These notes present a representative agent New Keynesian model with Epstein-Zin preferences and disaster shocks and borrow from Fernández-Villaverde and Levintal (2018) "Solution Methods for Models with Rare Disasters", Kekre and Lenel (2020) "Monetary Policy, Redistribution, and Risk Premia", and de Groot et al. (2020) "Valuation Risk Revalued."

1 Model

1.1 Household

The model admits a representative agent, so I directly write households' problem as the representative agent's. The representative household chooses consumption C_t, labor supply L_t, next-period nominal bond holdings B_t, and next-period capital holdings K_t to maximize, in the cashless limit, the Epstein-Zin preferences

$$V_t = \left((1 - \exp(\eta_{\beta,t})\beta) \left(C_t \mathcal{L}(L_t) \right)^{1-\psi} + \exp(\eta_{\beta,t})\beta \mathbb{E}_t \left[(V_{t+1})^{1-\gamma} \right]^{\frac{1-\psi}{1-\gamma}} \right)^{\frac{1}{1-\psi}}, \tag{1}$$

where β is the time preference rate; ψ is the inverse intertemporal elasticity of substitution; γ is the risk aversion coefficient; and the labor disutility function[1] $\mathcal{L}(L_t)$ is

$$\mathcal{L}(L_t) = \left(1 + (\psi - 1) \exp(\eta_{l,t}) \overline{\nu} \frac{L_t^{1+\nu}}{1+\nu} \right)^{\frac{\psi}{1-\psi}}, \tag{2}$$

where $\eta_{l,t}$ is a shock to labor disutility, $\overline{\nu}$ is the disutility of labor, ν is the inverse Frisch elasticity subject to the budget constraint

$$C_t + \frac{B_t}{P_t} + Q_t K_t \leq W_t L_t + (R_{k,t} + (1-\delta)Q_t) \exp(\eta_{k,t}) K_{t-1} + R_{t-1} \frac{B_{t-1}}{P_t} + F_t + T_t. \tag{3}$$

The quantity P_t is the price of the final consumption good, Q_t the real price of capital, W_t the real wage, $R_{k,t}$ the gross real rental rate on capital, δ the rate of capital depreciation, $\eta_{k,t}$ a disaster shock,[2] R_t the gross nominal interest rate on bonds, F_t real profits from intermediate firms, and T_t real lump-sum transfers from the government. The budget constraint (3) indicates that households can choose to consume, save in bonds, or invest in units of the capital stock from income through labor, capital, bonds, intermediate firms, and government transfers. Markets are assumed complete, but securities are in zero net supply. Because there is a representative agent, I may omit the Arrow securities from the budget constraint.

My notation treats B_{t-1} and K_{t-1} as the stocks of bonds and capital present at time t, while B_t and K_t are the chosen stocks of bonds and capital for the following period. I adopt

[1] This functional form is proposed by Shimer (2010) (see Chapter 1.4). Kekre and Lenel (2020) adapt this functional form for Epstein-Zin preferences. I have additionally included shocks to the disutility of labor.

[2] I follow Kekre and Lenel (2020)'s specifications of the disaster shock, but I could also define an intermediate variable $\hat{K}_t$ and set $K_t = \hat{K}_t \exp(\eta_{k,t})$, following Fernández-Villaverde and Levintal (2018).

this notation so that all time t choices are dated at time t rather than having to differentiate between the predetermined time-t variables from the endogenous controls.

Solving the household's problem is the same as solving the maximization problem

$$\begin{aligned} V_t = \max_{C_t,L_t,B_t,K_t} & \left((1-\exp(\eta_{\beta,t})\beta)\left(C_t\mathcal{L}(L_t)\right)^{1-\psi} + \exp(\eta_{\beta,t})\beta\mathbb{E}_t\left[(V_{t+1})^{1-\gamma}\right]^{\frac{1-\psi}{1-\gamma}}\right)^{\frac{1}{1-\psi}} \\ & + \lambda_t\left(W_tL_t + (R_{k,t}+(1-\delta)Q_t)\exp(\eta_{k,t})K_{t-1} + R_{t-1}\frac{B_{t-1}}{P_t} + F_t + T_t\right) \\ & - \lambda_t\left(C_t + \frac{B_t}{P_t} + Q_tK_t\right). \end{aligned} \tag{4}$$

Define $\hat{V}_t = V_t^{1-\psi}$, and conjecture that V_t is a function of the state variables K_{t-1} and B_{t-1}, among other states (e.g. the realized shocks). The first-order conditions with respect to controls are

$$\begin{aligned} 0 &= \frac{1}{1-\psi}\hat{V}_t^{\frac{\psi}{1-\psi}}(1-\exp(\eta_{\beta,t})\beta)(1-\psi)C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi} - \lambda_t \\ 0 &= \frac{1}{1-\psi}\hat{V}_t^{\frac{\psi}{1-\psi}}(1-\exp(\eta_{\beta,t})\beta)(1-\psi)C_t^{1-\psi}\mathcal{L}(L_t)^{-\psi}\frac{\partial}{\partial L}\mathcal{L}(L_t) + \lambda_tW_t \\ 0 &= \frac{1}{1-\psi}\hat{V}_t^{\frac{\psi}{1-\psi}}\exp(\eta_{\beta,t})\beta\frac{1-\psi}{1-\gamma}\mathbb{E}_t[(V_{t+1})^{1-\gamma}]^{\frac{\gamma-\psi}{1-\gamma}}(1-\gamma)\mathbb{E}_t\left[V_{t+1}^{-\gamma}\frac{\partial V_{t+1}}{\partial B_t}\right] - \frac{\lambda_t}{P_t}, \\ 0 &= \frac{1}{1-\psi}\hat{V}_t^{\frac{\psi}{1-\psi}}\exp(\eta_{\beta,t})\beta\frac{1-\psi}{1-\gamma}\mathbb{E}_t[(V_{t+1})^{1-\gamma}]^{\frac{\gamma-\psi}{1-\gamma}}(1-\gamma)\mathbb{E}_t\left[V_{t+1}^{-\gamma}\frac{\partial V_{t+1}}{\partial K_t}\right] - \lambda_tQ_t. \end{aligned}$$

The envelope conditions for B_{t-1} and K_{t-1} are

$$\begin{aligned} \frac{\partial V_t}{\partial B_{t-1}} &= \lambda_t\frac{R_{t-1}}{P_t}, \\ \frac{\partial V_t}{\partial K_{t-1}} &= \lambda_t(R_{k,t}+(1-\delta)Q_t)\exp(\eta_{k,t}). \end{aligned}$$

The first two first-order conditions can be combined by isolating λ_t, which obtains the intratemporal consumption-labor condition

$$\begin{aligned} 0 &= C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi} + \frac{C_t^{1-\psi}}{W_t}\mathcal{L}(L_t)^{-\psi}\frac{\partial}{\partial L}\mathcal{L}(L_t) \\ &= \mathcal{L}(L_t) + \frac{C_t}{W_t}\frac{\partial}{\partial L}\mathcal{L}(L_t). \end{aligned}$$

The derivative of $\mathcal{L}(L_t)$ evaluates to

$$\begin{aligned} \frac{\partial}{\partial L}\mathcal{L}(L_t) &= \frac{\psi}{1-\psi}\left(1+(\psi-1)\exp(\eta_{l,t})\overline{\nu}\frac{L_t^{1+\nu}}{1+\nu}\right)^{\frac{2\psi-1}{1-\psi}}(\psi-1)\exp(\eta_{l,t})\overline{\nu}L_t^{\nu} \\ &= \frac{\psi-1}{1-\psi}\mathcal{L}(L_t)^{\frac{2\psi-1}{\psi}}\psi\exp(\eta_{l,t})\overline{\nu}L_t^{\nu} = -\mathcal{L}(L_t)^{\frac{2\psi-1}{\psi}}\psi\exp(\eta_{l,t})\overline{\nu}L_t^{\nu}, \end{aligned}$$

hence the intratemporal consumption-labor condition becomes

$$0 = 1 - \frac{C_t}{W_t}\psi \exp(\eta_{l,t})\overline{\nu} L_t^{\nu} \mathcal{L}(L_t)^{\frac{\psi-1}{\psi}}.$$

Re-arrange to acquire

$$W_t = \frac{\psi \exp(\eta_{l,t})\overline{\nu} C_t L_t^{\nu}}{\mathcal{L}(L_t)^{\frac{1-\psi}{\psi}}}. \tag{5}$$

Notice additionally that

$$\mathcal{L}(L_t)^{\frac{1-\psi}{\psi}} = 1 + (\psi - 1)\exp(\eta_{l,t})\overline{\nu}\frac{L_t^{1+\nu}}{1+\nu}.$$

The Euler equation for bonds can be obtained by combining the envelope condition for B_{t-1} with the first and third first-order conditions. Iterate the envelope condition for B_{t-1} forward by one period.

$$\frac{\partial V_{t+1}}{\partial B_t} = \lambda_{t+1}\frac{R_t}{P_{t+1}}.$$

Define

$$\mathcal{CE}_t = \mathbb{E}_t[V_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}} \tag{6}$$

as households' certainty equivalent. Substitute this expression and the iterated envelope condition into the third first-order condition.

$$\begin{aligned} 0 &= \frac{1}{1-\psi}\hat{V}_t^{\frac{\psi}{1-\psi}} \exp(\eta_{\beta,t})\beta(1-\psi)\mathcal{CE}_t^{\gamma-\psi}\mathbb{E}_t\left[V_{t+1}^{-\gamma}\lambda_{t+1}\frac{R_t}{P_{t+1}}\right] - \frac{\lambda_t}{P_t} \\ &= \hat{V}_t^{\frac{\psi}{1-\psi}} \exp(\eta_{\beta,t})\beta\mathcal{CE}_t^{\gamma-\psi}\mathbb{E}_t\left[V_{t+1}^{-\gamma}\lambda_{t+1}\frac{R_t}{P_{t+1}}\right] - \frac{\lambda_t}{P_t}. \end{aligned}$$

Observe that, from the first first-order condition,

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{\hat{V}_{t+1}^{\frac{\psi}{1-\psi}}(1-\exp(\eta_{\beta,t+1})\beta)C_{t+1}^{-\psi}\mathcal{L}(L_{t+1})^{1-\psi}}{\hat{V}_t^{\frac{\psi}{1-\psi}}(1-\exp(\eta_{\beta,t})\beta)C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi}},$$

and that $\hat{V}_t$ satisfies

$$\hat{V}_t^{\frac{\psi}{1-\psi}} = V_t^{\psi}.$$

Divide the third first-order condition by λ_t/P_t and substitute these quantities. Re-arrange to acquire

$$\begin{aligned} 1 &= \exp(\eta_{\beta,t})\beta\mathcal{CE}_t^{\gamma-\psi}\mathbb{E}_t\left[V_{t+1}^{-\gamma}R_t\frac{P_t}{P_{t+1}}V_{t+1}^{\psi}\frac{(1-\exp(\eta_{\beta,t+1})\beta)C_{t+1}^{-\psi}\mathcal{L}(L_{t+1})^{1-\psi}}{(1-\exp(\eta_{\beta,t})\beta)C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi}}\right] \\ &= \mathcal{CE}_t^{\gamma-\psi}\mathbb{E}_t\left[\exp(\eta_{\beta,t})\beta\frac{(1-\exp(\eta_{\beta,t+1})\beta)C_{t+1}^{-\psi}\mathcal{L}(L_{t+1})^{1-\psi}}{(1-\exp(\eta_{\beta,t})\beta)C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi}}V_{t+1}^{\psi-\gamma}R_t\frac{P_t}{P_{t+1}}\right]. \end{aligned}$$

Define the gross inflation rate and real stochastic discount factors as

$$\Pi_t = \frac{P_{t+1}}{P_t}, \tag{7}$$

$$M_{t,t+1} = \exp(\eta_{\beta,t})\beta \frac{1-\exp(\eta_{\beta,t+1})\beta}{1-\exp(\eta_{\beta,t})\beta} \frac{C_{t+1}^{-\psi}\mathcal{L}(L_{t+1})^{1-\psi}}{C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi}} \left(\frac{V_{t+1}}{\mathcal{CE}_t}\right)^{\psi-\gamma}, \tag{8}$$

where the two time subscripts in $M_{t,t+1}$ indicate that the real stochastic discount factor includes terms dated at times t and $t+1$. Using these definitions, the Euler equation for bonds becomes

$$1 = \mathbb{E}_t\left[M_{t,t+1}\frac{R_t}{\Pi_{t+1}}\right]. \tag{9}$$

Households' asset pricing equation for capital can be obtained using similar steps and will take the familiar form from consumption-based asset pricing. Iterate the envelope condition for K_{t-1} forward by one period.

$$\frac{\partial V_{t+1}}{\partial K_t} = \lambda_{t+1}(R_{K,t+1} + (1-\delta)Q_{t+1})\exp(\eta_{k,t+1}).$$

Substitute this expression and other quantities derived previously into the fourth first-order condition.

$$0 = V_t^{\psi}\exp(\eta_{\beta,t})\beta\mathcal{CE}_t^{\gamma-\psi}\mathbb{E}_t\left[V_{t+1}^{-\gamma}\lambda_{t+1}(R_{K,t+1}+(1-\delta)Q_{t+1})\exp(\eta_{k,t+1})\right] - \lambda_t Q_t.$$

Divide by λ_t and plug in λ_{t+1}/λ_t.

$$\begin{aligned}
0 &= V_t^{\psi}\exp(\eta_{\beta,t})\beta\mathcal{CE}_t^{\gamma-\psi} \\
&\quad \times \mathbb{E}_t\left[\frac{V_{t+1}^{\psi-\gamma}(1-\exp(\eta_{\beta,t+1})\beta)C_{t+1}^{-\psi}\mathcal{L}(L_{t+1})^{1-\psi}}{V_t^{\psi}(1-\exp(\eta_{\beta,t})\beta)C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi}}(R_{K,t+1}+(1-\delta)Q_{t+1})\exp(\eta_{k,t+1})\right] - Q_t \\
&= \mathbb{E}_t\left[M_{t,t+1}(R_{K,t+1}+(1-\delta)Q_{t+1})\exp(\eta_{k,t+1})\right] - Q_t \\
Q_t &= \mathbb{E}_t\left[M_{t,t+1}(R_{K,t+1}+(1-\delta)Q_{t+1})\exp(\eta_{k,t+1})\right].
\end{aligned}$$

In summary, households' optimality conditions are

$$V_t = \left((1-\exp(\eta_{\beta,t})\beta)(C_t\mathcal{L}(L_t))^{1-\psi} + \exp(\eta_{\beta,t})\beta\mathcal{CE}_t^{1-\psi}\right)^{\frac{1}{1-\psi}} \tag{10}$$

$$\mathcal{CE}_t = \mathbb{E}_t[V_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}, \tag{11}$$

$$W_t = \frac{\psi\exp(\eta_{l,t})\overline{\nu}C_tL_t^{\nu}}{\mathcal{L}(L_t)^{\frac{\psi-1}{\psi}}}, \tag{12}$$

$$\mathcal{L}(L_t) = \left(1+(\psi-1)\exp(\eta_{l,t})\overline{\nu}\frac{L_t^{1+\nu}}{1+\nu}\right)^{\frac{\psi}{1-\psi}}, \tag{13}$$

$$M_{t,t+1} = \exp(\eta_{\beta,t})\beta\frac{1-\exp(\eta_{\beta,t+1})\beta}{1-\exp(\eta_{\beta,t})\beta}\frac{C_{t+1}^{-\psi}\mathcal{L}(L_{t+1})^{1-\psi}}{C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi}}\left(\frac{V_{t+1}}{\mathcal{CE}_t}\right)^{\psi-\gamma}, \tag{14}$$

$$1 = \mathbb{E}_t\left[M_{t,t+1}\frac{R_t}{\Pi_{t+1}}\right], \tag{15}$$

$$Q_t = \mathbb{E}_t\left[M_{t,t+1}\left(R_{K,t+1}+(1-\delta)Q_{t+1}\right)\exp(\eta_{k,t+1})\right]. \tag{16}$$

1.2 Production

Final Producers There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} \tag{17}$$

where $\epsilon > 1$ so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t\left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j)Y_t(j)\,dj. \tag{18}$$

The FOC for $Y_t(j)$ is

$$0 = P_t\frac{\epsilon}{\epsilon-1}\left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}}\right)^{\frac{1}{\epsilon-1}}\frac{\epsilon-1}{\epsilon}Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j)$$

$$0 = \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}}\right)^{\frac{1}{\epsilon-1}}Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t}$$

$$0 = \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}}\right)^{-\frac{\epsilon}{\epsilon-1}}Y_t(j) - \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon}.$$

Re-arranging obtains

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon}Y_t. \tag{19}$$

Plugging this quantity into the identity

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j)\, dj$$

and simplifying yields the price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon}\, dj \right)^{\frac{1}{1-\epsilon}}. \tag{20}$$

Intermediate Producers Intermediate goods are producing according to the Cobb-Douglas technology

$$Y_t(j) = \hat{K}_t^{\alpha}(j)(\exp(\eta_{a,t}) L_t)^{1-\alpha}(j) - \kappa_y \exp(\eta_{A,t}), \tag{21}$$

where $\hat{K}_t = \exp(\eta_{k,t}) K_{t-1}$ and productivity $\exp(\eta a, t)$ follows the unit root process

$$\eta_{a,t} = \mu_a + \eta_{a,t-1} + \sigma_a \varepsilon_{a,t} + \kappa_a \eta_{k,t}. \tag{22}$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\begin{aligned} &\min_{\hat{K}_t(j), L_t(j)} R_{k,t} \hat{K}_t(j) + W_t L_t(j) \\ &\text{s.t.} \quad \hat{K}_t^{\alpha}(j)(\exp(\eta_{a,t}) L_t)^{1-\alpha}(j) - \kappa_y \exp(\eta_{a,t}) \geq \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t. \end{aligned} \tag{23}$$

The RHS of the inequality constraint is the demand from final goods producers for intermediate j. The Lagrangian is

$$\begin{aligned} \mathcal{H} = &\, R_{k,t} \hat{K}_t(j) + W_t L_t(j) \\ &+ MC_t(j) \left(\left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - \hat{K}_t^{\alpha}(j)(\exp(\eta_{a,t}) L_t)^{1-\alpha}(j) + \kappa_y \exp(\eta_{a,t}) \right), \end{aligned}$$

so the first-order conditions are

$$\begin{aligned} 0 &= R_{k,t} - MC_t(j) \alpha \exp(\eta_{a,t})^{1-\alpha} \left(\frac{L_t(j)}{\hat{K}_t(j)} \right)^{1-\alpha} \\ 0 &= W_t - MC_t(j)(1-\alpha) \exp(\eta_{a,t})^{1-\alpha} \left(\frac{\hat{K}_t(j)}{L_t(j)} \right)^{\alpha}, \end{aligned}$$

hence the optimal capital-labor ratio satisfies

$$\begin{aligned} \frac{R_{k,t}}{\alpha \exp(\eta_{a,t})^{1-\alpha} (\hat{K}_t(j)/L_t(j))^{\alpha-1}} &= \frac{W_t}{(1-\alpha) \exp(\eta_{a,t})^{1-\alpha} (\hat{K}_t(j)/L_t(j))^{\alpha}} \\ \frac{\hat{K}_t(j)}{L_t(j)} &= \frac{\alpha}{1-\alpha} \frac{W_t}{R_{k,t}}. \end{aligned}$$

Since the RHS does not vary with j, all firms choose the same capital-labor ratio. Given this optimal ratio, the marginal cost satisfies

$$\begin{aligned} MC_t &= \frac{R_{k,t}}{\alpha \exp(\eta_{a,t})^{1-\alpha}} \left(\frac{\hat{K}_t}{L_t} \right)^{1-\alpha} \\ &= \frac{R_{k,t}}{\alpha \exp(\eta_{a,t})^{1-\alpha}} \left(\frac{\alpha}{1-\alpha} \frac{W_t}{R_{k,t}} \right)^{1-\alpha} \\ &= \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^{\alpha} \frac{W_t^{1-\alpha} R_{k,t}^{\alpha}}{\exp(\eta_{a,t})^{1-\alpha}}. \end{aligned}$$

It follows that

$$\begin{aligned} R_{k,t}\hat{K}_t + W_t L_t &= \left(\frac{R_{k,t}}{\exp(\eta_{a,t})} \left(\frac{\hat{K}_t}{L_t} \right)^{1-\alpha} + \frac{W_t}{\exp(\eta_{a,t})} \left(\frac{L_t}{\hat{K}_t} \right)^{\alpha} \right) (\exp(\eta_{a,t}) \hat{K}_t^{\alpha} L_t^{1-\alpha}) \\ &= (\alpha MC_t + (1-\alpha) MC_t)\, Y_t(j) = MC_t Y_t(j). \end{aligned}$$

Therefore, (real) profits for an intermediate producer become

$$F_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - MC_t Y_t(j) - \kappa_y \exp(\eta_{a,t}). \tag{24}$$

In addition to the capital-labor choice, firms also have the chance to reset prices in every period with probability $1-\theta$. This problem can be written as

$$\begin{aligned} &\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} \\ &\times \left(\frac{P_t(j)}{P_{t+s}} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - \kappa_y \exp(\eta_{a,t}) \right), \end{aligned} \tag{25}$$

where I have imposed that intermediate output equals demand. The first-order condition is

$$\begin{aligned} 0 = (1-\epsilon) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} (P_{t+s})^{-(1-\epsilon)} Y_{t+s} \\ + \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta)^s \frac{\exp(\eta_{\beta,t+s})}{\exp(\eta_{\beta,t})} \frac{u'(C_{t+s})}{u'(C_t)} mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s} \end{aligned}$$

Divide by $P_t(j)^{-\epsilon}/(\exp(\eta_{\beta,t})u'(C_t))$, apply the abuse of notation that $\prod_{u=1}^{0}\Pi_{t+u}=1$, and re-arrange to obtain

$$\begin{aligned}
P_t(j) &= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})P_{t+s}^{\epsilon-1}Y_{t+s}}\\
&= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}P_t^{\epsilon}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})P_t^{\epsilon-1}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon-1}Y_{t+s}}\\
\frac{P_t(j)}{P_t} &= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon}Y_{t+s}}{\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})\left(\prod_{u=1}^{s}\Pi_{t+u}\right)^{\epsilon-1}Y_{t+s}}.
\end{aligned}$$

This expression gives the optimal (real) reset price $P_t^* \equiv P_t(j)/P_t$ (note that the RHS does not depend on j). Define

$$\overline{S}_{1,t} = \mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})mc_{t+s}Y_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+s}\right)^{\epsilon}, \tag{26}$$

$$\overline{S}_{2,t} = \mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta)^s\exp(\eta_{\beta,t+s})u'(C_{t+s})Y_{t+s}\left(\prod_{u=1}^{s}\Pi_{t+s}\right)^{\epsilon-1}. \tag{27}$$

Using these definitions, I may write the optimal reset price more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon-1}\frac{\overline{S}_{1,t}}{\overline{S}_{2,t}} \tag{28}$$

where $\overline{S}_{1,t}$ and $\overline{S}_{2,t}$ satisfy the recursions

$$\begin{aligned}
\overline{S}_{1,t} &= \exp(\eta_{\beta,t})u'(C_t)MC_tY_t + \theta\beta\mathbb{E}_t\Pi_{t+s}^{\epsilon}\overline{S}_{1,t+1}\\
\overline{S}_{2,t} &= \exp(\eta_{\beta,t})u'(C_t)Y_t + \theta\beta\mathbb{E}_t\Pi_{t+s}^{\epsilon-1}\overline{S}_{2,t+1}.
\end{aligned}$$

These recursions can be further rewritten as

$$\begin{aligned}
\frac{\overline{S}_{1,t}}{\exp(\eta_{\beta,t})u'(C_t)} &= MC_tY_t + \theta\beta\mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)}\Pi_{t+s}^{\epsilon}\frac{\overline{S}_{1,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})}\right]\\
\frac{\overline{S}_{2,t}}{\exp(\eta_{\beta,t})u'(C_t)} &= Y_t + \theta\beta\mathbb{E}_t\left[\frac{\exp(\eta_{\beta,t+1})u'(C_{t+1})}{\exp(\eta_{\beta,t})u'(C_t)}\Pi_{t+s}^{\epsilon-1}\frac{\overline{S}_{2,t+1}}{\exp(\eta_{\beta,t+1})u'(C_{t+1})}\right],
\end{aligned}$$

By defining $S_{1,t} \equiv \overline{S}_{1,t}/(\exp(\eta_{\beta,t})u'(C_t))$ and $S_{2,t} \equiv \overline{S}_{2,t}/(\exp(\eta_{\beta,t})u'(C_t))$, I can simplify these recursions into the form I use for the numerical solution.

From this section, we obtain the following five equilibrium conditions:

$$MC_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{W_t^{1-\alpha}R_{k,t}^{\alpha}}{\exp(\eta_{a,t})^{1-\alpha}}, \tag{29}$$

$$\frac{\exp(\eta_{k,t})K_{t-1}}{L_t} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_{k,t}}, \tag{30}$$

$$P_t^* = \frac{\epsilon}{\epsilon-1}\frac{S_{1,t}}{S_{2,t}}, \tag{31}$$

$$S_{1,t} = MC_tY_t + \theta\mathbb{E}_t[M_{t,t+1}\Pi_{t+1}^{\epsilon}S_{1,t+1}], \tag{32}$$

$$S_{2,t} = Y_t + \theta\mathbb{E}_t[M_{t,t+1}\Pi_{t+1}^{\epsilon-1}S_{2,t+1}]. \tag{33}$$

Capital Producers For expositional clarity, I model capital production as its own sector.[3] After intermediate firms finish using the time t stock of capital $\hat{K}_t = \exp(\eta_{k,t})K_{t-1}$, households sell their capital holdings to capital producers, who solve the problem

$$\max_{X_t}\left(\Phi\left(\frac{X_t}{\hat{K}_t}\right)-1\right)Q_t\hat{K}_t - X_t. \tag{34}$$

In other words, capital producers maximize the static profits from producing new capital since it costs them $Q_t\hat{K}_t$ to purchase $\hat{K}_t$ units of capital and X_t in investment to produce $\Phi(X_t/\hat{K}_t)Q_t\hat{K}_t$ in revenue. The solution to this problem yields the Tobin's Q equation

$$1 = \Phi'\left(\frac{X_t}{\exp(\eta_{k,t})K_{t-1}}\right)Q_t. \tag{35}$$

After households purchase capital back from capital producers, a fraction of the capital depreciates. Thus, the evolution of the aggregate capital stock is

$$K_t = \left(1-\delta+\Phi\left(\frac{X_t}{\exp(\eta_{k,t})K_{t-1}}\right)\right)\exp(\eta_{k,t})K_{t-1}, \tag{36}$$

where δ is the rate of depreciation and $\Phi(\cdot)$ is an investment technology satisfying standard conditions.[4]

1.3 Monetary Policy

I specify the monetary policy rule as the following Taylor rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\phi_r}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_{t-1}}\exp(-\mu_a)\right)^{\phi_y}\right)^{1-\phi_r}\exp(\eta_{r,t}) \tag{37}$$

Any proceeds from monetary policy are distributed as lump sum to the representative household.

[3] I could subsume capital production within the household problem by adding as an additional constraint $K_t = \Phi(X_t/\hat{K}_t)\hat{K}_t$.

[4] $\Phi'(\cdot) > 0$, $\Phi''(\cdot) < 0$, $\Phi(X_{ss}/K_{ss}) = \delta$, where X_{ss}/K_{ss} is the steady state investment rate.

1.4 Aggregation

The price level is currently characterized as the integral

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon}\, dj.$$

To represent the model entirely in terms of aggregates, notice that, without loss of generality, we may re-order the fraction θ of firms which cannot reset prices to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\theta)(P_t^*)^{1-\epsilon} + \int_{1-\theta}^1 P_{t-1}(j)^{1-\epsilon}\, dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\theta)(P_t^*)^{1-\epsilon} + \theta \int_0^1 P_{t-1}(j)^{1-\epsilon}\, dj = (1-\theta)(P_t^*)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon}.$$

Dividing by $P_{t-1}^{1-\epsilon}$ implies

$$\Pi_t^{1-\epsilon} = (1-\theta)(P_t^*\Pi_t)^{1-\epsilon} + \theta. \tag{38}$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$\begin{aligned} V_t^p &= \int_0^{1-\theta} (P_t^*)^{-\epsilon}\, dj + \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\epsilon} dj \\ &= \int_0^{1-\theta} (P_t^*\Pi_t)^{-\epsilon} \left(\frac{1}{\Pi_t}\right)^{-\epsilon} dj + \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} dj \\ &= (1-\theta)(P_t^*\Pi_t)^{-\epsilon}\Pi_t^\epsilon + \Pi_t^\epsilon \int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj. \end{aligned}$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\theta}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \theta \int_0^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \theta V_{t-1}^p.$$

Thus, we acquire

$$V_t^p = \Pi_t^\epsilon((1-\theta)(P_t^*\Pi_t)^{-\epsilon} + \theta V_{t-1}^p) \tag{39}$$

1.5 Equilibrium

To close the model, I need to specify the functional form for investment, remaining aggregate shocks, and market-clearing conditions.

1.5.1 Investment Function

Following Jermann (1998), I assume the investment function takes the concave form

$$\Phi\left(\frac{X_t}{\hat{K}_t}\right) = \frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{\hat{K}_t}\right)^{1-1/\chi} - \frac{\overline{X}}{\chi(\chi-1)} \tag{40}$$

where $\overline{X} = \delta\chi/(\chi+1)$ is the steady-state investment rate (per unit of capital). The first derivative of $\Phi(\cdot)$ w.r.t. $X_t/\hat{K}_t$ is

$$\Phi'\left(\frac{X_t}{\hat{K}_t}\right) = \overline{X}^{1/\chi}\left(\frac{X_t}{\hat{K}_t}\right)^{-1/\chi}. \tag{41}$$

This functional form implies the law of motion

$$\begin{aligned}
K_t &= \left(1-\delta+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{\hat{K}_t}\right)^{1-1/\chi} - \frac{\overline{X}}{\chi(\chi-1)}\right)\hat{K}_t \\
&= \left(1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{\hat{K}_t}\right)^{1-1/\chi} - \delta\left(1+\frac{1}{(\chi-1)(\chi+1)}\right)\right)\hat{K}_t \\
&= \left(1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{\hat{K}_t}\right)^{1-1/\chi} - \delta\left(\frac{\chi^2}{(\chi-1)(\chi+1)}\right)\right)\hat{K}_t \\
&= \left(1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{\hat{K}_t}\right)^{1-1/\chi} - \frac{\delta\chi^2}{\chi^2(1-1/\chi)(1+1/\chi)}\right)\hat{K}_t \\
&= \left(1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\left(\frac{X_t}{\hat{K}_t}\right)^{1-1/\chi} - \frac{\overline{X}}{1-1/\chi}\right)\hat{K}_t.
\end{aligned}$$

If $K_t = \hat{K}_t = K_{ss}$ and $X_{ss}/K_{ss} = \overline{X}$, then

$$1+\frac{\overline{X}^{1/\chi}}{1-1/\chi}\overline{X}^{1-1/\chi} - \frac{\overline{X}}{1-1/\chi} = 1+\frac{\overline{X}}{1-1/\chi} - \frac{\overline{X}}{1-1/\chi} = 1,$$

thus verifying the original conjecture that $\overline{X}$ represents the steady-state investment rate.

1.5.2 Exogenous Shocks

There are five shocks in the model: $\eta_{a,t}$, $\eta_{k,t}$, $\eta_{\beta,t}$, $\eta_{l,t}$, and $\eta_{r,t}$. The first shock has been specified as an AR(1) with a disaster component. I specify $\eta_{k,t}$ below. Without loss of generality, I assume the last three shocks follow AR(1) processes with persistence ρ_i and standard deviation σ_i.

There are multiple ways to model the disaster shock. The simplest approach is the approach taken by Kekre and Lenel (2020). The shock $\eta_{k,t}$ equals $\underline{\eta}_k < 0$ with probability p_t and zero with probability $1-p_t$. The probability of a disaster varies according to a two-state Markov process, which takes values $\underline{p}$ and $\overline{p}$, with $\underline{p} < \overline{p}$. The transition probability away from $\underline{p}$ ($\overline{p}$) toward $\overline{p}$ ($\underline{p}$) is $\overline{\rho}_p$ ($\underline{\rho}_p$). The values and transition probabilities of the Markov chain are restricted by the requirement that the unconditional mean of the Markov chain must be p, i.e.

$$\frac{\underline{\rho}_p \underline{p} + \overline{\rho}_p \overline{p}}{\underline{\rho}_p + \overline{\rho}_p} = p.$$

In this case, I construct martingale difference sequences for $\eta_{k,t}$ and p_t by defining $\varepsilon_{k,t} = \eta_{k,t} - \mathbb{E}_{t-1}[\eta_{k,t}]$ and $\varepsilon_{p,t} = p_t - \mathbb{E}_{t-1}[p_t]$. Then

$$\eta_{k,t+1} = \underline{\eta}_k p_t + \varepsilon_{k,t+1}, \tag{42}$$

$$\eta_{p,t+1} = \begin{cases} \underline{\rho}_p \underline{p} + (1-\underline{\rho}_p)\overline{p} & \text{if } p_t = \underline{p} \\ \overline{\rho}_p \overline{p} + (1-\overline{\rho}_p)\underline{p} & \text{if } p_t = \overline{p} \end{cases} + \varepsilon_{p,t+1}. \tag{43}$$

The second approach models the time variation in p_t as a discrete-time Cox-Ingersoll-Ross process:

$$p_{t+1} = (1-\rho_p)p + \rho_p p_t + \sqrt{p_t}\sigma_p \varepsilon_{p,t+1}. \tag{44}$$

In this case, $\varepsilon_{p,t+1} \sim N(0,1)$. The disaster shock $\eta_{k,t}$ still follows the martingale difference sequence $\tilde{\eta}_{k,t}$. The disadvantage of this approach is that p_{t+1} can move outside $[0,1]$, given a sufficiently large shock.

The third approach models the time variation in p_t in logs to avoid p_t moving below zero:

$$\log(p_{t+1}) = (1-\rho_p)\log(p) + \rho_p \log(p_t) + \sigma_p \varepsilon_{p,t+1}. \tag{45}$$

This approach still risks p_t increasing above 1.

The fourth approach models the disaster shock as an exponentially distributed shock $\eta_{k,t} \sim \text{Exponential}(p_t)$, where p_t is now the intensity of the exponential distribution. The evolution of p_t can be modeled in three ways, as before, but p_t may now increase above one.

The fifth approach uses time variation in the size of the shock rather than the probability. The disaster shock $\eta_{k,t}$ takes the form

$$\eta_{k,t} = \hat{\eta}_{k,t} p_t, \tag{46}$$

where $\tilde{\eta}_{k,t}$ is modeled as either a Bernoulli or an Exponential random variable and p_t evolves either according to the Cox-Ingersoll-Ross process or in logs.

1.5.3 Market Clearing

Markets must clear for capital, labor, bonds, final goods, and intermediate goods. The first three markets clear as a consequence of optimality conditions and the assumption that bonds have zero net supply. To clear the market for final goods, we set the sum of aggregate consumption demand C_t and investment demand X_t equal to aggregate supply Y_t net of fixed costs, which satisfies

$$\int_0^1 (\hat{K}_t^{\alpha}(L_t \exp(\eta_{a,t}))^{1-\alpha} - \kappa_y \exp(\eta_{a,t}))\, dj = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t\, dj$$
$$\hat{K}_t^{\alpha}(\exp(\eta_{a,t})L_t)^{1-\alpha} - \kappa_y \exp(\eta_{a,t}) = Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj = V_t^p Y_t.$$

Re-arranging yields the output market-clearing condition

$$C_t + X_t = Y_t, \tag{47}$$
$$Y_t = \frac{(\exp(\eta_{k,t})K_{t-1})^{\alpha}(\exp(\eta_{a,t})L_t)^{1-\alpha} - \kappa_y \exp(\eta_{a,t})}{V_t^p}. \tag{48}$$

It can be shown that $V_t^p \geq 1$ by applying Jensen's inequality. For our purposes, because the dimensionality of our model is not too large, we add the auxiliary Y_t variable, even though we could substitute it out of the system of equations.

1.5.4 Equilibrium Conditions

All together, the full set of equilibrium conditions are

$$
\begin{aligned}
V_t &= \left((1-\exp(\eta_{\beta,t})\beta)(C_t\mathcal{L}(L_t))^{1-\psi} + \exp(\eta_{\beta,t})\beta\mathcal{CE}_t^{1-\psi}\right)^{\frac{1}{1-\psi}} && (49)\\
\mathcal{CE}_t &= \mathbb{E}_t[V_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}, && (50)\\
W_t &= \frac{\psi\exp(\eta_{l,t})\overline{\nu}C_tL_t^{\nu}}{\mathcal{L}(L_t)^{\frac{\psi-1}{\psi}}}, && (51)\\
\mathcal{L}(L_t) &= \left(1+(\psi-1)\exp(\eta_{l,t})\overline{\nu}\frac{L_t^{1+\nu}}{1+\nu}\right)^{\frac{\psi}{1-\psi}}, && (52)\\
M_{t,t+1} &= \exp(\eta_{\beta,t})\beta\frac{1-\exp(\eta_{\beta,t+1})\beta}{1-\exp(\eta_{\beta,t})\beta}\frac{C_{t+1}^{-\psi}\mathcal{L}(L_{t+1})^{1-\psi}}{C_t^{-\psi}\mathcal{L}(L_t)^{1-\psi}}\left(\frac{V_{t+1}}{\mathcal{CE}_t}\right)^{\psi-\gamma}, && (53)\\
1 &= \mathbb{E}_t\left[M_{t,t+1}\frac{R_t}{\Pi_{t+1}}\right], && (54)\\
Q_t &= \mathbb{E}_t\left[M_{t,t+1}\left(R_{K,t+1}+(1-\delta)Q_{t+1}\right)\exp(\eta_{k,t+1})\right], && (55)\\
MC_t &= \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{W_t^{1-\alpha}R_{k,t}^{\alpha}}{\exp(\eta_{a,t})^{1-\alpha}}, && (56)\\
\frac{\exp(\eta_{k,t})K_{t-1}}{L_t} &= \frac{\alpha}{1-\alpha}\frac{W_t}{R_{k,t}}, && (57)\\
P_t^* &= \frac{\epsilon}{\epsilon-1}\frac{S_{1,t}}{S_{2,t}}, && (58)\\
S_{1,t} &= MC_tY_t + \theta\mathbb{E}_t[M_{t,t+1}\Pi_{t+1}^{\epsilon}S_{1,t+1}], && (59)\\
S_{2,t} &= Y_t + \theta\mathbb{E}_t[M_{t,t+1}\Pi_{t+1}^{\epsilon-1}S_{2,t+1}], && (60)\\
1 &= \Phi'\left(\frac{X_t}{\exp(\eta_{k,t})K_{t-1}}\right)Q_t, && (61)\\
K_t &= \left(1-\delta+\Phi\left(\frac{X_t}{\exp(\eta_{k,t})K_{t-1}}\right)\right)\exp(\eta_{k,t})K_{t-1}, && (62)\\
\Pi_t^{1-\epsilon} &= (1-\theta)(P_t^*\Pi_t)^{1-\epsilon}+\theta, && (63)\\
V_t^p &= \Pi_t^{\epsilon}((1-\theta)(P_t^*\Pi_t)^{-\epsilon}+\theta V_{t-1}^p), && (64)\\
\frac{R_t}{R} &= \left(\frac{R_{t-1}}{R}\right)^{\phi_r}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_{t-1}}\exp(-\mu_a)\right)^{\phi_y}\right)^{1-\phi_r}\exp(\eta_{r,t}), && (65)\\
C_t + X_t &= Y_t, && (66)\\
Y_t &= \frac{(\exp(\eta_{k,t})K_{t-1})^{\alpha}(\exp(\eta_{a,t})L_t)^{1-\alpha} - \kappa_y\exp(\eta_{a,t})}{V_t^p}, && (67)
\end{aligned}
$$

the three exogenous processes

$$\eta_{\beta,t+1} = \rho_\beta \eta_{\beta,t} + \sigma_\beta \varepsilon_{\beta,t+1}, \tag{68}$$
$$\eta_{l,t+1} = \rho_L \eta_{l,t} + \sigma_L \varepsilon_{L,t+1}, \tag{69}$$
$$\eta_{r,t+1} = \rho_R \eta_{r,t} + \sigma_R \varepsilon_{R,t+1}, \tag{70}$$

the process for productivity

$$\eta_{a,t} = \mu_a + \eta_{a,t-1} + \sigma_a \varepsilon_{a,t} + \kappa_a \eta_{k,t}, \tag{71}$$

and one of the proposed disaster processes in Section 1.5.2.

1.5.5 Stationary Equilibrium Conditions

Because of the unit root in $\eta_{a,t}$, the model is non-stationary. To obtain a stationary representation, define the transformations

$$\begin{aligned}
\tilde{C}_t &= \frac{C_t}{\exp(\eta_{a,t})}, \quad \tilde{V}_t = \frac{V_t}{\exp(\eta_{a,t})}, \quad \tilde{\mathcal{CE}}_t = \frac{\mathcal{CE}_t}{\exp(\eta_{a,t})}, \quad \tilde{K}_t = \frac{K_t}{\exp(\eta_{a,t})}, \\
\tilde{K}_{t-1} &= \frac{K_{t-1}}{\exp(\eta_{a,t})}, \quad \tilde{W}_t = \frac{W_t}{\exp(\eta_{a,t})}, \quad \tilde{X}_t = \frac{X_t}{\exp(\eta_{a,t})}, \quad \tilde{Y}_t = \frac{Y_t}{\exp(\eta_{a,t})}, \\
A_t &= \exp(\eta_{a,t} - \eta_{a,t-1} - \mu_a) = \exp(\sigma_a \varepsilon_{a,t} + \kappa_a \eta_{k,t}).
\end{aligned}$$

Most of the calculations for the stationary representation are straightforward, so I only show the work for the more complicated cases.

Substituting the transformations in the recursion for preferences (49) implies

$$\begin{aligned}
\exp(\eta_{a,t})\tilde{V}_t &= \left((1 - \exp(\eta_{\beta,})\beta)\beta \exp(\eta_{a,t})^{1-\psi}(\tilde{C}_t \mathcal{L}(L_t))^{1-\psi} + \exp(\eta_{\beta,t})\beta \exp(\eta_{a,t})^{1-\psi} \tilde{\mathcal{CE}}_t^{1-\psi}\right)^{\frac{1}{1-\psi}} \\
&= \exp(\eta_{a,t}) \left((1 - \exp(\eta_{\beta,})\beta)\beta(\tilde{C}_t \mathcal{L}(L_t))^{1-\psi} + \exp(\eta_{\beta,t})\beta \tilde{\mathcal{CE}}_t^{1-\psi}\right)^{\frac{1}{1-\psi}}.
\end{aligned}$$

The certainty equivalent definition (50) becomes

$$\begin{aligned}
\exp(\eta_{a,t})\tilde{\mathcal{CE}}_t &= \mathbb{E}_t[\tilde{V}_{t+1}^{1-\gamma} \eta_{a,t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}} \\
&= \mathbb{E}_t \left[\tilde{V}_{t+1}^{1-\gamma} \exp(\eta_{a,t+1})^{1-\gamma} \frac{\exp(\eta_{a,t})^{1-\gamma}}{\exp(\eta_{a,t})^{1-\gamma}}\right]^{\frac{1}{1-\gamma}} \\
&= \exp(\eta_{a,t})\mathbb{E}_t \left[\tilde{V}_{t+1}^{1-\gamma} \left(\frac{\exp(\eta_{a,t+1})}{\exp(\eta_{a,t})}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} \\
\tilde{\mathcal{CE}}_t &= \exp(\mu_a)\mathbb{E}_t \left[A_{t+1} \tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}.
\end{aligned}$$

The stochastic discount factor (53) becomes

$$
\begin{aligned}
M_{t,t+1} &= \exp(\eta_{\beta,t})\beta \frac{1-\exp(\eta_{\beta,t+1})\beta}{1-\exp(\eta_{\beta,t})\beta} \frac{\tilde{C}_{t+1}^{-\psi}}{\tilde{C}_t^{-\psi}} \left(\frac{\exp(\eta_{a,t+1})}{\exp(\eta_{a,t})}\right)^{\psi} \\
&\quad \times \frac{\mathcal{L}(L_{t+1})^{1-\psi}}{\mathcal{L}(L_t)^{1-\psi}} \left(\frac{\tilde{V}_{t+1}}{\tilde{\mathcal{CE}}_t}\right)^{\psi-\gamma} \left(\frac{\exp(\eta_{a,t+1})}{\exp(\eta_{a,t})}\right)^{-(\psi-\gamma)} \\
&= \exp(\eta_{\beta,t})\beta \frac{1-\exp(\eta_{\beta,t+1})\beta}{1-\exp(\eta_{\beta,t})\beta} \frac{\tilde{C}_{t+1}^{-\psi}}{\tilde{C}_t^{-\psi}} \frac{\mathcal{L}(L_{t+1})^{1-\psi}}{\mathcal{L}(L_t)^{1-\psi}} \left(\frac{\tilde{V}_{t+1}}{\tilde{\mathcal{CE}}_t}\right)^{\psi-\gamma} \left(\frac{\exp(\eta_{a,t+1})}{\exp(\eta_{a,t})}\right)^{\gamma} \\
&= \exp(\eta_{\beta,t})\beta \frac{1-\exp(\eta_{\beta,t+1})\beta}{1-\exp(\eta_{\beta,t})\beta} \frac{\tilde{C}_{t+1}^{-\psi}}{\tilde{C}_t^{-\psi}} \frac{\mathcal{L}(L_{t+1})^{1-\psi}}{\mathcal{L}(L_t)^{1-\psi}} \left(\frac{\tilde{V}_{t+1}}{\tilde{\mathcal{CE}}_t}\right)^{\psi-\gamma} (A_{t+1}\exp(\mu_a))^{\gamma}
\end{aligned}
$$

The recursions for the optimal price resetting problem become

$$
\begin{aligned}
\exp(\eta_{a,t})\tilde{S}_{1,t} &= MC_t \exp(\eta_{a,t})\tilde{Y}_t + \theta \mathbb{E}_t[M_{t,t+1}\Pi_{t+1}^{\epsilon}\tilde{S}_{1,t+1}\exp(\eta_{a,t+1})] \\
\tilde{S}_{1,t} &= MC_t \tilde{Y}_t + \exp(\mu_a)\theta \mathbb{E}_t[M_{t,t+1}A_{t+1}\Pi_{t+1}^{\epsilon}\tilde{S}_{1,t+1}] \\
\exp(\eta_{a,t})\tilde{S}_{2,t} &= \exp(\eta_{a,t})\tilde{Y}_t + \theta \mathbb{E}_t[M_{t,t+1}\Pi_{t+1}^{\epsilon-1}\tilde{S}_{2,t+1}\exp(\eta_{a,t+1})] \\
\tilde{S}_{2,t} &= \tilde{Y}_t + \exp(\mu_a)\theta \mathbb{E}_t[M_{t,t+1}A_{t+1}\Pi_{t+1}^{\epsilon-1}\tilde{S}_{2,t+1}].
\end{aligned}
$$

Then the stationary equilibrium conditions are

$$\tilde{V}_t = \left((1-\exp(\eta_{\beta,})\beta)\beta(\tilde{C}_t\mathcal{L}(L_t))^{1-\psi} + \exp(\eta_{\beta,t})\beta\tilde{\mathcal{CE}}_t^{1-\psi}\right)^{\frac{1}{1-\psi}}, \tag{72}$$

$$\tilde{\mathcal{CE}}_t = \exp(\mu_a)\mathbb{E}_t[A_{t+1}\tilde{V}_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}, \tag{73}$$

$$\tilde{W}_t = \frac{\psi\exp(\eta_{l,t})\overline{\nu}\tilde{C}_t L_t^{\nu}}{\mathcal{L}(L_t)^{\frac{\psi-1}{\psi}}}, \tag{74}$$

$$\mathcal{L}(L_t) = \left(1 + (\psi-1)\exp(\eta_{l,t})\overline{\nu}\frac{L_t^{1+\nu}}{1+\nu}\right)^{\frac{\psi}{1-\psi}}, \tag{75}$$

$$\begin{aligned} M_{t,t+1} &= \exp(\eta_{\beta,t})\beta\frac{1-\exp(\eta_{\beta,t+1})\beta}{1-\exp(\eta_{\beta,t})\beta}\frac{\tilde{C}_{t+1}^{-\psi}}{\tilde{C}_t^{-\psi}}\frac{\mathcal{L}(L_{t+1})^{1-\psi}}{\mathcal{L}(L_t)^{1-\psi}} \\ &\quad\times\left(\frac{\tilde{V}_{t+1}}{\tilde{\mathcal{CE}}_t}\right)^{\psi-\gamma}(A_{t+1}\exp(\mu_a))^{\gamma}, \end{aligned} \tag{76}$$

$$1 = \mathbb{E}_t\left[M_{t,t+1}\frac{R_t}{\Pi_{t+1}}\right], \tag{77}$$

$$Q_t = \mathbb{E}_t\left[M_{t,t+1}\left(R_{K,t+1} + (1-\delta)Q_{t+1}\right)\exp(\eta_{k,t+1})\right], \tag{78}$$

$$MC_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\tilde{W}_t^{1-\alpha}R_{k,t}^{\alpha}, \tag{79}$$

$$\frac{\exp(\eta_{k,t})\tilde{K}_{t-1}}{L_t} = \frac{\alpha}{1-\alpha}\frac{\tilde{W}_t}{R_{k,t}}, \tag{80}$$

$$P_t^* = \frac{\epsilon}{\epsilon-1}\frac{\tilde{S}_{1,t}}{\tilde{S}_{2,t}}, \tag{81}$$

$$\tilde{S}_{1,t} = MC_t\tilde{Y}_t + \exp(\mu_a)\theta\mathbb{E}_t[M_{t,t+1}A_{t+1}\Pi_{t+1}^{\epsilon}\tilde{S}_{1,t+1}], \tag{82}$$

$$\tilde{S}_{2,t} = \tilde{Y}_t + \exp(\mu_a)\theta\mathbb{E}_t[M_{t,t+1}A_{t+1}\Pi_{t+1}^{\epsilon-1}\tilde{S}_{2,t+1}], \tag{83}$$

$$1 = \Phi'\left(\frac{\tilde{X}_t}{\exp(\eta_{k,t})\tilde{K}_{t-1}}\right)Q_t, \tag{84}$$

$$\tilde{K}_t = \left(1-\delta+\Phi\left(\frac{\tilde{X}_t}{\exp(\eta_{k,t})\tilde{K}_{t-1}}\right)\right)\exp(\eta_{k,t})\tilde{K}_{t-1}, \tag{85}$$

$$\Pi_t^{1-\epsilon} = (1-\theta)(P_t^*\Pi_t)^{1-\epsilon} + \theta, \tag{86}$$

$$V_t^p = \Pi_t^{\epsilon}((1-\theta)(P_t^*\Pi_t)^{-\epsilon} + \theta V_{t-1}^p), \tag{87}$$

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\phi_r}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi}\left(\frac{\tilde{Y}_t}{\tilde{Y}_{t-1}}A_t\right)^{\phi_y}\right)^{1-\phi_r}\exp(\eta_{r,t}), \tag{88}$$

$$\tilde{C}_t + \tilde{X}_t = \tilde{Y}_t, \tag{89}$$

$$\tilde{Y}_t = \frac{(\exp(\eta_{k,t})\tilde{K}_{t-1})^{\alpha}(L_t)^{1-\alpha} - \kappa_y}{V_t^p}, \tag{90}$$

1.6 Deterministic Steady State

To provide an initial guess for the risk-adjusted linearization and to provide a verification that the model is coded correctly, I determine some reasonable guesses for the deterministic steady state.

Within this subsection, I denote the deterministic steady state values by an absence of a time subscript. The exogenous processes, by construction, have steady states of 0, i.e. $\eta_\beta = \eta_L = \eta_A = \eta_R = 0$. Further, $A = 1$.

Focusing now on the endogenous equilibrium conditions, from (51),

$$W = \frac{\varphi L^\nu}{C^{-\gamma}}.$$

From (14),

$$M = \beta.$$

From (61), the fact that $\overline{X}$ is the steady-state investment rate, and the fact that $\Phi'(\overline{X}) = 1$,

$$Q = 1.$$

From (55), first observing that,

$$\Phi(\overline{X}) = \frac{\overline{X}\chi}{\chi - 1} - \frac{\overline{X}}{\chi(\chi - 1)} = \overline{X}\frac{\chi^2 - 1}{\chi(\chi - 1)} = \overline{X}\frac{(\chi - 1)(\chi + 1)}{\chi(\chi - 1)} = \frac{\delta\chi}{\chi + 1}\frac{\chi + 1}{\chi} = \delta,$$

which ensures that K does indeed remain at steady state, it must be the cast that

$$\begin{aligned} 1 &= \beta(R_K + (1 - \delta + \Phi(\overline{X}) - \overline{X}) \\ R_K &= \frac{1}{\beta} + \overline{X} - 1. \end{aligned}$$

Equation (56) remains as it is but with time subscripts removed. From (59),

$$\tilde{S}_1 = MC \cdot Y + \theta\beta\Pi^\epsilon \tilde{S}_1 \Rightarrow \tilde{S}_1 = \frac{MC \cdot Y}{1 - \theta\beta\Pi^\epsilon}.$$

From (60),

$$\tilde{S}_2 = Y + \theta\beta\Pi^{\epsilon-1}\tilde{S}_2 \Rightarrow \tilde{S}_1 = \frac{Y}{1 - \theta\beta\Pi^{\epsilon-1}}.$$

Thus,

$$P^* = \frac{\epsilon}{\epsilon - 1} MC \frac{1 - \theta\Pi^{\epsilon-1}}{1 - \theta\Pi^\epsilon}.$$

From (63),

$$\Pi^{1-\epsilon} = (1-\theta)\left(P^*\Pi\right)^{1-\epsilon} + \theta$$

Note that P^* depends on MC and fundamental parameters, hence the above equation pins down MC, which then pins down the ratio of K to L. From (64),

$$\begin{aligned} V^p &= \Pi^{\epsilon}\left((1-\theta)(P^*\Pi)^{-\epsilon} + \theta V^p\right) \\ V^p &= \frac{(1-\theta)(P^*\Pi)^{-\epsilon}}{\Pi^{-\epsilon} - \theta} = \frac{(1-\theta)(P^*)^{-\epsilon}}{1-\theta\Pi^{\epsilon}}. \end{aligned}$$

From the Taylor rule (65), the steady state interest and inflation rates are R and Π, respectively, and from the Euler equation (54), R must satisfy

$$R = \frac{\Pi}{\beta}.$$

From (66),

$$C + X = Y.$$

From (67),

$$Y = \frac{K^{\alpha}L^{1-\alpha}}{V^p}.$$

As shown previously, the steady-state investment rate is $\overline{X}$, hence

$$X = \overline{X}K$$

Finally, I claim that the deterministic steady state reduces to a nonlinear equation in L. Using the aggregate supply and capital accumulation equations,

$$C + \overline{X}K = \frac{K^{\alpha}L^{1-\alpha}}{V^p}.$$

The optimal capital-labor ratio implies

$$\begin{aligned} K &= \frac{\alpha}{1-\alpha}\frac{W}{R_K}L, \\ C + \overline{X}K &= \left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\left(\frac{W}{R_K}\right)^{\alpha}\frac{L}{V^p}. \end{aligned}$$

The intratemporal condition for consumption and labor implies

$$\begin{aligned} K &= \frac{\alpha}{1-\alpha}\frac{\varphi L^{\nu}}{C^{-\gamma}R_K}L, \\ C + \delta K &= \left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\left(\frac{\varphi L^{\nu}}{C^{-\gamma}R_K}\right)^{\alpha}\frac{L}{V^p}. \end{aligned}$$

Given a guess for L, I can compute C using these two equations. Given C, I can compute W. Given the wage W, I can compute K and MC. Given the marginal cost MC, I can compute the inflation-related terms.

2 Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$
\begin{aligned}
0 &= \log \mathbb{E}_t \left[\exp\left(\xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1}\right)\right] \\
z_{t+1} &= \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t)\varepsilon_{t+1},
\end{aligned}
$$

where z_t are (predetermined) state variables and y_t are (nondetermined) jump variables.

For the remainder of this section, lower case variables are the logs of previously upper case variables, and with a small abuse of notation, let $s_{1,t} = \log(S_{1,t})$ and $s_{2,t} = \log(S_{2,t})$. Additionally, let $r_{k,t} = \log(R_{K,t})$ and $v_t = \log(V_t^p)$.

Equation (51) becomes

$$
\begin{aligned}
1 &= \varphi \exp(\eta_{l,t}) \frac{L_t^\nu}{C_t^{-\gamma} W_t} \\
0 &= \log \mathbb{E}_t \left[\exp\left(\underbrace{\log(\varphi) + \eta_{l,t} + \nu l_t - (-\gamma c_t + w_t)}_{\xi}\right)\right].
\end{aligned}
$$

Equation (14) will not be used in the system of equations for the risk-adjusted linearization, but it simplifies the other equations. Taking logs and re-arranging yields

$$
\begin{aligned}
0 &= \log(\beta) + \eta_{\beta,t+1} + (-\gamma c_{t+1}) - \eta_{\beta,t} - (-\gamma c_t) - m_{t+1} \\
m_{t+1} &= \underbrace{\log(\beta) - \eta_{\beta,t} + \gamma c_t}_{\xi} + \underbrace{\eta_{\beta,t+1} - \gamma c_{t+1}}_{\text{forward-looking}}.
\end{aligned}
$$

Equation (54) becomes

$$
0 = \log \mathbb{E}_t \left[\exp\left(\underbrace{r_t}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} - \underbrace{\pi_{t+1}}_{\text{forward-looking}}\right)\right].
$$

Equation (61) becomes

$$
0 = \log \mathbb{E}_t \left[\exp\left(q_t + \log\left(\Phi'\left(\exp(x_t - k_{t-1})\right)\right)\right)\right].
$$

For equation (55), observe that the RHS is not log-linear in the forward-looking variables. To handle this case, I define the new variable

$$
\Omega_t = R_{K,t} + Q_t \left(1 - \delta + \Phi\left(\frac{X_t}{K_{t-1}}\right) - \Phi'\left(\frac{X_t}{K_{t-1}}\right) \frac{X_t}{K_{t-1}}\right). \tag{91}
$$

Then (55) can be written as

$$
\begin{aligned}
1 &= \mathbb{E}_t\left[\frac{M_{t,t+1}\Omega_{t+1}}{Q_t}\right] \\
0 &= \log \mathbb{E}_t\left[\exp(m_{t+1}+\omega_{t+1}-q_t)\right].
\end{aligned}
$$

Equation (56) becomes

$$
0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{(1-\alpha)w_t + \alpha r_{k,t} - a_t - (1-\alpha)\log(1-\alpha) - \alpha\log(\alpha) - mc_t}_{\xi}\right)\right].
$$

Equation (57) becomes

$$
0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{k_{t-1} - l_t - \log\left(\frac{\alpha}{1-\alpha}\right) - (w_t - r_{k,t})}_{\xi}\right)\right].
$$

Like the stochastic discount factor, equation (58) will not be used in the system of equations, but it will be useful to simplify other equations. Taking logs yields

$$
p_t^* = \log\left(\frac{\epsilon}{\epsilon-1}\right) + s_{1,t} - s_{2,t}.
$$

Equation (59) becomes

$$
\begin{aligned}
&S_{1,t} - MC_t Y_t \\
&= \mathbb{E}_t\left[\exp\left(\log(\theta) + m_{t+1} + \epsilon\pi_{t+1} + s_{1,t+1}\right)\right] \\
0 &= \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) - \log(\exp(s_{1,t}) - \exp(mc_t)\exp(y_t))}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{\epsilon\pi_{t+1} + s_{1,t+1}}_{\text{forward-looking}}\right)\right]
\end{aligned}
$$

and equation (60) becomes

$$
0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\theta) - \log(\exp(s_{2,t}) - \exp(y_t))}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{(\epsilon-1)\pi_{t+1} + s_{2,t+1}}_{\text{forward-looking}}\right)\right].
$$

Equation (63) becomes

$$
\begin{aligned}
1 &= \frac{\Pi_t^{1-\epsilon}}{(1-\theta)(P_t^*\Pi_t)^{1-\epsilon} + \theta} \\
0 &= \log \mathbb{E}_t\left[\exp\left((1-\epsilon)\pi_t - \log((1-\theta)\exp((1-\epsilon)(p_t^*+\pi_t)) + \theta)\right)\right].
\end{aligned}
$$

Equation (64) becomes

$$
\begin{aligned}
1 &= \frac{V_t^p}{\Pi_t^\epsilon((1-\theta)(P_t^* \Pi_t)^{-\epsilon} + \theta V_{t-1}^p)} \\
0 &= \log \mathbb{E}_t \left[\exp\left(v_t - \epsilon \pi_t - \log((1-\theta)\exp(-\epsilon(p_t^* + \pi_t)) + \theta \exp(v_{t-1}))\right)\right].
\end{aligned}
$$

Equation (65) becomes

$$
0 = \log \mathbb{E}_t \left[\exp\left(\phi_r r_{t-1} + (1-\phi_r) r + (1-\phi_r)(\phi_\pi(\pi_t - \pi) + \phi_y(y_t - y_{t-1})) + \eta_{r,t} - r_t\right)\right].
$$

Equation (66) becomes

$$
0 = \log \mathbb{E}_t \left[\exp(y_t - \log(\exp(c_t) + \exp(x_t)))\right].
$$

Equation (67) becomes

$$
0 = \log \mathbb{E}_t \left[\exp(a_t + \alpha k_{t-1} + (1-\alpha) l_t - v_t - y_t)\right].
$$

Equation (62) becomes

$$
k_t = \log\left(1 + \frac{\overline{X}^{1/\chi}}{1 - 1/\chi}(\exp(x_t - k_{t-1}))^{1-1/\chi} - \frac{\overline{X}}{1-1/\chi}\right) + k_{t-1}.
$$

The autoregressive processes (68) to (70) remain as they are.

The jump variables are y_t, c_t, l_t, w_t, r_t, π_t, q_t, x_t, $r_{k,t}$, ω_t, mc_t, $s_{1,t}$, $s_{2,t}$, and v_t. The state variables are k_{t-1}, v_{t-1}, r_{t-1}, y_{t-1}, and the autoregressive processes. The equations defining the evolution of the lags v_{t-1}, r_{t-1}, and y_{t-1} are obtained by the formula $z_{(t-1)+1} = z_t$.

This system has three forward difference equations (55), (59), and (60). To ensure accuracy of the risk-adjusted linearization, I derive N-period ahead forward difference equations for all three.

First, redefine Ω_t as

$$
\Omega_t = 1 - \delta + \Phi\left(\frac{X_t}{K_{t-1}}\right) - \Phi'\left(\frac{X_t}{K_{t-1}}\right)\frac{X_t}{K_{t-1}}.
$$

Then we can write (55) recursively as

$$
\begin{aligned}
Q_t &= \mathbb{E}_t[M_{t,t+1}(R_{K,t+1} + Q_{t+1}\Omega_{t+1})] \\
&= \mathbb{E}_t[M_{t,t+1}R_{K,t+1} + \Omega_{t+1}M_{t,t+1}\mathbb{E}_{t+1}[M_{t,t+2}(R_{K,t+2} + Q_{t+2}\Omega_{t+1})]] \\
&= \mathbb{E}_t[M_{t,t+1}R_{K,t+1}] + \Omega_{t+1}\mathbb{E}_t\mathbb{E}_{t+1}[M_{t,t+1}M_{t,t+2}(R_{K,t+2} + Q_{t+2}\Omega_{t+2})].
\end{aligned}
$$

By the tower property,

$$
\begin{aligned}
Q_t &= \mathbb{E}_t[M_{t,t+1}R_{K,t+1}] + \Omega_{t+1}\mathbb{E}_t[M_{t,t+1}M_{t,t+2}(R_{K,t+2} + Q_{t+2}\Omega_{t+2})] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{2}\left(\prod_{u=1}^{s-1}\Omega_{t+u}\right)\left(\prod_{u=1}^{s}M_{t,t+u}\right)R_{K,t+s}\right) + M_{t,t+1}M_{t,t+2}Q_{t+2}\Omega_{t+1}\Omega_{t+2}\right] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{2}\left(\prod_{u=1}^{s-1}\Omega_{t+u}\right)\left(\prod_{u=1}^{s}M_{t,t+u}\right)R_{K,t+s}\right) + \prod_{s=1}^{2}(M_{t,t+s}\Omega_{t+s})\mathbb{E}_{t+2}[M_{t,t+3}(R_{K,t+3} + Q_{t+3}\Omega_{t+3})]\right] \\
&= \mathbb{E}_t\left[\left(\sum_{s=1}^{3}\left(\prod_{u=1}^{s-1}\Omega_{t+u}\right)\left(\prod_{u=1}^{s}M_{t,t+u}\right)R_{K,t+s}\right) + \prod_{s=1}^{3}(M_{t,t+s}\Omega_{t+s})Q_{t+3}\right]
\end{aligned}
$$

and so on, with the abuse of notation that $\prod_{u=1}^{0}\Omega_{t+u} = 1$. Given this recursive structure, define $D_{Q,t}^{(n)}$ and $P_{Q,t}^{(n)}$ as

$$
\begin{aligned}
D_{Q,t}^{(n)} &= \mathbb{E}_t\left[\Omega_{t+1}M_{t,t+1}D_{Q,t+1}^{(n-1)}\right] \\
P_{Q,t}^{(n)} &= \mathbb{E}_t\left[\Omega_{t+1}M_{t,t+1}P_{Q,t+1}^{(n-1)}\right]
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
D_{Q,t}^{(0)} &= \frac{R_{K,t}}{\Omega_t} \\
P_{Q,t}^{(0)} &= Q.
\end{aligned}
$$

Then I may write the N-period ahead recursive form of equation (55) as

$$
Q_t = \sum_{n=1}^{N} D_{Q,t}^{(n)} + P_{Q,t}^{(N)}.
$$

To see why this recursion works, it is simpler to first verify that $P_{Q,t}^{(3)}$ is correct:

$$
\begin{aligned}
P_{Q,t}^{(1)} &= \mathbb{E}_t\left[\Omega_{t+1}M_{t,t+1}Q_{t+1}\right] \\
P_{Q,t}^{(2)} &= \mathbb{E}_t\left[\Omega_{t+1}M_{t,t+1}(\mathbb{E}_{t+1}[\Omega_{t+2}M_{t,t+2}Q_{t+2}])\right] \\
&= \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\prod_{s=1}^{2}(\Omega_{t+s}M_{t,t+s})Q_{t+2}\right]\right] \\
&= \mathbb{E}_t\left[\prod_{s=1}^{2}(\Omega_{t+s}M_{t,t+s})Q_{t+2}\right].
\end{aligned}
$$

where the second equality for $P_{Q,t}^{(2)}$ follows from the fact that $M_{t,t+1}$ is measurable with respect to the information set at time $t+1$ and can therefore be moved insided the conditional expectation $\mathbb{E}_{t+1}[\cdot]$. Continuing for one more recursion, I have

$$\begin{aligned} P_{Q,t}^{(3)} &= \mathbb{E}_t\left[\Omega_{t+1}M_{t,t+1}\mathbb{E}_{t+1}\left[\prod_{s=1}^{2}(\Omega_{t+1+s}M_{t,t+1+s})Q_{t+3}\right]\right] \\ &= \mathbb{E}_t\left[\prod_{s=1}^{3}(\Omega_{t+s}M_{t,t+s})Q_{t+3}\right]. \end{aligned}$$

Similarly, for $D_{Q,t}$, I have

$$\begin{aligned} D_{Q,t}^{(1)} &= \mathbb{E}_t\left[\Omega_{t+1}M_{t,t+1}\frac{R_{K,t+1}}{\Omega_{t+1}}\right] = \mathbb{E}_t[M_{t,t+1}R_{K,t+1}] \\ D_{Q,t}^{(2)} &= \mathbb{E}_t[\Omega_{t+1}M_{t,t+1}\mathbb{E}_{t+1}[M_{t,t+2}R_{K,t+2}]] \\ &= \mathbb{E}_t[\Omega_{t+1}M_{t,t+1}M_{t,t+2}R_{K,t+2}] \\ D_{Q,t}^{(3)} &= \mathbb{E}_t[\Omega_{t+1}M_{t,t+1}\mathbb{E}_{t+1}[\Omega_{t+2}M_{t,t+2}M_{t,t+3}R_{K,t+3}]] \\ &= \mathbb{E}_t[\Omega_{t+1}\Omega_{t+2}M_{t,t+1}M_{t,t+2}M_{t,t+3}R_{K,t+3}]. \end{aligned}$$

Since $P_{Q,t}^{(n)}$ and $D_{Q,t}^{(n)}$ are time-t conditional expectations, they are measurable at time t, so they are not forward-looking variables. Thus, to get this version of (55) in the appropriate form, define $d_{q,n,t} = \log(D_{Q,t}^{(n)})$ and $p_{q,n,t} = \log(P_{Q,t}^{(n)})$, and use the following $2N+1$ equations:

$$0 = \log\mathbb{E}_t\left[\exp\left(\underbrace{q_t - \log\left(\sum_{n=1}^{N}\exp(d_{q,n,t}) + \exp(p_{q,N,t})\right)}_{\xi}\right)\right] \tag{92}$$

$$0 = \begin{cases} \log\mathbb{E}_t\left[\exp\left(\underbrace{-d_{q,n,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{\omega_{t+1} + d_{q,n-1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n > 1 \\ \log\mathbb{E}_t\left[\exp\left(\underbrace{-d_{q,1,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{r_{k,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n = 1, \end{cases} \tag{93}$$

$$0 = \begin{cases} \log\mathbb{E}_t\left[\exp\left(\underbrace{-p_{q,n,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{\omega_{t+1} + p_{q,n-1,t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n > 1 \\ \log\mathbb{E}_t\left[\exp\left(\underbrace{-p_{q,1,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{\omega_{t+1} + q_{t+1}}_{\text{forward-looking}}\right)\right] & \text{if } n = 1. \end{cases} \tag{94}$$

For (59), observe that

$$
\begin{aligned}
S_{1,t} &= MC_tY_t + \theta\mathbb{E}_t[M_{t,t+1}\Pi^{\epsilon}_{t+1}(MC_{t+1}Y_{t+1} + \theta\mathbb{E}_{t+1}[M_{t+2}\Pi^{\epsilon}_{t+2}\tilde{S}_{1,t+2}])] \\
&= MC_tY_t + \theta\mathbb{E}_t[M_{t,t+1}\Pi^{\epsilon}_{t+1}MC_{t+1}Y_{t+1} + \theta M_{t,t+1}\Pi^{\epsilon}_{t+1}M_{t+2}\Pi^{\epsilon}_{t+2}\tilde{S}_{1,t+2}] \\
&= MC_tY_t + \mathbb{E}_t\left[\sum_{s=1}^{1}(\theta^s\prod_{u=1}^{s}(M_{t,t+u}\Pi^{\epsilon}_{t+u}))MC_{t+s}Y_{t+s}\right] + \mathbb{E}_t\left[\prod_{s=1}^{2}(\theta M_{t,t+s}\Pi^{\epsilon}_{t+s})\tilde{S}_{1,t+2}\right].
\end{aligned}
$$

Thus, define $D^{(n)}_{S1,t}$ and $P^{(n)}_{S1,t}$ as the recursions

$$
\begin{aligned}
D^{(n)}_{S1,t} &= \mathbb{E}_t[\theta M_{t,t+1}\Pi^{\epsilon}_{t+1}D^{(n-1)}_{S1,t+1}], \\
P^{(n)}_{S1,t} &= \mathbb{E}_t[\theta M_{t,t+1}\Pi^{\epsilon}_{t+1}P^{(n-1)}_{S1,t+1}],
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
D^{(0)}_{S1,t} &= MC_tY_t \\
P^{(0)}_{S1,t} &= S_{1,t}.
\end{aligned}
$$

Given these definitions, it follows that

$$
\begin{aligned}
D^{(1)}_{S1,t} &= \mathbb{E}_t[\theta M_{t,t+1}\Pi^{\epsilon}_{t+1}MC_{t+1}Y_{t+1}] \\
P^{(1)}_{S1,t} &= \mathbb{E}_t[\theta M_{t,t+1}\Pi^{\epsilon}_{t+1}\tilde{S}_{1,t+1}] \\
P^{(2)}_{S1,t} &= \mathbb{E}_t[\theta M_{t,t+1}\Pi^{\epsilon}_{t+1}\mathbb{E}_{t+1}[\theta M_{t,t+2}\Pi^{\epsilon}_{t+2}\tilde{S}_{1,t+2}]] \\
&= \mathbb{E}_t[\theta^2 M_{t,t+1}\Pi^{\epsilon}_{t+1}M_{t,t+2}\Pi^{\epsilon}_{t+2}\tilde{S}_{1,t+2}].
\end{aligned}
$$

Thus, defining $d_{s1,t} = \log(D_{S1,t})$ and $p_{s1,t} = \log(P_{S1,t})$, the N-period ahead recursive form of

(59) results in the $2N+1$ equations

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{s_{1,t} - \log \left(\sum_{n=0}^{N-1} \exp(d_{s1,n,t}) + \exp(p_{s1,N,t}) \right)}_{\xi} \right) \right] \tag{95}$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) - d_{s1,n,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{\epsilon \pi_{t+1} + d_{s1,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n \geq 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{d_{s1,0,t} - mc_t - y_t}_{\xi} \right) \right] & \text{if } n = 0. \end{cases} \tag{96}$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) - p_{s1,n,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{\epsilon \pi_{t+1} + p_{s1,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) - p_{s1,1,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{\epsilon \pi_{t+1} + s_{1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1. \end{cases} \tag{97}$$

It is straightforward to show that a similar recursive form applies to (60):

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{s_{2,t} - \log \left(\sum_{n=0}^{N-1} \exp(d_{s2,n,t}) + \exp(p_{s2,N,t}) \right)}_{\xi} \right) \right] \tag{98}$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) - d_{s2,n,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{(\epsilon - 1)\pi_{t+1} + d_{s2,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n \geq 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{d_{s2,0,t} - y_t}_{\xi} \right) \right] & \text{if } n = 0. \end{cases} \tag{99}$$

$$0 = \begin{cases} \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) - p_{s2,n,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{(\epsilon - 1)\pi_{t+1} + p_{s2,n-1,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n > 1 \\ \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\theta) - p_{s2,1,t}}_{\xi} + \underbrace{m_{t+1}}_{\text{both}} + \underbrace{(\epsilon - 1)\pi_{t+1} + s_{2,t+1}}_{\text{forward-looking}} \right) \right] & \text{if } n = 1, \end{cases} \tag{100}$$

where terms and boundary conditions are analogously defined.