Imperial College London

NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF PHYSICS

Advanced Classical Physics

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Date: November 3, 2023

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1 Lagrangian Mechanics

1.1 Action Principle

1.1.1 Fermat's Principle

The idea of the 'principle of least action' has its origin in Fermat's principle in optics, according to which light follows the shortest optical path, i.e., the path of shortest time to reach its destination.

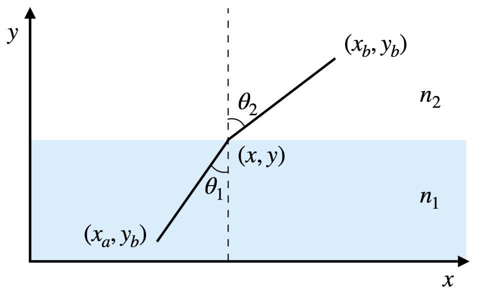

Figure 1: Snell's law of refraction for light passing through media of different indices of refraction.

According to figure (1), consider a light ray from point (x_a, y_a) to (x_b, y_b), the optical path length follows

$$T(x) = \frac{n_1}{c}\sqrt{(x_a - x)^2 + (y_a - y)^2} + \frac{n_2}{c}\sqrt{(x_b - x)^2 + (y_b - y)^2} \tag{1}$$

According to Fermat's principle, we need to find the minimum of this quantity.

$$\begin{aligned}\frac{\partial T(x)}{\partial x} &= \frac{n_1}{c}\frac{x - x_a}{\sqrt{(x_a - x)^2 + (y_a - y)^2}} + \frac{n_2}{c}\frac{x - x_b}{\sqrt{(x_b - x)^2 + (y_b - y)^2}} \\ &= -\frac{n_1}{c}\sin\theta_1 + \frac{n_2}{c}\sin\theta_2 = 0\end{aligned} \tag{2}$$

which shows the *Snell's law*

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{3}$$

What happens if the refractive index is a function of space $n(x, y)$? The Fermat's action becomes

$$T = \int \mathrm{d}T = \int \frac{n(x, y)}{c}\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} \tag{4}$$

Suppose we parameterise the trajectory of the particle by a monotonic parameter λ

$$\boldsymbol{r}(\lambda) = (x(\lambda), y(\lambda)) \tag{5}$$

such that

$$x(\lambda_a) = x_a, \quad x(\lambda_b) = x_b, \quad y(\lambda_a) = y_a, \quad y(\lambda_b) = y_b \tag{6}$$

The Fermat's action is

$$T = \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \frac{n(x(\lambda), y(\lambda))}{c}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2} \tag{7}$$

1.1.2 Euler-Lagrange Equations

In configuration space (the space of coordinates and velocities), the action S is defined as an integral over a function $L(x, \dot{x}, t)$ known as the Lagrangian, as

$$S[x(t)] = \int_{t_a}^{t_b} \mathrm{d}t L(x(t), \dot{x}(t), t) \tag{8}$$

For standard conservative systems the Lagrangian is simply the difference of the kinetic energy T and the potential energy V, i.e.

$$\boxed{L = T - V} \tag{9}$$

The actual physical trajectory is the function x that minimises the action subject to the boundary conditions $x(t_a) = x_a$ and $x(t_b) = x_b$. This perturbation changes the action by an amount δS given by

$$\delta S[x(t)] = \int_{t_a}^{t_b} \mathrm{d}t \left[\frac{\partial L}{\partial x}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}(t)\right] \tag{10}$$

In the second term, we may integrate by parts,

$$\int_{t_a}^{t_b} \mathrm{d}t \frac{\partial L}{\partial \dot{x}}\delta \dot{x}(t) = \left[\frac{\partial L}{\partial \dot{x}}\delta x(t)\right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x(t) = -\int_{t_a}^{t_b} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x(t) \tag{11}$$

Substituting this result into the expression (10) for δS, we have

$$\delta S = \int_{t_a}^{t_b} \mathrm{d}t \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x(t) \tag{12}$$

In order that S be stationary, we require

$$\boxed{\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0} \tag{13}$$

This is known as the *Euler-Lagrange equation*, and it is the equation of motion in the Lagrangian formulation of mechanics.

1.1.3 Back to Fermat

In the gauge $\lambda = y$, the total time was given to

$$T = \int_{y_a}^{y_b} \mathrm{d}y \frac{n(x(y), y)}{c}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1} \tag{14}$$

The Lagrangian is

$$L(x(y), \dot{x}(y), y) = \frac{n(x(y), y)}{c}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1} \tag{15}$$

The Euler-Lagrange equation for this problem is

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial L}{\partial \frac{\mathrm{d}x}{\mathrm{d}y}}\right) = \frac{\partial L}{\partial x} \tag{16}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[n(x,y)\frac{\frac{\mathrm{d}x}{\mathrm{d}y}}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}}\right] = \frac{\partial n(x,y)}{\partial x}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1} \tag{17}$$

it simplifies a lot in the case where the refractive index is just a function of y. Then

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[n(y)\frac{\frac{\mathrm{d}x}{\mathrm{d}y}}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}}\right] = 0 \tag{18}$$

which is easily solved as

$$n(y)\frac{\frac{\mathrm{d}x}{\mathrm{d}y}}{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}} = n(y)\frac{\mathrm{d}x}{\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}} = n(y)\sin(\theta(y)) = A \tag{19}$$

with A a constant.

1.2 Generalized coordinates and momenta

General coordinate q_i
General conjugate momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$\delta S = \int \mathrm{d}t\left(\sum_{i=1}^{N}\frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{N}\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\right) \tag{20}$$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad \forall i = 1, \cdots, N} \tag{21}$$

1.3 Conservation Laws

1.3.1 Momentum Conservation

For any generalised coordinate q_i, we define the generalised momentum p_i by

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}} \tag{22}$$

So we have

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\mathrm{d}p_i}{\mathrm{d}t} \tag{23}$$

Whenever the Lagrangian L does not depend *explicitly* on q_i, the corresponding generalised momentum p_i is conserved

$$\boxed{\frac{\partial L}{\partial q_i} = 0 \quad \Leftrightarrow \quad p_i \text{ is conserved.}} \tag{24}$$

1.3.2 Energy (or Hamiltonian) Conservation

First consider the total time derivative of the Lagrangian

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t}\end{aligned} \tag{25}$$

Rearranging this equation

$$\frac{\partial L}{\partial t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i p_i\dot{q}_i - L\right) = -\frac{\mathrm{d}H}{\mathrm{d}t} \tag{26}$$

Whenever the Lagrangian does not depend *explicitly* on t, the Hamiltonian is conserved

$$\boxed{\frac{\partial L}{\partial t} = 0 \quad \Leftrightarrow \quad H = \sum_i p_i\dot{q}_i - L \text{ is conserved.}} \tag{27}$$

1.4 Constraints and Number of degrees of Freedom

1.4.1 Kinetic Matrix

Using the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_j \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \sum_j \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} = \frac{\partial L}{\partial q_i} \tag{28}$$

We can write in this form

$$\boxed{\sum_j \mathcal{Z}^{ij}\ddot{q}_j + \mathcal{F}^i = 0} \tag{29}$$

where the *kinetic matrix* $\mathcal{Z}_{ij}$ and the vector $\mathcal{F}_i$ are both functions of the coordinates

$$\mathcal{Z}^{ij}(q_1, \cdots, q_N, \dot{q}_1, \cdots, \dot{q}_N, t) = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \mathcal{Z}^{ji} \tag{30}$$

$$\mathcal{F}^i(q_1, \cdots, q_N, \dot{q}_1, \cdots, \dot{q}_N, t) = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} - \frac{\partial L}{\partial q_i} \tag{31}$$

So long as the symmetric matrix $\mathcal{Z}_{ij}$ is non-degenerate, i.e. so long as $\det(\mathcal{Z}) \neq 0$

$$\ddot{q} = -(\mathcal{Z})^{-1}\mathcal{F} \tag{32}$$

There are cases however where the matrix $\mathcal{Z}$ is 'degenerate' and $\det(\mathcal{Z}) = 0$ means that not all coordinates q_i are independent.

1.4.2 Lagrange Multipliers

$$\tilde{L}(q_i, \dot{q}_i, \lambda, t) = L(q_i, \dot{q}_i, t) + \lambda f(q_i, \dot{q}_i, t) \tag{33}$$

The Euler-Lagrange equation for λ is given by

$$\frac{\delta \tilde{S}}{\delta \lambda} = \frac{\partial \tilde{L}}{\partial \lambda} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\lambda}}\right) = f(q_i, \dot{q}_i, t) \tag{34}$$

and therefore it imposes the constraint $f = 0$ independently of what λ actually is. The Euler-Lagrange equations for the original coordinates are

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = -\lambda(t)\frac{\partial f}{\partial q_i} \tag{35}$$

1.4.3 Example: Helter Skelter

A child of m slides down a helter skelter, the Lagrangian is given by

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \tag{36}$$

and

$$z = h - \alpha\theta, \qquad r = \beta\theta \tag{37}$$

where α and β are positive constants and h is the height of the helter skelter. We take the angle θ to go from 0 to infinity as the trajectory winds around multiple times. The constraints are

$$C_1 = z - h + \alpha\theta, \qquad C_2 = r - \beta\theta \tag{38}$$

1) **Method 1: Solve constraints**

Put constraints into the Lagrangian

$$L(\theta, t) = \frac{1}{2}m\left(\beta^2\dot{\theta}^2 + \beta^2\theta^2\dot{\theta}^2 + \alpha^2\dot{\theta}^2\right) - mg(h - \alpha\theta) \tag{39}$$

The Euler-Lagrange equation for θ is

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\alpha^2 + \beta^2\right) + \beta^2\theta^2\right]\dot{\theta} = m\beta^2\theta\dot{\theta}^2 + mg\alpha \tag{40}$$

Figure 2: A helter skelter.

2) **Method 2: Work in extended configuration space**

$$\tilde{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz + (\lambda_1 C_1 + \lambda_2 C_2) \tag{41}$$

The Euler-Lagrange equations are

- For r: $mr\dot{\theta}^2 - m\ddot{r} = -\lambda_2$
- For θ: $m\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = \alpha\lambda_1 - \beta\lambda_2$
- For z: $m\ddot{z} + mg = \lambda_1$
- For λ_1: $z = h - \alpha\theta$
- For λ_2: $r = \beta\theta$

So we have

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\alpha^2 + \beta^2\right) + \beta^2\theta^2\right]\dot{\theta} = m\beta^2\theta\dot{\theta}^2 + mg\alpha \tag{42}$$

1.5 Normal Modes

'Natural' System: We consider the situation where particles can move around their equilibrium positions. We assume that the system is described by N generalised coordinates q_i. We also assume that it is *natural*, which means that the kinetic energy is a *quadratic homogeneous function* of the generalised velocities. We can then write it as

$$T = \frac{1}{2}\sum_{ij} a_{ij}(q_1, \cdots, q_N)\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{\boldsymbol{q}}^T A\dot{\boldsymbol{q}} \tag{43}$$

Since the Lagrangian is given by

$$L = T - V(q_1, \cdots, q_N, t) \tag{44}$$

where the potential does not depend on the velocities $\dot{q}_i$. Then we have

$$\begin{aligned}
\mathcal{Z}^{ij} =& \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial^2}{\partial \dot{q}_i \partial \dot{q}_j}\left(\frac{1}{2} a_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta\right) \\
=& \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} a_{\alpha\beta} \frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i} \dot{q}_\beta + \frac{1}{2} a_{\alpha\beta} \dot{q}_\alpha \frac{\partial \dot{q}_\beta}{\partial \dot{q}_i}\right) \\
=& \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} a_{\alpha\beta} \delta_{i\alpha} \dot{q}_\beta + \frac{1}{2} a_{\alpha\beta} \dot{q}_\alpha \delta_{i\beta}\right) \\
=& \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} a_{i\beta} \dot{q}_\beta + \frac{1}{2} a_{\alpha i} \dot{q}_\alpha\right) \\
=& \frac{1}{2} a_{ij} \dot{q}_j + \frac{1}{2} a_{ji} \dot{q}_j = a_{ij}
\end{aligned} \tag{45}$$

the kinetic matrix is then given directly by the coefficients a_{ij}.

1.5.1 Equilibrium Points

From the calculation in above, we know that

$$\frac{\partial L}{\partial \dot{q}_i} = a_{ij} \dot{q}_j \tag{46}$$

The Euler-Lagrangian equation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(a_{ij} \dot{q}_j\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\left(a_{ij} \dot{q}_j\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2} a_{jk} \dot{q}_j \dot{q}_k\right) + \frac{\partial V}{\partial q_i} \\
=& a_{ij} \ddot{q}_j - \frac{1}{2} \frac{\partial a_{jk}}{\partial q_i} \dot{q}_j \dot{q}_k + \frac{\partial V}{\partial q_i} = 0
\end{aligned} \tag{47}$$

In equilibrium point, $\dot{q}_i$ is a constant, so we require

$$\left.\frac{\partial V}{\partial q_i}\right|_{q_i = q_{0i}} = 0, \qquad i = 1, 2, \cdots, N \tag{48}$$

1.5.2 Small Oscillations

We can write the coordinates as $q_i(t) = q_{0i} + \delta q_i(t)$. The action is given by

$$\begin{aligned}
S =& \int \mathrm{d}t \left[\frac{1}{2} a_{ij}(\boldsymbol{q}_0) \frac{\mathrm{d}}{\mathrm{d}t}(q_{0i} + \delta q_i) \frac{\mathrm{d}}{\mathrm{d}t}(q_{0j} + \delta q_j) - V(\boldsymbol{q}_0 + \delta \boldsymbol{q})\right] \\
=& \int \mathrm{d}t \left[\frac{1}{2} a_{ij}(\boldsymbol{q}_0) \delta \dot{q}_i \delta \dot{q}_j - \left(V(\boldsymbol{q}_0) + \left.\frac{\partial V}{\partial q_i}\right|_{\boldsymbol{q}_0} \delta q_i + \frac{1}{2} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}_0} \delta q_i \delta q_j + \mathcal{O}(\delta q^3)\right)\right] \\
=& \int \mathrm{d}t \left[\frac{1}{2} a_{ij}(\boldsymbol{q}_0) \delta \dot{q}_i \delta \dot{q}_j - \frac{1}{2} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}_0} \delta q_i \delta q_j\right] \\
=& \int \mathrm{d}t \left[\frac{1}{2} a_{ij}(\boldsymbol{q}_0) \delta \dot{q}_i \delta \dot{q}_j - \frac{1}{2} b_{ij}(\boldsymbol{q}_0) \delta q_i \delta q_j\right] - \int \mathrm{d}t V(\boldsymbol{q}_0)
\end{aligned} \tag{49}$$

where

$$b_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} \tag{50}$$

The second term is just a constant, so and so for small fluctuations all that matters is the quadratic part, which we refer to as the action for the quadratic fluctuations

$$\begin{aligned} S_{(2)} &= \int \mathrm{d}t \left(\frac{1}{2} a_{ij} \delta \dot{q}_i \delta \dot{q}_j - \frac{1}{2} b_{ij} \delta q_i \delta q_j \right) \\ &= \int \mathrm{d}t \left(\frac{1}{2} \delta \dot{\boldsymbol{q}}^T \mathbf{A} \delta \dot{\boldsymbol{q}} - \frac{1}{2} \delta \boldsymbol{q}^T \mathbf{B} \delta \boldsymbol{q} \right) \end{aligned} \tag{51}$$

where the matrix $\mathbf{A}$ and $\mathbf{B}$ have the components $\mathbf{A}_{ij} = a_{ij}$ and $\mathbf{B}_{ij} = b_{ij}$.

The Euler-Lagrange equation for the fluctuations is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L_{(2)}}{\partial \delta \dot{q}_i} \right) = \frac{\partial L}{\partial \delta q_i} \tag{52}$$

which is

$$a_{ij} \delta \ddot{q}_j = -b_{ij} \delta q_j \qquad \text{or} \qquad \mathbf{A} \delta \ddot{\boldsymbol{q}} = -\mathbf{B} \delta \boldsymbol{q} \tag{53}$$

If $\det(\mathbf{A}) \neq 0$, then

$$\delta \ddot{\boldsymbol{q}} = -\mathbf{K} \delta \boldsymbol{q} \tag{54}$$

with

$$\mathbf{K} = \mathbf{A}^{-1} \mathbf{B} \tag{55}$$

1.5.3 Gram-Schmidt Diagonalization

Since the matrix $\mathbf{A}$ is symmetric (we may assume this from the outset), we can always diagonalize it using a $N \times N$ orthogonal matrix $\mathbf{O}$ ($\mathbf{O}^T \mathbf{O} = \mathbb{1}$) so that

$$\mathbf{A}_D = \mathbf{O}^T \mathbf{A} \mathbf{O} \tag{56}$$

with $\mathbf{A}_D$ diagonal. The kinetic term

$$\begin{aligned} T =& \frac{1}{2} \delta \dot{\boldsymbol{q}}^T \mathbf{A} \delta \dot{\boldsymbol{q}} = \frac{1}{2} \delta \dot{\boldsymbol{q}}^T \mathbf{O} \mathbf{A}_D \mathbf{O}^T \delta \dot{\boldsymbol{q}} = \frac{1}{2} \delta \dot{\tilde{\boldsymbol{q}}}^T \mathbf{A}_D \delta \dot{\tilde{\boldsymbol{q}}} \\ =& \frac{1}{2} \delta \dot{\tilde{q}}_i (a_D)_{ij} \delta \dot{\tilde{q}}_j \\ =& \frac{1}{2} (a_D)_{11} \delta \dot{\tilde{q}}_1^2 + \frac{1}{2} (a_D)_{22} \delta \dot{\tilde{q}}_2^2 + \cdots + \frac{1}{2} (a_D)_{NN} \delta \dot{\tilde{q}}_N^2 \end{aligned} \tag{57}$$

where

$$\delta \tilde{\boldsymbol{q}} = \mathbf{O}^T \delta \boldsymbol{q} \tag{58}$$

To put the kinetic term in diagonal and *normalized* form, we require

$$\delta q_i = \sqrt{(a_D)_{ii}^{-1}} \delta Q_i \tag{59}$$

This is the statement that there is a diagonal matrix $\mathbf{W}$ whose matrix elements are

$$\mathbf{W}_{ij} = \sqrt{(a_D)^{-1}_{ii}}\delta_{ij} \tag{60}$$

such that

$$\delta\tilde{\boldsymbol{q}} = \mathbf{W}\delta\boldsymbol{Q} \tag{61}$$

for which

$$\mathbf{W}^T\mathbf{A}_D\mathbf{W} = \mathbb{1} \tag{62}$$

These two operations amount to saying that there is a matrix $\mathbf{S} = \mathbf{OW}$ such that

$$\mathbf{S}^T\mathbf{A}_D\mathbf{S} = \mathbb{1} \tag{63}$$

the action for the fluctuations becomes

$$\begin{aligned} S_{(2)} &= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\mathbf{S}^T\mathbf{A}\mathbf{S}\delta\dot{\boldsymbol{Q}} - \frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\mathbf{S}^T\mathbf{B}\mathbf{S}\delta\dot{\boldsymbol{Q}}\right) \\ &= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\delta\dot{\boldsymbol{Q}} - \frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\mathbf{k}\delta\dot{\boldsymbol{Q}}\right) \end{aligned} \tag{64}$$

where

$$\mathbf{k} = \mathbf{S}^T\mathbf{B}\mathbf{S} \tag{65}$$

This is the canonically normalized form for the fluctuations.

1.5.4 Cholesky Decomposition

The Cholesky decomposition of a *real Hermitian positive*-definite[1] matrix $\mathbf{A}$, is a decomposition of the form

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T \tag{66}$$

where $\mathbf{L}$ is a left triangular matrix with real and positive diagonal entries

$$\mathbf{L} = \begin{pmatrix} \# & 0 & 0 & 0 \\ \# & \# & 0 & 0 \\ \# & \# & \# & 0 \\ \# & \# & \# & \# \end{pmatrix} \tag{67}$$

So the kinetic term can be written as

$$T = \frac{1}{2}\delta\dot{\boldsymbol{q}}\mathbf{A}\delta\dot{\boldsymbol{q}} = \frac{1}{2}\delta\dot{\boldsymbol{q}}\mathbf{L}\mathbf{L}^T\delta\dot{\boldsymbol{q}} = \frac{1}{2}\delta\dot{\boldsymbol{Q}}^T\delta\dot{\boldsymbol{Q}} \tag{68}$$

where

$$\delta\boldsymbol{Q} = \mathbf{L}^T\delta\boldsymbol{q} \tag{69}$$

[1]'Positive' means the eigenvalues of $\mathbf{A}$ are all positive.

And the action

$$S = \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{\boldsymbol{q}}\mathbf{A}\delta\dot{\boldsymbol{q}} - \frac{1}{2}\delta\boldsymbol{q}^T\mathbf{B}\delta\boldsymbol{q} \right)$$
$$= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{\boldsymbol{Q}}\delta\dot{\boldsymbol{Q}} - \frac{1}{2}\delta\boldsymbol{Q}^T\mathbf{k}\delta\boldsymbol{Q} \right) \tag{70}$$
$$= \int \mathrm{d}t \left(\frac{1}{2}\delta\dot{Q}_i\delta\dot{Q}_i - \frac{1}{2}k_{ij}\delta Q_i\delta Q_j \right)$$

where

$$\mathbf{k} = \mathbf{L}^{-1}\mathbf{B}(\mathbf{L}^T)^{-1} \tag{71}$$

The Euler-Lagrange equation for Q_i is given by

$$\delta\ddot{Q}_i = -k_{ij}\delta Q \qquad \text{or} \qquad \delta\ddot{\boldsymbol{Q}} = -\mathbf{k}\delta\boldsymbol{Q} \tag{72}$$

We look for solutions of the form

$$\delta\boldsymbol{Q} = \mathrm{e}^{i\omega t}\delta\boldsymbol{Q}_\omega \tag{73}$$

then we have the eigenfunction equation with the matrix $\mathbf{k}$ and eigenvalue ω^2.

$$\mathbf{k}\delta\boldsymbol{Q}_\omega = \omega^2\delta\boldsymbol{Q}_\omega \tag{74}$$

which means that each normal coordinate $\boldsymbol{Q}_{\omega_\alpha}$ oscillates independently of all others with its own normal frequency ω_α^2.

1.5.5 Example: Double Pendulum

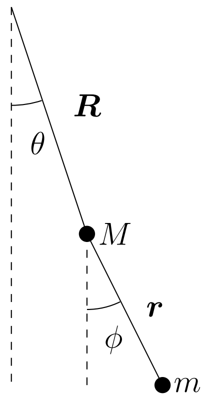

Figure 3: Double pendulum

Consider a double pendulum, with a second pendulum hanging from the first as depicted in Fig.(3). The kinetic energy of the system is

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}m(\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}})^2$$
$$= \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}m\left(\dot{\boldsymbol{R}}^2 + \dot{\boldsymbol{r}}^2 + 2\dot{\boldsymbol{R}}\dot{\boldsymbol{r}}\right) \tag{75}$$
$$= \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + mRr\dot{\theta}\dot{\phi}\cos(\phi - \theta)$$

The potential energy is

$$\begin{aligned} V =& Mg(-R\cos\theta) + mg(-R\cos\theta - r\cos\phi) \\ =& -(M+m)gR\cos\theta - mgr\cos\phi \end{aligned} \tag{76}$$

The equilibrium solution is $\theta_0 = \phi_0 = 0$, so θ and ϕ can be expressed by

$$\theta(t) = \delta\theta(t), \qquad \phi(t) = \delta\phi(t) \tag{77}$$

then we have

$$\begin{aligned} L_{(2)} =& \frac{1}{2}(M+m)R^2\delta\dot{\theta}^2 + \frac{1}{2}mr^2\delta\dot{\phi}^2 + mRr\delta\dot{\theta}\delta\dot{\phi} - \frac{1}{2}(M+m)gR\delta^2\theta - \frac{1}{2}mgr\delta\phi^2 \\ =& \frac{1}{2}MR^2\delta\dot{\theta}^2 + \frac{1}{2}m\left(R\delta\dot{\theta} + r\delta\dot{\phi}\right)^2 - \frac{1}{2}(M+m)gR\delta^2\theta - \frac{1}{2}mgr\delta\phi^2 \end{aligned} \tag{78}$$

We define that

$$\delta Q_1 = \sqrt{M}R\delta\theta, \qquad \delta Q_2 = \sqrt{m}(R\delta\theta + r\delta\phi) \tag{79}$$

so the Lagrangian becomes

$$L_{(2)} = \frac{1}{2}\delta\dot{Q}_1^2 + \frac{1}{2}\delta\dot{Q}_2^2 - \frac{1}{2}\frac{(M+m)g}{MR}\delta Q_1^2 - \frac{1}{2}\frac{mg}{Mr}(\delta Q_1 - \delta Q_2)^2 \tag{80}$$

and the potential energy is given by

$$V = -\frac{1}{2}\delta\boldsymbol{Q}^T\mathbf{k}\delta\boldsymbol{Q} = -\frac{1}{2}\begin{pmatrix}\delta Q_1 & \delta Q_2\end{pmatrix}\mathbf{k}\begin{pmatrix}\delta Q_1 \\ \delta Q_2\end{pmatrix} \tag{81}$$

where

$$\mathbf{k} = \begin{pmatrix} \frac{(M+m)g}{MR} + \frac{mg}{Mr} & -\frac{mg}{Mr} \\ -\frac{mg}{Mr} & \frac{mg}{Mr} \end{pmatrix} = \frac{mg}{Mr}\begin{pmatrix} \frac{(M+m)r}{mR} + 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{82}$$

To simplify $\mathbf{k}$, we choose $R = r$ and $M = m$, then we have

$$\mathbf{k} = \frac{g}{R}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \tag{83}$$

The eigenvalues ω_1^2 and ω_2^2 satisfies

$$\det(\mathbf{k} - \omega^2\mathbb{1}) = \frac{g}{R}\left[(3-\omega^2)(1-\omega^2) - 1\right] = 0 \tag{84}$$

solve this and get

$$\omega_{1,2}^2 = (2 \pm \sqrt{2})\frac{g}{R} > 0 \tag{85}$$

the associated eigenvectors are $\boldsymbol{Q}_\pm = (1 \pm \sqrt{2}, -1)$. Go back to the original coordinates θ, ϕ, we know that

$$Q_1 \sim \theta, \qquad Q_2 \sim \theta + \phi \tag{86}$$

so the normal modes are

$$(\theta, \phi)_\pm = (1 \pm \sqrt{2}, -(2 \pm \sqrt{2})) \sim (1, \pm\sqrt{2}) \tag{87}$$

1.6 Symmetries and Noether's Theorem

Let's parameterize the action by arbitrary coordinates q_i

$$S = \int \mathrm{d}t L(q, \dot{q}, t) \tag{88}$$

now we have

$$\begin{aligned} \delta S &= \int \mathrm{d}t \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) \\ &= \underbrace{\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) \Bigg|_{t_1}^{t_2}}_{\text{boundary term}} + \int \mathrm{d}t \underbrace{\left[\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right]}_{\text{Euler-Lagrange equation}} \delta q_i \\ &= \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) \Bigg|_{t_1}^{t_2} \end{aligned} \tag{89}$$

Euler-Lagrange equations follow the principle of least action. So $\delta S = 0$ up to boundary terms.

1.6.1 Discrete Global Symmetry

Let's consider a simple example, which is a harmonic oscillator

$$S[q(t)] = \int \mathrm{d}t \left(\frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right) \tag{90}$$

consider the transition $q' = -q$, and

$$S[q'(t)] = \int \mathrm{d}t \left(\frac{1}{2} m \dot{q}'^2 - \frac{1}{2} m \omega^2 q'^2 \right) = \int \mathrm{d}t \left(\frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right) = S[q(t)] \tag{91}$$

1.6.2 Continuous Global Symmetry

Let's consider a general infinitesimal transform

$$\boxed{\delta q_i = F_i(q, \dot{q}) \delta \lambda} \tag{92}$$

then the Lagrangian transforms as

$$\begin{aligned} \delta L(q, \dot{q}, t) =& L(q + F\delta\lambda, \dot{q} + \dot{F}\delta\lambda) - L(q, \dot{q}) \\ =& \frac{\partial L}{\partial q_i} F_i \delta\lambda + \frac{\partial L}{\partial \dot{q}_i} \dot{F}_i \delta\lambda \\ =& \left(\frac{\partial L}{\partial q_i} F_i + \frac{\partial L}{\partial \dot{q}_i} \dot{F}_i \right) \delta\lambda \end{aligned} \tag{93}$$

If this is a symmetry, then this must be a total derivative then

$$\boxed{\frac{\partial L}{\partial q_i} F_i + \frac{\partial L}{\partial \dot{q}_i} \dot{F}_i = \frac{\mathrm{d}A}{\mathrm{d}t}} \tag{94}$$

and the integration

$$\delta S = \int \mathrm{d}t \delta L = \int_{t_1}^{t_2} \mathrm{d}t \frac{\mathrm{d}A}{\mathrm{d}t} \delta\lambda = \underbrace{[A(t_2) - A(t_1)]}_{\text{boundary term}} \delta\lambda = 0 \tag{95}$$

1.6.3 Noether's theorem

Noether performed a clever trick. She make λ a function of time

$$\delta q_i = F_i(q_i, \dot{q}_i)\delta\lambda(t) \tag{96}$$

now we have

$$\begin{aligned}
\delta S &= \int \mathrm{d}t \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_i)\right] \\
&= \int \mathrm{d}t \left[\frac{\partial L}{\partial q_i}F_i\delta\lambda + \frac{\partial L}{\partial \dot{q}_i}\dot{F}_i\delta\lambda + \frac{\partial L}{\partial \dot{q}_i}F_i\frac{\mathrm{d}}{\mathrm{d}t}(\delta\lambda)\right] \\
&= \int \mathrm{d}t \left[\frac{\mathrm{d}A}{\mathrm{d}t}\delta\lambda - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}F_i\right)\delta\lambda\right] + \left(\frac{\partial L}{\partial \dot{q}}F_i\delta\lambda\right)\Bigg|_{t_1}^{t_2} \\
&= \int \mathrm{d}t \left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}F_i - A\right)\delta\lambda\right] + \left(\frac{\partial L}{\partial \dot{q}}F_i\delta\lambda\right)\Bigg|_{t_1}^{t_2} \\
&= \int \mathrm{d}t \left(-\frac{\mathrm{d}G}{\mathrm{d}t}\delta\lambda\right) + \left(\frac{\partial L}{\partial \dot{q}}F_i\delta\lambda\right)\Bigg|_{t_1}^{t_2} = 0
\end{aligned} \tag{97}$$

where

$$\boxed{G = \frac{\partial L}{\partial \dot{q}}F - A} \tag{98}$$

so we have

$$\boxed{\frac{\mathrm{d}G}{\mathrm{d}t} = 0 \quad \text{(up to the boundary term)}} \tag{99}$$

Theorem 1
Noether's theorem: For every continuous symmetry, there exist a conserved quantity G conserved in time.

Example
Consider the action

$$S = \int \mathrm{d}t \frac{1}{2}m\dot{q}^2 \tag{100}$$

and the transform $q' = q + \lambda$, *i.e.*, $\delta q = \delta\lambda$ with $F = 1$ and $A = 0$.

$$G = \frac{\partial L}{\partial \dot{q}}F - A = \frac{\partial L}{\partial \dot{q}} = p \tag{101}$$

which means the corresponding generalized momentum component p to be a constant of motion.

1.6.4 Hamiltonian as the Noether Charge for Time Translations

Consider a system which is time translation invariant meaning that the action is invariant (un to boundary terms) under the symmetry $t \to t+\delta t$. Such a transformation induces a change of coordinates

$$\delta q_i(t) = q_i(t+\lambda) - q_i(t) = \dot{q}_i \delta\lambda \tag{102}$$

Similarly the Lagrangian transforms as

$$\delta L = L(q(t+\lambda), \dot{q}(t+\lambda), t+\lambda) - L(q, \dot{q}, t) = \frac{\mathrm{d}L}{\mathrm{d}t}\delta\lambda \tag{103}$$

from which we infer

$$A = L \tag{104}$$

Thus the conserved charge implied by Noether's theorem associated with the symmetry of time translation invariance is

$$G = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = p_i\dot{q}_i - L = H \tag{105}$$

which we recognize to be the Hamiltonian.