1 Optimal Binary Search Tree Problem (BST)

$$f(S) = \begin{cases} \min_{\alpha \in S}\{f(S_l) + f(S_r) + r(\alpha, S)\} & \text{if } S \neq \emptyset \\ 0 & \text{if } S = \emptyset, \end{cases} \tag{1}$$

where $S_l = \{x \in S : x < \alpha\}$;
$S_r = \{x \in S : x > \alpha\}$;

$$r(\alpha, S) = \sum_{x \in S} p(x).$$

Using an alternative base case the DP functional equation can be expressed as

$$f(S) = \begin{cases} \min_{\alpha \in S}\{f(S_l) + f(S_r) + r(\alpha, S)\} & \text{if } |S| > 1 \\ p(x) & \text{if } S = \{x\}, \end{cases}$$

The goal is to compute $f(X)$.

$$f(i,j) = \begin{cases} \min_{k \in \{i,\ldots,j\}}\{f(i,k-1) + f(k+1,j) + \sum_{l=i}^{j} p_l\} & \text{if } i \leq j \\ 0 & \text{if } i > j. \end{cases} \tag{2}$$

Using an alternative base case the DP functional equation can be expressed as

$$f(i,j) = \begin{cases} \min_{k \in \{i,\ldots,j\}}\{f(i,k-1) + f(k+1,j) + \sum_{l=i}^{j} p_l\} & \text{if } i < j \\ p_i & \text{if } i = j. \end{cases}$$

In this second model the goal is to compute $f(0, n-1)$.
X=(A,B,C,D,E).
P(x)=(.25,.05,.2,.4,.1).

2 Optimal Covering Problem (COV) or (SCA)

$$f(j,l) = \begin{cases} \min_{d \in \{j-2,\ldots,l-1\}}\{(l-d)c_l + f(j-1,d)\} & \text{if } j > 1 \\ (l+1)c_l & \text{if } j = 1. \end{cases}$$

The goal is to compute $f(n, k-1)$.
k=10.
(c0,...,c9)=(1,4,5,7,8,12,13,18,19,21).
n=3

3 Discounted Profits Problem (DPP) or (FPP)

$$f(t,b) = \begin{cases} \max\limits_{x_t \in \{0,\ldots,b\}} \{ & r(x_t) - c(x_t, b) \\ & + \frac{1}{1+y} f(t+1, \lfloor s(b - x_t) \rfloor)\} \quad \text{if } t \le T \\ 0 & \quad \text{if } t = T+1. \end{cases}$$

and the goal is to compute $f(1, b_1)$.
r(x)=3x c(x,b)=2x s=2 b1=10

4 Edit Distance Problem (EDP)

$$\begin{aligned} c(D) &= c_D && \text{for deleting any character, at any position} \\ c(I) &= c_I && \text{for inserting any character, at any position} \\ c(R) &= \begin{cases} 0 & \text{if } x_i = y_j \text{ (matching characters)} \\ c_R & \text{if } x_i \ne y_j \text{ (a true replacement)} \end{cases} \end{aligned}$$

$$f(X_i, Y_j) = \begin{cases} jD & \text{if } i = 0 \\ iI & \text{if } j = 0 \\ \min\limits_{d \in \{D,I,R\}} \{ f(t(X_i, Y_j, d)) + c(d) \} & \text{if } i > 0 \text{ and } j > 0. \end{cases} \tag{3}$$

where the transformation function is defined by

$$\begin{aligned} t(X_i, Y_j, D) &= (X_{i-1}, Y_j) \\ t(X_i, Y_j, I) &= (X_i, Y_{j-1}) \\ t(X_i, Y_j, R) &= (X_{i-1}, Y_{j-1}). \end{aligned}$$

The goal is to compute $f(x, y)$, the cost of a minimal cost edit sequence.
$c_D = c_I = c_R = 1$
x="CAN"
y="ANN"

5 Integer Linear Programming (ILP)

$$f(j, S) = \begin{cases} \max\limits_{x_{j+1} \in D} \{ c_{j+1} x_{j+1} + f(j+1, S \cup \{(j+1, x_{j+1})\}) \} & \text{if } j < n \\ 0 & \text{if } j = n. \end{cases} \tag{4}$$

and the goal becomes to compute $f(0, \emptyset)$.

$$f(j, y_1, \ldots, y_m) = \begin{cases} \max\limits_{x_{j+1} \in D} \{ c_{j+1} x_{j+1} \\ \quad + f(j+1, y_1 - a_{1,j+1} x_{j+1}, \ldots, y_m - a_{m,j+1} x_{j+1}) \} & \text{if } j < n \\ 0 & \text{if } j = n. \end{cases} \tag{4}$$

and the goal becomes to compute $f(0, b_1, \dots, b_m)$.

$$D = D(j, y_1, \dots, y_m) = \{0, \dots, \min\{\lfloor \frac{y_1}{a_{1,j}} \rfloor, \dots, \lfloor \frac{y_m}{a_{m,j}} \rfloor\}\}$$

where a term $\frac{y_i}{0}$ (with a zero in the denominator) should be interpreted as ∞

Let $c = (3, 5)$, $b = (4, 12, 18)$, and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{pmatrix}.$$

6 Investment: Winning in Las Vegas Problem (INVEST WLV)

$$f(n, s_n) = \begin{cases} \max\limits_{x_n \in \{0,\dots,s_n\}} \{ & (1-p) f(n+1, s_n - x_n) \\ & +p f(n+1, s_n + x_n)\} \quad \text{if } n \leq R \\ 0 & \text{if } n > R \text{ and } s_n < t \\ 1 & \text{if } n > R \text{ and } s_n \geq t \end{cases}$$

and the goal of the computation is $f(1, s_1)$.

p=2/3 r=3 s1=3 t=5

7 0/1 Knapsack Problem (KS01)

$$f(i, w) = \begin{cases} 0 & \text{if } i = -1 \text{ and } 0 \leq w \leq c \\ -\infty & \text{if } i = -1 \text{ and } w < 0 \\ \max\limits_{x_i \in \{0,1\}} \{x_i v_i + f(i-1, w - x_i w_i)\} & \text{if } i \geq 0. \end{cases} \tag{5}$$

The goal is to compute $f(n-1, c)$.

$$D = D(i, w) = \begin{cases} \{0\} & \text{if } w_i > w \\ \{0, 1\} & \text{if } w_i \leq w. \end{cases}$$

$$f(i, w) = \begin{cases} 0 & \text{if } i = -1 \text{ and } 0 \leq w \leq c \\ \max\limits_{x_i \in D} \{x_i v_i + f(i-1, w - x_i w_i)\} & \text{if } i \geq 0. \end{cases} \tag{6}$$

c=22 n=3 V={25,24,15} W={18,15,10}

8 Longest Common Subsequence (LCS)

Depending on the current state (X_i, Y_j) the *current* decision set D is defined as

$$D(X_i, Y_j) = \begin{cases} \{d_1, d_2\} & \text{if last char. of } (X_i, Y_j) \text{ don't match } (x_i \neq y_j) \\ \{d_{12}\} & \text{if last characters of } (X_i, Y_j) \text{ match } (x_i = y_j). \end{cases}$$

The transformation function is defined by

$$\begin{aligned} t(X_i, Y_j, d_1) &= (X_{i-1}, Y_j) \\ t(X_i, Y_j, d_2) &= (X_i, Y_{j-1}) \\ t(X_i, Y_j, d_{12}) &= (X_{i-1}, Y_{j-1}). \end{aligned}$$

The reward function is defined by

$$\begin{aligned} r(X_i, Y_j, d_1) &= 0 \\ r(X_i, Y_j, d_2) &= 0 \\ r(X_i, Y_j, d_{12}) &= 1. \end{aligned}$$

Now the DP functional equation can be expressed as

$$f(X_i, Y_j) = \begin{cases} 0 & \text{if } X_i = \epsilon \text{ or } Y_j = \epsilon \\ \max\limits_{d \in D(X_i, Y_j)} \{f(t(X_i, Y_j, d)) + r(X_i, Y_j, d)\} & \text{otherwise,} \end{cases} \tag{7}$$

where ϵ denotes the empty string.

$$f(X_i, Y_j) = \begin{cases} 0 & \text{if } X_i = \epsilon \text{ or } Y_j = \epsilon \\ \max\{ \quad f(X_{i-1}, Y_{j-1}) + \delta_{x_i, y_j}, & \\ \qquad\qquad f(X_{i-1}, Y_j), f(X_i, Y_{j-1})\} & \text{otherwise,} \end{cases} \tag{8}$$

where

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

denotes Kronecker's delta.

goal f(X7,Y6)

X=(a,b,c,b,d,a,b)

Y=(b,d,c,a,b,a)

$X_i = prefix(X, i)$

9 Longest Simple Path Problem (LSP)

$$f(S, v) = \begin{cases} \max\limits_{d \notin S} \{f(S \cup \{d\}, d) + c_{v,d}\} & \text{if } v \neq t \\ 0 & \text{if } v = t \end{cases}$$

where the length of a maximal length path is computed as $f(\{s\}, s)$.

$$C = \begin{pmatrix} -\infty & 1 & -\infty & 1 \\ 1 & -\infty & 1 & -\infty \\ -\infty & 1 & -\infty & 1 \\ 1 & -\infty & 1 & -\infty \end{pmatrix}$$

s=0

10 Matrix Chain Multiplication Problem (MCM)

$$f(i,j) = \begin{cases} \min\limits_{k \in \{i,\ldots,j-1\}} \{f(i,k) + f(k+1,j) + d_{i-1}d_kd_j\} & \text{if } i < j \\ 0 & \text{if } i = j. \end{cases}$$

goal=f(1,n)
Ai has dimension d$_{i-1}xd_i$
D={3,4,5,2,2}
n=4

11 Optimal Distribution Problem (ODP) (OFP)

$$f(i,x) = \min_{a_i}\{y_i(a_i) + f(i+1, x + c_i(a_i))\}.$$

The base case is

$$f(3,x) = \begin{cases} \infty & \text{if } x < 6 \\ 0 & \text{if } x \geq 6. \end{cases}$$

An alternative base case

$$f(2,x) = \begin{cases} \infty & \text{if } 0 \leq x \leq 1 \\ 22 & \text{if } x = 2 \\ 16 & \text{if } 3 \leq x \leq 5 \\ 0 & \text{if } 6 \leq x \leq 7. \end{cases}$$

goal=f(0,0)
c$_0 = \{0,1,2,3\} c_1 = \{0,1,2,3,4\} c_2 = \{0,3,4\}$
y$_0 = \{0,4,12,21\} y_1 = \{0,6,11,16,20\} y_2 = \{0,16,22\}$

12 Optimal Permutation Problem (PERM) (OST)

$$f(S) = \min_{x \in S}\{l(x) \cdot |S| + f(S - \{x\})\}$$

with the base case $f(\emptyset) = 0$.
goal=f(X)
X={A,B,C}
L={5,3,2}

13 Production: Reject Allowances Problem (PROD RAP)

$$f(n) = \begin{cases} \min\limits_{x_n \in \{0,\ldots,L\}} \{K(x_n) + c_m x_n + p^{x_n} f(n+1)\} & \text{if } n \leq R \\ c_p & \text{if } n = R+1. \end{cases}$$

where the function

$$K(x_n) = \begin{cases} 0 & \text{if } x_n = 0 \\ c_s & \text{if } x_n > 0 \end{cases}$$

p=0.5 L=5 R=3 c$_m = 3c_s = 3c_p = 16$

14 Reliability Design Problem (RDP)

$$f(i,x) = \max_{m_i \in M(i,x)} \{(1-(1-r_i)^{m_i}) \cdot f(i-1, x - c_i m_i)\}$$

with the base cases $f(-1,x) = 1.0$ where $x \leq b$. The goal is $f(n-1,b)$.

C={30,15,20} R={.9,.8,.5} B=105

M(i,x)={1,...,u(i,x)}

u(i,x)=floor(x=sum(c$_j$)$/c_i$)

15 Stagecoach Problem (SCP)

$$f(g,x) = \begin{cases} \min\limits_{d \in V_{g+1}} \{f(g+1,d) + c_{x,d}\} & \text{if } x < n-1 \\ 0 & \text{if } x = n-1. \end{cases} \tag{9}$$

$$C = \begin{pmatrix}
\infty & 550 & 900 & 770 & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & 680 & 790 & 1050 & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & 580 & 760 & 660 & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & 510 & 700 & 830 & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & 610 & 790 & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & 540 & 940 & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & 790 & 270 & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1030 \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1390 \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty
\end{pmatrix}$$

goal=f(0,0)

16 Shortest Path in an Acyclic Graph (SPA)

$$f(x) = \begin{cases} \min\limits_{d \in V}\{f(d) + c_{x,d}\} & \text{if } x < n-1 \\ 0 & \text{if } x = n-1. \end{cases} \tag{10}$$

The goal becomes to compute $f(0)$.

$$C = \begin{pmatrix} \infty & 3 & 5 & \infty \\ \infty & \infty & 1 & 8 \\ \infty & \infty & \infty & 5 \\ \infty & \infty & \infty & \infty \end{pmatrix}$$

17 Shortest Path in an Cyclic Graph (SPC)

$$f(x, S) = \begin{cases} \min\limits_{d \notin S}\{f(d, S \cup \{d\}) + c_{x,d}\} & \text{if } x < n-1 \\ 0 & \text{if } x = n-1. \end{cases} \tag{11}$$

goal $= f(0, \{0\})$.

$$f(x, i) = \begin{cases} \min\limits_{d}\{f(d, i-1) + c_{x,d}\} & \text{if } x < n-1 \text{ and } i > 0 \\ \infty & \text{if } x < n-1 \text{ and } i = 0 \\ 0 & \text{if } x = n-1. \end{cases} \tag{12}$$

$$\begin{pmatrix} \infty & 3 & 5 & \infty \\ \infty & \infty & 1 & 8 \\ \infty & 2 & \infty & 5 \\ \infty & \infty & \infty & \infty \end{pmatrix}$$

goal $= f(0, 3) = 9$.

18 Traveling Salesman Problem (TSP)

$$f(v, S) = \begin{cases} \min\limits_{d \notin S}\{f(d, S \cup \{d\}) + c_{v,d}\} & \text{if } |S| < n \\ c_{v,s} & \text{if } |S| = n \end{cases} \tag{13}$$

goal is $f(s, \{s\})$ where $s \in V$. s=0, V = {0,1,2,3,4}

$$f(v, S) = \begin{cases} \min\limits_{d \in S}\{f(d, S - \{d\}) + c_{v,d}\} & \text{if } |S| > 1 \\ c_{v,s} & \text{if } S = \emptyset \end{cases} \tag{14}$$

goal is $f(s, V - \{s\})$ where $s \in V$. s=0, V = {0,1,2,3,4}

$$C = \begin{pmatrix} 0 & 1 & 8 & 9 & 60 \\ 2 & 0 & 12 & 3 & 50 \\ 7 & 11 & 0 & 6 & 14 \\ 10 & 4 & 5 & 0 & 15 \\ 61 & 51 & 13 & 16 & 0 \end{pmatrix}$$