Imperial College London

NOTES

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DEPARTMENT OF PHYSICS

Quantum Optics

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1 A Quantum Mechanics Atom in a Classical Light Field

An atom is described by the Hamiltonian

$$H_a = \frac{p^2}{2m} + V(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) \tag{1}$$

If the atom is interacting with a classical electro-magnetic field, the Hamiltonian is replaced by

$$H_{\boldsymbol{A}} = -\frac{\hbar^2}{2m}\left(\nabla - i\frac{\rho}{\hbar}\boldsymbol{A}\right)^2 + V(\boldsymbol{r}) \tag{2}$$

Many problems are fomulated in terms of a Hamiltonian of the form

$$H_{\boldsymbol{E}} = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) - \rho\boldsymbol{E}\cdot\boldsymbol{r} \tag{3}$$

In most case, the Hamiltonian can be expressed as

$$H = H_{\text{atom}} + H_{\text{interaction}} \tag{4}$$

1.1 Dynamics of Atom in Light-Field

1.1.1 The Propagator

We define the propagator $U(t)$ via the relation

$$|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle \tag{5}$$

for any solution $|\Psi(t)\rangle$ of the Schrödinger equation. By definition, the propagator satisfies the initial condition $U(0) = \mathbb{1}$.
The propagator also satisfies a Schrödinger equation.

$$i|\dot{\Psi}(t)\rangle = i\dot{U}(t)\,|\Psi(0)\rangle = HU(t)\,|\Psi(0)\rangle \tag{6}$$

so that

$$\boxed{i\dot{U} = HU} \tag{7}$$

The ad-joint $U^\dagger(t)$ satisfies

$$-i\dot{U}^\dagger = U^\dagger H^\dagger = U^\dagger H \tag{8}$$

According to this relations, we have

$$i\frac{\partial}{\partial t}\left(UU^\dagger\right) = i\dot{U}U^\dagger + iU\dot{U}^\dagger = HUU^\dagger - UU^\dagger H = [H, UU^\dagger] \tag{9}$$

With the initial condition $U(0)U^\dagger(0) = \mathbb{1}$, this is solved for

$$U(t)U^\dagger(t) = U^\dagger(t)U(t) = \mathbb{1} \tag{10}$$

1.1.2 Perturbation Theory

The Schrödinger equation together with the initial condition $U(0) = \mathbb{1}$ can be rewritten as the integral equation

$$\begin{aligned} U(t) &= \mathbb{1} + \int_0^t \mathrm{d}t' \dot{U}(t') = \mathbb{1} - i \int_0^t \mathrm{d}t' H(t') U(t') \\ &= \mathbb{1} - i \int_0^t \mathrm{d}t' H(t') \left[\mathbb{1} - i \int_0^{t'} \mathrm{d}t'' H(t'') U(t'') \right] \\ &= \mathbb{1} - i \int_0^t \mathrm{d}t' H(t') - \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t') H(t'') U(t'') \end{aligned} \tag{11}$$

For sufficiently short times this can be approximated as

$$U(t) \simeq \mathbb{I} - i \int_0^t \mathrm{d}t' H(t') - \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t') H(t'') \tag{12}$$

For this to be a good approximation, it is essential that ‘magnitude’ of H is sufficiently **small**. It is therefore important to work in a suitable frame. Rather than solving the Schrödinger equation $i\dot{U} = HU$ for U, we can try to solve for V defined via the relation

$$U = U_0 V \tag{13}$$

in terms of a unitary U_0 that we are **free** to choose. The Schrödinger equation

$$i\dot{U} = i\dot{U}_0 V + iU_0 \dot{V} = HU_0 V \tag{14}$$

can now be solved for $\dot{V}$ what yields

$$i\dot{V} = U_0^\dagger H U_0 V - iU_0^\dagger \dot{U}_0 V = \left(U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0 \right) V = \tilde{H} V \tag{15}$$

with the new Hamiltonian

$$\boxed{\tilde{H} = U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0} \tag{16}$$

The goal is then to find U_0 such that the time-dependent perturbation theory is a good approximation.

1.1.3 Atom-Light Hamiltonian

Let’s consider an atom with Hamiltonian H_0 and interaction Hamiltonian H_I

$$H_0 = \sum_j \omega_j \left|\psi_j\right\rangle \left\langle\psi_j\right| \tag{17}$$

$$H_I = \sum_{j,k} \left|\psi_j\right\rangle \left\langle\psi_j\right| H_I \left|\psi_k\right\rangle \left\langle\psi_k\right| = \sum_{j,k} \left\langle\psi_j\right| H_I \left|\psi_k\right\rangle \left|\psi_j\right\rangle \left\langle\psi_k\right| = \sum_{j,k} h_{jk} \left|\psi_j\right\rangle \left\langle\psi_k\right| \tag{18}$$

We choose

$$U_0(t) = \exp(-iH_0 t) = \sum_j \mathrm{e}^{-i\omega_j t} \left|\psi_j\right\rangle \left\langle\psi_j\right| \tag{19}$$

such that $-iU_0^\dagger \dot{U}_0 = -H_0$. Then the transforemed Hamiltonian reads

$$\begin{aligned}
\tilde{H} &= U_0^\dagger (H_0 + H_I) U_0 - iU_0^\dagger \dot{U}_0 \\
&= U_0^\dagger H_0 U_0 + U_0^\dagger H_I U_0 - iU_0^\dagger \dot{U}_0 \\
&= U_0^\dagger H_I U_0 \\
&= \sum_l \mathrm{e}^{i\omega_l t} \left|\psi_l\right\rangle \left\langle\psi_l\right| \sum_{jk} h_{jk} \left|\psi_j\right\rangle \left\langle\psi_k\right| \exp(-iH_0 t) \\
&= \sum_{jk} h_{jk} \exp(i\omega_k t) \left|\psi_j\right\rangle \left\langle\psi_k\right| \exp(-i\omega_k t) \\
&= \sum_{jk} h_{jk} \exp(i(\omega_j - \omega_k)t) \left|\psi_j\right\rangle \left\langle\psi_k\right|
\end{aligned} \tag{20}$$

where $h_{jk} = \left\langle\psi_j\right| H_{jk} \left|\psi_k\right\rangle$ is the oscillating term with frequency ν, and $\cos \nu t = (\mathrm{e}^{i\nu t} + \mathrm{e}^{-i\nu t})/2$. The oscillating functions result in a vanishing integral in the integration $\int_0^t \mathrm{d}t' \cdots$. If we choose the ground state and another selected eigenstate, we can approximate the atom as a two-level system.

1.1.4 The Pauli Matrices

The Pauli matrices satisfies the relation

$$[\sigma_\alpha, \sigma_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad \{\sigma_\alpha, \sigma_\beta\} = 0 \tag{21}$$

In terms of the eigenstates $|g\rangle$ and $|e\rangle$, we have

$$\sigma_z |g\rangle = -|g\rangle, \qquad \sigma_z |e\rangle = |e\rangle \tag{22}$$

$$\sigma_x |g\rangle = |e\rangle, \qquad \sigma_x |e\rangle = |g\rangle \tag{23}$$

$$\sigma_y |g\rangle = -i|e\rangle, \qquad \sigma_y |e\rangle = i|g\rangle \tag{24}$$

1.2 Hamiltonian and Propagator of a Two-Level Atom

The Hamiltonian of a two-level atom (see Fig.1) is given by

$$\begin{aligned}
H &= \omega_g |g\rangle\langle g| + \omega_e |e\rangle\langle e| \\
&= \frac{\omega_e + \omega_g}{2}(|g\rangle\langle g| + |e\rangle\langle e|) + \frac{\omega_e - \omega_g}{2}(|e\rangle\langle e| - |g\rangle\langle g|) \\
&= \frac{\omega_e + \omega_g}{2}\mathbb{1} + \frac{\omega}{2}\sigma_z \simeq \frac{\omega}{2}\sigma_z
\end{aligned} \tag{25}$$

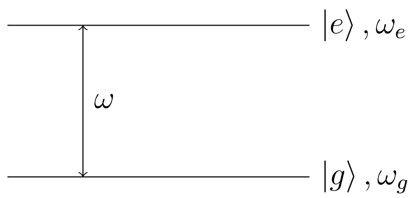

Figure 1: A two-level atom with resonance frequency ω.

The corresponding propagator reads

$$U(t) = \exp\Big(-i\frac{\omega}{2}\sigma_z t\Big) = \mathbb{1}\cos\left(\frac{\omega}{2}t\right) - i\sigma_z \sin\left(\frac{\omega}{2}t\right) \tag{26}$$

Back to the Schrödinger equation, we have

$$\begin{aligned} i\dot{U} &= -i\frac{\omega}{2}\mathbb{1}\sin\left(\frac{\omega}{2}t\right) + \frac{\omega}{2}\sigma_z\cos\left(\frac{\omega}{2}t\right) \\ &= \frac{\omega}{2}\sigma_z\left[\mathbb{1}\cos\left(\frac{\omega}{2}t\right) - i\sigma_z\sin\left(\frac{\omega}{2}t\right)\right] = HU \end{aligned} \tag{27}$$

1.3 The Two-Level Atom in a Monochromatic Light Field

The Hamiltonian for the atom interacting with a light field in two-level approximation reads

$$H = \frac{\omega}{2}\sigma_z + \Omega_R\sigma_x\cos(\nu t) \tag{28}$$

The prefactor Ω_R is called *Rabi-frequency*; it is proportional to the intensity of the light field. And ν is frequency of light.

In order to find the solution of the Schrödinger equation, it is helpful to consider the transformation

$$U_0 = \exp\Big(-i\frac{\eta}{2}\sigma_z t\Big) \tag{29}$$

and

$$\dot{U}_0 = -i\frac{\eta}{2}\sigma_z\exp\Big(-i\frac{\eta}{2}\sigma_z t\Big) = -i\frac{\eta}{2}\sigma_z U_0 \tag{30}$$

So we have

$$U_0^\dagger\dot{U}_0 = -i\frac{\eta}{2}\sigma_z \tag{31}$$

Then we construct the transformed Hamiltonian $\tilde{H}$. With $U_0^\dagger\sigma_z U_0 = \sigma_z$ and $U_0^\dagger\sigma_x U_0 = \sigma_+\mathrm{e}^{i\eta t} + \sigma_-\mathrm{e}^{-i\eta t}$, the explicit form of H reads

$$\begin{aligned} \tilde{H} &= U_0^\dagger H U_0 - iU_0^\dagger\dot{U}_0 \\ &= \frac{\omega-\eta}{2}\sigma_z + \Omega_R\left(\sigma_+\mathrm{e}^{i\eta t} + \sigma_-\mathrm{e}^{-i\eta t}\right)\cos(\nu t) \\ &= \frac{\omega-\eta}{2}\sigma_z + \frac{\Omega_R}{2}\left[\sigma_+\mathrm{e}^{i(\eta-\nu)t} + \sigma_-\mathrm{e}^{-i(\eta-\nu)t}\right] + \frac{\Omega_R}{2}\left[\sigma_+\mathrm{e}^{i(\eta+\nu)t} + \sigma_-\mathrm{e}^{-i(\eta+\nu)t}\right] \end{aligned} \tag{32}$$

After the *rotating wave approximation (RWA)*, we have

$$H' = \frac{\omega-\eta}{2}\sigma_z + \frac{\Omega_R}{2}\left[\sigma_+\mathrm{e}^{i(\eta-\nu)t} + \sigma_-\mathrm{e}^{-i(\eta-\nu)t}\right] \tag{33}$$

Let's consider the case $\eta = \nu$,

$$H' = \frac{\omega-\nu}{2}\sigma_z + \frac{\Omega_R}{2}(\sigma_+ + \sigma_-) = \frac{\omega-\nu}{2}\sigma_z + \frac{1}{2}\Omega_R\sigma_x \tag{34}$$

The associated propagator

$$\boxed{\begin{aligned}\exp(-iH't) &= \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - \frac{2i}{\Omega_G}H'\sin\left(\frac{1}{2}\Omega_G t\right) \\ &= \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - i\left(\frac{\omega-\nu}{\Omega_G}\sigma_z + \frac{\Omega_R}{\Omega_G}\sigma_x\right)\sin\left(\frac{1}{2}\Omega_G t\right)\end{aligned}} \tag{35}$$

where

$$\Omega_G = \sqrt{(\omega-\nu)^2 + \Omega_R^2} \tag{36}$$

is called *generalised Rabi frequency*. It is helpful to notice

$$(H')^2 = \frac{(\omega-\nu)^2}{4}\sigma_z^2 + \frac{1}{4}\Omega_R^2\sigma_x^2 + \frac{1}{4}(\omega-\nu)\Omega_R\{\sigma_z,\sigma_x\} = \frac{1}{4}\Omega_G^2\mathbb{1} \tag{37}$$

which implies that $\mathbb{1} = \frac{4}{\Omega_G^2}(H')^2$. Taking the time-derivative yields

$$\begin{aligned} i\frac{\partial}{\partial t}\exp(-iH't) &= -i\frac{\Omega_G}{2}\mathbb{I}\sin\left(\frac{1}{2}\Omega_G t\right) + H'\cos\left(\frac{1}{2}\Omega_G t\right) \\ &= H'\cos\left(\frac{1}{2}\Omega_G t\right) - i\frac{\Omega_G}{2}\frac{4}{\Omega_G^2}(H')^2\sin\left(\frac{1}{2}\Omega_G t\right) \\ &= H'\left[\mathbb{I}\cos\left(\frac{1}{2}\Omega_G t\right) - i\frac{2}{\Omega_G}H'\sin\left(\frac{1}{2}\Omega_G t\right)\right] \\ &= H'\exp(-iH't) \end{aligned} \tag{38}$$

The expression given in eqn.(35) is the correct solution of the Schrödinger equation with the Hamiltonian H'.

1.3.1 Resonant Driving

If the light-field is on resonance with the atomic transition, *i.e.* $\omega - \nu = 0$, this simplifies to

$$\exp(-iH't) = \mathbb{1}\cos\left(\frac{1}{2}\Omega_R t\right) - i\sigma_x\sin\left(\frac{1}{2}\Omega_R t\right) \tag{39}$$

Applying this to the ground state as initial state yields

$$\exp(-iH't)\,|g\rangle = \cos\left(\frac{1}{2}\Omega_R t\right)|g\rangle - i\sin\left(\frac{1}{2}\Omega_R t\right)|e\rangle \tag{40}$$

Together with the factor U_0, we have

$$\begin{aligned} &\exp\Big(-i\frac{\omega}{2}\sigma_z t\Big)\exp(-iH't)\,|g\rangle \\ &= \exp\Big(i\frac{\omega}{2}t\Big)\cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\exp\Big(-i\frac{\omega}{2}t\Big)\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle \\ &= \exp\Big(i\frac{\omega}{2}t\Big)\left[\cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\exp(-i\omega t)\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle\right] \end{aligned} \tag{41}$$

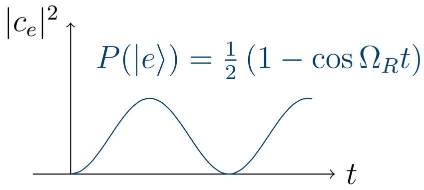

Figure 2: The probability to find the atom in the excited state in Rabi oscillation.

The probability to find the atom in the excited state or ground state is given by

$$|c_e(t)|^2 = \left[\sin\left(\frac{\Omega_R}{2}t\right)\right]^2 = \frac{1}{2}(1 - \cos\Omega_R t) \tag{42}$$

$$|c_g(t)|^2 = \left[\cos\left(\frac{\Omega_R}{2}t\right)\right]^2 = \frac{1}{2}(1 + \cos\Omega_R t) \tag{43}$$

They are called Rabi oscillation.

1.3.2 Off-Resonant Driving

If the light field is far off-resonant, *i.e.* $|\nu - \omega| \gg \Omega_R$, the approximations

$$\frac{\omega - \nu}{\Omega_G} = \frac{\omega - \nu}{\sqrt{(\omega - \nu)^2 + \Omega_R^2}} \simeq \frac{\nu - \omega}{|\nu - \omega|} = \pm 1 \tag{44}$$

$$\frac{\Omega_R}{\Omega_G} = \frac{\Omega_R}{\sqrt{(\omega - \nu)^2 + \Omega_R^2}} \simeq \frac{\Omega_R}{|\nu - \omega|} \ll 1 \tag{45}$$

so the propagator eqn.(35) becomes

$$\exp(-iH't) = \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - i\sin\left(\frac{1}{2}\Omega_G t\right) \tag{46}$$

Let Ω_G do a Taylor expansion at $\Omega_R = 0$

$$\Omega_G = |\nu - \omega| + \frac{\Omega_R^2}{2|\nu - \omega|} + \mathcal{O}(\Omega_R^4) \tag{47}$$

So we have

$$\Omega_G - (\omega - \nu) \simeq \frac{\Omega_R^2}{2|\delta|} \tag{48}$$

with the detuning $\delta = \omega - \nu$

1.3.3 Ramsey

In the case of $\nu = \omega$, we found the propagator

$$U_x = \mathbb{1}\cos\left(\frac{1}{2}\Omega_R t\right) - i\sigma_x \sin\left(\frac{1}{2}\Omega_R t\right) \tag{49}$$

in the interaction picture. For a duration $T = \frac{\pi}{2\Omega_R}$, this reduce to

$$U_x(T) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_x) \tag{50}$$

Assuming the atom initially in its ground state $|g\rangle$, we obtain

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}(|g\rangle - i\,|e\rangle) \tag{51}$$

A measurement of the population of the eigenstates would yield 50% ground state and 50% excited state.

$$H_\phi = \frac{\omega}{2}\sigma_z + \Omega_R\sigma_x \cos(\nu t + \phi) \tag{52}$$

the associated propagator

$$U_\phi(T) = \frac{1}{\sqrt{2}}\left[\mathbb{1} - i(\cos\phi\sigma_x + \sin\phi\sigma_y)\right] \tag{53}$$

Applying $U_\phi(T)$ to the state $|\Psi(T)\rangle$

$$\begin{aligned} |\Psi(2T)\rangle =& U_\phi(T)\,|\Psi(T)\rangle \\ =& \frac{1}{2}\left[\mathbb{1} - i(\cos\phi\sigma_x + \sin\phi\sigma_y)\right](|g\rangle - i\,|e\rangle) \\ =& -i\left(\exp\left(i\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)|g\rangle + \exp\left(-i\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)|e\rangle\right) \end{aligned} \tag{54}$$

The probability to find the atom in the ground state or the excited state thus oscillates with ϕ.