NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF PHYSICS

Advanced Classical Physics

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1 Lagrangian Mechanics

1.1 Action Principle

1.1.1 Fermat's Principle

The idea of the 'principle of least action' has its origin in Fermat's principle in optics, according to which light follows the shortest optical path, i.e., the path of shortest time to reach its destination.

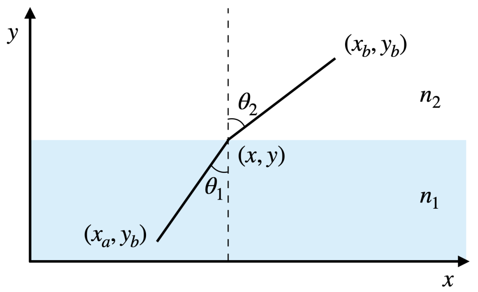

Figure 1: Snell's law of refraction for light passing through media of different indices of refraction.

According to figure (1), consider a light ray from point (x_a, y_a) to (x_b, y_b), the optical path length follows

$$T(x) = \frac{n_1}{c}\sqrt{(x_a - x)^2 + (y_a - y)^2} + \frac{n_2}{c}\sqrt{(x_b - x)^2 + (y_b - y)^2} \tag{1}$$

According to Fermat's principle, we need to find the minimum of this quantity.

$$\begin{aligned} \frac{\partial T(x)}{\partial x} &= \frac{n_1}{c}\frac{x - x_a}{\sqrt{(x_a - x)^2 + (y_a - y)^2}} + \frac{n_2}{c}\frac{x - x_b}{\sqrt{(x_b - x)^2 + (y_b - y)^2}} \\ &= -\frac{n_1}{c}\sin\theta_1 + \frac{n_2}{c}\sin\theta_2 = 0 \end{aligned} \tag{2}$$

which shows the *Snell's law*

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{3}$$

What happens if the refractive index is a function of space $n(x, y)$? The Fermat's action becomes

$$T = \int \mathrm{d}T = \int \frac{n(x, y)}{c}\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} \tag{4}$$

Suppose we parameterise the trajectory of the particle by a monotonic parameter λ

$$\boldsymbol{r}(\lambda) = (x(\lambda), y(\lambda)) \tag{5}$$

such that

$$x(\lambda_a) = x_a, \quad x(\lambda_b) = x_b, \quad y(\lambda_a) = y_a, \quad y(\lambda_b) = y_b \tag{6}$$

The Fermat's action is

$$T = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \frac{n(x(\lambda), y(\lambda))}{c}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2} \tag{7}$$

1.1.2 Euler-Lagrange Equations

In configuration space (the space of coordinates and velocities), the action S is defined as an integral over a function $L(x, \dot{x}, t)$ known as the Lagrangian, as

$$S[x(t)] = \int_{t_a}^{t_b} \mathrm{d}t L(x(t), \dot{x}(t), t) \tag{8}$$

For standard conservative systems the Lagrangian is simply the difference of the kinetic energy T and the potential energy V, i.e.

$$\boxed{L = T - V} \tag{9}$$

The actual physical trajectory is the function x that minimises the action subject to the boundary conditions $x(t_a) = x_a$ and $x(t_b) = x_b$. This perturbation changes the action by an amount δS given by

$$\delta S[x(t)] = \int_{t_a}^{t_b} \mathrm{d}t \left[\frac{\partial L}{\partial x}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}(t)\right] \tag{10}$$

In the second term, we may integrate by parts,

$$\int_{t_a}^{t_b} \mathrm{d}t \frac{\partial L}{\partial \dot{x}}\delta \dot{x}(t) = \left[\frac{\partial L}{\partial \dot{x}}\delta x(t)\right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x(t) = -\int_{t_a}^{t_b} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x(t) \tag{11}$$

Substituting this result into the expression (10) for δS, we have

$$\delta S = \int_{t_a}^{t_b} \mathrm{d}t \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x(t) \tag{12}$$

In order that S be stationary, we require

$$\boxed{\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0} \tag{13}$$

This is known as the *Euler-Lagrange equation*, and it is the equation of motion in the Lagrangian formulation of mechanics.

1.1.3 Back to Fermat

In the gauge $\lambda = y$, the total time was given to

$$T = \int_{y_a}^{y_b} \mathrm{d}y \frac{n(x(y), y)}{c}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1} \tag{14}$$

The Lagrangian is

$$L(x(y), \dot{x}(y), y) = \frac{n(x(y), y)}{c}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1} \tag{15}$$

The Euler-Lagrange equation for this problem is

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial L}{\partial \frac{\mathrm{d}x}{\mathrm{d}y}}\right) = \frac{\partial L}{\partial x} \tag{16}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[n(x,y)\frac{\frac{\mathrm{d}x}{\mathrm{d}y}}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}}\right] = \frac{\partial n(x,y)}{\partial x}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1} \tag{17}$$

it simplifies a lot in the case where the refractive index is just a function of y. Then

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[n(y)\frac{\frac{\mathrm{d}x}{\mathrm{d}y}}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}}\right] = 0 \tag{18}$$

which is easily solved as

$$n(y)\frac{\frac{\mathrm{d}x}{\mathrm{d}y}}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}} = n(y)\frac{\mathrm{d}x}{\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}} = n(y)\sin(\theta(y)) = A \tag{19}$$

with A a constant.

1.2 Generalized coordinates and momenta

General coordinate q_i
General conjugate momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$\delta S = \int \mathrm{d}t \left(\sum_{i=1}^{N}\frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{N}\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\right) \tag{20}$$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad \forall i = 1,\cdots,N} \tag{21}$$

1.3 Conservation Laws

1.3.1 Momentum Conservation

For any generalised coordinate q_i, we define the generalised momentum p_i by

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}} \tag{22}$$

So we have

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\mathrm{d}p_i}{\mathrm{d}t} \tag{23}$$

Whenever the Lagrangian L does not depend *explicitly* on q_i, the corresponding generalised momentum p_i is conserved

$$\boxed{\frac{\partial L}{\partial q_i} = 0 \quad \Leftrightarrow \quad p_i \text{ is conserved.}} \tag{24}$$

1.3.2 Energy (or Hamiltonian) Conservation

First consider the total time derivative of the Lagrangian

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t} \end{aligned} \tag{25}$$

Rearranging this equation

$$\frac{\partial L}{\partial t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i p_i\dot{q}_i - L\right) = -\frac{\mathrm{d}H}{\mathrm{d}t} \tag{26}$$

Whenever the Lagrangian does not depend *explicitly* on t, the Hamiltonian is conserved

$$\boxed{\frac{\partial L}{\partial t} = 0 \quad \Leftrightarrow \quad H = \sum_i p_i\dot{q}_i - L \text{ is conserved.}} \tag{27}$$

1.4 Constraints and Number of degrees of Freedom

1.4.1 Kinetic Matrix

Using the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_j \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \sum_j \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} = \frac{\partial L}{\partial q_i} \tag{28}$$

We can write in this form

$$\boxed{\sum_j \mathcal{Z}^{ij}\ddot{q}_j + \mathcal{F}^i = 0} \tag{29}$$

where the *kinetic matrix* $\mathcal{Z}_{ij}$ and the vector $\mathcal{F}_i$ are both functions of the coordinates

$$\mathcal{Z}^{ij}(q_1, \cdots, q_N, \dot{q}_1, \cdots, \dot{q}_N, t) = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \mathcal{Z}^{ji} \tag{30}$$

$$\mathcal{F}^i(q_1, \cdots, q_N, \dot{q}_1, \cdots, \dot{q}_N, t) = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} - \frac{\partial L}{\partial q_i} \tag{31}$$

So long as the symmetric matrix $\mathcal{Z}_{ij}$ is non-degenerate, i.e. so long as $\det(\mathcal{Z}) \neq 0$

$$\ddot{q} = -(\mathcal{Z})^{-1}\mathcal{F} \tag{32}$$

There are cases however where the matrix $\mathcal{Z}$ is 'degenerate' and $\det(\mathcal{Z}) = 0$ means that not all coordinates q_i are independent.

1.4.2 Lagrange Multipliers

$$\tilde{L}(q_i, \dot{q}_i, \lambda, t) = L(q_i, \dot{q}_i, t) + \lambda f(q_i, \dot{q}_i, t) \tag{33}$$

The Euler-Lagrange equation for λ is given by

$$\frac{\delta \tilde{S}}{\delta \lambda} = \frac{\partial \tilde{L}}{\partial \lambda} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\lambda}}\right) = f(q_i, \dot{q}_i, t) \tag{34}$$

and therefore it imposes the constraint $f = 0$ independently of what λ actually is. The Euler-Lagrange equations for the original coordinates are

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = -\lambda(t)\frac{\partial f}{\partial q_i} \tag{35}$$

1.4.3 Example: Helter Skelter

A child of m slides down a helter skelter, the Lagrangian is given by

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \tag{36}$$

and

$$z = h - \alpha\theta, \qquad r = \beta\theta \tag{37}$$

where α and β are positive constants and h is the height of the helter skelter. We take the angle θ to go from 0 to infinity as the trajectory winds around multiple times. The constraints are

$$C_1 = z - h + \alpha\theta, \qquad C_2 = r - \beta\theta \tag{38}$$

1) **Method 1: Solve constraints**

   Put constraints into the Lagrangian

$$L(\theta, t) = \frac{1}{2}m\left(\beta^2\dot{\theta}^2 + \beta^2\theta^2\dot{\theta}^2 + \alpha^2\dot{\theta}^2\right) - mg(h - \alpha\theta) \tag{39}$$

   The Euler-Lagrange equation for θ is

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\alpha^2 + \beta^2\right) + \beta^2\theta^2\right]\dot{\theta} = m\beta^2\theta\dot{\theta}^2 + mg\alpha \tag{40}$$

Figure 2: A helter skelter.

2) **Method 2: Work in extended configuration space**

$$\tilde{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz + (\lambda_1 C_1 + \lambda_2 C_2) \tag{41}$$

The Euler-Lagrange equations are

- For r: $mr\dot{\theta}^2 - m\ddot{r} = -\lambda_2$
- For θ: $m\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = \alpha\lambda_1 - \beta\lambda_2$
- For z: $m\ddot{z} + mg = \lambda_1$
- For λ_1: $z = h - \alpha\theta$
- For λ_2: $r = \beta\theta$

So we have

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\alpha^2 + \beta^2\right) + \beta^2\theta^2\right]\dot{\theta} = m\beta^2\theta\dot{\theta}^2 + mg\alpha \tag{42}$$

1.5 Normal Modes

'Natural' System: We consider the situation where particles can move around their equilibrium positions. We assume that the system is described by N generalised coordinates q_i. We also assume that it is *natural*, which means that the kinetic energy is a *quadratic homogeneous function* of the generalised velocities. We can then write it as

$$T = \frac{1}{2}\sum_{ij} a_{ij}(q_1, \cdots, q_N)\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{\boldsymbol{q}}^T A\dot{\boldsymbol{q}} \tag{43}$$

Since the Lagrangian is given by

$$L = T - V(q_1, \cdots, q_N, t) \tag{44}$$

where the potential does not depend on the velocities $\dot{q}_i$. Then we have

$$\begin{aligned}\mathcal{Z}^{ij} =&\frac{\partial^2 L}{\partial\dot{q}_i\partial\dot{q}_j} = \frac{\partial^2 T}{\partial\dot{q}_i\partial\dot{q}_j} = \frac{\partial^2}{\partial\dot{q}_i\partial\dot{q}_j}\left(\frac{1}{2}a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta\right)\\ =&\frac{\partial}{\partial\dot{q}_j}\left(\frac{1}{2}a_{\alpha\beta}\frac{\partial\dot{q}_\alpha}{\partial\dot{q}_i}\dot{q}_\beta + \frac{1}{2}a_{\alpha\beta}\dot{q}_\alpha\frac{\partial\dot{q}_\beta}{\partial\dot{q}_i}\right)\\ =&\frac{\partial}{\partial\dot{q}_j}\left(\frac{1}{2}a_{\alpha\beta}\delta_{i\alpha}\dot{q}_\beta + \frac{1}{2}a_{\alpha\beta}\dot{q}_\alpha\delta_{i\beta}\right)\\ =&\frac{\partial}{\partial\dot{q}_j}\left(\frac{1}{2}a_{i\beta}\dot{q}_\beta + \frac{1}{2}a_{\alpha i}\dot{q}_\alpha\right)\\ =&\frac{1}{2}a_{ij}\dot{q}_j + \frac{1}{2}a_{ji}\dot{q}_j = a_{ij}\end{aligned} \tag{45}$$

the kinetic matrix is then given directly by the coefficients a_{ij}.

1.5.1 Equilibrium Points

From the calculation in above, we know that

$$\frac{\partial L}{\partial\dot{q}_i} = a_{ij}\dot{q}_j \tag{46}$$

The Euler-Lagrangian equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} =&\frac{\mathrm{d}}{\mathrm{d}t}\left(a_{ij}\dot{q}_j\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i}\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\left(a_{ij}\dot{q}_j\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}a_{jk}\dot{q}_j\dot{q}_k\right) + \frac{\partial V}{\partial q_i}\\ =&a_{ij}\ddot{q}_j - \frac{1}{2}\frac{\partial a_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k + \frac{\partial V}{\partial q_i} = 0\end{aligned} \tag{47}$$

In equilibrium point, $\dot{q}_i$ is a constant, so we require

$$\left.\frac{\partial V}{\partial q_i}\right|_{q_i=q_{0i}} = 0, \qquad i = 1, 2, \cdots, N \tag{48}$$

1.5.2 Small Oscillations

We can write the coordinates as $q_i(t) = q_{0i} + \delta q_i(t)$. The action is given by

$$\begin{aligned}S =& \int\mathrm{d}t\left[\frac{1}{2}a_{ij}(\boldsymbol{q}_0)\frac{\mathrm{d}}{\mathrm{d}t}(q_{0i}+\delta q_i)\frac{\mathrm{d}}{\mathrm{d}t}(q_{0j}+\delta q_j) - V(\boldsymbol{q}_0+\delta\boldsymbol{q})\right]\\ =& \int\mathrm{d}t\left[\frac{1}{2}a_{ij}(\boldsymbol{q}_0)\delta\dot{q}_i\delta\dot{q}_j - \left(V(\boldsymbol{q}_0) + \left.\frac{\partial V}{\partial q_i}\right|_{\boldsymbol{q}_0}\delta q_i + \frac{1}{2}\left.\frac{\partial^2 V}{\partial q_i\partial q_j}\right|_{\boldsymbol{q}_0}\delta q_i\delta q_j + \mathcal{O}(\delta q^3)\right)\right]\\ =& \int\mathrm{d}t\left[\frac{1}{2}a_{ij}(\boldsymbol{q}_0)\delta\dot{q}_i\delta\dot{q}_j - \frac{1}{2}\left.\frac{\partial^2 V}{\partial q_i\partial q_j}\right|_{\boldsymbol{q}_0}\delta q_i\delta q_j\right]\\ =& \int\mathrm{d}t\left[\frac{1}{2}a_{ij}(\boldsymbol{q}_0)\delta\dot{q}_i\delta\dot{q}_j - \frac{1}{2}b_{ij}(\boldsymbol{q}_0)\delta q_i\delta q_j\right] - \int\mathrm{d}tV(\boldsymbol{q}_0)\end{aligned} \tag{49}$$

where

$$b_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} \tag{50}$$

The second term is just a constant, so and so for small fluctuations all that matters is the quadratic part, which we refer to as the action for the quadratic fluctuations

$$\begin{aligned} S_{(2)} &= \int \mathrm{d}t \left(\frac{1}{2} a_{ij} \delta\dot{q}_i \delta\dot{q}_j - \frac{1}{2} b_{ij} \delta q_i \delta q_j \right) \\ &= \int \mathrm{d}t \left(\frac{1}{2} \delta\dot{\boldsymbol{q}}^T \mathbf{A} \delta\dot{\boldsymbol{q}} - \frac{1}{2} \delta\boldsymbol{q}^T \mathbf{B} \delta\boldsymbol{q} \right) \end{aligned} \tag{51}$$

where the matrix $\mathbf{A}$ and $\mathbf{B}$ have the components $\mathbf{A}_{ij} = a_{ij}$ and $\mathbf{B}_{ij} = b_{ij}$.

The Euler-Lagrange equation for the fluctuations is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L_{(2)}}{\partial \delta\dot{q}_i} \right) = \frac{\partial L}{\partial \delta q_i} \tag{52}$$

which is

$$a_{ij} \delta\ddot{q}_j = -b_{ij} \delta q_j \qquad \text{or} \qquad \mathbf{A} \delta\ddot{\boldsymbol{q}} = -\mathbf{B} \delta\boldsymbol{q} \tag{53}$$

If $\det(\mathbf{A}) \neq 0$, then

$$\delta\ddot{\boldsymbol{q}} = -\mathbf{K} \delta\boldsymbol{q} \tag{54}$$

with

$$\mathbf{K} = \mathbf{A}^{-1} \mathbf{B} \tag{55}$$

1.5.3 Gram-Schmidt Diagonalization

Since the matrix $\mathbf{A}$ is symmetric (we may assume this from the outset), we can always diagonalize it using a $N \times N$ orthogonal matrix $\mathbf{O}$ ($\mathbf{O}^T \mathbf{O} = \mathbb{1}$) so that

$$\mathbf{A}_D = \mathbf{O}^T \mathbf{A} \mathbf{O} \tag{56}$$

with $\mathbf{A}_D$ diagonal. The kinetic term

$$\begin{aligned} T =& \frac{1}{2} \delta\dot{\boldsymbol{q}}^T \mathbf{A} \delta\dot{\boldsymbol{q}} = \frac{1}{2} \delta\dot{\boldsymbol{q}}^T \mathbf{O} \mathbf{A}_D \mathbf{O}^T \delta\dot{\boldsymbol{q}} = \frac{1}{2} \delta\dot{\tilde{\boldsymbol{q}}}^T \mathbf{A}_D \delta\dot{\tilde{\boldsymbol{q}}} \\ =& \frac{1}{2} \delta\dot{\tilde{q}}_i (a_D)_{ij} \delta\dot{\tilde{q}}_j \\ =& \frac{1}{2} (a_D)_{11} \delta\dot{\tilde{q}}_1^2 + \frac{1}{2} (a_D)_{22} \delta\dot{\tilde{q}}_2^2 + \cdots + \frac{1}{2} (a_D)_{NN} \delta\dot{\tilde{q}}_N^2 \end{aligned} \tag{57}$$

where

$$\delta\tilde{\boldsymbol{q}} = \mathbf{O}^T \delta\boldsymbol{q} \tag{58}$$

To put the kinetic term in diagonal and *normalized* form, we require

$$\delta q_i = \sqrt{(a_D)_{ii}^{-1}} \delta Q_i \tag{59}$$

This is the statement that there is a diagonal matrix $\mathbf{W}$ whose matrix elements are

$$\mathbf{W}_{ij} = \sqrt{(a_D)^{-1}_{ii}}\delta_{ij} \tag{60}$$

such that

$$\delta\tilde{\boldsymbol{q}} = \mathbf{W}\delta\boldsymbol{Q} \tag{61}$$

for which

$$\mathbf{W}^T\mathbf{A}_D\mathbf{W} = \mathbb{1} \tag{62}$$

These two operations amount to saying that there is a matrix $\mathbf{S} = \mathbf{OW}$ such that

$$\mathbf{S}^T\mathbf{A}_D\mathbf{S} = \mathbb{1} \tag{63}$$

the action for the fluctuations becomes

$$\begin{aligned} S_{(2)} &= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\mathbf{S}^T\mathbf{A}\mathbf{S}\delta\dot{\boldsymbol{Q}} - \frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\mathbf{S}^T\mathbf{B}\mathbf{S}\delta\dot{\boldsymbol{Q}}\right) \\ &= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\delta\dot{\boldsymbol{Q}} - \frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\mathbf{k}\delta\dot{\boldsymbol{Q}}\right) \end{aligned} \tag{64}$$

where

$$\mathbf{k} = \mathbf{S}^T\mathbf{B}\mathbf{S} \tag{65}$$

This is the canonically normalized form for the fluctuations.

1.5.4 Cholesky Decomposition

The Cholesky decomposition of a *real Hermitian positive*-definite[1] matrix $\mathbf{A}$, is a decomposition of the form

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T \tag{66}$$

where $\mathbf{L}$ is a left triangular matrix with real and positive diagonal entries

$$\mathbf{L} = \begin{pmatrix} \# & 0 & 0 & 0 \\ \# & \# & 0 & 0 \\ \# & \# & \# & 0 \\ \# & \# & \# & \# \end{pmatrix} \tag{67}$$

So the kinetic term can be written as

$$T = \frac{1}{2}\delta\dot{\boldsymbol{q}}\mathbf{A}\delta\dot{\boldsymbol{q}} = \frac{1}{2}\delta\dot{\boldsymbol{q}}\mathbf{L}\mathbf{L}^T\delta\dot{\boldsymbol{q}} = \frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\delta\dot{\boldsymbol{Q}} \tag{68}$$

where

$$\delta\boldsymbol{Q} = \mathbf{L}^T\delta\boldsymbol{q} \tag{69}$$

[1]'Positive' means the eigenvalues of $\mathbf{A}$ are all positive.

And the action

$$\begin{aligned} S &= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{\boldsymbol{q}}\mathbf{A}\delta\dot{\boldsymbol{q}} - \frac{1}{2}\delta\boldsymbol{q}^T\mathbf{B}\delta\boldsymbol{q} \right) \\ &= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{\boldsymbol{Q}}\delta\dot{\boldsymbol{Q}} - \frac{1}{2}\delta\boldsymbol{Q}^T\mathbf{k}\delta\boldsymbol{Q} \right) \\ &= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{Q}_i\delta\dot{Q}_i - \frac{1}{2}k_{ij}\delta Q_i\delta Q_j \right) \end{aligned} \tag{70}$$

where

$$\mathbf{k} = \mathbf{L}^{-1}\mathbf{B}(\mathbf{L}^T)^{-1} \tag{71}$$

The Euler-Lagrange equation for Q_i is given by

$$\delta\ddot{Q}_i = -k_{ij}\delta Q \qquad \text{or} \qquad \delta\ddot{\boldsymbol{Q}} = -\mathbf{k}\delta\boldsymbol{Q} \tag{72}$$

We look for solutions of the form

$$\delta\boldsymbol{Q} = \mathrm{e}^{i\omega t}\delta\boldsymbol{Q}_\omega \tag{73}$$

then we have the eigenfunction equation with the matrix $\mathbf{k}$ and eigenvalue ω^2.

$$\mathbf{k}\delta\boldsymbol{Q}_\omega = \omega^2\delta\boldsymbol{Q}_\omega \tag{74}$$

which means that each normal coordinate $\boldsymbol{Q}_{\omega_\alpha}$ oscillates independently of all others with its own normal frequency ω_α^2.

1.5.5 Example: Double Pendulum

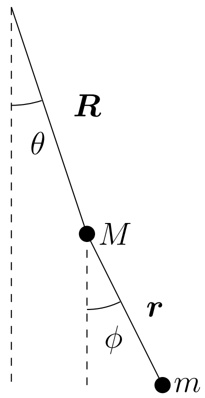

Figure 3: Double pendulum

Consider a double pendulum, with a second pendulum hanging from the first as depicted in Fig.(3). The kinetic energy of the system is

$$\begin{aligned} T =& \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}m(\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}})^2 \\ =& \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}m\left(\dot{\boldsymbol{R}}^2 + \dot{\boldsymbol{r}}^2 + 2\dot{\boldsymbol{R}}\dot{\boldsymbol{r}}\right) \\ =& \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + mRr\dot{\theta}\dot{\phi}\cos(\phi - \theta) \end{aligned} \tag{75}$$

The potential energy is

$$\begin{aligned}V =& Mg(-R\cos\theta) + mg(-R\cos\theta - r\cos\phi)\\ =& -(M+m)gR\cos\theta - mgr\cos\phi\end{aligned} \tag{76}$$

The equilibrium solution is $\theta_0 = \phi_0 = 0$, so θ and ϕ can be expressed by

$$\theta(t) = \delta\theta(t), \qquad \phi(t) = \delta\phi(t) \tag{77}$$

then we have

$$\begin{aligned}L_{(2)} =& \frac{1}{2}(M+m)R^2\delta\dot\theta^2 + \frac{1}{2}mr^2\delta\dot\phi^2 + mRr\delta\dot\theta\delta\dot\phi - \frac{1}{2}(M+m)gR\delta^2\theta - \frac{1}{2}mgr\delta\phi^2\\ =& \frac{1}{2}MR^2\delta\dot\theta^2 + \frac{1}{2}m\left(R\delta\dot\theta + r\delta\dot\phi\right)^2 - \frac{1}{2}(M+m)gR\delta^2\theta - \frac{1}{2}mgr\delta\phi^2\end{aligned} \tag{78}$$

We define that

$$\delta Q_1 = \sqrt{M}R\delta\theta, \qquad \delta Q_2 = \sqrt{m}(R\delta\theta + r\delta\phi) \tag{79}$$

so the Lagrangian becomes

$$L_{(2)} = \frac{1}{2}\delta\dot Q_1^2 + \frac{1}{2}\delta\dot Q_2^2 - \frac{1}{2}\frac{(M+m)g}{MR}\delta Q_1^2 - \frac{1}{2}\frac{mg}{Mr}(\delta Q_1 - \delta Q_2)^2 \tag{80}$$

and the potential energy is given by

$$V = -\frac{1}{2}\delta\boldsymbol{Q}^T\mathbf{k}\delta\boldsymbol{Q} = -\frac{1}{2}\begin{pmatrix}\delta Q_1 & \delta Q_2\end{pmatrix}\mathbf{k}\begin{pmatrix}\delta Q_1\\ \delta Q_2\end{pmatrix} \tag{81}$$

where

$$\mathbf{k} = \begin{pmatrix}\frac{(M+m)g}{MR} + \frac{mg}{Mr} & -\frac{mg}{Mr}\\ -\frac{mg}{Mr} & \frac{mg}{Mr}\end{pmatrix} = \frac{mg}{Mr}\begin{pmatrix}\frac{(M+m)r}{mR}+1 & -1\\ -1 & 1\end{pmatrix} \tag{82}$$

To simplify $\mathbf{k}$, we choose $R = r$ and $M = m$, then we have

$$\mathbf{k} = \frac{g}{R}\begin{pmatrix}3 & -1\\ -1 & 1\end{pmatrix} \tag{83}$$

The eigenvalues ω_1^2 and ω_2^2 satisfies

$$\det(\mathbf{k} - \omega^2\mathbb{1}) = \frac{g}{R}\left[(3-\omega^2)(1-\omega^2) - 1\right] = 0 \tag{84}$$

solve this and get

$$\omega_{1,2}^2 = (2\pm\sqrt{2})\frac{g}{R} > 0 \tag{85}$$

the associated eigenvectors are $\boldsymbol{Q}_\pm = (1\pm\sqrt{2}, -1)$. Go back to the original coordinates θ, ϕ, we know that

$$Q_1 \sim \theta, \qquad Q_2 \sim \theta + \phi \tag{86}$$

so the normal modes are

$$(\theta,\phi)_\pm = (1\pm\sqrt{2}, -(2\pm\sqrt{2})) \sim (1, \pm\sqrt{2}) \tag{87}$$

1.6 Symmetries and Noether's Theorem

Let's parameterize the action by arbitrary coordinates q_i

$$S = \int \mathrm{d}t L(q, \dot{q}, t) \tag{88}$$

now we have

$$\begin{aligned}\delta S &= \int \mathrm{d}t \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right) \\ &= \underbrace{\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i \right)\Big|_{t_1}^{t_2}}_{\text{boundary term}} + \int \mathrm{d}t \underbrace{\left[\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right]}_{\text{Euler-Lagrange equation}} \delta q_i \\ &= \left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i \right)\Big|_{t_1}^{t_2}\end{aligned} \tag{89}$$

Euler-Lagrange equations follow the principle of least action. So $\delta S = 0$ up to boundary terms.

1.6.1 Discrete Global Symmetry

Let's consider a simple example, which is a harmonic oscillator

$$S[q(t)] = \int \mathrm{d}t \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \right) \tag{90}$$

consider the transition $q' = -q$, and

$$S[q'(t)] = \int \mathrm{d}t \left(\frac{1}{2}m\dot{q}'^2 - \frac{1}{2}m\omega^2 q'^2 \right) = \int \mathrm{d}t \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \right) = S[q(t)] \tag{91}$$

1.6.2 Continuous Global Symmetry

Let's consider a general infinitesimal transform

$$\boxed{\delta q_i = F_i(q, \dot{q})\delta\lambda} \tag{92}$$

then the Lagrangian transforms as

$$\begin{aligned}\delta L(q, \dot{q}, t) =& L(q + F\delta\lambda, \dot{q} + \dot{F}\delta\lambda) - L(q, \dot{q}) \\ =& \frac{\partial L}{\partial q_i}F_i\delta\lambda + \frac{\partial L}{\partial \dot{q}_i}\dot{F}_i\delta\lambda \\ =& \left(\frac{\partial L}{\partial q_i}F_i + \frac{\partial L}{\partial \dot{q}_i}\dot{F}_i \right)\delta\lambda\end{aligned} \tag{93}$$

If this is a symmetry, then this must be a total derivative then

$$\boxed{\frac{\partial L}{\partial q_i}F_i + \frac{\partial L}{\partial \dot{q}_i}\dot{F}_i = \frac{\mathrm{d}A}{\mathrm{d}t}} \tag{94}$$

and the integration

$$\delta S = \int \mathrm{d}t\delta L = \int_{t_1}^{t_2} \mathrm{d}t \frac{\mathrm{d}A}{\mathrm{d}t}\delta\lambda = \underbrace{[A(t_2) - A(t_1)]}_{\text{boundary term}}\delta\lambda = 0 \tag{95}$$

1.6.3 Noether's theorem

Noether performed a clever trick. She make λ a function of time

$$\delta q_i = F_i(q_i, \dot{q}_i)\delta\lambda(t) \tag{96}$$

now we have

$$\begin{aligned}
\delta S &= \int \mathrm{d}t \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_i) \right] \\
&= \int \mathrm{d}t \left[\frac{\partial L}{\partial q_i}F_i\delta\lambda + \frac{\partial L}{\partial \dot{q}_i}\dot{F}_i\delta\lambda + \frac{\partial L}{\partial \dot{q}_i}F_i\frac{\mathrm{d}}{\mathrm{d}t}(\delta\lambda) \right] \\
&= \int \mathrm{d}t \left[\frac{\mathrm{d}A}{\mathrm{d}t}\delta\lambda - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}F_i \right)\delta\lambda \right] + \left(\frac{\partial L}{\partial \dot{q}}F_i\delta\lambda \right)\Bigg|_{t_1}^{t_2} \\
&= \int \mathrm{d}t \left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}F_i - A \right)\delta\lambda \right] + \left(\frac{\partial L}{\partial \dot{q}}F_i\delta\lambda \right)\Bigg|_{t_1}^{t_2} \\
&= \int \mathrm{d}t \left(-\frac{\mathrm{d}G}{\mathrm{d}t}\delta\lambda \right) + \left(\frac{\partial L}{\partial \dot{q}}F_i\delta\lambda \right)\Bigg|_{t_1}^{t_2} = 0
\end{aligned} \tag{97}$$

where

$$\boxed{G = \frac{\partial L}{\partial \dot{q}}F - A} \tag{98}$$

so we have

$$\boxed{\frac{\mathrm{d}G}{\mathrm{d}t} = 0 \quad \text{(up to the boundary term)}} \tag{99}$$

Theorem 1
Noether's theorem: For every continuous symmetry, there exist a conserved quantity G conserved in time.

Example
Consider the action

$$S = \int \mathrm{d}t \frac{1}{2}m\dot{q}^2 \tag{100}$$

and the transform $q' = q + \lambda$, *i.e.*, $\delta q = \delta\lambda$ with $F = 1$ and $A = 0$.

$$G = \frac{\partial L}{\partial \dot{q}}F - A = \frac{\partial L}{\partial \dot{q}} = p \tag{101}$$

which means the corresponding generalized momentum component p to be a constant of motion.

1.6.4 Hamiltonian as the Noether Charge for Time Translations

Consider a system which is time translation invariant meaning that the action is invariant (un to boundary terms) under the symmetry $t \to t+\delta t$. Such a transformation induces a change of coordinates

$$\delta q_i(t) = q_i(t+\lambda) - q_i(t) = \dot{q}_i \delta\lambda \tag{102}$$

Similarly the Lagrangian transforms as

$$\delta L = L(q(t+\lambda), \dot{q}(t+\lambda), t+\lambda) - L(q, \dot{q}, t) = \frac{\mathrm{d}L}{\mathrm{d}t}\delta\lambda \tag{103}$$

from which we infer

$$A = L \tag{104}$$

Thus the conserved charge implied by Noether's theorem associated with the symmetry of time translation invariance is

$$G = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = p_i\dot{q}_i - L = H \tag{105}$$

which we recognize to be the Hamiltonian.

2 Hamiltonian

2.1 Hamiltonian Formulation

In Lagrangian, we use coordinates and velocities $(q, \dot{q})$ to parameterise the system, while in Hamiltonian, we use coordinates and momenta (q, p) to parameterise the system.

$$\boxed{\begin{array}{lcr} \text{Lagrangian} & & \text{Hamiltonian} \\ L = L(q, \dot{q}, t) & \longleftrightarrow & H = H(q, p, t) \\ & p = \dfrac{\partial L}{\partial \dot{q}} & \end{array}}$$

2.1.1 From Lagrangian to Hamiltonian

The total derivative of Lagrangian can be expressed as

$$\begin{aligned} \mathrm{d}L(q, \dot{q}, t) =& \frac{\partial L}{\partial q_i}\mathrm{d}q_i + \frac{\partial L}{\partial \dot{q}_i}\mathrm{d}\dot{q}_i + \frac{\partial L}{\partial t}\mathrm{d}t \\ =& \frac{\partial L}{\partial q_i}\mathrm{d}q_i + p_i\mathrm{d}\dot{q}_i + \frac{\partial L}{\partial t}\mathrm{d}t \end{aligned} \tag{106}$$

We define the *Hamiltonian function* H by Legendre transformation

$$\boxed{H = \sum_i p_i\dot{q}_i - L} \tag{107}$$

2.1.2 Hamilton's Equations

The total derivative of Hamiltonian can be expressed as

$$\begin{aligned} \mathrm{d}H =& -\frac{\partial L}{\partial q_i}\mathrm{d}q_i + \dot{q}_i\mathrm{d}p_i - \frac{\partial L}{\partial t}\mathrm{d}t \\ =& \frac{\partial H}{\partial q_i}\mathrm{d}q_i + \frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial t}\mathrm{d}t \end{aligned} \tag{108}$$

Comparing the two expressions, we have

$$\boxed{\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i, \qquad \frac{\partial H}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}} \tag{109}$$

These results are known as *Hamilton's equations*.

2.1.3 Conservation of the Hamiltonian

We can also calculate the time derivative of the Hamiltonian

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} =& \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) \\ =& \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = \frac{\partial H}{\partial t} \end{aligned} \tag{110}$$

which we summarize below

$$\boxed{\frac{\partial H}{\partial t} = \frac{\mathrm{d}H}{\mathrm{d}t}} \tag{111}$$

We can conclude that if the Hamiltonian has mo explicit time dependence (*i.e.* $\partial H/\partial t = 0$), then the Hamiltonian is conserved.

2.2 Hamilton's Principle of Least Action

The phase space action is

$$S = \int \mathrm{d}t \left(\sum_i p_i \frac{\mathrm{d}q_i}{\mathrm{d}t} - H(q,p,t) \right) \tag{112}$$

Let us now perform the variation of this action in which we regard $q_i(t)$ and $p_i(t)$ as independent functions

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \mathrm{d}t \sum_i \left(\delta p_i \frac{\mathrm{d}q_i}{\mathrm{d}t} + p_i \frac{\mathrm{d}\delta q_i}{\mathrm{d}t} - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i \right) \\ &= \sum_i p_i \delta q_i \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathrm{d}t \sum_i \left[\delta p_i \left(\frac{\mathrm{d}q_i}{\mathrm{d}t} - \frac{\partial H}{\partial p_i} \right) + \left(-\frac{\mathrm{d}p_i}{\mathrm{d}t} - \frac{\partial H}{\partial q_i} \right) \delta q_i \right] \end{aligned} \tag{113}$$

Thus demanding that $\delta S = 0$ for variations which vanish as the boundary $\delta q_i(t_1) = \delta q_i(t_2) = 0$, which implies Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{114}$$

2.3 Poisson Brackets

The total time derivative of $A(q,p,t)$ is

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} =& \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i \right) \\ =& \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \\ =& \frac{\partial A}{\partial t} + \{A, H\} \end{aligned} \tag{115}$$

where the quantity

$$\{A, H\} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \tag{116}$$

is known as *Poisson bracket*. The Poisson brackets for coordinates and momenta

$$\{q_i, q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k} \right) = 0 \tag{117}$$

$$\{p_i, p_j\} = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0 \tag{118}$$

$$\{q_i, p_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = \sum_k \delta_{ik}\delta_{jk} = \delta_{ij} \tag{119}$$

We can also write Hamilton's equations in terms of Poisson brackets as

$$\dot{p}_i = \{p_i, H\}, \qquad \dot{q}_i = \{q_i, H\} \tag{120}$$

2.3.1 Poisson Brackets and Quantum Mechanics

Compare the Poisson brackets in classical mechanics and the canonical commutation relations in quantum mechanics

Classic	Quantum
$\{q_i, q_j\} = 0$	$[\hat{q}_i, \hat{q}_j] = 0$
$\{p_i, p_j\} = 0$	$[\hat{p}_i, \hat{p}_j] = 0$
$\{q_i, p_j\} = \delta_{ij}$	$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$

Table 1: The Poisson brackets in classical mechanics and the canonical commutation relations in quantum mechanics

2.3.2 Properties of Poisson Brackets

1. Anti-symmetric
$$\{A, B\} = -\{B, A\} \tag{121}$$

2.
$$\{A, BC\} = \{A, B\}C + B\{A, C\} \tag{122}$$

3. Jacobi identity
$$\{A, \{B, C\}\} + \{B, \{C, A\}\}, \{C, \{A, B\}\} = 0 \tag{123}$$

2.4 Canonical Transformations

The canonical transformation means a transformation which preserve Poisson brackets. Considering the transformation

$$q \to Q(q, p), \qquad p \to P(q, p) \tag{124}$$

they take the same Poisson brackets, *i.e.*

$$\{f, g\}_{Q,P} = \{f, g\}_{q,p} \tag{125}$$

2.4.1 Infinitesimal Canonical Transformations

The general form of an infinitesimal canonical transformation is

$$\delta q_i = \{q_i, G(q,p,t)\}\delta\lambda, \qquad \delta p_i = \{p_i, G(q,p,t)\}\delta\lambda \tag{126}$$

We define the phase space coordinates

$$\rho_I = \begin{cases} q_I & I = 1, \cdots, N \\ p_{I-N} & I = N+1, \cdots, 2N \end{cases} \tag{127}$$

then the form of the Poisson brackets are

$$\begin{aligned}\{\rho_I, \rho_J\} =& \Omega_{IJ} \\ =& \begin{cases} 0 & 1 \leq I \leq N, 1 \leq J \leq N \\ \delta_{I,J-N} & 1 \leq I \leq N, N+1 \leq J \leq 2N \\ -\delta_{(I-N)J} & N+1 \leq I \leq 2N, 1 \leq J \leq N \\ 0 & N+1 \leq I \leq 2N, N+1 \leq J \leq 2N \end{cases}\end{aligned} \tag{128}$$

Consider a infinitesimal change of coordinates $\rho_I \to \rho_I + \delta\rho_O$. The Poisson bracket will change as

$$\delta\rho_I, \rho_J = \delta\Omega_{IJ} = \{\rho_I, \delta\rho_J\} + \{\delta\rho_I, \rho_J\} = 0 \tag{129}$$

According to the Jacobi identity $\{A, \{B, C\} + \{B, \{C, A\}\} + \{C, \{A, B\}\}$, we have

$$\{\rho_I, \{\rho_J, G\}\} + \{\rho_J, \{G, \rho_I\}\} + \{G, \underbrace{\{\rho_I, \rho_J\}}_{=\Omega_{IJ}}\} = 0 \tag{130}$$

thus

$$\boxed{\{\rho_I, \{\rho_J, G\}\} + \{\rho_J, \{G, \rho_I\}\} = 0} \tag{131}$$

for any $G(q,p,t)$.

2.4.2 Noether Charge as the Generator

In Lagrange frame, the infinitesimal form of the continuous global symmetry transformation is

$$\delta q_i = F_i(q, \dot{q})\delta\lambda \tag{132}$$

Under such a global transformation we assume the Lagrangian varies as

$$\delta L = \frac{\mathrm{d}A}{\mathrm{d}t}\delta\lambda \tag{133}$$

and the corresponding generator expressed as

$$G = \sum_i \frac{\partial L}{\partial \dot{q}_i} F_i - A \tag{134}$$

In phase space

$$\delta q_i = F_i(q,p)\delta\lambda, \qquad \delta p_i = G_i(q,p)\delta\lambda, \qquad \delta L = \frac{\mathrm{d}A}{\mathrm{d}t}\delta\lambda \tag{135}$$

so the generator

$$\boxed{\begin{aligned} G &= \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}F_i(p,q) + \frac{\partial L}{\partial \dot{p}_i}G_i(p,q)\right) - A \\ &= \sum_i p_i F_i(p,q) - A \end{aligned}} \tag{136}$$

2.4.3 Generator of Rotations in two dimensions

Consider a two-dimensional problem with Cartesian coordinates (x, y)

$$H = \frac{p_x^2 + p_y^2}{2m} + V(\sqrt{x^2, y^2}) \tag{137}$$

Consider an infinitesimal rotation $\delta\theta$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\delta\theta & -\sin\delta\theta \\ \sin\delta\theta & \cos\delta\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y\delta\theta \\ y + x\delta\theta \end{pmatrix} \tag{138}$$

so we have the changes

$$\delta x = -y\delta\theta, \qquad \delta y = x\delta\theta \tag{139}$$

Since L is invariant *i.e.* $\delta L = 0$, the generator is easily seen to be

$$G = p_x F_x + p_y F_y = p_x(-y) + p_y x = xp_y - yp_x = L_z \tag{140}$$

To check explicitly

$$\delta x = \frac{\partial G}{\partial p_x}\delta\lambda = -y\delta\lambda, \qquad \delta y = \frac{\partial G}{\partial p_y}\delta\lambda = x\delta\lambda \tag{141}$$

$$\delta p_x = -\frac{\partial G}{\partial x}\delta\lambda = -p_y\delta\lambda, \qquad \delta p_y = -\frac{\partial G}{\partial y}\delta\lambda = p_x\delta\lambda \tag{142}$$

2.4.4 Generator of Time translations

Noether charge associated with time translations is the Hamiltonian, so we choose $G = H$ as the generator.

$$\delta q_i = \{q_i, H\}\delta t = \dot{q}_i\delta t, \qquad \delta p_i = \{p_i, H\}\delta t = \dot{p}_i\delta t \tag{143}$$

2.5 Hamilton-Jacobi Formulation

2.5.1 Hamilton-Jacobi Equation

If we evaluate the action

$$S = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \left[\sum_{i=1}^{n} p_i \frac{\mathrm{d}q_i}{\mathrm{d}\lambda} - H(q,p,t)\frac{\mathrm{d}t}{\mathrm{d}\lambda} \right] \tag{144}$$

with the variation $q_i \to q_i + \delta q_i$, $p_i \to p_i + \delta p_i$ and $t \to t + \delta t$. Follows by the chain rule that

$$\begin{aligned}
\delta S &= \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \left(\delta p_i \frac{\mathrm{d}q_i}{\mathrm{d}\lambda} + p_i \frac{\mathrm{d}}{\mathrm{d}\lambda}\delta q_i - \frac{\partial H}{\partial q_i}\delta q_i \frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\partial H}{\partial p_i}\delta p_i \frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\partial H}{\partial t}\delta t \frac{\mathrm{d}t}{\mathrm{d}\lambda} - H\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta t \right) \\
&= \left(\sum_{i=1}^{N} p_i \delta q_i \right) \Bigg|_{\lambda_1}^{\lambda_2} - (H\delta t)\big|_{\lambda_1}^{\lambda_2} + \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sum_{i=1}^{N} \delta p_i \left(\frac{\mathrm{d}q_i}{\mathrm{d}\lambda} - \frac{\partial H}{\partial p_i}\frac{\mathrm{d}t}{\mathrm{d}\lambda} \right) \\
&\quad + \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sum_{i=1}^{N} \delta q_i \left(-\frac{\mathrm{d}p_i}{\mathrm{d}\lambda} - \frac{\partial H}{\partial q_i}\frac{\mathrm{d}t}{\mathrm{d}\lambda} \right) + \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \delta t \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\partial H}{\mathrm{d}\lambda} \right) \\
&= \left(\sum_{i=1}^{N} p_i \delta q_i \right) \Bigg|_{\lambda_1}^{\lambda_2} - (H\delta t)\big|_{\lambda_1}^{\lambda_2} \\
&= \sum_{i=1}^{N} p_i(t_2)\delta q_i(t_2) - \sum_{i=1}^{N} p_i(t_1)\delta q_i(t_1) - H(t_2)\delta t_2 + H(t_1)\delta t_1
\end{aligned} \tag{145}$$

The total derivative of S

$$\mathrm{d}S = \sum_{i=1}^{N} \frac{\partial S}{\partial q_i(t_2)}\mathrm{d}q_i(t_2) + \frac{\partial S}{\partial t_2}\mathrm{d}t_2 + \sum_{i=1}^{N} \frac{\partial S}{\partial q_i(t_1)}\mathrm{d}q_i(t_1) + \frac{\partial S}{\partial t_2}\mathrm{d}t_1 \tag{146}$$

Comparing the equations we see that

$$\boxed{p_i(t_2) = \frac{\partial S}{\partial q_i(t_2)}, \quad H(t_2) = -\frac{\partial S}{\partial t_2}, \quad p_i(t_1) = -\frac{\partial S}{\partial q_i(t_1)}, \quad H(t_1) = \frac{\partial S}{\partial t_1}} \tag{147}$$

For example, given a Hamiltonian for a non-relativistic particle in one dimension

$$H = \frac{p}{2m} + V(x) \tag{148}$$

Using the above equations at time $t = t_2$ we have

$$\boxed{-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x)} \tag{149}$$

This is known as the *Hamilton-Jacobi equation*.

2.5.2 Action and Quantum Mechanics

The Schrödinger equation

$$\frac{i\hbar}{\psi}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2} + V \tag{150}$$

Here we define $\psi = \mathrm{e}^{\frac{iS}{\hbar}}$, the equations becomes

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{i\hbar}{2m}\frac{\partial^2 S}{\partial x^2} + V \tag{151}$$

when $\hbar \to 0$ (classical limit), the equation regresses to the classical case, has the same formation with Hamilton-Jacobi equation.

2.5.3 Constants of Motion

The variation of the action

$$\delta S = \left(\sum_i p_i(t_2)\delta q_i(t_2) - H(t_2)\delta t_2\right) - \left(\sum_i p_i(t_1)\delta q_i(t_1) - H(t_1)\delta t_1\right) \tag{152}$$

Consider a system in which the Hamiltonian is independent of t

$$\frac{\partial H}{\partial t} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}H}{\mathrm{d}t} = 0,\ H(t_1) = H(t_2) = E \tag{153}$$

Now define the new action via the Legendre transform with respect to the initial data

$$\tilde{S} = S - Et_1 \tag{154}$$

then

$$\begin{aligned}
\delta\tilde{S} =& \delta S - t_1\delta E - E\delta t_1 \\
=& \sum_i p_i(t_2)\delta q_i(t_2) - E\delta t_2 - \sum_i p_i(t_1)\delta q_i(t_1) + E\delta t_1 - t_1\delta E - E\delta t_1 \\
=& \sum_i p_i(t_2)\delta q_i(t_2) - E\delta t_2 - \sum_i p_i(t_1)\delta q_i(t_1) - t_1\delta E \\
=& \tilde{S}(q_i(t_2), t_2; q_i(t_1), E)
\end{aligned} \tag{155}$$

We thus conclude that

$$\boxed{\frac{\delta\tilde{S}}{\delta E} = -t_1} \tag{156}$$

Example
The simplest example is a free particle in one dimension

$$H = \frac{p^2}{2m} = E, \qquad q = x \tag{157}$$

Since the energy is conserved, we will perform a Legendre transformation with respect to the initial time

$$\tilde{S} = S - Et_1 \tag{158}$$

and we have

$$\frac{\partial \tilde{S}}{\partial x_2} = p, \qquad \frac{\partial \tilde{S}}{\partial t_2} = -E \tag{159}$$

whose solutions are

$$\tilde{S} = -Et_2 + W(x_2, x_1) \tag{160}$$

$$E = \frac{1}{2m}\left(\frac{\partial \tilde{S}}{\partial x_2}\right)^2 = \frac{1}{2m}\left(\frac{\partial W}{\partial x_2}\right)^2 \tag{161}$$

The equation is easy to solve as

$$W = \pm\sqrt{2mE}(x_2 - x_1) \tag{162}$$

hence

$$\tilde{S} = -Et_2 \pm \sqrt{2mE}(x_2 - x_1) \tag{163}$$

So the action is

$$\begin{aligned} S =& -E(t_2 - t_1) \pm \sqrt{2mE}(x_2 - x_1) \\ =& -\frac{p^2}{2m}(t_2 - t_1) + p(x_2 - x_1) \end{aligned} \tag{164}$$

2.5.4 Perturbation Theory

Consider a potential of the form

$$V = V_0 + gV_1 \tag{165}$$

where $|gV_1| \ll |V_0|$. The Hamilton-Jacobi equation for V_0

$$-\frac{\partial S_0}{\partial t} = \frac{1}{2m}\left(\frac{\partial S_0}{\partial x}\right)^2 + V_0(x) \tag{166}$$

Now if we write $S = S_0 + gS'$ and substitute into the full Hamilton-Jacobi equation

$$\frac{\partial S'}{\partial t} + \frac{1}{m}\frac{\partial S_0}{\partial x}\frac{\partial S'}{\partial x} = -\frac{g}{2m}\left(\frac{\partial S'}{\partial x}\right)^2 - V_1(x) \approx -V_1(x) \tag{167}$$

We can develop a perturbative expansion

$$S = S_0 + gS' = S_0 + gS_1 + g^2 S_2 + g^3 S_3 + \cdots = \sum_{n=0}^{\infty} g^n S_n \tag{168}$$

Example

Consider the simple case where $V_0 = 0$ so that S_0 is the action for a free particle.

$$S_0(x,t;x_0,t_0) = -\frac{p^2}{2m}(t-t_0) + p(x-x_0) \tag{169}$$

so we have

$$\frac{\partial S_0}{\partial x} = p \tag{170}$$

and the equation for first order perturbation S_1 is

$$\frac{\partial S_1}{\partial t} + \frac{p}{m}\frac{\partial S_1}{\partial x} = -V_1(x) \tag{171}$$

Perform the change of variables $x = X + \frac{pt}{m}$

$$\begin{aligned}
\mathrm{d}S_1 &= \left(\frac{\partial S_1}{\partial x}\right)_t \mathrm{d}x + \left(\frac{\partial S_1}{\partial t}\right)_x \mathrm{d}t \\
&= \left(\frac{\partial S_1}{\partial X}\right)_t \left(\mathrm{d}X + \frac{p}{m}\mathrm{d}t\right) + \left(\frac{\partial S_1}{\partial t}\right)_X \mathrm{d}t \\
&= \left(\frac{\partial S_1}{\partial X}\right)_t \mathrm{d}X + \left[\frac{p}{m}\left(\frac{\partial S_1}{\partial X}\right)_t + \left(\frac{\partial S_1}{\partial t}\right)_X\right] \mathrm{d}t \\
&= \left(\frac{\partial S_1}{\partial X}\right)_t \mathrm{d}X - V_1(X)\mathrm{d}t
\end{aligned} \tag{172}$$

so

$$\left(\frac{\partial S_1}{\partial t}\right)_X = -V_1\left(x + \frac{pt}{m}, t\right) \tag{173}$$

$$S_1 = -\int_{t_0}^{t} \mathrm{d}t V_1\left(x + \frac{pt}{m}, t\right) \tag{174}$$

The next order perturbation is then

$$\frac{\partial S_2}{\partial t} = -\frac{1}{2m}\left(\frac{\partial S_1}{\partial X}\right)^2 \tag{175}$$

whose solution is

$$S_2 = -\frac{1}{2m}\int_{t_0}^{t} \mathrm{d}t' \left(\int_{t_0}^{t'} \mathrm{d}t' V_1(X + pt'/m, t')\right)^2 \tag{176}$$

Together, the solution of the HJ equation to second order in perturbation is

$$\begin{aligned}
S = &-\frac{p^2}{2m}(t-t_0) + p(x-x_0) - g\int_{t_0}^{t} \mathrm{d}t V_1\left(x + \frac{pt}{m}, t\right) \\
&- \frac{g^2}{2m}\int_{t_0}^{t} \mathrm{d}t' \left(\int_{t_0}^{t'} \mathrm{d}t' V_1(X + pt'/m, t')\right)^2 + \cdots
\end{aligned} \tag{177}$$

2.6 Constraints

The action in phase space

$$S = \int \mathrm{d}t \left[\sum_i p_i \frac{\mathrm{d}q}{\mathrm{d}t} - H(q,p) \right] \tag{178}$$

include a set of constraints $C_\alpha(q,p) = 0$ with Lagrangian multiples λ_α, and the action becomes

$$S = \int \mathrm{d}t \left[\sum_i p_i \frac{\mathrm{d}q}{\mathrm{d}t} - H(q,p) - \sum_\alpha \lambda_\alpha C_\alpha \right] \tag{179}$$

The effective Hamiltonian

$$H^*(q,p) = H(q,p) + \sum_\alpha \lambda_\alpha C_\alpha \tag{180}$$

The equations of motion

$$\boxed{\dot{q}_i = \{q_i, H^*\} = \frac{\partial H^*}{\partial p_i}, \qquad \dot{p}_i = \{p_i, H^*\} = -\frac{\partial H^*}{\partial q_i}} \tag{181}$$

Consider a system with one constraint $C_1(q,p) = 0$

$$S = \int \mathrm{d}t \left(\sum_i p_i \dot{q}_i - H - \lambda_1 C_1 \right) \tag{182}$$

then

$$\dot{C}_1 = \{C_1, H^*\} = \{C_1, H + \lambda_1 C_1\} = \{C_1, H\} = C_2 \tag{183}$$

The secondary constraint C_2 not automatically equals to 0. We can write the secondary effective Hamiltonian as

$$H^{**} = H + \lambda_1 C_1 + \lambda_2 C_2 \tag{184}$$

and

$$\dot{C}_1 = \{C_1, H\} + \lambda_1 \{C_1, C_1\} + \lambda_2 \{C_1, C_2\} = C_2 + \lambda_2 \{C_1, C_2\} \tag{185}$$

$$\dot{C}_2 = \{C_2, H\} + \lambda_1 \{C_2, C_1\} + \lambda_2 \{C_2, C_2\} = \{C_2, H\} + \lambda_1 \{C_2, C_1\} \tag{186}$$

If $\{C_1, C_2\} = 0$ and $\{C_2, H\} \neq 0$, then we write the tertiary constraint $C_3 = \{C_2, H\}$ and repeat the process.
If $\{C_1, C_2\} \neq 0$ and $\{C_2, H\} \neq 0$, then we have

$$\boxed{\lambda_1 = \frac{\{C_2, H\}}{\{C_1, C_2\}}, \qquad \lambda_2 = 0} \tag{187}$$

Example
Consider the 2D harmonic oscillator

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega^2 y^2 \tag{188}$$

with constrain $C_1 = p_y - \frac{1}{2}\beta x^2 = 0$.

$$C_2 = \{C_1, H\} = -\frac{\partial C_1}{\partial p_y}\frac{\partial H}{\partial y} + \frac{\partial C_1}{\partial x}\frac{\partial H}{\partial p_x} = -\omega^2 y - \beta x p_x \tag{189}$$

The secondary constraint C_2 not automatically equals to 0.

$$\{C_2, H\} = \frac{\partial C_2}{\partial y}\frac{\partial H}{\partial p_y} + \frac{\partial C_2}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial C_2}{\partial p_x}\frac{\partial H}{\partial x} = -\omega^2 p_y - \beta p_x^2 + \beta\omega^2 x^2 \tag{190}$$

$$\{C_1, C_2\} = \beta^2 x^2 + \omega^2 > 0 \tag{191}$$

So we have

$$\lambda_1 = \frac{-\omega^2 p_y - \beta p_x^2 + \beta\omega^2 x^2}{\beta^2 x^2 + \omega^2}, \qquad \lambda_2 = 0 \tag{192}$$

2.6.1 Variations at Fixed Energy

We choose to focus on paths which have a fixed energy $H = E$, and the constrained action is

$$S = \int_{t_1}^{t_2} \mathrm{d}t \left[\boldsymbol{p}\cdot\dot{\boldsymbol{r}} - \frac{\boldsymbol{p}^2}{2m} - \lambda\left(\frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}) - E\right)\right] \tag{193}$$

with

$$\dot{\boldsymbol{r}} = \{\boldsymbol{r}, H^*\} = -\frac{\partial H^*}{\partial \boldsymbol{p}} = (1+\lambda)\frac{\boldsymbol{p}}{m} \quad \Rightarrow \quad \boldsymbol{p} = \frac{m}{1+\lambda}\dot{\boldsymbol{r}} \tag{194}$$

and substituting back in gives

$$S = \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{m}{2(1+\lambda)}\dot{\boldsymbol{r}}^2 - (1+\lambda)V(\boldsymbol{r}) + \lambda E\right] \tag{195}$$

The equation for λ is

$$\frac{m}{2(1+\lambda)^2}\dot{\boldsymbol{r}}^2 = E - V \quad \Rightarrow \quad 1+\lambda = \sqrt{\frac{m\dot{\boldsymbol{r}}^2}{2(E-V)}} \tag{196}$$

Substituting back in we obtain

$$\begin{aligned} S &= \int_{t_1}^{t_2} \mathrm{d}t(-E) + \int_{t_1}^{t_2} \mathrm{d}t\sqrt{2m(E-V)\dot{\boldsymbol{r}}^2} \\ &= -E(t_2 - t_1) + \int_{t_1}^{t_2} \mathrm{d}t\sqrt{2m(E-V)\dot{\boldsymbol{r}}^2} \end{aligned} \tag{197}$$

2.7 Dynamics in Phase Space

For example if the system is described by

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} = G\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, t\right) = G(x,y,z,t) \tag{198}$$

which introduce variables

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ G(x,y,z,t) \end{pmatrix} \tag{199}$$

2.7.1 First Order Systems

Example
Consider the *logistic* equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx - \sigma x^2 \tag{200}$$

which is equivalent to

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{x(k-\sigma x)} = \frac{1}{k}\left(\frac{1}{x} + \frac{\sigma}{k-\sigma x}\right) \tag{201}$$

$$t = \frac{1}{k}\left(\ln x - \ln(k-\sigma x)\right)\Big|_{x_0}^{x} = \frac{1}{k}\ln\left(\frac{x}{k-\sigma x}\right)\Big|_{x_0}^{x} = \frac{1}{k}\ln\left(\frac{x}{x_0}\frac{k-\sigma x_0}{k-\sigma x}\right) \tag{202}$$

and the solution is

$$x(t) = \frac{kx_0}{\sigma x_0 + (k-\sigma x_0)\exp(-kt)} \tag{203}$$

for initial condition $x(0) = x_0 > 0$. For the *critical point* $\dot{x} = 0$, we have two solutions $x_c = 0$ and $x_c = k/\sigma$. Let $x(t) = x_c + \epsilon\delta x(t)$

$$\begin{aligned} \dot{x} =& k(x_c + \epsilon\delta x(t)) - \sigma(x_c + \epsilon\delta x(t))^2 \\ =& \left(kx_c - \sigma x_c^2\right) + \epsilon(k\delta x - 2\sigma x_c\delta x) - \mathcal{O}(\epsilon^2) \\ =& \epsilon(k - 2\sigma x_c)\delta x \end{aligned} \tag{204}$$

So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x) = (k - 2\sigma x_c)\delta x \tag{205}$$

Look at linear stability of each critical point

- $x_c = 0 \quad \Rightarrow \quad \delta\dot{x} = k\delta x \quad \Rightarrow \quad \delta x(t) = \mathrm{e}^{kt}\delta x(0)$ (repeller)
- $x_c = k/\sigma \quad \Rightarrow \quad \delta\dot{x} = -k\delta x \quad \Rightarrow \quad \delta x(t) = \mathrm{e}^{-kt}\delta x(0)$ (attractor)

We summarise the procedure

$$\dot{x} = F(x) \tag{206}$$

1. Find critical point $F(x_c) = 0$
2. Perturb around $x = x_c + \epsilon\delta x$, and

$$\delta\dot{x} = F'(x_c)\delta x \tag{207}$$

3.
   - $\Re[F'(x_c)] > 0 \to$ unstable (repeller)
   - $\Re[F'(x_c)] < 0 \to$ stable (attractor)
   - $\Re[F'(x_c)] = 0$ and $\Im[F'(x_c)] \neq 0 \to$ oscillatory (libration)

2.7.2 Second Order Systems (2D Phase Space)

For the second order system, a critical point $\boldsymbol{x}_c = \{x_c, y_c\}$ is a point at which both F and G vanish $F(x_c, y_c) = G(x_c, y_c) = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix} \tag{208}$$

Everywhere except at critical point, the slope of a trajectory in the phase plane is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{G(x,y)}{F(x,y)} \tag{209}$$

Example
Consider the second order differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{1}{m}\frac{\mathrm{d}V}{\mathrm{d}x}, \quad y = \dot{x} \tag{210}$$

which can expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -\frac{1}{m}\frac{\mathrm{d}V}{\mathrm{d}x} \end{pmatrix} \tag{211}$$

At the critical point

$$y = 0, \qquad \frac{\mathrm{d}V}{\mathrm{d}x} = 0 \tag{212}$$

Consider small displacements from critical points (x_c, y_c)

$$x = x_c + \delta x, \qquad y = 0 + \delta y \tag{213}$$

so we have

$$\delta\dot{x} = \delta y \tag{214}$$

$$\begin{aligned}
\delta\dot{y} &= -\frac{1}{m}\frac{\mathrm{d}V(x_c + \delta x)}{\mathrm{d}x} \\
&= -\frac{1}{m}\left[\frac{\mathrm{d}V(x_c)}{\mathrm{d}x} + \frac{\mathrm{d}^2 V(x_c)}{\mathrm{d}x^2}\delta x + \mathcal{O}((\delta x)^2)\right] \\
&= -\frac{1}{m}\frac{\mathrm{d}^2 V(x_c)}{\mathrm{d}x^2}\delta x
\end{aligned} \tag{215}$$

- If
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\delta x) = -\frac{1}{m}\frac{\mathrm{d}^2 V(x_c)}{\mathrm{d}x^2}\delta x = -\omega^2 \delta x \tag{216}$$
then
$$\delta x = A\mathrm{e}^{\pm i\omega t} \quad \text{libration} \tag{217}$$
- If
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\delta x) = -\frac{1}{m}\frac{\mathrm{d}^2 V(x_c)}{\mathrm{d}x^2}\delta x = \Omega^2 \delta x \tag{218}$$
then
$$\delta x = A\mathrm{e}^{\pm \Omega t} \quad \text{unstable} \tag{219}$$

We summarize the procedure for 2D phase space
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix} \tag{220}$$

1. Find critical points $F(x_c, y_c) = G(x_c, y_c) = 0$
2. Perturb around critical points $x = x_c + \delta x$ and $y = y_c + \delta y$
$$F(x_c + \delta x, y_c + \delta y) = F(x_c, y_c) + \delta x \frac{\partial}{\partial x} F(x_c, y_c) + \delta y \frac{\partial}{\partial y} F(x_c, y_c) \tag{221}$$
$$G(x_c + \delta x, y_c + \delta y) = G(x_c, y_c) + \delta x \frac{\partial}{\partial x} G(x_c, y_c) + \delta y \frac{\partial}{\partial y} G(x_c, y_c) \tag{222}$$
which can be expressed as
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \partial_x F & \partial_y F \\ \partial_x G & \partial_y G \end{pmatrix}\Bigg|_{(x_c,y_c)} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \mathbf{M}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \lambda \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \tag{223}$$
So we have
$$\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = \mathrm{e}^{\lambda t}\begin{pmatrix} \delta x(0) \\ \delta y(0) \end{pmatrix} \tag{224}$$
where λ is the eigenvalue of $\mathbf{M}$.
3. - $\Re[\lambda] > 0 \to$ unstable
   - $\Re[\lambda] \leq 0 \to$ stable
   - $\Re[\lambda] = 0$ and $\Im[\lambda] \neq 0 \to$ libration

3 Rigid Bodies

We define the total mass

$$M = \sum_a m_a \tag{225}$$

and the position of *center of mass*

$$\boldsymbol{R} = \frac{\sum_a m_a \boldsymbol{r}_a}{M} \tag{226}$$

The *total momentum* of the system

$$\boldsymbol{P} = \sum_a \boldsymbol{p}_a = \sum_a m_a \dot{\boldsymbol{r}}_a = M\dot{\boldsymbol{R}} \tag{227}$$

The *total angular momentum* of the system

$$\boldsymbol{L} = \sum_a m_a \boldsymbol{r}_a \times \dot{\boldsymbol{r}}_a 09 \tag{228}$$

The position $\boldsymbol{r}_a$ can be expressed as $\boldsymbol{r}_a = \boldsymbol{R} + \boldsymbol{r}_a^*$ and $\sum_a m_a \boldsymbol{r}_a^* = 0$. The total angular angular

$$\begin{aligned}
\boldsymbol{L} =& \sum_a m_a(\boldsymbol{R} + \boldsymbol{r}_a^*) \times (\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}}_a^*) \\
=& \sum_a m_a \boldsymbol{R} \times \dot{\boldsymbol{R}} + \sum_a m_a \boldsymbol{R} \times \dot{\boldsymbol{r}}_a^* + \sum_a m_a \boldsymbol{r}_a^* \times \dot{\boldsymbol{R}} + \sum_a m_a \boldsymbol{r}_a^* \times \dot{\boldsymbol{r}}_a^* \\
=& \sum_a m_a \boldsymbol{R} \times \dot{\boldsymbol{R}} + \sum_a m_a \boldsymbol{r}_a^* \times \dot{\boldsymbol{r}}_a^* \\
=& \boldsymbol{L}_{\text{com}} + \boldsymbol{L}^*
\end{aligned} \tag{229}$$

where

$$\boldsymbol{L}^* = \sum_a m_a \boldsymbol{r}_a^* \times \dot{\boldsymbol{r}}_a^* \tag{230}$$

is called the angular momentum about the center of mass.

3.1 Action for a Rotating Rigid Body

The kinetic energy of the center of mass

$$\begin{aligned}
T =& \sum_a \frac{1}{2} m_a \dot{\boldsymbol{r}}_a^2 = \sum_a \frac{1}{2} m_a (\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}}^*)^2 \\
=& \sum_a \frac{1}{2} m_a \dot{\boldsymbol{R}}^2 + \sum_a m_a \dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{r}}_a^* + \sum_a \frac{1}{2} m_a \dot{\boldsymbol{r}}_a^{*2} \\
=& \underbrace{\frac{1}{2} M \dot{\boldsymbol{R}}^2}_{\text{com kinetic energy}} + \underbrace{\sum_a \frac{1}{2} m_a \dot{\boldsymbol{r}}_a^{*2}}_{\text{rotational kinetic energy}}
\end{aligned} \tag{231}$$

Since the body is assumed rigid, all it can do is rotate relative to its center of mass motion. Let $\boldsymbol{\omega}$ denote the angular velocity of the rotation, then we have

$$\boxed{\dot{\boldsymbol{r}}_a^* = \boldsymbol{\omega} \times \boldsymbol{r}_a^*} \tag{232}$$

this preserves all the scalar products

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\boldsymbol{r}_a^* - \boldsymbol{r}_b^*|^2) = 2(\boldsymbol{r}_a^* - \boldsymbol{r}_b^*) \cdot (\dot{\boldsymbol{r}}_a^* - \dot{\boldsymbol{r}}_b^*) = 2(\boldsymbol{r}_a^* - \boldsymbol{r}_b^*) \cdot (\boldsymbol{\omega} \times (\boldsymbol{r}_a^* - \boldsymbol{r}_b^*)) = 0 \tag{233}$$

Putting this together, the action is

$$\boxed{S = \int \mathrm{d}t \left[\frac{1}{2} M \dot{\boldsymbol{R}}^2 + \frac{1}{2} \sum_a m_a (\boldsymbol{\omega} \times \boldsymbol{r}_a^*)^2\right]} \tag{234}$$

3.1.1 Rotation around a Pivot

If we assume that the pivot point is taken at $\boldsymbol{r} = 0$, then we have the constraints

$$\dot{\boldsymbol{R}} = \boldsymbol{\omega} \times \boldsymbol{R} \tag{235}$$

When this is true we have

$$\dot{\boldsymbol{r}}_a = \boldsymbol{\omega} \times \boldsymbol{r}_a \tag{236}$$

and to the action can be evaluated as

$$\boxed{S = \int \mathrm{d}t \sum_a \frac{1}{2} m_a (\boldsymbol{\omega} \times \boldsymbol{r}_a)^2} \tag{237}$$

3.1.2 Switch Vector to Index Notation

We have known the relation

$$(\boldsymbol{A} \times \boldsymbol{B})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_j B_k \tag{238}$$

Hence

$$\begin{aligned}
(\boldsymbol{\omega} \times \boldsymbol{r}_a^*)^2 &= \sum_{i=1}^{3} (\boldsymbol{\omega} \times \boldsymbol{r}_a^*)_i^2 = \sum_{i=1}^{3} \left(\sum_{j,k=1}^{3} \varepsilon_{ijk} \omega_j r_{ak}^*\right) \left(\sum_{l,m=1}^{3} \varepsilon_{ilm} \omega_l r_{am}^*\right) \\
&= \sum_{j,k,l,m=1}^{3} \left(\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{ilm}\right) \omega_j r_{ak}^* \omega_l r_{am}^* \\
&= \sum_{j,k,l,m=1}^{3} \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) \omega_j r_{ak}^* \omega_l r_{am}^* \\
&= \sum_{j,l=1}^{3} \omega_j \omega_l \sum_{k,m=1}^{3} \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) r_{ak}^* r_{am}^* \\
&= \sum_{j,l=1}^{3} \omega_j \omega_l \left(\delta_{jl} \boldsymbol{r}_a^{*2} - r_{al}^* r_{aj}^*\right) = \sum_{i,j=1}^{3} \omega_i \omega_j \left(\delta_{ij} \boldsymbol{r}_a^{*2} - r_{ai}^* r_{aj}^*\right)
\end{aligned} \tag{239}$$

Define *the momenta of inertia*

$$\boxed{I_{ij} = \sum_a m_a \left(\delta_{ij} \boldsymbol{r}_a^2 - r_{ai} r_{aj}\right)} \tag{240}$$

the equivalent quantity defined relative to the center of mass

$$\boxed{I_{ij}^* = \sum_a m_a \left(\delta_{ij} \boldsymbol{r}_a^{*2} - r_{ai}^* r_{aj}^*\right)} \tag{241}$$

With this notation, we can rewrite the kinetic energy T and the action S

$$\boxed{T = \frac{1}{2} \sum_{i,j=1}^{3} \omega_i I_{ij} \omega_j} \tag{242}$$

$$\boxed{S = \int \mathrm{d}t \frac{1}{2} \omega_i I_{ij} \omega_j} \tag{243}$$