# Physics GRE Notes

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## Contents



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# 1 Classical Mechanics - 20%

## 1.1 Kinematics

### 1.1.1 Linear Motion

The basic kinematic equations of motion under constaant acceleration $a$

$$v = v_0 + at \tag{1.1}$$

$$v^2 = v_0^2 + 2a(x - x_0) \tag{1.2}$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2 \tag{1.3}$$

### 1.1.2 Circular Motion

Centripetal acceleration

$$a = \frac{v^2}{r} = \omega^2 r \tag{1.4}$$

Angular velocity

$$\omega = \frac{v}{r} \tag{1.5}$$

## 1.2 Newton's Laws

### 1.2.1 Newton's Law of Motion

1. An object at rest stays at rest unless acted on by an outside force.
2. $\mathbf{F} = m\mathbf{a}$
3. Every action has an equal and opposite reaction $\mathbf{F}_{\mathrm{AB}} = -\mathbf{F}_{\mathrm{BA}}$

### 1.2.2 Momentum

$$\mathbf{p} = m\mathbf{v} \tag{1.6}$$

**Example 1: Collisions**

- **Elastic Collision**

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2' \tag{1.7}$$

$$\frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2 = \frac{1}{2}m_1(\mathbf{v}_1')^2 + \frac{1}{2}m_2(\mathbf{v}_2')^2 \tag{1.8}$$

- **Inelastic Collision**: Collision in which the kinetic energy of the system is not conserved.
- **Completely Inelastic Collision**: Collision in which both the particales stick together after the collision.

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v}' \tag{1.9}$$

**Example 2: Rocket Motion**

$$(m + \mathrm{d}m)v = (v + \mathrm{d}v)m + V\mathrm{d}m \tag{1.10}$$

$$m\mathrm{d}v + (V - v)\mathrm{d}m = 0 \tag{1.11}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = 0 \tag{1.12}$$

$u$ represents the speed of the rocket's exhaust relative to the rocket.

### 1.2.3 Impulse

$$\Delta\mathbf{p} = \mathbf{J} = \int_{t_1}^{t_2} \mathbf{F}\mathrm{d}t = \mathbf{F}_{\mathrm{avg}}(t_2 - t_1) \tag{1.13}$$

## 1.3 Work & Energy

### 1.3.1 Kinetic Energy

$$K = \frac{1}{2}mv^2 \tag{1.14}$$

### 1.3.2 The Work-Energy Theorem

The net work done is given by

$$W_{\mathrm{net}} = K_f - K_i \tag{1.15}$$

### 1.3.3 Work

$$W = \int_{x_i}^{x_f} F(x)\mathrm{d}x \tag{1.16}$$

### 1.3.4 Potential Energy

$$F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x} \tag{1.17}$$

For conservative force, the potential energy is

$$V(x) = V(x_0) - \int_{x_0}^{x} F(x')\mathrm{d}x' \tag{1.18}$$

**Example: Spring**

Potential energy of a spring

$$V(x) = -\int_0^x (-kx')\mathrm{d}x' = \frac{1}{2}kx^2 \tag{1.19}$$

### 1.3.5 Hooke's Law

$$F = -kx \tag{1.20}$$

where $k$ is a constant.

- Parallel

$$k_{\mathrm{tot}} = k_1 + k_2 \tag{1.21}$$

- Series

$$\frac{1}{k_{\mathrm{tot}}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{1.22}$$

## 1.4 Oscillatory Motion

### 1.4.1 Simple Harmonic Motion

$$\ddot{x} + \omega^2 x = 0 \tag{1.23}$$

And the equation of simple harmonic motion is

$$x(t) = A\sin(\omega t + \delta) \tag{1.24}$$

- **Spring**

$$m\ddot{x} + kx = 0 \tag{1.25}$$

$$\omega = \sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \tag{1.26}$$

- **Pendulum**

$$ml\ddot{\theta} + mg\theta = 0 \tag{1.27}$$

$$\omega = \sqrt{\frac{g}{l}} \qquad T = 2\pi\sqrt{\frac{l}{g}} \tag{1.28}$$

- **Compound Pendulum**

$$I\ddot{\theta} + mgl\theta = 0 \tag{1.29}$$

$$\omega = \sqrt{\frac{mgl}{I}} \qquad T = 2\pi\sqrt{\frac{I}{mgl}} \tag{1.30}$$

### 1.4.2 Total Energy of an Oscillating System

$$E = K.E. + P.E. = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \tag{1.31}$$

### 1.4.3 Damped Harmonic Motion

$$\mathbf{F}_d = -b\mathbf{v} \tag{1.32}$$

where $b$ os the damping coefficient. The equation of motion for a damped oscillating system becomes

$$\ddot{x}^2 + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \tag{1.33}$$

Solving this we get

$$x = Ae^{-\beta t}\sin(\omega' t + \delta) \tag{1.34}$$

where

$$\beta = \frac{b}{2m} \tag{1.35}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \beta^2} \tag{1.36}$$

### 1.4.4 Driven Oscillation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t \tag{1.37}$$

The amplitude $D$ of an undamped oscillator of natural frequency $\omega_D$ subject to a driving force at frequency is proportional as follows:

$$D \propto \frac{1}{|\omega_0^2 - \omega^2|} \tag{1.38}$$

- **Free oscillation** $\omega^2 = k/m$
- **Damped oscillation**
  - overdamping
  - critical damping
  - underdamping ${\omega'}^2 = \omega_0^2 - \beta^2$
- **Driven oscillation** $\omega_R^2 = \omega_0^2 - 2\beta^2$

### 1.4.5 Small Oscillations

$$V(x) = V(x_\mathrm{e}) + \frac{1}{2}k(x - x_\mathrm{e})^2 \tag{1.39}$$

where

$$k = \left[\frac{\mathrm{d}^2V(x)}{\mathrm{d}x^2}\right]_{x=x_\mathrm{e}} \geq 0 \tag{1.40}$$

## 1.5 Rotational Motion about a Fixed Axis

### 1.5.1 Moment of Inertia

$$I = \int r^2 \mathrm{d}m \tag{1.41}$$

parallel axis theorem

$$I = I_{\mathrm{cm}} + md^2 \tag{1.42}$$

Moments of inertia about center of mass

- **Rod** $\frac{1}{12}Ml^2$
- **Disc** $\frac{1}{2}MR^2$
- **Sphere** $\frac{2}{5}MR^2$

### 1.5.2 Kinetic Energy in Rolling

$$K_{\mathrm{rot}} = \frac{1}{2}I_{\mathrm{contact}}\omega^2 = \frac{1}{2}\left(I_{\mathrm{cm}} + md^2\right)\omega^2 = \frac{1}{2}I_{\mathrm{cm}}\omega^2 + \frac{1}{2}mv^2 \tag{1.43}$$

### 1.5.3 Angular Momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = I\omega \tag{1.44}$$

### 1.5.4 Torque

$$\tau = \mathbf{r} \times \mathbf{F} = I\alpha = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \tag{1.45}$$

**Example: Roll Down without Slipping**

Equations of motion

$$mg\sin\theta - f = ma \tag{1.46}$$

$$fR = I\alpha \tag{1.47}$$

$$a = R\alpha \tag{1.48}$$

Conservation of energy

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{1.49}$$

$$v = \omega R \tag{1.50}$$

### 1.5.5 Matrix Transformations

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.51}$$

## 1.6 Dynamics of Systems of Particles

Position vector of a system of particles

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_N\mathbf{r}_N}{M} \tag{1.52}$$

Velocity vector of a system of particle

$$\mathbf{V} = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_N\mathbf{v}_N}{M} \tag{1.53}$$

Acceleration vector of a system of particle

$$\mathbf{A} = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \cdots + m_N\mathbf{a}_N}{M} \tag{1.54}$$

## 1.7 Central Forces and Celestial Mechanics

### 1.7.1 Newton's Law of Universal Gravitation

$$\mathbf{F} = -\left(\frac{GMm}{r^2}\right)\hat{r} \tag{1.55}$$

### 1.7.2 Potential Energy of a Gravitational Force

$$V(r) = -\frac{GMm}{r} \tag{1.56}$$

### 1.7.3 Escape Speed and Orbits

The energy of an orbiting body is

$$E = T + V = \frac{1}{2}mv^2 - \frac{GMm}{r} \tag{1.57}$$

The escape speed becomes

$$E = T + V = \frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{R_{\text{e}}} = 0 \tag{1.58}$$

Solving for $v_{\text{esc}}$ we find

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R_{\text{e}}}} \tag{1.59}$$

### 1.7.4 Kepler's Laws

1. The orbit of every planet is an ellipse with the sun at a focus.
2. A line joining a planet and the sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital period is directly proportional to the cube of the semi-major axis of its orbit.

$$T^2 \propto R^3 \tag{1.60}$$

### 1.7.5 Types of Orbits

- **Circular Orbit** $E = V_{\text{min}}$

$$m\frac{v^2}{r} = \frac{GMm}{r^2} \tag{1.61}$$

  The orbital velocity is

$$v = \sqrt{\frac{GM}{r}} \tag{1.62}$$

- **Elliptic Orbit** $V_{\text{min}} < E < 0$
- **Parabolic Orbit** $E = 0$

$$v = v_{\text{esc}} = \sqrt{\frac{2GM}{r}} \tag{1.63}$$

- **Hyperbolic Orbit** $E > 0$

## 1.8 Fluid Dynamics

### 1.8.1 Buoyant Force

When an object is fully or partially immersed, the buoyant force is equal to the weight of fluid displaced.

### 1.8.2 Equation of Continuity

$$\rho v A = \text{constant} \tag{1.64}$$

### 1.8.3 Bernoulli's Equation

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant} \tag{1.65}$$

## 1.9 Non-inertial Reference Frames

Coriolis' force

$$\mathbf{F}_\text{C} = -2m\omega \times \mathbf{v} \tag{1.66}$$

Objects deflect to the right in the Northern Hemisphere and deflect to the left in the Southern Hemisphere, because of the earth's rotation.

## 1.10 Lagrangian Mechanics

### 1.10.1 Lagrange's Function

$$\mathcal{L}(q, \dot{q}, t) = T - V \tag{1.67}$$

The Lagranagian of a particle described by the polar coordinates $r$ and $\theta$ is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V \tag{1.68}$$

The action $S$ is defined as

$$\mathcal{S} = \int \mathcal{L}\mathrm{d}t \tag{1.69}$$

### 1.10.2 Euler-Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1.70}$$

The conjugate momentum $p$ is defined as

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{1.71}$$

### 1.10.3 Hamiltonian

The Hamiltonian is defined as

$$H(q, p, t) = \sum_i p_i \dot{q}_i - \mathcal{L} \tag{1.72}$$

The Hamiltonian equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{1.73}$$

## 1.11 Conservation Theorem

- **Conservation of Linear Momentum**: The component of linear momentum in a direction in which the forces vanishes is constant in time.

  $$\dot{\mathbf{p}} = \mathbf{F} = 0 \tag{1.74}$$

  If the generalized coordinate $q_i$ is cyclic, then the corresponding generalized momentum component $p_i$ to be a constant of motion.

  $$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant} \tag{1.75}$$

- **Conservation of Angular Momentum**: The angular momentum of a particle of a particle subject to no torque is constant.

  $$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1.76}$$

  $$\dot{\mathbf{L}} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = (\dot{\mathbf{r}} \times \mathbf{p}) + (\mathbf{r} \times \dot{\mathbf{p}}) = \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \mathbf{F} = \tau = 0 \tag{1.77}$$

- **Conservation of Energy**: The total energy of a particle in a conservative field is a constant in time.

$$\mathbf{F} = -\nabla U \tag{1.78}$$

  The Lagrangian must be independent of time and the potential energy must independent of velocities.

$$H = \sum_i p_i \dot{q}_i - \mathcal{L} = 2T - \mathcal{L} = T + V = E = \text{constant} \tag{1.79}$$

# 2 Eletromagnetism - 18%

## 2.1 Electrostatics

### 2.1.1 Cooulomb's Law

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{\mathbf{r}} \tag{2.1}$$

where $\varepsilon_0$ is the permitivity of free space, where

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2N^{-1}m^{-2}} \tag{2.2}$$

### 2.1.2 The Electric Field

The electric field of a point charge

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \tag{2.3}$$

The electric field of a continuous distribute charge

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\mathbf{r}}}{r^2}\mathrm{d}q \tag{2.4}$$

### 2.1.3 Gauss' Law

The electric field through aa surface is

$$\Phi = \oint \mathrm{d}\Phi = \oint \mathbf{E}\cdot \mathrm{d}\mathbf{A} \tag{2.5}$$

The electric flux through a closed surface encloses a net charge

$$\oint \mathbf{E}\cdot \mathrm{d}\mathbf{A} = \frac{Q_{\mathrm{enc}}}{\varepsilon_0} \tag{2.6}$$

- **Electric Field due to a line of charge**

$$\Phi = E\cdot 2\pi r h = \frac{\lambda h}{\varepsilon_0} \tag{2.7}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{2.8}$$

- **Electric Field in a solid non-conducting sphere**

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \tag{2.9}$$

$$\Phi = E\cdot 4\pi r^2 = \frac{\rho\cdot\frac{4}{3}\pi r^3}{\varepsilon_0} = \frac{Q}{\varepsilon_0}\frac{r^3}{R^3} \tag{2.10}$$

$$E = \frac{Qr}{4\pi\varepsilon_0 R^3} \tag{2.11}$$

### 2.1.4 Electric Potential

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_a^b \mathbf{E}\cdot\mathrm{d}\mathbf{l} \tag{2.12}$$

$$\mathbf{E} = -\nabla V \tag{2.13}$$

## 2.2 Currents and DC Circuits

### 2.2.1 Current

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{2.14}$$

### 2.2.2 Current Density

$$I = \iint \mathbf{J} \cdot \mathrm{d}\mathbf{A} \tag{2.15}$$

Current density of moving charges is

$$\mathbf{J} = n_e q \mathbf{v} \tag{2.16}$$

### 2.2.3 Resistance and Ohm's Law

$$R = \frac{V}{I} \tag{2.17}$$

### 2.2.4 Resistivity and Conductivity

$$R = \rho \frac{L}{A} \tag{2.18}$$

where $\rho$ is called resistivity. The Ohm's law can be written as

$$\mathbf{J} = \sigma \mathbf{E} \tag{2.19}$$

where $\sigma$ is conductivity which can be expressed as

$$\sigma = \frac{1}{\rho} \tag{2.20}$$

### 2.2.5 Power

$$P = VI \tag{2.21}$$

## 2.3 Magnetic Fields in Free Space

### 2.3.1 The Biot-Savart Law

A steady current refers to a continuous flow that has been going on forever, without change and without charge piling up anywhere, which means

$$\nabla \cdot \mathbf{J} = 0 \tag{2.22}$$

The magnetic field of a steady line current is given by the Biot-Savart law:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \tag{2.23}$$

### 2.3.2 Ampere's Law

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0 \int \mathbf{J} \cdot \mathrm{d}\mathbf{A} \tag{2.24}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2.25}$$

where $\mu_0$ is called the permeability of free space.

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{NA}^{-2} \tag{2.26}$$

## 2.4 Lorentz Force

The magnetic force in a charge $Q$, moving with velocity $v$ in a magnetic field $\mathbf{B}$, is

$$\mathbf{F}_{\text{mag}} = q(\mathbf{v} \times \mathbf{B}) \tag{2.27}$$

This is known as Lorentz force law. In the presence of both electric and magnetic fields, the net force on $Q$ would be

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.28}$$

## 2.5 Maxwell's Equations and their Applications

### 2.5.1 Faraday's Law

$$\mathcal{E} = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \mathbf{B} \cdot \mathrm{d}\mathbf{A} \tag{2.29}$$

### 2.5.2 Maxwell's Equations

**Integral Form**

- **Gauss' Law for Electric Fields**

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q}{\varepsilon_0} \tag{2.30}$$

- **Gauss' Law for Magnetic Fields**

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0 \tag{2.31}$$

- **Ampere's Law**

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \oint \mathbf{E} \cdot \mathrm{d}\mathbf{A} \tag{2.32}$$

- **Faraday's Law**

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \oint \mathbf{B} \cdot \mathrm{d}\mathbf{A} \tag{2.33}$$

**Differential Form**

- **Gauss' Law for Electric Fields**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{2.34}$$

- **Gauss' Law for Magnetic Fields**

$$\nabla \cdot \mathbf{B} = 0 \tag{2.35}$$

- **Ampere's Law**

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.36}$$

- **Faraday's Law**

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} \tag{2.37}$$

## 2.6 Electromagnetic Waves

### 2.6.1 Speed of Propagation of a EM Wave

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \tag{2.38}$$

### 2.6.2 Relationship between E and B Fields

$$E = cB \tag{2.39}$$

$$\mathbf{E} \cdot \mathbf{B} = 0 \tag{2.40}$$

### 2.6.3 Energy Density of an EM Wave

$$w = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} + \frac{1}{2}\mathbf{B} \cdot \mathbf{H} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \tag{2.41}$$

### 2.6.4 Poynting's Vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{2.42}$$

## 2.7 Magnetic and Electric Fields in Matter

### 2.7.1 Polarization and Magnetization

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = K\varepsilon_0 \mathbf{E} \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} = \frac{\mathbf{B}}{\mu} \tag{2.43}$$

Displacement current

$$\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t} \tag{2.44}$$

### 2.7.2 Boundary Conditions

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f \qquad \hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \tag{2.45}$$

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \qquad \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \alpha_f \tag{2.46}$$

$$\hat{\mathbf{n}} \cdot (\mathbf{P}_2 - \mathbf{P}_1) = -\sigma_p \qquad \hat{\mathbf{n}} \times (\mathbf{M}_2 - \mathbf{M}_1) = \alpha_M \tag{2.47}$$

## 2.8 AC Circuits

Impedance:

- **Resistors**: $X_R = R$
- **Capacitors**: $X_C = 1/\omega C$
- **Inductors**: $X_L = \omega L$

### 2.8.1 RC Circuits

$$\mathcal{E} = IR + \frac{Q}{C} \tag{2.48}$$

The voltage of a capacitor follows an exponential decay

$$V(t) = V_0 \exp\left(-\frac{t}{RC}\right) \tag{2.49}$$

### 2.8.2 RL Circuits

$$\mathcal{E} = IR + L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{2.50}$$

The differential equation can be solved such that

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\tau_L}}\right) \tag{2.51}$$

where $\tau_L = L/R$, which means the time to fall to $1/e$ of its original value.

### 2.8.3 RLC Circuits

$$Z = R + j(X_L - X_C) \tag{2.52}$$

The current will be maximized when the inductive and capacitive reactances are equal in magnitude but cancel each other out due to being 180° out of phase.

$$\omega L = \frac{1}{\omega C} \qquad \Rightarrow \qquad \omega = \frac{1}{\sqrt{LC}} \tag{2.53}$$

## 2.9 Electronic Elements

### 2.9.1 Resistors

The voltage $V$ across a resister with resistance $R$ and current $I$ is given by

$$V = IR \tag{2.54}$$

Given an tube of length $L$, across sectional area $A$, and resistivity $\rho$ the resistance is given by

$$R = \rho\frac{L}{A} \tag{2.55}$$

### 2.9.2 Capacitors

The capacitance $C$ of a capacitor with charge $Q$ and potential $V$ is given by

$$C = \frac{Q}{V} \tag{2.56}$$

The energy stored by a capacitor is given by

$$\int q\mathrm{d}V = \int q\frac{\mathrm{d}q}{C} = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \tag{2.57}$$

Consider a **parallel plate capacitor**

$$V = \frac{\sigma d}{\varepsilon} \tag{2.58}$$

Thus

$$C = \frac{\varepsilon A}{d} \tag{2.59}$$

The capacitance of a combination of $n$ capacitors is given by

- Capacitors in series

$$\frac{1}{C_{eq}} = \sum_{i=1}^{n}\frac{1}{C_i} \tag{2.60}$$

- Capacitors in parallel

$$C_{eq} = \sum_{i=1}^{n} C_i \tag{2.61}$$

### 2.9.3 Inductors

The inductance

$$L = \frac{\Phi_B}{I} = \frac{N\phi_B}{I} \tag{2.62}$$

$$\mathcal{E} = -L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{2.63}$$

The energy stored by a inductor is given by

$$\int(-\mathcal{E}I)\mathrm{d}t = -L\int I\mathrm{d}I = \frac{1}{2}LI^2 \tag{2.64}$$

# 3 Optics & Wave Phonomena - 9%

## 3.1 Wave Properties

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} \tag{3.1}$$

$$v = \frac{1}{\sqrt{\varepsilon\mu}} \tag{3.2}$$

Phase and group velocity are given by

$$v_p = \frac{\omega}{k} \qquad\qquad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} \tag{3.3}$$

## 3.2 Superposition

$$\psi(x,t) = \sum_i \psi_i(x,t) \tag{3.4}$$

$$I(x,t) = |\psi(x,t)|^2 = \left|\sum_i \psi_i(x,t)\right|^2 \tag{3.5}$$

## 3.3 The Propagation of Wave

### 3.3.1 Rayleigh Scattering

When scatter invloving particles smaller than a wavelength, the intensity of the scattering light was proportional to $1/\lambda^4$.

### 3.3.2 Reflection

$$\theta_i = \theta_r \tag{3.6}$$

### 3.3.3 Refraction

**Snell's law**

$$n_i \sin\theta_i = n_t \sin\theta_t \tag{3.7}$$

$$\sin\theta_c = \frac{n_1}{n_2} \tag{3.8}$$

where $\theta_c$ is the critical angle.

## 3.4 Interference

### 3.4.1 Young's Double Slit Experiment

The **maximum** positions in Young's double slit experiment are given by

$$d\sin\theta = m\lambda \tag{3.9}$$

While the **minima** are given by

$$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda \tag{3.10}$$

Alternatively using small angles it is easy enough to show that the angular positions of the peaks are

$$\theta_m = \frac{m\lambda}{d} \tag{3.11}$$

### 3.4.2 Rayleigh Criterion

The resolving power of a telescope is

$$\theta_R = 1.22\frac{\lambda}{D} \tag{3.12}$$

where $D$ is the diameter of the aperture.

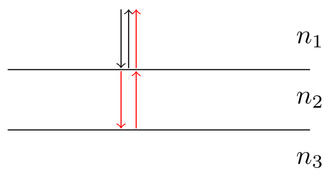
### 3.4.3 Michelson's Interferometer

$$\Delta N = \frac{2L}{\lambda} \tag{3.13}$$

### 3.4.4 Newton's Rings

$$r = \sqrt{R\left(m + \frac{1}{2}\right)\lambda} \tag{3.14}$$

### 3.4.5 Thin films




- If $n_1 < n_2 < n_3$ or $n_1 > n_2 > n_3$

$$2n_2 d = m\lambda \tag{3.15}$$

- If $n_2 < n_1, n_3$ or $n_2 > n_1, n_3$

$$2n_2 d = \left(m + \frac{1}{2}\right)\lambda \tag{3.16}$$

## 3.5 Diffraction

### 3.5.1 Single Slit Diffraction

The angle at which **maxima** occur is given by

$$a \sin\theta = \left(m + \frac{1}{2}\right)\lambda \tag{3.17}$$

The angle in which **minima** occur is given by

$$a \sin\theta = m\lambda \tag{3.18}$$

### 3.5.2 Diffraction Grating

$$d \sin\theta = m\lambda \tag{3.19}$$

where $d$ is the spacing between the slits. If instead the question gives the number of slits ($N$) per unit length ($L$) then

$$\frac{L}{N}\sin\theta = m\lambda \tag{3.20}$$

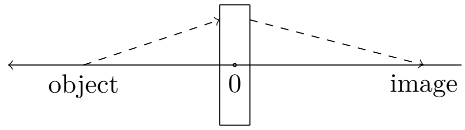
### 3.5.3 Bragg Diffraction

$$2d \sin\theta = m\lambda \tag{3.21}$$

## 3.6 Geometrical Optics

### 3.6.1 Images

Choose the convention that object distances are positive quantities and image distances are negative quantities.




- **Spherical mirror**

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f} \tag{3.22}$$

- **Spherically refracting surface**

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{r} \tag{3.23}$$

  where $r$ is the radius of curvature of the surface. The focal length

$$f = \frac{1}{2}r \tag{3.24}$$

- **Thin lens**

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{i} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{3.25}$$

  where $r_1$ is the radius of curvature of the lens surface closest to the light source and $r_2$ is the radius of curvature of the lens surface fatherest from the light source.
  ***Sign convention:*** *the radius curvature is positive if the center of spherical surface lies to the right of the lens.*

## 3.7 Polarization

Unpolarized light that went through a linear polarizer has its intensity reduced by a factor of two

$$I = \frac{1}{2}I_0 \tag{3.26}$$

Linearly polarized light entering a second linear polarizer has an intensity distribution given by Malus's law

$$I = I_0 \cos^2 \theta \tag{3.27}$$

where $\theta$ is the angle between the initial polarication axis and the polarization axis of the second polarizer.

## 3.8 Non-Relativistic Doppler Effect

The non-relativistic doppler shift is dependent on whether the source (s) or the detector (D) is moving, the equation is

$$f_D = \left(\frac{v \pm v_D}{v \mp v_s}\right) f_s \tag{3.28}$$

- $v$ is the propagation speed of waves in the medium;
- $v_D$ is the speed of the detictor relative to the medium, added to $v$ if the detector is moving towards the source, subtracted if the detector is moving away from the source;
- $v_s$ is the speed of the source relative to the medium, added to $v$ if the source is moving away from the detector, subtracted if the source is moving towards the detector.

# 4 Thermodynamics & Statistical Mechanics - 10%

## 4.1 Laws of Thermodynamics

- **The zeroth law**: Two systems, each separately in thermal equilibrium with a third, are in equilibrium with each other.
- **The first law**: $\mathrm{d}U = đQ + đW = S\mathrm{d}T - p\mathrm{d}V$, where $\mathrm{d}U$ is the internal energy of the system, $đQ$ is the energy added to the system and $đW$ is the work done on the system.
- **The second law**: $\mathrm{d}S \geq 0$
- **The third law**: $\lim\limits_{T\to 0} S(T) = 0$

## 4.2 Thermodynamic Processes

- **Isothermal Process**: Process which occurs at a constant *temperature*, $\Delta T = 0$.

$$đQ = -đW = p\mathrm{d}V \tag{4.1}$$

- **Isobaric Process**: Process which occurs at a constant *pressure*.
- **Adiabatic Process**: Process which occurs with no energy transferred as heat, $\Delta Q = 0$.

$$\mathrm{d}U = đW \qquad \Rightarrow \qquad C_V \mathrm{d}T = -p\mathrm{d}V \tag{4.2}$$

  For an ideal gas we have $C_p = C_V + nR$

$$\gamma = \frac{C_p}{C_V} = 1 + \frac{nR}{C_V} \tag{4.3}$$

  hence

$$TV^{\gamma-1} = \text{constant} \tag{4.4}$$

$$p^{1-\gamma}T^{\gamma} = \text{constant} \tag{4.5}$$

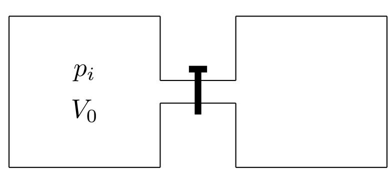
$$pV^{\gamma} = \text{constant} \tag{4.6}$$

- **Isochoric Process**: Process which occurs at a constant *volume*.

### 4.2.1 Carnot Engine

$$\eta = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{T_c}{T_h} \tag{4.7}$$

### 4.2.2 Joule Expansion




Since the system is isolated from its surroundings, $\Delta U = 0$. Hence $\Delta T = 0$

$$\Delta S = \int_i^f \frac{đQ}{T} = \int_i^f \frac{p\mathrm{d}V}{T} = nR\int_{V_i}^{V_f} \frac{\mathrm{d}V}{V} = nR\ln 2 \tag{4.8}$$

## 4.3 Equation of State

### 4.3.1 Ideal Gases

$$pV = nRT \tag{4.9}$$

### 4.3.2 van der Waals Gas

$$\left(p + \frac{a}{V_m^2}\right)(V_m - b) = RT \tag{4.10}$$

## 4.4 Kinetic Theory

### 4.4.1 The Maxwell-Boltzmann Distribution

- **The velocity distribution**

$$\langle v_x \rangle = \int_{-\infty}^{\infty} v_x g(v_x)\mathrm{d}v_x = 0 \tag{4.11}$$

$$\langle |v_x| \rangle = 2\int_{0}^{\infty} v_x g(v_x)\mathrm{d}v_x = \sqrt{\frac{2k_BT}{\pi m}} \tag{4.12}$$

$$\langle v_x^2 \rangle = \int_{-\infty}^{\infty} v_x^2 g(v_x)\mathrm{d}v_x = \frac{k_BT}{m} \tag{4.13}$$

- **The speed distribution**

$$\langle v \rangle = \int_{0}^{\infty} v f(v)\mathrm{d}v = \sqrt{\frac{8k_BT}{\pi m}} \tag{4.14}$$

$$v_{\max} = \sqrt{\frac{2k_BT}{m}} \tag{4.15}$$

$$v_{\mathrm{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_BT}{m}} \tag{4.16}$$

### 4.4.2 The Mean Free Path

$$\lambda = \frac{1}{\sqrt{2}n\sigma} = \frac{1}{\sqrt{2}\pi d^2 n} \tag{4.17}$$

where $\sigma$ is the cross section, $d$ is the diameter of the molecule and $n$ is the number density of the gas.

## 4.5 Ensembles

- **The microcanonical ensemble** $(N, V, E)$: an ensemble of systems that each have the same fixed energy.
- **The canonical ensemble** $(N, V, T)$: an ensemble of systems, each of which can exchange its energy with a large reservoir of heat. This fixes the temperature $T$ of the system.
- **The grand canonical ensemble** $(\mu, V, T)$: an ensemble of systems, each of which can exchange both energy and particles with a large reservoir. This fixes the temperature $T$ and the chemical potential $\mu$ of the system.

## 4.6 Statistical Concepts and Calculation of Thermodynamic Properties

### 4.6.1 Probability Distribution

- **Maxwell-Boltzmann distribution**

$$\langle n_r \rangle = \frac{1}{e^{\beta(\varepsilon_r - \mu)}} \tag{4.18}$$

- **Bose-Einstein distribution**

$$\mathcal{Z}_r = \sum_{n_r} e^{-\beta n_r(\varepsilon_r - \mu)} = \frac{1}{1 - e^{-\beta(\varepsilon_r - \mu)}} \tag{4.19}$$

$$\ln \mathcal{Z} = \ln \prod_r \mathcal{Z}_r = \sum_r \ln \mathcal{Z}_r = -\sum_r \ln\left[1 - e^{-\beta(\varepsilon_r - \mu)}\right] \tag{4.20}$$

$$N = \frac{1}{\beta}\frac{\partial}{\partial \mu}\ln \mathcal{Z} = \sum_r \frac{1}{e^{\beta(\varepsilon_r - \mu)} - 1} = \sum_r \langle n_r \rangle \tag{4.21}$$

The average number of particles in the state $|r\rangle$ is

$$\langle n_r \rangle = \frac{1}{e^{\beta(\varepsilon_r - \mu)} - 1} \tag{4.22}$$

- **Fermi-Dirac distribution**
  *Pauli exlusion principle* shows that fermions cannot sit in the same state, each state can either be empty or simply occupied, so that $\{n_s\} = \{0, 1\}$

$$\mathcal{Z}_r = \sum_{n_r=0,1} e^{-\beta n_r(\varepsilon_r - \mu)} = 1 + e^{-\beta(\varepsilon_r - \mu)} \tag{4.23}$$

$$\ln \mathcal{Z} = \ln \prod_r \mathcal{Z}_r = \sum_r \ln \mathcal{Z}_r = \sum_r \ln \left[1 + e^{-\beta(\varepsilon_r - \mu)}\right] \tag{4.24}$$

$$N = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln \mathcal{Z} = \sum_r \frac{1}{e^{\beta(\varepsilon_r - \mu)} + 1} = \sum_r \langle n_r \rangle \tag{4.25}$$

The average number of particles in the state $|r\rangle$ is

$$\langle n_r \rangle = \frac{1}{e^{\beta(\varepsilon_r - \mu)} + 1} \tag{4.26}$$

### 4.6.2 Equipartition

If the energy of a classical system is the sum of $n$ quadratic modes, and that system is in contact with a heat reservoir at temperature $T$, the mean energy of the system is given by $n \times \frac{1}{2} k_B T$.

- **Monatomic** gas with only translation: $3/2 k_B T$ (3 translational degrees of freedom)
- **Diatomic** gas with translation and rotation: $5/2 k_B T$ (3 translational and 2 rotational degrees of freedom)
- **Diatomic** gas with transition, rotation and vibration: $7/2 k_B T$ (3 translational, 2 rotational and 2 vibration degrees of freedom)

### 4.6.3 The Partition Function

Steps to solving statistical mechanics problems:

1. Write down the partition function $Z$

$$Z = \sum_i e^{-\beta E_i} \tag{4.27}$$

2. Go through some standard procedures to obtain the functions of state you want from $Z$.

$$P_i = \frac{e^{-\beta E_i}}{Z} \tag{4.28}$$

- **Internal energy** $U$

$$U = \sum_i E_i P_i = k_B T^2 \frac{\mathrm{d} \ln Z}{\mathrm{d} T} \tag{4.29}$$

- **Entropy** $S$

$$S = -k_B \sum_i P_i \ln P_i = \frac{U}{T} + k_B \ln Z \tag{4.30}$$

- **Helmholtz function** $F = U - TS$

$$F = -k_B T \ln Z \tag{4.31}$$

$$Z = e^{-\beta F} \tag{4.32}$$

#### 4.6.4 Combining Partition Function

- If the $N$ particles are *distinguishable*

$$Z_N = (Z_1)^N \tag{4.33}$$

- If the $N$ particles are *indistinguishable*

$$Z_N = \frac{(Z_1)^N}{N!} \tag{4.34}$$

#### 4.6.5 Thermodynamic Potentials

- **Internal energy** $U$

$$\mathrm{d}U = S\mathrm{d}T - p\mathrm{d}V + \mu\mathrm{d}N \tag{4.35}$$

- **Helmholtz free energy** $F = U - TS$

$$\mathrm{d}F = -T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N \tag{4.36}$$

- **Enthalpy** $H = U + pV$

$$\mathrm{d}U = S\mathrm{d}T + V\mathrm{d}p + \mu\mathrm{d}N \tag{4.37}$$

- **Gibbs free energy** $G = U - TS + pV$

$$\mathrm{d}U = -T\mathrm{d}S + V\mathrm{d}p + \mu\mathrm{d}N \tag{4.38}$$

#### 4.6.6 The Heat Capacity

- **Classical gases**

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V \tag{4.39}$$

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_V + \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p \tag{4.40}$$

  For an ideal gas

$$C_V = \frac{3}{2}R \qquad C_p = \frac{5}{2}R \qquad \gamma = \frac{C_p}{C_v} = \frac{5}{3} \tag{4.41}$$

- **Phonons**
  - **The Einstein model** assumes that all vibrational modes of the solid have the same frequency $\omega_E$.

$$C = 3R\frac{x^2e^x}{(e^x-1)^2} \tag{4.42}$$

    where $x = \Theta_E/T$ and $\Theta_E = \hbar\omega_E/k_B$.
    * As $T \to 0$, $x \to \infty$ and $C \to 3Rx^2e^{-x}$.
    * As $T \to \infty$, $x \to 0$ and $C \to 3R$.
  - **The Debye model** assumes a distribution of frequencies with an upper frequency limit $\omega_D$.

$$C = \frac{9R}{x_D^3}\int_0^{x_D}\frac{x^4e^x\mathrm{d}x}{(e^x-1)^2} \tag{4.43}$$

    where $x = \hbar\beta\omega$ and $x_D = \hbar\beta\omega_D$.
    * As $T \to 0$, $x \to \infty$ and $C \to \frac{12\pi^4R}{5}\left(\frac{T}{\Theta_D}\right)^3$.
    * As $T \to \infty$, $x \to 0$ and $C \to 3R$.

### 4.6.7 Entropy

Entropy is defined as

$$S = k_B \ln \Omega = \frac{\partial}{\partial T}(k_B T \ln \mathcal{Z}) \tag{4.44}$$

$$\mathrm{d}S = \frac{\mathrm{d}Q_{\mathrm{rev}}}{T} \tag{4.45}$$

so that

$$\Delta S = S(B) - S(A) = \int_A^B \frac{\mathrm{d}Q_{\mathrm{rev}}}{T} \tag{4.46}$$

and $S$ is a function of state. For an adiabatic process (a reversible adiathermal process) we have that

$$\mathrm{d}Q_{\mathrm{rev}} = 0 \tag{4.47}$$

Hence an adiabatic process involves no change in entropy (the process is also called **isentropic**).

Temperature an be defined in terms of entropy

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V} \tag{4.48}$$

## 4.7 Phase Transition

- First-order transition
- Second-order transition

### 4.7.1 Latent Heat

$$L = \Delta Q_{\mathrm{rev}} = T_c(S_2 - S_1) \tag{4.49}$$

### 4.7.2 Phase Diagrams

- **The critical point**: point at which there is no distinction between liquid and vapour.
- **The triple point**: the one value of $T$ and $p$ at which all three phases coexist.
- **Criticle isotherm**: an isothermal line on a PV diagram which just touches a liquid-gas boundary.

# 5 Quantum Mechanics - 12%

## 5.1 Fundamental Concepts

### 5.1.1 History

1. **Plank energy quantization**: $E = h\nu$
2. **Einstein quantum light assumption**: $E = \hbar\omega$, $p = \hbar k$
3. **Bohr quantization**: $L = n\hbar$
4. **de Broglie matter wave**: $E = h\nu = \hbar\omega$, $p = h/\lambda$

### 5.1.2 Wave Function

1. $\psi(x)$ is always continuous.
2. $\mathrm{d}\psi(x)/\mathrm{d}x$ is continuous except at points where the potential is infinite.
3. $\int |\psi(x)|^2 \mathrm{d}x = 1$

### 5.1.3 Probability Current Density

$$\mathbf{J} = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) \tag{5.1}$$

## 5.2 Schrodinger Equation

$$H\psi = E\psi \tag{5.2}$$

The time dependent Schrodinger's equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \tag{5.3}$$

The time independent schrodinger's equation is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x) \tag{5.4}$$

where

$$\psi(x,t) = \psi(x)e^{-iE_n t/\hbar} \tag{5.5}$$

### 5.2.1 Infinite Square Wells

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < a \\ \infty & \text{for } |x| > a \end{cases} \tag{5.6}$$

In the regine $(0, a)$, we get

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = E\psi \tag{5.7}$$

This differential is of the form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0 \tag{5.8}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{5.9}$$

The wave function

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \tag{5.10}$$

The energy eigenvalues are

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{5.11}$$

### 5.2.2 Harmonic Oscillators

$$V(x) = \frac{1}{2}m\omega^2 x^2 \tag{5.12}$$

The energies of the harmonic oscillator are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{5.13}$$

The ground state of the harmonic oscillator is

$$\psi_0 = \left(\frac{\beta^2}{\pi}\right)^{1/4} e^{\beta^2 x^2/2} \tag{5.14}$$

where $\beta = \sqrt{m\omega/\hbar}$.

### 5.2.3 Finite Square Well

$$V(x) = \begin{cases} -V_0 & \text{for } 0 < x < a \\ 0 & \text{for } |x| > a \end{cases} \tag{5.15}$$

The wave function

$$\psi(x) = \begin{cases} Ae^{\kappa x} & \text{for } x < 0 \\ B\cos kx & \text{for } 0 < x < a \\ Ce^{-\kappa x} & \text{for } x > a \end{cases} \tag{5.16}$$

### 5.2.4 Hydrogenic Atoms

$$\psi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_{lm}(\theta, \phi) \tag{5.17}$$

$$E_n = -\frac{m_e}{2n^2}\left(\frac{Ze^2}{4\pi\varepsilon_0\hbar}\right)^2 \tag{5.18}$$

## 5.3 Spin

The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.19}$$

The spin operator is given by

$$S = \frac{\hbar}{2}\sigma \tag{5.20}$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y \tag{5.21}$$

## 5.4 Angular Momentum

$$L^2\psi = l(l+1)\hbar^2\psi \tag{5.22}$$

$$L_z\psi = m_l\hbar\psi \tag{5.23}$$

- $n = 1, 2, 3, \cdots$ is the principle quantum number and controls the radial wavefunction as well as the energy of an orbital.
- $l = 0, 1, \cdots, n-1$ is the orbital quantum number which controls the ratial wavelfunction and angular wavefuntion.
- $m = -l, \cdots, l$ is the magnetic quantum number and controls the angular wavefuntion.

## 5.5 Wave Function Symmetry

Consider a system of two spin half particles. The system can be in either the singlet state (antisymmetric state) *e.g. Fermion: electron, positron, proton, neutron, ...*

$$|00\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{5.24}$$

or in one of the triplet states (symmetric states) *e.g. Boson: deuteron, ...*)

$$|10\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{5.25}$$

$$|1-1\rangle = |\downarrow\downarrow\rangle \tag{5.26}$$

$$|11\rangle = |\uparrow\uparrow\rangle \tag{5.27}$$

## 5.6 Time Independent Non-Degenerate Perturbation Theory

$$H = H_0 + H' \tag{5.28}$$

The first order energy perturbation by $H'$ on state $\psi_n$ is given by

$$E_n^{(1)} = \left\langle \psi_n^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle \tag{5.29}$$

The pertubed wave function (first order) is given by

$$\psi_n^{(1)} = \sum_{k \neq n} \frac{\left\langle \psi_k^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}} \psi_k^{(0)} \tag{5.30}$$

The second order perturbation is given by

$$E_n^{(2)} = \sum_{k \neq n} \frac{\left| \left\langle \psi_k^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle \right|^2}{E_n^{(0)} - E_k^{(0)}} \tag{5.31}$$

# 6 Atomic Phycics - 10%

## 6.1 General Knowledge

The common notation for a particular isotope of an element is

$$^{A}E \tag{6.1}$$

where $E$ represents the element abbreviation and $A$ is the atomic mass in atomic mass units. Since the mass of the electrons is so small $A$ is typically given by

$$A = Z + N \tag{6.2}$$

where $Z$ is the number of protons and $N$ is the number of neutrons. The nucleus $X$ is written as

$$^{A}_{Z}X \tag{6.3}$$

## 6.2 Bohr Model

Borh made 2 assumptions:

1. The classical circular orbits are replaced by stationary states. These stationary states take discreet values.
2. The energy of these stationary states are determined by their angular momentum which must take on quantized values of $\hbar$.

$$L = n\hbar \tag{6.4}$$

We can find the angular momentum of a circular orbit

$$L = m_e v_n r_n = n\hbar \tag{6.5}$$

The cetripetal force is equal to the Coulomb force

$$\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n^2} = \frac{m_e v_n^2}{r_n} \tag{6.6}$$

Thus

$$r_n = a_0 n^2 \tag{6.7}$$

where $a_0$ is the Bohr radius

$$a_0 = 0.53 \times 10^{-10} \text{ m} \tag{6.8}$$

And the energy

$$E_n = \frac{1}{2}m_e v_n^2 - \frac{Ze^2}{4\pi\varepsilon_0 r_n} = -\frac{Ze^2}{8\pi\varepsilon_0 r_n} = -\frac{m_e}{2n^2}\left(\frac{Ze^2}{4\pi\varepsilon_0\hbar}\right)^2 = -13.6\frac{Z^2}{n^2} \text{ eV} \tag{6.9}$$

## 6.3 Atomic Spectra

### 6.3.1 Rydberg's Equation

$$\frac{1}{\lambda} = R_H\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \tag{6.10}$$

where $R_H$ is the Rydberg contant.

- Visible region: Balmer series $m = 2$
- Infrared region: Paschen series $m = 3$
- Ultraviolet region: Lyman series $m = 1$

## 6.4 Selection Rules

The selection rules for transition between states designated by $n, l, m$ are given by

- $\Delta l = \pm 1$
- $\Delta m_l = 0, \pm 1$
- $\Delta m_s = 0$
- $\Delta j = 0, \pm 1$

## 6.5 Blackbody Radiation

### 6.5.1 Plank Formula

$$u(\nu, T) = \frac{8\pi\hbar}{c^3}\frac{\nu^3}{e^{h\nu/k_BT} - 1} \tag{6.11}$$

- **Classical**: $h\nu < k_BT$

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3}k_BT \tag{6.12}$$

- **Quantum :** $h\nu > k_BT$

$$u(\nu, T) = \frac{8\pi h}{c^3}\nu^3 e^{-h\nu/k_BT} \tag{6.13}$$

### 6.5.2 Stefan-Boltzmann's Law

$$P(T) = \sigma T^4 \tag{6.14}$$

### 6.5.3 Wein's Displacement Law

$$\lambda_{\text{max intensity}} = \frac{2.9 \times 10^{-3} \text{ m} \cdot \text{K}}{T} \tag{6.15}$$

## 6.6 X-Rays

### 6.6.1 Bragg Condition

$$2d \sin\theta = m\lambda \tag{6.16}$$

### 6.6.2 The Compton Effect

The Compton Effect deals with the scattering of monochromatic X-rays by atomic targets and the observation that the wavelength of the scattered X-ray is greater than the incident radiation.

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) = \lambda_c(1 - \cos\theta) \tag{6.17}$$

where $\lambda_c$ is the Compton wavelength

$$\lambda_c = \frac{h}{m_e c} = 2.427 \times 10^{-12} \text{ m} \tag{6.18}$$

## 6.7 Atoms in Electric and Magnetic Fields

### 6.7.1 The Cyclotron Frenquency

$$F_B = \frac{mv^2}{R} = qvB \tag{6.19}$$

Solving for $R$ we get

$$R = \frac{mv}{qB} \tag{6.20}$$

and the cyclotron frenquency

$$T = \frac{2\pi m}{qB} \qquad f = \frac{qB}{2\pi m} \tag{6.21}$$

### 6.7.2 Zeeman Effect

The energy change is given from electromagnetism

$$U = -\mu \cdot \mathbf{B} \tag{6.22}$$

The perturbing Hamiltonian is thus

$$H' = -\mu \cdot \mathbf{B} = -\frac{ge}{2m}\mathbf{J} \cdot \mathbf{B} \tag{6.23}$$

The frequency shift

$$\Delta\nu = \frac{eB}{4\pi m_e} \tag{6.24}$$

### 6.7.3 Franck-Hertz Experiment

The Franck-Hertz experiment showed that the energy levels of mercury are quantized confirming quantum theory.

# 7 Special Relativity - 6%

1. The laws of physics are the same in all inertial frames.
2. The speed of light is the same in all inertial frames.

We can define

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad \beta = \frac{u}{c} \tag{7.1}$$

## 7.1 Time Dilation

$$\Delta t' = \gamma \Delta t \tag{7.2}$$

where $\Delta t$ is the time measured at rest relative to the observer, $\Delta t'$ is the time measured in motion relative to the observer.

## 7.2 Length Contraction

$$L' = \frac{L}{\gamma} \tag{7.3}$$

where $L$ is the length of an object observed at rest relative to the observer and $L'$ is the length of the object moving at a speed $u$ relative to the observer.

## 7.3 Energy and Momentum

### 7.3.1 Relativistic Momentum and Energy

- Relativistic momentum $p = \gamma m_0 v$
- Relativistic energy $E = \gamma m_0 c^2$, $E^2 = m_0^2 c^4 + p^2 c^2$
- Relativistic kinetic energy $K = m_0 c^2(\gamma - 1)$
- Energy of photon (massless) $E = pc$

### 7.3.2 Lorentz Transformation (Momentum and Energy)

$$p'_x = \gamma \left( p_x - \beta \frac{E}{c} \right) \tag{7.4}$$

$$p'_y = p_y \tag{7.5}$$

$$p'_z = p_z \tag{7.6}$$

$$\frac{E'}{c} = \gamma \left( \frac{E}{c} - \beta p_x \right) \tag{7.7}$$

## 7.4 Four-Vectors and Lorentz Transformation

$$\begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix} \tag{7.8}$$

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ i\frac{E'}{c} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ i\frac{E}{c} \end{pmatrix} \tag{7.9}$$

## 7.5 Space-Time Interval

$$c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{7.10}$$

We define the space-time interval $\Delta s$

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{7.11}$$

Space-time intervals may be categorized into three types depending on their separation.

- **Time-like interval**: If $(\Delta s)^2 > 0$, the two events occur in the same place but at different times.
- **Light-like interval**: If $(\Delta s)^2 = 0$, the two events are connected by a signal moving at light speed.
- **Space-like interval**: If $(\Delta s)^2 < 0$, the two events occur at the same time (simultaneously) but are seperated spatially.

## 7.6 Lorentz Transformation of Electric and Magnetic Field

Given motion along the $x$ axis:

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y) \tag{7.12}$$

$$B'_x = B_x \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \tag{7.13}$$

## 7.7 Velocity Addition

$$x' = \gamma(x - ut) \qquad t' = \gamma\left(t - \frac{u}{c^2}x\right) \tag{7.14}$$

$$v' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x - u\mathrm{d}t}{\mathrm{d}t - \frac{u}{c^2}\mathrm{d}x} = \frac{v - u}{1 - \frac{uv}{c^2}} \tag{7.15}$$

## 7.8 Relativistic Doppler Formula

$$r = \sqrt{\frac{1-\beta}{1+\beta}} \tag{7.16}$$

- **Red-shift** (source receding)

$$\nu_{\text{receding}} = r\nu_0 \tag{7.17}$$

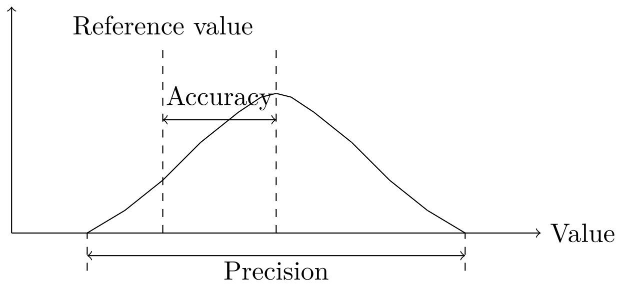
- **Blue-shift** (source approaching)

$$\nu_{\text{receding}} = \frac{\nu_0}{r} \tag{7.18}$$

# 8 Laboratory Methods - 6%

## 8.1 Data and Error Analysis

### 8.1.1 Accuracy and Precision




### 8.1.2 Propagating Uncertainties

$$y = f(x_1, x_2, \cdots, x_n) \tag{8.1}$$

The uncertainty

$$\delta y = \sqrt{\sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \delta x_i \right)^2} \tag{8.2}$$

Given $N$ measurements of the form $x_i \pm \sigma_i$, the weight of measurement $i$ is defined as

$$w_i = \frac{1}{\sigma_i^2} \tag{8.3}$$

With this definition we can define the weighted average as

$$x_{\text{wav}} = \frac{\sum_i w_i x_i}{\sum_i w_i} \tag{8.4}$$

The uncertainty in this value is

$$\sigma_{\text{wav}} = \frac{1}{\sqrt{\sum_i w_i}} \tag{8.5}$$

### 8.1.3 Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \tag{8.6}$$

The standard deviation from the mean value $\lambda$ is

$$\sigma = \sqrt{\lambda} \tag{8.7}$$

It the experiment is repeated $N$ times then the standard deviation is

$$\sigma_N = \frac{\sigma}{\sqrt{N}} \tag{8.8}$$

### 8.1.4 Counting Statistics

Let's assume that for a particular experiment, we are making counting measurements for a radioactive source. In this experiment, we recored $N$ counts in time $T$. The counting rate for this trial is $R = N/T$. This rate should be close ti the average rate $\bar{R}$. The standard deviation or the uncertainty of our count is a simply called the $\sqrt{N}$ rule. So

$$\sigma = \sqrt{N} \tag{8.9}$$

And the number of counts is $N \pm \sqrt{N}$. The uncertainty can be expressed as

$$\frac{\delta R}{R} = \frac{1}{\sqrt{N}} \tag{8.10}$$

## 8.2 Instrumentation

- **Thermocouple gauge**: measurement of the degree of a vacuum by a thermocouple gauge is based primarily on the decrease in thermal conductivity of a gas with decreasing pressure.
- **Propotional counter**: ionization by collisions.
- **Work hardening**: tangling of dislocation lines.

## 8.3 Radiation Detection

## 8.4 Interaction of Charged Particles with Matter

## 8.5 Lasers and Optical Interferometers

The properties of a laser are

1. Light is coherent.
2. Light is monochromatic.
3. Light has minimal divergence.
4. Light has a high intensity.

- **Diode laser**: a laser formed with a semiconducting active medium. The semiconductor is typically a p-n junction that is injected with electric current.
- **Gas laser**: a laser where a free gas is the active medium. An electric current is run through the gas to excite the atoms.

## 8.6 Fundamental Applications of Probability and Statistics

# 9 Specialized Topics - 9%

## 9.1 Particle Physics

### 9.1.1 Elementary Particles

- **Hadrons**: Particles on which the strong force acts.
- **Leptons** are elementary particlea of half-integer spin (spin-1/2) that does not undergo strong interactions.

<table>
<tr><th rowspan="2">Particle or Antiparticle name</th><th rowspan="2">Symbol</th><th rowspan="2">Charge</th><th colspan="3">Lepton flavor number</th><th rowspan="2">Mass (MeV/$c^2$)</th></tr>
<tr><th>$L_e$</th><th>$L_\mu$</th><th>$L_\tau$</th></tr>
<tr><td>Electron</td><td>$e^-$</td><td>-1</td><td>+1</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">0.511</td></tr>
<tr><td>Positron</td><td>$e^+$</td><td>+1</td><td>-1</td></tr>
<tr><td>Muon</td><td>$\mu^-$</td><td>-1</td><td rowspan="2">0</td><td>+1</td><td rowspan="2">0</td><td rowspan="2">105.66</td></tr>
<tr><td>Antimuon</td><td>$\mu^+$</td><td>+1</td><td>-1</td></tr>
<tr><td>Tau</td><td>$\tau^-$</td><td>-1</td><td rowspan="2">0</td><td rowspan="2">0</td><td>+1</td><td rowspan="2">1776.84</td></tr>
<tr><td>Antitau</td><td>$\tau^+$</td><td>+1</td><td>-1</td></tr>
<tr><td>Electron neutrino</td><td>$\nu_e$</td><td rowspan="6">0</td><td>+1</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2"><0.0000022</td></tr>
<tr><td>Electron antineutrino</td><td>$\bar{\nu}_e$</td><td>-1</td></tr>
<tr><td>Muon nuetrino</td><td>$\nu_\mu$</td><td rowspan="2">0</td><td>+1</td><td rowspan="2">0</td><td rowspan="2"><0.17</td></tr>
<tr><td>Muon antineutrino</td><td>$\bar{\nu}_\mu$</td><td>-1</td></tr>
<tr><td>Tau neutrino</td><td>$\nu_\tau$</td><td rowspan="2">0</td><td rowspan="2">0</td><td>+1</td><td rowspan="2"><15.5</td></tr>
<tr><td>Tau anteneutrino</td><td>$\bar{\nu}_\tau$</td><td>-1</td></tr>
</table>

### 9.1.2 Radiation

- **Cherenkov radiation**: Radiation produced when a charged particle passes through a dielectric medium at a speed greater then the phase velocity of light in that medium ($c/n$).
- **Bremeetrahlung radiation**: Radiation produced by a decelerating charge.

### 9.1.3 Radioactive Decay

- $\alpha$ **decay**

$$^A_Z\mathrm{X} \rightarrow ^{A-4}_{Z-2}\mathrm{Y} + ^4_2\mathrm{He} \tag{9.1}$$

- $\beta$ **decay** (weak interaction)

$$\beta^- : \ ^A_Z\mathrm{X} \rightarrow \ ^A_{Z+1}\mathrm{Y} + e^- + \bar{\nu}_e \tag{9.2}$$

$$\beta^+ : \ ^A_Z\mathrm{X} \rightarrow \ ^A_{Z-1}\mathrm{Y} + e^+ + \bar{\nu}_e \tag{9.3}$$

- $\gamma$ **decay**

$$^A_Z\mathrm{X} \rightarrow \ ^A_Z\mathrm{X} + \gamma \tag{9.4}$$

### 9.1.4 Half-life

For an exponential decay

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N \tag{9.5}$$

Solving this, we have

$$N = N_0 e^{-\lambda t} \tag{9.6}$$

The half-life, $\tau$, is

$$\tau = \frac{1}{\lambda} \tag{9.7}$$

If there are several kinds of decay, then

$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2} + \cdots \frac{1}{\tau_n} \tag{9.8}$$

### 9.1.5 Nuclear Binding Energy

$$U_i - U_f = K \tag{9.9}$$

### 9.1.6 Fission and Fusion

- **Fission**

$$n + {}^{A}N \to {}^{A+1}N \to L + M \tag{9.10}$$

- **Fusion**
  The sun undergoes fusion in a 4-step

$$2({}^{1}H + {}^{1}H \to {}^{2}H + e^{+} + \nu) \tag{9.11}$$

$$2(e^{+} + e^{-} \to 2\gamma) \tag{9.12}$$

$$2({}^{2}H + {}^{1}H \to {}^{3}He + \gamma) \tag{9.13}$$

$${}^{3}He + {}^{3}He \to {}^{4}He + {}^{1}H + {}^{1}H \tag{9.14}$$

### 9.1.7 Detectors

If the detector is of length $L$ and the particles have speed $v$, the the resolving time must be

$$t_{\text{res}} \leq \frac{L}{v} \tag{9.15}$$

## 9.2 Solid State Physics

### 9.2.1 Reciprocal Lattice

$$\mathbf{b}_1 = 2\pi \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3} \tag{9.16}$$

### 9.2.2 Bragg Diffraction

$$2d \sin\theta = n\lambda \tag{9.17}$$

### 9.2.3 Free Electron Gas

$$E = \frac{\hbar^2 k^2}{2m} \tag{9.18}$$

The Fermi energy is

$$E_F = \frac{\hbar^2 k_F^2}{2m} \tag{9.19}$$

where $k_F$ is the Fermi momentum which is equal to

$$k_F = (3n\pi^2)^{1/3} \tag{9.20}$$

### 9.2.4 Effective Mass

$$\frac{1}{m^*} = \frac{1}{\hbar^2} \frac{\mathrm{d}^2 E}{\mathrm{d}k^2} \tag{9.21}$$

### 9.2.5 Semiconductor

When a semiconductor is 'cold', all its electrons are tightly held by their atoms. When the substance is heated, the energy liberates some electrons and he substance has some free electrons; it conducts. The more energy the more electrons freed. So we are looking for a relationship where **the conductivity increases with temperature**.

- **Intrinsic semiconductor**

$$n = p = \sqrt{N_+ N_-} \exp\left(-\frac{E_g}{2k_B T}\right) \tag{9.22}$$

- **Impurity semiconductor**
  - **N-type**: +5
  - **P-type**: +3

### 9.2.6 Superconductor

According to the BCS theory, the attraction between Copper pairs in a super conductor is due to interactions with the ionic lattice.

# 10 Mathematics

$$\nabla \times (\nabla U) = 0 \tag{10.1}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{10.2}$$

$$\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta \tag{10.3}$$

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = r\mathrm{d}\mathrm{d}\phi\mathrm{d}z \tag{10.4}$$

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2 \sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{10.5}$$