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NOTES

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DEPARTMENT OF PHYSICS

Quantum Optics

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Part I
Theory

1   A Quantum Mechanics Atom in a Classical Light Field

An atom is described by the Hamiltonian

$$H_a = \frac{p^2}{2m} + V(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) \tag{1}$$

If the atom is interacting with a classical electro-magnetic field, the Hamiltonian is replaced by

$$H_{\boldsymbol{A}} = -\frac{\hbar^2}{2m}\left(\nabla - i\frac{\rho}{\hbar}\boldsymbol{A}\right)^2 + V(\boldsymbol{r}) \tag{2}$$

Many problems are fomulated in terms of a Hamiltonian of the form

$$H_{\boldsymbol{E}} = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) - \rho\boldsymbol{E}\cdot\boldsymbol{r} \tag{3}$$

In most case, the Hamiltonian can be expressed as

$$H = H_{\text{atom}} + H_{\text{interaction}} \tag{4}$$

1.1   Dynamics of Atom in Light-Field

1.1.1   The Propagator

We define the propagator $U(t)$ via the relation

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle \tag{5}$$

for any solution $|\Psi(t)\rangle$ of the Schrödinger equation. By definition, the propagator satisfies the initial condition $U(0) = \mathbb{1}$.
The propagator also satisfies a Schrödinger equation.

$$i|\dot{\Psi}(t)\rangle = i\dot{U}(t)|\Psi(0)\rangle = HU(t)|\Psi(0)\rangle \tag{6}$$

so that

$$\boxed{i\dot{U} = HU} \tag{7}$$

The ad-joint $U^\dagger(t)$ satisfies

$$-i\dot{U}^\dagger = U^\dagger H^\dagger = U^\dagger H \tag{8}$$

According to this relations, we have

$$i\frac{\partial}{\partial t}\left(UU^\dagger\right) = i\dot{U}U^\dagger + iU\dot{U}^\dagger = HUU^\dagger - UU^\dagger H = [H, UU^\dagger] \tag{9}$$

With the initial condition $U(0)U^\dagger(0) = \mathbb{1}$, this is solved for

$$U(t)U^\dagger(t) = U^\dagger(t)U(t) = \mathbb{1} \tag{10}$$

1.1.2 Perturbation Theory

The Schrödinger equation together with the initial condition $U(0) = \mathbb{1}$ can be rewritten as the integral equation

$$
\begin{aligned}
U(t) &= \mathbb{1} + \int_0^t \mathrm{d}t' \dot{U}(t') = \mathbb{1} - i \int_0^t \mathrm{d}t' H(t')U(t') \\
&= \mathbb{1} - i \int_0^t \mathrm{d}t' H(t') \left[\mathbb{1} - i \int_0^{t'} \mathrm{d}t'' H(t'')U(t'') \right] \\
&= \mathbb{1} - i \int_0^t \mathrm{d}t' H(t') - \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t')H(t'')U(t'')
\end{aligned}
\tag{11}
$$

For sufficiently short times this can be approximated as

$$
U(t) \simeq \mathbb{I} - i \int_0^t \mathrm{d}t' H(t') - \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t')H(t'')
\tag{12}
$$

For this to be a good approximation, it is essential that 'magnitude' of H is sufficiently **small**. It is therefore important to work in a suitable frame. Rather than solving the Schrödinger equation $i\dot{U} = HU$ for U, we can try to solve for V defined via the relation

$$
U = U_0 V
\tag{13}
$$

in terms of a unitary U_0 that we are **free** to choose. The Schrödinger equation

$$
i\dot{U} = i\dot{U}_0 V + iU_0 \dot{V} = HU_0 V
\tag{14}
$$

can now be solved for $\dot{V}$ what yields

$$
i\dot{V} = U_0^\dagger H U_0 V - iU_0^\dagger \dot{U}_0 V = \left(U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0 \right) V = \tilde{H}V
\tag{15}
$$

with the new Hamiltonian

$$
\boxed{\tilde{H} = U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0}
\tag{16}
$$

The goal is then to find U_0 such that the time-dependent perturbation theory is a good approximation.

1.1.3 Atom-Light Hamiltonian

Let's consider an atom with Hamiltonian H_0 and interaction Hamiltonian H_I

$$
H_0 = \sum_j \omega_j \left| \psi_j \right\rangle \left\langle \psi_j \right|
\tag{17}
$$

$$
H_I = \sum_{j,k} \left| \psi_j \right\rangle \left\langle \psi_j \right| H_I \left| \psi_k \right\rangle \left\langle \psi_k \right| = \sum_{j,k} \left\langle \psi_j \right| H_I \left| \psi_k \right\rangle \left| \psi_j \right\rangle \left\langle \psi_k \right| = \sum_{j,k} h_{jk} \left| \psi_j \right\rangle \left\langle \psi_k \right|
\tag{18}
$$

We choose

$$
U_0(t) = \exp(-iH_0 t) = \sum_j \mathrm{e}^{-i\omega_j t} \left| \psi_j \right\rangle \left\langle \psi_j \right|
\tag{19}
$$

such that $-iU_0^\dagger \dot{U}_0 = -H_0$. Then the transforemed Hamiltonian reads

$$
\begin{aligned}
\tilde{H} &= U_0^\dagger (H_0 + H_I) U_0 - iU_0^\dagger \dot{U}_0 \\
&= U_0^\dagger H_0 U_0 + U_0^\dagger H_I U_0 - iU_0^\dagger \dot{U}_0 \\
&= U_0^\dagger H_I U_0 \\
&= \sum_l e^{i\omega_l t} |\psi_l\rangle \langle\psi_l| \sum_{jk} h_{jk} |\psi_j\rangle \langle\psi_k| \exp(-iH_0 t) \\
&= \sum_{jk} h_{jk} \exp(i\omega_k t) |\psi_j\rangle \langle\psi_k| \exp(-i\omega_k t) \\
&= \sum_{jk} h_{jk} \exp(i(\omega_j - \omega_k)t) |\psi_j\rangle \langle\psi_k|
\end{aligned}
\tag{20}
$$

where $h_{jk} = \langle\psi_j| H_{jk} |\psi_k\rangle$ is the oscillating term with frequency ν, and $\cos\nu t = (e^{i\nu t} + e^{-i\nu t})/2$. The oscillating functions result in a vanishing integral in the integration $\int_0^t \mathrm{d}t' \cdots$. If we choose the ground state and another selected eigenstate, we can approximate the atom as a two-level system.

1.1.4   The Pauli Matrices

The Pauli matrices satisfies the relation

$$
[\sigma_\alpha, \sigma_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad \{\sigma_\alpha, \sigma_\beta\} = 0
\tag{21}
$$

In terms of the eigenstates $|g\rangle$ and $|e\rangle$, we have

$$
\sigma_z |g\rangle = - |g\rangle, \qquad \sigma_z |e\rangle = |e\rangle
\tag{22}
$$

$$
\sigma_x |g\rangle = |e\rangle, \qquad \sigma_x |e\rangle = |g\rangle
\tag{23}
$$

$$
\sigma_y |g\rangle = -i |e\rangle, \qquad \sigma_y |e\rangle = i |g\rangle
\tag{24}
$$

1.2   Hamiltonian and Propagator of a Two-Level Atom

The Hamiltonian of a two-level atom (see Fig.1) is given by

$$
\begin{aligned}
H &= \omega_g |g\rangle \langle g| + \omega_e |e\rangle \langle e| \\
&= \frac{\omega_e + \omega_g}{2} (|g\rangle \langle g| + |e\rangle \langle e|) + \frac{\omega_e - \omega_g}{2} (|e\rangle \langle e| - |g\rangle \langle g|) \\
&= \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\omega}{2}\sigma_z \simeq \frac{\omega}{2}\sigma_z
\end{aligned}
\tag{25}
$$
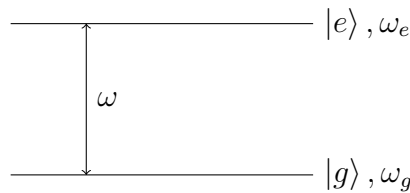
Figure 1: A two-level atom with resonance frequency ω.

The corresponding propagator reads

$$U(t) = \exp\left(-i\frac{\omega}{2}\sigma_z t\right) = \mathbb{1}\cos\left(\frac{\omega}{2}t\right) - i\sigma_z\sin\left(\frac{\omega}{2}t\right) \tag{26}$$

Back to the Schrödinger equation, we have

$$
\begin{aligned}
i\dot{U} &= -i\frac{\omega}{2}\mathbb{1}\sin\left(\frac{\omega}{2}t\right) + \frac{\omega}{2}\sigma_z\cos\left(\frac{\omega}{2}t\right) \\
&= \frac{\omega}{2}\sigma_z\left[\mathbb{1}\cos\left(\frac{\omega}{2}t\right) - i\sigma_z\sin\left(\frac{\omega}{2}t\right)\right] = HU
\end{aligned}
\tag{27}
$$

1.3 The Two-Level Atom in a Monochromatic Light Field

The Hamiltonian for the atom interacting with a light field in two-level approximation reads

$$H = \frac{\omega}{2}\sigma_z + \Omega_R\sigma_x\cos(\nu t) \tag{28}$$

The prefactor Ω_R is called *Rabi-frequency*; it is proportional to the intensity of the light field. And ν is frequency of light.

In order to find the solution of the Schrödinger equation, it is helpful to consider the transformation

$$U_0 = \exp\left(-i\frac{\eta}{2}\sigma_z t\right) \tag{29}$$

and

$$\dot{U}_0 = -i\frac{\eta}{2}\sigma_z\exp\left(-i\frac{\eta}{2}\sigma_z t\right) = -i\frac{\eta}{2}\sigma_z U_0 \tag{30}$$

So we have

$$U_0^\dagger\dot{U}_0 = -i\frac{\eta}{2}\sigma_z \tag{31}$$

Then we construct the transformed Hamiltonian $\tilde{H}$. With $U_0^\dagger\sigma_z U_0 = \sigma_z$ and $U_0^\dagger\sigma_x U_0 = \sigma_+e^{i\eta t} + \sigma_-e^{-i\eta t}$, the explicit form of H reads

$$
\begin{aligned}
\tilde{H} &= U_0^\dagger H U_0 - iU_0^\dagger\dot{U}_0 \\
&= \frac{\omega-\eta}{2}\sigma_z + \Omega_R\left(\sigma_+e^{i\eta t} + \sigma_-e^{-i\eta t}\right)\cos(\nu t) \\
&= \frac{\omega-\eta}{2}\sigma_z + \frac{\Omega_R}{2}\left[\sigma_+e^{i(\eta-\nu)t} + \sigma_-e^{-i(\eta-\nu)t}\right] + \frac{\Omega_R}{2}\left[\sigma_+e^{i(\eta+\nu)t} + \sigma_-e^{-i(\eta+\nu)t}\right]
\end{aligned}
\tag{32}
$$

After the *rotating wave approximation (RWA)*, we have

$$H' = \frac{\omega-\eta}{2}\sigma_z + \frac{\Omega_R}{2}\left[\sigma_+e^{i(\eta-\nu)t} + \sigma_-e^{-i(\eta-\nu)t}\right] \tag{33}$$

Let's consider the case $\eta = \nu$,

$$H' = \frac{\omega-\nu}{2}\sigma_z + \frac{\Omega_R}{2}\left(\sigma_+ + \sigma_-\right) = \frac{\omega-\nu}{2}\sigma_z + \frac{1}{2}\Omega_R\sigma_x \tag{34}$$

The associated propagator

$$
\boxed{
\begin{aligned}
\exp(-iH't) &= \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - \frac{2i}{\Omega_G}H'\sin\left(\frac{1}{2}\Omega_G t\right) \\
&= \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - i\left(\frac{\omega-\nu}{\Omega_G}\sigma_z + \frac{\Omega_R}{\Omega_G}\sigma_x\right)\sin\left(\frac{1}{2}\Omega_G t\right)
\end{aligned}
}
\tag{35}
$$

where

$$
\Omega_G = \sqrt{(\omega-\nu)^2 + \Omega_R^2} \tag{36}
$$

is called *generalised Rabi frequency*. It is helpful to notice

$$
(H')^2 = \frac{(\omega-\nu)^2}{4}\sigma_z^2 + \frac{1}{4}\Omega_R^2\sigma_x^2 + \frac{1}{4}(\omega-\nu)\Omega_R\{\sigma_z,\sigma_x\} = \frac{1}{4}\Omega_G^2\mathbb{1} \tag{37}
$$

which implies that $\mathbb{1} = \frac{4}{\Omega_G^2}(H')^2$. Taking the time-derivative yields

$$
\begin{aligned}
i\frac{\partial}{\partial t}\exp(-iH't) &= -i\frac{\Omega_G}{2}\mathbb{1}\sin\left(\frac{1}{2}\Omega_G t\right) + H'\cos\left(\frac{1}{2}\Omega_G t\right) \\
&= H'\cos\left(\frac{1}{2}\Omega_G t\right) - i\frac{\Omega_G}{2}\frac{4}{\Omega_G^2}(H')^2\sin\left(\frac{1}{2}\Omega_G t\right) \\
&= H'\left[\mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - i\frac{2}{\Omega_G}H'\sin\left(\frac{1}{2}\Omega_G t\right)\right] \\
&= H'\exp(-iH't)
\end{aligned}
\tag{38}
$$

The expression given in eqn.(35) is the correct solution of the Schrödinger equation with the Hamiltonian H'.

1.3.1 Resonant Driving

If the light-field is on resonance with the atomic transition, *i.e.* $\omega - \nu = 0$, this simplifies to

$$
\exp(-iH't) = \mathbb{1}\cos\left(\frac{1}{2}\Omega_R t\right) - i\sigma_x\sin\left(\frac{1}{2}\Omega_R t\right) \tag{39}
$$

Applying this to the ground state as initial state yields

$$
\exp(-iH't)\,|g\rangle = \cos\left(\frac{1}{2}\Omega_R t\right)|g\rangle - i\sin\left(\frac{1}{2}\Omega_R t\right)|e\rangle \tag{40}
$$

Together with the factor U_0, we have

$$
\begin{aligned}
&\exp\left(-i\frac{\omega}{2}\sigma_z t\right)\exp(-iH't)\,|g\rangle \\
&= \exp\left(i\frac{\omega}{2}t\right)\cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\exp\left(-i\frac{\omega}{2}t\right)\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle \\
&= \exp\left(i\frac{\omega}{2}t\right)\left[\cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\exp(-i\omega t)\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle\right]
\end{aligned}
\tag{41}
$$
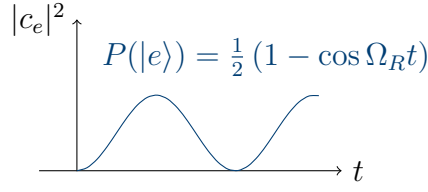
Figure 2: The probability to find the atom in the excited state in Rabi oscillation.

The probability to find the atom in the excited state or ground state is given by

$$|c_e(t)|^2 = \left[\sin\left(\frac{\Omega_R}{2}t\right)\right]^2 = \frac{1}{2}(1 - \cos\Omega_R t) \tag{42}$$

$$|c_g(t)|^2 = \left[\cos\left(\frac{\Omega_R}{2}t\right)\right]^2 = \frac{1}{2}(1 + \cos\Omega_R t) \tag{43}$$

They are called Rabi oscillation.

1.3.2   Off-Resonant Driving

If the light field is far off-resonant, *i.e.* $|\nu - \omega| \gg \Omega_R$, the approximations

$$\frac{\omega - \nu}{\Omega_G} = \frac{\omega - \nu}{\sqrt{(\omega - \nu)^2 + \Omega_R^2}} \simeq \frac{\nu - \omega}{|\nu - \omega|} = \pm 1 \tag{44}$$

$$\frac{\Omega_R}{\Omega_G} = \frac{\Omega_R}{\sqrt{(\omega - \nu)^2 + \Omega_R^2}} \simeq \frac{\Omega_R}{|\nu - \omega|} \ll 1 \tag{45}$$

so the propagator eqn.(35) becomes

$$\exp(-iH't) = \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - i\sin\left(\frac{1}{2}\Omega_G t\right) \tag{46}$$

Let Ω_G do a Taylor expansion at $\Omega_R = 0$

$$\Omega_G = |\nu - \omega| + \frac{\Omega_R^2}{2|\nu - \omega|} + \mathcal{O}(\Omega_R^4) \tag{47}$$

So we have

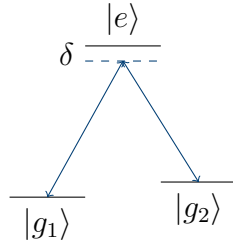$$\Omega_G - (\omega - \nu) \simeq \frac{\Omega_R^2}{2|\delta|} \tag{48}$$

with the detuning $\delta = \omega - \nu$.

1.3.3   Ramsey

In the case of $\nu = \omega$, we found the propagator

$$U_x = \mathbb{1}\cos\left(\frac{1}{2}\Omega_R t\right) - i\sigma_x \sin\left(\frac{1}{2}\Omega_R t\right) \tag{49}$$

Figure 3: The three-level atom.

in the interaction picture. For a duration $T = \frac{\pi}{2\Omega_R}$, this reduce to

$$U_x(T) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_x) \tag{50}$$

Assuming the atom initially in its ground state $|g\rangle$, we obtain

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}(|g\rangle - i\,|e\rangle) \tag{51}$$

A measurement of the population of the eigenstates would yield 50% ground state and 50% excited state.

$$H_\phi = \frac{\omega}{2}\sigma_z + \Omega_R\sigma_x\cos(\nu t + \phi) \tag{52}$$

the associated propagator

$$U_\phi(T) = \frac{1}{\sqrt{2}}\left[\mathbb{1} - i(\cos\phi\sigma_x + \sin\phi\sigma_y)\right] \tag{53}$$

Applying $U_\phi(T)$ to the state $|\Psi(T)\rangle$

$$\begin{aligned}
|\Psi(2T)\rangle =& U_\phi(T)\,|\Psi(T)\rangle \\
=& \frac{1}{2}\left[\mathbb{1} - i(\cos\phi\sigma_x + \sin\phi\sigma_y)\right](|g\rangle - i\,|e\rangle) \\
=& -i\left(\exp\left(i\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)|g\rangle + \exp\left(-i\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)|e\rangle\right)
\end{aligned} \tag{54}$$

The probability to find the atom in the ground state or the excited state thus oscillates with ϕ.

1.4 The Three-Level Atom

$$H_1 = \frac{\Omega_R}{2\sqrt{2}}\left(|e\rangle\langle g_1|\,\mathrm{e}^{i\delta t} + |g_1\rangle\langle e|\,\mathrm{e}^{-i\delta t}\right) \tag{55}$$

$$H_2 = \frac{\Omega_R}{2\sqrt{2}}\left(|e\rangle\langle g_2|\,\mathrm{e}^{i\delta t} + |g_2\rangle\langle e|\,\mathrm{e}^{-i\delta t}\right) \tag{56}$$

$$H = H_1 + H_2 = \frac{\Omega_R}{2} \left(|e\rangle \frac{\langle g_1| + \langle g_2|}{\sqrt{2}} \mathrm{e}^{i\delta t} + \frac{|g_1\rangle + |g_2\rangle}{\sqrt{2}} \langle e| \mathrm{e}^{-i\delta t} \right)$$

$$= \frac{\Omega_R}{2} \left(|e\rangle \langle g| \mathrm{e}^{i\delta t} + |g\rangle \langle e| \mathrm{e}^{-i\delta t} \right)$$

$$= \frac{\Omega_R}{4} \left[(\sigma_x - i\sigma_y)(\cos \delta t + i \sin \delta t) + (\sigma_x + i\sigma_y)(\cos \delta t - i \sin \delta t) \right]$$

$$= \frac{\Omega_R}{2} \left(\sigma_x \cos \delta t + \sigma_y \sin \delta t \right)$$

(57)

where $|g\rangle = (|g_1\rangle + |g_2\rangle)/\sqrt{2}$. The propagator

$$U(t) \simeq \mathbb{I} - i \int_0^t \mathrm{d}t_1 H(t_1) - \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 H(t_1) H(t_2)$$

(58)

Now we calculate $U(t)$ at $t = 2\pi/\delta$. The first order

$$U^{(1)} \left(\frac{2\pi}{\delta} \right) = -i\frac{\Omega_R}{2} \int_0^{2\pi/\delta} \mathrm{d}t_1 \left(\sigma_x \cos \delta t_1 + \sigma_y \sin \delta t_1 \right) = 0$$

(59)

and the second

$$U^{(2)} \left(\frac{2\pi}{\delta} \right) = -\frac{\Omega_R^2}{4} \int_0^{2\pi/\delta} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \left(\sigma_x \cos \delta t_1 + \sigma_y \sin \delta t_1 \right) \left(\sigma_x \cos \delta t_2 + \sigma_y \sin \delta t_2 \right)$$

(60)

To do so, we need some integrals

$$\int_0^{\frac{2\pi}{\delta}} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cos(\delta t_1) \cos(\delta t_2) = 0$$

(61)

$$\int_0^{\frac{2\pi}{\delta}} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \sin(\delta t_1) \sin(\delta t_2) = 0$$

(62)

$$\int_0^{\frac{2\pi}{\delta}} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cos(\delta t_1) \sin(\delta t_2) = -\frac{1}{2\delta} \frac{2\pi}{\delta}$$

(63)

$$\int_0^{\frac{2\pi}{\delta}} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \sin(\delta t_1) \cos(\delta t_2) = \frac{1}{2\delta} \frac{2\pi}{\delta}$$

(64)

We thus obtain the perturbative expression

$$U \left(\frac{2\pi}{\delta} \right) \simeq \mathbb{1} + \frac{\Omega_R^2}{4} \frac{1}{2\delta} [\sigma_x, \sigma_y] \frac{2\pi}{\delta} = \mathbb{1} + i\frac{\Omega_R^2}{4\delta} \sigma_z \frac{2\pi}{\delta}$$

(65)

The propagator looks as if it was induced by the Hamiltonian

$$H_e = -\frac{\Omega_R^2}{4\delta} \sigma_z = \frac{\Omega_e}{2} \sigma_z$$

(66)

Now we can return to the explicit three levels, and obtain

$$H_e = \frac{\Omega_e}{2} (|e\rangle \langle e| - |g\rangle \langle g|)$$

$$= \frac{\Omega_e}{2} \left(|e\rangle \langle e| - \frac{1}{2} (|g_1\rangle \langle g_1| + |g_1\rangle \langle g_2| + |g_2\rangle \langle g_1| + |g_2\rangle \langle g_2|) \right)$$

(67)

The eigenstates are $|e\rangle , |g\rangle$ and $(|g_1\rangle - |g_2\rangle)\sqrt{2}$

$$H_e |e\rangle = \frac{\Omega_e}{2} |e\rangle \tag{68}$$

$$H_e |g\rangle = -\frac{\Omega_e}{2} |g\rangle \tag{69}$$

$$H_e \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}} = 0 \tag{70}$$

with the eigenvalues $\frac{\Omega_e}{2}$, $-\frac{\Omega_e}{2}$ and 0, we can now express the propagator as

$$\exp(-iH_e t) = \exp\left(-i\frac{\Omega_e}{2}t\right) |e\rangle \langle e| + \exp\left(i\frac{\Omega_e}{2}t\right) |g\rangle \langle g| + \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}} \frac{\langle g_1| - \langle g_2|}{\sqrt{2}} \tag{71}$$

Applying this to the initial state $|g_1\rangle$ yields

$$\begin{aligned}
\exp(-iH_e t) |g_1\rangle &= \exp\left(i\frac{\Omega_e}{2}t\right) \frac{|g_1\rangle + |g_2\rangle}{2} + \frac{|g_1\rangle - |g_2\rangle}{2} \\
&= \frac{1}{2}\left(\exp\left(i\frac{\Omega_e}{2}t\right) + 1\right) |g_1\rangle + \frac{1}{2}\left(\exp\left(i\frac{\Omega_e}{2}t\right) - 1\right) |g_1\rangle \\
&= \exp\left(i\frac{\Omega_e}{4}t\right)\left(\cos\left(\frac{\Omega_e}{4}t\right)|g_1\rangle + i\sin\left(\frac{\Omega_e}{4}t\right)|g_2\rangle\right)
\end{aligned} \tag{72}$$

1.5 Bloch Equations

The Bloch equations

$$\langle\Psi| \mathbb{1} |\Psi\rangle = 1 \tag{73}$$

$$\langle\Psi| \sigma_x |\Psi\rangle = S_x \tag{74}$$

$$\langle\Psi| \sigma_y |\Psi\rangle = S_y \tag{75}$$

$$\langle\Psi| \sigma_z |\Psi\rangle = S_z \tag{76}$$

that define the Bloch vector

$$\boldsymbol{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \tag{77}$$

For the state $|g\rangle$ one obtains

$$\langle g| \sigma_x |g\rangle = \langle g|e\rangle = 0 \tag{78}$$

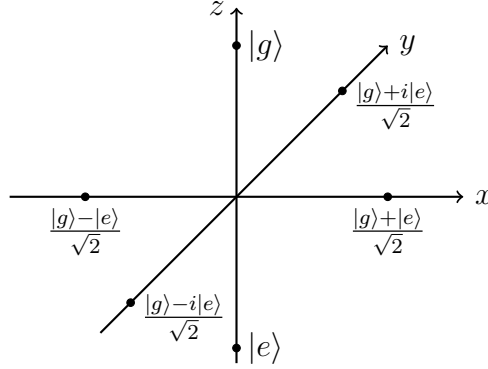$$\langle g| \sigma_y |g\rangle = -i \langle g|e\rangle = 0 \tag{79}$$

$$\langle g| \sigma_z |g\rangle = -\langle g|g\rangle = -1 \tag{80}$$

For the state $|+\rangle$ one obtains

$$\langle +| \sigma_x |+\rangle = \langle +|+\rangle = 1 \tag{81}$$

$$\langle +| \sigma_y |+\rangle = i \langle +|-\rangle = 0 \tag{82}$$

$$\langle +| \sigma_z |+\rangle = -\langle +|-\rangle = 0 \tag{83}$$

Figure 4: The eigenstates of $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ lie on the x, y and z axis.

1.6 Dynamics of the Bloch Vector

Instead of the Schrödinger equation, we can describe the system dynamics in terms of an equation of motion for the Bloch vector $\boldsymbol{S}$. In addition to the Schrödinger equation

$$\left|\dot{\Psi}\right\rangle = -iH\left|\Psi\right\rangle \tag{84}$$

For the state vector $\left|\Psi\right\rangle$

$$\begin{aligned}
\dot{S}_x &= \left\langle\dot{\Psi}\right|\sigma_x\left|\Psi\right\rangle + \left\langle\Psi\right|\sigma_x\left|\dot{\Psi}\right\rangle \\
&= i\left\langle\Psi\right|H\sigma_x\left|\Psi\right\rangle - i\left\langle\Psi\right|\sigma_xH\left|\Psi\right\rangle \\
&= i\left\langle\Psi\right|[H,\sigma_x]\left|\Psi\right\rangle
\end{aligned} \tag{85}$$

To simplify this, we can express the Hamiltonian in Pauli matrices

$$H = \sum_j \frac{\omega_j}{2}\sigma_j \tag{86}$$

so that

$$\dot{S}_x = i\sum_j \frac{\omega_j}{2}\left\langle\Psi\right|[\sigma_j,\sigma_x]\left|\Psi\right\rangle = \omega_y\sigma_z - \omega_z\sigma_y \tag{87}$$

Similarly

$$\dot{S}_y = \omega_z\sigma_x - \omega_x\sigma_z \tag{88}$$

$$\dot{S}_z = \omega_x\sigma_y - \omega_y\sigma_x \tag{89}$$

In terms of vector notation, we have

$$\dot{\boldsymbol{S}} = \boldsymbol{\omega} \times \boldsymbol{S} \tag{90}$$

and

$$\frac{\partial}{\partial t}|\boldsymbol{S}|^2 = \dot{\boldsymbol{S}}\boldsymbol{S} + \boldsymbol{S}\dot{\boldsymbol{S}} = (\boldsymbol{\omega} \times \boldsymbol{S})\boldsymbol{S} + \boldsymbol{S}(\boldsymbol{\omega} \times \boldsymbol{S}) = 0 \tag{91}$$

1.7   Averages over Different States

Consider the expectation of an observable A

$$\langle A \rangle = \langle \Psi | A | \Psi \rangle = \sum_j p_j \langle \Psi_j | A | \Psi_j \rangle \tag{92}$$

We can thus define the Bloch vector

$$\boldsymbol{S} = \sum_j p_j \boldsymbol{S}_j \tag{93}$$

for the ensemble average. Now we recall the Ramsey experiment

$$H_\phi = \frac{\omega}{2} \sigma_z + \Omega_R \sigma_x \cos(\nu t + \phi) \tag{94}$$

Consider the state $|\Psi\rangle = |\pm\rangle = (|e\rangle \pm |g\rangle)/\sqrt{2}$

$$
\begin{aligned}
\boldsymbol{S} = \frac{1}{2}\boldsymbol{S}_{|g\rangle} + \frac{1}{2}\boldsymbol{S}_{|e\rangle} &= \frac{1}{2} \langle g | \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} |g\rangle + \frac{1}{2} \langle e | \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} |e\rangle \\
&= \frac{1}{2} \langle g | \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} |g\rangle + \frac{1}{2} \langle e | \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} |e\rangle = 0
\end{aligned} \tag{95}
$$

and $\dot{\boldsymbol{S}} = \boldsymbol{\omega} \times \boldsymbol{S} = 0$. The Bloch vector for the ensemble average is thus stationary.

2 Harmonic Oscillator

The Hamiltonian of one dimension harmonic oscillator

$$
\begin{aligned}
H =& \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 \\
=& \hbar\omega \left(\frac{P^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar} X^2 \right) \\
=& \frac{1}{2}\hbar\omega \left(\hat{p}^2 + \hat{x}^2 \right)
\end{aligned}
\tag{96}
$$

in terms of the unitless operators

$$
\hat{x} = \sqrt{\frac{m\omega}{\hbar}} X, \qquad \hat{p} = \frac{1}{\sqrt{m\hbar\omega}} P
\tag{97}
$$

which satisfy the commutation relation

$$
[\hat{x}, \hat{p}] = \frac{1}{\hbar}[X, P] = i
\tag{98}
$$

Creation and annihilation operators are defined as

$$
\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})
\tag{99}
$$

with

$$
[a, a^\dagger] = \frac{1}{2}[\hat{x} + i\hat{p}, \hat{x} - i\hat{p}] = \frac{1}{2}([\hat{x}, -i\hat{p}] + [i\hat{p}, \hat{x}]) = 1
\tag{100}
$$

$\hat{x}$ and $\hat{p}$ can be expressed as

$$
\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})
\tag{101}
$$

The Hamiltonian becomes

$$
\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)
\tag{102}
$$

For the commute relation $[a, a^\dagger] = 1$, one obtains

$$
a^\dagger a a \ket{\mu} = (aa^\dagger - 1)a \ket{\mu} = a(a^\dagger a - 1) \ket{\mu} = (\mu - 1)a \ket{\mu}
\tag{103}
$$

so that

$$
\bra{\mu} a^\dagger a \ket{\mu} = \mu \quad \Rightarrow \quad a \ket{\mu} = \sqrt{\mu} \ket{\mu - 1}
\tag{104}
$$

and similarly,

$$
a^\dagger \ket{\mu} = \sqrt{\mu + 1} \ket{\mu + 1}
\tag{105}
$$

In real space representation, the annihilation operator and creation operator read

$$
a \to \frac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right), \qquad a^\dagger \to \frac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right)
\tag{106}
$$

The wave function satisfying

$$
\frac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right) \phi_0(x) = 0 \quad \Rightarrow \quad \phi_0(x) \sim \exp\left(-\frac{x^2}{2} \right)
\tag{107}
$$

$$
\phi_1(x) \sim \frac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right) \exp\left(-\frac{x^2}{2} \right) = 2x \exp\left(-\frac{x^2}{2} \right)
\tag{108}
$$

3 Quantisation of the Electromagnetic Field

The Maxwell equations read

$$\nabla \cdot \boldsymbol{E} = 0 \tag{109}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{110}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{111}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{112}$$

In terms of divergence and curl

$$\nabla \cdot \boldsymbol{Q} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \tag{113}$$

$$\nabla \times \boldsymbol{Q} = -\left(\frac{\partial Q_y}{\partial z} - \frac{\partial Q_z}{\partial y}\right)\boldsymbol{e}_x - \left(\frac{\partial Q_z}{\partial x} - \frac{\partial Q_x}{\partial z}\right)\boldsymbol{e}_y - \left(\frac{\partial Q_x}{\partial y} - \frac{\partial Q_y}{\partial x}\right)\boldsymbol{e}_z \tag{114}$$

Let's start with a simple ansatz

$$\boldsymbol{E} = f(t)\sin kz \boldsymbol{e}_x \tag{115}$$

$$\boldsymbol{B} = g(t)\cos kz \boldsymbol{e}_y \tag{116}$$

The Maxwell equations imply

$$\nabla \cdot \boldsymbol{E} = \nabla \cdot \boldsymbol{B} = 0 \tag{117}$$

$$\nabla \times \boldsymbol{E} = \frac{\partial E_x}{\partial z}\boldsymbol{e}_y = f(t)k\cos kz \boldsymbol{e}_y$$
$$= -\frac{\partial \boldsymbol{B}}{\partial t} = -\dot{g}(t)\cos kz \boldsymbol{e}_y \tag{118}$$

$$\nabla \times \boldsymbol{B} = -\frac{\partial B_y}{\partial z}\boldsymbol{e}_x = g(t)k\sin kz \boldsymbol{e}_x$$
$$= \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{c^2}\dot{f}(t)\sin kz \boldsymbol{e}_x \tag{119}$$

This requires the differential equations

$$f(t)k = -\dot{g}(t), \qquad g(t)k = \frac{1}{c^2}\dot{f}(t) \tag{120}$$

So we have

$$\ddot{f}(t) = c^2 k\dot{g}(t) = -c^2 k^2 f(t) = \nu_k^2 f(t) \tag{121}$$

where $\nu_k = ck$ is called the linear dispersion. We can now quantise the electric field as

$$\boldsymbol{E} = \sqrt{\frac{\hbar\nu}{\varepsilon_0 V}}\left(\hat{a}e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t}\right)\sin kz \boldsymbol{e}_x \tag{122}$$

The creation and annihilation operators satisfy

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \tag{123}$$

We can also define creation and annihilation operators for wave packets

$$\hat{a}_\phi = \int \mathrm{d}k \phi(k) \hat{a}_k, \qquad \hat{a}_\phi^\dagger = \int \mathrm{d}k \phi^*(k) \hat{a}_k^\dagger \tag{124}$$

and the commutator

$$[\hat{a}_\phi, \hat{a}_\psi^\dagger] = \int \mathrm{d}k \mathrm{d}k' \phi(k) \psi^*(k') [\hat{a}_k, \hat{a}_{k'}^\dagger] = \int \mathrm{d}k \phi(k) \psi^*(k) \tag{125}$$

4 Jaynes-Cummings

we can now consider a two-level system interacting with a single mode of the quantised electro-magnetic field. The Hamiltonian reads

$$H = \frac{\omega}{2}\sigma_z + \frac{1}{2}\Omega_R\sigma_x\left(ae^{-i\nu t} + a^\dagger e^{i\nu t}\right) \tag{126}$$

1) $U_S = \exp\left(i\nu a^\dagger a t\right)$, the Hamiltonian reads

$$H_S = U_S^\dagger H U_S - iU_S^\dagger \dot{U}_S = \frac{\omega}{2}\sigma_z + \nu a^\dagger a + \frac{1}{2}\Omega_R\sigma_x(a + a^\dagger) \tag{127}$$

2) $U_I = \exp\left(-i\frac{\omega}{t}\sigma_z t\right)$

$$\begin{aligned}
H_I &= U_I^\dagger H U_I - iU_I^\dagger \dot{U}_I \\
&= \frac{1}{2}\Omega_R\left(\sigma_+ e^{i\omega t} + \sigma_- e^{-i\omega t}\right)\left(ae^{-i\nu t} + a^\dagger e^{i\nu t}\right) \\
&= \frac{1}{2}\Omega_R\left[\sigma_+ a e^{i(\omega-\nu)t} + \sigma_+ a^\dagger e^{i(\omega+\nu)t} + \sigma_- a e^{-i(\omega+\nu)t} + \sigma_- a^\dagger e^{-i(\omega-\nu)t}\right] \\
&\approx \frac{1}{2}\Omega_R\left[\sigma_+ a e^{i(\omega-\nu)t} + \sigma_- a^\dagger e^{-i(\omega-\nu)t}\right]
\end{aligned} \tag{128}$$

This Hamiltonian contains four elementary process

- $\sigma_+ a$: atom absorbs a photon and gets excited.
- $\sigma_+ a^\dagger$: atom emits a photon and gets excited.
- $\sigma_- a$: atom absorbs a photon and gets de-excited.
- $\sigma_- a^\dagger$: atom emits a photon and gets de-excited.

4.1 Two-Dimensional Subspaces

The Hamiltonian (in rotating wave approximation) in lab frame reads

$$H = \frac{\omega}{2}\sigma_z + \nu a^\dagger a + \frac{1}{2}\Omega_R(\sigma_+ a + \sigma_- a^\dagger) \tag{129}$$

and

$$H\left|g,\mu\right\rangle = \left(-\frac{\omega}{2} + \mu\nu\right)\left|g,\mu\right\rangle + \frac{1}{2}\Omega_R\sqrt{\mu}\left|e,\mu-1\right\rangle \tag{130}$$

$$H\left|e,\mu-1\right\rangle = \frac{1}{2}\Omega_R\sqrt{\mu}\left|g,\mu\right\rangle + \left(\frac{\omega}{2} + (\mu-1)\nu\right)\left|e,\mu-1\right\rangle \tag{131}$$

In terms of the basis $\{\left|g,\mu\right\rangle, \left|e,\mu-1\right\rangle\}$ we can express this as the matrix

$$\begin{pmatrix} -\frac{\omega-\nu}{2} + \left(\mu - \frac{1}{2}\right)\nu & \frac{1}{2}\Omega_R\sqrt{\mu} \\ \frac{1}{2}\Omega_R\sqrt{\mu} & \frac{\omega-\nu}{2} + \left(\mu - \frac{1}{2}\right)\nu \end{pmatrix} \tag{132}$$

or, in terms of Pauli-matrices as

$$H = -\frac{\omega - \nu}{2}\sigma_z + \frac{1}{2}\Omega_R\sqrt{\mu}\sigma_x + \left(\mu - \frac{1}{2}\right)\nu\mathbb{1} \tag{133}$$

In the case of resonance between atom and light-field, this reduces to

$$H(\nu = \omega) = \frac{1}{2}\Omega_R\sqrt{\mu}\sigma_x + \left(\mu - \frac{1}{2}\right)\nu\mathbb{1} \tag{134}$$

with eigenstates

$$\frac{1}{\sqrt{2}}(|g, \mu\rangle \pm |e, \mu - 1\rangle) \tag{135}$$

4.2 The Lambda-System

The Hamiltonian of the Lambda-system interacting with a single-mode quantum field in rotating wave approximation reads

$$
\begin{aligned}
H =& \omega\,|e\rangle\,\langle e| + 0(|g_1\rangle\,\langle g_1| + |g_2\rangle\,\langle g_2|) + \nu a^\dagger a \\
&+ \frac{1}{2\sqrt{2}}\Omega_R\left(|g_1\rangle\,\langle e|\,a^\dagger + |g_2\rangle\,\langle e|\,a^\dagger + |e\rangle\,\langle g_1|\,a + |e\rangle\,\langle g_2|\,a\right) \\
=& \omega\,|e\rangle\,\langle e| + \frac{1}{2\sqrt{2}}\Omega_R\left[(|g_1\rangle\,\langle e| + |g_2\rangle\,\langle e|)a^\dagger + (|e\rangle\,\langle g_1| + |e\rangle\,\langle g_2|)a\right]
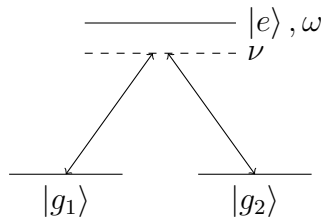\end{aligned} \tag{136}
$$

In the interaction picture we have

$$H_I = \frac{1}{2\sqrt{2}}\Omega_R\left[(|g_1\rangle\,\langle e| + |g_2\rangle\,\langle e|)a^\dagger \mathrm{e}^{-i\delta t} + (|e\rangle\,\langle g_1| + |e\rangle\,\langle g_2|)a\mathrm{e}^{i\delta t}\right] \tag{137}$$

with the detuning $\delta = \omega - \nu$. According to the perturbation theory

$$U = \mathbb{1} - i\int_0^t \mathrm{d}t'\,H(t') - \frac{1}{2}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\,H_I(t')H_I(t'') + \cdots \tag{138}$$

thus we have to consider the second order

$$
\begin{aligned}
H_I(t')H_I(t'') =& \frac{1}{8}\Omega_R^2\Big[(|g_1\rangle\,\langle e| + |g_2\rangle\,\langle e|)(|e\rangle\,\langle g_1| + |e\rangle\,\langle g_2|)a^\dagger a\mathrm{e}^{-i\delta(t'-t'')} \\
&+ (|e\rangle\,\langle g_1| + |e\rangle\,\langle g_2|)(|g_1\rangle\,\langle e| + |g_2\rangle\,\langle e|)aa^\dagger \mathrm{e}^{i\delta(t'-t'')}\Big] \\
=& \frac{1}{8}\Omega_R^2\left(|g_1\rangle\,\langle g_1| + |g_2\rangle\,\langle g_2| + |g_1\rangle\,\langle g_2| + |g_2\rangle\,\langle g_1|\right)a^\dagger a\mathrm{e}^{-i\delta(t'-t'')} \\
&+ \frac{1}{4}\Omega_R^2\,|e\rangle\,\langle e|\,(a^\dagger a + 1)\mathrm{e}^{i\delta(t'-t'')}
\end{aligned} \tag{139}
$$

Figure 5: The Lambda-system

5   Coherent States

For fock states $|\mu\rangle$, the expectation values for x and p vanish

$$\langle\mu|\,x\,|\mu\rangle = \frac{1}{\sqrt{2}}(\langle\mu|\,a\,|\mu\rangle + \langle\mu|\,a^\dagger\,|\mu\rangle) = 0 \tag{140}$$

$$\langle\mu|\,p\,|\mu\rangle = \frac{-i}{\sqrt{2}}(\langle\mu|\,a\,|\mu\rangle - \langle\mu|\,a^\dagger\,|\mu\rangle) = 0 \tag{141}$$

and the fluctuations

$$\langle\mu|\,x^2\,|\mu\rangle = \frac{1}{2}(\langle\mu|\,a^2\,|\mu\rangle + \langle\mu|\,aa^\dagger\,|\mu\rangle + \langle\mu|\,a^\dagger a\,|\mu\rangle + \langle\mu|\,a^\dagger a^\dagger\,|\mu\rangle) = \mu + \frac{1}{2} \tag{142}$$

$$\langle\mu|\,p^2\,|\mu\rangle = -\frac{1}{2}(\langle\mu|\,a^2\,|\mu\rangle - \langle\mu|\,aa^\dagger\,|\mu\rangle - \langle\mu|\,a^\dagger a\,|\mu\rangle + \langle\mu|\,a^\dagger a^\dagger\,|\mu\rangle) = \mu + \frac{1}{2} \tag{143}$$

For the ground state

$$\langle 0|\,x\,|0\rangle = \langle 0|\,p\,|0\rangle = 0 \tag{144}$$

$$\langle 0|\,x^2\,|0\rangle = \langle 0|\,p^2\,|0\rangle = \frac{1}{2} \tag{145}$$

This yields

$$\Delta x \Delta p = (\langle 0|\,x^2\,|0\rangle - (\langle 0|\,x\,|0\rangle)^2)(\langle 0|\,p^2\,|0\rangle - (\langle 0|\,p\,|0\rangle)^2) = \frac{1}{4} \tag{146}$$

which is the minimal allowed uncertainty. We can generate different states with the same uncertainty, by displacing the vacuum in phase space. The *displacement operator* is defined as

$$\boxed{D(\alpha) = \exp\!\big(\alpha a^\dagger - \alpha^* a\big)} \tag{147}$$

The coherent state

$$\begin{aligned}
|\alpha\rangle &= D(\alpha)\,|0\rangle = \exp\!\big(\alpha a^\dagger - \alpha^* a\big)\,|0\rangle \\
&= \exp\!\left(-\frac{1}{2}|\alpha|^2\right)\exp\!\big(\alpha a^\dagger\big)\exp(-\alpha^* a)\,|0\rangle \\
&= \exp\!\left(-\frac{1}{2}|\alpha|^2\right)\exp\!\big(\alpha a^\dagger\big)\,|0\rangle \\
&= \exp\!\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu (a^\dagger)^\mu}{\mu!}\,|0\rangle \\
&= \exp\!\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}}\,|\mu\rangle
\end{aligned} \tag{148}$$

The probability to find μ photons is thus given by the Poisson distribution.

$$P(\mu) = \exp\!\big(-|\alpha|^2\big)\frac{(|\alpha|^2)^\mu}{\mu!} \tag{149}$$

For the expectation value of x and p with respect to a general state $|\Psi\rangle$ one has

$$(\langle\Psi|\,D^\dagger(\alpha))x(D(\alpha)\,|\Psi\rangle) = \langle\Psi|\,(D^\dagger(\alpha)xD(\alpha))\,|\Psi\rangle \tag{150}$$

$$(\langle\Psi|\,D^\dagger(\alpha))p(D(\alpha)\,|\Psi\rangle) = \langle\Psi|\,(D^\dagger(\alpha)pD(\alpha))\,|\Psi\rangle \tag{151}$$

then we calculate

$$D^\dagger(\alpha)xD(\alpha) = x + \frac{\alpha + \alpha^*}{\sqrt{2}} = x + x_0 \tag{152}$$

$$D^\dagger(\alpha)pD(\alpha) = p - i\frac{\alpha - \alpha^*}{\sqrt{2}} = p + p_0 \tag{153}$$

We verify the uncertainty in position and momentum of any coherent state

$$\begin{aligned}\langle\alpha|\,x^2\,|\alpha\rangle - (\langle\alpha|\,x\,|\alpha\rangle)^2 &= \langle0|\,D^\dagger(\alpha)x^2D(\alpha)\,|0\rangle - (\langle0|\,D^\dagger(\alpha)xD(\alpha)\,|0\rangle)^2 \\ &= \langle0|\,x^2\,|0\rangle - (\langle0|\,x\,|0\rangle)^2\end{aligned} \tag{154}$$

$$\begin{aligned}\langle\alpha|\,p^2\,|\alpha\rangle - (\langle\alpha|\,p\,|\alpha\rangle)^2 &= \langle0|\,D^\dagger(\alpha)p^2D(\alpha)\,|0\rangle - (\langle0|\,D^\dagger(\alpha)pD(\alpha)\,|0\rangle)^2 \\ &= \langle0|\,p^2\,|0\rangle - (\langle0|\,p\,|0\rangle)^2\end{aligned} \tag{155}$$

5.1   Coherent States in Real-Space Representation

The character of the displacement operator can be exemplified in the real-space representation of wave functions.

$$\Psi_\alpha(x) = \langle x|\alpha\rangle \propto \exp\left(-\frac{1}{2}(x - x_0)^2 + ip_0x - \frac{i}{2}x_0p_0\right) \tag{156}$$

with $x_0 = (\alpha + \alpha^*)/\sqrt{2}$ and $p_0 = (\alpha - \alpha^*)/(\sqrt{2}i)$. It is convenient to define the vector

$$|\alpha, \tau\rangle = \exp\left[(\alpha a^\dagger - \alpha^*a)\tau\right]|0\rangle \tag{157}$$

with additional scalar parameter τ. It satisfies the differential equation

$$\frac{\partial\,|\alpha, \tau\rangle}{\partial\tau} = (\alpha a^\dagger - \alpha^*a)\,|\alpha, \tau\rangle \tag{158}$$

The real-space representation of the operator $(\alpha a^\dagger - \alpha^*a)$ reads

$$\frac{1}{\sqrt{2}}\left[\alpha\left(x - \frac{\partial}{\partial x}\right) - \alpha^*\left(x + \frac{\partial}{\partial x}\right)\right] = \frac{\alpha - \alpha^*}{\sqrt{2}}x - \frac{\alpha + \alpha^*}{\sqrt{2}}\frac{\partial}{\partial x} = ip_0x - x_0\frac{\partial}{\partial x} \tag{159}$$

We thus need to solve the equation

$$\frac{\partial\Phi}{\partial\tau} = \left(ip_0x - x_0\frac{\partial}{\partial x}\right)\Phi \tag{160}$$

with the Ansatz

$$\Phi(\tau) = \exp\left(-\frac{1}{2}(x - f_x)^2 + if_px - i\varphi\right) \tag{161}$$

The initial conditions are $f_x(0) = f_p(0) = \varphi(0) = 0$. The derivatives

$$\frac{\partial \Phi(\tau)}{\partial \tau} = \left((x - f_x)\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau}x - i\frac{\partial \varphi}{\partial \tau} \right) \Phi(\tau) \tag{162}$$

$$\frac{\partial \Phi(\tau)}{\partial x} = \left(-(x - f_x) + if_p \right) \Phi(\tau) \tag{163}$$

This yields

$$(x - f_x)\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau}x - i\frac{\partial \varphi}{\partial \tau} = ip_0 x - x_0\left(-(x - f_x) + if_p \right) \tag{164}$$

Collect all terms proportional to x

$$\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau} = ip_0 + x_0 \tag{165}$$

This is solved for

$$\frac{\partial f_x}{\partial \tau} = x_0 \quad \Rightarrow \quad f_x = x_0\tau \tag{166}$$

$$\frac{\partial f_p}{\partial \tau} = p_0 \quad \Rightarrow \quad f_p = p_0\tau \tag{167}$$

Collect all terms do not contain x yields

$$-f_x\frac{\partial f_x}{\partial \tau} - i\frac{\partial \varphi}{\partial \tau} = -x_0 f_x - ix_0 f_p \tag{168}$$

which is solved for

$$\varphi(\tau) = \frac{1}{2}x_0 p_0 \tau^2 \tag{169}$$

With $\tau = 1$, this gives the phase factor $\exp\left(-\frac{i}{2}x_0 p_0\right)$.

5.2   Dynamics of Coherent States

For the dynamics induced by $U_0(t) = \exp\left(-i\nu a^\dagger a t\right)$, one obtains

$$\begin{aligned}
U_0(t)\left|\alpha\right\rangle &= U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)\left|0\right\rangle \\
&= U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)U_0^\dagger(t)U_0(t)\left|0\right\rangle \\
&= U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)U_0^\dagger(t)\left|0\right\rangle \\
&= \exp\left[\alpha U_0(t)a^\dagger U_0^\dagger(t) - \alpha^* U_0(t)aU_0^\dagger(t)\right]\left|0\right\rangle \\
&= \exp\left(\alpha a^\dagger \mathrm{e}^{-i\nu t} + \alpha^* a\mathrm{e}^{i\nu t}\right)\left|0\right\rangle \\
&= D\left(\alpha\mathrm{e}^{-i\nu t}\right)\left|0\right\rangle = \left|\alpha\mathrm{e}^{-i\nu t}\right\rangle
\end{aligned} \tag{170}$$

5.3   Light-Matter Interaction with Coherent States

Coherent states are eigenstates to the annihilation operator a.

$$
\begin{aligned}
a\,|\alpha\rangle &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}} a\,|\mu\rangle \\
&= \alpha \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_\mu \frac{\alpha^{\mu-1}}{\sqrt{(\mu-1)!}}\,|\mu-1\rangle = \alpha\,|\alpha\rangle
\end{aligned}
\tag{171}
$$

Similarly

$$
\langle\alpha|\,a^\dagger = \alpha^*\,\langle\alpha|
\tag{172}
$$

Coherent states are not orthogonal to each other

$$
\begin{aligned}
\langle\alpha|\beta\rangle &= \left(\exp\left(-\frac{|\alpha|^2}{2}\right) \sum_\mu \frac{(\alpha^*)^\mu}{\sqrt{\mu!}}\,\langle\mu|\right)\left(\exp\left(-\frac{|\beta|^2}{2}\right) \sum_\nu \frac{\beta^\nu}{\sqrt{\nu!}}\,|\nu\rangle\right) \\
&= \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right) \sum_{\mu,\nu} \frac{(\alpha^*)^\mu \beta^\nu}{\sqrt{\mu!}\sqrt{\nu!}}\,\langle\mu|\nu\rangle \\
&= \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right) \sum_\mu \frac{(\alpha^*\beta)^\mu}{\mu!}
\end{aligned}
\tag{173}
$$

Now we want to find the eigenvector $|\Psi\rangle$ of $a^\dagger$

$$
a^\dagger\,|\Psi\rangle = \lambda\,|\Psi\rangle = \left|\tilde\Psi\right\rangle
\tag{174}
$$

The normalised vector

$$
\frac{|\tilde\Psi\rangle}{\sqrt{\langle\tilde\Psi|\tilde\Psi\rangle}}
\tag{175}
$$

and

$$
\left|\frac{\langle\Psi|\tilde\Psi\rangle}{\sqrt{\langle\tilde\Psi|\tilde\Psi\rangle}}\right| = 1
\tag{176}
$$

Normalising $a^\dagger\,|\alpha\rangle$ yields

$$
\frac{a^\dagger\,|\alpha\rangle}{\sqrt{\langle\alpha|\,aa^\dagger\,|\alpha\rangle}} = \frac{a^\dagger\,|\alpha\rangle}{\sqrt{|\alpha^2|+1}}
\tag{177}
$$

and

$$
\frac{\langle\alpha|\,a^\dagger\,|\alpha\rangle}{\sqrt{\langle\alpha|\,aa^\dagger\,|\alpha\rangle}} = \frac{\alpha^*}{\sqrt{|\alpha^2|+1}}
\tag{178}
$$

In the limit $|\alpha|\to\infty$

$$
\frac{\alpha^*}{\sqrt{|\alpha^2|+1}} \to \frac{\alpha^*}{|\alpha|}
\tag{179}
$$

with

$$\left| \frac{\alpha^*}{|\alpha|} \right| = 1 \tag{180}$$

The relation

$$a^\dagger \left| \alpha \right\rangle \simeq \alpha^* \left| \alpha \right\rangle \tag{181}$$

is thus a good approximation for $|\alpha| \ll 1$.

6   Coherent States

For fock states $|\mu\rangle$, the expectation values for x and p vanish

$$\langle\mu|\,x\,|\mu\rangle = \frac{1}{\sqrt{2}}(\langle\mu|\,a\,|\mu\rangle + \langle\mu|\,a^\dagger\,|\mu\rangle) = 0 \tag{182}$$

$$\langle\mu|\,p\,|\mu\rangle = \frac{-i}{\sqrt{2}}(\langle\mu|\,a\,|\mu\rangle - \langle\mu|\,a^\dagger\,|\mu\rangle) = 0 \tag{183}$$

and the fluctuations

$$\langle\mu|\,x^2\,|\mu\rangle = \frac{1}{2}(\langle\mu|\,a^2\,|\mu\rangle + \langle\mu|\,aa^\dagger\,|\mu\rangle + \langle\mu|\,a^\dagger a\,|\mu\rangle + \langle\mu|\,a^\dagger a^\dagger\,|\mu\rangle) = \mu + \frac{1}{2} \tag{184}$$

$$\langle\mu|\,p^2\,|\mu\rangle = -\frac{1}{2}(\langle\mu|\,a^2\,|\mu\rangle - \langle\mu|\,aa^\dagger\,|\mu\rangle - \langle\mu|\,a^\dagger a\,|\mu\rangle + \langle\mu|\,a^\dagger a^\dagger\,|\mu\rangle) = \mu + \frac{1}{2} \tag{185}$$

For the ground state

$$\langle0|\,x\,|0\rangle = \langle0|\,p\,|0\rangle = 0 \tag{186}$$

$$\langle0|\,x^2\,|0\rangle = \langle0|\,p^2\,|0\rangle = \frac{1}{2} \tag{187}$$

This yields

$$\Delta x \Delta p = (\langle0|\,x^2\,|0\rangle - (\langle0|\,x\,|0\rangle)^2)(\langle0|\,p^2\,|0\rangle - (\langle0|\,p\,|0\rangle)^2) = \frac{1}{4} \tag{188}$$

which is the minimal allowed uncertainty. We can generate different states with the same uncertainty, by displacing the vacuum in phase space. The *displacement operator* is defined as

$$\boxed{D(\alpha) = \exp\big(\alpha a^\dagger - \alpha^* a\big)} \tag{189}$$

The coherent state

$$\begin{aligned}
|\alpha\rangle &= D(\alpha)\,|0\rangle = \exp\big(\alpha a^\dagger - \alpha^* a\big)\,|0\rangle \\
&= \exp\left(-\frac{1}{2}|\alpha|^2\right)\exp\big(\alpha a^\dagger\big)\exp(-\alpha^* a)\,|0\rangle \\
&= \exp\left(-\frac{1}{2}|\alpha|^2\right)\exp\big(\alpha a^\dagger\big)\,|0\rangle \\
&= \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu (a^\dagger)^\mu}{\mu!}\,|0\rangle \\
&= \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}}\,|\mu\rangle
\end{aligned} \tag{190}$$

The probability to find μ photons is thus given by the Poisson distribution.

$$P(\mu) = \exp\big(-|\alpha|^2\big)\frac{(|\alpha|^2)^\mu}{\mu!} \tag{191}$$

For the expectation value of x and p with respect to a general state $|\Psi\rangle$ one has

$$(\langle\Psi| D^\dagger(\alpha))x(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)xD(\alpha))|\Psi\rangle \tag{192}$$

$$(\langle\Psi| D^\dagger(\alpha))p(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)pD(\alpha))|\Psi\rangle \tag{193}$$

then we calculate

$$D^\dagger(\alpha)xD(\alpha) = x + \frac{\alpha + \alpha^*}{\sqrt{2}} = x + x_0 \tag{194}$$

$$D^\dagger(\alpha)pD(\alpha) = p - i\frac{\alpha - \alpha^*}{\sqrt{2}} = p + p_0 \tag{195}$$

We verify the uncertainty in position and momentum of any coherent state

$$\begin{aligned}\langle\alpha| x^2 |\alpha\rangle - (\langle\alpha| x |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)x^2 D(\alpha) |0\rangle - (\langle 0| D^\dagger(\alpha)xD(\alpha) |0\rangle)^2 \\ &= \langle 0| x^2 |0\rangle - (\langle 0| x |0\rangle)^2\end{aligned} \tag{196}$$

$$\begin{aligned}\langle\alpha| p^2 |\alpha\rangle - (\langle\alpha| p |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)p^2 D(\alpha) |0\rangle - (\langle 0| D^\dagger(\alpha)pD(\alpha) |0\rangle)^2 \\ &= \langle 0| p^2 |0\rangle - (\langle 0| p |0\rangle)^2\end{aligned} \tag{197}$$

6.1 Coherent States in Real-Space Representation

The character of the displacement operator can be exemplified in the real-space representation of wave functions.

$$\Psi_\alpha(x) = \langle x|\alpha\rangle \propto \exp\left(-\frac{1}{2}(x - x_0)^2 + ip_0 x - \frac{i}{2}x_0 p_0\right) \tag{198}$$

with $x_0 = (\alpha + \alpha^*)/\sqrt{2}$ and $p_0 = (\alpha - \alpha^*)/(\sqrt{2}i)$. It is convenient to define the vector

$$|\alpha, \tau\rangle = \exp\left[(\alpha a^\dagger - \alpha^* a)\tau\right] |0\rangle \tag{199}$$

with additional scalar parameter τ. It satisfies the differential equation

$$\frac{\partial |\alpha, \tau\rangle}{\partial\tau} = (\alpha a^\dagger - \alpha^* a) |\alpha, \tau\rangle \tag{200}$$

The real-space representation of the operator $(\alpha a^\dagger - \alpha^* a)$ reads

$$\frac{1}{\sqrt{2}}\left[\alpha\left(x - \frac{\partial}{\partial x}\right) - \alpha^*\left(x + \frac{\partial}{\partial x}\right)\right] = \frac{\alpha - \alpha^*}{\sqrt{2}}x - \frac{\alpha + \alpha^*}{\sqrt{2}}\frac{\partial}{\partial x} = ip_0 x - x_0\frac{\partial}{\partial x} \tag{201}$$

We thus need to solve the equation

$$\frac{\partial\Phi}{\partial\tau} = \left(ip_0 x - x_0\frac{\partial}{\partial x}\right)\Phi \tag{202}$$

with the Ansatz

$$\Phi(\tau) = \exp\left(-\frac{1}{2}(x - f_x)^2 + if_p x - i\varphi\right) \tag{203}$$

The initial conditions are $f_x(0) = f_p(0) = \varphi(0) = 0$. The derivatives

$$\frac{\partial \Phi(\tau)}{\partial \tau} = \left((x - f_x)\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau}x - i\frac{\partial \varphi}{\partial \tau} \right) \Phi(\tau) \tag{204}$$

$$\frac{\partial \Phi(\tau)}{\partial x} = \left(-(x - f_x) + if_p \right) \Phi(\tau) \tag{205}$$

This yields

$$(x - f_x)\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau}x - i\frac{\partial \varphi}{\partial \tau} = ip_0 x - x_0 \left(-(x - f_x) + if_p \right) \tag{206}$$

Collect all terms proportional to x

$$\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau} = ip_0 + x_0 \tag{207}$$

This is solved for

$$\frac{\partial f_x}{\partial \tau} = x_0 \quad \Rightarrow \quad f_x = x_0 \tau \tag{208}$$

$$\frac{\partial f_p}{\partial \tau} = p_0 \quad \Rightarrow \quad f_p = p_0 \tau \tag{209}$$

Collect all terms do not contain x yields

$$-f_x\frac{\partial f_x}{\partial \tau} - i\frac{\partial \varphi}{\partial \tau} = -x_0 f_x - ix_0 f_p \tag{210}$$

which is solved for

$$\varphi(\tau) = \frac{1}{2}x_0 p_0 \tau^2 \tag{211}$$

With $\tau = 1$, this gives the phase factor $\exp\left(-\frac{i}{2}x_0 p_0\right)$.

6.2   Dynamics of Coherent States

For the dynamics induced by $U_0(t) = \exp\left(-i\nu a^\dagger a t\right)$, one obtains

$$\begin{aligned}
U_0(t)\,|\alpha\rangle &= U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)|0\rangle \\
&= U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)U_0^\dagger(t)U_0(t)\,|0\rangle \\
&= U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)U_0^\dagger(t)\,|0\rangle \\
&= \exp\left[\alpha U_0(t)a^\dagger U_0^\dagger(t) - \alpha^* U_0(t)aU_0^\dagger(t)\right]|0\rangle \\
&= \exp\left(\alpha a^\dagger e^{-i\nu t} + \alpha^* a e^{i\nu t}\right)|0\rangle \\
&= D\left(\alpha e^{-i\nu t}\right)|0\rangle = \left|\alpha e^{-i\nu t}\right\rangle
\end{aligned} \tag{212}$$

6.3   Light-Matter Interaction with Coherent States

Coherent states are eigenstates to the annihilation operator a.

$$
\begin{aligned}
a\ket{\alpha} &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}} a\ket{\mu} \\
&= \alpha \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_\mu \frac{\alpha^{\mu-1}}{\sqrt{(\mu-1)!}} \ket{\mu-1} = \alpha\ket{\alpha}
\end{aligned}
\tag{213}
$$

Similarly

$$
\bra{\alpha} a^\dagger = \alpha^* \bra{\alpha}
\tag{214}
$$

Coherent states are not orthogonal to each other

$$
\begin{aligned}
\braket{\alpha|\beta} &= \left(\exp\left(-\frac{|\alpha|^2}{2}\right) \sum_\mu \frac{(\alpha^*)^\mu}{\sqrt{\mu!}} \bra{\mu}\right) \left(\exp\left(-\frac{|\beta|^2}{2}\right) \sum_\nu \frac{\beta^\nu}{\sqrt{\nu!}} \ket{\nu}\right) \\
&= \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right) \sum_{\mu,\nu} \frac{(\alpha^*)^\mu \beta^\nu}{\sqrt{\mu!}\sqrt{\nu!}} \braket{\mu|\nu} \\
&= \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right) \sum_\mu \frac{(\alpha^*\beta)^\mu}{\mu!}
\end{aligned}
\tag{215}
$$

Now we want to find the eigenvector $\ket{\Psi}$ of $a^\dagger$

$$
a^\dagger \ket{\Psi} = \lambda \ket{\Psi} = \ket{\tilde{\Psi}}
\tag{216}
$$

The normalised vector

$$
\frac{\ket{\tilde{\Psi}}}{\sqrt{\braket{\tilde{\Psi}|\tilde{\Psi}}}}
\tag{217}
$$

and

$$
\left|\frac{\braket{\Psi|\tilde{\Psi}}}{\sqrt{\braket{\tilde{\Psi}|\tilde{\Psi}}}}\right| = 1
\tag{218}
$$

Normalising $a^\dagger \ket{\alpha}$ yields

$$
\frac{a^\dagger\ket{\alpha}}{\sqrt{\bra{\alpha} a a^\dagger \ket{\alpha}}} = \frac{a^\dagger\ket{\alpha}}{\sqrt{|\alpha^2|+1}}
\tag{219}
$$

and

$$
\frac{\bra{\alpha} a^\dagger \ket{\alpha}}{\sqrt{\bra{\alpha} a a^\dagger \ket{\alpha}}} = \frac{\alpha^*}{\sqrt{|\alpha^2|+1}}
\tag{220}
$$

In the limit $|\alpha| \to \infty$

$$
\frac{\alpha^*}{\sqrt{|\alpha^2|+1}} \to \frac{\alpha^*}{|\alpha|}
\tag{221}
$$

with

$$\left| \frac{\alpha^*}{|\alpha|} \right| = 1 \tag{222}$$

The relation

$$a^\dagger \left| \alpha \right\rangle \simeq \alpha^* \left| \alpha \right\rangle \tag{223}$$

is thus a good approximation for $|\alpha| \ll 1$.

Part II
Technique

7 The Wigner Quasi-Probability Distribution

Let's start with a classical property that we would like to be fulfilled

$$\int_{-\infty}^{\infty} \mathrm{d}PW(X,P) = \Pr(X), \qquad \int_{-\infty}^{\infty} \mathrm{d}XW(X,P) = \Pr(P) \tag{224}$$

The probability distribution of any quadrature is called a 'marginal'. We can generalize the marginal equations above into a single expression to include rotation of the harmonic oscillator in its phase space. We then have

$$\Pr(X,\theta) = \langle X| U(\theta)\rho U^{\dagger}(\theta) |X\rangle = \int_{-\infty}^{\infty} \mathrm{d}PW(X\cos\theta - P\sin\theta, X\sin\theta + P\cos\theta) \tag{225}$$

where $U(\theta) = \exp\!\left(-i\theta a^{\dagger}a\right)$ is the rotation operator.

7.1 A Derivation of Wigner's Classic Formula

To start our derivation we introduce two quantities. First the 'characteristic function', *i.e.* the two-dimensional Fourier transform of the Wigner function

$$\tilde{W}(U,V) = \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}PW(X,P)\mathrm{e}^{-iUX-iVP} \tag{226}$$

second, the Fourier-transformed probability distribution

$$\tilde{\Pr}(\xi,\theta) = \int_{-\infty}^{\infty} \mathrm{d}X\Pr(X,\theta)\mathrm{e}^{-i\xi X} \tag{227}$$

We use the second part of the eqn.(225)

$$\begin{aligned}
\tilde{\Pr}(\xi,\theta) &= \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}PW(X\cos\theta - P\sin\theta, X\sin\theta + P\cos\theta)\mathrm{e}^{-i\xi X} \\
&= \tilde{W}(\xi\cos\theta, \xi\sin\theta)
\end{aligned} \tag{228}$$

using the first part of the eqn.(225)

$$\begin{aligned}
\tilde{\Pr}(\xi,\theta) &= \int_{-\infty}^{\infty} \mathrm{d}X \langle X| U(\theta)\rho U^{\dagger}(\theta) |X\rangle \, \mathrm{e}^{-i\xi X} \\
&= \int_{-\infty}^{\infty} \mathrm{d}X \langle X| \rho U^{\dagger}(\theta)\mathrm{e}^{-i\xi X}U(\theta) |X\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}X \langle X| \rho\exp(-iX\xi\cos\theta - iP\xi\sin\theta) |X\rangle \\
&= \mathrm{Tr}(\rho\exp(-iX\xi\cos\theta - iP\xi\sin\theta))
\end{aligned} \tag{229}$$

Let $U = \xi\cos\theta$ and $V = \xi\sin\theta$. We thus have our next important result

$$\boxed{\tilde{W}(U,V) = \mathrm{Tr}(\rho\exp(-iUX - iVP))} \tag{230}$$

Using the Baker-Campbell-Hausdorff formula

$$\exp(-iUX - iVP) = \exp(iUV/2)\exp(-iUX)\exp(-iVP) \tag{231}$$

so

$$
\begin{aligned}
\tilde{W}(U,V) &= \exp(iUV/2)\int_{-\infty}^{\infty} \mathrm{d}X \, \langle X| \, \rho \exp(-iUX)\exp(-iVP) \, |X\rangle \\
&= \exp(iUV/2)\int_{-\infty}^{\infty} \mathrm{d}X \, \langle X| \, \rho \exp(-iUX) \, |X + V\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}Q \, \mathrm{e}^{-iUQ} \, \langle Q - V/2| \, \rho \, |Q + V/2\rangle \qquad (X = Q - V/2)
\end{aligned}
\tag{232}
$$

Lastly, we do a inverse-Fourier transform to obtain the Wigner function

$$
\begin{aligned}
W(X,P) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}U\int_{-\infty}^{\infty}\mathrm{d}V\,\tilde{W}(U,V)\mathrm{e}^{iUX+iVP} \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}U\int_{-\infty}^{\infty}\mathrm{d}V\int_{-\infty}^{\infty}\mathrm{d}Q\left\langle Q - \frac{V}{2}\right|\rho\left|Q + \frac{V}{2}\right\rangle\mathrm{e}^{-iUQ+iUX+iVP} \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}V\int_{-\infty}^{\infty}\mathrm{d}Q\left\langle Q - \frac{V}{2}\right|\rho\left|Q + \frac{V}{2}\right\rangle\mathrm{e}^{iVP}\int_{-\infty}^{\infty}\mathrm{d}U\,\mathrm{e}^{iU(X-Q)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}V\int_{-\infty}^{\infty}\mathrm{d}Q\left\langle Q - \frac{V}{2}\right|\rho\left|Q + \frac{V}{2}\right\rangle\mathrm{e}^{iVP}\delta(X-Q) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}V\,\mathrm{e}^{iPV}\left\langle X - \frac{V}{2}\right|\rho\left|X + \frac{V}{2}\right\rangle
\end{aligned}
\tag{233}
$$

This equation is Wigner's now famous formula

$$\boxed{W(X,P) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}V\,\mathrm{e}^{iPV}\left\langle X - \frac{V}{2}\right|\rho\left|X + \frac{V}{2}\right\rangle} \tag{234}$$

7.2   Properties of the Wigner Function

- $W(X,P)$ is normalized, *i.e.* $\int_{-\infty}^{\infty}\mathrm{d}X\int_{-\infty}^{\infty}\mathrm{d}PW(X,P) = 1$.

- $W(X,P)$ is real, *i.e.* $W(X,P) = W^*(X,P)$.

- If $W(X,P)$ has any negative regions, then the state is non-classical.

- The overlap $\mathrm{Tr}(AB) = 2\pi\int_{-\infty}^{\infty}\mathrm{d}X\int_{-\infty}^{\infty}\mathrm{d}PW_A(X,P)W_B(X,P)$.

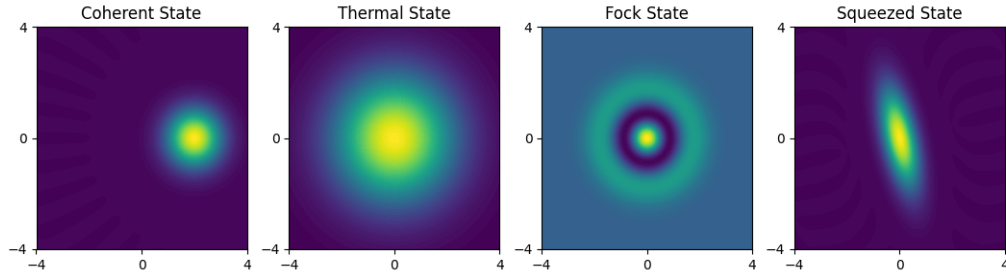## 7.3   Examples of the Wigner Function

See in fig.6.

Figure 6: Examples of Wigner function.

8   Optical Homodyne and Heterodyne Detection

8.1   Balanced Homodyne Detection

Two coherent light beams with complex amplitudes α, β transform through the beam-splitter. In classical optics

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = B \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{235}$$

In quantum optics, the complex amplitudes α, β correspond to the annihilation operators $\hat{a}$ and $\hat{b}$ of the incident fields. For $50 : 50$ beam-splitter

$$\begin{pmatrix} \hat{a}' \\ \hat{b}' \end{pmatrix} = B \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{a} + i\hat{b} \\ \hat{b} + i\hat{a} \end{pmatrix} \tag{236}$$
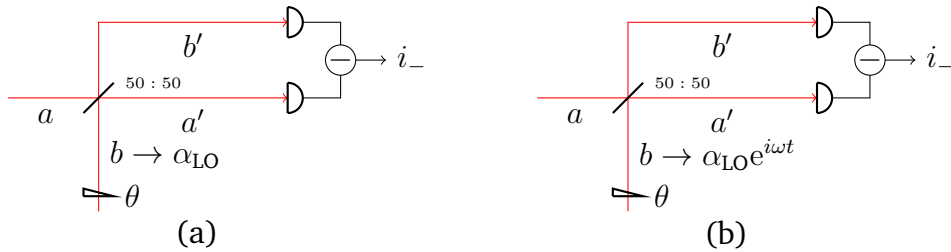
Then we calculate the number operator observed by the photondiodes

$$\hat{a}'^{\dagger}\hat{a}' = \frac{1}{2}(\hat{a}^{\dagger} - i\hat{b}^{\dagger})(\hat{a} + i\hat{b}) = \frac{1}{2}(\hat{a}^{\dagger}\hat{a} + i\hat{a}^{\dagger}\hat{b} - i\hat{b}^{\dagger}\hat{a} + \hat{b}^{\dagger}\hat{b}) \tag{237}$$

$$\hat{b}'^{\dagger}\hat{b}' = \frac{1}{2}(\hat{b}^{\dagger} - i\hat{a}^{\dagger})(\hat{b} + i\hat{a}) = \frac{1}{2}(\hat{a}^{\dagger}\hat{a} - i\hat{a}^{\dagger}\hat{b} + i\hat{b}^{\dagger}\hat{a} + \hat{b}^{\dagger}\hat{b}) \tag{238}$$

In a balanced detector, these two photo-currents are subtracted, yielding the 'difference current'

$$i_- \propto b'^{\dagger}b' - a'^{\dagger}a' = ib^{\dagger}a - ia^{\dagger}b \tag{239}$$

Figure 7: (a) Schematic for an optical homodyne detector. Mode b is put in a strong coherent state α_{LO} and is mixed on a beam-splitter with mode a that we wish to measure. (b) Schematic for an optical heterodyne detection.

Now we consider a is the signal and b is the reference, and it is also called a *local oscillator* (LO). We assume that the LO is powerful enough to be treated classically, *i.e.* we neglect totally the quantum fluctuations of the LO.

$$b \;\to\; \alpha_{\mathrm{LO}} = |\alpha_{\mathrm{LO}}|\mathrm{e}^{i\pi/2}\mathrm{e}^{i\theta} \tag{240}$$

where we introduce the phase θ in a convenient way to absorb the factor of i that came from our convention for the phase in the beam-splitter operator. After this transformation, the difference current becomes

$$
\begin{aligned}
i_- &\propto |\alpha_{\mathrm{LO}}|(a\mathrm{e}^{-i\theta} + a^\dagger \mathrm{e}^{i\theta}) \\
&= |\alpha_{\mathrm{LO}}|\left(a(\cos\theta - i\sin\theta) + a^\dagger(\cos\theta + i\sin\theta)\right) \\
&= |\alpha_{\mathrm{LO}}|\left((a + a^\dagger)\cos\theta + i(a^\dagger - a)\sin\theta\right) \\
&= \sqrt{2}|\alpha_{\mathrm{LO}}|(\hat{X}\cos\theta + \hat{P}\sin\theta) \\
&= \sqrt{2}|\alpha_{\mathrm{LO}}|\hat{X}_\theta
\end{aligned}
\tag{241}
$$

A balanced homodyne detector measures the quadrature component $\hat{X}_\theta$.
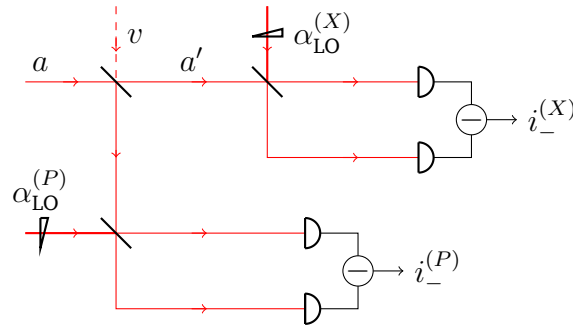
8.2 Heterodyne Detection

The heterodyne detection is similar to the homodyne detection however the LO has a different frequency.

$$
\begin{aligned}
i_- &\propto |\alpha_{\mathrm{LO}}|\left(a\mathrm{e}^{-i\theta}\mathrm{e}^{-i\omega t} + a^\dagger \mathrm{e}^{i\theta}\mathrm{e}^{i\omega t}\right) \\
&\propto \sqrt{2}|\alpha_{\mathrm{LO}}|\left(X\cos(\omega t + \theta) + P\sin(\omega t + \theta)\right)
\end{aligned}
\tag{242}
$$

8.3 Dual Homodyne Detection

From the discussions above, we can write the difference currents $i_-^{(X)}$ and $i_-^{(P)}$

$$
\begin{aligned}
i_-^{(X)} &\propto |\alpha_{\mathrm{LO}}^{(X)}|(\hat{a}' + \hat{a}'^\dagger) = \frac{1}{\sqrt{2}}|\alpha_{\mathrm{LO}}^{(X)}|(\hat{a} + i\hat{v} + \hat{a}^\dagger - i\hat{v}^\dagger) \\
&= |\alpha_{\mathrm{LO}}^{(X)}|\left[\frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) - \frac{i}{\sqrt{2}}(\hat{v}^\dagger - \hat{v})\right] = |\alpha_{\mathrm{LO}}^{(X)}|(\hat{X}_a - \hat{P}_v)
\end{aligned}
\tag{243}
$$

Figure 8: Dual homodyne detection. 'v' denotes vacuum fluctuations.

$$
\begin{aligned}
i_-^{(P)} &\propto |\alpha_{\mathrm{LO}}^{(P)}|(\hat{v}' + \hat{v}'^{\dagger}) = \frac{1}{\sqrt{2}}|\alpha_{\mathrm{LO}}^{(P)}|(\hat{v} + i\hat{a} + \hat{v}^{\dagger} - i\hat{a}^{\dagger}) \\
&= |\alpha_{\mathrm{LO}}^{(P)}| \left[\frac{1}{\sqrt{2}}(\hat{v} + \hat{v}^{\dagger}) - \frac{i}{\sqrt{2}}(\hat{a}^{\dagger} - \hat{a}) \right] = |\alpha_{\mathrm{LO}}^{(P)}|(\hat{X}_v - \hat{P}_a)
\end{aligned}
\tag{244}
$$

9   Photon Counting Statistics

- Sub-Poissonian statistics: $\Delta^2 n < \bar{n}$

- Poissonian statistics: $\Delta^2 n = \bar{n}$

- Super-Poissonian statistics: $\Delta^2 n > \bar{n}$

9.1   Poissonian Statistics

Coherent states have Poissonian statistics

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle \tag{245}$$

with amplitude

$$\langle n|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \tag{246}$$

Then we get the probability

$$\mathrm{Pr}(n) = |\langle n|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \tag{247}$$

The mean photon number in a coherent state is

$$\bar{n} = \langle\alpha| a^\dagger a |\alpha\rangle = |\alpha|^2 \tag{248}$$

so we can write

$$\boxed{\mathrm{Pr}(n) = \mathrm{e}^{-\bar{n}} \frac{\bar{n}^n}{n!}} \tag{249}$$

The expression is the *Poisson* distribution. Now we calculate the variance

$$\begin{aligned}
\Delta^2 n &= \langle n^2\rangle - \langle n\rangle^2 \\
&= \langle\alpha| (a^\dagger a)^2 |\alpha\rangle - \langle\alpha| a^\dagger a |\alpha\rangle^2 \\
&= |\alpha|^2 \langle\alpha| a a^\dagger |\alpha\rangle - \langle\alpha| a^\dagger a |\alpha\rangle^2 \\
&= |\alpha|^2 \langle\alpha| (1 + a^\dagger a) |\alpha\rangle - \langle\alpha| a^\dagger a |\alpha\rangle^2 = |\alpha|^2 = \bar{n}
\end{aligned} \tag{250}$$

9.2   Super-Poissonian Statistics

One of the example is the thermal state. Consider the Boltzmann distribution

$$\mathrm{Pr}(n) = \frac{\exp(-n\hbar\omega/k_B T)}{\sum_{n=0}^{\infty} \exp(-n\hbar\omega/k_B T)} = \frac{x^n}{\sum_{n=0}^{\infty} x^n} \tag{251}$$

where $x = \exp(-\hbar\omega/k_B T)$. If $\hbar\omega \gg k_B T$, *i.e.* x is small, then $\sum_n x^n = 1/(1-x)$. The probability becomes

$$\mathrm{Pr}(n) = (1-x)x^n \tag{252}$$

The mean photon number

$$\bar{n} = \sum_n nx^n(1-x) = \frac{x}{1-x} = \frac{1}{\exp(\hbar\omega/k_B T) - 1} \tag{253}$$

from this, we have

$$x = \frac{\bar{n}}{1 + \bar{n}} \tag{254}$$

so

$$\boxed{\Pr(n) = \frac{1}{1 + \bar{n}} \left(\frac{\bar{n}}{1 + \bar{n}}\right)^n} \tag{255}$$

which is called the *Bose-Einstein* distribution. Now we calculate the variance

$$\Delta^2 n = \bar{n} + \bar{n}^2 \tag{256}$$

Thus, we see that a thermal state exhibits super-Poissonian statistics.

9.3   Sub-Poissonian Statistics

Consider the Fock state $|n\rangle$. The Fock states have a mean photon number

$$\bar{n} = \langle n| a^\dagger a |n\rangle = n \tag{257}$$

and a variance

$$\Delta^2 n = \langle n| (a^\dagger a)^2 |n\rangle - \langle n| a^\dagger a |n\rangle = 0 \tag{258}$$

10   Hanbury Brown - Twiss Measurements

- Bunched light: $g^{(2)} > 1$

- Coherent light: $g^{(2)} = 1$

- Antibunched light: $g^{(2)} < 1$

10.1   Classical Second-Order Intensity Correlations

Let's start with the classical 'second-order correlation function'

$$g^{(2)} = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle} \tag{259}$$

The intensity of classical field can be written as

$$I(t) = \bar{I} + \Delta I(t) \tag{260}$$

where $\langle I(t)\rangle = \langle I(t+\tau)\rangle = \bar{I}$. The light field has a coherence time τ_c, such that $\tau \gg \tau_c$, intensity fluctuations will be uncorrelated, *i.e.* $\langle \Delta I(t)\Delta I(t+\tau)\rangle = 0$. In this case

$$\begin{aligned} g^{(2)}(\tau \gg \tau_c) &= \frac{\langle(\bar{I} + \Delta I(t)\rangle\langle\bar{I} + \Delta I(t+\tau)\rangle}{\bar{I}^2} \\ &= \frac{\bar{I}^2 + \langle\bar{I}\Delta I(t+\tau)\rangle + \langle\Delta I(t)\bar{I}\rangle + \langle\Delta I(t)\Delta I(t+\tau)\rangle}{\bar{I}^2} = 1 \end{aligned} \tag{261}$$

Consider $\tau < \tau_c$. Of particular interest, is the case where $\tau = 0$ and we have

$$g^{(2)}(0) = \frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2} \geq 1 \tag{262}$$

which is a key result of the classical analysis of the second order correlation function and indicates that the light tends to 'bunch' together.
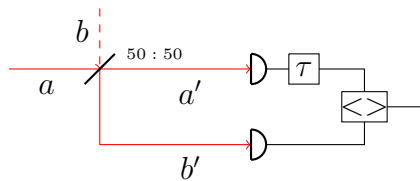
Figure 9: The Hanbury Brown - Twiss experimental setup to measure intensity correlations.

10.2   Quantum Second-Order Correlations

Let's now look at the second-order correlation function in the quantum picture of light. The setup is in fig.9. The second-order correlation function can be expressed as

$$
\begin{aligned}
g^{(2)}(\tau) =& \frac{\langle n_1(t) n_2(t+\tau) \rangle}{\langle n_1(t) \rangle \langle n_2(t+\tau) \rangle} \\
=& \frac{\langle a'^\dagger(t) b'^\dagger(t+\tau) b'(t+\tau) a'(t) \rangle}{\langle a'^\dagger(t) a'(t) \rangle \langle b'^\dagger(t+\tau) b'(t+\tau) \rangle}
\end{aligned}
\tag{263}
$$

Consider $\tau = 0$, then

$$
\langle a'^\dagger(t) a'(t) \rangle = \frac{1}{2}\langle (a^\dagger - ib^\dagger)(a + ib) \rangle = \frac{1}{2}\langle a^\dagger a \rangle = \frac{1}{2}\langle n \rangle
\tag{264}
$$

$$
\langle b'^\dagger(t) b'(t) \rangle = \frac{1}{2}\langle (b^\dagger - ia^\dagger)(b + ia) \rangle = \frac{1}{2}\langle b^\dagger b \rangle = \frac{1}{2}\langle n \rangle
\tag{265}
$$

$$
\begin{aligned}
\langle a'^\dagger(t) b'^\dagger(t+\tau) b'(t+\tau) a'(t) \rangle =& \frac{1}{4}\langle (a^\dagger - ib^\dagger)(b^\dagger - ia^\dagger)(b + ia)(a + ib) \rangle \\
=& \frac{1}{4}\langle a^\dagger a^\dagger a a \rangle = \frac{1}{4}\langle a^\dagger (a a^\dagger - 1) a \rangle \\
=& \frac{1}{4}\langle a^\dagger a (a^\dagger a - 1) \rangle = \frac{1}{4}\langle n(n-1) \rangle
\end{aligned}
\tag{266}
$$

so we have

$$
g^{(2)}(0) = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2}
\tag{267}
$$

Notably, for a single photon input $g^{(2)} = 0$, which indicate strong antibunching and is highly non-classical.

11   The Hong-Ou-Mandel Effect

Consider the interference of two single photons each incident on a $50:50$ beam-splitter as shown in Fig.10. The input is $|00\rangle$, and the output

$$
\begin{aligned}
|\psi\rangle = \hat{B}\,|1\rangle\,|1\rangle = \hat{B}\hat{a}^\dagger\hat{b}^\dagger\,|0\rangle\,|0\rangle =& \hat{B}\hat{a}^\dagger\hat{B}^\dagger\hat{B}\hat{b}^\dagger\hat{B}^\dagger\hat{B}\,|0\rangle\,|0\rangle \\
=& \left(\hat{B}\hat{a}^\dagger\hat{B}^\dagger\right)\left(\hat{B}\hat{b}^\dagger\hat{B}^\dagger\right)|0\rangle\,|0\rangle
\end{aligned}
\tag{268}
$$

The unitary beam-splitter operator

$$
\hat{B} = \exp\left[i\frac{\theta}{2}\left(a^\dagger b + a b^\dagger\right)\right]
\tag{269}
$$

and we have

$$
\hat{B}^\dagger\hat{a}\hat{B} = \hat{a}\cos\frac{\theta}{2} + i\hat{b}\sin\frac{\theta}{2}
\tag{270}
$$

$$
\hat{B}^\dagger\hat{b}\hat{B} = \hat{b}\cos\frac{\theta}{2} + i\hat{a}\sin\frac{\theta}{2}
\tag{271}
$$

and we note that

$$
\hat{B}^\dagger\left(\frac{\theta}{2}\right) = \hat{B}\left(-\frac{\theta}{2}\right)
\tag{272}
$$

For the $50:50$ beam-splitter, $\theta = \pi/2$, so we have

$$
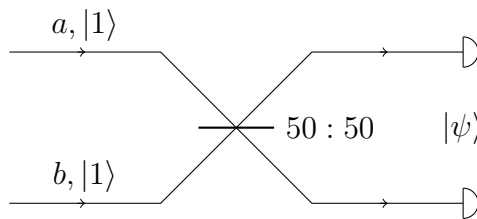\left(\hat{B}\hat{a}\hat{B}^\dagger\right)^\dagger = \hat{B}\hat{a}^\dagger\hat{B}^\dagger = \hat{a}\cos\left(-\frac{\pi}{4}\right) + i\hat{b}\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\hat{a} - i\hat{b}\right)
\tag{273}
$$

$$
\left(\hat{B}\hat{b}\hat{B}^\dagger\right)^\dagger = \hat{B}\hat{b}^\dagger\hat{B}^\dagger = \hat{b}\cos\left(-\frac{\pi}{4}\right) + i\hat{a}\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\hat{b} - i\hat{a}\right)
\tag{274}
$$

hence

$$
\hat{B}\hat{a}\hat{B}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right)
\tag{275}
$$

$$
\hat{B}\hat{b}\hat{B}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{b}^\dagger + i\hat{a}^\dagger\right)
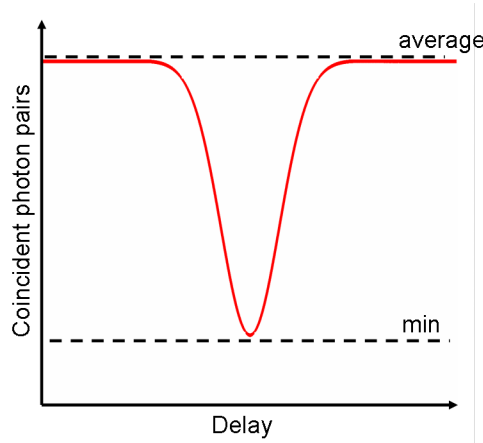\tag{276}
$$

Figure 10: Two photon interference.

So the output state would be

$$
\begin{aligned}
|\psi\rangle =& \frac{1}{2}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right)\left(\hat{b}^\dagger + i\hat{a}^\dagger\right)|0\rangle\,|0\rangle \\
=& \frac{1}{2}\left(\hat{a}^\dagger\hat{b}^\dagger + i\hat{a}^{\dagger 2} + i\hat{b}^{\dagger 2} - \hat{b}^\dagger\hat{a}^\dagger\right)|0\rangle\,|0\rangle \\
=& \frac{1}{2}\left(i\hat{a}^{\dagger 2} + i\hat{b}^{\dagger 2}\right)|0\rangle\,|0\rangle \\
=& \frac{1}{\sqrt{2}}(|2\rangle\,|0\rangle + |0\rangle\,|2\rangle)
\end{aligned}
\tag{277}
$$

The 'global phase' factor i can be omitted as this does not change the prediction of measurement probabilities.

Figure 11: The Hong–Ou–Mandel dig.

Customarily the Hong–Ou–Mandel effect is observed using two photondetectors monitoring the output modes of the beam splitter. The coincidence rate of the detectors will drop to zero when the identical input photons overlap perfectly in time. This is called the Hong–Ou–Mandel dip, or HOM dip.