Revision for 2024 MMP Exam

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(*) means very important.

1 Vector Spaces and Tensors

(1) **Summation convention**

$$C_{ij} = \sum_k A_{ik}B_{kj} = A_{ik}B_{kj} \tag{1}$$

- Free indices: i, j
- Dummy index: k

Note: in any one term of an expression, indices may appear 0, 1, 2 times.

(2) **Levi-Civita symbol**

$$\varepsilon_{ijk} = \begin{cases} 1, & (i,j,k) = (1,2,3),(2,3,1),(3,1,2) \\ -1, & (i,j,k) = (3,2,1),(2,1,3),(1,3,2) \\ 0, & \text{Otherwise} \end{cases} \tag{2}$$

Two important relations:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \quad \Leftrightarrow \quad c_i = \varepsilon_{ijk}a_jb_k \tag{3}$$

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{4}$$

(3) **Tensor calculus**

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = (\partial_i, \partial_j, \partial_k) \tag{5}$$

- Gradient

$$(\boldsymbol{\nabla}\phi)_i = \partial_i\phi \tag{6}$$

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- Divergence
$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \partial_i F_i \tag{7}$$
- Curl
$$(\boldsymbol{\nabla} \times \boldsymbol{F})_i = \varepsilon_{ijk} \partial_j F_k \tag{8}$$

(4) **Transforms under rotations (*)**
The rotation matrix $\mathbf{L}$ is orthogonal, with
$$L_{ij} L_{ik} = L_{ji} L_{ki} = \delta_{jk} \tag{9}$$

- A scalar $\phi(x)$
$$\phi(x) \to \phi'(x') = \phi(x) \tag{10}$$
- A vector $v_i(x)$
$$v_i(x) \to v_i'(x') = L_{ij} v_j(x) \tag{11}$$
- A rank 2 tensor $T_{ij}(x)$
$$T_{ij}(x) \to T_{ij}'(x') = L_{il} L_{jm} T_{lm}(x) \tag{12}$$

2 Green Functions

(1) **Wronskian**
Consider homogeneous second order differential equations
$$y'' + p(x) y' + q(x) y = 0 \tag{13}$$
and $y_1(x)$ and $y_2(x)$ are linearly independent solutions with non-vanishing Wronskian
$$\boxed{W(x) = y_1 y_2' - y_1' y_2} \tag{14}$$
Note: $W \neq 0 \;\Leftrightarrow\; y_1$ and y_2 are independent.
Here are two important consequences
$$y_2(x) = y_1(x) \int^x \frac{W(\tilde{x})}{y_1^2(\tilde{x})} \mathrm{d}\tilde{x} \tag{15}$$
$$W(x) = \pm c \exp\left[- \int^x p(\tilde{x}) \mathrm{d}\tilde{x} \right] \tag{16}$$

(2) **More generally**
$$y'' + p(x) y' + q(x) y = f(x) \tag{17}$$
Then $y(x) = a y_1(x) + b y_2(x) + y_0(x)$ is a solution of the inhomogeneous ODE. The particular integral
$$y_0 = u(x) y_1(x) + v(x) y_2(x) \tag{18}$$

subject to the constraint

$$\boxed{u'y_1 + v'y_2 = 0} \tag{19}$$

and the ODE is simplified as

$$\boxed{u'y_1' + v'y_2' = f} \tag{20}$$

So the solutions are

$$u'(x) = -\frac{y_2(x)f(x)}{W(x)}, \qquad v'(x) = \frac{y_1(x)f(x)}{W(x)} \tag{21}$$

(3) **Green function** (*)

$$\boxed{\mathcal{L}_x G(x,\tilde{x}) = \delta(x-\tilde{x})} \tag{22}$$

$G(x,\tilde{x})$ is continuous at $x = \tilde{x}$

$$[G(x,\tilde{x})]_{x\to\tilde{x}_-}^{x\to\tilde{x}_+} = 0 \tag{23}$$

$\frac{\partial}{\partial x}G(x,\tilde{x})$ has a unit discontinuity at $x = \tilde{x}$

$$\left[\frac{\partial G(x,\tilde{x})}{\partial x}\right]_{x\to\tilde{x}_-}^{x\to\tilde{x}_+} = 1 \tag{24}$$

(4) **Boundary conditions**
The particular integral of the OED

$$y_0 = \int_\alpha^\beta \mathrm{d}\tilde{x} G(x,\tilde{x})f(\tilde{x}) \tag{25}$$

a) **Homogeneous initial conditions** $y(\alpha) = y'(\alpha) = 0$.

- For $x < \tilde{x}$, $G(x,\tilde{x}) = 0$ is a solution of the homogeneous equation that satisfies the boundary conditions $G(\alpha,\tilde{x}) = G'(\alpha,\tilde{x}) = 0$.
- For $x > \tilde{x}$, $G(x,\tilde{x}) = A(\tilde{x})y_1(x) + B(\tilde{x})y_2(x)$.
  i) G is continuous at $x = \tilde{x}$

$$A(\tilde{x})y_1(\tilde{x}) + B(\tilde{x})y_2(\tilde{x}) = 0 \tag{26}$$

  ii) G' has a unit discontinuity at $x = \tilde{x}$

$$A(\tilde{x})y_1'(\tilde{x}) + B(\tilde{x})y_2'(\tilde{x}) = 1 \tag{27}$$

  The solution is

$$A(\tilde{x}) = -\frac{y_2(\tilde{x})}{W(\tilde{x})}, \qquad B(\tilde{x}) = \frac{y_1(\tilde{x})}{W(\tilde{x})} \tag{28}$$

b) **Homogeneous two-point boundary Conditions** $y(\alpha) = y(\beta) = 0$.

$$G(x,\tilde{x}) = \begin{cases} A(\tilde{x})y_1(x) + B(\tilde{x})y_2(x), & x < \tilde{x} \\ C(\tilde{x})y_1(x) + D(\tilde{x})y_2(x), & x > \tilde{x} \end{cases} \tag{29}$$

- Boundary conditions: $G(\alpha, \tilde{x}) = G(\beta, \tilde{x}) = 0$

$$A(\tilde{x})y_1(\alpha) + B(\tilde{x})y_2(\alpha) = B(\tilde{x})y_2(\alpha) = 0 \quad \Rightarrow \quad B(\tilde{x}) = 0 \tag{30}$$

$$C(\tilde{x})y_1(\beta) + D(\tilde{x})y_2(\beta) = C(\tilde{x})y_1(\beta) = 0 \quad \Rightarrow \quad C(\tilde{x}) = 0 \tag{31}$$

so we have

$$G(x, \tilde{x}) = \begin{cases} A(\tilde{x})y_1(x), & \alpha < x < \tilde{x} \\ D(\tilde{x})y_2(x), & \tilde{x} < x < \beta \end{cases} \tag{32}$$

- Continuity of G and unit discontinuity of G' at $x = \tilde{x}$

$$A(\tilde{x})y_1(\tilde{x}) - D(\tilde{x})y_2(\tilde{x}) = 0 \tag{33}$$

$$A(\tilde{x})y_1'(\tilde{x}) - D(\tilde{x})y_2'(\tilde{x}) = 1 \tag{34}$$

so we have

$$A(\tilde{x}) = \frac{y_2(\tilde{x})}{W(\tilde{x})}, \qquad D(\tilde{x}) = \frac{y_1(\tilde{x})}{W(\tilde{x})} \tag{35}$$

3 Sturm-Liouville Theory

(1) **Self-adjoint form** (*)

$$\boxed{\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + \sigma(x)} \tag{36}$$

where $\rho(x) > 0$ and $x \in (a, b)$.
Note: being in self-adjoint form does NOT mean that a differential operator is self-adjoint, that depend on the operator and the specific Hilbert space.

(2) **Self-adjoint** (*)
A second order linear differential operator $\mathcal{L}$ is self-adjoint on Hilbert space $\mathcal{H}$ if

$$\boxed{\langle u, \mathcal{L}v\rangle = \langle v, \mathcal{L}u\rangle^*, \quad \forall u, v \in \mathcal{H}} \tag{37}$$

Consider $\mathcal{L}$ as in self-adjoint form,

$$\begin{aligned}
\langle u, \mathcal{L}v\rangle &= \int_a^b u^* \left[-(\rho v')' + \sigma v\right] \mathrm{d}x \\
&= - u^*\rho v'\big|_a^b + \int_a^b \left(u^{*\prime}\rho v' + u^*\sigma v\right) \mathrm{d}x \\
&= - u^*\rho v'\big|_a^b + u^{*\prime}\rho v\big|_a^b + \int_a^b \left(-(u^{*\prime}\rho)'v + u^*\sigma v\right) \mathrm{d}x \\
&=(-u^*\rho v' + u^{*\prime}\rho v)\big|_a^b + \int_a^b \left(-(u^{*\prime}\rho)' + u^*\sigma\right) v\mathrm{d}x \\
&=(-u^*\rho v' + u^{*\prime}\rho v)\big|_a^b + \left[\int_a^b \left(-(u'\rho)' + u\sigma\right) v^*\mathrm{d}x\right]^* \\
&=(-u^*\rho v' + u^{*\prime}\rho v)\big|_a^b + \langle v, \mathcal{L}u\rangle^*
\end{aligned} \tag{38}$$

$\mathcal{L}$ is self-adjoint on $\mathcal{H}$ if

$$\boxed{\rho\left[u^{*\prime}v - u^{*}v'\right]_a^b = 0} \tag{39}$$

(3) **Weight function** (*)
Consider the most generally operator

$$\boxed{\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(A(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - B(x)\frac{\mathrm{d}}{\mathrm{d}x} + C(x)} \tag{40}$$

where $x \in (a, b)$. A, B, C are real and $A(x) > 0$. We want to find the weight function, w, such that $\mathcal{L} = w\tilde{\mathcal{L}}$ is in self-adjoint form, *i.e.*

$$-(\rho y')' + \sigma y = w\left[-(Ay')' - By' + Cy\right] \tag{41}$$

gives

$$\frac{w'}{w} = \frac{B}{A}, \qquad \rho = Aw, \qquad \sigma = Cw \tag{42}$$

then we can choose $w(x)$ such that

$$\boxed{w(x) = \exp\left[\int_a^x \frac{B(\tilde{x})}{A(\tilde{x})}\mathrm{d}\tilde{x}\right]} \tag{43}$$

where $w(a) = 1$.

(4) **Eigenfunctions and Eigenvalues** (*)

$$\tilde{\mathcal{L}}y = \lambda y \quad \Rightarrow \quad \boxed{\mathcal{L}y = \lambda\omega y} \tag{44}$$

here, y is an eigenfunction of the self-adjoint operator $\mathcal{L}$ with eigenvalue λ and weight w.

a) The eigenvalues λ are real.

b) The eigenfunctions y with distinct eigenvalues are orthogonal.

Proof: Consider two eigenfunctions, y_i and y_j of $\tilde{\mathcal{L}}$ with eigenvalues λ_i and λ_j respectively. They are also eigenfunctions of $\mathcal{L}$ with eigenvalues λ_i and λ_j and weight w. $\mathcal{L}$ is self-adjoint, then

$$\langle y_i, \mathcal{L}y_j\rangle = \lambda_j\langle y_i, wy_j\rangle = \lambda_j\langle y_i, y_j\rangle_w \tag{45}$$

$$\langle y_i, \mathcal{L}y_j\rangle = \langle y_j, \mathcal{L}y_i\rangle^* = \lambda_i^*\langle y_j, wy_i\rangle^* = \lambda_i^*\langle y_i, y_j\rangle_w \tag{46}$$

Compare the two equations, we have

$$(\lambda_j - \lambda_i^*)\langle y_i, y_j\rangle_w = 0 \tag{47}$$

- For $i = j$, we have $(\lambda_i - \lambda_i^*)\|y_i\|_w^2 = 0 \;\Rightarrow\; \lambda_i = \lambda_i^*$. λ_i is real.

- For $i \neq j$, we have $(\lambda_j - \lambda_i)\langle y_i, y_j\rangle_w = 0 \;\Rightarrow\; \langle y_i, y_j\rangle_w = 0$, *i.e.*, the eigenfunctions are orthogonal with weight $w(x)$.

(5) **Eigenfunction Expansions**
A function $f(x)$ can be written as

$$f(x) = \sum_n f_n y_n(x) \tag{48}$$

where y_n is the eigenfunction of $\mathcal{L}$ with the weight w. f_n is the expansion coefficient and

$$f_n = \langle y_n(x), f(x)\rangle_w = \int_a^b y_n^*(x) w(x) f(x) \mathrm{d}x \tag{49}$$

(6) **Monic polynomial**

$$y_n = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{50}$$

4 Integral Transforms

(1) **Fourier transform**

$$\boxed{\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x} \tag{51}$$

Convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) \mathrm{d}y \tag{52}$$

Properties

a) $\mathcal{F}[f_1 * f_2] = \sqrt{2\pi}\tilde{f}_1(k)\tilde{f}_2(k)$
b) $\mathcal{F}[f_1(x)f_2(x)] = \frac{1}{\sqrt{2\pi}}\tilde{f}_1(k) * \tilde{f}_2(k)$

(2) **Laplace transform**

$$\boxed{\hat{f}(s) = \int_0^{\infty} f(t) \mathrm{e}^{-st} \mathrm{d}t} \tag{53}$$

Convolution

$$(f * g)(t) = \int_0^t f(t') g(t-t') \mathrm{d}t' \tag{54}$$

Properties

a) $\mathcal{L}[f_1 * f_2] = \hat{f}_1(s)\hat{f}_2(s)$
b) $\mathcal{L}[f^{(n)}(t)] = s^n \hat{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
c) $\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} \hat{f}(s)$

5 Complex Analysis

$$z = x + iy, \qquad f(x,y) = u(x,y) + iv(x,y) \tag{55}$$

(1) **Cauchy-Riemann equations** (*)

$$f(x,y) \text{ is analytic in a domain } D \quad \Leftrightarrow \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Derivative (*)

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial z}{\partial x} = f', \qquad \frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial z}{\partial y} = if' \tag{56}$$

which means

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \quad \Rightarrow \quad i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \tag{57}$$

(2) **Harmonic functions $\nabla^2 f = 0$** (*)

$$f(x,y) \text{ is analytic in domain } D \quad \Leftrightarrow \quad u(x,y) \text{ and } v(x,y) \text{ are harmonic}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(3) **Multi-valued functions Example:**

$$f(z) = z^{1/2} \tag{58}$$

a) Define **branches** of $f(z)$ (*)
We define the principal branch of $\arg(z)$ to be $0 \leq \mathrm{Arg}(z) < 2\pi$. The two branches of $f(z)$

$$F_1(z) = |z|^{1/2}\mathrm{e}^{i\theta/2} \tag{59}$$

$$F_2(z) = |z|^{1/2}\mathrm{e}^{i\theta/2+i\pi} = -|z|^{1/2}\mathrm{e}^{i\theta/2} \tag{60}$$

where $\theta = \mathrm{Arg}(z)$. Each branch has a brunch cut along the negative real axis where it is discontinuous.

b) Explain how the two branches form a **Riemann surface**:

- The first Riemann sheet is a copy of the complex z-plane with on which $f(z)$ is defined to be equal to $F_1(z)$ and with a branch cut along the negative real axis.
- The second Riemann sheet is a copy of the complex z-plane with on which $f(z)$ is defined to be equal to $F_2(z)$ and with a branch cut along the negative real axis.
- The Riemann surface is the union of these two Riemann sheets glued across the branch cuts.

(4) **Cauchy's theorem** (*) If $f(z)$ is analytic everywhere on and within a closed contour C

$$\oint_C f(z)\mathrm{d}z = 0 \tag{61}$$

Proof (*) From Green's theorem in the plane, P and Q are functions of x and y, and C is a closed contour in the $x-y$ plane, then

$$\oint_C (P\mathrm{d}x + Q\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y \tag{62}$$

so we have

$$\begin{aligned}
\oint_C f(z)\mathrm{d}z &= \oint_C (u+iv)(\mathrm{d}x + i\mathrm{d}y) \\
&= \oint_C [(u+iv)\mathrm{d}x + (-v+iu)\mathrm{d}y] \\
&= \iint_D \left[\frac{\partial}{\partial x}(-v+iu) - \frac{\partial}{\partial y}(u+iv)\right] \mathrm{d}x\mathrm{d}y \\
&= \iint_D \left[\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] \mathrm{d}x\mathrm{d}y = 0
\end{aligned} \tag{63}$$

(5) **Cauchy's integral theorem**: If $f(z)$ is analytic within and on a closed contour C and z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z \tag{64}$$

(6) **Residue**: Let f has an isolated singularity at z_0, then the residue of f at z_0 is

$$\mathrm{Res}_f(z_0) = \frac{1}{2\pi i}\oint_{C_{z_0}} f(z)\mathrm{d}z \tag{65}$$

where C_{z_0} is a closed contour *s.t.* z_0 is inside and $f(z)$ is analytic inside except at z_0. If $f(z)$ has a pole of order m at z_0, then

$$f(z) = \frac{g(z)}{(z-z_0)^m} \tag{66}$$

and

$$\mathrm{Res}_f(z_0) = \frac{1}{2\pi i}\oint_C \frac{g(z)}{(z-z_0)^m}\mathrm{d}z = \frac{1}{(m-1)!}\left.\frac{\mathrm{d}^{m-1}g(z)}{\mathrm{d}z^{m-1}}\right|_{z=z_0} \tag{67}$$

(7) **Residue theorem** (*) Let C is a closed contour, $f(z)$ is a function that is analytic on C and inside C except at $z = z_1, \cdots, z_N$. Then

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{Res}_f(z_k) \tag{68}$$

(8) **Jordan's lemma** (*)

$$I(R) = \int_{C_R} \mathrm{e}^{i\alpha z} f(z)\mathrm{d}z \tag{69}$$

where $\alpha > 0$ ($\alpha < 0$) and C_R is a semicircle of radius R in the upper (lower) half-plane. Let $M(R)$ be the maximum value of $f(z)$ on C_R. If $M(R) \to 0$ as $R \to \infty$, so dose $I(R)$.

(9) **Inverse Laplace transform** (*)

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\mathrm{e}^{st}\mathrm{d}s = \sum_i \mathrm{Res}(a_i) \tag{70}$$

6 Calculus of Variations

(1) **Euler-Lagrange equation** (*)

$$I[y] = \int_{x_A}^{x_B} f(x, y, y')\mathrm{d}x \tag{71}$$

Varying y slightly

$$\begin{aligned} I[y+\delta y] &= \int_{x_A}^{x_B} f(x, y+\delta y, y'+\delta y')\mathrm{d}x \\ &= \int_{x_A}^{x_B} \left[f(x,y,y') + \delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} + \mathcal{O}(\delta y^2)\right]\mathrm{d}x \end{aligned} \tag{72}$$

then

$$\begin{aligned} \delta I[y] &= \int_{x_A}^{x_B} \left[\delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'}\right]\mathrm{d}x \\ &= \left(\delta y \frac{\partial f}{\partial y'}\right)_{x_A}^{x_B} + \int_{x_A}^{x_B} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y \mathrm{d}x \\ &= \int_{x_A}^{x_B} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y \mathrm{d}x = 0 \end{aligned} \tag{73}$$

so

$$\boxed{\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0} \tag{74}$$

(2) **Beltrami identity** (*)

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} \\ &= \frac{\partial f}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} \\ &= \frac{\partial f}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} y'\right) \end{aligned} \tag{75}$$

Suppose f has no explicit dependence on x, *i.e.*, $\partial f/\partial x = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f - \frac{\partial f}{\partial y'}y'\right) = 0 \tag{76}$$

which means

$$\boxed{f - \frac{\partial f}{\partial y'}y' = \text{const}} \tag{77}$$