These notes for the New Keynesian model follow Eric Sims's notes. Section 1 solves the equilibrium conditions of the New Keynesian model, and Section 2 transforms the equilibrium conditions into the desired form for a risk-adjusted linearization.

# 1 Model

## 1.1 Household

Households solve the problem

$$\max_{C_t,N_t,B_{t+1},M_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left( \frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \log\left(\frac{M_t}{P_t}\right)\right)$$

subject to the budget constraint

$$P_t C_t + B_{t+1} + M_t - M_{t-1} \leq W_t N_t + \Pi_t + (1+i_{t-1})B_t.$$

In this model, households have demand for money $M_t$, which is also the numeraire. The price of goods in terms of money is $P_t$. The stock of nominal bonds a households has is $B_t$. Note that $B_t$ will be pre-determined at period $t$ while $M_t$ will not be ($M_{t-1}$ is pre-determined). The Lagrangian for the household is

$$\begin{aligned}
\mathcal{L} &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \log\left(\frac{M_t}{P_t}\right)\right] \\
&\quad + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_t (P_t C_t + B_{t+1} + M_t - M_{t-1} - W_t N_t + \Pi_t + (1+i_{t-1})B_t)\right],
\end{aligned}$$

which implies first-order conditions

$$\begin{aligned}
0 &= C_t^{-\sigma} - \lambda_t P_t \\
0 &= -\psi N_t^{\eta} + \lambda_t W_t \\
0 &= -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1}(1+i_t) \\
0 &= \theta \frac{1}{M_t} - \lambda_t + \beta \mathbb{E}_t \lambda_{t+1}.
\end{aligned}$$

The first two equations can be combined by isolating $\lambda_t$. Using $\lambda_t = C_t^{-\sigma}/P_t$, we can obtain the Euler equation for households and an equation relating money balances to consumption.

$$\begin{aligned}
C_t^{-\sigma} \frac{W_t}{P_t} &= \psi N_t^{\eta}, \\
C_t^{-\sigma} &= \beta \mathbb{E}_t C_{t+1}^{-\sigma}(1+i_t), \\
\theta \left(\frac{M_t}{P_t}\right)^{-1} &= \frac{i_t}{1+i_t} C_t^{-\sigma}.
\end{aligned}$$

## 1.2 Production

**Final Producers** There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 1$ so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) \, dj.$$

The FOC for $Y_t(j)$ is

$$\begin{aligned}
0 &= P_t \frac{\epsilon}{\epsilon-1} \left( \int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} \frac{\epsilon-1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) \\
0 &= \left( \int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t} \\
0 &= \left( \int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) - \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} \\
Y_t(j) &= \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.
\end{aligned}$$

Plugging this quantity into the identity

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j) \, dj$$

and simplifying yields the price index

$$P_t = \left( \int_0^1 P_t(j)^{1-\epsilon} \, dj \right)^{\frac{1}{1-\epsilon}}.$$

**Intermediate Producers** Intermediate goods are producing according to the linear technology

$$Y_t(j) = A_t N_t(j).$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\min_{N_t(j)} W_t N_t(j) \qquad \text{s.t.} \qquad A_t N_t(j) \geq \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

The Lagrangian is

$$\mathcal{L} = W_t N_t(j) + \varphi_t(j) \left( \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - A_t N_t(j) \right),$$

so the first-order condition is

$$0 = W_t - \varphi_t(j) A_t \Rightarrow \varphi_t(j) = \frac{W_t}{A_t}.$$

The multiplier $\varphi_t$ can be interpreted as the nominal marginal cost. Let $mc_t$ be the real marginal cost. Then profits for an intermediate producer is

$$\Pi_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - mc_t Y_t(j).$$

In addition to the labor choice, firms also have the chance to reset prices in every period with probability $1 - \phi$. This problem can be written as

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} \left( \frac{P_t(j)}{P_{t+s}} \left( \frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left( \frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} \right),$$

where I have imposed that output equals demand. The first-order condition is

$$\begin{aligned} 0 = (1-\epsilon) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} (P_{t+s})^{-(1-\epsilon)} Y_{t+s} \\ + \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s} \end{aligned}$$

Divide by $P_t(j)^{-\epsilon}/u'(C_t)$ and re-arrange to obtain

$$P_t(j) = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s}) P_{t+s}^{\epsilon-1} Y_{t+s}}.$$

This expression gives the optimal reset price $P_t^*$, which we can write more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{X_{1,t}}{X_{2,t}}$$

where

$$\begin{aligned} X_{1,t} &= u'(C_t) mc_t P_t^{\epsilon} Y_t + \phi\beta \mathbb{E}_t X_{1,t+1} \\ X_{2,t} &= u'(C_t) P_t^{\epsilon-1} Y_t + \phi\beta \mathbb{E}_t X_{2,t+1}. \end{aligned}$$

## 1.3 Equilibrium and Aggregation

To close the model, I assume that the log of technology $A_t$ follows the AR(1)

$$\log A_t = \rho_a \log A_{t_1} + \varepsilon_{a,t},$$

and the growth rate in the log money supply follows the AR(1)

$$\Delta \log M_t = (1-\rho_m)\pi + \rho_m \Delta \log M_{t-1} + \varepsilon_{m,t},$$

where $\pi$ is the steady-state rate of inflation. Note that this specification ensures that money balances grow at the same rate as the price level, which ensures real balances are stationary. To re-write the money growth equation in real terms, note that

$$\log(m_t) = \log(M_t) - \log(P_t) \Rightarrow \Delta \log(m_t) = \log(m_t) - \log(m_{t-1}) = \Delta \log(M_t) - \log(1+\pi_t),$$

hence

$$\Delta \log(m_t) = (1-\rho_m)\pi + \rho_m \Delta \log(m_{t-1}) + \rho_m \log(1+\pi_{t-1}) - \log(1+\pi_t) + \varepsilon_{m,t}.$$

In equilibrium, bond-holding must be zero, hence

$$C_t = w_t N_t + \frac{\Pi_t}{P_t}.$$

Real dividends $\Pi_t$ satisfy the accounting identity

$$\begin{aligned}\frac{\Pi_t}{P_t} &= \int_0^1 \left(\frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j)\right) dj \\ &= \int_0^1 \frac{P_t(j)}{P_t} Y_t(j)\, dj - w_t \int_0^1 N_t(j)\, dj\end{aligned}$$

where $w_t = W_t/P_t$. Aggregate labor supply $N_t$ equals aggregate labor demand in equilibrium, and market-clearing for consumption requires

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j)\, dj = \int_0^1 \frac{P_t(j)}{P_t} \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t\, dj = P_t^{\epsilon-1} Y_t \int_0^1 P_t(j)^{1-\epsilon}\, dj = Y_t$$

since $\int_0^1 P_t(j)^{1-\epsilon}\, dj = P_t^{1-\epsilon}$.

The quantity $Y_t$ is aggregate output, so we must have

$$\begin{aligned}\int_0^1 A_t N_t(j)\, dj &= \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t\, dj \\ A_t N_t &= Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj = v_t Y_t.\end{aligned}$$

Thus, aggregate output is

$$Y_t = \frac{A_t N_t}{v_t}.$$

It can be shown that $v_t \geq 1$ by applying Jensen's inequality.
Finally, recall that

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon}\, dj.$$

In each period, a fraction $\phi$ cannot change their price. Without loss of generality, we may re-order these firms to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon}\, dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \phi \int_0^1 P_{t-1}(j)^{1-\epsilon}\, dj = (1-\phi)(P_t^*)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}.$$

Dividing by $P_{t-1}^{1-\epsilon}$ implies

$$(1+\pi_t)^{1-\epsilon} = (1-\phi)(1+\pi_t^*)^{1-\epsilon} + \phi$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$\begin{aligned}
v_t &= \int_0^{1-\phi} \left(\frac{P_t^*}{P_t}\right)^{-\epsilon} dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\epsilon} dj \\
&= \int_0^{1-\phi} \left(\frac{P_t^*}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} dj \\
&= (1-\phi)(1+\pi_t^*)^{-\epsilon}(1+\pi_t)^{\epsilon} + (1+\pi_t)^{\epsilon} \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj.
\end{aligned}$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \phi \int_0^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon} dj = \phi v_{t-1}.$$

Thus, we acquire

$$\begin{aligned}
v_t &= (1-\phi)(1+\pi_t^*)^{-\epsilon}(1+\pi_t)^{\epsilon} + \phi(1+\pi_t)^{\epsilon} v_{t-1} \\
&= (1+\pi_t)^{\epsilon}((1-\phi)(1+\pi_t^*)^{-\epsilon} + \phi v_{t-1}).
\end{aligned}$$

To finish, we need to derive an expression characterizing $\pi_t^*$. Define

$$x_{1,t} \equiv \frac{X_{1,t}}{P_t^{\epsilon}}, \qquad x_{2,t} \equiv \frac{X_{2,t}}{P_t^{\epsilon-1}}.$$

It follows that

$$\begin{aligned}
x_{1,t} &= u'(C_t)mc_tY_t + \phi\beta\mathbb{E}_t\frac{X_{1,t+1}}{P_t^{\epsilon}} \\
&= u'(C_t)mc_tY_t + \phi\beta\mathbb{E}_t\left[\frac{X_{1,t+1}}{P_{t+1}^{\epsilon}}\frac{P_{t+1}^{\epsilon}}{P_t^{\epsilon}}\right] \\
&= u'(C_t)mc_tY_t + \phi\beta\mathbb{E}_t[x_{1,t+1}(1+\pi_{t+1})^{\epsilon}] \\
x_{2,t} &= C_t^{-\sigma}Y_t + \phi\beta\mathbb{E}_t\frac{X_{2,t+1}}{P_t^{\epsilon-1}} \\
&= C_t^{-\sigma}Y_t + \phi\beta\mathbb{E}_t\left[\frac{X_{2,t+1}}{P_{t+1}^{\epsilon-1}}\frac{P_{t+1}^{\epsilon-1}}{P_t^{\epsilon-1}}\right] \\
&= C_t^{-\sigma}Y_t + \phi\beta\mathbb{E}_t[x_{2,t+1}(1+\pi_{t+1})^{\epsilon-1}].
\end{aligned}$$

Further,

$$\frac{X_{1,t}}{X_{2,t}} = \frac{x_{1,t}}{x_{2,t}}P_t,$$

hence

$$\begin{aligned}
P_t^* &= \frac{\epsilon}{\epsilon-1}\frac{x_{1,t}}{x_{2,t}}P_t \\
(1+\pi_t^*) &= \frac{\epsilon}{\epsilon-1}\frac{x_{1,t}}{x_{2,t}}(1+\pi_t)
\end{aligned}$$

All together, the full set of equilibrium conditions are

$$
\begin{aligned}
C_t^{-\sigma} &= \beta \mathbb{E}_t \left[ C_{t+1}^{-\sigma} \frac{(1+i_t)}{1+\pi_{t+1}} \right] \\
C_t^{-\sigma} &= \psi \frac{N_t^\eta}{w_t} \\
m_t &= \theta \frac{1+i_t}{i_t} C_t^\sigma \\
mc_t &= \frac{w_t}{A_t} \\
C_t &= Y_t \\
Y_t &= \frac{A_t N_t}{v_t} \\
v_t &= (1+\pi_t)^\epsilon ((1-\phi)(1+\pi_t^*)^{-\epsilon} + \phi v_{t-1}) \\
(1+\pi_t)^{1-\epsilon} &= (1-\phi)(1+\pi_t^*)^{1-\epsilon} + \phi \\
(1+\pi_t^*) &= \frac{\epsilon}{\epsilon-1} \frac{x_{1,t}}{x_{2,t}} (1+\pi_t) \\
x_{1,t} &= C_t^{-\sigma} mc_t Y_t + \phi\beta \mathbb{E}_t [x_{1,t+1}(1+\pi_{t+1})^\epsilon] \\
x_{2,t} &= C_t^{-\sigma} Y_t + \phi\beta \mathbb{E}_t [x_{2,t+1}(1+\pi_{t+1})^{\epsilon-1}] \\
\log A_t &= \rho_a \log(A_{t-1}) + \varepsilon_{a,t} \\
\Delta \log(m_t) &= (1-\rho_m)\pi + \rho_m \Delta \log(m_{t-1}) + \rho_m \log(1+\pi_{t-1}) - \log(1+\pi_t) + \varepsilon_{m,t} \\
\Delta \log m_t &= \log m_t - \log m_{t-1},
\end{aligned}
$$

which comprise 14 equations in 14 aggregate variables

$$(C_t, i_t, \pi_t, N_t, w_t, m_t, mc_t, A_t, Y_t, v_t, \pi_t^*, x_{1,t}, x_{2,t}, \Delta \log m_t).$$

Alternatively, the money growth equation can be replaced by the Taylor rule

$$\log(1+i_t) = (1-\rho_i)\log(1+i) + \rho_i \log(1+i_{t-1}) + (1-\rho_i)\phi_\pi(\log(1+\pi_t) - \log(1+\pi)) + \varepsilon_{i,t},$$

and the third equation relating money demand to consumption could also be ignored. To reduce the number of equations, we utilize this specification. Furthermore, we can also substitute $1+\pi_t^*$ to remove $\pi_t^*$ from the aggregate variables.

# 2 Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$
\begin{aligned}
0 &= \log \mathbb{E}_t \left[ \exp\left( \xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1} \right) \right] \\
z_{t+1} &= \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t)\varepsilon_{t+1},
\end{aligned}
$$

where $z_t$ are (predetermined) state variables and $y_t$ are (nondetermined) jump variables. For the remainder of this section, lower case variables are the logs of previously upper case variables, and variables with a tilde are the logs of previously lower case variables (e.g. the real wage $w_t$)

The first equation becomes

$$\begin{aligned}
1 &= \beta \mathbb{E}_t \left[ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{1+i_t}{1+\pi_{t+1}} \right] \\
0 &= \log \mathbb{E}_t \left[ \exp\left( \log(\beta) - \sigma(c_{t+1} - c_t) + \tilde{i}_t - \tilde{\pi}_{t+1} \right) \right] \\
&= \log \mathbb{E}_t \left[ \exp\left( \underbrace{\log(\beta) + \sigma c_t + \tilde{i}_t}_{\xi} - \underbrace{\sigma c_{t+1} - \tilde{\pi}_{t+1}}_{\text{Forward-Looking}} \right) \right],
\end{aligned}$$

where $c_t = \log(C_t)$, $\tilde{i}_t = \log(1+i_t)$, and $\tilde{\pi}_t = \log(1+\pi_{t+1})$.

The second equation becomes

$$\begin{aligned}
1 &= \psi \frac{N_t^{\eta}}{C_t^{-\sigma} w_t} \\
0 &= \log \mathbb{E}_t \left[ \exp\left( \underbrace{\log(\psi) + \eta n_t - (-\sigma c_t + \hat{w}_t)}_{\xi} \right) \right],
\end{aligned}$$

where $n_t = \log(N_t)$ and $\hat{w}_t = \log(w_t)$.

The third equation becomes

$$\begin{aligned}
1 &= \frac{w_t}{A_t m c_t} \\
0 &= \log \mathbb{E}_t \left[ \exp\left( \hat{w}_t - a_t - \tilde{mc}_t \right) \right].
\end{aligned}$$

The fourth and fifth equation become

$$0 = \log \mathbb{E}_t \left[ \exp\left( c_t - a_t - n_t + \hat{v}_t \right) \right].$$

The sixth equation becomes

$$0 = \hat{v}_t - \epsilon \tilde{\pi}_t - \log((1-\phi)\exp(\tilde{\pi}_t^*)^{-\epsilon} + \phi \exp(\hat{v}_{t-1})),$$

where $\hat{v}_{t-1}$ will be treated as an additional state variable, i.e. if $a_t = \hat{v}_t$ is a jump variable and $b_t = \hat{v}_{t-1}$ is a state variable, then

$$b_{t+1} = a_t.$$

The seventh equation becomes

$$0 = (1-\epsilon)\tilde{\pi}_t - \log((1-\phi)\exp(\tilde{\pi}_t^*)^{1-\epsilon} + \phi).$$

The eighth equation becomes

$$0 = \tilde{\pi}_t^* - \log\left(\frac{\epsilon}{\epsilon - 1}\right) - \tilde{\pi}_t - (\hat{x}_{1,t} - \hat{x}_{2,t})).$$

By plugging this expression for $\tilde{\pi}_t^*$ into the previous two equations, we can also remove one more variable from the system.

The ninth and tenth equation become

$$1 = \mathbb{E}_t\left[\phi\beta\frac{x_{1,t+1}(1+\pi_{t+1})}{x_{1,t} - C_t^{-\sigma} mc_t A_t N_t / v_t}\right]$$

$$0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{1,t}) - \exp((1-\sigma)c_t + \tilde{m}c_t))}_{\xi} + \hat{x}_{1,t+1} + \epsilon\tilde{\pi}_{t+1}\right)\right]$$

$$0 = \log \mathbb{E}_t\left[\exp\left(\underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{2,t}) - \exp((1-\sigma)c_t))}_{\xi} + \hat{x}_{2,t+1} + (\epsilon - 1)\tilde{\pi}_{t+1}\right)\right],$$

where the fact that $x_{1,t}$ and $x_{2,t}$ must both be positive implies $x_{1,t} - C_t^{-\sigma} mc_t Y_t$ and $x_{2,t} - C_t^{-\sigma} Y_t$ are both positive, as the expectations on the RHS are also both positive.

For the monetary policy rule, we use $\tilde{i}_{t-1} \equiv \log(1 + i_{t-1})$ and $\varepsilon_{i,t}$ as states and treat $i_t$ as a jump variable, hence

$$\tilde{i}_t = (1 - \rho_i)\tilde{i} + \rho_i \tilde{i}_{t-1} + (1 - \rho_i)\phi_\pi(\tilde{\pi}_t - \tilde{\pi}) + \varepsilon_{i,t}.$$

This formulation allows us to treat the policy rule as an expectational equation.

The above nine equations comprise the expectational equations. The following four equations comprise the states:

$$\begin{aligned}
a_{t+1} &= \rho_a a_t + \varepsilon_{a,t+1} \\
\hat{v}_{(t-1)+1} &= \hat{v}_t \\
\tilde{i}_{(t-1)+1} &= \tilde{i}_t \\
\varepsilon_{i,t+1} &= \varepsilon_{i,t+1}
\end{aligned}$$

To conclude, note that we have three forward difference equations. The first is the Euler equation, which can be expressed as