

NOTES

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DEPARTMENT OF PHYSICS

Mathematical Methods for Physicists

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Contents

1	Vector Spaces and Tensors	4
1.1	vector spaces	4
1.1.1	Definition of a Vector Space	4
1.1.2	Linear Independence	4
1.1.3	Basis Vectors	5
1.1.4	Inner Product	5
1.1.5	Orthogonality	5
1.2	Matrices	6
1.2.1	Summation Convention	6
1.2.2	Recall Special Square Matrices	6
1.2.3	Levi-Civita Symbol (Alternating Tensor or Epsilon Tensor) . .	7
1.2.4	Eigenvalues, Eigenvectors and Diagonalization	7
1.3	Scalars, Vectors and Tensors in 3d Space	7
1.4	Transformations under Rotations	8
1.4.1	Transformation of Vectors	8
1.4.2	Transformation of Rank-Two Tensors	8
1.5	Tensor Calculus	8
1.5.1	The Gradient Operator	8
2	Green Functions	9
2.1	Introduction	9
2.2	Variation of Parameters	9
2.2.1	Homogeneous Initial Conditions	10
2.2.2	Inhomogeneous Initial Conditions	11
2.2.3	Homogeneous Two-Point Boundary Conditions	11
2.3	Green Function More Generally	12
2.3.1	Homogeneous Initial Conditions	13
2.3.2	Homogeneous Two-Point Boundary Conditions	14
2.3.3	Higher Dimensions, More Variables	14
3	Hilbert Spaces	16
3.1	Sturm-Liouville Theory	16
3.1.1	Self-Adjoint Differential Operators	16
3.1.2	Boundary Conditions	17
3.1.3	Weight Functions	17
3.1.4	Eigenfunctions and Eigenvalues	18
3.1.5	Eigenfunction Expansions	19
3.1.6	Green Functions Revisited	20
3.1.7	Eigenfunction Expansions for Solving ODEs	20
3.2	Legendre Polynomials	21
3.2.1	Two Examples	21
3.2.2	Legendre's Equation	22
3.3	Spherical Harmonics	23

4	Integral Transforms	25
4.1	Fourier Series	25
4.2	Fourier Transforms	25
4.2.1	Definition and Notation	25
4.2.2	Dirac Delta-Function	26
4.2.3	Properties of the Fourier Transform	26
4.2.4	Parseval's Theorem	27
4.2.5	Convolution Theorem	27
4.2.6	Examples of Fourier Transform	28
4.3	The Applications of Fourier Transforms in Physics	29
4.3.1	Diffraction Through an Aperture	29
4.3.2	Double Slit Diffraction	29
4.3.3	Diffusion Equation	30
4.4	Laplace Transforms	31
4.4.1	Properties	31
4.4.2	Examples	32
4.4.3	Convolution Theorem for Laplace Transforms	32
5	Complex Analysis	33
5.1	Complex Functions of a Complex Variable	33
5.2	Continuity, Differentiability and Analyticity	33
5.2.1	Definitions	33
5.2.2	The Cauchy-Riemann Conditions	34
5.2.3	Harmonic Functions	35
5.3	Multi-Valued Functions	36
5.4	Integration of Complex Functions	36
5.4.1	Contours	36
5.4.2	Cauchy's Theorem	37
5.4.3	Path Independence	37
5.4.4	Contour Deformation	38
5.4.5	Cauchy's Integral Theorem	38
5.4.6	Derivatives of Analytic Functions	39
5.4.7	Fourier Transform of a Gaussian	40
5.5	Power Series Representations of Complex Functions	41
5.5.1	Taylor Series	41
5.5.2	Singularities	41
5.6	Contour Integration using the Residue Theorem	42
5.6.1	The Residue Theorem	42
5.6.2	Contour Integration Examples	43
5.6.3	Inverting Laplace Transforms	44
6	Calculus of Variations	46
6.1	The Euler-Lagrange Problem	46

1 Vector Spaces and Tensors

1.1 vector spaces

1.1.1 Definition of a Vector Space

Definition. A real (complex) vector space is a set \mathbb{V} - whose elements are called vectors - together with two operations called addition (+) and scalar multiplication such that

1. \mathbb{V} is closed under **addition**: $\forall \underline{u}, \underline{v} \in \mathbb{V} \Rightarrow \underline{u} + \underline{v} \in \mathbb{V}$.
2. \mathbb{V} is closed under **scalar multiplication**: $\forall \underline{u} \in \mathbb{V}$ and \forall scalar $\lambda \Rightarrow \lambda \underline{u} \in \mathbb{V}$.
3. There exists a null or zero vector $\underline{0}$ such that $\underline{u} + \underline{0} = \underline{u}$.
4. Each vector \underline{u} has a corresponding negative vector $-\underline{u}$ such that: $\underline{u} + (-\underline{u}) = \underline{0}$.
5. The addition operation satisfies: $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ and $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$.
6. Scalar multiplication satisfies: $a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}$, $a(b\underline{u}) = (ab)\underline{u}$

Example. 3 component real column vectors

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

2 component complex vectors

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$$

1.1.2 Linear Independence

Definition. A set of n non-zero vectors $\{u_1, u_2, \dots, u_n\}$ in a vector space is linearly independent if

$$\sum_{i=1}^n a_i u_i = 0 \quad \Rightarrow \quad a_i = 0 \quad \forall i$$

Otherwise we say $\{u_1, u_2, \dots, u_n\}$ is linearly dependent.

Let N be the maximum number of linearly independent vectors in \mathbb{V} , then N is the dimension of \mathbb{V} .

Definition. A subspace, \mathbb{W} , of a vector space \mathbb{V} is a subset of \mathbb{V} that is itself a vector space.

1.1.3 Basis Vectors

Any set of n linearly independent vectors $\{u_i\}$ in an n -dimension vector space \mathbb{V} is a *basis* for \mathbb{V} . Any vector v in \mathbb{V} can be represented as a linear combination of the basis vectors

$$v = \sum_{i=1}^n a_i u_i$$

1.1.4 Inner Product

Definition. An inner product on a **real vector space** \mathbb{V} , is a **real number** $\langle \underline{u}, \underline{v} \rangle$ for every pair of vectors \underline{u} and \underline{v} . The inner product has the following properties

1. $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$
2. $\langle \underline{u}, a\underline{v}_1 + b\underline{v}_2 \rangle = a\langle \underline{u}, \underline{v}_1 \rangle + b\langle \underline{u}, \underline{v}_2 \rangle$
3. $\langle \underline{v}, \underline{v} \rangle \geq 0$
4. Define $\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$. Then $\|\underline{v}\| = 0 \Rightarrow \underline{v} = \underline{0}$

Definition. An inner product on a **complex space** \mathbb{V} , is a **real number** $\langle u, v \rangle$ for every ordered pair of vectors u and v . The inner product has the following properties

1. $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle^*$
2. $\langle \underline{u}, a\underline{v}_1 + b\underline{v}_2 \rangle = a\langle \underline{u}, \underline{v}_1 \rangle + b\langle \underline{u}, \underline{v}_2 \rangle$
 $\langle a\underline{u}_1 + b\underline{u}_2, \underline{v} \rangle = a^*\langle \underline{v}, \underline{u}_1 \rangle^* + b^*\langle \underline{v}, \underline{u}_2 \rangle^* = a^*\langle \underline{u}_1, \underline{v} \rangle + b^*\langle \underline{u}_2, \underline{v} \rangle$
3. $\langle \underline{v}, \underline{v} \rangle \geq 0$
4. Define $\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$. Then $\|\underline{v}\| = 0 \Rightarrow \underline{v} = \underline{0}$

Example.

$$\mathbb{R}^3 = \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right\rangle = ad + be + cf, \quad \mathbb{C}^2 = \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = a^*c + b^*d$$

1.1.5 Orthogonality

Two vectors are said to be **orthogonal** if their inner product is zero, i.e.

$$\langle \underline{u}, \underline{v} \rangle = 0 \tag{1}$$

A set of vectors $\{\underline{e}_1, \dots, \underline{e}_n\}$ is **orthonormal** if

$$\langle \underline{e}_i, \underline{e}_j \rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{2}$$

where δ_{ij} is named as Kronecker delta.

1.2 Matrices

A $m \times n$ matrix is an array of numbers with m rows and n columns.

1.2.1 Summation Convention

The expression for the elements of $C = AB$ is

$$C_{ij} = \sum_k A_{ik} B_{kj} \quad (3)$$

and this may be written as

$$C_{ij} = A_{ik} B_{kj} \quad (4)$$

where it is implicitly assumed that there is a summation over the repeated index k . This shorthand is known as the *Einstein summation convention*. In this expression, k is called a *dummy index*, and i and j are called as *free indices*.

There are three basic rules to index notation:

1. In any one term of an expression, indexes may appear only once, twice or not at all.
2. A index that appears only once on one side of an expression must also appear once on the other side. It is called a *free index*.
3. A index that appears twice is summed over. It is called a *dummy index*.

1.2.2 Recall Special Square Matrices

- **Unit matrix.**

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (5)$$

- **Unitary matrix.** U is unitary if $UU^\dagger = U^\dagger U = \mathbb{I}$
- **Symmetric and anti-symmetric matrices.** S is symmetric, if $S^T = S$ or, alternatively, $S_{ij} = S_{ji}$. A is anti-symmetric if $A^T = -A$ or, alternatively, $A_{ij} = -A_{ji}$.
- **Hermitian and anti-Hermitian matrices.** These may be thought of as the complex generalisations of symmetric and anti-symmetric matrices. H is Hermitian if $H^\dagger = H$ or, alternatively, $H_{ij} = H_{ji}^*$. A is anti-Hermitian if $A^\dagger = -A$ or, alternatively, $A_{ij} = -A_{ji}^*$.
- **Orthogonal matrix.** R is orthogonal, if it satisfies

$$R^T R = R R^T = \mathbb{I} \quad \Leftrightarrow \quad R^T = R^{-1} \quad (6)$$

1.2.3 Levi-Civita Symbol (Alternating Tensor or Epsilon Tensor)

The Levi-Civita symbol has three indices, each of which can take value 1,2 or 3 and is defined as

$$\varepsilon_{ijk} = \begin{cases} 1, & (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2) \\ -1, & (i, j, k) = (3, 2, 1), (2, 1, 3), (1, 3, 2) \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

The alternating tensor can be used to write 3-d Euclidean vector (cross) products in index notation:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \quad \Leftrightarrow \quad c_i = \varepsilon_{ijk} a_j b_k \quad (8)$$

A useful identity involving the contraction of two alternating tensors is

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \quad (9)$$

Example. we can use it to prove the vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Proof.

$$\begin{aligned} [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i &= \varepsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k \\ &= \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m \\ &= (a_j c_j) b_i - (a_j b_j) c_i \\ &= (\mathbf{a} \cdot \mathbf{c}) [\mathbf{b}]_i - (\mathbf{a} \cdot \mathbf{b}) [\mathbf{c}]_i \end{aligned} \quad (10)$$

1.2.4 Eigenvalues, Eigenvectors and Diagonalization

An eigenvalue equation takes the form

$$A_{ij} x_j = \lambda x_i \quad (11)$$

where A_{ij} are the components of an $n \times n$ matrix, and x is an eigenvector with corresponding eigenvalue λ .

Form the $n \times n$ matrix M whose n columns are the vectors $\{e^{(1)}, \dots, e^{(n)}\}$. Then M is an orthogonal matrix and

$$M^\dagger A M = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad (12)$$

1.3 Scalars, Vectors and Tensors in 3d Space

- **Scalar** quantities have magnitude and are independent of the any direction.
- **Vector** quantities have magnitude and direction.
- **Rank-two tensor** quantities are an extension of the concept of a vector, and each has two indices.

$$J_i = \sigma_{ij} E_j \quad (13)$$

1.4 Transformations under Rotations

1.4.1 Transformation of Vectors

The two sets of components of x are related by an orthonal matrix L and $\det(L) = 1$

$$x'_i = L_{ij}x_j \quad (14)$$

Set of all such matrices form $SO(3)$ group.

1.4.2 Transformation of Rank-Two Tensors

A rank-two tensor transforms as

$$T'_{ij}(x') = L_{ip}L_{jq}T_{pq}(x) \quad \Leftrightarrow \quad T' = LTL^T \quad (15)$$

For higher rank tensor,

$$T'_{ijk\dots}(x') = L_{ip}L_{jq}L_{kr}\cdots T_{pqr\dots}(x) \quad (16)$$

1.5 Tensor Calculus

1.5.1 The Gradient Operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \quad (17)$$

The definitions of grad, div and curl in Cartesian coordinates may be expressed using index notation:

$$[\nabla\phi]_i = \partial_i\phi \quad (18)$$

$$\nabla \cdot \mathbf{F} = \partial_i F_i \quad (19)$$

$$(\nabla \times \mathbf{F})_i = \varepsilon_{ijk}\partial_j F_k \quad (20)$$

where we have used the convenient shorthand $\partial_i = \frac{\partial}{\partial x_i}$.

2 Green Functions

2.1 Introduction

Green functions are an invaluable tool for the solution of inhomogeneous differential equations. Here we consider the second-order linear *ordinary differential equation* (ODE) with some boundary conditions. \mathcal{L} is a linear second order differential operator, and

$$\mathcal{L}_x[y(x)] = \left[\frac{d}{dx^2} + p(x) \frac{d}{dx} + q(x) \right] y(x) = f(x) \quad (21)$$

The range of the parameter x is $x \in [\alpha, \beta]$ where α might be finite or $-\infty$ and β might be finite or $+\infty$. $f(x)$ is a known function. If $f(x) = 0$, the ordinary is **homogeneous**; while when $f(x) \neq 0$, the equation is **inhomogeneous**.

Suppose that we know $y_1(x), y_2(x)$ are solutions of $\mathcal{L}_x[y(x)] = 0$, and they are linearly independent.

2.2 Variation of Parameters

From the assumptions above, we know that

$$y(x) = ay_1(x) + by_2(x) \quad (22)$$

is a set of $\mathcal{L}_x[y(x)] = 0$ for any constant a and b , and

$$y(x) = ay_1(x) + by_2(x) + y_0(x) \quad (23)$$

is a solution of $\mathcal{L}_x[y(x)] = f(x)$. y_0 is called particular integral, and is any solution of $\mathcal{L}_x[y(x)] = f(x)$.

Imposing the boundary conditions of a particular problem will result in equations for the numbers a and b in the general solution. These equations can be solved for a and b . Two boundary conditions will give two equations for the two unknown constants a and b .

We assume that the particular integral of ode is given by

$$y_0(x) = u(x)y_1(x) + v(x)y_2(x) \quad (24)$$

and the differential

$$y'_0 = u'y_1 + uy'_1 + v'y_2 + vy'_2 \quad (25)$$

$$y''_0 = u''y_1 + 2u'y'_1 + uy''_1 + v''y_2 + 2v'y'_2 + vy''_2 \quad (26)$$

Substituting these expressions into the eqn.(21)

$$\begin{aligned} \mathcal{L}_x[y_0(x)] &= u''y_1 + 2u'y'_1 + uy''_1 + v''y_2 + 2v'y'_2 + vy''_2 \\ &\quad + p(u'y_1 + uy'_1 + v'y_2 + vy'_2) + q(uy_1 + vy_2) \\ &= u(y''_1 + py'_1 + qy_1) + v(y''_2 + py'_2 + qy_2) \\ &\quad + u''y_1 + 2u'y'_1 + v''y_2 + 2v'y'_2 + p(u'y_1 + v'y_2) \\ &= u''y_1 + 2u'y'_1 + v''y_2 + 2v'y'_2 + p(u'y_1 + v'y_2) \end{aligned} \quad (27)$$

Therefore, we will vary these parameters subject to the constraint

$$\boxed{u'y_1 + v'y_2 = 0} \quad (28)$$

and

$$u''y_1 + u'y'_1 + v''y_2 + v'y'_2 = 0 \quad (29)$$

Substituting these expressions into the differential equation (21), after some rearrangement, gives

$$\boxed{u'y'_1 + v'y'_2 = f} \quad (30)$$

So we have

$$\begin{cases} u'y'_1 + v'y'_2 = f \\ u'y_1 + v'y_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = M \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \quad (31)$$

then

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ f \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} y'_2 & -y_2 \\ -y'_1 & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} \quad (32)$$

where $W(x)$ is the *Wronskian*, and

$$W(x) = \det(M) = y_1y'_2 - y_2y'_1 \quad (33)$$

So the solutions are

$$u'(x) = -\frac{y_2(x)f(x)}{W(x)}, \quad v'(x) = \frac{y_1(x)f(x)}{W(x)} \quad (34)$$

2.2.1 Homogeneous Initial Conditions

The boundary conditions $y(\alpha) = y'(\alpha) = 0$ are called *homogeneous initial conditions*. Integrating eqn.(34) gives

$$u(x) = -\int_{\alpha}^x d\tilde{x} \frac{y_2(\tilde{x})f(\tilde{x})}{W(\tilde{x})}, \quad v(x) = \int_{\alpha}^x d\tilde{x} \frac{y_1(\tilde{x})f(\tilde{x})}{W(\tilde{x})} \quad (35)$$

then

$$y_0(x) = u(x)y_1(x) + v(x)y_2(x) = \int_{\alpha}^x d\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) \quad (36)$$

satisfies $y_0(\alpha) = y'_0(\alpha) = 0$. So $y = y_0$ is a solution of the ODE with boundary conditions $y(\alpha) = y'(\alpha) = 0$.

$$\begin{aligned} y(x) &= \int_{\alpha}^x d\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) + \int_x^{\beta} d\tilde{x} \cdot 0 \\ &= \int_{\alpha}^{\beta} G(x, \tilde{x}) f(\tilde{x}) d\tilde{x} \end{aligned} \quad (37)$$

where we have defined the *Green Function*

$$G(x, \tilde{x}) = \begin{cases} 0, & x < \tilde{x} \\ \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})}, & x > \tilde{x} \end{cases} \quad (38)$$

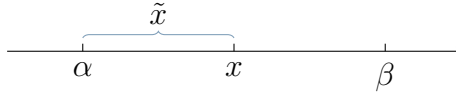


Figure 1: The range of variable x in the problem is $x \in [\alpha, \beta]$.

2.2.2 Inhomogeneous Initial Conditions

Consider more general initial conditions of the form $y(\alpha) = c_1$, $y'(\alpha) = c_2$. Choose a function $g(x)$ s.t. $g(\alpha) = c_1$ and $g'(\alpha) = c_2$. Define

$$Y(x) = y(x) - g(x) \quad (39)$$

which satisfies $Y(\alpha) = Y'(\alpha) = 0$, and $\mathcal{L}_x Y(x) = F(x)$, where

$$F(x) = f(x) - \mathcal{L}_x g(x) = f(x) - g''(x) - p(x)g'(x) - q(x)g(x) \quad (40)$$

Then we can solve for Y as before and that will give us $y(x) = Y(x) + g(x)$.

2.2.3 Homogeneous Two-Point Boundary Conditions

Consider homogeneous two-point boundary conditions $y(\alpha) = y(\beta) = 0$. A solution to eqn.(21) satisfies $y(\alpha) = 0$ is

$$y(x) = \int_{\alpha}^x d\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) + ay_1(x) + by_2(x) \quad (41)$$

We choose $y_1(\alpha) = y_2(\beta) = 0$. Setting $y(\alpha) = 0$ gives

$$y(\alpha) = y_0(\alpha) + ay_1(\alpha) + by_2(\alpha) = by_2(\alpha) = 0 \quad \Rightarrow \quad b = 0 \quad (42)$$

Similarly, setting $y(\beta) = 0$ gives

$$\begin{aligned} y(\beta) &= y_0(\beta) + ay_1(\beta) + by_2(\beta) \\ &= - \int_{\alpha}^{\beta} d\tilde{x} \frac{y_1(\beta)y_2(\tilde{x})}{W(\tilde{x})} f(\tilde{x}) + ay_1(\beta) = 0 \quad \Rightarrow \quad a = \int_{\alpha}^{\beta} d\tilde{x} \frac{y_2(\tilde{x})}{W(\tilde{x})} f(\tilde{x}) \end{aligned} \quad (43)$$

which may be substituted in to the solution to give

$$\begin{aligned} y(x) &= \int_{\alpha}^x d\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) + ay_1(x) \\ &= \int_{\alpha}^x d\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) + \int_{\alpha}^{\beta} d\tilde{x} \frac{y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) \\ &= \int_{\alpha}^x d\tilde{x} \frac{y_1(\tilde{x})y_2(x)}{W(\tilde{x})} f(\tilde{x}) + \int_x^{\beta} d\tilde{x} \frac{y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) \\ &= \int_{\alpha}^{\beta} d\tilde{x} G(x, \tilde{x}) f(\tilde{x}) \end{aligned} \quad (44)$$

where we have defined the *Green Function*

$$G(x, \tilde{x}) = \begin{cases} \frac{y_1(\tilde{x})y_2(x)}{W(\tilde{x})}, & \alpha \leq \tilde{x} < x \\ \frac{y_2(\tilde{x})y_1(x)}{W(\tilde{x})}, & x < \tilde{x} \leq \beta \end{cases} \quad (45)$$

Consider $G(x, \tilde{x})$ as a function of x at a fixed value of $\tilde{x} \in [\alpha, \beta]$, which has several properties

1. When $x \neq \tilde{x}$

$$\mathcal{L}_x[G(x, \tilde{x})] = 0 \quad (46)$$

2. $G(x, \tilde{x})$ is continuous at $x = \tilde{x}$

$$\lim_{\varepsilon \rightarrow 0} [G(x, \tilde{x})]_{x=\tilde{x}-\varepsilon}^{x=\tilde{x}+\varepsilon} = \lim_{\varepsilon \rightarrow 0} \left[\frac{y_1(\tilde{x})y_2(\tilde{x}+\varepsilon)}{W(\tilde{x})} - \frac{y_2(\tilde{x})y_1(\tilde{x}-\varepsilon)}{W(\tilde{x})} \right] = 0 \quad (47)$$

3. $\frac{\partial}{\partial x}G(x, \tilde{x})$ has a unit discontinuity at $x = \tilde{x}$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[\frac{\partial G(x, \tilde{x})}{\partial x} \right]_{x=\tilde{x}-\varepsilon}^{x=\tilde{x}+\varepsilon} &= \lim_{\varepsilon \rightarrow 0} \left[\frac{y_1(\tilde{x})y_2'(\tilde{x}+\varepsilon)}{W(\tilde{x})} - \frac{y_2(\tilde{x})y_1'(\tilde{x}-\varepsilon)}{W(\tilde{x})} \right] \\ &= \frac{W(\tilde{x})}{W(\tilde{x})} = 1 \end{aligned} \quad (48)$$

2.3 Green Function More Generally

Let $G(x, \tilde{x})$ be a function that satisfies

$$\boxed{\mathcal{L}_x[G(x, \tilde{x})] = \delta(x - \tilde{x})} \quad (49)$$

$\delta(x)$ is the *Dirac delta-function* which satisfies

1. $\delta(x) = 0$ when $x \neq 0$

2. $\delta(x) = \delta(-x)$

3. $\int_a^b \delta(x - x_0)f(x)dx = \begin{cases} 0, & x_0 \notin [a, b] \\ f(x_0), & x_0 \in [a, b] \end{cases}$

$G(x, \tilde{x})$ is called a *Green function* for the differential operator \mathcal{L}_x . If $G(x, \tilde{x})$ satisfies eqn.(49), then so does $G(x, \tilde{x}) + Y(x)$, where $\mathcal{L}_x[Y(x)] = 0$.

Now define

$$\boxed{y_0(x) = \int_{\alpha}^{\beta} d\tilde{x} G(x, \tilde{x}) f(\tilde{x})} \quad (50)$$

is a solution of $\mathcal{L}_x[y(x)] = f(x)$. Which can be verified by operating on both sides with \mathcal{L}_x , i.e.

$$\mathcal{L}_x[y_0] = \int_{\alpha}^{\beta} d\tilde{x} \mathcal{L}_x[G(x, \tilde{x})] f(\tilde{x}) = \int_{\alpha}^{\beta} d\tilde{x} \delta(x - \tilde{x}) f(\tilde{x}) = f(x) \quad (51)$$

$f(x)$ is a “linear combination” of delta-function spikes at each $x = \tilde{x}$ with coefficient $f(\tilde{x})$. So y is a continuous linear combination of $G(x, \tilde{x})$ responses

$$y_0(x) = \int_{\alpha}^{\beta} d\tilde{x} G(x, \tilde{x}) f(\tilde{x}) \quad (52)$$

This is called *linear response*.

We can now solve for a and b using the boundary conditions that y satisfies.

2.3.1 Homogeneous Initial Conditions

The boundary conditions are $y(\alpha) = y'(\alpha) = 0$. If $G(\alpha, \tilde{x}) = G'(\alpha, \tilde{x}) = 0$, then

$$y(x) = \int_{\alpha}^{\beta} d\tilde{x} G(x, \tilde{x}) f(\tilde{x}) \quad (53)$$

is the solution of the ode and satisfies the boundary conditions. Now, let's look for the Green function with the right boundary conditions.

1. For $x < \tilde{x}$, $\mathcal{L}_x[G(x, \tilde{x})] = 0$. $G(x, \tilde{x}) = 0$ is a solution of the homogeneous equation that satisfies the boundary conditions that $G(\alpha, \tilde{x}) = G'(\alpha, \tilde{x}) = 0$. So for $x < \tilde{x}$

$$G(x, \tilde{x}) = 0 \quad (54)$$

2. For $x \geq \tilde{x}$, $\mathcal{L}_x[G(x, \tilde{x})] = 0$. $G(x, \tilde{x})$ equals some linear combination of $y_1(x)$ and $y_2(x)$

$$G(x, \tilde{x}) = A(\tilde{x})y_1(x) + B(\tilde{x})y_2(x) \quad (55)$$

We can find A and B by using the properties of G :

- (i) G is continuous at $x = \tilde{x}$

$$A(\tilde{x})y_1(\tilde{x}) + B(\tilde{x})y_2(\tilde{x}) = 0 \quad (56)$$

- (ii) G' has a unit discontinuity at $x = \tilde{x}$

$$A(\tilde{x})y_1'(\tilde{x}) + B(\tilde{x})y_2'(\tilde{x}) = 0 \quad (57)$$

The solution is

$$A(\tilde{x}) = -\frac{y_2(\tilde{x})}{W(\tilde{x})}, \quad B(\tilde{x}) = \frac{y_1(\tilde{x})}{W(\tilde{x})} \quad (58)$$

where W is the Wronskian of y_1 and y_2 .

So we have

$$G(x, \tilde{x}) = \begin{cases} 0, & x < \tilde{x} \\ \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})}, & x > \tilde{x} \end{cases} \quad (59)$$

which agrees with that calculated before.

2.3.2 Homogeneous Two-Point Boundary Conditions

The boundary conditions are $y(\alpha) = y(\beta) = 0$. The Green Function should satisfies

$$G(\alpha, \tilde{x}) = G(\beta, \tilde{x}) = 0 \quad (60)$$

We assume y_1 and y_2 are linear independent solutions of homogeneous equation, and we choose $y_1(\alpha) = y_2(\beta) = 0$.

$$G(x, \tilde{x}) = \begin{cases} A(\tilde{x})y_1(x) + B(\tilde{x})y_2(x), & \alpha < x < \tilde{x} \\ C(\tilde{x})y_1(x) + D(\tilde{x})y_2(x), & \tilde{x} < x < \beta \end{cases} \quad (61)$$

1. Boundary conditions: $G(\alpha, \tilde{x}) = G(\beta, \tilde{x}) = 0$

$$A(\tilde{x})y_1(\alpha) + B(\tilde{x})y_2(\alpha) = B(\tilde{x})y_2(\alpha) = 0 \Rightarrow B(\tilde{x}) = 0 \quad (62)$$

$$C(\tilde{x})y_1(\beta) + D(\tilde{x})y_2(\beta) = C(\tilde{x})y_1(\beta) = 0 \Rightarrow C(\tilde{x}) = 0 \quad (63)$$

so we have

$$G(x, \tilde{x}) = \begin{cases} A(\tilde{x})y_1(x), & \alpha < x < \tilde{x} \\ D(\tilde{x})y_2(x), & \tilde{x} < x < \beta \end{cases} \quad (64)$$

2. Continuity of G and unit discontinuity of G' at $x = \tilde{x}$

$$A(\tilde{x})y_1(\tilde{x}) - D(\tilde{x})y_2(\tilde{x}) = 0 \quad (65)$$

$$A(\tilde{x})y_1'(\tilde{x}) - D(\tilde{x})y_2'(\tilde{x}) = 0 \quad (66)$$

so we have

$$A(\tilde{x}) = \frac{y_2(\tilde{x})}{W(\tilde{x})}, \quad D(\tilde{x}) = \frac{y_1(\tilde{x})}{W(\tilde{x})} \quad (67)$$

The final result

$$G(x, \tilde{x}) = \begin{cases} \frac{y_2(\tilde{x})y_1(x)}{W(\tilde{x})}, & \alpha < x < \tilde{x} \\ \frac{y_1(\tilde{x})y_2(x)}{W(\tilde{x})}, & \tilde{x} < x < \beta \end{cases} \quad (68)$$

which agrees with that calculated before.

2.3.3 Higher Dimensions, More Variables

Consider a second order linear differential operator \mathcal{L} on function $y(x_1, x_2, x_3)$, then

$$\mathcal{L}y = f(x_1, x_2, x_3) \quad (69)$$

and

$$\mathcal{L}G(\underline{x}, \underline{\tilde{x}}) = \delta^{(3)}(\underline{x} - \underline{\tilde{x}}) = \delta(x_1 - \tilde{x}_1)\delta(x_2 - \tilde{x}_2)\delta(x_3 - \tilde{x}_3) \quad (70)$$

Let R be a 3-d region in 3-d Euclidean space

$$\int_R d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3 \delta^{(2)}(\underline{x} - \underline{\tilde{x}}) f(\underline{\tilde{x}}) = \begin{cases} f(\underline{x}), & \underline{x} \in R \\ 0 & \underline{x} \notin R \end{cases} \quad (71)$$

Example. The most famous example is

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 \quad (72)$$

and the Green function satisfies

$$\nabla^2 G(\underline{x}, \underline{\tilde{x}}) = \delta(\underline{x} - \underline{\tilde{x}}) \quad (73)$$

Consider the Poisson equation for the scalar electric potential $\phi(\underline{x})$ in terms of the scalar charge density $\rho(\underline{x})$:

$$\nabla^2 \phi(\underline{x}) = -\frac{\rho(\underline{x})}{\varepsilon} \quad (74)$$

and

$$\phi(x) = \int d\tilde{x} G(\underline{x}, \underline{\tilde{x}}) \left[-\frac{\rho(\underline{\tilde{x}})}{\varepsilon} \right] \quad (75)$$

is a solution of Poisson's equation. The Green function for the Poisson equation that satisfying the boundary condition $G(\underline{x}, \underline{\tilde{x}}) \rightarrow 0$ as $|\underline{x}| \rightarrow \infty$ is

$$G(\underline{x}, \underline{\tilde{x}}) = \frac{1}{4\pi|\underline{x} - \underline{\tilde{x}}|} \quad (76)$$

where $|\underline{x} - \underline{\tilde{x}}| = \sqrt{(x_1 - \tilde{x}_1)^2 + (x_2 - \tilde{x}_2)^2 + (x_3 - \tilde{x}_3)^2}$.

3 Hilbert Spaces

Definition. A Hilbert space is an infinite dimensional complex vector space with inner product $\langle \cdot, \cdot \rangle$ and a infinite countable orthonormal basis $\{u_1, u_2, u_3, \dots\}$. The Hilbert space we will look at in this chapter will be a vector space of complex function of a real variable $x \in [a, b]$ with

1. an inner product

$$\langle f, g \rangle = \int_a^b f^*(x)g(x)dx \quad (77)$$

Functions $f(x)$ and $g(x)$ are orthogonal if $\langle f, g \rangle = 0$. The *norm* of f is given by $\|f\| = \sqrt{\langle f, f \rangle}$, and $f(x)$ may be normalised in $\hat{f} = f/\|f\|$. If $\langle y_i, y_j \rangle = \delta_{ij}$, then the set of $\{y_1, y_2, y_3, \dots\}$ is orthogonal.

2. Let $\{y_1, y_2, y_3, \dots\}$ be an orthogonal basis, then any function $f(x) \in \mathcal{H}$ can be expanded

$$f(x) = \sum_{i=1}^{\infty} f_i y_i(x), \quad f_i \in \mathbb{C} \quad (78)$$

Then we have

$$\langle y_k, f \rangle = \langle y_k, \sum_{i=1}^{\infty} f_i y_i \rangle = \sum_{i=1}^{\infty} f_i \langle y_k, y_i \rangle = \sum_{i=1}^{\infty} f_i \delta_{ik} = f_k \quad (79)$$

3.1 Sturm-Liouville Theory

The theory of inhomogeneous differential equations of form $\mathcal{L}y(x) = f(x)$ on $x \in [a, b]$ where \mathcal{L} is second order, linear and **self-adjoint**.

3.1.1 Self-Adjoint Differential Operators

Consider

$$\mathcal{L} = -\frac{d}{dx} \left[\rho(x) \frac{d}{dx} \right] + \sigma(x) \quad (80)$$

and

$$\mathcal{L}y = -\frac{d}{dx} \left(\rho \frac{dy}{dx} \right) + \sigma y = -(\rho y')' + \sigma y \quad (81)$$

where $\rho(x)$ and $\sigma(x)$ are real valued and defined on $x \in [a, b]$ and $\rho(x) > 0$ on $x \in (a, b)$. Such an operator is said to be in *self-adjoint form*¹.

Definition. A second order linear differential operator \mathcal{D} is self-adjoint on Hilbert space \mathcal{H} if

$$\langle u, \mathcal{D}v \rangle = \langle v, \mathcal{D}u \rangle^*, \quad \forall u, v \in \mathcal{H} \quad (82)$$

¹being in self-adjoint form does not mean that a differential operator is self-adjoint, that depend on the operator and the specific Hilbert space.

c.f. the definition of a Hermitian matrix $M : M_{ij} = M_{ji}^*$.

Consider \mathcal{L} as in eqn.(80),

$$\begin{aligned}
 \langle u, \mathcal{L}v \rangle &= \int_a^b u^* [-(\rho v')' + \sigma v] dx \\
 &= -u^* \rho v' \Big|_a^b + \int_a^b (u^{*'} \rho v' + u^* \sigma v) dx \\
 &= -u^* \rho v' \Big|_a^b + u^{*'} \rho v \Big|_a^b + \int_a^b (-(u^{*'} \rho)' v + u^* \sigma v) dx \\
 &= (-u^* \rho v' + u^{*'} \rho v) \Big|_a^b + \int_a^b (-(u^{*'} \rho)' + u^* \sigma) v dx \\
 &= (-u^* \rho v' + u^{*'} \rho v) \Big|_a^b + \left[\int_a^b (-(u' \rho)' + u \sigma) v^* dx \right]^* \\
 &= (-u^* \rho v' + u^{*'} \rho v) \Big|_a^b + \langle v, \mathcal{L}u \rangle^*
 \end{aligned} \tag{83}$$

So \mathcal{L} is self-adjoint on \mathcal{H} if

$$\rho(u^{*'} v - u^* v') \Big|_a^b = 0 \tag{84}$$

3.1.2 Boundary Conditions

1. if $\rho(a) = \rho(b) = 0$ and $u(a)u(b)$ is finite for all $u \in \mathcal{H}$, then \mathcal{L} is self-adjoint.
2. if $u(a) = u(b)$ and $u'(a) = u'(b)$ for all $u \in \mathcal{H}$, and $\rho(a) = \rho(b)$, then \mathcal{L} is self-adjoint. \mathcal{H} is set of functions of periodic boundary conditions.
3. If $u(a) = u(b) = 0$ for all $u \in \mathcal{H}$, then \mathcal{L} is self-adjoint. This is a special case of

$$\begin{cases} C_1 u(a) + C_2 u'(a) = 0 \\ D_1 u(b) + D_2 u'(b) = 0 \end{cases} \tag{85}$$

Note that these examples of boundary conditions that work are preserved under taking linear combinations

3.1.3 Weight Functions

Any second order linear differential operator can be put into self-adjoint form. Consider the most general operator

$$\tilde{\mathcal{L}} = -\frac{d}{dx} \left(A(x) \frac{d}{dx} \right) - B(x) \frac{d}{dx} + C(x) \tag{86}$$

where A, B, C are real and $A(x) > 0$ for $x \in [a, b]$.

Claim that there exists a function $w(x) > 0$ such that $w\tilde{\mathcal{L}}$ can be written in self-adjoint form i.e.

$$w(x) [-(Ay')' - By' + Cy] = -(\rho y')' + \sigma y \tag{87}$$

rearranging this

$$-w(Ay')' - Bwy' + Cwy = -(\rho y')' + \sigma y \quad (88)$$

so we have

$$\begin{cases} -(Aw y')' + w' Ay' - Bwy' = -(\rho y')' \\ Cwy = \sigma y \end{cases} \quad (89)$$

then

$$\frac{w'}{w} = \frac{B}{A}, \quad Aw = \rho, \quad Cw = \sigma \quad (90)$$

We choose $w(x)$ such that

$$w(x) = \exp \left[\int_a^x \frac{B(\tilde{x})}{A(\tilde{x})} d\tilde{x} \right] \quad (91)$$

where $w(a) = 1$.

Definition. The inner product with weight w

$$\langle f, g \rangle_w = \langle f, wg \rangle = \int_a^b f^*(x)w(x)g(x)dx = \langle wf, g \rangle \quad (92)$$

w is real.

3.1.4 Eigenfunctions and Eigenvalues

Consider the inhomogeneous eigenfunction equation

$$\tilde{\mathcal{L}}y = \lambda y \quad (93)$$

we may define an operator in self-adjoint form $\mathcal{L} = w\tilde{\mathcal{L}}$ and eigenfunction equation becomes

$$\boxed{\mathcal{L}y = \lambda wy} \quad (94)$$

A solution is called an eigenfunction of \mathcal{L} with eigenvalue λ and weight $w(x)$. We claim that

1. The eigenvalues of eqn.(94) are real.
2. The eigenfunctions of eqn.(94) with distinct eigenvalues are orthogonal.

Consider two eigenfunctions, y_i and y_j of $\tilde{\mathcal{L}}$ with eigenvalues λ_i and λ_j respectively. They are also eigenfunctions of \mathcal{L} with eigenvalues λ_i and λ_j and weight w . Then we have

$$\mathcal{L}y_i = \lambda_i wy_i \quad (95)$$

$$\langle y_j, \mathcal{L}y_i \rangle = \lambda_i \langle y_j, wy_i \rangle \quad (96)$$

$$\langle y_j, \mathcal{L}y_i \rangle^* = \lambda_i^* \langle y_j, wy_i \rangle^* \quad (\text{take complex conjugate}) \quad (97)$$

$$\langle y_i, \mathcal{L}y_j \rangle = \lambda_j^* \langle y_i, wy_j \rangle = \lambda_j^* \langle y_i, y_j \rangle_w \quad (\text{use self-adjointness}) \quad (98)$$

$$\mathcal{L}y_j = \lambda_j w y_j \quad (99)$$

$$\langle y_i, \mathcal{L}y_j \rangle = \lambda_j \langle y_i, w y_j \rangle = \lambda_j \langle y_i, y_j \rangle_w \quad (100)$$

Compare eqn.(98) and eqn.(100), we find

$$(\lambda_i^* - \lambda_j) \langle y_i, y_j \rangle_w = 0 \quad (101)$$

- For $i = j$ we have

$$(\lambda_i^* - \lambda_i) \|y_i\|_w^2 = 0 \quad (102)$$

so, if we have non-zero eigenfunctions, then $\lambda_i^* = \lambda_i$, i.e., the eigenvalues are real.

- For $i \neq j$ we have

$$(\lambda_i - \lambda_j) \langle y_i, y_j \rangle_w = 0 \quad (103)$$

so, if we are considering distinct eigenvalues, then $\langle y_i, y_j \rangle_w = 0$, i.e., the eigenfunctions are orthogonal with weight $w(x)$.

3.1.5 Eigenfunction Expansions

Theorem. The eigenvalues of a self-adjoint operator with w form a discrete, infinite sequence $\lambda_1, \lambda_2, \lambda_3, \dots$ such that $|\lambda_n| \rightarrow \infty$ as $n \rightarrow \infty$, and that the corresponding eigenfunctions with weight w , f_1, f_2, f_3, \dots form a *complete orthonormal basis* for functions on $[a, b]$ in the Hilbert space. So any function $g \in \mathcal{H}$ can be expanded as

$$g(x) = \sum_n g_n f_n(x), \quad g_n \in \mathbb{C} \quad (104)$$

where

$$g_n = \langle f_n, g \rangle_w = \int_a^b f_n^*(x) w(x) g(x) dx \quad (105)$$

Substituting into the expansion we find

$$\begin{aligned} g(x) &= \sum_n \int_a^b d\tilde{x} [f_n^*(\tilde{x}) w(\tilde{x}) g(\tilde{x})] f_n(x) \\ &= \int_a^b d\tilde{x} g(\tilde{x}) \left[w(\tilde{x}) \sum_n f_n(x) f_n^*(\tilde{x}) \right] \\ &= \int_a^b d\tilde{x} \delta(x - \tilde{x}) g(\tilde{x}) \end{aligned} \quad (106)$$

where

$$\boxed{\delta(x - \tilde{x}) = w(\tilde{x}) \sum_n f_n(\tilde{x}) f_n^*(\tilde{x})} \quad (107)$$

Let $u \in \mathcal{H}$, consider the expression

$$\begin{aligned} \int_a^b |u|^2 \omega dx &= \langle u, u \rangle_w = \left\langle \sum_n u_n f_n(x), \sum_m u_m f_m(x) \right\rangle_w \\ &= \sum_{n,m} u_n^* u_m \langle f_n, f_m \rangle_w = \sum_{n,m} u_n^* u_m \delta_{nm} = \sum_n |u_n|^2 \end{aligned} \quad (108)$$

which is *Parseval's identity* in the case with a weight function $w(x)$

$$\langle u, u \rangle_w = \sum_n |u_n|^2 \quad (109)$$

3.1.6 Green Functions Revisited

If $\{y_n\}$ are a set of orthonormal eigenfunctions of self-adjoint operator \mathcal{L} with weight w with corresponding eigenvalues $\{\lambda_n\}$, then the Green function for \mathcal{L} is given by

$$G(x, \tilde{x}) = \sum_n \frac{y_n(x) y_n^*(\tilde{x})}{\lambda_n} \quad (110)$$

To prove this, we apply \mathcal{L} to $G(x, \tilde{x})$

$$\begin{aligned} \mathcal{L}_x[G(x, \tilde{x})] &= \sum_n \frac{\mathcal{L}_x[y_n(x)] y_n^*(\tilde{x})}{\lambda_n} \\ &= \sum_n w(x) y_n(x) y_n^*(\tilde{x}) \\ &= \frac{\omega(x)}{\omega(\tilde{x})} \left[\omega(\tilde{x}) \sum_n y_n(x) y_n^*(\tilde{x}) \right] \\ &= \delta(x - \tilde{x}) \quad \square \end{aligned} \quad (111)$$

3.1.7 Eigenfunction Expansions for Solving ODEs

As an example, consider the differential equation

$$\mathcal{L}y - \nu y = f \quad (112)$$

with some boundary conditions. \mathcal{L} is a self-adjoint operator with weight function $w = 1$ and $\{y_n\}$ are eigenfunctions. Suppose \mathcal{L} has eigenvalues λ_n , and corresponding eigenfunctions $\{y_n\}$, satisfying the same boundary conditions. Let

$$y(x) = \sum_n a_n y_n(x), \quad f(x) = \sum_n f_n y_n(x) \quad (113)$$

Substituting into the original equation, we find

$$\begin{aligned} \mathcal{L} \sum_n a_n y_n - \nu \sum_n a_n y_n &= \sum_n f_n y_n \\ \Rightarrow \sum_n (a_n \lambda_n - \nu a_n) y_n &= \sum_n f_n y_n \\ \Rightarrow (a_n \lambda_n - \nu a_n) &= f_n \end{aligned} \quad (114)$$

So that

$$a_n = \frac{f_n}{\lambda_n - \nu}, \quad (\lambda_n \neq \nu) \quad (115)$$

so that the solution is given by

$$y(x) = \sum_n \frac{f_n}{\lambda_n - \nu} y_n(x) \quad (116)$$

3.2 Legendre Polynomials

3.2.1 Two Examples

Example. Let

$$\mathcal{L} = -\frac{d^2}{dx^2}, \quad x \in [0, 2\pi R] \quad (117)$$

with boundary conditions $y(0) = y(2\pi R) = 0$. Then the eigenfunction equation becomes

$$-y_n'' = \lambda_n y_n \quad (118)$$

and the eigenfunctions and the corresponding eigenvalues are

$$y_n = \sin\left(\frac{n}{2R}x\right), \quad \lambda_n = \left(\frac{n}{2R}\right)^2, \quad n = 1, 2, 3, \dots \quad (119)$$

Example. Let

$$\mathcal{L} = -\frac{d^2}{dx^2}, \quad x \in [0, 2\pi R] \quad (120)$$

with boundary conditions $y(0) = y(2\pi R)$ and $y'(0) = y'(2\pi R)$.

$$-y_m'' = \lambda_m y_m \quad (121)$$

and the eigenfunctions and the corresponding eigenvalues are

$$y_m = \exp\left(i\frac{m}{R}x\right), \quad \lambda_m = \left(\frac{m}{R}\right)^2, \quad m \in \mathbb{Z} \quad (122)$$

When $m = 0$, there's the extra 'zero mode' of y_0 is a constant with eigenvalue 0.

$$\boxed{-\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} y \right] = \lambda y} \quad (123)$$

Assume that eigenfunctions of eqn.(123) are polynomials

$$y_n(x) = x^{m_n} + a_{m_n-1}x^{m_n-1} + \cdots + a_1x + a_0 \quad (124)$$

substituting this to the eigenfunction equation, we have

$$m_n(m_n + 1) = \lambda \quad (125)$$

So eigenvalues take form

$$\lambda = l(l+1), \quad l \in \mathbb{N} \quad (126)$$

We can label the eigenfunctions by l

- $l = 0, y_0(x) = 1$
- $l = 1, y_1(x) = x + a_0$
- $l = 2, y_2(x) = x^2 + a_1x + a_0$

$$\int_{-1}^1 y_l^*(x) y_{l'}(x) dx = \delta_{ll'} \quad (127)$$

3.2.2 Legendre's Equation

Legendre's equation

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \quad \text{with } x \in [-1, 1] \quad (128)$$

arises is a number of contexts in science, for example in the solution of Laplace's equation in spherical coordinates. This equation can be put into the form of a self-adjoint eigenvalue problem with $\rho = 1 - x^2$, $\sigma = 0$, $w = 1$ and $\lambda = l(l+1)$.

$$\boxed{-\frac{d}{dx} [(1-x^2)y'] = l(l+1)y} \quad (129)$$

or

$$\mathcal{L}y = l(l+1)y \quad (130)$$

where

$$\mathcal{L} = -\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} \right] \quad (131)$$

is self-adjoint on a Hilbert space of functions that are finite at ± 1 . Assume that eigenfunctions of eqn.(129) are polynomials

$$y_n(x) = x^{m_n} + a_{m_n-1}x^{m_n-1} + \cdots + a_1x + a_0 \quad (132)$$

Substituting the polynomial solution y_n into eqn.(129), then thinking about equation coefficients of partial of x . The highest power m_n satisfies the relation

$$m_n(m_n + 1) = \lambda \quad (133)$$

So eigenvalues take form

$$\lambda = l(l + 1), \quad l \in \mathbb{N} \quad (134)$$

and can label eigenfunctions by l

- $l = 0, y_0(x) = 1$
- $l = 1, y_1(x) = x + a_0$
- $l = 2, y_2(x) = x^2 + a_1x + a_0$

3.3 Spherical Harmonics

Laplace's equation in spherical coordinates is given by

$$\nabla^2 f(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 f}{\partial \phi^2} \quad (135)$$

If we take

$$f(r, \theta, \phi) = r^l e^{im\phi} \Theta(\theta) \quad (136)$$

as an *ansatz*, where $l \in \mathbb{N}$ and $m \in \mathbb{Z}$, then Laplace's equation becomes

$$l(l + 1)e^{im\phi} \Theta(\theta) + \frac{e^{im\phi}}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{\Theta}{\sin \theta} m^2 e^{im\phi} = 0 \quad (137)$$

Rearrange this, we have

$$\sin^2 \theta l(l + 1) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = m^2 \quad (138)$$

Let $u = \cos \theta$ and $\Theta(\theta) = P(u)$, where $u \in [-1, 1]$, we have

$$\frac{d}{d\theta} = \frac{d}{du} \frac{du}{d\theta} = -\sin \theta \frac{d}{du} \quad (139)$$

Then the equation becomes

$$-\left[(1 - u^2) P' \right]_{\text{self-adjoint form}} + \frac{m^2}{1 - u^2} P = l(l + 1) P \quad (140)$$

with $\rho = 1 - u^2$, $\sigma = \frac{m^2}{1 - u^2}$, $w = 1$ and $\lambda = l(l + 1)$. Now the differential operators depend on m , and there will be a different set of indefinite solutions for each m . This can show that we get non-singular solutions if $l \in \mathbb{N}$ and $m \in [-l, l]$. The solutions are called *associated Legendre polynomials* $P_l^m(u)$, which is a basis set for functions of u on $[-1, 1]$.

The orthogonality

$$\int_{-1}^1 P_l^m(u) P_{l'}^m(u) du = \frac{2(l+m)!}{(2l+1)(l-m)} \delta_{ll'} \quad (141)$$

Similarly, the equation can be expressed as

$$\underset{\text{self-adjoint form}}{-[(1-u^2)P']' - l(l+1)P} = -\frac{m^2}{1-u^2}P \quad (142)$$

with $\rho = 1 - u^2$, $\sigma = -l(l+1)$ and $w = \frac{1}{1-u^2}$. This show that

$$\int_{-1}^1 \frac{P_l^m(u) P_{l'}^{m'}(u)}{1-u^2} du = \frac{(l+m)!}{m(l-m)} \delta_{mm'} \quad (143)$$

Finally we get

$$Y_l^m = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}, \quad l \in \mathbb{N}, -l \leq m \leq l \quad (144)$$

they are solutions of $\nabla^2 Y_l^m = 0$, and form an orthogonal basis of function on \mathbb{S}^2

$$\delta_{ll'} \delta_{mm'} = \int_0^{2\pi} \int_0^\pi Y_l^m(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) \sin \theta d\theta d\phi \quad (145)$$

So any function f can be expressed as

$$f(\theta, \phi) = \sum_l \sum_{m=-l}^l f_{lm} Y_l^m(\theta, \phi) \quad (146)$$

where

$$f_{lm} = \int_{\mathbb{S}^2} Y_l^{m*} f d\Omega \quad (147)$$

4 Integral Transforms

4.1 Fourier Series

Consider $f(x)$ has a period of $2\pi R$, we can express $f(x)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} f_n y_n(x) \quad (148)$$

where

$$y_n(x) = \frac{1}{\sqrt{2\pi R}} e^{inx/R} \quad (149)$$

and we have

$$\langle y_n, y_m \rangle = \int_0^{2\pi R} y_n^* y_m dx = \delta_{nm} \quad (150)$$

We choose $x \in [-\pi R, \pi R]$, then

$$\begin{aligned} f_n &= \int_{-\pi R}^{\pi R} y_n^*(x) f(x) dx \\ &= \frac{1}{\sqrt{2\pi R}} \int_{-\pi R}^{\pi R} e^{-inx/R} f(x) dx \\ &= \frac{1}{\sqrt{2\pi R}} \int_{-\pi R}^{\pi R} e^{-ik_n x} f(x) dx \end{aligned} \quad (151)$$

where $k_n = n/R$, $x \in (-\infty, \infty)$. Let $R \rightarrow \infty$ and k_n take the real continuous values from $-\infty$ to ∞ , we define that

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \quad (152)$$

for f satisfying $\int_{-\infty}^{\infty} |f| dx$ is finite. $\tilde{f}(k)$ is the *Fourier transform* of $f(x)$.

4.2 Fourier Transforms

4.2.1 Definition and Notation

Definition. Fourier transform

$$\boxed{\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx} \quad (153)$$

The *inverse Fourier transform* is defined as

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk} \quad (154)$$

In other words, this operation on $\tilde{f}(k)$ is the inverse Fourier transform and we can define

$$\text{FT}^{-1}[\text{FT}(f)] = f \quad \Rightarrow \quad \text{FT}^{-1}\text{FT} = \mathbb{1} \quad (155)$$

4.2.2 Dirac Delta-Function

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-ikx'} dx' \right] e^{ikx} dk \\
&= \int_{-\infty}^{\infty} f(x') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk \right] dx' \\
&= \int_{-\infty}^{\infty} f(x') \delta(x - x') dx'
\end{aligned} \tag{156}$$

where we have defined the *Dirac delta-function*

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk \tag{157}$$

4.2.3 Properties of the Fourier Transform

1. If $f(x)$ is a real function $[f(x)]^* = f(x)$

$$\tilde{f}^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-k)x} f^*(x) dx = \tilde{f}(-k) \tag{158}$$

- If $f(x)$ is an even function $f(-x) = f(x)$, then $\tilde{f}(x)$ is a pure real function.

Proof. Define $y = -x$, then

$$\tilde{f}^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iky} f(-y) d(-y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iky} f(y) dy = \tilde{f}(k) \tag{159}$$

- If $f(x)$ is an odd function $f(-x) = -f(x)$, then $\tilde{f}(x)$ is a pure imaginary function.

Proof. Define $y = -x$, then

$$\tilde{f}^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iky} f(-y) d(-y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iky} f(y) dy = -\tilde{f}(k) \tag{160}$$

2. Differentiation

$$\text{TF}[f^{(n)}(x)] = (ik)^n \tilde{f}(k) \tag{161}$$

Proof. Consider the first order derivative

$$\begin{aligned}
\text{TF}[f'(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} f'(x) \\
&= \frac{1}{\sqrt{2\pi}} [f(x) e^{-ikx}]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) (-ik) e^{-ikx} \\
&= ik \tilde{f}(k)
\end{aligned} \tag{162}$$

3. Multiplication by x

$$\text{FT}[xf(x)] = i \frac{d}{dk} \tilde{f}(k) \quad (163)$$

4. Rigid shift of coordinate

$$\text{FT}[f(x - a)] = e^{-ika} \tilde{f}(k) \quad (164)$$

Proof. Define $y = x - a$, then

$$\begin{aligned} \text{FT}[f(x - a)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x - a) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ika} e^{-ik(x-a)} f(x - a) d(x - a) \\ &= \frac{1}{\sqrt{2\pi}} e^{-ika} \int_{-\infty}^{\infty} e^{-iky} f(y) dy = e^{-ika} \tilde{f}(k) \end{aligned} \quad (165)$$

4.2.4 Parseval's Theorem

Parseval's theorem for Fourier transforms states that

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk} \quad (166)$$

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|^2 dx &= \int_{-\infty}^{\infty} f(x) f^*(x) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \tilde{f}(k) \tilde{f}^*(k') e^{i(k-k')x} \right] dx \\ &= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \tilde{f}(k) \tilde{f}^*(k') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} dx \right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(k') \delta(k - k') dk dk' \\ &= \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(k) dk = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk \end{aligned} \quad (167)$$

4.2.5 Convolution Theorem

The convolution of f and g is defined as

$$\boxed{f * g = \int_{-\infty}^{\infty} f(y) g(x - y) dy} \quad (168)$$

with claims

1. $f * g = g * f$

2. $f * \delta = f$

The convolution theorem can be stated in two, equivalent forms.

1.

$$\begin{aligned} \text{FT}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} \int_{-\infty}^{\infty} dy f(y) g(x-y) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-iky} f(y) \int_{-\infty}^{\infty} dx e^{-ik(x-y)} g(x-y) \\ &= \sqrt{2\pi} \tilde{f}(k) \tilde{g}(k) = \sqrt{2\pi} \text{FT}[f] \text{FT}[g] \end{aligned} \quad (169)$$

2.

$$\text{FT}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}} \tilde{f}(k) * \tilde{g}(k) \quad (170)$$

4.2.6 Examples of Fourier Transform

1. Constant function $f(x) = 1$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} dx = \sqrt{2\pi} \delta(k) \quad (171)$$

2. Single frequency/wavenumber mode $f(x) = e^{ik_0 x}$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_0 x} e^{-ikx} dx = \sqrt{2\pi} \delta(k - k_0) \quad (172)$$

3. Dirac delta-function $f(x) = \delta(x - x_0)$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - x_0) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} e^{-ikx_0} \quad (173)$$

4. Gaussian function $f(x) = \frac{1}{\sigma(2\pi)^{1/4}} e^{-x^2/4\sigma^2}$

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma(2\pi)^{1/4}} \int_{-\infty}^{\infty} e^{-x^2/4\sigma^2} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma(2\pi)^{1/4}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{4\sigma^2} - ikx\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma(2\pi)^{1/4}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{4\sigma^2} (x + ik2\sigma)^2 - k^2\sigma^2\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma(2\pi)^{1/4}} e^{-k^2\sigma^2} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{4\sigma^2} (x + ik2\sigma)^2\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma(2\pi)^{1/4}} e^{-k^2\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{4\sigma^2} x'^2\right) dx' \\ &= \frac{\sqrt{2\sigma}}{(2\pi)^{1/4}} e^{-k^2\sigma^2} \end{aligned} \quad (174)$$

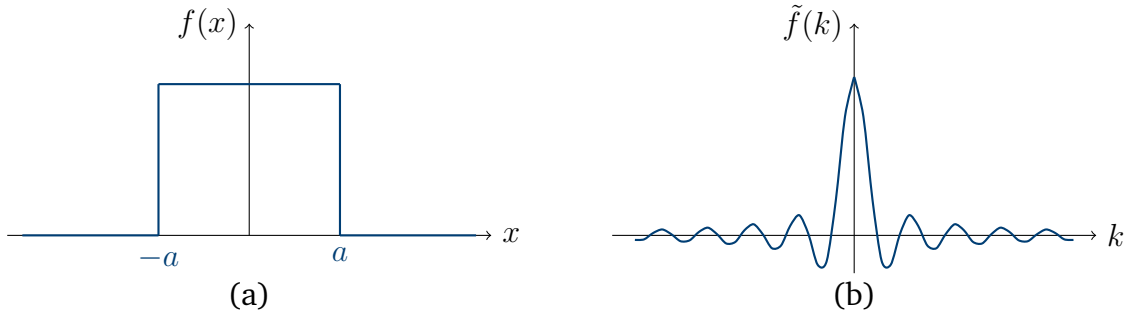


Figure 2: Top-hat function.

5. Top-hat function $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}$

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{ik} e^{-ikx} \right]_{-a}^a \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin(ka)}{k} = a \sqrt{\frac{2}{\pi}} \text{sinc}(ak) \end{aligned} \quad (175)$$

4.3 The Applications of Fourier Transforms in Physics

4.3.1 Diffraction Through an Aperture

The geometry for Fraunhofer diffraction see Fig.3(a). For small values of θ we have $\theta \approx \sin \theta \approx \tan \theta = \frac{X}{D}$. The aperture function is given by a top-hat

$$h(x) = \begin{cases} 1 & |x| < \frac{a}{2} \\ 0 & |x| \geq \frac{a}{2} \end{cases} \quad (176)$$

so we have

$$\tilde{h}(k) = \frac{a}{\sqrt{2\pi}} \text{sinc}\left(\frac{ak}{2}\right) \quad (177)$$

The intensity $I(k)$ of light observed in the diffraction pattern is the square of the Fourier transform of the aperture function $f(x)$

$$I(x = X) = I\left(k_x = \frac{2\pi X}{\lambda D}\right) = |\tilde{h}(k_x)|^2 = \frac{a^2}{2\pi} \text{sinc}^2\left(\frac{a\pi X}{2\lambda D}\right) \quad (178)$$

4.3.2 Double Slit Diffraction

The aperture function is given by

$$h(x) = f(x)g(x) \quad (179)$$

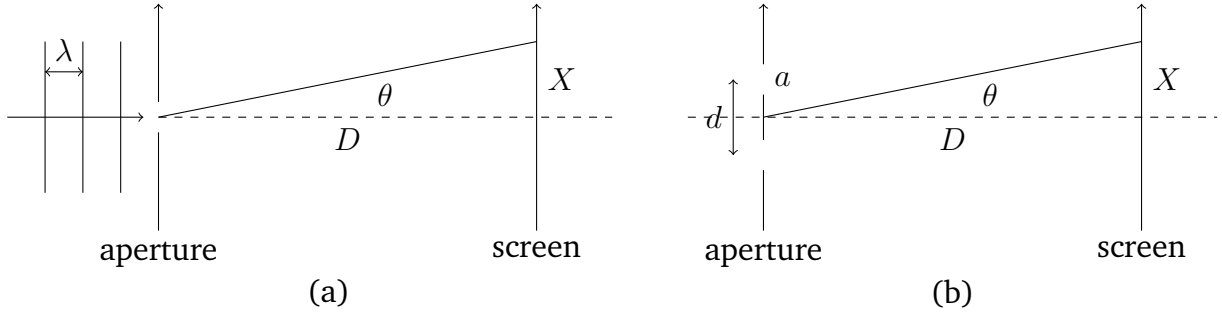


Figure 3: Geometry for Fraunhofer diffraction. (a) Diffraction through an aperture. (b) Double slit diffraction.

where

$$f(x) = \delta\left(x - \frac{d}{2}\right) + \delta\left(x + \frac{d}{2}\right) \quad (180)$$

and $g(x)$ is single aperture function. And

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} \left[\delta\left(x - \frac{d}{2}\right) + \delta\left(x + \frac{d}{2}\right) \right] \\ &= \frac{1}{\sqrt{2\pi}} (e^{-ikd/2} + e^{ikd/2}) = \sqrt{\frac{2}{\pi}} \cos\left(\frac{kd}{2}\right) \end{aligned} \quad (181)$$

so we have

$$\begin{aligned} \text{TF}(f * g) &= \sqrt{2\pi} \tilde{f}(k) \tilde{g}(k) \\ &= \sqrt{2\pi} \sqrt{\frac{2}{\pi}} \cos\left(\frac{kd}{2}\right) \frac{a}{\sqrt{2\pi}} \text{sinc}\left(\frac{ak}{2}\right) \\ &= \sqrt{\frac{2}{\pi}} a \text{sinc}\left(\frac{ak}{2}\right) \cos\left(\frac{kd}{2}\right) \end{aligned} \quad (182)$$

and the intensity on the screen is given by

$$I(k) = \frac{2a^2}{\pi} \text{sinc}^2\left(\frac{ak}{2}\right) \cos^2\left(\frac{kd}{2}\right) \quad (183)$$

4.3.3 Diffusion Equation

Consider an infinite, one-dimensional conducting bar. The flow of heat is determined by the diffusion equation

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2} \quad (184)$$

where θ is the heat distribution. The boundary conditions on this problem is $\theta(\pm\infty, t = 0)$ and $\theta(x, t = 0) = \delta(x)$.

$$\frac{\partial}{\partial t} \tilde{\theta}(k, t) = D(ik)^2 \tilde{\theta}(k, t) = -Dk^2 \tilde{\theta}(k, t) \quad (185)$$

the solution is

$$\tilde{\theta}(k, t) = \tilde{\theta}(k, 0)e^{-Dk^2t} = \text{FT}[\delta(x)]e^{-Dk^2t} = \frac{1}{\sqrt{2\pi}}e^{-Dk^2t} \quad (186)$$

So we have

$$\begin{aligned} \theta(x, t) &= \text{FT}^{-1}[\tilde{\theta}(k, t)] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} e^{-Dk^2t} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp \left[-Dt \left(k - \frac{ix}{2Dt} \right)^2 - \frac{x^2}{4Dt} \right] dk \\ &= \frac{1}{2\pi} e^{-x^2/4Dt} \int_{-\infty}^{\infty} e^{-Dtq^2} dq \\ &= \frac{1}{2\pi} \sqrt{\frac{\pi}{Dt}} e^{-x^2/4Dt} \left(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \right) \end{aligned} \quad (187)$$

Hence the final result

$$\theta(x, t) = \frac{1}{2\sqrt{\pi Dt}} e^{-x^2/4Dt} \quad (188)$$

4.4 Laplace Transforms

Laplace transforms is useful for initial value problem where $f(t)$ only exists for $t \geq 0$.

$$\mathcal{L}[f(t)] = \hat{f}(s) = \int_0^{\infty} dt e^{-st} f(t) \quad (189)$$

where s is a complex variable and $\text{Re}(S) > 0$ is required for the convergence of the integral.

4.4.1 Properties

- $\mathcal{L}[f'(t)] = s\hat{f}(s) - f(0)$

Proof.

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^{\infty} dt e^{-st} f'(t) \\ &= e^{-st} f(t) \Big|_0^{\infty} + s \int_0^{\infty} dt e^{-st} f(t) = s\hat{f}(s) - f(0) \end{aligned} \quad (190)$$

- $\mathcal{L}[f^{(n)}(t)] = s^n \hat{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$
- $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \hat{f}(s)$

Proof.

$$\begin{aligned} (-1)^n \frac{d^n}{ds^n} \hat{f}(s) &= (-1)^n \frac{d^n}{ds^n} \int_0^{\infty} dt e^{-st} f(t) \\ &= (-1)^n \int_0^{\infty} dt (-t)^n e^{-st} f(t) \\ &= \int_0^{\infty} dt e^{-st} t^n f(t) = \mathcal{L}[t^n f(t)] \end{aligned} \quad (191)$$

4.4.2 Examples

- $\mathcal{L}[1] = \frac{1}{s}$
- $\mathcal{L}[e^{-at}] = \frac{1}{s+a}$
- $\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$
- $\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$
- $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$
- $\mathcal{L}[te^{-at}] = \frac{1}{(s+a)^2}$
- $\mathcal{L}[e^{at}f(t)] = \hat{f}(s-a)$

4.4.3 Convolution Theorem for Laplace Transforms

A convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as

$$f_1 * f_2 = \int_{-\infty}^{\infty} f_1(t')f_2(t-t')dt' \quad (192)$$

If f_1 and f_2 vanish for $t < 0$, then

$$f_1 * f_2 = \int_0^t f_1(t')f_2(t-t')dt' \quad (193)$$

If we apply the Laplace transform

$$\begin{aligned} \mathcal{L}[f_1 * f_2] &= \int_0^{\infty} dt e^{-st} \int_0^t f_1(t')f_2(t-t')dt' \\ &= \int_0^{\infty} dt' f_1(t') \int_{t'}^{\infty} dt e^{-st} f_2(t-t') \\ &= \int_0^{\infty} dt' e^{-st'} f_1(t') \int_{t'}^{\infty} dt e^{-s(t-t')} f_2(t-t') \\ &= \tilde{f}_1(s)\tilde{f}_2(s) \end{aligned} \quad (194)$$

Example. Consider the differential equation

$$f'' + 5f' + 6f = 0 \quad (195)$$

with boundary conditions $f'(0) = f(0) = 0$. Apply the Laplace transform on the equation, we have

$$s^2 \hat{f} - sf(0) - f'(0) + 5[s\hat{f} - f(0)] + 6\hat{f} = \hat{f}(s^2 + 5s + 6) = \frac{1}{s} \quad (196)$$

rearranging this, we have

$$\hat{f}(s) = \frac{1}{s(s+2)(s+3)} = \frac{1}{6s} - \frac{1}{2} \frac{1}{s+2} + \frac{1}{3} \frac{1}{s+3} \quad (197)$$

So

$$f(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} \quad (198)$$

5 Complex Analysis

5.1 Complex Functions of a Complex Variable

A complex number $z = x + iy$ can be mapped to another complex number $w = f(z) = u(x, y) + iv(x, y)$. It is often useful to use the ‘polar representation’ of complex numbers where

$$z = re^{i\theta} \quad (199)$$

where $r = |z| = \sqrt{x^2 + y^2}$ is called the modulus of z and $\theta = \arg(z)$ is called the argument of z . $\arg(z)$ can be made unambiguous by a choice of ‘branch’. We will write the principal branch as $\text{Arg}(z)$, which is values $-\pi < \text{Arg}(z) \leq \pi$.

Example.

$$1) f(z) = |z| = \sqrt{x^2 + y^2}$$

$$2) f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$3) f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$4) f(z) = z^{1/3} = r^{1/3}e^{(i\theta+2\pi in)/3} = \begin{cases} r^{1/3} \exp\left(\frac{i\theta}{3}\right) \\ r^{1/3} \exp\left(\frac{i\theta}{3} + \frac{2\pi i}{3}\right) \\ r^{1/3} \exp\left(\frac{i\theta}{3} + \frac{4\pi i}{3}\right) \end{cases}$$

Complex functions defined as power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (200)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \quad (201)$$

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \quad (|z| < 1) \quad (202)$$

5.2 Continuity, Differentiability and Analyticity

5.2.1 Definitions

Definition. $f(z)$ is *continuous* at $z = z_0$ if $\forall \varepsilon > 0$, there exists a $\delta > 0$, such that, if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$. We also say

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \quad (203)$$

Definition. $f(z)$ is *differentiable* at $z = z_0$ if $\exists F \in \mathbb{C}$ such that

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = F \quad (204)$$

we say $f'(z_0) = (df/dz)|_{z_0} = F$.

Definition. A subset $D \subset \mathbb{C}$ is *open* if for every $z \in D$, there is an open disc centred at z entirely contained in D .

Definition. A function $f(z)$ is *analytic* at z_0 if $f(z)$ is differentiable everywhere in an open domain containing z_0 ; if $f(z)$ is NOT analytic at z_0 we say $f(z)$ is *singular* at z_0 .

Example. $f(z) = z^2$ and $z = z_0 + \delta z$

$$\lim_{\delta z \rightarrow 0} \frac{(z_0 + \delta z)^2 - z_0^2}{\delta z} = 2z_0 \quad (205)$$

$f(z) = z^2$ is differentiable everywhere in \mathbb{C} . So we say $f(z)$ is analytic in \mathbb{C} and $f(z)$ is entire.

Example. $f(z) = z^* = x - iy$ and $z = z_0 + \delta z$

$$\lim_{\delta z \rightarrow 0} \frac{(z_0 + \delta z)^* - z_0^*}{\delta z} = \lim_{\delta z \rightarrow 0} \frac{\delta z^*}{\delta z} = e^{-2i\theta} \quad (206)$$

$f(z) = z^*$ is not differentiable anywhere so $f(z)$ is not analytic in \mathbb{C} .

5.2.2 The Cauchy-Riemann Conditions

In this section we ask: *Under what conditions is a complex function $f(z) = u(x, y) + iv(x, y)$ analytic in a domain D ?* Let us assume that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist in D , i.e., $f(z)$ is analytic in D .

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} \frac{df}{dz} = f', \quad \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} \frac{df}{dz} = if' \quad (207)$$

which shows

$$i \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \Rightarrow i \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) = \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \quad (208)$$

Rearranging this, now we get the *Cauchy-Riemann equations*

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad (209)$$

It is a theorem that $f(z)$ is analytic if and only if Cauchy-Riemann equations hold in D .

Example. $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$. In this function, $u = x^2 - y^2$ and $v = 2xy$.

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y \quad (210)$$

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x \quad (211)$$

satisfy the C-R equations.

Example. $f(z) = x = (z + z^*)/2$. In this function, $u = x$ and $v = 0$, so we have

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0 \quad (212)$$

C-R equations fail.

Example. $f(z) = x^2 + y^2 = zz^*$ with $u = x^2 + y^2$ and $v = 0$.

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad (213)$$

So $f(z)$ satisfies C-R equations at $x = y = 0$ but nowhere else.

Theorem. $f(z)$ is analytic at $z = z_0$ if and only if $f(z)$ has a power series expansion around $z = z_0$ that converges in an open neighborhood of z_0 .

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots = \sum_{k=0}^{\infty} c_k(z - z_0)^k \quad (214)$$

with

$$c_k = \frac{f^{(k)}(z_0)}{k!} \quad (215)$$

5.2.3 Harmonic Functions

Definition. $g(x, y)$ is harmonic if $\nabla^2 g = 0$.

Now we look at C-R equations

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \frac{\partial v}{\partial y} = \frac{\partial}{\partial y} \frac{\partial v}{\partial x} = -\frac{\partial}{\partial y} \frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \quad (216)$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0 \quad (217)$$

$u(x, y)$ is harmonic. Similarly, $v(x, y)$ is harmonic. We conclude that if $f = u + iv$ is analytic, u and v are *conjugate* harmonic functions.

Example. Consider the real function $u(x, y) = \cos x \cosh y$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0 \quad (218)$$

hence u is harmonic. Then we find the conjugate harmonic function $v(x, y)$. Using the C-R equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -\sin x \cosh y \Rightarrow v = -\sin x \sinh y + c_1(y) \quad (219)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\cos x \sinh y \Rightarrow v = -\sin x \sinh y + c_2(x) \quad (220)$$

so that $c_1 = c_2 = c$ and $v(x, y) = -\sin x \sinh y + c$, where c is a constant. Hence

$$f(z) = \cos x \cosh y - i \sin x \sinh y + \tilde{c} \quad (221)$$

is analytic by construction.

5.3 Multi-Valued Functions

Example. $f(z) = z^{1/3}$. There are three related branches of $z^{1/3}$

$$\begin{cases} F_1(z) = r^{1/3}e^{i\theta/3} \\ F_2(z) = r^{1/3}e^{i\theta/3+2\pi i/3} \\ F_3(z) = r^{1/3}e^{i\theta/3+4\pi i/3} \end{cases} \quad (222)$$

with $\theta \in [-\pi, \pi]$. Each one is single valued, but discontinuous along the negative real axis. If we glue sheets together on the branch cuts, then the three sheets form a *Riemann surface*. $f(z) = z^{1/3}$ is defined on the Riemann surface on the following way

$$f(z) = F_i(z) \quad \text{on sheet } i \quad (223)$$

$f(z)$ is single valued and continuous on the Riemann surface.

Example. $f(z) = z^{1/2}$: 2 branches and 2 Riemann sheets.

Example. $f(z) = z^{1/n}$: n branches and n Riemann sheets.

Example. $f(z) = \ln z = \ln(re^{i\theta})$ not defined at $z = 0$.

$$f(z) = \ln r + i\theta + 2\pi in \quad (224)$$

has one branch for each integer n .

Example. $f(z) = (z - z_0)^{1/3}$. A *branch point* is a point that cannot be encircled without moving on to a different sheet of the Riemann surface of $f(z)$.

Example. $f(z) = (z - a)^{1/2}(z - b)^{1/2}$, $a, b \in \mathbb{R}$. The function has two branch points a and b , the branch cuts must begin or end there (see Fig.4).

5.4 Integration of Complex Functions

5.4.1 Contours

We focus on *contour integrals*, $\int_C f(z)dz$, along lines or paths C in the complex plane.

Example. Evaluate $\int_C z dz$ along (i) $y = x^2$ and (ii) $y = x$.

$$\int_C z dz = \int_C (x + iy)(dx + idy) = \int_C (x dx - y dy) + i \int_C (y dx + x dy) \quad (225)$$

$$(i) \int_0^1 \int (x dx - 2x^3 dx) + i \int_0^1 (x^2 dx + 2x^2 dx) = i$$

$$(ii) \int_0^1 (x dx - x dx) + i \int_0^1 (x dx + x dx) = i$$



Figure 4: The two possible ways to place branch cuts for $f(z) = (z - a)^{1/2}(z - b)^{1/2}$, and they form the same Riemann surface.

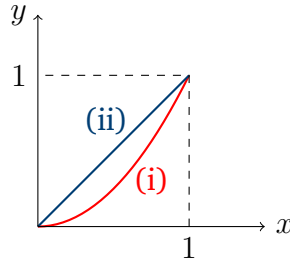


Figure 5: The two paths, (i) $y = x^2$ and (ii) $y = x$, along with the function $f(z)$ is to be integrated in the example.

5.4.2 Cauchy's Theorem

Theorem. Cauchy's theorem. If $f(z)$ is analytic everywhere on and within a closed contour C

$$\oint_C f(z)dz = 0 \quad (226)$$

Theorem. Green's theorem in the plane. P and Q are functions of x and y , and C is a closed contour in the $x - y$ plane, then

$$\oint_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \quad (227)$$

Proof. Proof of Cauchy's theorem

$$\begin{aligned} \oint_C f(z)dz &= \oint_C (u(x, y) + iv(x, y))(dx + idy) \\ &= \oint_C (udx - vdy) + i \oint_C (vdx + udy) \\ &= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy = 0 \end{aligned} \quad (228)$$

5.4.3 Path Independence

Theorem. Let C_1 and C_2 be two contours from z_a to z_b . If $f(z)$ is analytic on C_1 and C_2 and the region between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \quad (229)$$

Proof. Consider closed contour $C = C_1 - C_2$. By Cauchy's theorem

$$\oint_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0 \quad (230)$$

5.4.4 Contour Deformation

Theorem. If C_1 and C_2 are closed contours, and C_1 can be deformed into C_2 entirely in a region where $f(z)$ is analytic, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz \quad (231)$$

Proof. Choose line segment AB as shown in the Fig.6. Consider $C = C_1 + \overline{BA} - C_2 + \overline{AB}$. By Cauchy's theorem

$$\begin{aligned} \oint_C f(z)dz &= \left(\int_{C_1} + \int_{\overline{BA}} - \int_{C_2} + \int_{\overline{AB}} \right) f(z)dz \\ &= \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0 \end{aligned} \quad (232)$$

Example. Evaluate $\oint_{|z|=1} \frac{1}{z} dz$. Deform the contour into a small circle, radius r , centred on the origin, then

$$\oint_{|z|=1} \frac{1}{z} dz = \oint_{|z|=r} e^{-i\theta} i e^{i\theta} d\theta = 2\pi i \quad (233)$$

5.4.5 Cauchy's Integral Theorem

Theorem. If $f(z)$ is analytic within and on a closed contour C and z_0 is any point within C , then

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) \quad (234)$$

or

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \quad (235)$$

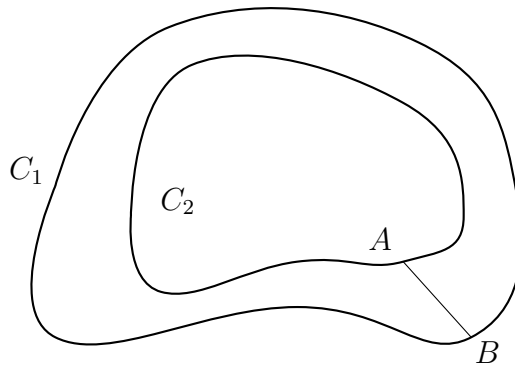
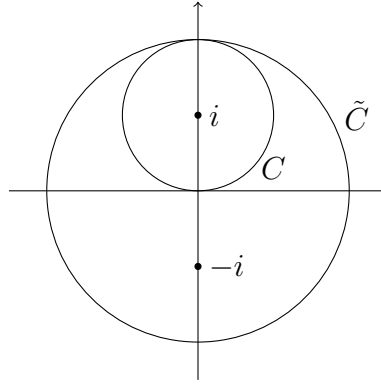


Figure 6: Caption

Figure 7: The contour C and \tilde{C} .

Proof. The integral is analytic within and on C except at $z = z_0$. Let C_r be a small circle around z_0 , i.e. $C_r : z = z_0 + re^{i\theta}$, then

$$\begin{aligned} \oint_C \frac{f(z)}{z - z_0} dz &= \oint_{C_r} \frac{f(z)}{z - z_0} dz = \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= i \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \lim_{r \rightarrow 0} i \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = 2\pi i f(z_0) \end{aligned} \quad (236)$$

Example. Consider the integral

$$\oint \frac{\sin z}{z^2 + 1} dz = \oint_C \frac{\sin z}{(z + i)(z - i)} dz \quad (237)$$

and consider the closed contour C and \tilde{C} , which are showed in Fig.7. For the contour C , We choose

$$f(z) = \frac{\sin z}{z + i} \quad (238)$$

Then

$$\oint_C \frac{\sin z}{z^2 + 1} dz = \oint_C \frac{f(z)}{z - i} dz = 2\pi i \frac{\sin i}{2i} = \pi i \sinh 1 \quad (239)$$

\tilde{C} is a circle of radius 2 centred at origin, so

$$\begin{aligned} \oint_{\tilde{C}} \frac{\sin z}{z^2 + 1} dz &= \oint_{\tilde{C}} \frac{\sin z}{(z + i)(z - i)} dz = \frac{i}{2} \oint_{\tilde{C}} \left(\frac{\sin z}{z + i} - \frac{\sin z}{z - i} \right) dz \\ &= -\pi (\sin(-i) - \sin(i)) = 2\pi i \sinh 1 \end{aligned} \quad (240)$$

5.4.6 Derivatives of Analytic Functions

Cauchy's integral theorem gives

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \quad (241)$$

If we differentiate both sides of Cauchy's integral formula with respect to z_0 , interchanging the orders of integration and differentiation, we get

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz \quad (242)$$

Similarly,

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3} dz \quad (243)$$

\vdots

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (244)$$

Example.

$$\oint_C \frac{1}{z^n} dz \quad \text{with} \quad C : |z| = r \quad (245)$$

- $n = 1, \oint_C (1/z) dz = 2\pi i$
- $n \geq 2, \oint_C (1/z) dz = 0$

5.4.7 Fourier Transform of a Gaussian

We have known that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(x'+ia)^2} dx' \quad (246)$$

where a is a real number. Now we use Cauchy's theorem to prove it.

Proof.

$$I_1 = \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{C_1} e^{-z^2} dz \quad (247)$$

$$I_2 = \int_{-\infty}^{\infty} e^{-(x'+ia)^2} dx' = \int_{C_2} e^{-z^2} dz \quad (248)$$

where C_1 is the whole x -axis and C_2 is the line parallel to the x -axis at $z = x + ia$. Let's assume $a > 0$. To begin with, we construct a closed contour $C_R = C_{2R} + E_R^+ - C_{1R} + E_R^-$ (see in Fig.8).

$$\oint_{C_R} e^{-z^2} dz = 0 \quad (249)$$

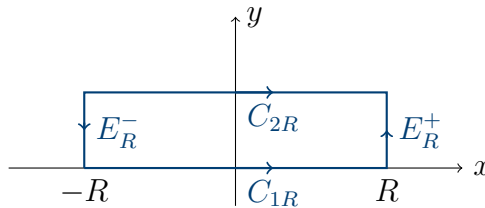


Figure 8: The contour C_R .

for any R . When $R \rightarrow \infty$, then

$$\lim_{R \rightarrow \infty} \oint_{C_R} e^{-z^2} dz = \lim_{R \rightarrow \infty} \left(\int_{C_1} + \int_{E_R^+} - \int_{C_2} + \int_{E_R^-} \right) e^{-z^2} dz = I_1 - I_2 = 0 \quad (250)$$

5.5 Power Series Representations of Complex Functions

5.5.1 Taylor Series

$f(z)$ is analytic at z_0 if it has a Taylor series in a neighbourhood of z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (251)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (252)$$

5.5.2 Singularities

If $f(z)$ is analytic except at specific points in the complex plane, those points are called isolated singularities or *poles*.

Example.

$$f(z) = \frac{e^z}{(z - 5)(z + i)(z - (1 + i))^2} \quad (253)$$

has isolated singularities at $z = 5, i, 1 + i$.

There two types of singularities:

1. $f(z)$ has a *pole of order* m ($m \geq 1$) at z_0 if there exists a $g(z)$ which is analytic at z_0 and $g(z_0) \neq 0$ s.t.

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad (254)$$

This implies $f(z)$ has a power series except around z_0

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n \quad (255)$$

Poles of order 1 are called *single poles*.

2. $f(z)$ has an *essential singularity* at z_0 if $f(z)$ has a power series except around $z = z_0$ with infinitely many negative powers

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad (256)$$

Example.

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z} \right)^n \quad (257)$$

5.6 Contour Integration using the Residue Theorem

5.6.1 The Residue Theorem

Definition. Let f has an isolated singularity at z_0 , then the residue of f at z_0 is

$$\text{Res}_f(z_0) = \frac{1}{2\pi i} \oint_{C_{z_0}} f(z) dz \quad (258)$$

where C_{z_0} is a closed contour s.t. z_0 is inside and $f(z)$ is analytic inside except at z_0 . If $f(z)$ has a pole of order m at z_0 , then

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad (259)$$

and

$$\text{Res}_f(z_0) = \frac{1}{2\pi i} \oint_C \frac{g(z)}{(z - z_0)^m} dz = \frac{1}{(m-1)!} \left. \frac{d^{m-1}g(z)}{dz^{m-1}} \right|_{z=z_0} \quad (260)$$

Example.

$$(1) f(z) = 1/(z - z_0)$$

$$\text{Res}_f(z_0) = 1 \quad (261)$$

$$(2) f(z) = \sin z / (1 + z)^2$$

$$\text{Res}_f(-1) = \left. \frac{d \sin z}{dz} \right|_{z=-1} = \cos(-1) = \cos 1 \quad (262)$$

Theorem. Let C is a closed contour, $f(z)$ is a function that is analytic on C and inside C except at $z = z_1, \dots, z_N$. Then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^N \text{Res}_f(z_k) \quad (263)$$

Proof. By Cauchy's theorem

$$\oint_{\tilde{C}} f(z) dz = \oint_C f(z) dz - 2\pi i \sum_{k=1}^N \text{Res}_f(z_k) = 0 \quad (264)$$

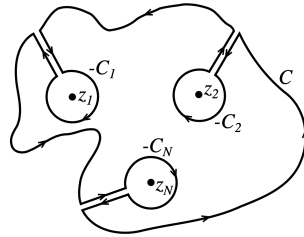


Figure 9: The contour C used in the proof of the residue theorem.

5.6.2 Contour Integration Examples

Example. 1

$$\begin{aligned}
I &= \oint_{|z|=1} e^{1/z} dz \\
&= \oint_{|z|=1} \left[1 + \frac{1}{z} + \frac{1}{2!} \left(\frac{1}{z} \right)^2 + \cdots \right] dz = 2\pi i
\end{aligned} \tag{265}$$

Example. 2

$$\begin{aligned}
I &= \oint_{|z|=3} \frac{z+2}{2z^2+1} dz = \oint_{|z|=3} \frac{z+2}{2(z^2+1/2)} dz \\
&= \oint_{|z|=3} \frac{z+2}{2(z+\frac{i}{\sqrt{2}})(z-\frac{i}{\sqrt{2}})} dz \\
&= 2\pi i \left[\text{Res} \left(\frac{i}{\sqrt{2}} \right) + \text{Res} \left(-\frac{i}{\sqrt{2}} \right) \right] \\
&= 2\pi i \left[\frac{\frac{i}{\sqrt{2}}+2}{2(\frac{i}{\sqrt{2}}+\frac{i}{\sqrt{2}})} + \frac{-\frac{i}{\sqrt{2}}+2}{2(-\frac{i}{\sqrt{2}}-\frac{i}{\sqrt{2}})} \right] = \pi i
\end{aligned} \tag{266}$$

Example. 3

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} \tag{267}$$

Consider the contour $C = C_R + S_R$ (see in fig.10(a)), we have

$$\begin{aligned}
I &= \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{(x^2+1)(x^2+9)} \\
&= \lim_{R \rightarrow \infty} \oint_C \frac{dz}{(z+i)(z-i)(z+3i)(z-3i)} - \lim_{R \rightarrow \infty} \int_{C_R} \frac{dz}{(z^2+1)(z^2+9)} \\
&= 2\pi i [\text{Res}(i) + \text{Res}(3i)] - 0 \\
&= 2\pi i \left(\frac{1}{16i} + \frac{1}{-48i} \right) = \frac{\pi}{12}
\end{aligned} \tag{268}$$

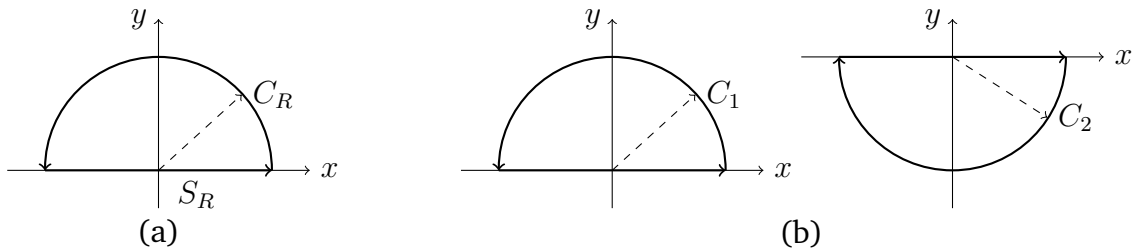


Figure 10: (a) The contour for example 3. (b) The contour for example 4.

Example. 4

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx = \int_{x\text{-axis}} \frac{\cos z}{z^2 + 1} dz \\
&= \int_{x\text{-axis}} \frac{e^{iz}}{2(z+i)(z-i)} dz + \int_{x\text{-axis}} \frac{e^{-iz}}{2(z+i)(z-i)} dz \\
&= I_1 + I_2
\end{aligned} \tag{269}$$

Consider the contour C_1 and C_2 (see in fig.10(b))

$$\begin{aligned}
I_1 &= \lim_{R \rightarrow \infty} \oint_C \frac{e^{iz}}{2(z+i)(z-i)} dz - \lim_{R \rightarrow \infty} \int_{C_1} \frac{e^{iz}}{2(z+i)(z-i)} dz \\
&= 2\pi i \operatorname{Res}(i) - \lim_{R \rightarrow \infty} \int_{C_1} \frac{e^{ix-y}}{2(z+i)(z-i)} dz \\
&= 2\pi i \frac{e^{-1}}{4i} - 0 = \frac{\pi}{2} e^{-1}
\end{aligned} \tag{270}$$

$$\begin{aligned}
I_2 &= - \lim_{R \rightarrow \infty} \oint_C \frac{e^{-iz}}{2(z+i)(z-i)} dz - \lim_{R \rightarrow \infty} \int_{C_2} \frac{e^{-iz}}{2(z+i)(z-i)} dz \\
&= -2\pi i \operatorname{Res}(-i) - \lim_{R \rightarrow \infty} \int_{C_2} \frac{e^{-ix+y}}{2(z+i)(z-i)} dz \\
&= -2\pi i \frac{e^{-1}}{-4i} - 0 = \frac{\pi}{2} e^{-1}
\end{aligned} \tag{271}$$

So we have

$$I = I_1 + I_2 = \pi e^{-1} \tag{272}$$

5.6.3 Inverting Laplace Transforms

Suppose

$$F(s) = \text{LT}[f(t)] = \int_0^s f(t) e^{-st} dt \tag{273}$$

where $t > 0$, then

$$f(t) = \text{LT}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) e^{st} ds \tag{274}$$

where γ is chosen big enough so all singularities of $F(s)$ are in the left of contour. Construct a semicircle contour C_R (see in fig.11). So we have

$$f(t) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \left(\oint_C - \int_{C_R} \right) F(s) e^{st} ds \tag{275}$$

where

$$\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \oint_C F(s) e^{st} ds = \sum_k \operatorname{Res}(s_k) \tag{276}$$

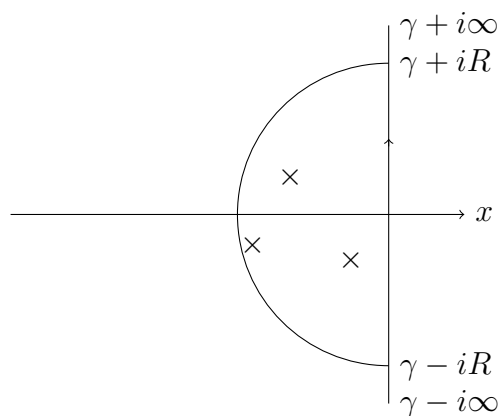


Figure 11: The contour for inverting Laplace transforms.

$$\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{C_R} F(s) e^{xt+iyt} ds = 0 \quad (277)$$

Hence when $t > 0$

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) e^{st} ds = \sum_k \text{Res}(s_k) \quad (278)$$

6 Calculus of Variations

$$f(x + \delta x) = f(x) + \delta f'(x) + \frac{\delta x^2}{2} f''(x) + \dots \quad (279)$$

at stationary point $x = x_0$, $f'(x_0) = 0$

$$\delta f(x) = f(x + \delta x) - f(x) = \mathcal{O}(\delta x^2) \quad (280)$$

$$\delta I = I[y + \delta y] - I[y] \quad (281)$$

seek y_0 such that $\delta I[y_0] = \mathcal{O}(\delta y^2)$

6.1 The Euler-Lagrange Problem

Let y be a function of variable $y(x)$

$$I[y] = \int_{x_A}^{x_B} f(x, y(x), y'(x)) dx \quad (282)$$

where f is a function of 3 arguments, and $x_A, x_B, y(x_A), y(x_B)$ are fixed. Euler-Lagrange problem is to find $y(x)$ such that $\delta I = \mathcal{O}(\delta y^2)$ at $y(x)$. Then we say y extremes I , or y is a stationary function of I , or I is stationary at y .

$$I[y + \delta y] = \int_{x_A}^{x_B} f(x, y(x) + \delta y(x), y' + \delta y'(x)) dx \quad (283)$$

$$\begin{aligned} \delta I &= I[y + \delta y] - I[y] \\ &= \int_{x_A}^{x_B} \left(\delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} \right) dx + \mathcal{O}(\delta y^2) \\ &= \int_{x_A}^{x_B} \left(\delta y \frac{\partial f}{\partial y} \right) dx + \left[\delta y \frac{\partial f}{\partial y'} \right]_{x_A}^{x_B} - \int_{x_A}^{x_B} \delta y \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx + \mathcal{O}(\delta y^2) \\ &= \int_{x_A}^{x_B} \delta y \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] dx + \left[\delta y \frac{\partial f}{\partial y'} \right]_{x_A}^{x_B} + \mathcal{O}(\delta y^2) \end{aligned} \quad (284)$$

$\delta I = \mathcal{O}(\delta y^2)$ if and only if

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0} \quad (285)$$

for $x_A \leq x \leq x_B$. This equation is called *Euler-Lagrange equation*.

Example.

$$f(x, y, y') = (1 + x^2)y'^2 - y^4 \quad (286)$$

is stationary if

$$-4y^3 - \frac{d}{dx} [(1 + x^2)2y'] = 0 \quad (287)$$