These notes for the New Keynesian model follow Eric Sims's notes. Section ?? solves the equilibrium conditions of the New Keynesian model, and Section ?? transforms the equilibrium conditions into the desired form for a risk-adjusted linearization.

1 Model

1.1 Household

Households solve the problem

$$\max_{C_t, N_t, B_{t+1}, M_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \log \left(\frac{M_t}{P_t} \right) \right)$$

subject to the budget constraint

$$P_t C_t + B_{t+1} + M_t - M_{t-1} \leq W_t N_t + \Pi_t + (1 + i_{t-1}) B_t.$$

In this model, households have demand for money M_t, which is also the numeraire. The price of goods in terms of money is P_t. The stock of nominal bonds a households has is B_t. Note that B_t will be pre-determined at period t while M_t will not be (M_{t-1} is pre-determined). The Lagrangian for the household is

$$\begin{aligned}
\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \log \left(\frac{M_t}{P_t} \right) \right] \\
+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_t (P_t C_t + B_{t+1} + M_t - M_{t-1} - W_t N_t + \Pi_t + (1 + i_{t-1}) B_t) \right],
\end{aligned}$$

which implies first-order conditions

$$\begin{aligned}
0 &= C_t^{-\sigma} - \lambda_t P_t \\
0 &= -\psi N_t^{\eta} + \lambda_t W_t \\
0 &= -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1} (1 + i_t) \\
0 &= \theta \frac{1}{M_t} - \lambda_t + \beta \mathbb{E}_t \lambda_{t+1}.
\end{aligned}$$

The first two equations can be combined by isolating λ_t. Using $\lambda_t = C_t^{-\sigma}/P_t$, we can obtain the Euler equation for households and an equation relating money balances to consumption.

$$\begin{aligned}
C_t^{-\sigma} \frac{W_t}{P_t} &= \psi N_t^{\eta}, \\
C_t^{-\sigma} &= \beta \mathbb{E}_t C_{t+1}^{-\sigma} (1 + i_t), \\
\theta \left(\frac{M_t}{P_t} \right)^{-1} &= \frac{i_t}{1 + i_t} C_t^{-\sigma}.
\end{aligned}$$

1.2 Production

Final Producers There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 1$ so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j)\, dj.$$

The FOC for $Y_t(j)$ is

$$\begin{aligned}
0 &= P_t \frac{\epsilon}{\epsilon-1} \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} \frac{\epsilon-1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) \\
0 &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t} \\
0 &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} \\
Y_t(j) &= \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.
\end{aligned}$$

Plugging this quantity into the identity

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j)\, dj$$

and simplifying yields the price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon}\, dj \right)^{\frac{1}{1-\epsilon}}.$$

Intermediate Producers Intermediate goods are producing according to the linear technology

$$Y_t(j) = A_t N_t(j).$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\min_{N_t(j)} W_t N_t(j) \qquad \text{s.t.} \qquad A_t N_t(j) \geq \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

The Lagrangian is

$$\mathcal{L} = W_t N_t(j) + \varphi_t(j) \left(\left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - A_t N_t(j) \right),$$

so the first-order condition is

$$0 = W_t - \varphi_t(j) A_t \Rightarrow \varphi_t(j) = \frac{W_t}{A_t}.$$

The multiplier φ_t can be interpreted as the nominal marginal cost. Let mc_t be the real marginal cost. Then profits for an intermediate producer is

$$\Pi_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - mc_t Y_t(j).$$

In addition to the labor choice, firms also have the chance to reset prices in every period with probability $1 - \phi$. This problem can be written as

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} \left(\frac{P_t(j)}{P_{t+s}} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} \right),$$

where I have imposed that output equals demand. The first-order condition is

$$\begin{aligned} 0 = (1-\epsilon) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} (P_{t+s})^{-(1-\epsilon)} Y_{t+s} \\ + \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s} \end{aligned}$$

Divide by $P_t(j)^{-\epsilon}/u'(C_t)$ and re-arrange to obtain

$$P_t(j) = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s}) P_{t+s}^{\epsilon-1} Y_{t+s}}.$$

This expression gives the optimal reset price P_t^*, which we can write more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{X_{1,t}}{X_{2,t}}$$

where

$$\begin{aligned} X_{1,t} &= u'(C_t) mc_t P_t^{\epsilon} Y_t + \phi\beta \mathbb{E}_t X_{1,t+1} \\ X_{2,t} &= u'(C_t) P_t^{\epsilon-1} Y_t + \phi\beta \mathbb{E}_t X_{2,t+1}. \end{aligned}$$

1.3 Equilibrium and Aggregation

To close the model, I assume that the log of technology A_t follows the AR(1)

$$\log A_t = \rho_a \log A_{t_1} + \varepsilon_{a,t},$$

and the growth rate in the log money supply follows the AR(1)

$$\Delta \log M_t = (1-\rho_m)\pi + \rho_m \Delta \log M_{t-1} + \varepsilon_{m,t},$$

where π is the steady-state rate of inflation. Note that this specification ensures that money balances grow at the same rate as the price level, which ensures real balances are stationary. To re-write the money growth equation in real terms, note that

$$\log(m_t) = \log(M_t) - \log(P_t) \Rightarrow \Delta \log(m_t) = \log(m_t) - \log(m_{t-1}) = \Delta \log(M_t) - \log(1+\pi_t),$$

hence

$$\Delta \log(m_t) = (1-\rho_m)\pi + \rho_m \Delta \log(m_{t-1}) + \rho_m \log(1+\pi_{t-1}) - \log(1+\pi_t) + \varepsilon_{m,t}.$$

In equilibrium, bond-holding must be zero, hence

$$C_t = w_t N_t + \frac{\Pi_t}{P_t}.$$

Real dividends Π_t satisfy the accounting identity

$$\begin{aligned}\frac{\Pi_t}{P_t} &= \int_0^1 \left(\frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j) \right) dj \\ &= \int_0^1 \frac{P_t(j)}{P_t} Y_t(j)\, dj - w_t \int_0^1 N_t(j)\, dj\end{aligned}$$

where $w_t = W_t/P_t$. Aggregate labor supply N_t equals aggregate labor demand in equilibrium, and market-clearing for consumption requires

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j)\, dj = \int_0^1 \frac{P_t(j)}{P_t} \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t\, dj = P_t^{\epsilon-1} Y_t \int_0^1 P_t(j)^{1-\epsilon}\, dj = Y_t$$

since $\int_0^1 P_t(j)^{1-\epsilon}\, dj = P_t^{1-\epsilon}$.

The quantity Y_t is aggregate output, so we must have

$$\begin{aligned}\int_0^1 A_t N_t(j)\, dj &= \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t\, dj \\ A_t N_t &= Y_t \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} dj = v_t Y_t.\end{aligned}$$

Thus, aggregate output is

$$Y_t = \frac{A_t N_t}{v_t}.$$

It can be shown that $v_t \geq 1$ by applying Jensen's inequality.
Finally, recall that

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon}\, dj.$$

In each period, a fraction ϕ cannot change their price. Without loss of generality, we may re-order these firms to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon}\, dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \phi \int_0^1 P_{t-1}(j)^{1-\epsilon}\, dj = (1-\phi)(P_t^*)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}.$$

Dividing by $P_{t-1}^{1-\epsilon}$ implies

$$(1+\pi_t)^{1-\epsilon} = (1-\phi)(1+\pi_t^*)^{1-\epsilon} + \phi$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$\begin{aligned}
v_t &= \int_0^{1-\phi} \left(\frac{P_t^*}{P_t}\right)^{-\epsilon} dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\epsilon} dj \\
&= \int_0^{1-\phi} \left(\frac{P_t^*}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} dj \\
&= (1-\phi)(1+\pi_t^*)^{-\epsilon}(1+\pi_t)^{\epsilon} + (1+\pi_t)^{\epsilon} \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj.
\end{aligned}$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \phi \int_0^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \phi v_{t-1}.$$

Thus, we acquire

$$\begin{aligned}
v_t &= (1-\phi)(1+\pi_t^*)^{-\epsilon}(1+\pi_t)^{\epsilon} + \phi(1+\pi_t)^{\epsilon} v_{t-1} \\
&= (1+\pi_t)^{\epsilon}((1-\phi)(1+\pi_t^*)^{-\epsilon} + \phi v_{t-1}).
\end{aligned}$$

To finish, we need to derive an expression characterizing π_t^*. Define

$$x_{1,t} \equiv \frac{X_{1,t}}{P_t^{\epsilon}}, \qquad x_{2,t} \equiv \frac{X_{2,t}}{P_t^{\epsilon-1}}.$$

It follows that

$$\begin{aligned}
x_{1,t} &= u'(C_t)mc_tY_t + \phi\beta\mathbb{E}_t\frac{X_{1,t+1}}{P_t^{\epsilon}} \\
&= u'(C_t)mc_tY_t + \phi\beta\mathbb{E}_t\left[\frac{X_{1,t+1}}{P_{t+1}^{\epsilon}}\frac{P_{t+1}^{\epsilon}}{P_t^{\epsilon}}\right] \\
&= u'(C_t)mc_tY_t + \phi\beta\mathbb{E}_t[x_{1,t+1}(1+\pi_{t+1})] \\
x_{2,t} &= C_t^{-\sigma}Y_t + \phi\beta\mathbb{E}_t\frac{X_{2,t+1}}{P_t^{\epsilon-1}} \\
&= C_t^{-\sigma}Y_t + \phi\beta\mathbb{E}_t\left[\frac{X_{2,t+1}}{P_{t+1}^{\epsilon-1}}\frac{P_{t+1}^{\epsilon-1}}{P_t^{\epsilon-1}}\right] \\
&= C_t^{-\sigma}Y_t + \phi\beta\mathbb{E}_t[x_{2,t+1}(1+\pi_{t+1})].
\end{aligned}$$

Further,

$$\frac{X_{1,t}}{X_{2,t}} = \frac{x_{1,t}}{x_{2,t}}P_t,$$

hence

$$\begin{aligned}
P_t^* &= \frac{\epsilon}{\epsilon-1}\frac{x_{1,t}}{x_{2,t}}P_t \\
(1+\pi_t^*) &= \frac{\epsilon}{\epsilon-1}\frac{x_{1,t}}{x_{2,t}}(1+\pi_t)
\end{aligned}$$

All together, the full set of equilibrium conditions are

$$\begin{aligned}
C_t^{-\sigma} &= \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} \frac{(1+i_t)}{1+\pi_{t+1}} \right] \\
C_t^{-\sigma} &= \psi \frac{N_t^{\eta}}{w_t} \\
m_t &= \theta \frac{1+i_t}{i_t} C_t^{\sigma} \\
mc_t &= \frac{w_t}{A_t} \\
C_t &= Y_t \\
Y_t &= \frac{A_t N_t}{v_t} \\
v_t &= (1+\pi_t)^{\epsilon}((1-\phi)(1+\pi_t^*)^{-\epsilon} + \phi v_{t-1}) \\
(1+\pi_t)^{1-\epsilon} &= (1-\phi)(1+\pi_t^*)^{1-\epsilon} + \phi \\
(1+\pi_t^*) &= \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}} (1+\pi_t) \\
x_{1,t} &= C_t^{-\sigma} mc_t Y_t + \phi \beta \mathbb{E}_t [x_{1,t+1}(1+\pi_{t+1})] \\
x_{2,t} &= C_t^{-\sigma} Y_t + \phi \beta \mathbb{E}_t [x_{2,t+1}(1+\pi_{t+1})] \\
\log A_t &= \rho_a \log(A_{t-1}) + \varepsilon_{a,t} \\
\Delta \log(m_t) &= (1-\rho_m)\pi + \rho_m \Delta \log(m_{t-1}) + \rho_m \log(1+\pi_{t-1}) - \log(1+\pi_t) + \varepsilon_{m,t} \\
\Delta \log m_t &= \log m_t - \log m_{t-1},
\end{aligned}$$

which comprise 14 equations in 14 aggregate variables

$$(C_t, i_t, \pi_t, N_t, w_t, m_t, mc_t, A_t, Y_t, v_t, \pi_t^*, x_{1,t}, x_{2,t}, \Delta \log m_t).$$

Alternatively, the money growth equation can be replaced by the Taylor rule

$$\log(1+i_t) = (1-\rho_i)\log(1+i) + \rho_i \log(1+i_{t-1}) + (1-\rho_i)\phi_\pi(\log(1+\pi_t) - \log(1+\pi)) + \varepsilon_{i,t},$$

and the third equation relating money demand to consumption could also be ignored. To reduce the number of equations, we utilize this specification. Furthermore, we can also substitute $1+\pi_t^*$ to remove π_t^* from the aggregate variables.

2 Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$\begin{aligned}
0 &= \log \mathbb{E}_t \left[\exp\left(\xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1} \right) \right] \\
z_{t+1} &= \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t)\varepsilon_{t+1},
\end{aligned}$$

where z_t are (predetermined) state variables and y_t are (nondetermined) jump variables. For the remainder of this section, lower case variables are the logs of previously upper case variables, and variables with a tilde are the logs of previously lower case variables (e.g. the real wage w_t)

The first equation becomes

$$\begin{aligned}
1 &= \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{1+i_t}{1+\pi_{t+1}} \right] \\
0 &= \log \mathbb{E}_t \left[\exp\left(\log(\beta) - \sigma(c_{t+1} - c_t) + \tilde{i}_t - \tilde{\pi}_{t+1} \right) \right] \\
&= \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\beta) + \sigma c_t + \tilde{i}_t}_{\xi} - \underbrace{\sigma c_{t+1} - \tilde{\pi}_{t+1}}_{\text{Forward-Looking}} \right) \right],
\end{aligned}$$

where $c_t = \log(C_t)$, $\tilde{i}_t = \log(1+i_t)$, and $\tilde{\pi}_t = \log(1+\pi_{t+1})$.

The second equation becomes

$$\begin{aligned}
1 &= \psi \frac{N_t^{\eta}}{C_t^{-\sigma} w_t} \\
0 &= \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\psi) + \eta n_t - (-\sigma c_t + \hat{w}_t)}_{\xi} \right) \right],
\end{aligned}$$

where $n_t = \log(N_t)$ and $\hat{w}_t = \log(w_t)$.

The third equation becomes

$$\begin{aligned}
1 &= \frac{w_t}{A_t mc_t} \\
0 &= \log \mathbb{E}_t \left[\exp\left(\hat{w}_t - a_t - \tilde{mc}_t \right) \right].
\end{aligned}$$

The fourth and fifth equation become

$$0 = \log \mathbb{E}_t \left[\exp\left(c_t - a_t - n_t + \hat{v}_t \right) \right].$$

The sixth equation becomes

$$0 = \hat{v}_t - \epsilon \tilde{\pi}_t - \log((1-\phi)\exp(\tilde{\pi}_t^*)^{-\epsilon} + \phi \exp(\hat{v}_{t-1})),$$

where $\hat{v}_{t-1}$ will be treated as an additional state variable, i.e. if $a_t = \hat{v}_t$ is a jump variable and $b_t = \hat{v}_{t-1}$ is a state variable, then

$$b_{t+1} = a_t.$$

The seventh equation becomes

$$0 = (1-\epsilon)\tilde{\pi}_t - \log((1-\phi)\exp(\tilde{\pi}_t^*)^{1-\epsilon} + \phi).$$

The eighth equation becomes

$$0 = \tilde{\pi}_t^* - \log\left(\frac{\epsilon}{\epsilon - 1}\right) - \tilde{\pi}_t - (\hat{x}_{1,t} - \hat{x}_{2,t})).$$

By plugging this expression for $\tilde{\pi}_t^*$ into the previous two equations, we can also remove one more variable from the system.

The ninth and tenth equation become

$$1 = \mathbb{E}_t\left[\phi\beta\frac{x_{1,t+1}(1+\pi_{t+1})}{x_{1,t} - C_t^{-\sigma} mc_t A_t N_t / v_t}\right]$$
$$0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{1,t}) - \exp((1-\sigma)c_t + \tilde{mc}_t))}_{\xi} + \hat{x}_{1,t+1} + \tilde{\pi}_{t+1}\right)\right]$$
$$0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{2,t}) - \exp((1-\sigma)c_t))}_{\xi} + \hat{x}_{2,t+1} + \tilde{\pi}_{t+1}\right)\right],$$

where the fact that $x_{1,t}$ and $x_{2,t}$ must both be positive implies $x_{1,t} - C_t^{-\sigma} mc_t Y_t$ and $x_{2,t} - C_t^{-\sigma} Y_t$ are both positive, as the expectations on the RHS are also both positive.

For the monetary policy rule, we use $\tilde{i}_{t-1} \equiv \log(1 + i_{t-1})$ and $\varepsilon_{i,t}$ as states and treat i_t as a jump variable, hence

$$\tilde{i}_t = (1 - \rho_i)\tilde{i} + \rho_i \tilde{i}_{t-1} + (1 - \rho_i)\phi_\pi(\tilde{\pi}_t - \tilde{\pi}) + \varepsilon_{i,t}.$$

This formulation allows us to treat the policy rule as an expectational equation.

The above nine equations comprise the expectational equations. The following four equations comprise the states:

$$\begin{aligned}
a_{t+1} &= \rho_a a_t + \varepsilon_{a,t+1} \\
\hat{v}_{(t-1)+1} &= \hat{v}_t \\
\tilde{i}_{(t-1)+1} &= \tilde{i}_t \\
\varepsilon_{i,t+1} &= \varepsilon_{i,t+1}
\end{aligned}$$