Imperial College London

NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF PHYSICS

Quantum Optics

Author:
Chen Huang

Email:
chen.huang23@imperial.ac.uk

Date: November 25, 2023

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Theory

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An atom is described by the Hamiltonian

$$H_a = \frac{p^2}{2m} + V(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) \tag{1}$$

If the atom is interacting with a classical electro-magnetic field, the Hamiltonian is replaced by

$$H_{\boldsymbol{A}} = -\frac{\hbar^2}{2m}\left(\nabla - i\frac{\rho}{\hbar}\boldsymbol{A}\right)^2 + V(\boldsymbol{r}) \tag{2}$$

Many problems are fomulated in terms of a Hamiltonian of the form

$$H_{\boldsymbol{E}} = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) - \rho\boldsymbol{E}\cdot\boldsymbol{r} \tag{3}$$

In most case, the Hamiltonian can be expressed as

$$H = H_{\text{atom}} + H_{\text{interaction}} \tag{4}$$

1.1 Dynamics of Atom in Light-Field

1.1.1 The Propagator

We define the propagator $U(t)$ via the relation

$$|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle \tag{5}$$

for any solution $|\Psi(t)\rangle$ of the Schrödinger equation. By definition, the propagator satisfies the initial condition $U(0) = \mathbb{1}$.
The propagator also satisfies a Schrödinger equation.

$$i|\dot{\Psi}(t)\rangle = i\dot{U}(t)\,|\Psi(0)\rangle = HU(t)\,|\Psi(0)\rangle \tag{6}$$

so that

$$\boxed{i\dot{U} = HU} \tag{7}$$

The ad-joint $U^\dagger(t)$ satisfies

$$-i\dot{U}^\dagger = U^\dagger H^\dagger = U^\dagger H \tag{8}$$

According to this relations, we have

$$i\frac{\partial}{\partial t}\left(UU^\dagger\right) = i\dot{U}U^\dagger + iU\dot{U}^\dagger = HUU^\dagger - UU^\dagger H = [H, UU^\dagger] \tag{9}$$

With the initial condition $U(0)U^\dagger(0) = \mathbb{1}$, this is solved for

$$U(t)U^\dagger(t) = U^\dagger(t)U(t) = \mathbb{1} \tag{10}$$

1.1.2 Perturbation Theory

The Schrödinger equation together with the initial condition $U(0) = \mathbb{1}$ can be rewritten as the integral equation

$$\begin{aligned} U(t) &= \mathbb{1} + \int_0^t \mathrm{d}t' \dot{U}(t') = \mathbb{1} - i \int_0^t \mathrm{d}t' H(t') U(t') \\ &= \mathbb{1} - i \int_0^t \mathrm{d}t' H(t') \left[\mathbb{1} - i \int_0^{t'} \mathrm{d}t'' H(t'') U(t'') \right] \\ &= \mathbb{1} - i \int_0^t \mathrm{d}t' H(t') - \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t') H(t'') U(t'') \end{aligned} \tag{11}$$

For sufficiently short times this can be approximated as

$$U(t) \simeq \mathbb{I} - i \int_0^t \mathrm{d}t' H(t') - \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t') H(t'') \tag{12}$$

For this to be a good approximation, it is essential that 'magnitude' of H is sufficiently **small**. It is therefore important to work in a suitable frame. Rather than solving the Schrödinger equation $i\dot{U} = HU$ for U, we can try to solve for V defined via the relation

$$U = U_0 V \tag{13}$$

in terms of a unitary U_0 that we are **free** to choose. The Schrödinger equation

$$i\dot{U} = i\dot{U}_0 V + iU_0 \dot{V} = HU_0 V \tag{14}$$

can now be solved for $\dot{V}$ what yields

$$i\dot{V} = U_0^\dagger H U_0 V - iU_0^\dagger \dot{U}_0 V = \left(U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0 \right) V = \tilde{H} V \tag{15}$$

with the new Hamiltonian

$$\boxed{\tilde{H} = U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0} \tag{16}$$

The goal is then to find U_0 such that the time-dependent perturbation theory is a good approximation.

1.1.3 Atom-Light Hamiltonian

Let's consider an atom with Hamiltonian H_0 and interaction Hamiltonian H_I

$$H_0 = \sum_j \omega_j \left|\psi_j\right\rangle \left\langle\psi_j\right| \tag{17}$$

$$H_I = \sum_{j,k} \left|\psi_j\right\rangle \left\langle\psi_j\right| H_I \left|\psi_k\right\rangle \left\langle\psi_k\right| = \sum_{j,k} \left\langle\psi_j\right| H_I \left|\psi_k\right\rangle \left|\psi_j\right\rangle \left\langle\psi_k\right| = \sum_{j,k} h_{jk} \left|\psi_j\right\rangle \left\langle\psi_k\right| \tag{18}$$

We choose

$$U_0(t) = \exp(-iH_0 t) = \sum_j \mathrm{e}^{-i\omega_j t} \left|\psi_j\right\rangle \left\langle\psi_j\right| \tag{19}$$

such that $-iU_0^\dagger \dot{U}_0 = -H_0$. Then the transforemed Hamiltonian reads

$$\begin{aligned}
\tilde{H} &= U_0^\dagger (H_0 + H_I) U_0 - iU_0^\dagger \dot{U}_0 \\
&= U_0^\dagger H_0 U_0 + U_0^\dagger H_I U_0 - iU_0^\dagger \dot{U}_0 \\
&= U_0^\dagger H_I U_0 \\
&= \sum_l \mathrm{e}^{i\omega_l t} \left|\psi_l\right\rangle \left\langle\psi_l\right| \sum_{jk} h_{jk} \left|\psi_j\right\rangle \left\langle\psi_k\right| \exp(-iH_0 t) \\
&= \sum_{jk} h_{jk} \exp(i\omega_k t) \left|\psi_j\right\rangle \left\langle\psi_k\right| \exp(-i\omega_k t) \\
&= \sum_{jk} h_{jk} \exp(i(\omega_j - \omega_k)t) \left|\psi_j\right\rangle \left\langle\psi_k\right|
\end{aligned} \tag{20}$$

where $h_{jk} = \left\langle\psi_j\right| H_{jk} \left|\psi_k\right\rangle$ is the oscillating term with frequency ν, and $\cos \nu t = (\mathrm{e}^{i\nu t} + \mathrm{e}^{-i\nu t})/2$. The oscillating functions result in a vanishing integral in the integration $\int_0^t \mathrm{d}t' \cdots$. If we choose the ground state and another selected eigenstate, we can approximate the atom as a two-level system.

1.1.4 The Pauli Matrices

The Pauli matrices satisfies the relation

$$[\sigma_\alpha, \sigma_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad \{\sigma_\alpha, \sigma_\beta\} = 0 \tag{21}$$

In terms of the eigenstates $|g\rangle$ and $|e\rangle$, we have

$$\sigma_z |g\rangle = -|g\rangle, \qquad \sigma_z |e\rangle = |e\rangle \tag{22}$$

$$\sigma_x |g\rangle = |e\rangle, \qquad \sigma_x |e\rangle = |g\rangle \tag{23}$$

$$\sigma_y |g\rangle = -i|e\rangle, \qquad \sigma_y |e\rangle = i|g\rangle \tag{24}$$

1.2 Hamiltonian and Propagator of a Two-Level Atom

The Hamiltonian of a two-level atom (see Fig.1) is given by

$$\begin{aligned}
H &= \omega_g |g\rangle\langle g| + \omega_e |e\rangle\langle e| \\
&= \frac{\omega_e + \omega_g}{2}(|g\rangle\langle g| + |e\rangle\langle e|) + \frac{\omega_e - \omega_g}{2}(|e\rangle\langle e| - |g\rangle\langle g|) \\
&= \frac{\omega_e + \omega_g}{2}\mathbb{1} + \frac{\omega}{2}\sigma_z \simeq \frac{\omega}{2}\sigma_z
\end{aligned} \tag{25}$$

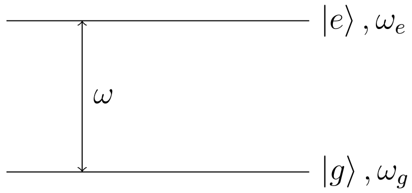

Figure 1: A two-level atom with resonance frequency ω.

The corresponding propagator reads

$$U(t) = \exp\Big(-i\frac{\omega}{2}\sigma_z t\Big) = \mathbb{1}\cos\left(\frac{\omega}{2}t\right) - i\sigma_z \sin\left(\frac{\omega}{2}t\right) \tag{26}$$

Back to the Schrödinger equation, we have

$$\begin{aligned} i\dot{U} &= -i\frac{\omega}{2}\mathbb{1}\sin\left(\frac{\omega}{2}t\right) + \frac{\omega}{2}\sigma_z\cos\left(\frac{\omega}{2}t\right) \\ &= \frac{\omega}{2}\sigma_z\left[\mathbb{1}\cos\left(\frac{\omega}{2}t\right) - i\sigma_z\sin\left(\frac{\omega}{2}t\right)\right] = HU \end{aligned} \tag{27}$$

1.3 The Two-Level Atom in a Monochromatic Light Field

The Hamiltonian for the atom interacting with a light field in two-level approximation reads

$$H = \frac{\omega}{2}\sigma_z + \Omega_R\sigma_x\cos(\nu t) \tag{28}$$

The prefactor Ω_R is called *Rabi-frequency*; it is proportional to the intensity of the light field. And ν is frequency of light.

In order to find the solution of the Schrödinger equation, it is helpful to consider the transformation

$$U_0 = \exp\Big(-i\frac{\eta}{2}\sigma_z t\Big) \tag{29}$$

and

$$\dot{U}_0 = -i\frac{\eta}{2}\sigma_z\exp\Big(-i\frac{\eta}{2}\sigma_z t\Big) = -i\frac{\eta}{2}\sigma_z U_0 \tag{30}$$

So we have

$$U_0^\dagger\dot{U}_0 = -i\frac{\eta}{2}\sigma_z \tag{31}$$

Then we construct the transformed Hamiltonian $\tilde{H}$. With $U_0^\dagger\sigma_z U_0 = \sigma_z$ and $U_0^\dagger\sigma_x U_0 = \sigma_+\mathrm{e}^{i\eta t} + \sigma_-\mathrm{e}^{-i\eta t}$, the explicit form of H reads

$$\begin{aligned} \tilde{H} &= U_0^\dagger H U_0 - iU_0^\dagger\dot{U}_0 \\ &= \frac{\omega-\eta}{2}\sigma_z + \Omega_R\left(\sigma_+\mathrm{e}^{i\eta t} + \sigma_-\mathrm{e}^{-i\eta t}\right)\cos(\nu t) \\ &= \frac{\omega-\eta}{2}\sigma_z + \frac{\Omega_R}{2}\left[\sigma_+\mathrm{e}^{i(\eta-\nu)t} + \sigma_-\mathrm{e}^{-i(\eta-\nu)t}\right] + \frac{\Omega_R}{2}\left[\sigma_+\mathrm{e}^{i(\eta+\nu)t} + \sigma_-\mathrm{e}^{-i(\eta+\nu)t}\right] \end{aligned} \tag{32}$$

After the *rotating wave approximation (RWA)*, we have

$$H' = \frac{\omega-\eta}{2}\sigma_z + \frac{\Omega_R}{2}\left[\sigma_+\mathrm{e}^{i(\eta-\nu)t} + \sigma_-\mathrm{e}^{-i(\eta-\nu)t}\right] \tag{33}$$

Let's consider the case $\eta = \nu$,

$$H' = \frac{\omega-\nu}{2}\sigma_z + \frac{\Omega_R}{2}\left(\sigma_+ + \sigma_-\right) = \frac{\omega-\nu}{2}\sigma_z + \frac{1}{2}\Omega_R\sigma_x \tag{34}$$

The associated propagator

$$\boxed{\begin{aligned}\exp(-iH't) &= \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - \frac{2i}{\Omega_G}H'\sin\left(\frac{1}{2}\Omega_G t\right) \\ &= \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - i\left(\frac{\omega-\nu}{\Omega_G}\sigma_z + \frac{\Omega_R}{\Omega_G}\sigma_x\right)\sin\left(\frac{1}{2}\Omega_G t\right)\end{aligned}} \tag{35}$$

where

$$\Omega_G = \sqrt{(\omega-\nu)^2 + \Omega_R^2} \tag{36}$$

is called *generalised Rabi frequency*. It is helpful to notice

$$(H')^2 = \frac{(\omega-\nu)^2}{4}\sigma_z^2 + \frac{1}{4}\Omega_R^2\sigma_x^2 + \frac{1}{4}(\omega-\nu)\Omega_R\{\sigma_z,\sigma_x\} = \frac{1}{4}\Omega_G^2\mathbb{1} \tag{37}$$

which implies that $\mathbb{1} = \frac{4}{\Omega_G^2}(H')^2$. Taking the time-derivative yields

$$\begin{aligned} i\frac{\partial}{\partial t}\exp(-iH't) &= -i\frac{\Omega_G}{2}\mathbb{I}\sin\left(\frac{1}{2}\Omega_G t\right) + H'\cos\left(\frac{1}{2}\Omega_G t\right) \\ &= H'\cos\left(\frac{1}{2}\Omega_G t\right) - i\frac{\Omega_G}{2}\frac{4}{\Omega_G^2}(H')^2\sin\left(\frac{1}{2}\Omega_G t\right) \\ &= H'\left[\mathbb{I}\cos\left(\frac{1}{2}\Omega_G t\right) - i\frac{2}{\Omega_G}H'\sin\left(\frac{1}{2}\Omega_G t\right)\right] \\ &= H'\exp(-iH't) \end{aligned} \tag{38}$$

The expression given in eqn.(35) is the correct solution of the Schrödinger equation with the Hamiltonian H'.

1.3.1 Resonant Driving

If the light-field is on resonance with the atomic transition, *i.e.* $\omega - \nu = 0$, this simplifies to

$$\exp(-iH't) = \mathbb{1}\cos\left(\frac{1}{2}\Omega_R t\right) - i\sigma_x\sin\left(\frac{1}{2}\Omega_R t\right) \tag{39}$$

Applying this to the ground state as initial state yields

$$\exp(-iH't)\,|g\rangle = \cos\left(\frac{1}{2}\Omega_R t\right)|g\rangle - i\sin\left(\frac{1}{2}\Omega_R t\right)|e\rangle \tag{40}$$

Together with the factor U_0, we have

$$\begin{aligned} &\exp\Big(-i\frac{\omega}{2}\sigma_z t\Big)\exp(-iH't)\,|g\rangle \\ &= \exp\Big(i\frac{\omega}{2}t\Big)\cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\exp\Big(-i\frac{\omega}{2}t\Big)\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle \\ &= \exp\Big(i\frac{\omega}{2}t\Big)\left[\cos\left(\frac{\Omega_R}{2}t\right)|g\rangle - i\exp(-i\omega t)\sin\left(\frac{\Omega_R}{2}t\right)|e\rangle\right] \end{aligned} \tag{41}$$

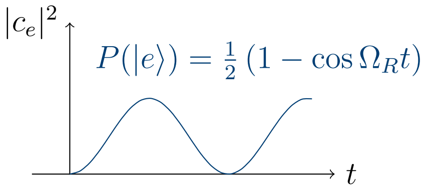

Figure 2: The probability to find the atom in the excited state in Rabi oscillation.

The probability to find the atom in the excited state or ground state is given by

$$|c_e(t)|^2 = \left[\sin\left(\frac{\Omega_R}{2}t\right)\right]^2 = \frac{1}{2}(1 - \cos\Omega_R t) \tag{42}$$

$$|c_g(t)|^2 = \left[\cos\left(\frac{\Omega_R}{2}t\right)\right]^2 = \frac{1}{2}(1 + \cos\Omega_R t) \tag{43}$$

They are called Rabi oscillation.

1.3.2 Off-Resonant Driving

If the light field is far off-resonant, *i.e.* $|\nu - \omega| \gg \Omega_R$, the approximations

$$\frac{\omega - \nu}{\Omega_G} = \frac{\omega - \nu}{\sqrt{(\omega-\nu)^2 + \Omega_R^2}} \simeq \frac{\nu - \omega}{|\nu - \omega|} = \pm 1 \tag{44}$$

$$\frac{\Omega_R}{\Omega_G} = \frac{\Omega_R}{\sqrt{(\omega-\nu)^2 + \Omega_R^2}} \simeq \frac{\Omega_R}{|\nu - \omega|} \ll 1 \tag{45}$$

so the propagator eqn.(35) becomes

$$\exp(-iH't) = \mathbb{1}\cos\left(\frac{1}{2}\Omega_G t\right) - i\sin\left(\frac{1}{2}\Omega_G t\right) \tag{46}$$

Let Ω_G do a Taylor expansion at $\Omega_R = 0$

$$\Omega_G = |\nu - \omega| + \frac{\Omega_R^2}{2|\nu - \omega|} + \mathcal{O}(\Omega_R^4) \tag{47}$$

So we have

$$\Omega_G - (\omega - \nu) \simeq \frac{\Omega_R^2}{2|\delta|} \tag{48}$$

with the detuning $\delta = \omega - \nu$.

1.3.3 Ramsey

In the case of $\nu = \omega$, we found the propagator

$$U_x = \mathbb{1}\cos\left(\frac{1}{2}\Omega_R t\right) - i\sigma_x\sin\left(\frac{1}{2}\Omega_R t\right) \tag{49}$$

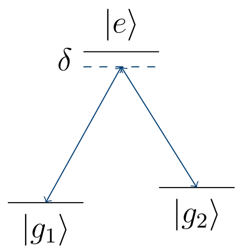

Figure 3: The three-level atom.

in the interaction picture. For a duration $T = \frac{\pi}{2\Omega_R}$, this reduce to

$$U_x(T) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_x) \tag{50}$$

Assuming the atom initially in its ground state $|g\rangle$, we obtain

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}(|g\rangle - i\,|e\rangle) \tag{51}$$

A measurement of the population of the eigenstates would yield 50% ground state and 50% excited state.

$$H_\phi = \frac{\omega}{2}\sigma_z + \Omega_R \sigma_x \cos(\nu t + \phi) \tag{52}$$

the associated propagator

$$U_\phi(T) = \frac{1}{\sqrt{2}}\left[\mathbb{1} - i(\cos\phi\sigma_x + \sin\phi\sigma_y)\right] \tag{53}$$

Applying $U_\phi(T)$ to the state $|\Psi(T)\rangle$

$$\begin{aligned} |\Psi(2T)\rangle =& U_\phi(T)\,|\Psi(T)\rangle \\ =& \frac{1}{2}\left[\mathbb{1} - i(\cos\phi\sigma_x + \sin\phi\sigma_y)\right](|g\rangle - i\,|e\rangle) \\ =& -i\left(\exp\left(i\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)|g\rangle + \exp\left(-i\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)|e\rangle\right) \end{aligned} \tag{54}$$

The probability to find the atom in the ground state or the excited state thus oscillates with ϕ.

1.4 The Three-Level Atom

$$H_1 = \frac{\Omega_R}{2\sqrt{2}}\left(|e\rangle\langle g_1|\,\mathrm{e}^{i\delta t} + |g_1\rangle\langle e|\,\mathrm{e}^{-i\delta t}\right) \tag{55}$$

$$H_2 = \frac{\Omega_R}{2\sqrt{2}}\left(|e\rangle\langle g_2|\,\mathrm{e}^{i\delta t} + |g_2\rangle\langle e|\,\mathrm{e}^{-i\delta t}\right) \tag{56}$$

$$
\begin{aligned}
H = H_1 + H_2 &= \frac{\Omega_R}{2}\left(|e\rangle \frac{\langle g_1| + \langle g_2|}{\sqrt{2}} \mathrm{e}^{i\delta t} + \frac{|g_1\rangle + |g_2\rangle}{\sqrt{2}} \langle e| \, \mathrm{e}^{-i\delta t} \right) \\
&= \frac{\Omega_R}{2}\left(|e\rangle \langle g| \, \mathrm{e}^{i\delta t} + |g\rangle \langle e| \, \mathrm{e}^{-i\delta t} \right) \\
&= \frac{\Omega_R}{4}\left[(\sigma_x - i\sigma_y)(\cos\delta t + i\sin\delta t) + (\sigma_x + i\sigma_y)(\cos\delta t - i\sin\delta t) \right] \\
&= \frac{\Omega_R}{2}\left(\sigma_x \cos\delta t + \sigma_y \sin\delta t \right)
\end{aligned} \tag{57}
$$

where $|g\rangle = (|g_1\rangle + |g_2\rangle)/\sqrt{2}$. The propagator

$$
U(t) \simeq \mathbb{I} - i\int_0^t \mathrm{d}t_1 H(t_1) - \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 H(t_1) H(t_2) \tag{58}
$$

Now we calculate $U(t)$ at $t = 2\pi/\delta$. The first order

$$
U^{(1)}\left(\frac{2\pi}{\delta}\right) = -i\frac{\Omega_R}{2}\int_0^{2\pi/\delta} \mathrm{d}t_1 \left(\sigma_x \cos\delta t_1 + \sigma_y \sin\delta t_1 \right) = 0 \tag{59}
$$

and the second

$$
U^{(2)}\left(\frac{2\pi}{\delta}\right) = -\frac{\Omega_R^2}{4}\int_0^{2\pi/\delta} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \left(\sigma_x \cos\delta t_1 + \sigma_y \sin\delta t_1 \right)\left(\sigma_x \cos\delta t_2 + \sigma_y \sin\delta t_2 \right) \tag{60}
$$

To do so, we need some integrals

$$
\int_0^{\frac{2\pi}{\delta}} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cos(\delta t_1)\cos(\delta t_2) = 0 \tag{61}
$$

$$
\int_0^{\frac{2\pi}{\delta}} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \sin(\delta t_1)\sin(\delta t_2) = 0 \tag{62}
$$

$$
\int_0^{\frac{2\pi}{\delta}} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cos(\delta t_1)\sin(\delta t_2) = -\frac{1}{2\delta}\frac{2\pi}{\delta} \tag{63}
$$

$$
\int_0^{\frac{2\pi}{\delta}} \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \sin(\delta t_1)\cos(\delta t_2) = \frac{1}{2\delta}\frac{2\pi}{\delta} \tag{64}
$$

We thus obtain the perturbative expression

$$
U\left(\frac{2\pi}{\delta}\right) \simeq \mathbb{1} + \frac{\Omega_R^2}{4}\frac{1}{2\delta}[\sigma_x, \sigma_y]\frac{2\pi}{\delta} = \mathbb{1} + i\frac{\Omega_R^2}{4\delta}\sigma_z\frac{2\pi}{\delta} \tag{65}
$$

The propagator looks as if it was induced by the Hamiltonian

$$
H_e = -\frac{\Omega_R^2}{4\delta}\sigma_z = \frac{\Omega_e}{2}\sigma_z \tag{66}
$$

Now we can return to the explicit three levels, and obtain

$$
\begin{aligned}
H_e =& \frac{\Omega_e}{2}(|e\rangle\langle e| - |g\rangle\langle g|) \\
=& \frac{\Omega_e}{2}\left(|e\rangle\langle e| - \frac{1}{2}(|g_1\rangle\langle g_1| + |g_1\rangle\langle g_2| + |g_2\rangle\langle g_1| + |g_2\rangle\langle g_2|) \right)
\end{aligned} \tag{67}
$$

The eigenstates are $|e\rangle$, $|g\rangle$ and $(|g_1\rangle - |g_2\rangle)\sqrt{2}$

$$H_e |e\rangle = \frac{\Omega_e}{2} |e\rangle \tag{68}$$

$$H_e |g\rangle = -\frac{\Omega_e}{2} |g\rangle \tag{69}$$

$$H_e \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}} = 0 \tag{70}$$

with the eigenvalues $\frac{\Omega_e}{2}$, $-\frac{\Omega_e}{2}$ and 0, we can now express the propagator as

$$\exp(-iH_e t) = \exp\left(-i\frac{\Omega_e}{2}t\right) |e\rangle \langle e| + \exp\left(i\frac{\Omega_e}{2}t\right) |g\rangle \langle g| + \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}} \frac{\langle g_1| - \langle g_2|}{\sqrt{2}} \tag{71}$$

Applying this to the initial state $|g_1\rangle$ yields

$$\begin{aligned} \exp(-iH_e t) |g_1\rangle =& \exp\left(i\frac{\Omega_e}{2}t\right) \frac{|g_1\rangle + |g_2\rangle}{2} + \frac{|g_1\rangle - |g_2\rangle}{2} \\ =& \frac{1}{2}\left(\exp\left(i\frac{\Omega_e}{2}t\right) + 1\right) |g_1\rangle + \frac{1}{2}\left(\exp\left(i\frac{\Omega_e}{2}t\right) - 1\right) |g_1\rangle \\ =& \exp\left(i\frac{\Omega_e}{4}t\right)\left(\cos\left(\frac{\Omega_e}{4}t\right) |g_1\rangle + i\sin\left(\frac{\Omega_e}{4}t\right) |g_2\rangle\right) \end{aligned} \tag{72}$$

1.5 Bloch Equations

The Bloch equations

$$\langle\Psi| \mathbb{1} |\Psi\rangle = 1 \tag{73}$$

$$\langle\Psi| \sigma_x |\Psi\rangle = S_x \tag{74}$$

$$\langle\Psi| \sigma_y |\Psi\rangle = S_y \tag{75}$$

$$\langle\Psi| \sigma_z |\Psi\rangle = S_z \tag{76}$$

that define the Bloch vector

$$\boldsymbol{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \tag{77}$$

For the state $|g\rangle$ one obtains

$$\langle g| \sigma_x |g\rangle = \langle g|e\rangle = 0 \tag{78}$$

$$\langle g| \sigma_y |g\rangle = -i \langle g|e\rangle = 0 \tag{79}$$

$$\langle g| \sigma_z |g\rangle = - \langle g|g\rangle = -1 \tag{80}$$

For the state $|+\rangle$ one obtains

$$\langle +| \sigma_x |+\rangle = \langle +|+\rangle = 1 \tag{81}$$

$$\langle +| \sigma_y |+\rangle = i \langle +|-\rangle = 0 \tag{82}$$

$$\langle +| \sigma_z |+\rangle = - \langle +|-\rangle = 0 \tag{83}$$

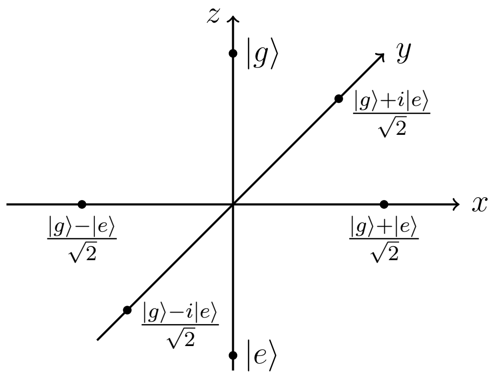

Figure 4: The eigenstates of $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ lie on the x, y and z axis.

1.6 Dynamics of the Bloch Vector

Instead of the Schrödinger equation, we can describe the system dynamics in terms of an equation of motion for the Bloch vector $\boldsymbol{S}$. In addition to the Schrödinger equation

$$\left|\dot{\Psi}\right\rangle = -iH\left|\Psi\right\rangle \tag{84}$$

For the state vector $|\Psi\rangle$

$$\begin{aligned}\dot{S}_x =& \left\langle\dot{\Psi}\right|\sigma_x\left|\Psi\right\rangle + \left\langle\Psi\right|\sigma_x\left|\dot{\Psi}\right\rangle \\ =& i\left\langle\Psi\right|H\sigma_x\left|\Psi\right\rangle - i\left\langle\Psi\right|\sigma_x H\left|\Psi\right\rangle \\ =& i\left\langle\Psi\right|[H,\sigma_x]\left|\Psi\right\rangle\end{aligned} \tag{85}$$

To simplify this, we can express the Hamiltonian in Pauli matrices

$$H = \sum_j \frac{\omega_j}{2}\sigma_j \tag{86}$$

so that

$$\dot{S}_x = i\sum_j \frac{\omega_j}{2}\left\langle\Psi\right|[\sigma_j,\sigma_x]\left|\Psi\right\rangle = \omega_y\sigma_z - \omega_z\sigma_y \tag{87}$$

Similarly

$$\dot{S}_y = \omega_z\sigma_x - \omega_x\sigma_z \tag{88}$$

$$\dot{S}_z = \omega_x\sigma_y - \omega_y\sigma_x \tag{89}$$

In terms of vector notation, we have

$$\dot{\boldsymbol{S}} = \boldsymbol{\omega}\times\boldsymbol{S} \tag{90}$$

and

$$\frac{\partial}{\partial t}|\boldsymbol{S}|^2 = \dot{\boldsymbol{S}}\boldsymbol{S} + \boldsymbol{S}\dot{\boldsymbol{S}} = (\boldsymbol{\omega}\times\boldsymbol{S})\boldsymbol{S} + \boldsymbol{S}(\boldsymbol{\omega}\times\boldsymbol{S}) = 0 \tag{91}$$

1.7 Averages over Different States

Consider the expectation of an observable A

$$\langle A\rangle = \langle\Psi| A |\Psi\rangle = \sum_j p_j \langle\Psi_j| A |\Psi_j\rangle \tag{92}$$

We can thus define the Bloch vector

$$\boldsymbol{S} = \sum_j p_j \boldsymbol{S}_j \tag{93}$$

for the ensemble average. Now we recall the Ramsey experiment

$$H_\phi = \frac{\omega}{2}\sigma_z + \Omega_R \sigma_x \cos(\nu t + \phi) \tag{94}$$

Consider the state $|\Psi\rangle = |\pm\rangle = (|e\rangle \pm |g\rangle)/\sqrt{2}$

$$\begin{aligned}\boldsymbol{S} = \frac{1}{2}\boldsymbol{S}_{|g\rangle} + \frac{1}{2}\boldsymbol{S}_{|e\rangle} =& \frac{1}{2}\langle g| \begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix} |g\rangle + \frac{1}{2}\langle e| \begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix} |e\rangle \\ =& \frac{1}{2}\langle g| \begin{pmatrix}0\\ 0\\ -1\end{pmatrix} |g\rangle + \frac{1}{2}\langle e| \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} |e\rangle = 0\end{aligned} \tag{95}$$

and $\dot{\boldsymbol{S}} = \boldsymbol{\omega} \times \boldsymbol{S} = 0$. The Bloch vector for the ensemble average is thus stationary.

2 Harmonic Oscillator

The Hamiltonian of one dimension harmonic oscillator

$$\begin{aligned} H =& \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 \\ =& \hbar\omega\left(\frac{P^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar}X^2\right) \\ =& \frac{1}{2}\hbar\omega\left(\hat{p}^2 + \hat{x}^2\right) \end{aligned} \tag{96}$$

in terms of the unitless operators

$$\hat{x} = \sqrt{\frac{m\omega}{\hbar}}X, \qquad \hat{p} = \frac{1}{\sqrt{m\hbar\omega}}P \tag{97}$$

which satisfy the commutation relation

$$[\hat{x}, \hat{p}] = \frac{1}{\hbar}[X, P] = i \tag{98}$$

Creation and annihilation operators are defined as

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}) \tag{99}$$

with

$$[a, a^\dagger] = \frac{1}{2}[\hat{x} + i\hat{p}, \hat{x} - i\hat{p}] = \frac{1}{2}([\hat{x}, -i\hat{p}] + [i\hat{p}, \hat{x}]) = 1 \tag{100}$$

$\hat{x}$ and $\hat{p}$ can be expressed as

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}) \tag{101}$$

The Hamiltonian becomes

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{102}$$

For the commute relation $[a, a^\dagger] = 1$, one obtains

$$a^\dagger a a\left|\mu\right\rangle = (aa^\dagger - 1)a\left|\mu\right\rangle = a(a^\dagger a - 1)\left|\mu\right\rangle = (\mu - 1)a\left|\mu\right\rangle \tag{103}$$

so that

$$\left\langle\mu\right| a^\dagger a\left|\mu\right\rangle = \mu \quad \Rightarrow \quad a\left|\mu\right\rangle = \sqrt{\mu}\left|\mu - 1\right\rangle \tag{104}$$

and similarly,

$$a^\dagger\left|\mu\right\rangle = \sqrt{\mu + 1}\left|\mu + 1\right\rangle \tag{105}$$

In real space representation, the annihilation operator and creation operator read

$$a \to \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \qquad a^\dagger \to \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right) \tag{106}$$

The wave function satisfying

$$\frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\phi_0(x) = 0 \quad \Rightarrow \quad \phi_0(x) \sim \exp\left(-\frac{x^2}{2}\right) \tag{107}$$

$$\phi_1(x) \sim \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\exp\left(-\frac{x^2}{2}\right) = 2x\exp\left(-\frac{x^2}{2}\right) \tag{108}$$

3 Quantisation of the Electromagnetic Field

The Maxwell equations read

$$\nabla \cdot \boldsymbol{E} = 0 \tag{109}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{110}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{111}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{112}$$

In terms of divergence and curl

$$\nabla \cdot \boldsymbol{Q} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \tag{113}$$

$$\nabla \times \boldsymbol{Q} = -\left(\frac{\partial Q_y}{\partial z} - \frac{\partial Q_z}{\partial y}\right)\boldsymbol{e}_x - \left(\frac{\partial Q_z}{\partial x} - \frac{\partial Q_x}{\partial z}\right)\boldsymbol{e}_y - \left(\frac{\partial Q_x}{\partial y} - \frac{\partial Q_y}{\partial x}\right)\boldsymbol{e}_z \tag{114}$$

Let's start with a simple ansatz

$$\boldsymbol{E} = f(t)\sin kz\boldsymbol{e}_x \tag{115}$$

$$\boldsymbol{B} = g(t)\cos kz\boldsymbol{e}_y \tag{116}$$

The Maxwell equations imply

$$\nabla \cdot \boldsymbol{E} = \nabla \cdot \boldsymbol{B} = 0 \tag{117}$$

$$\begin{aligned}\nabla \times \boldsymbol{E} =& \frac{\partial E_x}{\partial z}\boldsymbol{e}_y = f(t)k\cos kz\boldsymbol{e}_y \\ =& -\frac{\partial \boldsymbol{B}}{\partial t} = -\dot{g}(t)\cos kz\boldsymbol{e}_y\end{aligned} \tag{118}$$

$$\begin{aligned}\nabla \times \boldsymbol{B} =& -\frac{\partial B_y}{\partial z}\boldsymbol{e}_x = g(t)k\sin kz\boldsymbol{e}_x \\ =& \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{c^2}\dot{f}(t)\sin kz\boldsymbol{e}_x\end{aligned} \tag{119}$$

This requires the differential equations

$$f(t)k = -\dot{g}(t), \qquad g(t)k = \frac{1}{c^2}\dot{f}(t) \tag{120}$$

So we have

$$\ddot{f}(t) = c^2k\dot{g}(t) = -c^2k^2f(t) = \nu_k^2 f(t) \tag{121}$$

where $\nu_k = ck$ is called the linear dispersion. We can now quantise the electric field as

$$\boldsymbol{E} = \sqrt{\frac{\hbar\nu}{\varepsilon_0 V}}\left(\hat{a}\mathrm{e}^{-i\nu t} + \hat{a}^\dagger\mathrm{e}^{i\nu t}\right)\sin kz\boldsymbol{e}_x \tag{122}$$

The creation and annihilation operators satisfy

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \tag{123}$$

We can also define creation and annihilation operators for wave packets

$$\hat{a}_\phi = \int \mathrm{d}k\phi(k)\hat{a}_k, \qquad \hat{a}^\dagger_\phi = \int \mathrm{d}k\phi^*(k)\hat{a}^\dagger_k \tag{124}$$

and the commutator

$$[\hat{a}_\phi, \hat{a}^\dagger_\psi] = \int \mathrm{d}k\mathrm{d}k'\phi(k)\psi^*(k')[\hat{a}_k, \hat{a}^\dagger_{k'}] = \int \mathrm{d}k\phi(k)\psi^*(k) \tag{125}$$

4 Jaynes-Cummings

we can now consider a two-level system interacting with a single mode of the quantised electro-magnetic field. The Hamiltonian reads

$$H = \frac{\omega}{2}\sigma_z + \frac{1}{2}\Omega_R\sigma_x\left(a\mathrm{e}^{-i\nu t} + a^\dagger\mathrm{e}^{i\nu t}\right) \tag{126}$$

1) $U_S = \exp\bigl(i\nu a^\dagger a t\bigr)$, the Hamiltonian reads

$$H_S = U_S^\dagger H U_S - iU_S^\dagger\dot{U}_S = \frac{\omega}{2}\sigma_z + \nu a^\dagger a + \frac{1}{2}\Omega_R\sigma_x(a + a^\dagger) \tag{127}$$

2) $U_I = \exp\bigl(-i\frac{\omega}{t}\sigma_z t\bigr)$

$$\begin{aligned} H_I =& U_I^\dagger H U_I - iU_I^\dagger\dot{U}_I \\ =& \frac{1}{2}\Omega_R\left(\sigma_+\mathrm{e}^{i\omega t} + \sigma_-\mathrm{e}^{-i\omega t}\right)\left(a\mathrm{e}^{-i\nu t} + a^\dagger\mathrm{e}^{i\nu t}\right) \\ =& \frac{1}{2}\Omega_R\left[\sigma_+ a\mathrm{e}^{i(\omega-\nu)t} + \sigma_+ a^\dagger\mathrm{e}^{i(\omega+\nu)t} + \sigma_- a\mathrm{e}^{-i(\omega+\nu)t} + \sigma_- a^\dagger\mathrm{e}^{-i(\omega-\nu)t}\right] \\ \approx& \frac{1}{2}\Omega_R\left[\sigma_+ a\mathrm{e}^{i(\omega-\nu)t} + \sigma_- a^\dagger\mathrm{e}^{-i(\omega-\nu)t}\right] \end{aligned} \tag{128}$$

This Hamiltonian contains four elementary process

- $\sigma_+ a$: atom absorbs a photon and gets excited.
- $\sigma_+ a^\dagger$: atom emits a photon and gets excited.
- $\sigma_- a$: atom absorbs a photon and gets de-excited.
- $\sigma_- a^\dagger$: atom emits a photon and gets de-excited.

4.1 Two-Dimensional Subspaces

The Hamiltonian (in rotating wave approximation) in lab frame reads

$$H = \frac{\omega}{2}\sigma_z + \nu a^\dagger a + \frac{1}{2}\Omega_R(\sigma_+ a + \sigma_- a^\dagger) \tag{129}$$

and

$$H\left|g,\mu\right\rangle = \left(-\frac{\omega}{2} + \mu\nu\right)\left|g,\mu\right\rangle + \frac{1}{2}\Omega_R\sqrt{\mu}\left|e,\mu-1\right\rangle \tag{130}$$

$$H\left|e,\mu-1\right\rangle = \frac{1}{2}\Omega_R\sqrt{\mu}\left|g,\mu\right\rangle + \left(\frac{\omega}{2} + (\mu-1)\nu\right)\left|e,\mu-1\right\rangle \tag{131}$$

In terms of the basis $\{\left|g,\mu\right\rangle, \left|e,\mu-1\right\rangle\}$ we can express this as the matrix

$$\begin{pmatrix} -\frac{\omega-\nu}{2} + \left(\mu - \frac{1}{2}\right)\nu & \frac{1}{2}\Omega_R\sqrt{\mu} \\ \frac{1}{2}\Omega_R\sqrt{\mu} & \frac{\omega-\nu}{2} + \left(\mu - \frac{1}{2}\right)\nu \end{pmatrix} \tag{132}$$

or, in terms of Pauli-matrices as

$$H = -\frac{\omega-\nu}{2}\sigma_z + \frac{1}{2}\Omega_R\sqrt{\mu}\sigma_x + \left(\mu - \frac{1}{2}\right)\nu\mathbb{1} \tag{133}$$

In the case of resonance between atom and light-field, this reduces to

$$H(\nu=\omega) = \frac{1}{2}\Omega_R\sqrt{\mu}\sigma_x + \left(\mu - \frac{1}{2}\right)\nu\mathbb{1} \tag{134}$$

with eigenstates

$$\frac{1}{\sqrt{2}}(|g,\mu\rangle \pm |e,\mu-1\rangle) \tag{135}$$

4.2 The Lambda-System

The Hamiltonian of the Lambda-system interacting with a single-mode quantum field in rotating wave approximation reads

$$\begin{aligned} H =&\omega\,|e\rangle\langle e| + 0(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|) + \nu a^\dagger a \\ &+ \frac{1}{2\sqrt{2}}\Omega_R\left(|g_1\rangle\langle e|\,a^\dagger + |g_2\rangle\langle e|\,a^\dagger + |e\rangle\langle g_1|\,a + |e\rangle\langle g_2|\,a\right) \\ =&\omega\,|e\rangle\langle e| + \frac{1}{2\sqrt{2}}\Omega_R\left[(|g_1\rangle\langle e| + |g_2\rangle\langle e|)a^\dagger + (|e\rangle\langle g_1| + |e\rangle\langle g_2|)a\right] \end{aligned} \tag{136}$$

In the interaction picture we have

$$H_I = \frac{1}{2\sqrt{2}}\Omega_R\left[(|g_1\rangle\langle e| + |g_2\rangle\langle e|)a^\dagger\mathrm{e}^{-i\delta t} + (|e\rangle\langle g_1| + |e\rangle\langle g_2|)a\mathrm{e}^{i\delta t}\right] \tag{137}$$

with the detuning $\delta = \omega - \nu$. According to the perturbation theory

$$U = \mathbb{1} - i\int_0^t \mathrm{d}t' H(t') - \frac{1}{2}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H_I(t')H_I(t'') + \cdots \tag{138}$$

thus we have to consider the second order

$$\begin{aligned} H_I(t')H_I(t'') =&\frac{1}{8}\Omega_R^2\Big[(|g_1\rangle\langle e| + |g_2\rangle\langle e|)(|e\rangle\langle g_1| + |e\rangle\langle g_2|)a^\dagger a\mathrm{e}^{-i\delta(t'-t'')} \\ &+(|e\rangle\langle g_1| + |e\rangle\langle g_2|)(|g_1\rangle\langle e| + |g_2\rangle\langle e|)aa^\dagger\mathrm{e}^{i\delta(t'-t'')}\Big] \\ =&\frac{1}{8}\Omega_R^2\left(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2| + |g_1\rangle\langle g_2| + |g_2\rangle\langle g_1|\right)a^\dagger a\mathrm{e}^{-i\delta(t'-t'')} \\ &+ \frac{1}{4}\Omega_R^2\,|e\rangle\langle e|\,(a^\dagger a + 1)\mathrm{e}^{i\delta(t'-t'')} \end{aligned} \tag{139}$$

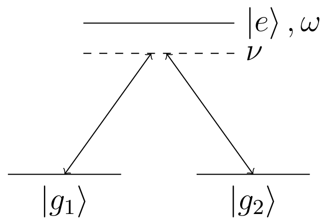

Figure 5: The Lambda-system

5 Coherent States

For fock states $|\mu\rangle$, the expectation values for x and p vanish

$$\langle\mu| x |\mu\rangle = \frac{1}{\sqrt{2}}(\langle\mu| a |\mu\rangle + \langle\mu| a^\dagger |\mu\rangle) = 0 \tag{140}$$

$$\langle\mu| p |\mu\rangle = \frac{-i}{\sqrt{2}}(\langle\mu| a |\mu\rangle - \langle\mu| a^\dagger |\mu\rangle) = 0 \tag{141}$$

and the fluctuations

$$\langle\mu| x^2 |\mu\rangle = \frac{1}{2}(\langle\mu| a^2 |\mu\rangle + \langle\mu| aa^\dagger |\mu\rangle + \langle\mu| a^\dagger a |\mu\rangle + \langle\mu| a^\dagger a^\dagger |\mu\rangle) = \mu + \frac{1}{2} \tag{142}$$

$$\langle\mu| p^2 |\mu\rangle = -\frac{1}{2}(\langle\mu| a^2 |\mu\rangle - \langle\mu| aa^\dagger |\mu\rangle - \langle\mu| a^\dagger a |\mu\rangle + \langle\mu| a^\dagger a^\dagger |\mu\rangle) = \mu + \frac{1}{2} \tag{143}$$

For the ground state

$$\langle 0| x |0\rangle = \langle 0| p |0\rangle = 0 \tag{144}$$

$$\langle 0| x^2 |0\rangle = \langle 0| p^2 |0\rangle = \frac{1}{2} \tag{145}$$

This yields

$$\Delta x \Delta p = (\langle 0| x^2 |0\rangle - (\langle 0| x |0\rangle)^2)(\langle 0| p^2 |0\rangle - (\langle 0| p |0\rangle)^2) = \frac{1}{4} \tag{146}$$

which is the minimal allowed uncertainty. We can generate different states with the same uncertainty, by displacing the vacuum in phase space. The *displacement operator* is defined as

$$\boxed{D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right)} \tag{147}$$

The coherent state

$$\begin{aligned} |\alpha\rangle =& D(\alpha) |0\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right) |0\rangle \\ =& \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp\left(\alpha a^\dagger\right) \exp(-\alpha^* a) |0\rangle \\ =& \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp\left(\alpha a^\dagger\right) |0\rangle \\ =& \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_\mu \frac{\alpha^\mu (a^\dagger)^\mu}{\mu!} |0\rangle \\ =& \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}} |\mu\rangle \end{aligned} \tag{148}$$

The probability to find μ photons is thus given by the Poisson distribution.

$$P(\mu) = \exp\left(-|\alpha|^2\right) \frac{(|\alpha|^2)^\mu}{\mu!} \tag{149}$$

For the expectation value of x and p with respect to a general state $|\Psi\rangle$ one has

$$(\langle\Psi| D^\dagger(\alpha))x(D(\alpha)|\Psi\rangle) = \langle\Psi|(D^\dagger(\alpha)xD(\alpha))|\Psi\rangle \tag{150}$$

$$(\langle\Psi| D^\dagger(\alpha))p(D(\alpha)|\Psi\rangle) = \langle\Psi|(D^\dagger(\alpha)pD(\alpha))|\Psi\rangle \tag{151}$$

then we calculate

$$D^\dagger(\alpha)xD(\alpha) = x + \frac{\alpha+\alpha^*}{\sqrt{2}} = x + x_0 \tag{152}$$

$$D^\dagger(\alpha)pD(\alpha) = p - i\frac{\alpha-\alpha^*}{\sqrt{2}} = p + p_0 \tag{153}$$

We verify the uncertainty in position and momentum of any coherent state

$$\begin{aligned}\langle\alpha| x^2 |\alpha\rangle - (\langle\alpha| x |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)x^2D(\alpha)|0\rangle - (\langle 0| D^\dagger(\alpha)xD(\alpha)|0\rangle)^2 \\ &= \langle 0| x^2 |0\rangle - (\langle 0| x |0\rangle)^2\end{aligned} \tag{154}$$

$$\begin{aligned}\langle\alpha| p^2 |\alpha\rangle - (\langle\alpha| p |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)p^2D(\alpha)|0\rangle - (\langle 0| D^\dagger(\alpha)pD(\alpha)|0\rangle)^2 \\ &= \langle 0| p^2 |0\rangle - (\langle 0| p |0\rangle)^2\end{aligned} \tag{155}$$

5.1 Coherent States in Real-Space Representation

The character of the displacement operator can be exemplified in the real-space representation of wave functions.

$$\Psi_\alpha(x) = \langle x|\alpha\rangle \propto \exp\left(-\frac{1}{2}(x-x_0)^2 + ip_0x - \frac{i}{2}x_0p_0\right) \tag{156}$$

with $x_0 = (\alpha+\alpha^*)/\sqrt{2}$ and $p_0 = (\alpha-\alpha^*)/(\sqrt{2}i)$. It is convenient to define the vector

$$|\alpha,\tau\rangle = \exp\left[(\alpha a^\dagger - \alpha^* a)\tau\right]|0\rangle \tag{157}$$

with additional scalar parameter τ. It satisfies the differential equation

$$\frac{\partial|\alpha,\tau\rangle}{\partial\tau} = (\alpha a^\dagger - \alpha^* a)|\alpha,\tau\rangle \tag{158}$$

The real-space representation of the operator $(\alpha a^\dagger - \alpha^* a)$ reads

$$\frac{1}{\sqrt{2}}\left[\alpha\left(x - \frac{\partial}{\partial x}\right) - \alpha^*\left(x + \frac{\partial}{\partial x}\right)\right] = \frac{\alpha-\alpha^*}{\sqrt{2}}x - \frac{\alpha+\alpha^*}{\sqrt{2}}\frac{\partial}{\partial x} = ip_0x - x_0\frac{\partial}{\partial x} \tag{159}$$

We thus need to solve the equation

$$\frac{\partial\Phi}{\partial\tau} = \left(ip_0x - x_0\frac{\partial}{\partial x}\right)\Phi \tag{160}$$

with the Ansatz

$$\Phi(\tau) = \exp\left(-\frac{1}{2}(x-f_x)^2 + if_px - i\varphi\right) \tag{161}$$

The initial conditions are $f_x(0) = f_p(0) = \varphi(0) = 0$. The derivatives

$$\frac{\partial \Phi(\tau)}{\partial \tau} = \left((x - f_x) \frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} x - i \frac{\partial \varphi}{\partial \tau} \right) \Phi(\tau) \tag{162}$$

$$\frac{\partial \Phi(\tau)}{\partial x} = \left(-(x - f_x) + i f_p \right) \Phi(\tau) \tag{163}$$

This yields

$$(x - f_x) \frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} x - i \frac{\partial \varphi}{\partial \tau} = i p_0 x - x_0 \left(-(x - f_x) + i f_p \right) \tag{164}$$

Collect all terms proportional to x

$$\frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} = i p_0 + x_0 \tag{165}$$

This is solved for

$$\frac{\partial f_x}{\partial \tau} = x_0 \quad \Rightarrow \quad f_x = x_0 \tau \tag{166}$$

$$\frac{\partial f_p}{\partial \tau} = p_0 \quad \Rightarrow \quad f_p = p_0 \tau \tag{167}$$

Collect all terms do not contain x yields

$$-f_x \frac{\partial f_x}{\partial \tau} - i \frac{\partial \varphi}{\partial \tau} = -x_0 f_x - i x_0 f_p \tag{168}$$

which is solved for

$$\varphi(\tau) = \frac{1}{2} x_0 p_0 \tau^2 \tag{169}$$

With $\tau = 1$, this gives the phase factor $\exp\left(-\frac{i}{2} x_0 p_0\right)$.

5.2 Dynamics of Coherent States

For the dynamics induced by $U_0(t) = \exp\left(-i\nu a^\dagger a t\right)$, one obtains

$$\begin{aligned} U_0(t) \left|\alpha\right\rangle =& U_0(t) \exp\left(\alpha a^\dagger - \alpha^* a\right) \left|0\right\rangle \\ =& U_0(t) \exp\left(\alpha a^\dagger - \alpha^* a\right) U_0^\dagger(t) U_0(t) \left|0\right\rangle \\ =& U_0(t) \exp\left(\alpha a^\dagger - \alpha^* a\right) U_0^\dagger(t) \left|0\right\rangle \\ =& \exp\left[\alpha U_0(t) a^\dagger U_0^\dagger(t) - \alpha^* U_0(t) a U_0^\dagger(t)\right] \left|0\right\rangle \\ =& \exp\left(\alpha a^\dagger \mathrm{e}^{-i\nu t} + \alpha^* a \mathrm{e}^{i\nu t}\right) \left|0\right\rangle \\ =& D\left(\alpha \mathrm{e}^{-i\nu t}\right) \left|0\right\rangle = \left|\alpha \mathrm{e}^{-i\nu t}\right\rangle \end{aligned} \tag{170}$$

5.3 Light-Matter Interaction with Coherent States

Coherent states are eigenstates to the annihilation operator a.

$$\begin{aligned} a\left|\alpha\right\rangle =& \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}}a\left|\mu\right\rangle \\ =&\alpha\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^{\mu-1}}{\sqrt{(\mu-1)!}}\left|\mu-1\right\rangle = \alpha\left|\alpha\right\rangle \end{aligned} \tag{171}$$

Similarly

$$\left\langle\alpha\right| a^\dagger = \alpha^* \left\langle\alpha\right| \tag{172}$$

Coherent states are not orthogonal to each other

$$\begin{aligned} \langle\alpha|\beta\rangle =& \left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_\mu \frac{(\alpha^*)^\mu}{\sqrt{\mu!}}\left\langle\mu\right|\right)\left(\exp\left(-\frac{|\beta|^2}{2}\right)\sum_\nu \frac{\beta^\nu}{\sqrt{\nu!}}\left|\nu\right\rangle\right) \\ =& \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_{\mu,\nu}\frac{(\alpha^*)^\mu\beta^\nu}{\sqrt{\mu!}\sqrt{\nu!}}\langle\mu|\nu\rangle \\ =& \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_\mu \frac{(\alpha^*\beta)^\mu}{\mu!} \end{aligned} \tag{173}$$

Now we want to find the eigenvector $\left|\Psi\right\rangle$ of $a^\dagger$

$$a^\dagger\left|\Psi\right\rangle = \lambda\left|\Psi\right\rangle = \left|\tilde{\Psi}\right\rangle \tag{174}$$

The normalised vector

$$\frac{|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}} \tag{175}$$

and

$$\left|\frac{\langle\Psi|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}}\right| = 1 \tag{176}$$

Normalising $a^\dagger\left|\alpha\right\rangle$ yields

$$\frac{a^\dagger\left|\alpha\right\rangle}{\sqrt{\left\langle\alpha\right| aa^\dagger\left|\alpha\right\rangle}} = \frac{\alpha^\dagger\left|\alpha\right\rangle}{\sqrt{|\alpha^2|+1}} \tag{177}$$

and

$$\frac{\left\langle\alpha\right| a^\dagger\left|\alpha\right\rangle}{\sqrt{\left\langle\alpha\right| aa^\dagger\left|\alpha\right\rangle}} = \frac{\alpha^*}{\sqrt{|\alpha^2|+1}} \tag{178}$$

In the limit $|\alpha|\to\infty$

$$\frac{\alpha^*}{\sqrt{|\alpha^2|+1}} \to \frac{\alpha^*}{|\alpha|} \tag{179}$$

with

$$\left|\frac{\alpha^*}{|\alpha|}\right| = 1 \tag{180}$$

The relation

$$a^\dagger |\alpha\rangle \simeq \alpha^* |\alpha\rangle \tag{181}$$

is thus a good approximation for $|\alpha| \ll 1$.

6 Coherent States

For fock states $|\mu\rangle$, the expectation values for x and p vanish

$$\langle\mu| x |\mu\rangle = \frac{1}{\sqrt{2}}(\langle\mu| a |\mu\rangle + \langle\mu| a^\dagger |\mu\rangle) = 0 \tag{182}$$

$$\langle\mu| p |\mu\rangle = \frac{-i}{\sqrt{2}}(\langle\mu| a |\mu\rangle - \langle\mu| a^\dagger |\mu\rangle) = 0 \tag{183}$$

and the fluctuations

$$\langle\mu| x^2 |\mu\rangle = \frac{1}{2}(\langle\mu| a^2 |\mu\rangle + \langle\mu| aa^\dagger |\mu\rangle + \langle\mu| a^\dagger a |\mu\rangle + \langle\mu| a^\dagger a^\dagger |\mu\rangle) = \mu + \frac{1}{2} \tag{184}$$

$$\langle\mu| p^2 |\mu\rangle = -\frac{1}{2}(\langle\mu| a^2 |\mu\rangle - \langle\mu| aa^\dagger |\mu\rangle - \langle\mu| a^\dagger a |\mu\rangle + \langle\mu| a^\dagger a^\dagger |\mu\rangle) = \mu + \frac{1}{2} \tag{185}$$

For the ground state

$$\langle 0| x |0\rangle = \langle 0| p |0\rangle = 0 \tag{186}$$

$$\langle 0| x^2 |0\rangle = \langle 0| p^2 |0\rangle = \frac{1}{2} \tag{187}$$

This yields

$$\Delta x \Delta p = (\langle 0| x^2 |0\rangle - (\langle 0| x |0\rangle)^2)(\langle 0| p^2 |0\rangle - (\langle 0| p |0\rangle)^2) = \frac{1}{4} \tag{188}$$

which is the minimal allowed uncertainty. We can generate different states with the same uncertainty, by displacing the vacuum in phase space. The *displacement operator* is defined as

$$\boxed{D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right)} \tag{189}$$

The coherent state

$$\begin{aligned}
|\alpha\rangle =& D(\alpha) |0\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right) |0\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp\left(\alpha a^\dagger\right) \exp(-\alpha^* a) |0\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp\left(\alpha a^\dagger\right) |0\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_\mu \frac{\alpha^\mu (a^\dagger)^\mu}{\mu!} |0\rangle \\
=& \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}} |\mu\rangle
\end{aligned} \tag{190}$$

The probability to find μ photons is thus given by the Poisson distribution.

$$P(\mu) = \exp\left(-|\alpha|^2\right) \frac{(|\alpha|^2)^\mu}{\mu!} \tag{191}$$

For the expectation value of x and p with respect to a general state $|\Psi\rangle$ one has

$$(\langle\Psi| D^\dagger(\alpha))x(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)xD(\alpha))|\Psi\rangle \tag{192}$$

$$(\langle\Psi| D^\dagger(\alpha))p(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)pD(\alpha))|\Psi\rangle \tag{193}$$

then we calculate

$$D^\dagger(\alpha)xD(\alpha) = x + \frac{\alpha+\alpha^*}{\sqrt{2}} = x + x_0 \tag{194}$$

$$D^\dagger(\alpha)pD(\alpha) = p - i\frac{\alpha-\alpha^*}{\sqrt{2}} = p + p_0 \tag{195}$$

We verify the uncertainty in position and momentum of any coherent state

$$\begin{aligned}\langle\alpha| x^2 |\alpha\rangle - (\langle\alpha| x |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)x^2D(\alpha)|0\rangle - (\langle 0| D^\dagger(\alpha)xD(\alpha)|0\rangle)^2 \\ &= \langle 0| x^2 |0\rangle - (\langle 0| x |0\rangle)^2\end{aligned} \tag{196}$$

$$\begin{aligned}\langle\alpha| p^2 |\alpha\rangle - (\langle\alpha| p |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)p^2D(\alpha)|0\rangle - (\langle 0| D^\dagger(\alpha)pD(\alpha)|0\rangle)^2 \\ &= \langle 0| p^2 |0\rangle - (\langle 0| p |0\rangle)^2\end{aligned} \tag{197}$$

6.1 Coherent States in Real-Space Representation

The character of the displacement operator can be exemplified in the real-space representation of wave functions.

$$\Psi_\alpha(x) = \langle x|\alpha\rangle \propto \exp\left(-\frac{1}{2}(x-x_0)^2 + ip_0x - \frac{i}{2}x_0p_0\right) \tag{198}$$

with $x_0 = (\alpha+\alpha^*)/\sqrt{2}$ and $p_0 = (\alpha-\alpha^*)/(\sqrt{2}i)$. It is convenient to define the vector

$$|\alpha,\tau\rangle = \exp\left[(\alpha a^\dagger - \alpha^* a)\tau\right]|0\rangle \tag{199}$$

with additional scalar parameter τ. It satisfies the differential equation

$$\frac{\partial|\alpha,\tau\rangle}{\partial\tau} = (\alpha a^\dagger - \alpha^* a)|\alpha,\tau\rangle \tag{200}$$

The real-space representation of the operator $(\alpha a^\dagger - \alpha^* a)$ reads

$$\frac{1}{\sqrt{2}}\left[\alpha\left(x-\frac{\partial}{\partial x}\right) - \alpha^*\left(x+\frac{\partial}{\partial x}\right)\right] = \frac{\alpha-\alpha^*}{\sqrt{2}}x - \frac{\alpha+\alpha^*}{\sqrt{2}}\frac{\partial}{\partial x} = ip_0x - x_0\frac{\partial}{\partial x} \tag{201}$$

We thus need to solve the equation

$$\frac{\partial\Phi}{\partial\tau} = \left(ip_0x - x_0\frac{\partial}{\partial x}\right)\Phi \tag{202}$$

with the Ansatz

$$\Phi(\tau) = \exp\left(-\frac{1}{2}(x-f_x)^2 + if_px - i\varphi\right) \tag{203}$$

The initial conditions are $f_x(0) = f_p(0) = \varphi(0) = 0$. The derivatives

$$\frac{\partial \Phi(\tau)}{\partial \tau} = \left((x - f_x)\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau}x - i\frac{\partial \varphi}{\partial \tau} \right) \Phi(\tau) \tag{204}$$

$$\frac{\partial \Phi(\tau)}{\partial x} = \left(-(x - f_x) + if_p\right) \Phi(\tau) \tag{205}$$

This yields

$$(x - f_x)\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau}x - i\frac{\partial \varphi}{\partial \tau} = ip_0x - x_0\left(-(x - f_x) + if_p\right) \tag{206}$$

Collect all terms proportional to x

$$\frac{\partial f_x}{\partial \tau} + i\frac{\partial f_p}{\partial \tau} = ip_0 + x_0 \tag{207}$$

This is solved for

$$\frac{\partial f_x}{\partial \tau} = x_0 \quad \Rightarrow \quad f_x = x_0\tau \tag{208}$$

$$\frac{\partial f_p}{\partial \tau} = p_0 \quad \Rightarrow \quad f_p = p_0\tau \tag{209}$$

Collect all terms do not contain x yields

$$-f_x\frac{\partial f_x}{\partial \tau} - i\frac{\partial \varphi}{\partial \tau} = -x_0f_x - ix_0f_p \tag{210}$$

which is solved for

$$\varphi(\tau) = \frac{1}{2}x_0p_0\tau^2 \tag{211}$$

With $\tau = 1$, this gives the phase factor $\exp\left(-\frac{i}{2}x_0p_0\right)$.

6.2 Dynamics of Coherent States

For the dynamics induced by $U_0(t) = \exp\left(-i\nu a^\dagger a t\right)$, one obtains

$$\begin{aligned}
U_0(t)\left|\alpha\right\rangle =& U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)\left|0\right\rangle \\
=& U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)U_0^\dagger(t)U_0(t)\left|0\right\rangle \\
=& U_0(t)\exp\left(\alpha a^\dagger - \alpha^* a\right)U_0^\dagger(t)\left|0\right\rangle \\
=& \exp\left[\alpha U_0(t)a^\dagger U_0^\dagger(t) - \alpha^* U_0(t)aU_0^\dagger(t)\right]\left|0\right\rangle \\
=& \exp\left(\alpha a^\dagger \mathrm{e}^{-i\nu t} + \alpha^* a\mathrm{e}^{i\nu t}\right)\left|0\right\rangle \\
=& D\left(\alpha\mathrm{e}^{-i\nu t}\right)\left|0\right\rangle = \left|\alpha\mathrm{e}^{-i\nu t}\right\rangle
\end{aligned} \tag{212}$$

6.3 Light-Matter Interaction with Coherent States

Coherent states are eigenstates to the annihilation operator a.

$$\begin{aligned} a\left|\alpha\right\rangle =&\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^\mu}{\sqrt{\mu!}}a\left|\mu\right\rangle \\ =&\alpha\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_\mu \frac{\alpha^{\mu-1}}{\sqrt{(\mu-1)!}}\left|\mu-1\right\rangle = \alpha\left|\alpha\right\rangle \end{aligned} \tag{213}$$

Similarly

$$\left\langle\alpha\right| a^\dagger = \alpha^*\left\langle\alpha\right| \tag{214}$$

Coherent states are not orthogonal to each other

$$\begin{aligned} \langle\alpha|\beta\rangle =& \left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_\mu \frac{(\alpha^*)^\mu}{\sqrt{\mu!}}\left\langle\mu\right|\right)\left(\exp\left(-\frac{|\beta|^2}{2}\right)\sum_\nu \frac{\beta^\nu}{\sqrt{\nu!}}\left|\nu\right\rangle\right) \\ =&\exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_{\mu,\nu}\frac{(\alpha^*)^\mu\beta^\nu}{\sqrt{\mu!}\sqrt{\nu!}}\langle\mu|\nu\rangle \\ =&\exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_\mu \frac{(\alpha^*\beta)^\mu}{\mu!} \end{aligned} \tag{215}$$

Now we want to find the eigenvector $\left|\Psi\right\rangle$ of $a^\dagger$

$$a^\dagger\left|\Psi\right\rangle = \lambda\left|\Psi\right\rangle = \left|\tilde{\Psi}\right\rangle \tag{216}$$

The normalised vector

$$\frac{|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}} \tag{217}$$

and

$$\left|\frac{\langle\Psi|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}}\right| = 1 \tag{218}$$

Normalising $a^\dagger\left|\alpha\right\rangle$ yields

$$\frac{a^\dagger\left|\alpha\right\rangle}{\sqrt{\left\langle\alpha\right| aa^\dagger\left|\alpha\right\rangle}} = \frac{\alpha^\dagger\left|\alpha\right\rangle}{\sqrt{|\alpha^2|+1}} \tag{219}$$

and

$$\frac{\left\langle\alpha\right| a^\dagger\left|\alpha\right\rangle}{\sqrt{\left\langle\alpha\right| aa^\dagger\left|\alpha\right\rangle}} = \frac{\alpha^*}{\sqrt{|\alpha^2|+1}} \tag{220}$$

In the limit $|\alpha|\to\infty$

$$\frac{\alpha^*}{\sqrt{|\alpha^2|+1}} \to \frac{\alpha^*}{|\alpha|} \tag{221}$$

with

$$\left|\frac{\alpha^*}{|\alpha|}\right| = 1 \tag{222}$$

The relation

$$a^\dagger |\alpha\rangle \simeq \alpha^* |\alpha\rangle \tag{223}$$

is thus a good approximation for $|\alpha| \ll 1$.

Part II
Technique

7 The Wigner Quasi-Probability Distribution

Let's start with a classical property that we would like to be fulfilled

$$\int_{-\infty}^{\infty} \mathrm{d}P W(X,P) = \mathrm{Pr}(X), \qquad \int_{-\infty}^{\infty} \mathrm{d}X W(X,P) = \mathrm{Pr}(P) \tag{224}$$

The probability distribution of any quadrature is called a 'marginal'. We can generalize the marginal equations above into a single expression to include rotation of the harmonic oscillator in its phase space. We then have

$$\mathrm{Pr}(X,\theta) = \langle X|\, U(\theta)\rho U^\dagger(\theta)\, |X\rangle = \int_{-\infty}^{\infty} \mathrm{d}P W(X\cos\theta - P\sin\theta, X\sin\theta + P\cos\theta) \tag{225}$$

where $U(\theta) = \exp\left(-i\theta a^\dagger a\right)$ is the rotation operator.

7.1 A Derivation of Wigner's Classic Formula

To start our derivation we introduce two quantities. First the 'characteristic function', *i.e.* the two-dimensional Fourier transform of the Wigner function

$$\tilde{W}(U,V) = \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}P W(X,P)\mathrm{e}^{-iUX-iVP} \tag{226}$$

second, the Fourier-transformed probability distribution

$$\tilde{\mathrm{Pr}}(\xi,\theta) = \int_{-\infty}^{\infty} \mathrm{d}X \mathrm{Pr}(X,\theta)\mathrm{e}^{-i\xi X} \tag{227}$$

We use the second part of the eqn.(225)

$$\begin{aligned}\tilde{\mathrm{Pr}}(\xi,\theta) =& \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}P W(X\cos\theta - P\sin\theta, X\sin\theta + P\cos\theta)\mathrm{e}^{-i\xi X} \\ =& \tilde{W}(\xi\cos\theta, \xi\sin\theta)\end{aligned} \tag{228}$$

using the first part of the eqn.(225)

$$\begin{aligned}\tilde{\mathrm{Pr}}(\xi,\theta) &= \int_{-\infty}^{\infty} \mathrm{d}X \,\langle X|\, U(\theta)\rho U^\dagger(\theta)\,|X\rangle\, \mathrm{e}^{-i\xi X} \\ &= \int_{-\infty}^{\infty} \mathrm{d}X \,\langle X|\, \rho U^\dagger(\theta)\mathrm{e}^{-i\xi X} U(\theta)\,|X\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}X \,\langle X|\, \rho \exp(-iX\xi\cos\theta - iP\xi\sin\theta)\,|X\rangle \\ &= \mathrm{Tr}(\rho\exp(-iX\xi\cos\theta - iP\xi\sin\theta))\end{aligned} \tag{229}$$

Let $U = \xi\cos\theta$ and $V = \xi\sin\theta$. We thus have our next important result

$$\boxed{\tilde{W}(U,V) = \mathrm{Tr}(\rho\exp(-iUX - iVP))} \tag{230}$$

Using the Baker-Campbell-Hausdorff formula

$$\exp(-iUX - iVP) = \exp(iUV/2)\exp(-iUX)\exp(-iVP) \tag{231}$$

so

$$\begin{aligned}\tilde{W}(U,V) &= \exp(iUV/2)\int_{-\infty}^{\infty} \mathrm{d}X \, \langle X|\,\rho\exp(-iUX)\exp(-iVP)\,|X\rangle \\ &= \exp(iUV/2)\int_{-\infty}^{\infty} \mathrm{d}X \, \langle X|\,\rho\exp(-iUX)\,|X+V\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}Q\mathrm{e}^{-iUQ} \, \langle Q-V/2|\,\rho\,|Q+V/2\rangle \qquad (X = Q - V/2)\end{aligned} \tag{232}$$

Lastly, we do a inverse-Fourier transform to obtain the Wigner function

$$\begin{aligned}W(X,P) =& \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}U \int_{-\infty}^{\infty} \mathrm{d}V \tilde{W}(U,V)\mathrm{e}^{iUX+iVP} \\ =& \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}U \int_{-\infty}^{\infty} \mathrm{d}V \int_{-\infty}^{\infty} \mathrm{d}Q \left\langle Q-\frac{V}{2}\right|\rho\left|Q+\frac{V}{2}\right\rangle \mathrm{e}^{-iUQ+iUX+iVP} \\ =& \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}V \int_{-\infty}^{\infty} \mathrm{d}Q \left\langle Q-\frac{V}{2}\right|\rho\left|Q+\frac{V}{2}\right\rangle \mathrm{e}^{iVP}\int_{-\infty}^{\infty} \mathrm{d}U\mathrm{e}^{iU(X-Q)} \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}V \int_{-\infty}^{\infty} \mathrm{d}Q \left\langle Q-\frac{V}{2}\right|\rho\left|Q+\frac{V}{2}\right\rangle \mathrm{e}^{iVP}\delta(X-Q) \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}V\mathrm{e}^{iPV} \left\langle X-\frac{V}{2}\right|\rho\left|X+\frac{V}{2}\right\rangle\end{aligned} \tag{233}$$

This equation is Wigner's now famous formula

$$\boxed{W(X,P) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}V\mathrm{e}^{iPV} \left\langle X-\frac{V}{2}\right|\rho\left|X+\frac{V}{2}\right\rangle} \tag{234}$$

7.2 Properties of the Wigner Function

7.3 Examples of the Wigner Function

Here can put my codes

8 Optical Homodyne and Heterodyne Detection

8.1 Balanced Homodyne Detection

Two coherent light beams with complex amplitudes α, β transform through the beam-splitter. In classical optics

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = B \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{235}$$

In quantum optics, the complex amplitudes α, β correspond to the annihilation operators $\hat{a}$ and $\hat{b}$ of the incident fields. For $50:50$ beam-splitter

$$\begin{pmatrix} \hat{a}' \\ \hat{b}' \end{pmatrix} = B \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{a} + i\hat{b} \\ \hat{b} + i\hat{a} \end{pmatrix} \tag{236}$$

Then we calculate the number operator observed by the photondiodes

$$\hat{a}'^{\dagger}\hat{a}' = \frac{1}{2}(\hat{a}^{\dagger} - i\hat{b}^{\dagger})(\hat{a} + i\hat{b}) = \frac{1}{2}(\hat{a}^{\dagger}\hat{a} + i\hat{a}^{\dagger}\hat{b} - i\hat{b}^{\dagger}\hat{a} + \hat{b}^{\dagger}\hat{b}) \tag{237}$$

$$\hat{b}'^{\dagger}\hat{b}' = \frac{1}{2}(\hat{b}^{\dagger} - i\hat{a}^{\dagger})(\hat{b} + i\hat{a}) = \frac{1}{2}(\hat{a}^{\dagger}\hat{a} - i\hat{a}^{\dagger}\hat{b} + i\hat{b}^{\dagger}\hat{a} + \hat{b}^{\dagger}\hat{b}) \tag{238}$$

In a balanced detector, these two photo-currents are subtracted, yielding the 'difference current'

$$i_{-} \propto b'^{\dagger}b' - a'^{\dagger}a' = ib^{\dagger}a - ia^{\dagger}b \tag{239}$$

Now we consider a is the signal and b is the reference, and it is also called a *local oscillator* (LO). We assume that the LO is powerful enough to be treated classically, *i.e.* we neglect totally the quantum fluctuations of the LO.

$$b \;\rightarrow\; \alpha_{\mathrm{LO}} = |\alpha_{\mathrm{LO}}| \mathrm{e}^{i\pi/2} \mathrm{e}^{i\theta} \tag{240}$$

where we introduce the phase θ in a convenient way to absorb the factor of i that came from our convention for the phase in the beam-splitter operator. After this

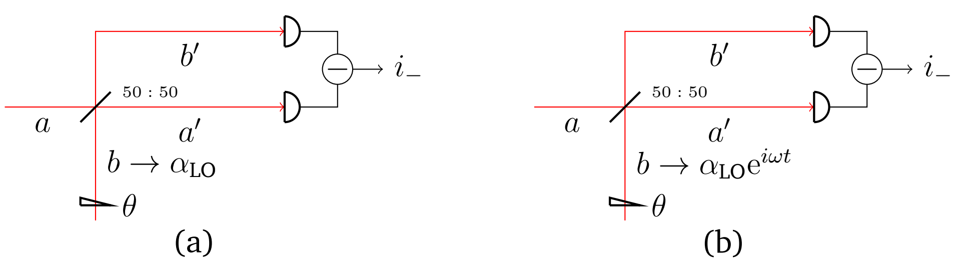

Figure 6: (a) Schematic for an optical homodyne detector. Mode b is put in a strong coherent state α_{LO} and is mixed on a beam-splitter with mode a that we wish to measure. (b) Schematic for an optical heterodyne detection.

transformation, the difference current becomes

$$
\begin{aligned}
i_- &\propto |\alpha_{\rm LO}|(a\mathrm{e}^{-i\theta} + a^\dagger \mathrm{e}^{i\theta}) \\
&= |\alpha_{\rm LO}| \left(a(\cos\theta - i\sin\theta) + a^\dagger(\cos\theta + i\sin\theta)\right) \\
&= |\alpha_{\rm LO}| \left((a + a^\dagger)\cos\theta + i(a^\dagger - a)\sin\theta\right) \\
&= \sqrt{2}|\alpha_{\rm LO}|(\hat{X}\cos\theta + \hat{P}\sin\theta) \\
&= \sqrt{2}|\alpha_{\rm LO}|\hat{X}_\theta
\end{aligned}
\tag{241}
$$

A balanced homodyne detector measures the quadrature component $\hat{X}_\theta$.

8.2 Heterodyne Detection

The heterodyne detection is similar to the homodyne detection however the LO has a different frequency.

$$
\begin{aligned}
i_- \propto &|\alpha_{\rm LO}| \left(a\mathrm{e}^{-i\theta}\mathrm{e}^{-i\omega t} + a^\dagger \mathrm{e}^{i\theta}\mathrm{e}^{i\omega t}\right) \\
\propto &\sqrt{2}|\alpha_{\rm LO}| \left(X\cos(\omega t + \theta) + P\sin(\omega t + \theta)\right)
\end{aligned}
\tag{242}
$$

8.3 Dual Homodyne Detection

From the discussions above, we can write the difference currents $i_-^{(X)}$ and $i_-^{(P)}$

$$
\begin{aligned}
i_-^{(X)} \propto &|\alpha_{\rm LO}^{(X)}|(\hat{a}' + \hat{a}'^\dagger) = \frac{1}{\sqrt{2}}|\alpha_{\rm LO}^{(X)}|(\hat{a} + i\hat{v} + \hat{a}^\dagger - i\hat{v}^\dagger) \\
= &|\alpha_{\rm LO}^{(X)}| \left[\frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) - \frac{i}{\sqrt{2}}(\hat{v}^\dagger - \hat{v})\right] = |\alpha_{\rm LO}^{(X)}|(\hat{X}_a - \hat{P}_v)
\end{aligned}
\tag{243}
$$

$$
\begin{aligned}
i_-^{(P)} \propto &|\alpha_{\rm LO}^{(P)}|(\hat{v}' + \hat{v}'^\dagger) = \frac{1}{\sqrt{2}}|\alpha_{\rm LO}^{(P)}|(\hat{v} + i\hat{a} + \hat{v}^\dagger - i\hat{a}^\dagger) \\
= &|\alpha_{\rm LO}^{(P)}| \left[\frac{1}{\sqrt{2}}(\hat{v} + \hat{v}^\dagger) - \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})\right] = |\alpha_{\rm LO}^{(P)}|(\hat{X}_v - \hat{P}_a)
\end{aligned}
\tag{244}
$$

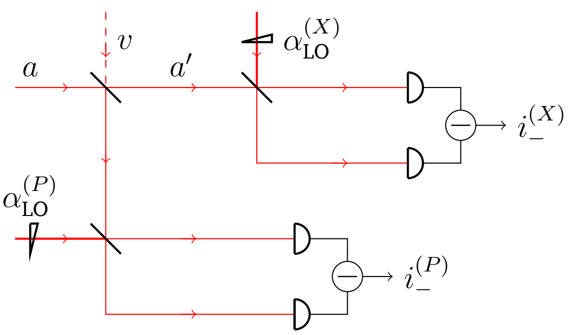

Figure 7: Dual homodyne detection. 'v' denotes vacuum fluctuations.

9 Optical Homodyne Tomography

This chapter build in your understanding of squeezing states

10 Photon Counting Statistics

- Sub-Poissonian statistics: $\Delta^2 n < \bar{n}$
- Poissonian statistics: $\Delta^2 n = \bar{n}$
- Super-Poissonian statistics: $\Delta^2 n > \bar{n}$

10.1 Poissonian Statistics

Coherent states have Poissonian statistics

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle \tag{245}$$

with amplitude

$$\langle n|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \tag{246}$$

Then we get the probability

$$\mathrm{Pr}(n) = |\langle n|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \tag{247}$$

The mean photon number in a coherent state is

$$\bar{n} = \langle\alpha| a^\dagger a |\alpha\rangle = |\alpha|^2 \tag{248}$$

so we can write

$$\boxed{\mathrm{Pr}(n) = \mathrm{e}^{-\bar{n}} \frac{\bar{n}^n}{n!}} \tag{249}$$

The expression is the *Poisson* distribution. Now we calculate the variance

$$\begin{aligned}
\Delta^2 n =& \langle n^2\rangle - \langle n\rangle^2 \\
=& \langle\alpha| (a^\dagger a)^2 |\alpha\rangle - \langle\alpha| a^\dagger a |\alpha\rangle^2 \\
=& |\alpha|^2 \langle\alpha| a a^\dagger |\alpha\rangle - \langle\alpha| a^\dagger a |\alpha\rangle^2 \\
=& |\alpha|^2 \langle\alpha| (1 + a^\dagger a) |\alpha\rangle - \langle\alpha| a^\dagger a |\alpha\rangle^2 = |\alpha|^2 = \bar{n}
\end{aligned} \tag{250}$$

10.2 Super-Poissonian Statistics

One of the example is the thermal state. Consider the Boltzmann distribution

$$\mathrm{Pr}(n) = \frac{\exp(-n\hbar\omega/k_B T)}{\sum_{n=0}^{\infty} \exp(-n\hbar\omega/k_B T)} = \frac{x^n}{\sum_{n=0}^{\infty} x^n} \tag{251}$$

where $x = \exp(-\hbar\omega/k_B T)$. If $\hbar\omega \gg k_B T$, *i.e.* x is small, then $\sum_n x^n = 1/(1-x)$. The probability becomes

$$\mathrm{Pr}(n) = (1-x)x^n \tag{252}$$

The mean photon number

$$\bar{n} = \sum_n n x^n (1-x) = \frac{x}{1-x} = \frac{1}{\exp(\hbar\omega/k_B T) - 1} \tag{253}$$

from this, we have

$$x = \frac{\bar{n}}{1+\bar{n}} \tag{254}$$

so

$$\boxed{\Pr(n) = \frac{1}{1+\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}} \right)^n} \tag{255}$$

which is called the *Bose-Einstein* distribution. Now we calculate the variance

$$\Delta^2 n = \bar{n} + \bar{n}^2 \tag{256}$$

Thus, we see that a thermal state exhibits super-Poissonian statistics.

10.3 Sub-Poissonian Statistics

Consider the Fock state $|n\rangle$. The Fock states have a mean photon number

$$\bar{n} = \langle n| a^\dagger a |n\rangle = n \tag{257}$$

and a variance

$$\Delta^2 n = \langle n| (a^\dagger a)^2 |n\rangle - \langle n| a^\dagger a |n\rangle = 0 \tag{258}$$