These notes loosely follow Fernández-Villaverde and Levintal (2018) “Solution Methods for Models with Rare Disasters” but omits several features, such as recursive preferences and disaster risk.

1 Model

1.1 Household

The model admits a representative agent, so I directly write households' problem as the representative agent's. The representative household solves, in the cashless limit,

$$\max_{C_t,L_t,B_t,X_t,K_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \exp(\eta_{L,t}) \frac{L_t^{1+\nu}}{1+\nu} \right) \tag{1}$$

subject to the budget constraint

$$C_t + \frac{B_t}{P_t} + X_t \leq W_t L_t + R_{K,t} K_{t-1} + R_{t-1} \frac{B_{t-1}}{P_t} + F_t + T_t, \tag{2}$$

where C_t is consumption, B_{t-1} nominal bonds, X_t investment, $W_{N,t}$ the real wage, L_t labor, $R_{KN,t}$ the gross real rental rate on capital, K_{t-1} capital, R_t the gross nominal interest rate on bonds, F_t real profits from firms, and T_t real lump-sum transfers from the government. The price of the final consumption good is P_t. Markets are assumed complete, but securities are in zero net supply. Because there is a representative agent, I may omit the Arrow securities from the budget constraint. My notation treats B_{t-1} and K_{t-1} as the stocks of bonds and capital present at time t, while B_t and K_t are the chosen stocks of bonds and capital for the following period. I adopt this notation so that all time t choices are dated at time t rather than having to differentiate between the predetermined time-t variables from the endogenous controls.

Investment for capital follows the law of motion

$$K_t = (1-\delta)K_{t-1} + \Phi\left(\frac{X_t}{K_{t-1}}\right) K_{t-1}. \tag{3}$$

The Lagrangian for the household is

$$\begin{aligned}
\mathcal{L} = {} & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \exp(\eta_{L,t}) \frac{L_t^{1+\nu}}{1+\nu} \right] \\
& + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \lambda_t \left[W_t L_t + R_{K,t} K_{t-1} + R_{t-1} \frac{B_{t-1}}{P_t} + F_t + T_t - C_t - \frac{B_t}{P_t} - X_t \right] \\
& + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{\beta,t}) \lambda_t Q_t \left[(1-\delta)K_{t-1} + \Phi\left(\frac{X_t}{K_{t-1}}\right) K_{t-1} - K_t \right],
\end{aligned}$$

which implies first-order conditions

$$
\begin{aligned}
0 &= C_t^{-\gamma} - \lambda_t \\
0 &= -\varphi \exp(\eta_{L,t}) L_t^{\nu} + \lambda_t W_t \\
0 &= -\exp(\eta_{\beta,t})\lambda_t + \beta \mathbb{E}_t[\exp(\eta_{\beta,t+1})\lambda_{t+1} R_t] \\
0 &= -\exp(\eta_{\beta,t})\lambda_t + \exp(\eta_{\beta,t})\lambda_t Q_t \Phi'\left(\frac{X_t}{K_{t-1}}\right) K_{t-1}\frac{1}{K_{t-1}} \\
0 &= -\exp(\eta_{\beta,t})\lambda_t Q_t + \beta \mathbb{E}_t\left[\exp(\eta_{\beta,t+1})\lambda_{t+1} R_{K,t+1}\right] \\
&\quad \beta \mathbb{E}_t\left[\exp(\eta_{\beta,t+1})\lambda_{t+1} Q_{t+1}\left(1 - \delta + \Phi'\left(\frac{X_{t+1}}{K_t}\right) K_t \left(-\frac{X_{t+1}}{K_t^2}\right) + \Phi\left(\frac{X_{t+1}}{K_t}\right)\right)\right]
\end{aligned}
$$

The first two equations can be combined by isolating λ_t, which obtains the intratemporal consumption-labor condition

$$
C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^{\nu},
$$

Using $\lambda_t = C_t^{-\gamma}$ and defining the gross inflation rate $\Pi_t \equiv P_t/P_{t-1}$, I can obtain the Euler equation for households

$$
\begin{aligned}
\exp(\eta_{\beta,t})\frac{C_t^{-\gamma}}{P_t} &= \beta \mathbb{E}_t\left[\exp(\eta_{\beta,t+1})\frac{C_{t+1}^{-\gamma}}{P_{t+1}} R_t\right] \\
1 &= \beta \mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})}\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}\frac{R_t}{\Pi_{t+1}}\right].
\end{aligned}
$$

I can further simplify the Euler equation by defining the (real) stochastic discount factor

$$
M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})}\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}
$$

After dividing through by $\exp(\eta_{\beta,t})\lambda_t$ and re-arranging, the investment condition becomes

$$
1 = Q_t \Phi'\left(\frac{X_t}{K_{t-1}}\right).
$$

Finally, after dividing through by $\exp(\eta_{\beta,t})\lambda_t$, the first-order condition for next-period capital is

$$
Q_t = \mathbb{E}_t\left[M_{t+1}\left(R_{K,t+1} + Q_{t+1}\left(1 - \delta + \Phi\left(\frac{X_{t+1}}{K_t}\right) - \Phi'\left(\frac{X_{t+1}}{K_t}\right)\frac{X_{t+1}}{K_t}\right)\right)\right].
$$

In summary, households' optimality conditions are

$$C_t^{-\gamma} W_t = \varphi \exp(\eta_{L,t}) L_t^{\nu}, \tag{4}$$

$$M_{t+1} = \beta \frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, \tag{5}$$

$$1 = \beta \mathbb{E}_t \left[M_{t+1} \frac{R_t}{\Pi_{t+1}} \right], \tag{6}$$

$$1 = Q_t \Phi' \left(\frac{X_t}{K_{t-1}} \right), \tag{7}$$

$$Q_t = \mathbb{E}_t \left[M_{t+1} \left(R_{K,t+1} + Q_{t+1} \left(1 - \delta + \Phi \left(\frac{X_{t+1}}{K_t} \right) - \Phi' \left(\frac{X_{t+1}}{K_t} \right) \frac{X_{t+1}}{K_t} \right) \right) \right]. \tag{8}$$

1.2 Production

Final Producers There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 1$ so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j)\, dj.$$

The FOC for $Y_t(j)$ is

$$\begin{aligned}
0 &= P_t \frac{\epsilon}{\epsilon - 1} \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} \frac{\epsilon - 1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) \\
0 &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t} \\
0 &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} \\
Y_t(j) &= \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.
\end{aligned}$$

Plugging this quantity into the identity

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j)\, dj$$

and simplifying yields the price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon}\, dj \right)^{\frac{1}{1-\epsilon}}.$$

Intermediate Producers Intermediate goods are producing according to the Cobb-Douglas technology

$$Y_t(j) = A_t K_{t-1}^{\alpha}(j) L_t^{1-\alpha}(j).$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\min_{K_{t-1}(j), L_t(j)} R_{K,t} K_{t-1}(j) + W_t L_t(j) \qquad \text{s.t.} \qquad A_t K_{t-1}^{\alpha}(j) L_t^{1-\alpha}(j) \geq \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t.$$

The RHS of the inequality constraint is the demand from final goods producers for intermediate j. The Lagrangian is

$$\mathcal{L} = R_{K,t} K_{t-1}(j) + W_t L_t(j) + MC_t(j) \left(\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t - A_t K_{t-1}^{\alpha}(j) L_t^{1-\alpha}(j) \right),$$

so the first-order conditions are

$$\begin{aligned} 0 &= R_{K,t} - MC_t(j) \alpha A_t \left(\frac{L_t(j)}{K_{t-1}(j)}\right)^{1-\alpha} \\ 0 &= W_t - MC_t(j)(1-\alpha) A_t \left(\frac{K_{t-1}(j)}{L_t(j)}\right)^{\alpha}, \end{aligned}$$

hence the optimal capital-labor ratio satisfies

$$\begin{aligned} \frac{R_{K,t}}{\alpha A_t (K_{t-1}(j)/L_t(j))^{\alpha-1}} &= \frac{W_t}{(1-\alpha) A_t (K_{t-1}(j)/L_t(j))^{\alpha}} \\ \frac{K_{t-1}(j)}{L_t(j)} &= \frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}}. \end{aligned}$$

Since the RHS does not vary with j, all firms choose the same capital-labor ratio. Given this optimal ratio, the marginal cost satisfies

$$\begin{aligned} MC_t &= \frac{R_{K,t}}{\alpha A_t} \left(\frac{K_{t-1}}{L_t}\right)^{1-\alpha} \\ &= \frac{R_{K,t}}{\alpha A_t} \left(\frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}}\right)^{1-\alpha} \\ &= \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{W_t^{1-\alpha} R_{K,t}^{\alpha}}{A_t}. \end{aligned}$$

It follows that

$$\begin{aligned} R_{K,t} K_{t-1} + W_t L_t &= \left(\frac{R_{K,t}}{A_t} \left(\frac{K_{t-1}}{L_t}\right)^{1-\alpha} + \frac{W_t}{A_t} \left(\frac{L_t}{K_{t-1}}\right)^{\alpha} \right) \left(A_t K_{t-1}^{\alpha} L_t^{1-\alpha}\right) \\ &= \left(\alpha MC_t + (1-\alpha) MC_t\right) Y_t(j) = MC_t Y_t(j). \end{aligned}$$

Therefore, (real) profits for an intermediate producer become

$$F_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - MC_t Y_t(j).$$

In addition to the capital-labor choice, firms also have the chance to reset prices in every period with probability $1-\theta$. This problem can be written as

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} \left(\frac{P_t(j)}{P_{t+s}} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} \right),$$

where I have imposed that intermediate output equals demand. The first-order condition is

$$\begin{aligned} 0 = (1-\epsilon) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} (P_{t+s})^{-(1-\epsilon)} Y_{t+s} \\ + \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s} \end{aligned}$$

Divide by $P_t(j)^{-\epsilon}/(\exp(\eta_{\beta,t}) u'(C_t))$, apply the abuse of notation that $\prod_{u=1}^{0} \Pi_{t+u} = 1$, and re-arrange to obtain

$$\begin{aligned} P_t(j) &= \frac{\epsilon}{\epsilon-1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) P_{t+s}^{\epsilon-1} Y_{t+s}} \\ &= \frac{\epsilon}{\epsilon-1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) mc_{t+s} P_t^{\epsilon} \left(\prod_{u=1}^{s} \Pi_{t+u}\right)^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) P_t^{\epsilon-1} \left(\prod_{u=1}^{s} \Pi_{t+u}\right)^{\epsilon-1} Y_{t+s}} \\ \frac{P_t(j)}{P_t} &= \frac{\epsilon}{\epsilon-1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) mc_{t+s} \left(\prod_{u=1}^{s} \Pi_{t+u}\right)^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) \left(\prod_{u=1}^{s} \Pi_{t+u}\right)^{\epsilon-1} Y_{t+s}}. \end{aligned}$$

This expression gives the optimal (real) reset price $P_t^* \equiv P_t(j)/P_t$ (note that the RHS does not depend on j). Define

$$\begin{aligned} S_{1,t} &= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) mc_{t+s} Y_{t+s} \left(\prod_{u=1}^{s} \Pi_{t+s} \right)^{\epsilon}, \\ S_{2,t} &= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \exp(\eta_{\beta,t+s}) u'(C_{t+s}) Y_{t+s} \left(\prod_{u=1}^{s} \Pi_{t+s} \right)^{\epsilon-1}. \end{aligned}$$

Using these definitions, I may write the optimal reset price more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon-1} \frac{S_{1,t}}{S_{2,t}}$$

where $S_{1,t}$ and $S_{2,t}$ satisfy the recursions

$$\begin{aligned} S_{1,t} &= \exp(\eta_{\beta,t}) u'(C_t) MC_t Y_t + \theta\beta \mathbb{E}_t \Pi_{t+s}^{\epsilon} S_{1,t+1} \\ S_{2,t} &= \exp(\eta_{\beta,t}) u'(C_t) Y_t + \theta\beta \mathbb{E}_t \Pi_{t+s}^{\epsilon-1} S_{2,t+1}. \end{aligned}$$

These recursions can be further rewritten as

$$\begin{aligned}
\frac{S_{1,t}}{\exp(\eta_{\beta,t})u'(C_t)} &= MC_tY_t + \theta\beta\mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)}\Pi_{t+s}^{\epsilon}\frac{S_{1,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})}\right] \\
\frac{S_{2,t}}{\exp(\eta_{\beta,t})u'(C_t)} &= Y_t + \theta\beta\mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)}\Pi_{t+s}^{\epsilon-1}\frac{S_{2,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})}\right],
\end{aligned}$$

By defining $\tilde{S}_{1,t} \equiv S_{1,t}/(\exp(\eta_{\beta,t})u'(C_t))$ and $\tilde{S}_{2,t} \equiv S_{2,t}/(\exp(\eta_{\beta,t})u'(C_t))$, I can simplify these recursions into the form I use for the numerical solution.

From this section, we obtain the following five equilibrium conditions:

$$MC_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{W_t^{1-\alpha}R_{K,t}^{\alpha}}{A_t}, \tag{9}$$

$$\frac{K_{t-1}}{L_t} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_{K,t}}, \tag{10}$$

$$P_t^* = \frac{\epsilon}{\epsilon-1}\frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, \tag{11}$$

$$\tilde{S}_{1,t} = MC_tY_t + \theta\mathbb{E}_t[M_{t+1}\Pi_{t+1}^{\epsilon}\tilde{S}_{1,t+1}], \tag{12}$$

$$\tilde{S}_{2,t} = Y_t + \theta\mathbb{E}_t[M_{t+1}\Pi_{t+1}^{\epsilon-1}\tilde{S}_{2,t+1}]. \tag{13}$$

1.3 Monetary Policy

I specify the monetary policy rule as the following Taylor rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\phi_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y}\right)^{1-\phi_R}\exp(\eta_{R,t}) \tag{14}$$

Any proceeds from monetary policy are distributed as lump sum to the representative household.

1.4 Aggregation

The price level is currently characterized as the integral

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon}\,dj.$$

To represent the model entirely in terms of aggregates, notice that, without loss of generality, we may re-order the fraction θ of firms which cannot reset prices to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\theta)(P_t^*)^{1-\epsilon} + \int_{1-\theta}^1 P_{t-1}(j)^{1-\epsilon}\,dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\theta)(P_t^*)^{1-\epsilon} + \theta \int_0^1 P_{t-1}(j)^{1-\epsilon}\, dj = (1-\theta)(P_t^*)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon}.$$

Dividing by $P_{t-1}^{1-\epsilon}$ implies

$$\Pi_t^{1-\epsilon} = (1-\theta)(P_t^* \Pi_t)^{1-\epsilon} + \theta. \tag{15}$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$\begin{aligned} V_t^p &= \int_0^{1-\theta} (P_t^*)^{-\epsilon}\, dj + \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\epsilon} dj \\ &= \int_0^{1-\theta} (P_t^* \Pi_t)^{-\epsilon} \left(\frac{1}{\Pi_t}\right)^{-\epsilon} dj + \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} dj \\ &= (1-\theta)(P_t^*\Pi_t)^{-\epsilon}\Pi_t^{\epsilon} + \Pi_t^{\epsilon} \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj. \end{aligned}$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \theta \int_0^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \theta V_{t-1}^p.$$

Thus, we acquire

$$V_t^p = \Pi_t^{\epsilon}((1-\theta)(P_t^*\Pi_t)^{-\epsilon} + \theta V_{t-1}^p) \tag{16}$$

1.5 Equilibrium

To close the model, I need to specify the functional form for investment, aggregate shocks, and market-clearing conditions.

Following Jermann (1998), I assume the investment function takes the concave form

$$\Phi\left(\frac{X_t}{K_{t-1}}\right) = \frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{K_{t-1}}\right)^{1-1/\chi} - \frac{\overline{X}}{\chi(\chi-1)} \tag{17}$$

where $\overline{X} = \delta\chi/(\chi+1)$ is the steady-state investment rate (per unit of capital). The first derivative of $\Phi(\cdot)$ w.r.t. X_t/K_{t-1} is

$$\Phi'\left(\frac{X_t}{K_{t-1}}\right) = \overline{X}^{1/\chi}\left(\frac{X_t}{K_{t-1}}\right)^{-1/\chi}. \tag{18}$$

This functional form implies the law of motion

$$
\begin{aligned}
K_t &= \left(1-\delta+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{K_{t-1}}\right)^{1-1/\chi}-\frac{\overline{X}}{\chi(\chi-1)}\right)K_{t-1}\\
&= \left(1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{K_{t-1}}\right)^{1-1/\chi}-\delta\left(1+\frac{1}{(\chi-1)(\chi+1)}\right)\right)K_{t-1}\\
&= \left(1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{K_{t-1}}\right)^{1-1/\chi}-\delta\left(\frac{\chi^2}{(\chi-1)(\chi+1)}\right)\right)K_{t-1}\\
&= \left(1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{K_{t-1}}\right)^{1-1/\chi}-\frac{\delta\chi^2}{\chi^2(1-1/\chi)(1+1/\chi)}\right)K_{t-1}\\
&= \left(1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{K_{t-1}}\right)^{1-1/\chi}-\frac{\overline{X}}{1-1/\chi}\right)K_{t-1}.
\end{aligned}
$$

If $K_t = K_{t-1} = K_{ss}$ and $X_{ss}/K_{ss} = \overline{X}$, then

$$
1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\overline{X}^{1-1/\chi}-\frac{\overline{X}}{1-1/\chi} = 1+\frac{\overline{X}}{1-1/\chi}-\frac{\overline{X}}{1-1/\chi} = 1,
$$

thus verifying the original conjecture that $\overline{X}$ represents the steady-state investment rate. There are four shocks in the model: $\eta_{A,t}$, $\eta_{\beta,t}$, $\eta_{L,t}$, and $\eta_{R,t}$. Without loss of generality, I assume all shocks follow AR(1) processes with persistence ρ_i and standard deviation σ_i.

Markets must clear for capital, labor, bonds, final goods, and intermediate goods, . The first three markets clear as a consequence of optimality conditions and the assumption that bonds have zero net supply. To clear the market for final goods, we set the sum of aggregate consumption demand C_t and investment demand X_t equal to aggregate supply Y_t, which satisfies

$$
\begin{aligned}
\int_0^1 A_t K_{t-1}^{\alpha} L_t^{1-\alpha}\,dj &= \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t\,dj\\
A_t K_{t-1}^{\alpha} L_t^{1-\alpha} &= Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj = V_t^p Y_t.
\end{aligned}
$$

Re-arranging yields the output market-clearing condition

$$
C_t + X_t = Y_t, \tag{19}
$$

$$
Y_t = \frac{A_t K_{t-1}^{\alpha} L_t^{1-\alpha}}{V_t^p}. \tag{20}
$$

It can be shown that $V_t^p \geq 1$ by applying Jensen's inequality. For our purposes, because the dimensionality of our model is not too large, we add the auxiliary Y_t variable, even though we could substitute it out of the system of equations.

All together, the full set of endogenous equilibrium conditions are

$$
\begin{aligned}
C_t^{-\gamma} W_t &= \varphi \exp(\eta_{L,t}) L_t^{\nu}, && (21)\\
M_{t+1} &= \beta \frac{\exp(\eta_{\beta,t+1})}{\exp(\eta_{\beta,t})} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}, && (22)\\
1 &= \mathbb{E}_t \left[M_{t+1} \frac{R_t}{\Pi_{t+1}} \right], && (23)\\
1 &= Q_t \Phi' \left(\frac{X_t}{K_{t-1}} \right), && (24)\\
Q_t &= \mathbb{E}_t \left[M_{t+1} \left(R_{K,t+1} + Q_{t+1} \left(1 - \delta + \Phi\left(\frac{X_{t+1}}{K_t} \right) - \Phi'\left(\frac{X_{t+1}}{K_t} \right) \frac{X_{t+1}}{K_t} \right) \right) \right], && (25)\\
MC_t &= \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^{\alpha} \frac{W_t^{1-\alpha} R_{K,t}^{\alpha}}{A_t}, && (26)\\
\frac{K_{t-1}}{L_t} &= \frac{\alpha}{1-\alpha} \frac{W_t}{R_{K,t}}, && (27)\\
P_t^* &= \frac{\epsilon}{\epsilon - 1} \frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, && (28)\\
\tilde{S}_{1,t} &= MC_t Y_t + \theta \mathbb{E}_t [M_{t+1} \Pi_{t+1}^{\epsilon} \tilde{S}_{1,t+1}], && (29)\\
\tilde{S}_{2,t} &= Y_t + \theta \mathbb{E}_t [M_{t+1} \Pi_{t+1}^{\epsilon-1} \tilde{S}_{2,t+1}], && (30)\\
\Pi_t^{1-\epsilon} &= (1-\theta)(P_t^* \Pi_t)^{1-\epsilon} + \theta, && (31)\\
V_t^p &= \Pi_t^{\epsilon}((1-\theta)(P_t^* \Pi_t)^{-\epsilon} + \theta V_{t-1}^p), && (32)\\
\frac{R_t}{R} &= \left(\frac{R_{t-1}}{R} \right)^{\phi_R} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_{t-1}} \right)^{\phi_y} \right)^{1-\phi_R} \exp(\eta_{R,t}), && (33)\\
C_t + X_t &= Y_t, && (34)\\
Y_t &= \frac{A_t K_{t-1}^{\alpha} L_t^{1-\alpha}}{V_t^p}, && (35)
\end{aligned}
$$

as well as the law of motion for capital

$$
K_t = \left(1 + \frac{\overline{X}^{1/\chi}}{1 - 1/\chi} \left(\frac{X_t}{K_{t-1}} \right)^{1-1/\chi} - \frac{\overline{X}}{1 - 1/chi} \right) K_{t-1} \tag{36}
$$

and the four exogenous processes

$$
\begin{aligned}
\eta_{\beta,t+1} &= \rho_\beta \eta_{\beta,t} + \sigma_\beta \varepsilon_{\beta,t+1}, && (37)\\
\eta_{L,t+1} &= \rho_L \eta_{L,t} + \sigma_L \varepsilon_{L,t+1}, && (38)\\
\eta_{A,t+1} &= \rho_A \eta_{A,t} + \sigma_A \varepsilon_{A,t+1}, && (39)\\
\eta_{R,t+1} &= \rho_R \eta_{R,t} + \sigma_R \varepsilon_{R,t+1}. && (40)
\end{aligned}
$$

1.6 Deterministic Steady State

To provide an initial guess for the risk-adjusted linearization and to provide a verification that the model is coded correctly, I determine some reasonable guesses for the deterministic steady state.

Within this subsection, I denote the deterministic steady state values by an absence of a time subscript. The exogenous processes, by construction, have steady states of 0, i.e. $\eta_\beta = \eta_L = \eta_A = \eta_R = 0$. Further, $A = 1$.

Focusing now on the endogenous equilibrium conditions, from (21),

$$W = \frac{\varphi L^\nu}{C^{-\gamma}}.$$

From (22),

$$M = \beta.$$

From (24), the fact that $\overline{X}$ is the steady-state investment rate, and the fact that $\Phi'(\overline{X}) = 1$,

$$Q = 1.$$

From (25), first observing that,

$$\Phi(\overline{X}) = \frac{\overline{X}\chi}{\chi - 1} - \frac{\overline{X}}{\chi(\chi - 1)} = \overline{X}\frac{\chi^2 - 1}{\chi(\chi - 1)} = \overline{X}\frac{(\chi - 1)(\chi + 1)}{\chi(\chi - 1)} = \frac{\delta\chi}{\chi + 1}\frac{\chi + 1}{\chi} = \delta,$$

which ensures that K does indeed remain at steady state, it must be the cast that

$$\begin{aligned} 1 &= \beta(R_K + (1 - \delta + \Phi(\overline{X}) - \overline{X}) \\ R_K &= \frac{1}{\beta} + \overline{X} - 1. \end{aligned}$$

Equation (26) remains as it is but with time subscripts removed. From (29),

$$\tilde{S}_1 = MC \cdot Y + \theta\beta\Pi^\epsilon \tilde{S}_1 \Rightarrow \tilde{S}_1 = \frac{MC \cdot Y}{1 - \theta\beta\Pi^\epsilon}.$$

From (30),

$$\tilde{S}_2 = Y + \theta\beta\Pi^{\epsilon-1}\tilde{S}_2 \Rightarrow \tilde{S}_1 = \frac{Y}{1 - \theta\beta\Pi^{\epsilon-1}}.$$

Thus,

$$P^* = \frac{\epsilon}{\epsilon - 1} MC \frac{1 - \theta\Pi^{\epsilon-1}}{1 - \theta\Pi^\epsilon}.$$

From (31),

$$\Pi^{1-\epsilon} = (1-\theta)\left(P^*\Pi\right)^{1-\epsilon} + \theta$$

Note that P^* depends on MC and fundamental parameters, hence the above equation pins down MC, which then pins down the ratio of K to L. From (32),

$$V^p = \Pi^\epsilon \left((1-\theta)(P^*\Pi)^{-\epsilon} + \theta V^p\right)$$
$$V^p = \frac{(1-\theta)(P^*\Pi)^{-\epsilon}}{\Pi^{-\epsilon} - \theta} = \frac{(1-\theta)(P^*)^{-\epsilon}}{1-\theta\Pi^\epsilon}.$$

From the Taylor rule (33), the steady state interest and inflation rates are R and Π, respectively, and from the Euler equation (23), R must satisfy

$$R = \frac{\Pi}{\beta}.$$

From (34),

$$C + X = Y.$$

From (35),

$$Y = \frac{K^\alpha L^{1-\alpha}}{V^p}.$$

As shown previously, the steady-state investment rate in (36) equals δ, hence,

$$X = \delta K.$$

Finally, I claim that the deterministic steady state reduces to a nonlinear equation in L. Using the aggregate supply and capital accumulation equations,

$$C + \delta K = \frac{K^\alpha L^{1-\alpha}}{V^p}.$$

The optimal capital-labor ratio implies

$$K = \frac{\alpha}{1-\alpha}\frac{W}{R_K}L,$$
$$C + \delta K = \left(\frac{\alpha}{1-\alpha}\right)^\alpha \left(\frac{W}{R_K}\right)^\alpha \frac{L}{V^p}.$$

The intratemporal condition for consumption and labor implies

$$K = \frac{\alpha}{1-\alpha}\frac{\varphi L^\nu}{C^{-\gamma}R_K}L,$$
$$C + \delta K = \left(\frac{\alpha}{1-\alpha}\right)^\alpha \left(\frac{\varphi L^\nu}{C^{-\gamma}R_K}\right)^\alpha \frac{L}{V^p}.$$

Given a guess for L, I can compute C using these two equations. Given C, I can compute W. Given the wage W, I can compute K and MC. Given the marginal cost MC, I can compute the inflation-related terms.

2 Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$
\begin{aligned}
0 &= \log \mathbb{E}_t \left[\exp\left(\xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1}\right)\right] \\
z_{t+1} &= \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t)\varepsilon_{t+1},
\end{aligned}
$$

where z_t are (predetermined) state variables and y_t are (nondetermined) jump variables. For the remainder of this section, lower case variables are the logs of previously upper case variables, and with a small abuse of notation, let $s_{1,t} = \log(\tilde{S}_{1,t})$ and $s_{2,t} = \log(\tilde{S}_{2,t})$. Additionally, let $r_{k,t} = \log(R_{K,t})$ and $v_t = \log(V_t^p)$.

Equation (21) becomes

$$
\begin{aligned}
1 &= \varphi \exp(\eta_{L,t}) \frac{L_t^\nu}{C_t^{-\gamma} W_t} \\
0 &= \log \mathbb{E}_t \left[\exp\left(\underbrace{\log(\varphi) + \eta_{L,t} + \nu l_t - (-\gamma c_t + w_t)}_{\xi}\right)\right].
\end{aligned}
$$

Equation (22) will not be used in the system of equations for the risk-adjusted linearization, but it simplifies the other equations. Taking logs and re-arranging yields

$$
\begin{aligned}
0 &= \log(\beta) + \eta_{\beta,t+1} + (-\gamma c_{t+1}) - \eta_{\beta,t} - (-\gamma c_t) - m_{t+1} \\
m_{t+1} &= \underbrace{\log(\beta) - \eta_{\beta,t} + \gamma c_t}_{\xi} + \underbrace{\eta_{\beta,t+1} - \gamma c_{t+1}}_{\text{forward-looking}}.
\end{aligned}
$$

Equation (23) becomes

$$
0 = \log \mathbb{E}_t \left[\exp\left(\underbrace{\log(\beta) + r_t}_{\xi} + \underbrace{m_{t+1} - \pi_{t+1}}_{\text{forward-looking}}\right)\right].
$$

Equation (24) becomes

$$
\begin{aligned}
&1 - Q_t \left(1 - \frac{3\chi}{2}\left(\frac{X_t}{X_{t-1}}\right)^2\right) \\
&= \mathbb{E}_t \left[M_{t+1} Q_{t+1} \chi \left(\frac{X_{t+1}}{X_t}\right)^3\right] \\
&\log\left(1 - \exp(q_t)\left(1 - \frac{3\chi}{2}\exp(2x_t - 2x_{t-1})\right)\right) \\
&= \log \mathbb{E}_t \left[\exp\left(m_{t+1} + q_{t+1} + \log(\chi) + 3x_{t+1} - 3x_t\right)\right] \\
0 &= \log \mathbb{E}_t \left[\frac{\exp\left(m_{t+1} + q_{t+1} + \log(\chi) + 3x_{t+1} - 3x_t\right)}{\exp\left(\log\left(1 - \exp(q_t)\left(1 - \frac{3\chi}{2}\exp(2x_t - 2x_{t-1})\right)\right)\right)}\right].
\end{aligned}
$$

Re-arranging yields

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\chi) - 3x_t - \log \left(1 - \exp(q_t) \left(1 - \frac{3\chi}{2} \exp(2x_t - 2x_{t-1}) \right) \right)}_{\xi} \right) \right]$$
$$+ \log \mathbb{E}_t \left[\exp(\underbrace{m_{t+1} + q_{t+1} + 3x_{t+1}}_{\text{forward-looking}}) \right].$$

Equation (25) becomes

$$1 = \mathbb{E}_t \left[\frac{M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))}{Q_t} \right]$$
$$0 = \log \mathbb{E}_t \left[\exp\left(m_{t+1} + \log(\exp(r_{k,t+1}) + (1-\delta)\exp(q_{t+1})) - q_t \right) \right].$$

This equation is not linear in the forward-looking variables, we define a new variable

$$\Omega_t = R_{K,t} + Q_t(1-\delta), \tag{41}$$

so that we may represent (25) in the required form for risk-adjusted linearizations as the following two equations:

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{-q_t}_{\xi} + \underbrace{m_{t+1} + \omega_{t+1}}_{\text{forward-looking}} \right) \right],$$
$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\omega_t - \log(\exp(r_{k,t}) + (1-\delta)\exp(q_t))}_{\xi} \right) \right],$$

Equation (26) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{(1-\alpha)w_t + \alpha r_{k,t} - a_t - (1-\alpha)\log(1-\alpha) - \alpha\log(\alpha) - mc_t}_{\xi} \right) \right].$$

Equation (27) becomes

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{k_t - l_t - \log\left(\frac{\alpha}{1-\alpha} \right) - (w_t - r_{k,t})}_{\xi} \right) \right].$$

Like the stochastic discount factor, equation (28) will not be used in the system of equations, but it will be useful to simplify other equations. Taking logs yields

$$p_t^* = \log\left(\frac{\epsilon}{\epsilon-1}\right) + s_{1,t} - s_{2,t}.$$

Equation (29) becomes

$$\begin{aligned}
&\tilde{S}_{1,t} - MC_t Y_t \\
&= \mathbb{E}_t\left[\exp\left(\log(\theta) + \log(\beta) + m_{t+1} + \epsilon\pi_{t+1} + s_{1,t+1}\right)\right]
\end{aligned}$$

$$0 = \log\mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - \log(\exp(s_{1,t}) - \exp(mc_t)\exp(y_t))}_{\xi} + \underbrace{m_{t+1} + \epsilon\pi_{t+1} + s_{1,t+1}}_{\text{forward-looking}}\right)\right]$$

and equation (30) becomes

$$0 = \log\mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - \log(\exp(s_{2,t}) - \exp(y_t))}_{\xi} + \underbrace{m_{t+1} + (\epsilon-1)\pi_{t+1} + s_{2,t+1}}_{\text{forward-looking}}\right)\right].$$

Equation (31) becomes

$$\begin{aligned}
1 &= \frac{\Pi_t^{1-\epsilon}}{(1-\theta)(P_t^*\Pi_t)^{1-\epsilon} + \theta} \\
0 &= \log\mathbb{E}_t\left[\exp\left((1-\epsilon)\pi_t - \log((1-\theta)\exp((1-\epsilon)(p_t^* + \pi_t)) + \theta)\right)\right].
\end{aligned}$$

Equation (32) becomes

$$\begin{aligned}
1 &= \frac{V_t^p}{\Pi_t^{\epsilon}((1-\theta)(P_t^*\Pi_t)^{-\epsilon} + \theta V_{t-1}^p)} \\
0 &= \log\mathbb{E}_t\left[\exp\left(v_t - \epsilon\pi_t - \log((1-\theta)\exp(-\epsilon(p_t^* + \pi_t)) + \theta\exp(v_{t-1}))\right)\right].
\end{aligned}$$

Equation (33) becomes

$$0 = \log\mathbb{E}_t\left[\exp\left(\phi_R r_{t-1} + (1-\phi_R)r + (1-\phi_R)(\phi_\pi(\pi_t - \pi) + \phi_y(y_t - y_{t-1})) + \eta_{R,t} - r_t\right)\right].$$

Equation (34) becomes

$$0 = \log\mathbb{E}_t\left[\exp(y_t - \log(\exp(c_t) + \exp(x_t)))\right].$$

Equation (35) becomes

$$0 = \log\mathbb{E}_t\left[\exp(a_t + \alpha k_t + (1-\alpha)l_t - v_t - y_t)\right].$$

Equation (36) becomes

$$k_{t+1} = \log\left((1-\delta)\exp(k_t) + \left(1 - \frac{\chi}{2}\exp(2(x_t - x_{t-1}))\right)\exp(x_t)\right).$$

The autoregressive processes (37) to (40) remain as they are.

The jump variables are y_t, c_t, l_t, w_t, r_t, π_t, q_t, x_t, $r_{k,t}$, ω_t, mc_t, $s_{1,t}$, $s_{2,t}$, and v_t. The state variables are k_t, x_{t-1}, v_{t-1}, r_{t-1}, y_{t-1}, and the autoregressive processes. The equations defining the evolution of the lags x_{t-1}, v_{t-1}, r_{t-1}, and y_{t-1} are obtained by the formula $z_{(t-1)+1} = z_t$.
This system has three forward difference equations (25), (29), and (30). To ensure accuracy of the risk-adjusted linearization, I derive N-period ahead forward difference equations for all three.

We can write (25) recursively as

$$\begin{aligned}
Q_t &= \mathbb{E}_t[M_{t+1}(R_{K,t+1} + Q_{t+1}(1-\delta))] \\
&= \mathbb{E}_t[M_{t+1}R_{K,t+1} + (1-\delta)M_{t+1}\mathbb{E}_{t+1}[M_{t+2}(R_{K,t+2} + Q_{t+2}(1-\delta))]] \\
&= \mathbb{E}_t[M_{t+1}R_{K,t+1}] + (1-\delta)\mathbb{E}_t\mathbb{E}_{t+1}[M_{t+1}M_{t+2}(R_{K,t+2} + Q_{t+2}(1-\delta))].
\end{aligned}$$

By the tower property,

$$\begin{aligned}
Q_t &= \mathbb{E}_t[M_{t+1}R_{K,t+1}] + (1-\delta)\mathbb{E}_t[M_{t+1}M_{t+2}(R_{K,t+2} + Q_{t+2}(1-\delta))] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{2}(1-\delta)^{s-1}\left(\prod_{u=1}^{s}M_{t+s}\right)R_{K,t+s}\right) + M_{t+1}M_{t+2}Q_{t+2}(1-\delta)^2\right] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{2}(1-\delta)^{s-1}\left(\prod_{u=1}^{s}M_{t+s}\right)R_{K,t+s}\right) + M_{t+1}M_{t+2}(1-\delta)^2\mathbb{E}_{t+2}[M_{t+3}(R_{K,t+3} + Q_{t+3}(1-\delta))]\right] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{3}(1-\delta)^{s-1}\left(\prod_{u=1}^{s}M_{t+s}\right)R_{K,t+s}\right) + \prod_{s=1}^{3}((1-\delta)M_{t+s})Q_{t+3}\right]
\end{aligned}$$

and so on. Given this recursive structure, define $D_{Q,t}^{(n)}$ and $P_{Q,t}^{(n)}$ as

$$\begin{aligned}
D_{Q,t}^{(n)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}D_{Q,t+1}^{(n-1)}\right] \\
P_{Q,t}^{(n)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}P_{Q,t+1}^{(n-1)}\right]
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
D_{Q,t}^{(0)} &= \frac{R_{K,t+1}}{1-\delta} \\
P_{Q,t}^{(0)} &= Q_{t+1}.
\end{aligned}$$

Then I may write the N-period ahead recursive form of equation (25) as

$$Q_t = \sum_{n=1}^{N} D_{Q,t}^{(n)} + P_{Q,t}^{(N)}.$$

To see why this recursion works, it is simpler to first verify that $P_{Q,t}^{(3)}$ is correct:

$$\begin{aligned}
P_{Q,t}^{(1)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}Q_{t+1}\right] \\
P_{Q,t}^{(2)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}(\mathbb{E}_{t+1}[(1-\delta)M_{t+2}Q_{t+2}])\right] \\
&= \mathbb{E}_t\left[\mathbb{E}_{t+1}[(1-\delta)^2 M_{t+1}M_{t+2}Q_{t+2}]\right] \\
&= \mathbb{E}_t\left[\prod_{s=1}^{2}((1-\delta)M_{t+s})Q_{t+2}\right],
\end{aligned}$$

where the second equality for $P_{Q,t}^{(2)}$ follows from the fact that M_{t+1} is measurable with respect to the information set at time $t+1$ and can therefore be moved insided the conditional expectation $\mathbb{E}_{t+1}[\cdot]$. Continuing for one more recursion, I have

$$\begin{aligned}
P_{Q,t}^{(3)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}\mathbb{E}_{t+1}\left[\prod_{s=1}^{2}((1-\delta)M_{t+1+s})Q_{t+3}\right]\right] \\
&= \mathbb{E}_t\left[\prod_{s=1}^{3}((1-\delta)M_{t+s})Q_{t+3}\right].
\end{aligned}$$

Similarly, for $D_{Q,t}$, I have

$$\begin{aligned}
D_{Q,t}^{(1)} &= \mathbb{E}_t\left[(1-\delta)M_{t+1}\frac{R_{K,t+1}}{1-\delta}\right] = \mathbb{E}_t[M_{t+1}R_{K,t+1}] \\
D_{Q,t}^{(2)} &= \mathbb{E}_t[(1-\delta)M_{t+1}\mathbb{E}_{t+1}[M_{t+2}R_{K,t+2}]] \\
&= \mathbb{E}_t[(1-\delta)M_{t+1}M_{t+2}R_{K,t+2}] \\
D_{Q,t}^{(3)} &= \mathbb{E}_t[(1-\delta)M_{t+1}\mathbb{E}_{t+1}[(1-\delta)M_{t+2}M_{t+3}R_{K,t+3}]] \\
&= \mathbb{E}_t[(1-\delta)^2 M_{t+1}M_{t+2}M_{t+3}R_{K,t+3}].
\end{aligned}$$

Since $P_{Q,t}^{(n)}$ and $D_{Q,t}^{(n)}$ are time-t conditional expectations, they are measurable at time t, so they are not forward-looking variables. Thus, to get this version of (25) in the appropriate form, define $d_{q,n,t} = \log(D^{(n)}_{Q,t})$ and $p_{q,n,t} = \log(P^{(n)}_{Q,t})$, and use the following $2N+1$ equations:

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{q_t - \log \left(\sum_{n=1}^{N} \exp(d_{q,n,t}) + \exp(p_{q,n,t}) \right)}_{\xi} \right) \right] \tag{42}$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(1-\delta) - d_{q,n,t}}_{\xi} + \underbrace{m_{t+1} + d_{q,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{-d_{q,1,t}}_{\xi} + \underbrace{m_{t+1} + r_{k,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1, \end{cases} \tag{43}$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(1-\delta) - p_{q,n,t}}_{\xi} + \underbrace{m_{t+1} + p_{q,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(1-\delta) - p_{q,1,t}}_{\xi} + \underbrace{m_{t+1} + q_{t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1. \end{cases} \tag{44}$$

For (29), observe that

$$\begin{aligned} \tilde{S}_{1,t} &= MC_t Y_t + \theta\beta \mathbb{E}_t[M_{t+1}\Pi^{\epsilon}_{t+1}(MC_{t+1}Y_{t+1} + \theta\beta\mathbb{E}_{t+1}[M_{t+2}\Pi^{\epsilon}_{t+2}\tilde{S}_{1,t+2}])] \\ &= MC_t Y_t + \theta\beta \mathbb{E}_t[M_{t+1}\Pi^{\epsilon}_{t+1}MC_{t+1}Y_{t+1} + \theta\beta M_{t+1}\Pi^{\epsilon}_{t+1}M_{t+2}\Pi^{\epsilon}_{t+2}\tilde{S}_{1,t+2}] \\ &= MC_t Y_t + \mathbb{E}_t \left[\sum_{s=1}^{1} ((\theta\beta)^s \prod_{u=1}^{s} (M_{t+u}\Pi^{\epsilon}_{t+u})) MC_{t+s} Y_{t+s} \right] + \mathbb{E}_t \left[\prod_{s=1}^{2} (\theta\beta M_{t+s}\Pi^{\epsilon}_{t+s}) \tilde{S}_{1,t+2} \right]. \end{aligned}$$

Thus, define $D^{(n)}_{S1,t}$ and $P^{(n)}_{S1,t}$ as the recursions

$$\begin{aligned} D^{(n)}_{S1,t} &= \mathbb{E}_t[\theta\beta M_{t+1}\Pi^{\epsilon}_{t+1} D^{(n-1)}_{S1,t+1}], \\ P^{(n)}_{S1,t} &= \mathbb{E}_t[\theta\beta M_{t+1}\Pi^{\epsilon}_{t+1} P^{(n-1)}_{S1,t+1}], \end{aligned}$$

with boundary conditions

$$\begin{aligned} D^{(0)}_{S1,t} &= MC_t Y_t \\ P^{(0)}_{S1,t} &= \tilde{S}_{1,t}. \end{aligned}$$

Given these definitions, it follows that

$$\begin{aligned}
D^{(1)}_{S1,t} &= \mathbb{E}_t[\theta\beta M_{t+1}\Pi^{\epsilon}_{t+1}MC_{t+1}Y_{t+1}] \\
P^{(1)}_{S1,t} &= \mathbb{E}_t[\theta\beta M_{t+1}\Pi^{\epsilon}_{t+1}\tilde{S}_{1,t+1}] \\
P^{(2)}_{S1,t} &= \mathbb{E}_t[\theta\beta M_{t+1}\Pi^{\epsilon}_{t+1}\mathbb{E}_{t+1}[\theta\beta M_{t+2}\Pi^{\epsilon}_{t+2}\tilde{S}_{1,t+2}]] \\
&= \mathbb{E}_t[(\theta\beta)^2 M_{t+1}\Pi^{\epsilon}_{t+1}M_{t+2}\Pi^{\epsilon}_{t+2}\tilde{S}_{1,t+2}].
\end{aligned}$$

Thus, defining $d_{s1,t} = \log(D_{S1,t})$ and $p_{s1,t} = \log(P_{S1,t})$, the N-period ahead recursive form of (29) results in the $2N+1$ equations

$$0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{s_{1,t} - \log\left(\sum_{n=0}^{N-1}\exp(d_{s1,n,t}) + \exp(p_{s1,n,t})\right)}_{\xi}\right)\right] \tag{45}$$

$$0 = \begin{cases} \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - d_{s1,n,t}}_{\xi} + \underbrace{m_{t+1} + \epsilon\pi_{t+1} + d_{s1,n-1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n \geq 1 \\ \log \mathbb{E}_t\left[\exp\left(\underbrace{d_{s1,0,t} - mc_t - y_t}_{\xi}\right)\right] & \text{if } n = 0. \end{cases} \tag{46}$$

$$0 = \begin{cases} \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - p_{s1,n,t}}_{\xi} + \underbrace{m_{t+1} + \epsilon\pi_{t+1} + p_{s1,n-1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n > 1 \\ \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) + \log(\beta) - p_{s1,1,t}}_{\xi} + \underbrace{m_{t+1} + s_{1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n = 1. \end{cases} \tag{47}$$

It is straightforward to show that a similar recursive form applies to (30):

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{s_{2,t} - \log \left(\sum_{n=0}^{N-1} \exp(d_{s2,n,t}) + \exp(p_{s2,n,t}) \right)}_{\xi} \right) \right] \tag{48}$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - d_{s2,n,t}}_{\xi} + \underbrace{m_{t+1} + (\epsilon - 1)\pi_{t+1} + d_{s2,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n \geq 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{d_{s2,0,t} - y_t}_{\xi} \right) \right] & \text{if } n = 0. \end{cases} \tag{49}$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - p_{s2,n,t}}_{\xi} + \underbrace{m_{t+1} + (\epsilon - 1)\pi_{t+1} + p_{s2,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) + \log(\beta) - p_{s2,1,t}}_{\xi} + \underbrace{m_{t+1} + s_{2,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1, \end{cases} \tag{50}$$

where terms and boundary conditions are analogously defined.