\epsilon}}{\Pi^{-\epsilon} - \theta} = \frac{(1 - \theta) (P^*)^{-\epsilon}}{1 - \theta \Pi^\epsilon}. \end{aligned}$$

From the Taylor rule (33), the steady state interest and inflation rates are R and Π , respectively, and from the Euler equation (23), R must satisfy

$$R = \frac{\Pi}{\beta}.$$

From (34),

$$C + X = Y.$$

From (35),

$$Y = \frac{K^\alpha L^{1-\alpha}}{V^p}.$$

From (36), steady-state investment satisfies

$$K - (1 - \delta)K = \left(1 - \frac{\chi}{2}\right) X \Rightarrow X = \frac{\delta K}{1 - \frac{\chi}{2}}.$$

I claim that the deterministic steady state reduces to a nonlinear equation in L . Using the aggregate supply and capital accumulation equations,

$$C + \frac{\delta K}{1 - \chi/2} = \frac{K^\alpha L^{1-\alpha}}{V^p}.$$

The optimal capital-labor ratio implies

$$K = \frac{\alpha}{1 - \alpha} \frac{W}{R_K} L,$$

$$C + \frac{\delta K}{1 - \chi/2} = \left(\frac{\alpha}{1 - \alpha}\right)^\alpha \left(\frac{W}{R_K}\right)^\alpha \frac{L}{V^p}.$$

The intratemporal condition for consumption and labor implies

$$K = \frac{\alpha}{1 - \alpha} \frac{\varphi L^\nu}{C^{-\gamma} R_K} L,$$

$$C + \frac{\delta K}{1 - \chi/2} = \left(\frac{\alpha}{1 - \alpha}\right)^\alpha \left(\frac{\varphi L^\nu}{C^{-\gamma} R_K}\right)^\alpha \frac{L}{V^p}.$$

Given a guess for L , I can compute C using these two equations. Given C , I can compute W . Given the wage W , I can compute K and MC . Given the marginal cost MC , I can compute the inflation-related terms.

2 Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$\begin{aligned} 0 &= \log \mathbb{E}_t [\exp (\xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1})] \\ z_{t+1} &= \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t) \varepsilon_{t+1}, \end{aligned}$$

where z_t are (predetermined) state variables and y_t are (nondetermined) jump variables. For the remainder of this section, lower case variables are the logs of previously upper case variables, and with a small abuse of notation, let $s_{1,t} = \log(\tilde{S}_{1,t})$ and $s_{2,t} = \log(\tilde{S}_{2,t})$. Additionally, let $r_{k,t} = \log(R_{K,t})$ and $v_t = \log(V_t^p)$.

Equation (21) becomes

$$\begin{aligned} 1 &= \varphi \exp(\eta_{L,t}) \frac{L_t^\nu}{C_t^{-\gamma} W_t} \\ 0 &= \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\varphi) + \eta_{L,t} + \nu l_t - (-\gamma c_t + w_t)}_{\xi} \right) \right]. \end{aligned}$$

Equation (22) will not be used in the system of equations for the risk-adjusted linearization, but it simplifies the other equations. Taking logs and re-arranging yields

$$\begin{aligned} 0 &= \log(\beta) + \eta_{\beta,t+1} + (-\gamma c_{t+1}) - \eta_{\beta,t} - (-\gamma c_t) - m_{t+1} \\ m_{t+1} &= \underbrace{\log(\beta) - \eta_{\beta,t} + \gamma c_t}_{\xi} + \underbrace{\eta_{\beta,t+1} - \gamma c_{t+1}}_{\text{forward-looking}}. \end{aligned}$$

Equation (23) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\beta) + r_t}_{\xi} + \underbrace{m_{t+1} - \pi_{t+1}}_{\text{forward-looking}} \right) \right].$$

Equation (24) becomes

$$\begin{aligned} &1 - \exp(\eta_{K,t}) Q_t \left(1 - \frac{3\chi}{2} \left(\frac{X_t}{X_{t-1}} \right)^2 \right) \\ &= \mathbb{E}_t \left[M_{t+1} \exp(\eta_{K,t+1}) Q_{t+1} \chi \left(\frac{X_{t+1}}{X_t} \right)^3 \right] \\ &\log \left(1 - \exp(\eta_{K,t} + q_t) \left(1 - \frac{3\chi}{2} \exp(2x_t - 2x_{t-1}) \right) \right) \\ &= \log \mathbb{E}_t [\exp(m_{t+1} + \eta_{K,t+1} + q_{t+1} + \log(\chi) + 3x_{t+1} - 3x_t)] \\ 0 &= \log \mathbb{E}_t \left[\frac{\exp(m_{t+1} + \eta_{K,t+1} + q_{t+1} + \log(\chi) + 3x_{t+1} - 3x_t)}{\exp(\log(1 - \exp(\eta_{K,t} + q_t) (1 - \frac{3\chi}{2} \exp(2x_t - 2x_{t-1}))))} \right]. \end{aligned}$$

Re-arranging yields

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\chi) - 3x_t - \log \left(1 - \exp(\eta_{K,t} + q_t) \left(1 - \frac{3\chi}{2} \exp(2x_t - 2x_{t-1}) \right) \right)}_{\xi} \right) \right] \\ + \log \mathbb{E}_t \left[\exp \left(\underbrace{m_{t+1} + \eta_{K,t+1} + q_{t+1} + 3x_{t+1}}_{\text{forward-looking}} \right) \right].$$

Equation (25) becomes

$$1 = \mathbb{E}_t \left[\frac{M_{t+1}(R_{K,t+1} + Q_{t+1}(1 - \delta))}{Q_t} \right] \\ 0 = \log \mathbb{E}_t [\exp(m_{t+1} + \log(\exp(r_{k,t+1}) + (1 - \delta) \exp(q_{t+1})) - q_t)].$$

This equation is not linear in the forward-looking variables, we define a new variable

$$\Omega_t = R_{K,t} + Q_t(1 - \delta), \quad (42)$$

so that we may represent (25) in the required form for risk-adjusted linearizations as the following two equations:

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{-q_t}_{\xi} + \underbrace{m_{t+1} + \omega_{t+1}}_{\text{forward-looking}} \right) \right], \\ 0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\omega_t - \log(\exp(r_{k,t}) + (1 - \delta) \exp(q_t))}_{\xi} \right) \right],$$

Equation (26) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{(1 - \alpha)w_t + \alpha r_{k,t} - a_t - (1 - \alpha) \log(1 - \alpha) - \alpha \log(\alpha) - mc_t}_{\xi} \right) \right].$$

Equation (27) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{k_t - l_t - \log \left(\frac{\alpha}{1 - \alpha} \right) - (w_t - r_{k,t})}_{\xi} \right) \right].$$

Like the stochastic discount factor, equation (28) will not be used in the system of equations, but it will be useful to simplify other equations. Taking logs yields

$$p_t^* = \log \left(\frac{\epsilon}{\epsilon - 1} \right) + s_{1,t} - s_{2,t}.$$

Equation (29) becomes

$$\begin{aligned} & \tilde{S}_{1,t} - MC_t Y_t \\ &= \mathbb{E}_t [\exp (\log (\theta) + \log (\beta) + m_{t+1} + \epsilon \pi_{t+1} + s_{1,t+1})] \\ 0 &= \log \mathbb{E}_t \left[\exp \left(\underbrace{\log (\theta) + \log (\beta) - \log (\exp (s_{1,t}) - \exp (m c_t) \exp (y_t))}_{\xi} + \underbrace{m_{t+1} + \epsilon \pi_{t+1} + s_{1,t+1}}_{\text{forward-looking}} \right) \right] \end{aligned}$$

and equation (30) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log (\theta) + \log (\beta) - \log (\exp (s_{2,t}) - \exp (y_t))}_{\xi} + \underbrace{m_{t+1} + (\epsilon - 1) \pi_{t+1} + s_{2,t+1}}_{\text{forward-looking}} \right) \right].$$

Equation (31) becomes

$$\begin{aligned} 1 &= \frac{\Pi_t^{1-\epsilon}}{(1-\theta)(P_t^* \Pi_t)^{1-\epsilon} + \theta} \\ 0 &= \log \mathbb{E}_t [\exp ((1-\epsilon) \pi_t - \log ((1-\theta) \exp ((1-\epsilon)(p_t^* + \pi_t)) + \theta))] . \end{aligned}$$

Equation (32) becomes

$$\begin{aligned} 1 &= \frac{V_t^p}{\Pi_t^\epsilon ((1-\theta)(P_t^* \Pi_t)^{-\epsilon} + \theta V_{t-1}^p)} \\ 0 &= \log \mathbb{E}_t [\exp (v_t - \epsilon \pi_t - \log ((1-\theta) \exp (-\epsilon(p_t^* + \pi_t)) + \theta \exp (v_{t-1})))]. \end{aligned}$$

Equation (33) becomes

$$0 = \log \mathbb{E}_t [\exp (\phi_R r_{t-1} + (1-\phi_R) r + (1-\phi_R)(\phi_\pi (\pi_t - \pi) + \phi_y (y_t - y_{t-1})) + \eta_{R,t} - r_t)].$$

Equation (34) becomes

$$0 = \log \mathbb{E}_t [\exp (y_t - \log (\exp (c_t) + \exp (x_t)))] .$$

Equation (35) becomes

$$0 = \log \mathbb{E}_t [\exp (a_t + \alpha k_t + (1-\alpha) l_t - v_t - y_t)].$$

Equation (36) becomes

$$k_{t+1} = \log \left((1 - \delta) \exp(k_t) + \exp(\eta_{K,t}) \left(1 - \frac{\chi}{2} \exp(2(x_t - x_{t-1})) \right) \exp(x_t) \right).$$

The autoregressive processes (37) to (41) remain as they are.

The jump variables are y_t , c_t , l_t , w_t , r_t , π_t , q_t , x_t , $r_{k,t}$, ω_t , mc_t , $s_{1,t}$, $s_{2,t}$, and v_t . The state variables are k_t , x_{t-1} , v_{t-1} , r_{t-1} , y_{t-1} , and the autoregressive processes. The equations defining the evolution of the lags x_{t-1} , v_{t-1} , r_{t-1} , and y_{t-1} are obtained by the formula $z_{(t-1)+1} = z_t$.

This system has three forward difference equations (25), (29), and (30). To ensure accuracy of the risk-adjusted linearization, I derive N -period ahead forward difference equations for all three.

We can write (25) recursively as

$$\begin{aligned} Q_t &= \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1 - \delta))] \\ &= \mathbb{E}_t[M_{t+1}R_{K,t+1} + (1 - \delta)M_{t+1}\mathbb{E}_{t+1}[M_{t+2}(R_{K,t+2} + Q_{t+2}(1 - \delta))]] \\ &= \mathbb{E}_t[M_{t+1}R_{K,t+1}] + (1 - \delta)\mathbb{E}_t\mathbb{E}_{t+1}[M_{t+1}M_{t+2}(R_{K,t+2} + Q_{t+2}(1 - \delta))]. \end{aligned}$$

By the tower property,

$$\begin{aligned} Q_t &= \mathbb{E}_t[M_{t+1}R_{K,t+1}] + (1 - \delta)\mathbb{E}_t[M_{t+1}M_{t+2}(R_{K,t+2} + Q_{t+2}(1 - \delta))] \\ &= \mathbb{E}_t \left[\left(\sum_{s=1}^2 (1 - \delta)^{s-1} \left(\prod_{u=1}^s M_{t+u} \right) R_{K,t+s} \right) + M_{t+1}M_{t+2}Q_{t+2}(1 - \delta)^2 \right] \\ &= \mathbb{E}_t \left[\left(\sum_{s=1}^2 (1 - \delta)^{s-1} \left(\prod_{u=1}^s M_{t+u} \right) R_{K,t+s} \right) + M_{t+1}M_{t+2}(1 - \delta)^2 \mathbb{E}_{t+2}[M_{t+3}(R_{K,t+3} + Q_{t+3}(1 - \delta))] \right] \\ &= \mathbb{E}_t \left[\left(\sum_{s=1}^3 (1 - \delta)^{s-1} \left(\prod_{u=1}^s M_{t+u} \right) R_{K,t+s} \right) + \prod_{s=1}^3 ((1 - \delta)M_{t+s})Q_{t+3} \right] \end{aligned}$$

and so on. Given this recursive structure, define $D_{Q,t}^{(n)}$ and $P_{Q,t}^{(n)}$ as

$$\begin{aligned} D_{Q,t}^{(n)} &= \mathbb{E}_t \left[(1 - \delta)M_{t+1}D_{Q,t+1}^{(n-1)} \right] \\ P_{Q,t}^{(n)} &= \mathbb{E}_t \left[(1 - \delta)M_{t+1}P_{Q,t+1}^{(n-1)} \right] \end{aligned}$$

with boundary conditions

$$\begin{aligned} D_{Q,t}^{(0)} &= \frac{R_{K,t+1}}{1 - \delta} \\ P_{Q,t}^{(0)} &= Q_{t+1}. \end{aligned}$$

Then I may write the N -period ahead recursive form of equation (25) as

$$Q_t = \sum_{n=1}^N D_{Q,t}^{(n)} + P_{Q,t}^{(N)}.$$

To see why this recursion works, it is simpler to first verify that $P_{Q,t}^{(3)}$ is correct:

$$\begin{aligned} P_{Q,t}^{(1)} &= \mathbb{E}_t [(1 - \delta)M_{t+1}Q_{t+1}] \\ P_{Q,t}^{(2)} &= \mathbb{E}_t [(1 - \delta)M_{t+1}(\mathbb{E}_{t+1}[(1 - \delta)M_{t+2}Q_{t+2}])] \\ &= \mathbb{E}_t [\mathbb{E}_{t+1}[(1 - \delta)^2 M_{t+1}M_{t+2}Q_{t+2}]] \\ &= \mathbb{E}_t \left[\prod_{s=1}^2 ((1 - \delta)M_{t+s})Q_{t+2} \right], \end{aligned}$$

where the second equality for $P_{Q,t}^{(2)}$ follows from the fact that M_{t+1} is measurable with respect to the information set at time $t + 1$ and can therefore be moved insided the conditional expectation $\mathbb{E}_{t+1}[\cdot]$. Continuing for one more recursion, I have

$$\begin{aligned} P_{Q,t}^{(3)} &= \mathbb{E}_t \left[(1 - \delta)M_{t+1} \mathbb{E}_{t+1} \left[\prod_{s=1}^2 ((1 - \delta)M_{t+1+s})Q_{t+3} \right] \right] \\ &= \mathbb{E}_t \left[\prod_{s=1}^3 ((1 - \delta)M_{t+s})Q_{t+3} \right]. \end{aligned}$$

Similarly, for $D_{Q,t}$, I have

$$\begin{aligned} D_{Q,t}^{(1)} &= \mathbb{E}_t \left[(1 - \delta)M_{t+1} \frac{R_{K,t+1}}{1 - \delta} \right] = \mathbb{E}_t [M_{t+1}R_{K,t+1}] \\ D_{Q,t}^{(2)} &= \mathbb{E}_t [(1 - \delta)M_{t+1} \mathbb{E}_{t+1} [M_{t+2}R_{K,t+2}]] \\ &= \mathbb{E}_t [(1 - \delta)M_{t+1}M_{t+2}R_{K,t+2}] \\ D_{Q,t}^{(3)} &= \mathbb{E}_t [(1 - \delta)M_{t+1} \mathbb{E}_{t+1} [(1 - \delta)M_{t+2}M_{t+3}R_{K,t+3}]] \\ &= \mathbb{E}_t [(1 - \delta)^2 M_{t+1}M_{t+2}M_{t+3}R_{K,t+3}]. \end{aligned}$$

Since $P_{Q,t}^{(n)}$ and $D_{Q,t}^{(n)}$ are time- t conditional expectations, they are measurable at time t , so they are not forward-looking variables. Thus, to get this version of (25) in the appropriate

form, define $d_{q,n,t} = \log(D_{Q,t}^{(n)})$ and $p_{q,n,t} = \log(P_{Q,t}^{(n)})$, and use the following $2N + 1$ equations:

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{q_t - \log \left(\sum_{n=1}^N \exp(d_{q,n,t}) + \exp(p_{q,n,t}) \right)}_{\xi} \right) \right] \quad (43)$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(1 - \delta) - d_{q,n,t}}_{\xi} + \underbrace{m_{t+1} + d_{q,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{-d_{q,1,t}}_{\xi} + \underbrace{m_{t+1} + r_{k,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1, \end{cases} \quad (44)$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(1 - \delta) - p_{q,n,t}}_{\xi} + \underbrace{m_{t+1} + p_{q,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(1 - \delta) - p_{q,1,t}}_{\xi} + \underbrace{m_{t+1} + q_{t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1. \end{cases} \quad (45)$$

For (29), observe that

$$\begin{aligned} \tilde{S}_{1,t} &= MC_t Y_t + \theta \beta \mathbb{E}_t [M_{t+1} \Pi_{t+1}^\epsilon (MC_{t+1} Y_{t+1} + \theta \beta \mathbb{E}_{t+1} [M_{t+2} \Pi_{t+2}^\epsilon \tilde{S}_{1,t+2}])] \\ &= MC_t Y_t + \theta \beta \mathbb{E}_t [M_{t+1} \Pi_{t+1}^\epsilon MC_{t+1} Y_{t+1} + \theta \beta M_{t+1} \Pi_{t+1}^\epsilon M_{t+2} \Pi_{t+2}^\epsilon \tilde{S}_{1,t+2}] \\ &= MC_t Y_t + \mathbb{E}_t \left[\sum_{s=1}^1 ((\theta \beta)^s \prod_{u=1}^s (M_{t+u} \Pi_{t+u}^\epsilon)) MC_{t+s} Y_{t+s} \right] + \mathbb{E}_t \left[\prod_{s=1}^2 (\theta \beta M_{t+s} \Pi_{t+s}^\epsilon) \tilde{S}_{1,t+2} \right]. \end{aligned}$$

Thus, define $D_{S1,t}^{(n)}$ and $P_{S1,t}^{(n)}$ as the recursions

$$\begin{aligned} D_{S1,t}^{(n)} &= \mathbb{E}_t [\theta \beta M_{t+1} \Pi_{t+1}^\epsilon D_{S1,t+1}^{(n-1)}], \\ P_{S1,t}^{(n)} &= \mathbb{E}_t [\theta \beta M_{t+1} \Pi_{t+1}^\epsilon P_{S1,t+1}^{(n-1)}], \end{aligned}$$

with boundary conditions

$$\begin{aligned} D_{S1,t}^{(0)} &= MC_t Y_t \\ P_{S1,t}^{(0)} &= \tilde{S}_{1,t}. \end{aligned}$$

Given these definitions, it follows that

$$\begin{aligned}
D_{S1,t}^{(1)} &= \mathbb{E}_t[\theta\beta M_{t+1}\Pi_{t+1}^\epsilon MC_{t+1}Y_{t+1}] \\
P_{S1,t}^{(1)} &= \mathbb{E}_t[\theta\beta M_{t+1}\Pi_{t+1}^\epsilon \tilde{S}_{1,t+1}] \\
P_{S1,t}^{(2)} &= \mathbb{E}_t[\theta\beta M_{t+1}\Pi_{t+1}^\epsilon \mathbb{E}_{t+1}[\theta\beta M_{t+2}\Pi_{t+2}^\epsilon \tilde{S}_{1,t+2}]] \\
&= \mathbb{E}_t[(\theta\beta)^2 M_{t+1}\Pi_{t+1}^\epsilon M_{t+2}\Pi_{t+2}^\epsilon \tilde{S}_{1,t+2}].
\end{aligned}$$

Thus, defining $d_{s1,t} = \log(D_{S1,t})$ and $p_{s1,t} = \log(P_{S1,t})$, the N -period ahead recursive form of (29) results in the $2N + 1$ equations

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{s_{1,t} - \log \left(\sum_{n=0}^{N-1} \exp(d_{s1,n,t}) + \exp(p_{s1,n,t}) \right)}_{\xi} \right) \right] \quad (46)$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - d_{s1,n,t}}_{\xi} + \underbrace{m_{t+1} + \epsilon\pi_{t+1} + d_{s1,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n \geq 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{d_{s1,0,t} - mc_t - y_t}_{\xi} \right) \right] & \text{if } n = 0. \end{cases} \quad (47)$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - p_{s1,n,t}}_{\xi} + \underbrace{m_{t+1} + \epsilon\pi_{t+1} + p_{s1,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - p_{s1,1,t}}_{\xi} + \underbrace{m_{t+1} + s_{1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1. \end{cases} \quad (48)$$

It is straightforward to show that a similar recursive form applies to (30):

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{s_{2,t} - \log \left(\sum_{n=0}^{N-1} \exp(d_{s2,n,t}) + \exp(p_{s2,n,t}) \right)}_{\xi} \right) \right] \quad (49)$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - d_{s2,n,t}}_{\xi} + \underbrace{m_{t+1} + (\epsilon - 1)\pi_{t+1} + d_{s2,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n \geq 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{d_{s2,0,t} - y_t}_{\xi} \right) \right] & \text{if } n = 0. \end{cases} \quad (50)$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - p_{s2,n,t}}_{\xi} + \underbrace{m_{t+1} + (\epsilon - 1)\pi_{t+1} + p_{s2,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - p_{s2,1,t}}_{\xi} + \underbrace{m_{t+1} + s_{2,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1, \end{cases} \quad (51)$$

where terms and boundary conditions are analogously defined.