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The Naive Bayes classifier requires the computation of the conditional probability:

$$P(\text{target}|\text{attributes}).$$

We can compute the above probability by applying Baye's Theorem to the easier to compute conditional probability:

$$P(\text{attributes}|\text{target}).$$

First we review some probability theory.

Example 1 (Review)

Pick a number at random between 1 and 10.

What is the probability the number is:

(a) prime? $P(\text{prime}) = ?$

(b) odd? $P(\text{odd}) = ?$

(c) prime and odd? $P(\text{prime} \cap \text{odd}) = ?$

(d) prime or odd? $P(\text{prime} \cup \text{odd}) = ?$

(e) prime given it is odd? $P(\text{prime}|\text{odd}) = ?$

(f) even given that it is odd? $P(\text{even}|\text{odd}) = ?$

Definition 2 (Conditional Probability)

The probability of event B given that event A has occurred is called the **conditional probability** of B **given** A and is written $P(B|A)$. By definition:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Definition 3 (Law of Multiplication)

The probability of event A and B simultaneously occurring is given by the **law of multiplication**:

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B).$$

Example 4 (Law of Multiplication)

Derive the law of multiplication from the definition of conditional probability.

Definition 5 (Independent Events)

Two events, A and B are independent events if

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B).$$

Example 6 (Independent Events)

Show that if A and B are independent events, then

$$P(A \cap B) = P(A)P(B).$$

Example 7 (Selecting Without Replacement)

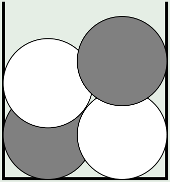
Two white and two black balls are in a box. Two balls are selected at random <u>without</u> replacement.

(a) Let B_1 be the event that the first ball selected is black. Let B_2 be the event that the second ball selected is black. Are B_1 and B_2 independent events?

(b) Determine the probability that the second ball is black given that the first ball selected was black.

(c) Determine the probability that the second ball is black.

(d) Determine the probability of selecting two black balls.



We have used the *law of total probability* in part (c) of the previous example. Before stating this law, we first define what a *partition* of the sample space is.

Definition 8 (Partition)

A **partition** of the sample space S is a sequence of *nonempty*, *mutually exclusive* sets, $B_1, B_2, \cdots B_n$ where

$$B_1 \cup B_2 \cup \cdots \cup B_n = S.$$

Example 9 (Partition)

One of the simplest partitions of a sample space S is

$$B,\ B^c$$

where B is any nonempty subset of S. (Check definition).

Definition 10 (Law of Total Probability)

Let B_i, $i = 1, \ldots n$, be a *partition* of S. Then, the **law of total probability** states that

$$
\begin{aligned}
P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n) \\
&= \sum_{i=1}^{n} P(A|B_i)P(B_i).
\end{aligned}
$$

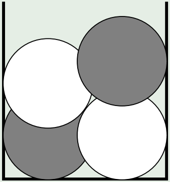
Example 11 (Law of Total Probability)

Assume $B,\ B^c$ is a partition of a sample space S. Show that

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

Example 12 (Selecting Balls)

A box contains two white and two black balls. A ball is selected at random and thrown away <u>without</u> looking at it. What is the probability the next ball selected is black?



Definition 13 (Baye's Theorem)

Let $B_i,\ i = 1, \ldots n$, be a *partition* of S. Assume $P(A) > 0$ and $P(B_i) > 0$ for $i = 1, 2, \ldots n$. Then **Baye's theorem** states that

$$
\begin{aligned}
P(B_k|A) &= \frac{P(A|B_k)P(B_k)}{P(A)} \\
&= \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1) + \cdots + P(A|B_n)P(B_n)}
\end{aligned}
$$

Example 14 (Fake Two-Headed Quarter)

Victor and Bonita have just won a trophy. They agree to flip a coin to determine who takes the trophy home. Victor has four quarters in his pocket, one of which is a fake two-headed quarter. Victor takes a quarter at random out of his pocket and flips it.

(a) Before the quarter lands, Bonita yells, *Its your fake two-headed quarter!* What is the probability that Bonita is correct?

(b) The quarter lands heads. Now what is the probability that Bonita is correct?

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Definition 16 (Prior vs Posterior Probabilities)

Assume A and B are events.

- $P(B)$ is called the **prior** probability of the event B.
- $P(B|A)$ is called the **posterior** (updated) probability of the event B given event A.

In the fake two-headed quarter example, $P(F)$ is called the prior probability and $P(F|H)$ is called the posterior probability.