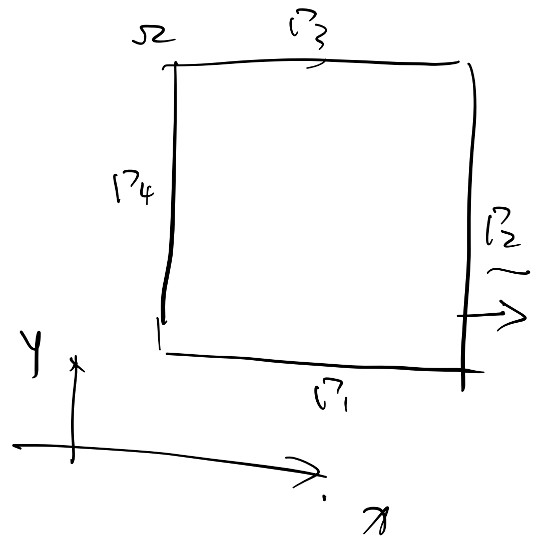
$$\begin{cases} \Delta u + p(x,y)u = f(x,y) & \text{on } \Omega. \\ u=0 & \text{on } \Gamma_4\cup\Gamma_1 \\ \partial_x u = g_1 & \text{on } \Gamma_2. \\ \partial_y u = g_2 & \text{on } \Gamma_3 \end{cases}$$



$p(x,y) = 2\pi.$

$f(x,y) = 2\pi^2 xy.$

$g_1(y) = y - \pi\sin(\pi y)$

$g_2(x) = x - \pi\sin(\pi x)$

弱变分:

$$\int_\Omega f(x,y)\cdot v(x,y)\,d\Omega = \int_\Omega (\Delta u + p\cdot u)\,v\,d\Omega$$

$$= \int_\Omega (\nabla u\cdot\nabla v + pu\cdot v)\,dxdy + \int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4} \partial_n u\cdot v\,ds$$

$$\int_{\Gamma_1\cup\Gamma_4} (\partial_n u\cdot v)\,ds = 0.$$

$$\int_{\Gamma_2} \partial_n u\cdot v\,ds = \int_{\Gamma_2} \partial_x u\cdot v\,dy = \int_{\Gamma_2} g_1\cdot v\,dy$$

$$\int_{\Gamma_3} \partial_n u\cdot v\,ds = \int_{\Gamma_3} \partial_y u\cdot v\,dx = \int_{\Gamma_3} g_3\cdot v\,dx$$

求 $u=0$ on $\Gamma_1\cup\Gamma_4$ 且 $u\in H^1(\Omega)$ 对 $\forall$

$v=0$ on $\Gamma_1\cup\Gamma_4$, $v\in H^1(\Omega)$ 满足.

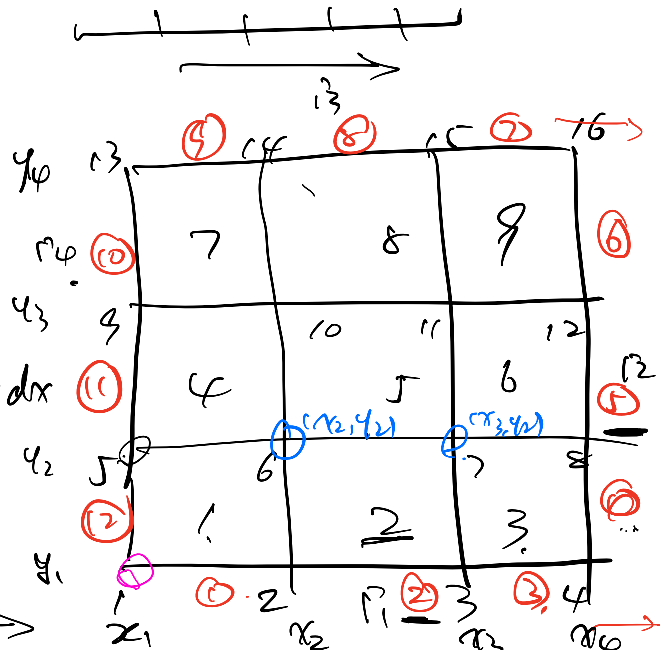
$$a(u,v) = \int_\Omega (-\nabla u\cdot\nabla v + p\,u\cdot v)\,dxdy$$

$$= \int_\Omega f(x,y)\,v\,d\Omega - \int_{\Gamma_2} g_1 v\,dy - \int_{\Gamma_3} g_2 v\,dx$$

近似.

$$u \approx \sum_{i=1}^{n} u_i \varphi_i$$

$$\approx \sum_{i=1}^{16} u_i \varphi_i$$



$$a(u,v) = \sum_{i=1}^{16} u_i\, a(\varphi_i, v)$$

$$= \int_\Omega f v\, d\Omega - \int_{\Gamma_2} g_1 v\, dy - \int_{\Gamma_3} g_2 v\, dx$$

取 $v = \varphi_i$, $i = 1, 2, \dots, 16$

$$\begin{bmatrix} a(\varphi_1,\varphi_1), a(\varphi_2,\varphi_1) \cdots, a(\varphi_{16},\varphi_1) \\ a(\varphi_1,\varphi_2), a(\varphi_2,\varphi_2) \cdots, a(\varphi_{16},\varphi_2) \\ \vdots \\ a(\varphi_1,\varphi_{16}), a(\varphi_2,\varphi_{16}) \cdots, a(\varphi_{16},\varphi_{16}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{16} \end{bmatrix} = \begin{bmatrix} \int_\Omega f \varphi_1 d\Omega - \int_{\Gamma_2} g_1 \varphi_1 dy - \int_{\Gamma_3} g_2 \varphi_1 dx \\ \int_\Omega f \varphi_2 d\Omega - \int_{\Gamma_2} g_1 \varphi_2 - \int_{\Gamma_3} g_2 \varphi_2 \\ \vdots \\ \int_\Omega f \varphi_{16} d\Omega - \int_{\Gamma_2} g_1 \varphi_{16} - \int_{\Gamma_3} g_2 \varphi_{16} \end{bmatrix}$$

$:= b$

Q1：解出的解是否为问题近似解.

：不是，注意对 1、2、3、4，5、9、13 还要做本质边界处理.

Q2：$\boxed{\int_\Omega f \varphi_1\, dxdy} - \int_{\Gamma_2} g_1 \varphi_1\, dy - \int_{\Gamma_3} g_2 \varphi_1\, dx$

：部分需处理（自然边界条件）

Q3：A 的具体形式

$$a(\varphi_1,\varphi_1) = \int_\Omega (P \varphi_1 \varphi_1 - \nabla\varphi_1 \nabla\varphi_1)\, dxdy$$

形成系数矩阵

For $i=1:1:9$

(以 5 个矩形块为例). $\varphi_6, \varphi_7, \varphi_{10}, \varphi_{11}$

$$K=\int_{I_5}(p\varphi_6\varphi_7-\nabla\varphi_6\cdot\nabla\varphi_7)\,dx\,dy$$

16. $$a(\varphi_6,\varphi_7)=\int_\Omega(p\varphi_6\varphi_7-\nabla\varphi_6\nabla\varphi_7)\,dx\,dy$$

$$=\int_{I_2}(p\varphi_6\varphi_7-\nabla\varphi_6\nabla\varphi_7)\,dx\,dy$$

$$+\int_{I_5}(p\varphi_6\varphi_7-\nabla\varphi_6\nabla\varphi_7)\,dx\,dy$$

即. $A(6,7)=A(6,7)+K$

$$I=\int_{I_5}f(x,y)\,\varphi_6\,dx\,dy$$

$$b(6)=\int_\Omega f(x,y)\varphi_6\,dx\,dy$$

$$=\int_{I_1\cup I_2\cup I_4\cup I_5}f(x,y)\,\varphi_6\,dx\,dy$$

$b(6)=b(6)+I$

End

处理自然边界条件

$$E=\begin{bmatrix}1&2&3&4&8&12&15&14&13&1&5&9\\2&3&4&8&12&16&16&15&14&5&9&13\\1&1&1&2&2&2&3&3&3&4&4&4\end{bmatrix}$$

for $i = 1 : 12$

if $E(3, i) == 2$?

則 ⑤ 边为例. 令 $\varphi_8.\ \varphi_{12}$.

$$K = -\int_{I_⑤} g_1\, \varphi_8(1, y)\, dy$$

$$b(8) = -\int_{\Gamma_2} g\cdot \varphi_8(1, y)\, dy$$

$$= -\int_{I_④} g_1 \varphi_8(1,y)\, dy - \int_{I_⑤} g_1 \varphi_8(1,y)\, dy$$

$$b(8) = b(8) + K$$

End

本质边界条件. $\varphi_1, \varphi_5, \varphi_9, \varphi_{13}, \varphi_{14}\ \varphi_{15}\ \varphi_{16}$

$$\sum_{i=1}^{16} u_i\, a(\varphi_i, \varphi_5) = \int_\Omega F \varphi_5\, dx - \int_{\Gamma_2} g_1 \varphi_5 - \int_{\Gamma_3} g_2 \varphi_5\, dx$$

修正. $u_5 = 0 \quad \Rightarrow \quad u_5 = \alpha$

$$\begin{array}{c} [\,0, \cdots 0,\ 1, 0 \cdots, 0\,] \\ \uparrow \\ 5 \end{array} \begin{bmatrix} u_1 \\ \vdots \\ u_{16} \end{bmatrix} = 0.$$

Q6: 先处理本质边界条件, 后处理自然边界条件, 是否可以?

算积分

$$K = \int_{I_5} (P\cdot \varphi_6 \cdot \varphi_7 - \nabla\varphi_6 \cdot \nabla\varphi_7)\, dx\, dy$$

用仿射变换.

$x = x_2 + h_x t \qquad t, s \in [0, 1]$

$$y = y_2 + h_y s.$$

dt

$$\varphi_6(x,y) = \frac{x_2+h_x-x}{h_x} \cdot \frac{y_2+h_y-y}{h_y}$$

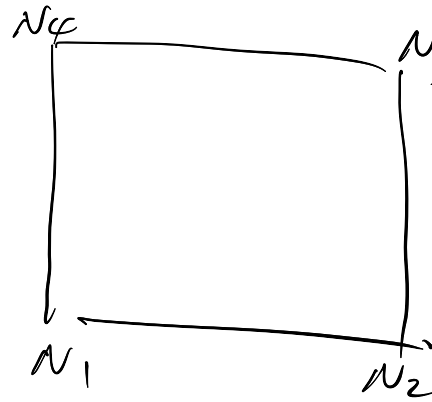
$$= \frac{x_2+h_x-x_2-h_x t}{h_x} \cdot \frac{y_2+h_y-y_2-h_y s}{h_y}$$

$$= (1-t)\,(1-s) = N_1$$

$$\varphi_7(x,y) = t\,(1-s) = N_2$$

$$\varphi_{10}(x,y) = (1-t)\,s = N_4$$

$$\varphi_{11}(x,y) = t\,s.$$



$$K_1 = \int_{I_0} P(x_2+h_x t, y_2+h_y s)\, \varphi_6(x_2+h_x t, y_2+h_y s) \cdot \varphi_7(x_2+h_x t, y_2+h_y s)\, h_x h_y\, ds$$

$$= \int_{I_0} P(x_2+h_x t, y_2+h_y s)\, N_1(t,s) \cdot N_2(t,s)\, h_x h_y\, dt\, ds$$

$$\varphi_6(x,y) = \varphi_6(x_2+h_x t, y_2+h_y s) = N_1(t,s)$$

$$\frac{\partial}{\partial x}\varphi_6 = \frac{\partial}{\partial t} N_1 \frac{\partial t}{\partial x} + \frac{\partial}{\partial s} N_1 \frac{\partial s}{\partial x}$$

$$= \frac{\partial}{\partial t} N_1 \frac{1}{h_x}$$

$$\frac{\partial}{\partial x}\varphi_7 = \frac{1}{h_x}\frac{\partial}{\partial t} N_2, \quad \frac{\partial}{\partial x}\varphi_{10} = \frac{1}{h_x}\frac{\partial}{\partial t} N_4, \quad \frac{\partial}{\partial x}\varphi_{11} = \frac{1}{h_x}\frac{\partial}{\partial t} N_3$$

$$\frac{\partial}{\partial y}\varphi_6 = \frac{1}{h_y}\frac{\partial}{\partial s} N_1, \quad \frac{\partial}{\partial x}\varphi_7 = \frac{1}{h_y}\frac{\partial}{\partial s} N_2, \quad \frac{\partial}{\partial y}\varphi_{10} = \frac{1}{h_y}\frac{\partial}{\partial s} N_4, \quad \frac{\partial}{\partial y}\varphi_{11} = \frac{1}{h_y}\frac{\partial}{\partial s} N_3$$

$$K_5 = -\int_{I_5} \nabla\varphi_6 \cdot \nabla\varphi_7 \, dx\,dy = -\int_{I_5} \left(\partial_x\varphi_6\, \partial_x\varphi_7 + \partial_y\varphi_6\, \partial_y\varphi_7\right) dx\,dy$$

$$= -\int_{I_0} \left[\left(\frac{1}{h_x}\partial_t N_1\right)\left(\frac{1}{h_x}\partial_t N_2\right) + \left(\frac{1}{h_y}\partial_s N_1\right)\left(\frac{1}{h_y}\partial_s N_2\right)\right] h_x h_y \, dt\,ds$$

$$K = \int_{I_0} \Big[p(x_2 + h_x t,\ y_2 + h_y s)\, N_1(t,s)\, N_2(t,s)$$

$$- \left(\frac{1}{h_x}\right)^2 \partial_t N_1\, \partial_t N_2 - \left(\frac{1}{h_y}\right)^2 \partial_s N_1\, \partial_s N_2 \Big]\, h_x h_y \, dt\,ds$$

考虑

$$\int_{I_0} f(x,y)\, dx\,dy$$

$$= \int_0^1 \int_0^1 f(x,y)\, dx\,dy$$

$$\approx \int_0^1 \left[\sum_{s=1}^{K_1} w_s^{(1)} f(x_s, y)\right] dy$$

$$= \sum_{s=1}^{K_1} w_s^{(1)} \int_0^1 f(x_s, y)\, dy$$

$$\approx \sum_{s=1}^{K_1} w_s^{(1)} \sum_{t=1}^{K_2} w_t^{(2)} f(x_s, y_t).$$

$$= \left[w_1^{(1)}, \cdots, w_{K_1}^{(1)}\right] \begin{bmatrix} f(x_1,y_1), & f(x_1,y_2), & \cdots, & f(x_1,y_{K_2}) \\ f(x_2,y_1), & f(x_2,y_2), & \cdots, & f(x_2,y_{K_2}) \\ \vdots & & & \\ f(x_{K_1},y_1), & f(x_{K_1},y_2), & \cdots, & f(x_{K_1},y_{K_1}) \end{bmatrix} \begin{bmatrix} w_1^{(1)} \\ \vdots \\ w_{K_2}^{(1)} \end{bmatrix}$$

$$F_1 = \begin{bmatrix} P(x_2+h_x t_1, y_2+h_y s_1)\cdot N_1(t_1,s_1)\, N_2(t_1,s_1) & \cdots, & P(x_2+h_x t_1, y_2+h_y s_{k_2})\, N_1(t_1,s_{k_2})\, N_2(t_1,s_{k_2}) \\ P(x_2+h_x t_2, y_2+h_y s_1)\, N_1(t_2,s_1)\, N_2(t_2,s_1) & , & P(x_2+h_x t_2, y_2+h_y s_{k_2})\, N_1(t_2,s_{k_2})\, N_2(t_2,s_{k_2}) \\ \vdots & & \vdots \\ P(x_2+h_x t_{k_1}, y_2+h_y s_1)\, N_1(t_{k_1},s_1)\, N_2(t_{k_1},s_1) & & P(x_2+h_x t_{k_1}, y_2+h_y s_{k_2})\, N_1(t_{k_1},s_{k_2})\, N_2(t_{k_1},s_{k_2}) \end{bmatrix}$$

$$= \begin{bmatrix} P(x_2+h_x t_1, y_2+h_y s_1) & \cdots, & P(x_2+h_x t_1, y_2+h_y s_{k_2}) \\ P(x_2+h_x t_2, y_2+h_y s_1) & & \\ \vdots & & \end{bmatrix} .\times \begin{bmatrix} N_1(t_1,s_1) & \cdots, & N_1(t_1,s_{k_2}) \\ N_1(t_2,s_1) & & \vdots \\ \vdots & & \\ N_1(t_{k_1},s_1) & & N_1(t_{k_1},s_{k_2}) \end{bmatrix}$$

$$.\times \begin{bmatrix} N_2(t_1,s_1) & \cdots, & N_2(t_1,s_{k_2}) \\ N_2(t_2,s_1) & & \\ \vdots & & \\ N_2(t_{k_1},s_1) & \cdots & N_2(t_{k_1},s_{k_2}) \end{bmatrix}$$

$$K_1 = \int_{I_0} P\, N_1 \cdot N_2\, h_x h_y \, dt\, ds$$

$$\approx [w_1^{(1)}, \cdots, w_{k_1}^{(1)}] \left(P .\times N_1 .\times N_2 \right) \begin{bmatrix} w_1^{(2)} \\ \vdots \\ w_{k_2}^{(2)} \end{bmatrix}$$

形式单元刚度矩阵

For $i=1:1:4$

For $j=1:1:4$

$$K(i,j) = [w_1^{(1)}, \cdots, w_{k_1}^{(1)}] \left[\overset{.*\Psi\{1,i\} .*\Psi\{1,j\}}{P .\times \underline{N_i} .\times \underline{N_j}} \right] \begin{bmatrix} w_1^{(2)} \\ \vdots \\ w_{k_2}^{(2)} \end{bmatrix} h_x h_y$$

$$- [w_1^{(1)}, \cdots, w_{k_1}^{(1)}] \left[\partial_t N_i .\times \partial_t N_j \right] \begin{bmatrix} w_1^{(2)} \\ \vdots \\ w_{k_2}^{(2)} \end{bmatrix} \frac{h_y}{h_x}$$

$$- [w_1^{(1)}, \cdots, w_{k_1}^{(1)}] \left[\partial_s N_i .\times \partial_s N_j \right] \begin{bmatrix} w_1^{(2)} \\ \vdots \\ w_{k_2}^{(2)} \end{bmatrix} \frac{h_x}{h_y}$$

① 把 N_1 矩阵做变形

$$N_1 = \begin{bmatrix} N_1(t_1,s_1), & N_1(t_1,s_2), \cdots, & N_1(t_1,s_{k_2}) \\ \vdots & & \\ N_1(t_{k_1},s_1) & \cdots & N_1(t_{k_1},s_{k_2}) \end{bmatrix}$$

$$= [N_1(t_1,s_1), N_1(t_1,s_2), \cdots, N_1(t_1,s_{k_2}), N_1(t_2,s_1) \cdots, N_1(t_2,s_{k_2}) \cdots, N_1(t_{k_1},s_1) \cdots, N_1(t_{k_1},s_{k_2})]$$

② $\psi(:,:,k) = N_k$

③ cell, $\psi = \text{cell}\{1,4\}$. $\psi\{1,k\} = N_k$

计算 $k = \int_{I_5} f(x,y)\, \varphi_6\, dx dy$

$$= \int_{I_0} \underline{f(x_2+h_x t, y_2+h_y s)\, N_1(t,s)}\, h_x h_y\, dt\, ds$$

$$\approx [w_1^{(1)} \cdots, w_{k_1}^{(1)}] \begin{bmatrix} f(x_2+h_x t_1, y_2+h_y s_1 \cdots, & f(x_2+h_x t_1, y_2+h_y s_{k_2} \\ \vdots & \\ f(x_2+h_x t_{k_1}, y_2+h_y s_1, \cdots, & f(x_2+h_x t_{k_1}, y_2+h_y s_{k_2}) \end{bmatrix}$$

$$h_x h_y \;.\!\times \begin{bmatrix} N_1(t_1,s_1) \cdots, & N_1(t_1,s_{k_2}) \\ \vdots & \\ N_1(t_{k_1},s_1) \cdots, & N_1(t_{k_1},s_{k_2}) \end{bmatrix} \begin{bmatrix} w_1^{(2)} \\ \vdots \\ w_{k_2}^{(2)} \end{bmatrix}$$

处理自然边界条件积分

$$k = \int_{I_{⑤}} g_1(y)\, \varphi_8(x,y)\, dy$$

$y = y_2 + h_y t$.

$$= \int_0^1 g_1(y_2+h_y t)\, \varphi_8(1, y_2+h_y t).\, h_y\, dt.$$

$$\varphi_8(x,y) = \frac{x - x_3}{h_x} \cdot \frac{y_2+h_y - y}{h_y} = \frac{\underline{x_4 - x_3}}{\underline{h_x}} \cdot \frac{y_2+h_y - (y_2+h_y t)}{h_y}$$

$= 1\cdot(1-t) = (1-t)$

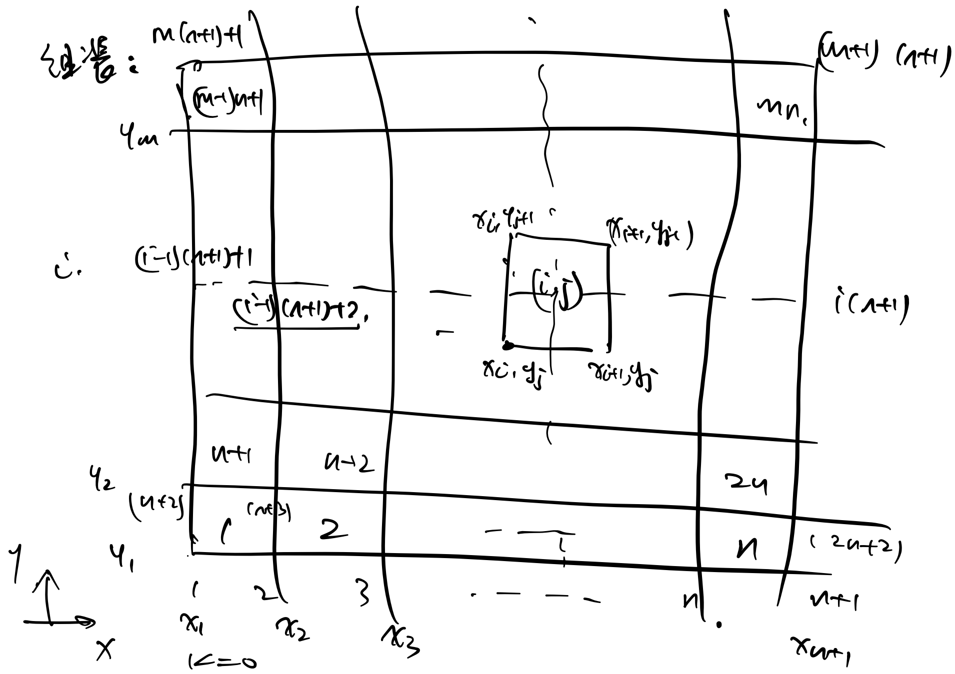
$\varphi_{12}(x,y) = \varphi_{12}(1, y_2+h_y t)$ $\qquad N_1(t) = 1-t.$

$= t$ $\qquad N_2(t) = t$

$= \int_0^1 g_1(y_2+h_y t) \, N_1(t) \, h_y \, dt.$

$= \sum_{k=1}^{K_1} w_k^{(1)} g_1(y_2+h_y t_k) N_1(t_k) h_y$

组装：



i 行，j 列. 全是.

$\underline{(j-1)(n+1)+i}$

for $i=1:1:n+1$ $\qquad$ x 方向

for $j=1:1:m+1$ $\qquad$ y 方向

$k=k+1$

$P(k,1) = x_i$

$P(k,2) = y_j$

End

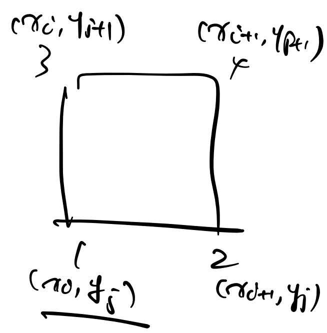
End.

形式 ~~单~~ 元 $\qquad$ $k=0.$

For $i=1:1:n.$ $\qquad$ x 方向

For $j=1:1:m$ $\qquad$ y 方向

$k=k+1$



$G(k,1) = (j-1)(n+1)+i$

$G(k,2) = (j-1)(n+1)+i+1$

$G(k,3) = j\cdot(n+1)+i$

$G(k,4) = j(n+1)+(i+1)$

End

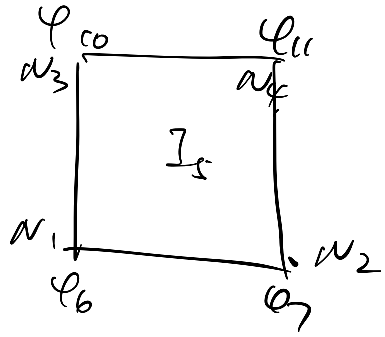
End

L^2 误差.

$$\int_{\Omega} |u_n - u|^2 \, dxdy$$

$$= \sum_{i=1}^{9} \int_{I_i} (u_n - u)^2 \, dxdy$$

$$= \sum_{i=1}^{9} \int_{I_i} \left| u_6 \varphi_6 + u_7 \varphi_7 + u_{10}\varphi_{10} + u_{11}\varphi_{11} - u \right|^2 dxdy$$



$$= \sum_{i=1}^{9} \int_{I_0} \Big| u_6 N_1(t,s) + u_7 N_2(t,s) + u_{10} N_3(t,s) + u_{11} N_4(t,s) - u(x_2 + h_x t,\, y_2 + h_y s) \Big|^2 h_x h_y \, dt\, ds$$