NOTES

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DEPARTMENT OF PHYSICS

Quantum Optics

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1.1 Dynamics of atom in light-field

1.1.1 The propagator

We define the propagator $\hat{U}(t)$ via the relation

$$|\Psi(t)\rangle = \hat{U}(t)\,|\Psi(0)\rangle\,. \tag{1}$$

Here, $|\Psi(t)\rangle$ represents any solution to the Schrödinger equation. The propagator adheres to the initial condition $\hat{U}(0) = \mathbb{1}$ and satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}\,|\Psi(t)\rangle = i\dot{\hat{U}}(t)\,|\Psi(0)\rangle = \hat{H}\hat{U}(t)\,|\Psi(0)\rangle\,. \tag{2}$$

This leads to the operator equation

$$\boxed{i\dot{\hat{U}} = \hat{H}\hat{U}.} \tag{3}$$

The adjoint operator $U^\dagger(t)$ satisfies

$$-i\dot{\hat{U}}^\dagger = \hat{U}^\dagger\hat{H}^\dagger = \hat{U}^\dagger\hat{H}. \tag{4}$$

These relations yield

$$i\frac{\partial}{\partial t}\left(\hat{U}\hat{U}^\dagger\right) = i\dot{\hat{U}}\hat{U}^\dagger + i\hat{U}\dot{\hat{U}}^\dagger = \hat{H}\hat{U}\hat{U}^\dagger - \hat{U}\hat{U}^\dagger\hat{H} = [\hat{H}, \hat{U}\hat{U}^\dagger]. \tag{5}$$

With the initial condition $\hat{U}(0)\hat{U}^\dagger(0) = \mathbb{1}$, we find

$$\hat{U}(t)\hat{U}^\dagger(t) = \hat{U}^\dagger(t)\hat{U}(t) = \mathbb{1}. \tag{6}$$

1.1.2 Perturbation theory

The Schrödinger equation, together with the initial condition $\hat{U}(0) = \mathbb{1}$, can be reformulated as the integral equation

$$\begin{aligned}
\hat{U}(t) &= \mathbb{1} + \int_0^t \mathrm{d}t'\dot{\hat{U}}(t') = \mathbb{1} - i\int_0^t \mathrm{d}t'\hat{H}(t')\hat{U}(t') \\
&= \mathbb{1} - i\int_0^t \mathrm{d}t'\hat{H}(t')\left[\mathbb{1} - i\int_0^{t'} \mathrm{d}t''\hat{H}(t'')\hat{U}(t'')\right] \\
&= \mathbb{1} - i\int_0^t \mathrm{d}t'\hat{H}(t') - \int_0^t \mathrm{d}t'\int_0^{t'} \mathrm{d}t''\hat{H}(t')\hat{H}(t'')\hat{U}(t'').
\end{aligned} \tag{7}$$

For sufficiently short times, this can be approximated as

$$\hat{U}(t) \simeq \mathbb{I} - i\int_0^t \mathrm{d}t'\hat{H}(t') - \int_0^t \mathrm{d}t'\int_0^{t'} \mathrm{d}t''\hat{H}(t')\hat{H}(t''). \tag{8}$$

For this approximation to be accurate, it is crucial that the 'magnitude' of H is suitably **small**. Hence, working in a suitable frame becomes essential. Instead of solving the Schrödinger equation $i\dot{U} = HU$ for U, we attempt to solve for V defined by

$$\hat{U} = \hat{U}_0 \hat{V}, \tag{9}$$

where $\hat{U}_0$ is a unitary operator that is freely chosen. The Schrödinger equation can now be solved for $\dot{V}$, yielding

$$i\dot{\hat{V}} = \hat{U}_0^\dagger \hat{H} \hat{U}_0 \hat{V} - i\hat{U}_0^\dagger \dot{\hat{U}}_0 \hat{V} = \left(\hat{U}_0^\dagger \hat{H} \hat{U}_0 - i\hat{U}_0^\dagger \dot{\hat{U}}_0\right) \hat{V} = \hat{\mathcal{H}} \hat{V}, \tag{10}$$

with the new Hamiltonian

$$\boxed{\hat{\mathcal{H}} = \hat{U}_0^\dagger \hat{H} \hat{U}_0 - i\hat{U}_0^\dagger \dot{\hat{U}}_0.} \tag{11}$$

The objective is to find $\hat{U}_0$ such that time-dependent perturbation theory provides a reliable approximation.

1.1.3 Atom-light Hamiltonian

Consider an atom with the Hamiltonian $\hat{H}_0$ and the interaction Hamiltonian $\hat{H}_I$ defined as follows

$$\hat{H}_0 = \sum_j \omega_j \left|\psi_j\right\rangle \left\langle\psi_j\right|, \quad \hat{H}_I = \sum_{j,k} h_{jk} \left|\psi_j\right\rangle \left\langle\psi_k\right|. \tag{12}$$

Here, $h_{jk} = \langle\psi_j| H_I |\psi_k\rangle$, and we choose the transformation $\hat{U}_0$ as

$$\hat{U}_0(t) = \exp\left(-i\hat{H}_0 t\right) = \sum_j \mathrm{e}^{-i\omega_j t} \left|\psi_j\right\rangle \left\langle\psi_j\right|. \tag{13}$$

This choice ensures that $-i\hat{U}_0^\dagger \dot{\hat{U}}_0 = -\hat{H}_0$. The transformed Hamiltonian, denoted as $\hat{\mathcal{H}}$, is given by

$$\begin{aligned} \hat{\mathcal{H}} &= \hat{U}_0^\dagger (\hat{H}_0 + \hat{H}_I) \hat{U}_0 - i\hat{U}_0^\dagger \dot{U}_0 = \hat{U}_0^\dagger \hat{H}_0 \hat{U}_0 + \hat{U}_0^\dagger \hat{H}_I \hat{U}_0 - i\hat{U}_0^\dagger \dot{\hat{U}}_0 = \hat{U}_0^\dagger \hat{H}_I \hat{U}_0 \\ &= \sum_l \mathrm{e}^{i\omega_l t} \left|\psi_l\right\rangle \left\langle\psi_l\right| \sum_{jk} h_{jk} \left|\psi_j\right\rangle \left\langle\psi_k\right| \sum_m \mathrm{e}^{-i\omega_m t} \left|\psi_m\right\rangle \left\langle\psi_m\right| \\ &= \sum_{jk} h_{jk} \exp[i(\omega_j - \omega_k)t] \left|\psi_j\right\rangle \left\langle\psi_k\right|. \end{aligned} \tag{14}$$

Here, h_{jk} represents the oscillating term with frequency ν, and we utilize $\cos\nu t = (\mathrm{e}^{i\nu t} + \mathrm{e}^{-i\nu t})/2$. The oscillating functions lead to a vanishing integral in the form $\int_0^t \mathrm{d}t'(\cdots)$.

1.2 A two-level atom in a monochromatic light field

Consider a two-level atom characterized by resonant frequencies ω_g and ω_e, with the Hamiltonian expressed as

$$\hat{H}_0 = \omega_g |g\rangle \langle g| + \omega_e |e\rangle \langle e| \simeq \frac{\omega}{2}\hat{\sigma}_z, \tag{15}$$

where $\omega = \omega_e - \omega_g$. The interaction of this atom with a monochromatic light field is described by the Hamiltonian

$$\hat{H}_I = \Omega_R \left(|e\rangle \langle g| + |g\rangle \langle e|\right) \cos(\nu t) = \Omega_R \hat{\sigma}_x \cos(\nu t). \tag{16}$$

Here, Ω_R represents the Rabi frequency, proportional to the light field's intensity, and ν is the frequency of the light. The total Hamiltonian for the two-level atom interacting with the monochromatic light field is given by

$$\hat{H} = \frac{\omega}{2}\hat{\sigma}_z + \Omega_R \hat{\sigma}_x \cos(\nu t). \tag{17}$$

To find the solution to the Schrödinger equation, we introduce the transformation

$$\hat{U}_0 = \exp\Big(-i\frac{\eta}{2}\hat{\sigma}_z t\Big), \tag{18}$$

where η is a parameter. This transformation yields the transformed Hamiltonian $\hat{\mathcal{H}}$

$$\begin{aligned}
\hat{\mathcal{H}} &= U_0^\dagger \hat{H} \hat{U}_0 - i\hat{U}_0^\dagger \dot{\hat{U}}_0 \\
&= \frac{\omega - \eta}{2}\hat{\sigma}_z + \Omega_R \left(\hat{\sigma}_+ \mathrm{e}^{i\eta t} + \hat{\sigma}_- \mathrm{e}^{-i\eta t}\right) \cos(\nu t) \\
&= \frac{\omega - \eta}{2}\hat{\sigma}_z + \frac{\Omega_R}{2}\left[\hat{\sigma}_+ \mathrm{e}^{i(\eta-\nu)t} + \hat{\sigma}_- \mathrm{e}^{-i(\eta-\nu)t}\right] + \frac{\Omega_R}{2}\left[\hat{\sigma}_+ \mathrm{e}^{i(\eta+\nu)t} + \hat{\sigma}_- \mathrm{e}^{-i(\eta+\nu)t}\right].
\end{aligned} \tag{19}$$

For the case of $\eta = \omega$ and under the rotating wave approximation (RWA), the transformed Hamiltonian becomes

$$\hat{\mathcal{H}}' = \frac{\Omega_R}{2}\left[\hat{\sigma}_+ \mathrm{e}^{i(\omega-\nu)t} + \hat{\sigma}_- \mathrm{e}^{-i(\omega-\nu)t}\right]. \tag{20}$$

Similarly, for $\eta = \nu$ under the RWA, the transformed Hamiltonian is given by

$$\hat{\mathcal{H}}' = \frac{\omega - \nu}{2}\hat{\sigma}_z + \frac{1}{2}\Omega_R \hat{\sigma}_x = \frac{1}{2}\begin{pmatrix} \omega - \nu & \Omega_R \\ \Omega_R & -(\omega - \nu) \end{pmatrix}. \tag{21}$$

The eigenvalues of $\hat{\mathcal{H}}'$ are then determined as

$$\lambda_\pm = \pm\frac{1}{2}\sqrt{(\omega - \nu)^2 + \Omega_R^2} = \pm\frac{\Omega_G}{2}, \tag{22}$$

where $\Omega_G = \sqrt{(\omega - \nu)^2 + \Omega_R^2}$ is referred to as the generalized Rabi frequency. The associated propagator is given by

$$\begin{aligned}
\exp\Big(-i\hat{\mathcal{H}}' t\Big) &= \mathbb{1}\cos\left(\frac{\Omega_G}{2}t\right) - i\frac{1}{\Omega_G/2}\hat{\mathcal{H}}' \sin\left(\frac{\Omega_G}{2}t\right) \\
&= \mathbb{1}\cos\left(\frac{\Omega_G}{2}t\right) - i\left(\frac{\omega - \nu}{\Omega_G}\hat{\sigma}_z + \frac{\Omega_R}{\Omega_G}\hat{\sigma}_x\right) \sin\left(\frac{\Omega_G}{2}t\right).
\end{aligned} \tag{23}$$

1.2.1 Resonant driving

In the scenario of resonant driving, where the light field matches the atomic transition frequency ($\omega - \nu = 0$), the system's evolution is described by the propagator

$$\exp\left(-i\hat{\mathcal{H}}'t\right) = \mathbb{1}\cos\left(\frac{\Omega_R}{2}t\right) - i\hat{\sigma}_x \sin\left(\frac{\Omega_R}{2}t\right). \tag{24}$$

Applying this to the ground state yields the evolved state

$$\exp\left(-i\hat{\mathcal{H}}'t\right)|g\rangle = \cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle. \tag{25}$$

When combined with the initial factor $\hat{U}_0$, the evolved state becomes

$$\begin{aligned}
&\exp\left(-i\frac{\omega}{2}\sigma_z t\right)\exp\left(-i\hat{\mathcal{H}}'t\right)|g\rangle \\
=&\exp\left(i\frac{\omega}{2}t\right)\cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\exp\left(-i\frac{\omega}{2}t\right)\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle \\
=&\exp\left(i\frac{\omega}{2}t\right)\left[\cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\exp(-i\omega t)\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle\right].
\end{aligned} \tag{26}$$

The probabilities of finding the atom in the excited state or ground state are given by

$$|c_e(t)|^2 = \sin^2\left(\frac{\Omega_R}{2}t\right) = \frac{1}{2}\left[1 - \cos(\Omega_R t)\right], \tag{27}$$

$$|c_g(t)|^2 = \cos^2\left(\frac{\Omega_R}{2}t\right) = \frac{1}{2}\left[1 + \cos(\Omega_R t)\right]. \tag{28}$$

These oscillations are known as Rabi oscillations.

1.2.2 Off-resonant driving

In the case of off-resonant driving, where the light field frequency deviates significantly from the atomic transition frequency ($|\nu - \omega| \gg \Omega_R$), we can make the following approximations:

$$\frac{\omega - \nu}{\Omega_G} = \frac{\omega - \nu}{\sqrt{(\omega-\nu)^2 + \Omega_R^2}} \approx \frac{\nu - \omega}{|\nu - \omega|} = \pm 1, \tag{29}$$

$$\frac{\Omega_R}{\Omega_G} = \frac{\Omega_R}{\sqrt{(\omega-\nu)^2 + \Omega_R^2}} \approx \frac{\Omega_R}{|\nu - \omega|} \ll 1. \tag{30}$$

As a result, the propagator simplifies to

$$\exp\left(-i\hat{\mathcal{H}}'t\right) \approx \mathbb{1}\cos\left(\frac{\Omega_G}{2}t\right) - i\hat{\sigma}_z \sin\left(\frac{\Omega_G}{2}t\right) = \exp\left(-i\frac{\Omega_G}{2}\hat{\sigma}_z t\right). \tag{31}$$

1.2.3 Ramsey

When $\nu = \omega$, the propagator in the interaction picture is given by

$$\hat{U}_x = \mathbb{1}\cos\left(\frac{1}{2}\Omega_R t\right) - i\hat{\sigma}_x \sin\left(\frac{1}{2}\Omega_R t\right). \tag{32}$$

For a duration $T = \frac{\pi}{2\Omega_R}$, this expression simplifies to

$$\hat{U}_x(T) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\hat{\sigma}_x). \tag{33}$$

Assuming the atom is initially in its ground state $|g\rangle$, the evolved state becomes

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}(|g\rangle - i\,|e\rangle). \tag{34}$$

A measurement of the population of the eigenstates at this point would yield a 50% probability of finding the system in the ground state and a 50% probability in the excited state.
Now, consider a Hamiltonian with an additional phase ϕ:

$$\hat{H}_\phi = \frac{\omega}{2}\hat{\sigma}_z + \Omega_R\hat{\sigma}_x\cos(\nu t + \phi). \tag{35}$$

The associated propagator is given by

$$\hat{U}_\phi(T) = \frac{1}{\sqrt{2}}\left[\mathbb{1} - i(\hat{\sigma}_x\cos\phi + \hat{\sigma}_y\sin\phi)\right]. \tag{36}$$

Applying $\hat{U}_\phi(T)$ to the state $|\Psi(T)\rangle$, we obtain

$$|\Psi(2T)\rangle = -i\exp\left(i\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)|g\rangle - i\exp\left(-i\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)|e\rangle. \tag{37}$$

The probability of finding the atom in the ground state or the excited state oscillates with ϕ.

1.3 The three-level atom

Consider a three-level atom, focusing on the transitions between states $|g_1\rangle$ and $|e\rangle$ as well as $|g_2\rangle$ and $|e\rangle$, both experiencing frequency detuning δ. The corresponding Hamiltonians are given by

$$\hat{H}_1 = \frac{\Omega_R}{2\sqrt{2}}\left(|e\rangle\langle g_1|\,\mathrm{e}^{i\delta t} + |g_1\rangle\langle e|\,\mathrm{e}^{-i\delta t}\right), \tag{38}$$

$$\hat{H}_2 = \frac{\Omega_R}{2\sqrt{2}}\left(|e\rangle\langle g_2|\,\mathrm{e}^{i\delta t} + |g_2\rangle\langle e|\,\mathrm{e}^{-i\delta t}\right). \tag{39}$$

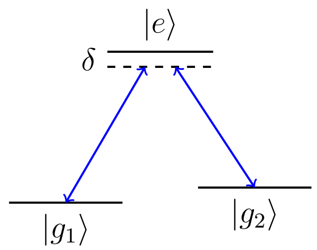

Figure 1: A three-level atom with energy levels $|g_1\rangle$, $|g_2\rangle$, and $|e\rangle$. The detuning is denoted by δ, and blue arrows indicate possible transitions between energy states.

where the factor $1/\sqrt{2}$ is taken for convenience. Combining these transitions, the total Hamiltonian becomes

$$\hat{H} = \frac{\Omega_R}{2}\left(|e\rangle\langle g|\,\mathrm{e}^{i\delta t} + |g\rangle\langle e|\,\mathrm{e}^{-i\delta t}\right), \tag{40}$$

where $|g\rangle = (|g_1\rangle + |g_2\rangle)/\sqrt{2}$. Simplifying further, we arrive at

$$\hat{H} = \frac{\Omega_R}{2}\left(\hat{\sigma}_x \cos\delta t + \hat{\sigma}_y \sin\delta t\right). \tag{41}$$

The propagator at time $t = 2\pi/\delta$ is approximately given by

$$\hat{U}(t) \simeq \mathbb{I} - i\int_0^t \mathrm{d}t_1 \hat{H}(t_1) - \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \hat{H}(t_1)\hat{H}(t_2). \tag{42}$$

Evaluating at $t = 2\pi/\delta$, the first-order term $U^{(1)}$ vanishes, and the second-order term $U^{(2)}$ involves integrals requiring careful computation

$$\hat{U}^{(2)}\left(\frac{2\pi}{\delta}\right) = -\frac{\Omega_R^2}{4}\int_0^{2\pi/\delta} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \left(\hat{\sigma}_x \cos\delta t_1 + \hat{\sigma}_y \sin\delta t_1\right)\left(\hat{\sigma}_x \cos\delta t_2 + \hat{\sigma}_y \sin\delta t_2\right). \tag{43}$$

Several necessary integrals are calculated, yielding the perturbative expression

$$\hat{U}\left(\frac{2\pi}{\delta}\right) \simeq \mathbb{1} + i\frac{\Omega_R^2}{4\delta}\hat{\sigma}_z\left(\frac{2\pi}{\delta}\right). \tag{44}$$

This expression mimics the influence of an effective Hamiltonian $\hat{H}_e = -\frac{\Omega_R^2}{4\delta}\hat{\sigma}_z = \frac{\Omega_e}{2}\hat{\sigma}_z$. Extending to the explicit three-level case, the effective Hamiltonian is expressed as

$$\hat{H}_e = \frac{\Omega_e}{2}(|e\rangle\langle e| - |g\rangle\langle g|). \tag{45}$$

With eigenstates $|e\rangle$, $|g\rangle$, and $(|g_1\rangle - |g_2\rangle)/\sqrt{2}$, the eigenvalues are $\frac{\Omega_e}{2}$, $-\frac{\Omega_e}{2}$, and 0. The corresponding propagator is given by

$$\exp\left(-i\hat{H}_e t\right) = \exp\left(-i\frac{\Omega_e}{2}t\right)|e\rangle\langle e| + \exp\left(i\frac{\Omega_e}{2}t\right)|g\rangle\langle g| + \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}}\frac{\langle g_1| - \langle g_2|}{\sqrt{2}}. \tag{46}$$

Applied to the initial state $|g_1\rangle$, it results in

$$\exp\left(-i\hat{H}_e t\right)|g_1\rangle = \exp\left(i\frac{\Omega_e}{4}t\right)\left[\cos\left(\frac{\Omega_e}{4}t\right)|g_1\rangle + i\sin\left(\frac{\Omega_e}{4}t\right)|g_2\rangle\right], \tag{47}$$

with the effective Rabi frequency $\Omega_e = -\Omega_R^2/2\delta$.

1.4 The Bloch sphere

1.4.1 The Bloch equations

The Pauli matrices satisfy the following relations

$$[\hat{\sigma}_\alpha, \hat{\sigma}_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\hat{\sigma}_\gamma, \qquad \{\hat{\sigma}_\alpha, \hat{\sigma}_\beta\} = 0. \tag{48}$$

Expressed in terms of the eigenstates $|g\rangle$ and $|e\rangle$

$$\hat{\sigma}_z |g\rangle = - |g\rangle, \qquad \hat{\sigma}_z |e\rangle = |e\rangle, \tag{49}$$
$$\hat{\sigma}_x |g\rangle = |e\rangle, \qquad \hat{\sigma}_x |e\rangle = |g\rangle, \tag{50}$$
$$\hat{\sigma}_y |g\rangle = -i |e\rangle, \qquad \hat{\sigma}_y |e\rangle = i |g\rangle. \tag{51}$$

The Bloch equations are defined as follows:

$$\langle\Psi| \mathbb{1} |\Psi\rangle = 1, \tag{52}$$
$$\langle\Psi| \hat{\sigma}_x |\Psi\rangle = S_x, \tag{53}$$
$$\langle\Psi| \hat{\sigma}_y |\Psi\rangle = S_y, \tag{54}$$
$$\langle\Psi| \hat{\sigma}_z |\Psi\rangle = S_z. \tag{55}$$

These equations define the Bloch vector

$$\boldsymbol{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}. \tag{56}$$

1.4.2 Dynamics of the Bloch vector

Instead of the Schrödinger equation, we can describe the system dynamics using an equation of motion for the Bloch vector $\boldsymbol{S}$. In addition to the Schrödinger equation

$$\frac{\partial}{\partial t} |\Psi\rangle = -i\hat{H} |\Psi\rangle. \tag{57}$$

For the state vector $|\Psi\rangle$

$$\begin{aligned} \dot{S}_x =& \frac{\partial \langle\Psi|}{\partial t}\hat{\sigma}_x |\Psi\rangle + \langle\Psi| \hat{\sigma}_x \frac{\partial |\Psi\rangle}{\partial t} \\ =& i \langle\Psi| \hat{H}\hat{\sigma}_x |\Psi\rangle - i \langle\Psi| \hat{\sigma}_x\hat{H} |\Psi\rangle \\ =& i \langle\Psi| [\hat{H}, \hat{\sigma}_x] |\Psi\rangle. \end{aligned} \tag{58}$$

To simplify, express the Hamiltonian in terms of Pauli matrices

$$\hat{H} = \sum_j \frac{\omega_j}{2}\hat{\sigma}_j, \tag{59}$$

so that

$$\dot{S}_x = i \sum_j \frac{\omega_j}{2} \langle\Psi| [\hat{\sigma}_j, \hat{\sigma}_x] |\Psi\rangle = \omega_y\hat{\sigma}_z - \omega_z\hat{\sigma}_y. \tag{60}$$

Similarly,

$$\dot{S}_y = \omega_z \hat{\sigma}_x - \omega_x \hat{\sigma}_z, \tag{61}$$

$$\dot{S}_z = \omega_x \sigma_y - \omega_y \sigma_x. \tag{62}$$

In vector notation

$$\dot{\boldsymbol{S}} = \boldsymbol{\omega} \times \boldsymbol{S}, \tag{63}$$

and

$$\frac{\partial}{\partial t}\|\boldsymbol{S}\|^2 = \dot{\boldsymbol{S}}\boldsymbol{S} + \boldsymbol{S}\dot{\boldsymbol{S}} = (\boldsymbol{\omega} \times \boldsymbol{S})\boldsymbol{S} + \boldsymbol{S}(\boldsymbol{\omega} \times \boldsymbol{S}) = 0. \tag{64}$$

1.4.3 Averages over different states

Consider the expectation of an observable $\langle A \rangle = \sum_j p_j \langle \Psi_j | \hat{A} | \Psi_j \rangle$. We can thus define the Bloch vector for the ensemble average

$$\langle \boldsymbol{S} \rangle = \sum_j p_j S_j. \tag{65}$$

2 Atom-light interactions - quantum theory

2.1 Quantum harmonic oscillator

The Hamiltonian for a one-dimensional harmonic oscillator is given in terms of the position operator $\hat{X}$ and the momentum operator $\hat{P}$ as follows

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 = \hbar\omega\left(\frac{\hat{P}^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar}\hat{X}^2\right) = \hbar\omega\frac{\hat{p}^2+\hat{x}^2}{2}, \tag{66}$$

where the dimensionless operators are defined as

$$\hat{x} = \sqrt{\frac{m\omega}{\hbar}}\hat{X}, \qquad \hat{p} = \frac{1}{\sqrt{m\hbar\omega}}\hat{P}. \tag{67}$$

These operators satisfy the commutation relation:

$$[\hat{x},\hat{p}] = \sqrt{\frac{m\omega}{\hbar}}\frac{1}{\sqrt{m\hbar\omega}}[\hat{X},\hat{P}] = \frac{1}{\hbar}[\hat{X},\hat{P}] = i. \tag{68}$$

The creation and annihilation operators are defined as

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x}+i\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x}-i\hat{p}). \tag{69}$$

They satisfy the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$. Additionally, $\hat{x}$ and $\hat{p}$ can be expressed in terms of these operators:

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger), \qquad \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger-\hat{a}). \tag{70}$$

Finally, the Hamiltonian takes the form

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right). \tag{71}$$

2.2 Quantization of the light field

Light is an electromagnetic field, and to quantize it, we start from the classical description based on Maxwell's equations

$$\nabla\cdot\boldsymbol{E} = 0, \quad \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \tag{72}$$

$$\nabla\cdot\boldsymbol{B} = 0, \quad \nabla\times\boldsymbol{B} = \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}. \tag{73}$$

The electromagnetic field is decomposed into standing wave modes, represented by an ansatz

$$\boldsymbol{E} = \sum_k A_k f_k(t)\sin(kz)\boldsymbol{e}_x, \tag{74}$$

$$\boldsymbol{B} = \sum_k A_k \dot{f}_k(t) \frac{\varepsilon_0 \mu_0}{k} \cos(kz) \boldsymbol{e}_y. \tag{75}$$

Here, f_k is the normal mode amplitude, $k = 2\pi n/L$ with $n = 1, 2, 3, \cdots$, and $\nabla \cdot \boldsymbol{E} = \nabla \cdot \boldsymbol{B} = 0$. The Maxwell equations imply the following wave equation

$$\ddot{f}(t) = -c^2 k^2 f(t) = -\nu_k^2 f(t), \tag{76}$$

where $\nu_k = ck$ is termed the linear dispersion. The Hamiltonian of the electromagnetic field is then expressed as

$$\begin{aligned} H =& \sum_k \frac{1}{2} \int \mathrm{d}\tau \left(\varepsilon_0 E_x^2 + \frac{B_y^2}{\mu_0} \right) \\ =& \sum_k \frac{\varepsilon_0 A_k^2}{2} \iint \mathrm{d}x \mathrm{d}y \int_0^L \left[f_k^2(t) \sin^2(kz) + \frac{1}{\nu_k^2} \dot{f}_k^2(t) \cos^2(kz) \right] \mathrm{d}z \\ =& \frac{1}{2} \sum_k \frac{\varepsilon_0 V A_k^2}{2\nu_k^2} \left[\nu_k^2 f_k^2(t) + \dot{f}_k^2(t) \right]. \end{aligned} \tag{77}$$

Comparing with the one-dimensional harmonic oscillator $\hat{H} = \frac{1}{2} m\omega^2 \hat{X}^2 + \frac{\hat{P}^2}{2m}$, we identify the following correspondence

$$m \leftrightarrow \frac{\varepsilon_0 V A_k^2}{2\nu_k^2}, \quad \omega \leftrightarrow \nu_k, \quad \hat{X} \leftrightarrow f, \quad \hat{P} = m\dot{f}. \tag{78}$$

$f(t)$ can be written in operator form

$$\hat{f}_k(t) = \frac{1}{A} \sqrt{\frac{\hbar \nu_k}{\varepsilon_0 V}} \left(\hat{a}_k \mathrm{e}^{-i\nu_k t} + \hat{a}_k^\dagger \mathrm{e}^{i\nu_k t} \right). \tag{79}$$

Now, the light field can be quantized as

$$\boldsymbol{E} = \sum_k \sqrt{\frac{\hbar \nu_k}{\varepsilon_0 V}} \left(\hat{a} \mathrm{e}^{-i\nu_k t} + \hat{a}^\dagger \mathrm{e}^{i\nu_k t} \right) \sin(kz) \boldsymbol{e}_x. \tag{80}$$

2.3 Fock states

With the commutation relation $[a, a^\dagger] = 1$, the action of the creation and annihilation operators on a Fock state $|\mu\rangle$ with eigenvalue μ for $\hat{a}^\dagger \hat{a}$ is explored

$$a^\dagger a a |\mu\rangle = (a a^\dagger - 1) a |\mu\rangle = a(a^\dagger a - 1) |\mu\rangle = (\mu - 1) a |\mu\rangle. \tag{81}$$

This implies that $\hat{a} |\mu\rangle$ is an eigenstate with the eigenvalue $\mu - 1$ of $\hat{a}^\dagger \hat{a}$. Additionally

$$\langle \mu | a^\dagger a | \mu \rangle = \mu \quad \Rightarrow \quad a |\mu\rangle = \sqrt{\mu} |\mu - 1\rangle. \tag{82}$$

And similarly

$$a^\dagger |\mu\rangle = \sqrt{\mu + 1} |\mu + 1\rangle. \tag{83}$$

The operator $\hat{a}^\dagger \hat{a}$ represents the photon number, and the state $|\mu\rangle$ is termed a Fock state or number state.

2.4 Jaynes-Cummings model

The interaction between a two-level system and a single mode of the quantized electromagnetic field, known as the Jaynes-Cummings model, is described by the Hamiltonian

$$\hat{H} = \frac{\omega}{2}\hat{\sigma}_z + \frac{1}{2}\Omega_R\hat{\sigma}_x\left(\hat{a}\mathrm{e}^{-i\nu t} + \hat{a}^\dagger\mathrm{e}^{i\nu t}\right). \tag{84}$$

We can Break down the evolution under specific transformations. In the lab frame, $\hat{U}_S = \exp\bigl(i\nu\hat{a}^\dagger\hat{a}t\bigr)$, the interaction Hamiltonian becomes

$$\hat{\mathcal{H}}_S = \hat{U}_S^\dagger\hat{H}\hat{U}_S - i\hat{U}_S^\dagger\dot{\hat{U}}_S = \frac{\omega}{2}\hat{\sigma}_z + \nu\hat{a}^\dagger\hat{a} + \frac{1}{2}\Omega_R\hat{\sigma}_x(\hat{a} + \hat{a}^\dagger). \tag{85}$$

In the interaction frame (or rotating frame), $\hat{U}_I = \exp\bigl(-i\frac{\omega}{2}\hat{\sigma}_z t\bigr)$, the interaction Hamiltonian is given by

$$\begin{aligned}\hat{\mathcal{H}}_I =&\hat{U}_I^\dagger\hat{H}\hat{U}_I - i\hat{U}_I^\dagger\dot{\hat{U}}_I = \frac{1}{2}\Omega_R\left(\hat{\sigma}_+\mathrm{e}^{i\omega t} + \hat{\sigma}_-\mathrm{e}^{-i\omega t}\right)\left(\hat{a}\mathrm{e}^{-i\nu t} + \hat{a}^\dagger\mathrm{e}^{i\nu t}\right)\\ =&\frac{1}{2}\Omega_R\left[\hat{\sigma}_+\hat{a}\mathrm{e}^{i(\omega-\nu)t} + \hat{\sigma}_+\hat{a}^\dagger\mathrm{e}^{i(\omega+\nu)t} + \hat{\sigma}_-\hat{a}\mathrm{e}^{-i(\omega+\nu)t} + \hat{\sigma}_-\hat{a}^\dagger\mathrm{e}^{-i(\omega-\nu)t}\right].\end{aligned} \tag{86}$$

This Hamiltonian involves four elementary processes

$\hat{\sigma}_+\hat{a}$: The atom absorbs a photon and becomes excited.

$\hat{\sigma}_+\hat{a}^\dagger$: The atom emits a photon and becomes excited.

$\hat{\sigma}_-\hat{a}$: The atom absorbs a photon and becomes de-excited.

$\hat{\sigma}_-\hat{a}^\dagger$: The atom emits a photon and becomes de-excited.

The last two processes violate energy conservation strongly; they are the quantum mechanical equivalent to the terms we had neglected in the case of a classical light field. If we neglect them in rotating wave approximation, we are left with the Hamiltonian

$$\hat{\mathcal{H}}_{IR} = \frac{1}{2}\Omega_R\left[\hat{\sigma}_+\hat{a}\mathrm{e}^{i(\omega-\nu)t} + \hat{\sigma}_+\hat{a}^\dagger\mathrm{e}^{i(\omega+\nu)t}\right] \tag{87}$$

Similarly, in the lab frame

$$\hat{\mathcal{H}}_{SR} = \frac{\omega}{2}\hat{\sigma}_z + \nu\hat{a}^\dagger\hat{a} + \frac{1}{2}\Omega_R(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger). \tag{88}$$

If we consider the states $|g,\mu\rangle$ and $|e,\mu-1\rangle$. Those states are eigenstates of $\hat{\mathcal{H}}_{SR}$, since

$$\hat{\mathcal{H}}_{SR}\,|g,\mu\rangle = \left(-\frac{\omega}{2} + \mu\nu\right)|g,\mu\rangle + \frac{1}{2}\Omega_R\sqrt{\mu}\,|e,\mu-1\rangle\,, \tag{89}$$

$$\hat{\mathcal{H}}_{SR}\,|e,\mu-1\rangle = \frac{1}{2}\Omega_R\sqrt{\mu}\,|g,\mu\rangle + \left(\frac{\omega}{2} + (\mu-1)\nu\right)|e,\mu-1\rangle\,. \tag{90}$$

In terms of the basis $\{|g,\mu\rangle\,,|e,\mu-1\rangle\}$ we can express this as the matrix

$$\begin{pmatrix} -\frac{\omega-\nu}{2} + \left(\mu-\frac{1}{2}\right)\nu & \frac{1}{2}\Omega_R\sqrt{\mu} \\ \frac{1}{2}\Omega_R\sqrt{\mu} & \frac{\omega-\nu}{2} + \left(\mu-\frac{1}{2}\right)\nu \end{pmatrix}, \tag{91}$$

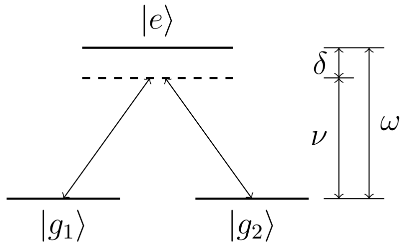

Figure 2: The Lambda-system

or, in terms of Pauli-matrices as

$$\hat{\mathcal{H}}_\mu = -\frac{\omega - \nu}{2}\hat{\sigma}_z + \frac{1}{2}\Omega_R\sqrt{\mu}\hat{\sigma}_x + \left(\mu - \frac{1}{2}\right)\nu\mathbb{1}. \tag{92}$$

In the case of resonance between atom and light-field, this reduces to

$$\hat{\mathcal{H}}_\mu(\nu = \omega) = \frac{1}{2}\Omega_R\sqrt{\mu}\sigma_x + \left(\mu - \frac{1}{2}\right)\nu\mathbb{1}, \tag{93}$$

with eigenstates

$$\frac{1}{\sqrt{2}}(|g, \mu\rangle \pm |e, \mu - 1\rangle). \tag{94}$$

2.5 The lambda System

The Hamiltonian of the Lambda-system interacting with a single-mode quantum field in rotating wave approximation is given by

$$\begin{aligned}\hat{H} =& \omega |e\rangle\langle e| + 0(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|) + \nu\hat{a}^\dagger\hat{a} \\ &+ \frac{1}{2\sqrt{2}}\Omega_R\left(|g_1\rangle\langle e|\hat{a}^\dagger + |g_2\rangle\langle e|\hat{a}^\dagger + |e\rangle\langle g_1|\hat{a} + |e\rangle\langle g_2|\hat{a}\right) \\ =& \omega|e\rangle\langle e| + \frac{1}{2\sqrt{2}}\Omega_R\left[(|g_1\rangle\langle e| + |g_2\rangle\langle e|)\hat{a}^\dagger + (|e\rangle\langle g_1| + |e\rangle\langle g_2|)\hat{a}\right].\end{aligned} \tag{95}$$

In the interaction picture, the effective Hamiltonian is expressed as

$$\hat{\mathcal{H}} = \frac{1}{2\sqrt{2}}\Omega_R\left[(|g_1\rangle\langle e| + |g_2\rangle\langle e|)\hat{a}^\dagger e^{-i\delta t} + (|e\rangle\langle g_1| + |e\rangle\langle g_2|)\hat{a}e^{i\delta t}\right]. \tag{96}$$

Applying perturbation theory up to the second order, the term $\hat{\mathcal{H}}(t')\hat{\mathcal{H}}(t'')$ is computed as follows

$$\begin{aligned}\hat{\mathcal{H}}(t')\hat{\mathcal{H}}(t'') =& \frac{1}{8}\Omega_R^2\Big[(|g_1\rangle\langle e| + |g_2\rangle\langle e|)(|e\rangle\langle g_1| + |e\rangle\langle g_2|)a^\dagger a e^{-i\delta(t'-t'')} \\ &+ (|e\rangle\langle g_1| + |e\rangle\langle g_2|)(|g_1\rangle\langle e| + |g_2\rangle\langle e|)aa^\dagger e^{i\delta(t'-t'')}\Big] \\ =& \frac{1}{8}\Omega_R^2\left(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2| + |g_1\rangle\langle g_2| + |g_2\rangle\langle g_1|\right)a^\dagger a e^{-i\delta(t'-t'')} \\ &+ \frac{1}{4}\Omega_R^2|e\rangle\langle e|(a^\dagger a + 1)e^{i\delta(t'-t'')}.\end{aligned} \tag{97}$$

3 Coherent states and squeezed states

3.1 Coherent states

Recall the one-dimensional harmonic oscillator, for fock states $|\mu\rangle$, the expectation values for $\hat{x}$ and $\hat{p}$ vanish

$$\langle\mu|\,\hat{x}\,|\mu\rangle = \frac{1}{\sqrt{2}}(\langle\mu|\,\hat{a}\,|\mu\rangle + \langle\mu|\,\hat{a}^\dagger\,|\mu\rangle) = 0, \tag{98}$$

$$\langle\mu|\,\hat{p}\,|\mu\rangle = \frac{-i}{\sqrt{2}}(\langle\mu|\,\hat{a}\,|\mu\rangle - \langle\mu|\,\hat{a}^\dagger\,|\mu\rangle) = 0, \tag{99}$$

and the fluctuations

$$\langle\mu|\,\hat{x}^2\,|\mu\rangle = \frac{1}{2}(\langle\mu|\,\hat{a}^2\,|\mu\rangle + \langle\mu|\,\hat{a}\hat{a}^\dagger\,|\mu\rangle + \langle\mu|\,\hat{a}^\dagger\hat{a}\,|\mu\rangle + \langle\mu|\,\hat{a}^\dagger\hat{a}^\dagger\,|\mu\rangle) = \mu + \frac{1}{2}, \tag{100}$$

$$\langle\mu|\,\hat{p}^2\,|\mu\rangle = -\frac{1}{2}(\langle\mu|\,\hat{a}^2\,|\mu\rangle - \langle\mu|\,\hat{a}\hat{a}^\dagger\,|\mu\rangle - \langle\mu|\,\hat{a}^\dagger\hat{a}\,|\mu\rangle + \langle\mu|\,\hat{a}^\dagger\hat{a}^\dagger\,|\mu\rangle) = \mu + \frac{1}{2}. \tag{101}$$

For the ground state

$$\langle 0|\,\hat{x}\,|0\rangle = \langle 0|\,\hat{p}\,|0\rangle = 0, \quad \langle 0|\,\hat{x}^2\,|0\rangle = \langle 0|\,\hat{p}^2\,|0\rangle = \frac{1}{2}. \tag{102}$$

This yields

$$\Delta\hat{x}\Delta\hat{p} = (\langle 0|\,\hat{x}^2\,|0\rangle - (\langle 0|\,\hat{x}\,|0\rangle)^2)(\langle 0|\,\hat{p}^2\,|0\rangle - (\langle 0|\,\hat{p}\,|0\rangle)^2) = \frac{1}{4}, \tag{103}$$

which is the minimal allowed uncertainty. We can generate different states with the same uncertainty. These states are called the coherent states, notated by $|\alpha\rangle$. α is defined to

$$\alpha = \hat{x} + i\hat{p}, \tag{104}$$

by displacing the vacuum in phase space. The *displacement operator* is defined as

$$\boxed{D(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right).} \tag{105}$$

The coherent state

$$\begin{aligned}
|\alpha\rangle =& D(\alpha)\,|0\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right)|0\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right)\exp\left(\alpha a^\dagger\right)\exp(-\alpha^* a)\,|0\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right)\exp\left(\alpha a^\dagger\right)|0\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu (a^\dagger)^\mu}{\mu!}\,|0\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu}{\mu!}\sqrt{\mu!}\,|\mu\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}}\,|\mu\rangle\,.
\end{aligned} \tag{106}$$

The probability to find μ photons is thus given by the Poisson distribution.

$$P(\mu) = \exp\bigl(-|\alpha|^2\bigr)\frac{(|\alpha|^2)^\mu}{\mu!}. \tag{107}$$

For the expectation value of x and p with respect to a general state $|\Psi\rangle$ one has

$$(\langle\Psi| D^\dagger(\alpha))x(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)xD(\alpha))|\Psi\rangle, \tag{108}$$

$$(\langle\Psi| D^\dagger(\alpha))p(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)pD(\alpha))|\Psi\rangle, \tag{109}$$

then we calculate

$$D^\dagger(\alpha)xD(\alpha) = x + \frac{\alpha+\alpha^*}{\sqrt{2}} = x + x_0, \tag{110}$$

$$D^\dagger(\alpha)pD(\alpha) = p - i\frac{\alpha-\alpha^*}{\sqrt{2}} = p + p_0. \tag{111}$$

We verify the uncertainty in position and momentum of any coherent state

$$\begin{aligned}\langle\alpha| x^2 |\alpha\rangle - (\langle\alpha| x |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)x^2D(\alpha)|0\rangle - (\langle 0| D^\dagger(\alpha)xD(\alpha)|0\rangle)^2 \\ &= \langle 0| x^2 |0\rangle - (\langle 0| x |0\rangle)^2,\end{aligned} \tag{112}$$

$$\begin{aligned}\langle\alpha| p^2 |\alpha\rangle - (\langle\alpha| p |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)p^2D(\alpha)|0\rangle - (\langle 0| D^\dagger(\alpha)pD(\alpha)|0\rangle)^2 \\ &= \langle 0| p^2 |0\rangle - (\langle 0| p |0\rangle)^2.\end{aligned} \tag{113}$$

3.2 Coherent states in real-space representation

The character of the displacement operator can be exemplified in the real-space representation of wave functions.

$$\Psi_\alpha(x) = \langle x|\alpha\rangle \propto \exp\left(-\frac{1}{2}(x-x_0)^2 + ip_0x - \frac{i}{2}x_0p_0\right), \tag{114}$$

with $x_0 = (\alpha+\alpha^*)/\sqrt{2}$ and $p_0 = (\alpha-\alpha^*)/(\sqrt{2}i)$. It is convenient to define the vector

$$|\alpha,\tau\rangle = \exp\bigl[(\alpha a^\dagger - \alpha^* a)\tau\bigr]|0\rangle, \tag{115}$$

with additional scalar parameter τ. It satisfies the differential equation

$$\frac{\partial|\alpha,\tau\rangle}{\partial\tau} = (\alpha a^\dagger - \alpha^* a)|\alpha,\tau\rangle. \tag{116}$$

The real-space representation of the operator $(\alpha a^\dagger - \alpha^* a)$ reads

$$\frac{1}{\sqrt{2}}\left[\alpha\left(x - \frac{\partial}{\partial x}\right) - \alpha^*\left(x + \frac{\partial}{\partial x}\right)\right] = \frac{\alpha-\alpha^*}{\sqrt{2}}x - \frac{\alpha+\alpha^*}{\sqrt{2}}\frac{\partial}{\partial x} = ip_0x - x_0\frac{\partial}{\partial x}. \tag{117}$$

We thus need to solve the equation

$$\frac{\partial\Phi}{\partial\tau} = \left(ip_0x - x_0\frac{\partial}{\partial x}\right)\Phi, \tag{118}$$

with the Ansatz

$$\Phi(\tau) = \exp\Big(-\frac{1}{2}(x-f_x)^2 + if_p x - i\varphi\Big). \tag{119}$$

The initial conditions are $f_x(0) = f_p(0) = \varphi(0) = 0$. The derivatives

$$\frac{\partial\Phi(\tau)}{\partial\tau} = \left((x-f_x)\frac{\partial f_x}{\partial\tau} + i\frac{\partial f_p}{\partial\tau}x - i\frac{\partial\varphi}{\partial\tau}\right)\Phi(\tau), \tag{120}$$

$$\frac{\partial\Phi(\tau)}{\partial x} = \left(-(x-f_x) + if_p\right)\Phi(\tau). \tag{121}$$

This yields

$$(x-f_x)\frac{\partial f_x}{\partial\tau} + i\frac{\partial f_p}{\partial\tau}x - i\frac{\partial\varphi}{\partial\tau} = ip_0 x - x_0\left(-(x-f_x) + if_p\right). \tag{122}$$

Collect all terms proportional to x

$$\frac{\partial f_x}{\partial\tau} + i\frac{\partial f_p}{\partial\tau} = ip_0 + x_0. \tag{123}$$

This is solved for

$$\frac{\partial f_x}{\partial\tau} = x_0 \quad\Rightarrow\quad f_x = x_0\tau, \tag{124}$$

$$\frac{\partial f_p}{\partial\tau} = p_0 \quad\Rightarrow\quad f_p = p_0\tau. \tag{125}$$

Collecting all terms do not contain x yields

$$-f_x\frac{\partial f_x}{\partial\tau} - i\frac{\partial\varphi}{\partial\tau} = -x_0 f_x - ix_0 f_p, \tag{126}$$

which is solved for

$$\varphi(\tau) = \frac{1}{2}x_0 p_0\tau^2. \tag{127}$$

With $\tau = 1$, this gives the phase factor $\exp\big(-\frac{i}{2}x_0p_0\big)$.

3.3 Dynamics of coherent states

For the dynamics induced by $U_0(t) = \exp\big(-i\nu a^\dagger a t\big)$, one obtains

$$\begin{aligned} U_0(t)\left|\alpha\right\rangle =& U_0(t)\exp\big(\alpha a^\dagger - \alpha^* a\big)\left|0\right\rangle \\ =& U_0(t)\exp\big(\alpha a^\dagger - \alpha^* a\big)U_0^\dagger(t)U_0(t)\left|0\right\rangle \\ =& U_0(t)\exp\big(\alpha a^\dagger - \alpha^* a\big)U_0^\dagger(t)\left|0\right\rangle \\ =& \exp\Big[\alpha U_0(t)a^\dagger U_0^\dagger(t) - \alpha^* U_0(t)aU_0^\dagger(t)\Big]\left|0\right\rangle \\ =& \exp\big(\alpha a^\dagger \mathrm{e}^{-i\nu t} + \alpha^* a\mathrm{e}^{i\nu t}\big)\left|0\right\rangle \\ =& D\left(\alpha\mathrm{e}^{-i\nu t}\right)\left|0\right\rangle = \left|\alpha\mathrm{e}^{-i\nu t}\right\rangle. \end{aligned} \tag{128}$$

3.4 Light-matter interaction with coherent states

Coherent states are eigenstates of the annihilation operator $\hat{a}$:

$$\begin{aligned}\hat{a}\,|\alpha\rangle =&\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}}\hat{a}\,|\mu\rangle \\ =&\alpha\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^{\mu-1}}{\sqrt{(\mu-1)!}}\,|\mu-1\rangle = \alpha\,|\alpha\rangle\,.\end{aligned} \tag{129}$$

Similarly,

$$\langle\alpha|\,a^\dagger = \alpha^*\,\langle\alpha|\,. \tag{130}$$

Coherent states are not orthogonal to each other

$$\begin{aligned}\langle\alpha|\beta\rangle =& \left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_\mu \frac{(\alpha^*)^\mu}{\sqrt{\mu!}}\,\langle\mu|\right)\left(\exp\left(-\frac{|\beta|^2}{2}\right)\sum_\nu \frac{\beta^\nu}{\sqrt{\nu!}}\,|\nu\rangle\right) \\ =&\exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_{\mu,\nu}\frac{(\alpha^*)^\mu\beta^\nu}{\sqrt{\mu!}\sqrt{\nu!}}\,\langle\mu|\nu\rangle \\ =&\exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_\mu \frac{(\alpha^*\beta)^\mu}{\mu!}.\end{aligned} \tag{131}$$

Now we want to find the eigenvector $|\Psi\rangle$ of $a^\dagger$. Suppose that

$$a^\dagger\,|\Psi\rangle = \lambda\,|\Psi\rangle = |\tilde{\Psi}\rangle. \tag{132}$$

The normalised vector is

$$\frac{|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}}, \tag{133}$$

and

$$\left|\frac{\langle\Psi|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}}\right| = 1. \tag{134}$$

Normalising $a^\dagger\,|\alpha\rangle$ yields

$$\frac{a^\dagger\,|\alpha\rangle}{\sqrt{\langle\alpha|\,aa^\dagger\,|\alpha\rangle}} = \frac{\alpha^\dagger\,|\alpha\rangle}{\sqrt{|\alpha^2|+1}} \tag{135}$$

and

$$\frac{\langle\alpha|\,a^\dagger\,|\alpha\rangle}{\sqrt{\langle\alpha|\,aa^\dagger\,|\alpha\rangle}} = \frac{\alpha^*}{\sqrt{|\alpha^2|+1}} \tag{136}$$

In the limit $|\alpha|\to\infty$

$$\frac{\alpha^*}{\sqrt{|\alpha^2|+1}} \to \frac{\alpha^*}{|\alpha|} \tag{137}$$

with

$$\left|\frac{\alpha^*}{|\alpha|}\right| = 1 \tag{138}$$

The relation

$$a^\dagger |\alpha\rangle \simeq \alpha^* |\alpha\rangle \tag{139}$$

is thus a good approximation for $|\alpha| \ll 1$.

3.5 Squeezed states

Suppose we have two operators $\hat{A}$ and $\hat{B}$ and they satisfy $[\hat{A}, \hat{B}] = i\hat{C}$, then one obtains the uncertainty $\Delta\hat{A}\Delta\hat{B} \geq |\langle\hat{C}\rangle|/2$. The minimal uncertain state of $\hat{A}$ and $\hat{B}$

$$\Delta\hat{A}\Delta\hat{B} = \frac{|\langle\hat{C}\rangle|}{2}. \tag{140}$$

It is noteworthy that the minimal uncertainty states are related to the group of operators considered. For example, coherent states are the minimal uncertainty states concerning $\hat{X}_1$ and $\hat{X}_2$.
From the minimal uncertainty states, we can define the squeezed states, or squeezed coherent states. If a state $|\xi\rangle$ with commutation relation $[\hat{A}, \hat{B}] = i\hat{C}$ satisfy

$$\Delta\hat{A} \leq \sqrt{\frac{|\langle\hat{C}\rangle|}{2}} \quad \text{or} \quad \Delta\hat{B} \leq \sqrt{\frac{|\langle\hat{C}\rangle|}{2}}, \tag{141}$$

then the state is called the squeezed state. It is clear that the coherent state is squeezed in some directions.
The process of squeezing, in general, is described in terms of the operator:

$$\hat{S}(\chi) = \exp\Big(\frac{\chi}{2}\left(\hat{a}^2 - (\hat{a}^\dagger)^2\right)\Big), \tag{142}$$

and we find $\hat{S}^\dagger(\chi) = \hat{S}(\chi)^{-1} = \hat{S}(-\chi)$. Using $\mathrm{e}^{\hat{A}}\hat{B}\mathrm{e}^{-\hat{A}} = \hat{B}+[\hat{A},\hat{B}]+\frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]]+\cdots$. We choose $\hat{A} = -\frac{\chi}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)$ and $\hat{B} = \hat{a} + \hat{a}^\dagger$, so $[\hat{A}, \hat{B}] = -\chi(\hat{a} + \hat{a}^\dagger) = -\chi\hat{B}$.

$$\hat{S}^\dagger(\chi)(\hat{a} + \hat{a}^\dagger)\hat{S}(\chi) = \hat{B} + (-\chi)\hat{B} + \frac{(-\chi)^2}{2!}\hat{B} + \cdots = (\hat{a} + \hat{a}^\dagger)\mathrm{e}^{-\chi}. \tag{143}$$

Similarly, if $\hat{B} = \hat{a} - \hat{a}^\dagger$, then $[\hat{A}, \hat{B}] = \chi(\hat{a} - \hat{a}^\dagger) = \chi\hat{B}$

$$\hat{S}^\dagger(\chi)(\hat{a} - \hat{a}^\dagger)\hat{S}(\chi) = \hat{B} + \chi\hat{B} + \frac{\chi^2}{2!}\hat{B} + \cdots = (\hat{a} - \hat{a}^\dagger)\mathrm{e}^{\chi}. \tag{144}$$

So regarding to the quantum harmonic oscillator

$$\hat{S}^\dagger(\chi)\hat{x}\hat{S}(\chi) = \hat{x}\mathrm{e}^{-\chi}, \qquad \hat{S}^\dagger(\chi)\hat{p}\hat{S}(\chi) = \hat{p}\mathrm{e}^{\chi}. \tag{145}$$

Regarding to the phasor diagram, the amplitude operator $\hat{X}_1$ and $\hat{X}_2$

$$\hat{S}^\dagger(\chi)\hat{X}_1\hat{S}(\chi) = \hat{X}_1\mathrm{e}^{-\chi}, \qquad \hat{S}^\dagger(\chi)\hat{X}_2\hat{S}(\chi) = \hat{X}_2\mathrm{e}^{\chi}. \tag{146}$$

Using the displacement operator $\hat{D}(\alpha)$ and squeezing operator $\hat{S}(\xi)$, we can generate the squeezed states, like the squeezed coherent state $|\alpha, \chi\rangle = \hat{D}(\alpha)\hat{S}(\chi)|0\rangle$, the coherent squeezed state $|\chi, \alpha\rangle = \hat{S}(\chi)\hat{D}(\alpha)|0\rangle$, and the vacuum squeezed state $|\chi\rangle = |\chi, \alpha = 0\rangle = \hat{S}(\chi)|0\rangle$.

4 The Wigner quasi-probability distribution

Let's start with a classical property that we would like to be fulfilled

$$\int_{-\infty}^{\infty} \mathrm{d}P W(X,P) = \mathrm{Pr}(X), \quad \int_{-\infty}^{\infty} \mathrm{d}X W(X,P) = \mathrm{Pr}(P). \tag{147}$$

The probability distribution of any quadrature is called a 'marginal'. We can generalize the marginal equations above into a single expression to include rotation of the harmonic oscillator in its phase space. We then have

$$\begin{aligned}\mathrm{Pr}(X,\theta) &= \langle X| U(\theta)\rho U^{\dagger}(\theta) |X\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}P W(X\cos\theta - P\sin\theta, X\sin\theta + P\cos\theta),\end{aligned} \tag{148}$$

where $U(\theta) = \exp\bigl(-i\theta a^{\dagger}a\bigr)$ is the rotation operator.

4.1 A derivation of Wigner's classic formula

To start our derivation, we introduce two quantities. The 'characteristic function', *i.e.* the two-dimensional Fourier transform of the Wigner function

$$\tilde{W}(U,V) = \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}P W(X,P)\mathrm{e}^{-iUX-iVP}, \tag{149}$$

and the Fourier-transformed probability distribution

$$\tilde{\mathrm{Pr}}(\xi,\theta) = \int_{-\infty}^{\infty} \mathrm{d}X \mathrm{Pr}(X,\theta)\mathrm{e}^{-i\xi X}. \tag{150}$$

We use the second part of the Eqn.(148) we have

$$\begin{aligned}\tilde{\mathrm{Pr}}(\xi,\theta) &= \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}P W(X\cos\theta - P\sin\theta, X\sin\theta + P\cos\theta)\mathrm{e}^{-i\xi X} \\ &= \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}P W(X,P)\mathrm{e}^{-i\xi X\cos\theta - i\xi P\sin\theta} = \tilde{W}(\xi\cos\theta, \xi\sin\theta).\end{aligned} \tag{151}$$

Using the first part of the Eqn.(148) we have

$$\begin{aligned}\tilde{\mathrm{Pr}}(\xi,\theta) &= \int_{-\infty}^{\infty} \mathrm{d}X \,\langle X| U(\theta)\rho U^{\dagger}(\theta) |X\rangle\, \mathrm{e}^{-i\xi X} \\ &= \int_{-\infty}^{\infty} \mathrm{d}X \,\langle X| \rho U^{\dagger}(\theta)\mathrm{e}^{-i\xi X} U(\theta) |X\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}X \,\langle X| \rho \exp(-iX\xi\cos\theta - iP\xi\sin\theta) |X\rangle \\ &= \mathrm{Tr}(\rho\exp(-iX\xi\cos\theta - iP\xi\sin\theta)).\end{aligned} \tag{152}$$

Letting $U = \xi\cos\theta$ and $V = \xi\sin\theta$, we thus have our next important result

$$\boxed{\tilde{W}(U,V) = \mathrm{Tr}(\rho\exp(-iUX - iVP)).} \tag{153}$$

Using the Baker-Campbell-Hausdorff formula

$$\exp(-iUX - iVP) = \exp(iUV/2)\exp(-iUX)\exp(-iVP), \tag{154}$$

we have

$$\begin{aligned}\tilde{W}(U,V) =& \exp(iUV/2)\int_{-\infty}^{\infty}\mathrm{d}X\,\langle X|\,\rho\exp(-iUX)\exp(-iVP)\,|X\rangle \\ =& \exp(iUV/2)\int_{-\infty}^{\infty}\mathrm{d}X\,\langle X|\,\rho\exp(-iUX)\,|X+V\rangle \\ =& \int_{-\infty}^{\infty}\mathrm{d}Q\,\langle Q-V/2|\,\rho\,|Q+V/2\rangle\exp(-iUQ),\end{aligned} \tag{155}$$

where $X = Q - V/2$. Lastly, we do a inverse-Fourier transform to obtain the Wigner function

$$\begin{aligned}W(X,P) =& \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}U\int_{-\infty}^{\infty}\mathrm{d}V\tilde{W}(U,V)\mathrm{e}^{iUX+iVP} \\ =& \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}U\int_{-\infty}^{\infty}\mathrm{d}V\int_{-\infty}^{\infty}\mathrm{d}Q\,\langle Q-V/2|\,\rho\,|Q+V/2\rangle \\ &\times\exp(-iUQ)\exp(iUX+iVP) \\ =& \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}U\int_{-\infty}^{\infty}\mathrm{d}V\int_{-\infty}^{\infty}\mathrm{d}Q\,\langle Q-V/2|\,\rho\,|Q+V/2\rangle \\ &\times\exp(-iUQ)\exp(-iUV/2)\exp(iUX)\exp(iVP) \\ =& \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}V\int_{-\infty}^{\infty}\mathrm{d}Q\,\langle Q-V/2|\,\rho\,|Q+V/2\rangle\exp(iVP) \\ &\times\int_{-\infty}^{\infty}\mathrm{d}U\exp(iU(X-Q)) \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}V\int_{-\infty}^{\infty}\mathrm{d}Q\,\langle Q-V/2|\,\rho\,|Q+V/2\rangle\exp(iVP)\delta(X-Q) \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}V\,\langle X-V/2|\,\rho\,|X+V/2\rangle\exp(iVP).\end{aligned} \tag{156}$$

This equation is Wigner's now famous formula

$$\boxed{W(X,P) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}V\mathrm{e}^{iPV}\left\langle X-\frac{V}{2}\right|\rho\left|X+\frac{V}{2}\right\rangle.} \tag{157}$$

4.2 Properties of the Wigner function

- $W(X,P)$ is normalized, *i.e.* $\int_{-\infty}^{\infty}\mathrm{d}X\int_{-\infty}^{\infty}\mathrm{d}PW(X,P) = 1$.

- $W(X,P)$ is real, *i.e.* $W(X,P) = W^*(X,P)$.
- If $W(X,P)$ has any negative regions, then the state is non-classical.
- The overlap $\mathrm{Tr}(AB) = 2\pi \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}P W_A(X,P) W_B(X,P)$.

5 Optical homodyne and heterodyne detection

5.1 Balanced homodyne detection

The Beam Splitter (BS) is one of the most important optical elements. It has two spatial input modes a and b and two output modes a' and b'. In quantum optics, the unitary beam-splitter operator is

$$\hat{B} = \exp\left[i\frac{\theta}{2}\left(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger\right)\right]. \tag{158}$$

The most commonly used BS is 50:50 BS with $\theta/2 = \pi/4$. Then we have

$$\hat{a}' = \hat{B}^\dagger\hat{a}\hat{B} = \frac{1}{\sqrt{2}}(\hat{a} + i\hat{b}), \tag{159}$$

$$\hat{b}' = \hat{B}^\dagger\hat{b}\hat{B} = \frac{1}{\sqrt{2}}(\hat{b} + i\hat{a}). \tag{160}$$

Then we calculate the number operator observed by the photondiodes

$$\hat{a}'^\dagger\hat{a}' = \frac{1}{2}(\hat{a}^\dagger - i\hat{b}^\dagger)(\hat{a} + i\hat{b}) = \frac{1}{2}(\hat{a}^\dagger\hat{a} + i\hat{a}^\dagger\hat{b} - i\hat{b}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}), \tag{161}$$

$$\hat{b}'^\dagger\hat{b}' = \frac{1}{2}(\hat{b}^\dagger - i\hat{a}^\dagger)(\hat{b} + i\hat{a}) = \frac{1}{2}(\hat{a}^\dagger\hat{a} - i\hat{a}^\dagger\hat{b} + i\hat{b}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}). \tag{162}$$

In a balanced detector, these two photocurrents are subtracted, yielding the "difference current"

$$i_- \propto \hat{b}'^\dagger\hat{b}' - \hat{a}'^\dagger\hat{a}' = i\hat{b}^\dagger\hat{a} - i\hat{a}^\dagger\hat{b}. \tag{163}$$

Now we consider mode $\hat{a}$ to be the signal and mode $\hat{b}$ is the reference, which is also called a **local oscillator** (LO). We assume that the LO is powerful enough to be treated classically, i.e., we can neglect totally the quantum fluctuations of the LO.

$$\hat{b} \;\rightarrow\; \alpha_{\mathrm{LO}} = |\alpha_{\mathrm{LO}}|\mathrm{e}^{i\pi/2}\mathrm{e}^{i\theta}. \tag{164}$$

Here, we introduce the phase ϕ in a convenient way to absorb the factor of i that came from our convention for the phase in the beam-splitter operator. After this

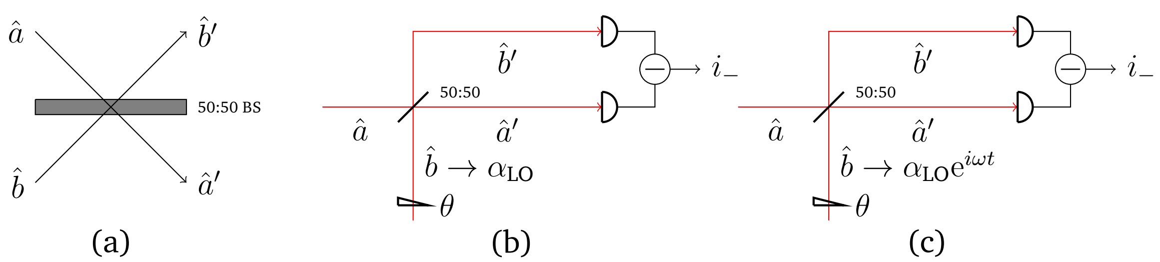

Figure 3: (a) The 50:50 beam splitter. (b) Schematic for an optical homodyne detector. The mode b is put in a strong coherent state α_{LO} and is mixed on a beam-splitter with mode a that we wish to measure. (c) Schematic for an optical heterodyne detection.

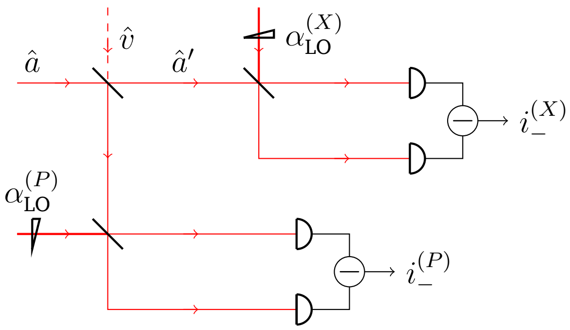

Figure 4: Dual homodyne detection. "v" denotes vacuum fluctuations.

transformation, the difference current becomes

$$\begin{aligned} i_- &\propto |\alpha_{\mathrm{LO}}|(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}) \\ &= |\alpha_{\mathrm{LO}}|\left((\hat{a}+\hat{a}^\dagger)\cos\theta + i(\hat{a}^\dagger - \hat{a})\sin\theta\right) \\ &= \sqrt{2}|\alpha_{\mathrm{LO}}|(\hat{X}\cos\theta + \hat{P}\sin\theta) \\ &= \sqrt{2}|\alpha_{\mathrm{LO}}|\hat{X}_\theta. \end{aligned} \tag{165}$$

where $\hat{X}_\theta = \hat{U}^\dagger(\theta)\hat{X}\hat{U}(\theta) = \hat{X}\cos\theta + \hat{P}\sin\theta = \frac{1}{\sqrt{2}}(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})$. In this way, a balanced homodyne detector measures the quadrature component $\hat{X}_\theta$.

5.2 Heterodyne detection

The heterodyne detection is similar to the homodyne detection; however the LO has a different frequency.

$$\begin{aligned} i_- \propto& |\alpha_{\mathrm{LO}}|\left(\hat{a}e^{-i\theta}e^{-i\omega t} + \hat{a}^\dagger e^{i\theta}e^{i\omega t}\right) \\ =& \sqrt{2}|\alpha_{\mathrm{LO}}|\left(\hat{X}\cos(\omega t+\theta) + \hat{P}\sin(\omega t+\theta)\right) \\ =& \sqrt{2}|\alpha_{\mathrm{LO}}|X_{\omega t+\theta}. \end{aligned} \tag{166}$$

We can thus see that the detector is rapidly oscillating between making a measurement of X, and P, and all angles in between, in time. Thus, it can be viewed as making a simultaneous measurement of X and P, when the difference frequency ω is much larger than any dynamics of interest in the signal field $\hat{a}$.

5.3 Dual homodyne detection

From the discussions above, we can write the difference currents $i_-^{(X)}$ and $i_-^{(P)}$

$$\begin{aligned} i_-^{(X)} \propto& \sqrt{2}|\alpha_{\mathrm{LO}}^{(X)}|\hat{X}_a = |\alpha_{\mathrm{LO}}^{(X)}|(\hat{a}' + \hat{a}'^\dagger) = |\alpha_{\mathrm{LO}}^{(X)}|\frac{1}{\sqrt{2}}(\hat{a} + i\hat{v} + \hat{a}^\dagger - i\hat{v}^\dagger) \\ =& |\alpha_{\mathrm{LO}}^{(X)}|\left[\frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger) + \frac{i}{\sqrt{2}}(\hat{v}-\hat{v}^\dagger)\right] = \sqrt{2}|\alpha_{\mathrm{LO}}^{(X)}|\frac{1}{\sqrt{2}}(\hat{X}_a - \hat{P}_v), \end{aligned} \tag{167}$$

$$\begin{aligned} i_{-}^{(P)} \propto & \sqrt{2}|\alpha_{\mathrm{LO}}^{(P)}|\hat{X}_v = |\alpha_{\mathrm{LO}}^{(P)}|(\hat{v}' + \hat{v}'^{\dagger}) = \frac{1}{\sqrt{2}}|\alpha_{\mathrm{LO}}^{(P)}|(\hat{v} + i\hat{a} + \hat{v}^{\dagger} - i\hat{a}^{\dagger}) \\ = & |\alpha_{\mathrm{LO}}^{(P)}|\left[\frac{1}{\sqrt{2}}(\hat{v} + \hat{v}^{\dagger}) + \frac{i}{\sqrt{2}}(\hat{a} - \hat{a}^{\dagger})\right] = \sqrt{2}|\alpha_{\mathrm{LO}}^{(P)}|\frac{1}{\sqrt{2}}(\hat{X}_v - \hat{P}_a). \end{aligned} \tag{168}$$

6 Photon counting statistics

A very useful way to categorize the photon statistics observed is to use the following three regimes:

- Sub-Poissonian statistics: $\Delta^2 n < \bar{n}$
- Poissonian statistics: $\Delta^2 n = \bar{n}$
- Super-Poissonian statistics: $\Delta^2 n > \bar{n}$

6.1 Poissonian statistics

Coherent states have Poissonian statistics

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle \,, \tag{169}$$

for photon number m. The amplitude

$$\langle n|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}. \tag{170}$$

Then we get the probability

$$\Pr(n) = |\langle n|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}. \tag{171}$$

The mean photon number in a coherent state is

$$\bar{n} = \langle\alpha| a^\dagger a |\alpha\rangle = |\alpha|^2, \tag{172}$$

so we can write

$$\boxed{\Pr(n) = \mathrm{e}^{-\bar{n}} \frac{\bar{n}^n}{n!}.} \tag{173}$$

This expression is called the *Poisson* distribution. Now we calculate the variance

$$\begin{aligned}
\Delta^2 n =& \sum_{n=0}^{\infty} (n-\bar{n})^2 \Pr(n) = \sum_{n=0}^{\infty} (n^2 - 2\bar{n}n + \bar{n}^2) \Pr(n) \\
=& \sum_{n=0}^{\infty} n^2 \Pr(n) - 2\bar{n} \sum_{n=0}^{\infty} n \Pr(n) + \bar{n}^2 \sum_{n=0}^{\infty} \Pr(n) \\
=& \sum_{n=0}^{\infty} n^2 \Pr(n) - \bar{n}^2 = \sum_{n=0}^{\infty} (n^2 - n + n) \Pr(n) - \bar{n}^2 \\
=& \sum_{n=0}^{\infty} n(n-1) \Pr(n) + \bar{n} - \bar{n}^2 = \bar{n}^2 \sum_{n=0}^{\infty} \mathrm{e}^{-\bar{n}} \frac{\bar{n}^{n-2}}{(n-2)!} + \bar{n} - \bar{n}^2 \\
=& \bar{n}^2 \sum_{n=0}^{\infty} \Pr(n-2) + \bar{n} - \bar{n}^2 = \bar{n},
\end{aligned} \tag{174}$$

or in a more “quantum optics style”

$$\begin{aligned}
\Delta^2 n =&\langle n^2\rangle - \langle n\rangle^2 \\
=&\left\langle\alpha\right| (a^\dagger a)^2 \left|\alpha\right\rangle - \left\langle\alpha\right| a^\dagger a \left|\alpha\right\rangle^2 \\
=&|\alpha|^2 \left\langle\alpha\right| a a^\dagger \left|\alpha\right\rangle - \left\langle\alpha\right| a^\dagger a \left|\alpha\right\rangle^2 \\
=&|\alpha|^2 \left\langle\alpha\right| (1 + a^\dagger a) \left|\alpha\right\rangle - \left\langle\alpha\right| a^\dagger a \left|\alpha\right\rangle^2 = |\alpha|^2 = \bar{n}.
\end{aligned} \tag{175}$$

6.2 Super-poissonian statistics

One of the example is the thermal state. Consider the Boltzmann distribution

$$\Pr(n) = \frac{\exp(-n\hbar\omega/k_B T)}{\sum_{n=0}^{\infty} \exp(-n\hbar\omega/k_B T)} = \frac{x^n}{\sum_{n=0}^{\infty} x^n}, \tag{176}$$

where $x = \exp(-\hbar\omega/k_B T)$. If $\hbar\omega \gg k_B T$, *i.e.* x is small, then $\sum_n x^n = 1/(1-x)$. The probability becomes

$$\Pr(n) = (1-x)x^n, \tag{177}$$

and we have known that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_n x^n = \sum_n n x^{n-1} = \frac{1}{(1-x)^2}. \tag{178}$$

So the mean photon number

$$\bar{n} = \sum_n n \Pr(n) = x(1-x) \sum_n n x^{n-1} = \frac{x}{1-x} = \frac{1}{\exp(\hbar\omega/k_B T) - 1}. \tag{179}$$

From this, we have

$$x = \frac{\bar{n}}{1+\bar{n}}, \tag{180}$$

so the distribution

$$\boxed{\Pr(n) = \frac{1}{1+\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n.} \tag{181}$$

which is called the *Bose-Einstein* distribution. Now we calculate the variance

$$\Delta^2 n = \bar{n} + \bar{n}^2 > \bar{n}. \tag{182}$$

Thus, we see that a thermal state exhibits super-Poissonian statistics.

6.3 Sub-poissonian statistics

Consider the Fock state $\left|n\right\rangle$. The Fock states have a mean photon number

$$\bar{n} = \left\langle n\right| a^\dagger a \left|n\right\rangle = n, \tag{183}$$

and a variance

$$\Delta^2 n = \left\langle n\right| (a^\dagger a)^2 \left|n\right\rangle - \left\langle n\right| a^\dagger a \left|n\right\rangle = 0. \tag{184}$$

7 Bunching and antibunching

In the previous section, we studied the statistics of the photon number for different states of light. In this section, we will continue to look at intensity and photon-counting measurements, but rather focus on how these quantities are correlated in time.
Central to this section is the second-order correlation function $g^{(2)}(\tau)$, which is defined below and describes how the intensity or photon number is correlated for a time-separation τ. We then have three categories of temporal correlations, which are related to, but describe different physical properties to the three categories studied in the previous section. These categories are:

- Bunched light: $g^{(2)}(0) > 1$.
- Coherent light: $g^{(2)}(0) = 1$.
- Antibunched light: $g^{(2)}(0) < 1$.

7.1 Classical second-order intensity correlations

Let's start with the classical "second-order correlation function"

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}. \tag{185}$$

The intensity of classical field can be written as

$$I(t) = \bar{I} + \Delta I(t), \tag{186}$$

where $\langle I(t)\rangle = \langle I(t+\tau)\rangle = \bar{I}$. The light field has a coherence time τ_c, such that when $\tau \gg \tau_c$, intensity fluctuations will be uncorrelated, *i.e.* $\langle \Delta I(t)\Delta I(t+\tau)\rangle = 0$. In this case

$$\begin{aligned} g^{(2)}(\tau \gg \tau_c) =& \frac{\langle \big(\bar{I} + \Delta I(t)\big)\big(\bar{I} + \Delta I(t+\tau)\big)\rangle}{\bar{I}^2} \\ =& \frac{\bar{I}^2 + \bar{I}\langle \Delta I(t+\tau)\rangle + \langle \Delta I(t)\rangle \bar{I} + \langle \Delta I(t)\Delta I(t+\tau)\rangle}{\bar{I}^2} = 1. \end{aligned} \tag{187}$$

Consider $\tau < \tau_c$. Of particular interest, is the case where $\tau = 0$ and we have

$$g^{(2)}(0) = \frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2} \geq 1, \tag{188}$$

which is a key result of the classical analysis of the second order correlation function and indicates that the light tends to 'bunch' together.

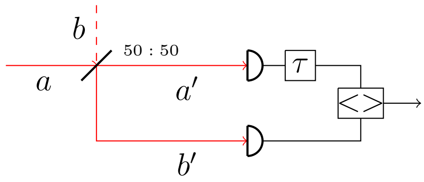

Figure 5: The Hanbury Brown - Twiss experimental setup to measure intensity correlations.

7.2 Quantum second-order correlations

Let's now look at the second-order correlation function in the quantum picture of light. The setup is in Fig. 5. The second-order correlation function can be expressed as

$$g^{(2)}(\tau) = \frac{\langle n_1(t) n_2(t+\tau)\rangle}{\langle n_1(t)\rangle\langle n_2(t+\tau)\rangle} = \frac{\langle a'^{\dagger}(t) b'^{\dagger}(t+\tau) b'(t+\tau) a'(t)\rangle}{\langle a'^{\dagger}(t) a'(t)\rangle\langle b'^{\dagger}(t+\tau) b'(t+\tau)\rangle}. \tag{189}$$

Consider $\tau = 0$

$$\langle a'^{\dagger}(t) a'(t)\rangle = \frac{1}{2}\langle (a^{\dagger} - i b^{\dagger})(a + i b)\rangle = \frac{1}{2}\langle a^{\dagger} a\rangle = \frac{1}{2}\langle n\rangle, \tag{190}$$

$$\langle b'^{\dagger}(t) b'(t)\rangle = \frac{1}{2}\langle (b^{\dagger} - i a^{\dagger})(b + i a)\rangle = \frac{1}{2}\langle a^{\dagger} a\rangle = \frac{1}{2}\langle n\rangle, \tag{191}$$

$$\begin{aligned}\langle a'^{\dagger}(t) b'^{\dagger}(t) b'(t) a'(t)\rangle &= \frac{1}{4}\langle (a^{\dagger} - i b^{\dagger})(b^{\dagger} - i a^{\dagger})(b + i a)(a + i b)\rangle \\ &= \frac{1}{4}\langle a^{\dagger} a^{\dagger} a a\rangle = \frac{1}{4}\langle a^{\dagger}(a a^{\dagger} - 1) a\rangle \\ &= \frac{1}{4}\langle a^{\dagger} a (a^{\dagger} a - 1)\rangle = \frac{1}{4}\langle n(n-1)\rangle,\end{aligned} \tag{192}$$

as mode b is in the vacuum state. So we have

$$g^{(2)}(0) = \frac{\langle n(n-1)\rangle}{\langle n\rangle^2}. \tag{193}$$

Notably, for a single photon input $g^{(2)}(0) = 0$, which indicate strong antibunching and is highly non-classical.

8 The Hong-Ou-Mandel effect

Consider the interference of two single photons each incident on a 50:50 beam-splitter as shown in Fig. 6. The input is $|1\rangle|1\rangle$, and the output is

$$\begin{aligned} |\psi\rangle = \hat{B}|1\rangle|1\rangle = \hat{B}\hat{a}^\dagger\hat{b}^\dagger|0\rangle|0\rangle =& \hat{B}\hat{a}^\dagger\hat{B}^\dagger\hat{B}\hat{b}^\dagger\hat{B}^\dagger\hat{B}|0\rangle|0\rangle \\ =& \left(\hat{B}\hat{a}^\dagger\hat{B}^\dagger\right)\left(\hat{B}\hat{b}^\dagger\hat{B}^\dagger\right)|0\rangle|0\rangle, \end{aligned} \tag{194}$$

where the unitary beam-splitter operator is

$$\hat{B} = \exp\left[i\frac{\theta}{2}\left(a^\dagger b + ab^\dagger\right)\right]. \tag{195}$$

We have

$$\hat{B}^\dagger\hat{a}\hat{B} = \hat{a}\cos\frac{\theta}{2} + i\hat{b}\sin\frac{\theta}{2}, \tag{196}$$

$$\hat{B}^\dagger\hat{b}\hat{B} = \hat{b}\cos\frac{\theta}{2} + i\hat{a}\sin\frac{\theta}{2}, \tag{197}$$

and we note that

$$\hat{B}^\dagger\left(\frac{\theta}{2}\right) = \hat{B}\left(-\frac{\theta}{2}\right). \tag{198}$$

For the 50:50 beam-splitter, $\theta/2 = \pi/4$, so we have

$$\left(\hat{B}\hat{a}\hat{B}^\dagger\right)^\dagger = \hat{B}\hat{a}^\dagger\hat{B}^\dagger = \hat{a}\cos\left(-\frac{\pi}{4}\right) + i\hat{b}\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\hat{a} - i\hat{b}\right), \tag{199}$$

$$\left(\hat{B}\hat{b}\hat{B}^\dagger\right)^\dagger = \hat{B}\hat{b}^\dagger\hat{B}^\dagger = \hat{b}\cos\left(-\frac{\pi}{4}\right) + i\hat{a}\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\hat{b} - i\hat{a}\right). \tag{200}$$

Hence

$$\hat{B}\hat{a}\hat{B}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right), \tag{201}$$

$$\hat{B}\hat{b}\hat{B}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{b}^\dagger + i\hat{a}^\dagger\right). \tag{202}$$

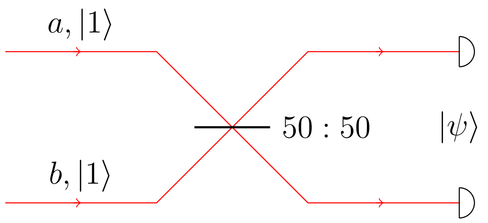

Figure 6: Two photon interference.

So the output state would be

$$\begin{aligned} |\psi\rangle =& \frac{1}{2}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right)\left(\hat{b}^\dagger + i\hat{a}^\dagger\right)|0\rangle\,|0\rangle \\ =& \frac{1}{2}\left(\hat{a}^\dagger\hat{b}^\dagger + i\hat{a}^{\dagger 2} + i\hat{b}^{\dagger 2} - \hat{b}^\dagger\hat{a}^\dagger\right)|0\rangle\,|0\rangle \\ =& \frac{1}{2}\left(i\hat{a}^{\dagger 2} + i\hat{b}^{\dagger 2}\right)|0\rangle\,|0\rangle \\ \simeq& \frac{1}{\sqrt{2}}(|2\rangle\,|0\rangle + |0\rangle\,|2\rangle). \end{aligned} \tag{203}$$

The "global phase" factor i can be omitted as this does not change the prediction of measurement probabilities.

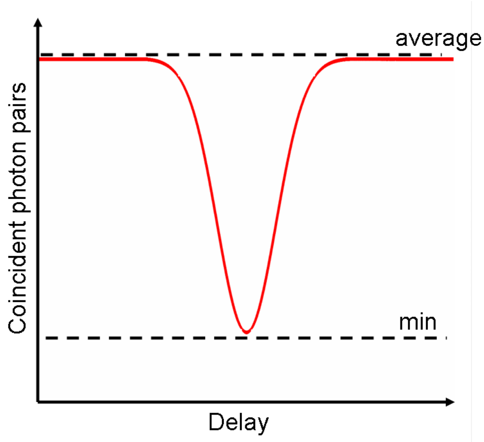

Figure 7: The Hong–Ou–Mandel dig.

Customarily the Hong–Ou–Mandel effect is observed using two photondetectors monitoring the output modes of the beam splitter. The coincidence rate of the detectors will drop to zero when the identical input photons overlap perfectly in time. This is called the Hong–Ou–Mandel dip, or HOM dip.

9 Cavity quantum optomechanics

An optical cavity must follows

$$N\frac{\lambda}{2} = L + x_M, \quad N = 1, 2, 3, \dots, \tag{204}$$

the optical cavity angular frequency is then

$$\omega_L(x_M) = \frac{2\pi c}{\lambda} = \frac{\pi c N}{L + x_M} = \frac{\pi c N}{L}\frac{1}{1 + x_M/L} = \langle\omega_L\rangle\frac{1}{1 + x_M/L}, \tag{205}$$

where $\langle\omega_L\rangle = \pi c N/L$ is the cavity frequency for $x_M = 0$. For $x_M/L \ll 1$, we have

$$\omega_L(x_M) \simeq \frac{\pi c N}{L}\left(1 - \frac{x_M}{L}\right) = \langle\omega_L\rangle\left(1 - \frac{x_M}{L}\right). \tag{206}$$

The Hamiltonian of the full system

$$\begin{aligned} \hat{H} &= \hbar\omega_M \hat{b}^\dagger\hat{b} + \hbar\omega_L(x_M)\hat{a}^\dagger\hat{a} \\ &\simeq \hbar\omega_M \hat{b}^\dagger\hat{b} + \hbar\langle\omega_L\rangle\hat{a}^\dagger\hat{a} - \hbar\langle\omega_L\rangle\frac{x_M}{L}\hat{a}^\dagger\hat{a}. \end{aligned} \tag{207}$$

We now introduce the dimensionless position quadrature operator for the mechanical motion

$$X_M = \frac{x_M}{x_0} = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger), \tag{208}$$

where $x_0 = \sqrt{\hbar/m\omega_M}$. So the Hamiltonian becomes

$$\hat{H} = \hbar\omega_M \hat{b}^\dagger\hat{b} + \hbar\langle\omega_L\rangle\hat{a}^\dagger\hat{a} - \hbar g_0 \hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger), \tag{209}$$

where g_0 is the coupling rate

$$g_0 = \langle\omega_L\rangle\frac{x_0}{\sqrt{2}L}. \tag{210}$$

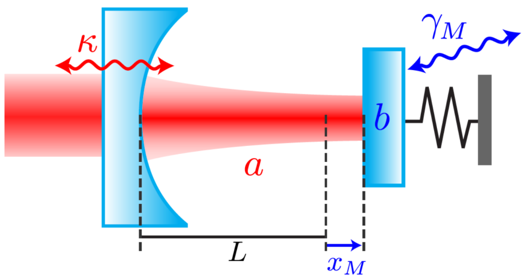

Figure 8: A Fabry-Perot optomechanical cavity comprising one large rigid input mirror and one smaller mirror that can move under the influence of radiation pressure. The cavity has a mean length L, the smaller mirror may be displaced from the mean position by x_M, and we describe the optical cavity field and the mechanical motion with annihilation operators $\hat{a}$, and $\hat{b}$, respectively.