

## NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF PHYSICS

---

# Quantum Optics

---

*Author:*

Chen Huang

*Email:*

chen.huang23@imperial.ac.uk

Date: November 20, 2023

## Contents

<b>1</b>	<b>A Quantum Mechanics Atom in a Classical Light Field</b>	<b>3</b>
1.1	Dynamics of Atom in Light-Field . . . . .	3
1.1.1	The Propagator . . . . .	3
1.1.2	Perturbation Theory . . . . .	4
1.1.3	Atom-Light Hamiltonian . . . . .	4
1.1.4	The Pauli Matrices . . . . .	5
1.2	Hamiltonian and Propagator of a Two-Level Atom . . . . .	5
1.3	The Two-Level Atom in a Monochromatic Light Field . . . . .	6
1.3.1	Resonant Driving . . . . .	7
1.3.2	Off-Resonant Driving . . . . .	8
1.3.3	Ramsey . . . . .	8
1.4	The Three-Level Atom . . . . .	9
1.5	Bloch Equations . . . . .	11
1.6	Dynamics of the Bloch Vector . . . . .	12
1.7	Averages over Different States . . . . .	13
<b>2</b>	<b>Harmonic Oscillator</b>	<b>14</b>
<b>3</b>	<b>Quantisation of the Electromagnetic Field</b>	<b>15</b>
<b>4</b>	<b>Jaynes-Cummings</b>	<b>17</b>
4.1	Two-Dimensional Subspaces . . . . .	17
4.2	The Lambda-System . . . . .	18
<b>5</b>	<b>Coherent States</b>	<b>19</b>
5.1	Coherent States in Real-Space Representation . . . . .	20
5.2	Dynamics of Coherent States . . . . .	21
5.3	Light-Matter Interaction with Coherent States . . . . .	22
<b>6</b>	<b>Coherent States</b>	<b>24</b>
6.1	Coherent States in Real-Space Representation . . . . .	25
6.2	Dynamics of Coherent States . . . . .	26
6.3	Light-Matter Interaction with Coherent States . . . . .	27

# 1 A Quantum Mechanics Atom in a Classical Light Field

An atom is described by the Hamiltonian

$$H_a = \frac{p^2}{2m} + V(\mathbf{r}) = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \quad (1)$$

If the atom is interacting with a classical electro-magnetic field, the Hamiltonian is replaced by

$$H_A = -\frac{\hbar^2}{2m} \left( \nabla - i\frac{\rho}{\hbar} \mathbf{A} \right)^2 + V(\mathbf{r}) \quad (2)$$

Many problems are fomulated in terms of a Hamiltonian of the form

$$H_E = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \rho \mathbf{E} \cdot \mathbf{r} \quad (3)$$

In most case, the Hamiltonian can be expressed as

$$H = H_{\text{atom}} + H_{\text{interaction}} \quad (4)$$

## 1.1 Dynamics of Atom in Light-Field

### 1.1.1 The Propagator

We define the propagator  $U(t)$  via the relation

$$|\Psi(t)\rangle = U(t) |\Psi(0)\rangle \quad (5)$$

for any solution  $|\Psi(t)\rangle$  of the Schrödinger equation. By definition, the propagator satisfies the initial condition  $U(0) = \mathbb{1}$ .

The propagator also satisfies a Schrödinger equation.

$$i|\dot{\Psi}(t)\rangle = i\dot{U}(t) |\Psi(0)\rangle = HU(t) |\Psi(0)\rangle \quad (6)$$

so that

$$\boxed{i\dot{U} = HU} \quad (7)$$

The ad-joint  $U^\dagger(t)$  satisfies

$$-i\dot{U}^\dagger = U^\dagger H^\dagger = U^\dagger H \quad (8)$$

According to this relations, we have

$$i\frac{\partial}{\partial t} (UU^\dagger) = i\dot{U}U^\dagger + iU\dot{U}^\dagger = HUU^\dagger - UU^\dagger H = [H, UU^\dagger] \quad (9)$$

With the initial condition  $U(0)U^\dagger(0) = \mathbb{1}$ , this is solved for

$$U(t)U^\dagger(t) = U^\dagger(t)U(t) = \mathbb{1} \quad (10)$$

### 1.1.2 Perturbation Theory

The Schrödinger equation together with the initial condition  $U(0) = \mathbb{1}$  can be rewritten as the integral equation

$$\begin{aligned}
 U(t) &= \mathbb{1} + \int_0^t dt' \dot{U}(t') = \mathbb{1} - i \int_0^t dt' H(t') U(t') \\
 &= \mathbb{1} - i \int_0^t dt' H(t') \left[ \mathbb{1} - i \int_0^{t'} dt'' H(t'') U(t'') \right] \\
 &= \mathbb{1} - i \int_0^t dt' H(t') - \int_0^t dt' \int_0^{t'} dt'' H(t') H(t'') U(t'')
 \end{aligned} \tag{11}$$

For sufficiently short times this can be approximated as

$$U(t) \simeq \mathbb{1} - i \int_0^t dt' H(t') - \int_0^t dt' \int_0^{t'} dt'' H(t') H(t'') \tag{12}$$

For this to be a good approximation, it is essential that ‘magnitude’ of  $H$  is sufficiently **small**. It is therefore important to work in a suitable frame. Rather than solving the Schrödinger equation  $i\dot{U} = HU$  for  $U$ , we can try to solve for  $V$  defined via the relation

$$U = U_0 V \tag{13}$$

in terms of a unitary  $U_0$  that we are **free** to choose. The Schrödinger equation

$$i\dot{U} = i\dot{U}_0 V + iU_0 \dot{V} = HU_0 V \tag{14}$$

can now be solved for  $\dot{V}$  what yields

$$i\dot{V} = U_0^\dagger H U_0 V - iU_0^\dagger \dot{U}_0 V = \left( U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0 \right) V = \tilde{H} V \tag{15}$$

with the new Hamiltonian

$$\boxed{\tilde{H} = U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0} \tag{16}$$

The goal is then to find  $U_0$  such that the time-dependent perturbation theory is a good approximation.

### 1.1.3 Atom-Light Hamiltonian

Let’s consider an atom with Hamiltonian  $H_0$  and interaction Hamiltonian  $H_I$

$$H_0 = \sum_j \omega_j |\psi_j\rangle \langle \psi_j| \tag{17}$$

$$H_I = \sum_{j,k} |\psi_j\rangle \langle \psi_j| H_I |\psi_k\rangle \langle \psi_k| = \sum_{j,k} \langle \psi_j| H_I |\psi_k\rangle |\psi_j\rangle \langle \psi_k| = \sum_{j,k} h_{jk} |\psi_j\rangle \langle \psi_k| \tag{18}$$

We choose

$$U_0(t) = \exp(-iH_0 t) = \sum_j e^{-i\omega_j t} |\psi_j\rangle \langle \psi_j| \tag{19}$$

such that  $-iU_0^\dagger \dot{U}_0 = -H_0$ . Then the transformed Hamiltonian reads

$$\begin{aligned}
 \tilde{H} &= U_0^\dagger (H_0 + H_I) U_0 - iU_0^\dagger \dot{U}_0 \\
 &= U_0^\dagger H_0 U_0 + U_0^\dagger H_I U_0 - iU_0^\dagger \dot{U}_0 \\
 &= U_0^\dagger H_I U_0 \\
 &= \sum_l e^{i\omega_l t} |\psi_l\rangle \langle \psi_l| \sum_{jk} h_{jk} |\psi_j\rangle \langle \psi_k| \exp(-iH_0 t) \\
 &= \sum_{jk} h_{jk} \exp(i\omega_k t) |\psi_j\rangle \langle \psi_k| \exp(-i\omega_k t) \\
 &= \sum_{jk} h_{jk} \exp(i(\omega_j - \omega_k)t) |\psi_j\rangle \langle \psi_k|
 \end{aligned} \tag{20}$$

where  $h_{jk} = \langle \psi_j | H_{jk} | \psi_k \rangle$  is the oscillating term with frequency  $\nu$ , and  $\cos \nu t = (e^{i\nu t} + e^{-i\nu t})/2$ . The oscillating functions result in a vanishing integral in the integration  $\int_0^t dt' \dots$ . If we choose the ground state and another selected eigenstate, we can approximate the atom as a two-level system.

### 1.1.4 The Pauli Matrices

The Pauli matrices satisfies the relation

$$[\sigma_\alpha, \sigma_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \quad \{\sigma_\alpha, \sigma_\beta\} = 0 \tag{21}$$

In terms of the eigenstates  $|g\rangle$  and  $|e\rangle$ , we have

$$\sigma_z |g\rangle = -|g\rangle, \quad \sigma_z |e\rangle = |e\rangle \tag{22}$$

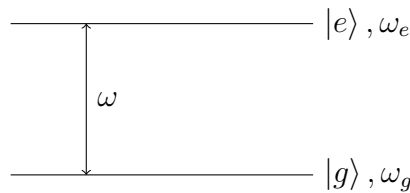
$$\sigma_x |g\rangle = |e\rangle, \quad \sigma_x |e\rangle = |g\rangle \tag{23}$$

$$\sigma_y |g\rangle = -i|e\rangle, \quad \sigma_y |e\rangle = i|g\rangle \tag{24}$$

## 1.2 Hamiltonian and Propagator of a Two-Level Atom

The Hamiltonian of a two-level atom (see Fig.1) is given by

$$\begin{aligned}
 H &= \omega_g |g\rangle \langle g| + \omega_e |e\rangle \langle e| \\
 &= \frac{\omega_e + \omega_g}{2} (|g\rangle \langle g| + |e\rangle \langle e|) + \frac{\omega_e - \omega_g}{2} (|e\rangle \langle e| - |g\rangle \langle g|) \\
 &= \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\omega}{2} \sigma_z \simeq \frac{\omega}{2} \sigma_z
 \end{aligned} \tag{25}$$



**Figure 1:** A two-level atom with resonance frequency  $\omega$ .

The corresponding propagator reads

$$U(t) = \exp\left(-i\frac{\omega}{2}\sigma_z t\right) = \mathbb{1} \cos\left(\frac{\omega}{2}t\right) - i\sigma_z \sin\left(\frac{\omega}{2}t\right) \quad (26)$$

Back to the Schrödinger equation, we have

$$\begin{aligned} i\dot{U} &= -i\frac{\omega}{2}\mathbb{1} \sin\left(\frac{\omega}{2}t\right) + \frac{\omega}{2}\sigma_z \cos\left(\frac{\omega}{2}t\right) \\ &= \frac{\omega}{2}\sigma_z \left[\mathbb{1} \cos\left(\frac{\omega}{2}t\right) - i\sigma_z \sin\left(\frac{\omega}{2}t\right)\right] = HU \end{aligned} \quad (27)$$

### 1.3 The Two-Level Atom in a Monochromatic Light Field

The Hamiltonian for the atom interacting with a light field in two-level approximation reads

$$H = \frac{\omega}{2}\sigma_z + \Omega_R \sigma_x \cos(\nu t) \quad (28)$$

The prefactor  $\Omega_R$  is called *Rabi-frequency*; it is proportional to the intensity of the light field. And  $\nu$  is frequency of light.

In order to find the solution of the Schrödinger equation, it is helpful to consider the transformation

$$U_0 = \exp\left(-i\frac{\eta}{2}\sigma_z t\right) \quad (29)$$

and

$$\dot{U}_0 = -i\frac{\eta}{2}\sigma_z \exp\left(-i\frac{\eta}{2}\sigma_z t\right) = -i\frac{\eta}{2}\sigma_z U_0 \quad (30)$$

So we have

$$U_0^\dagger \dot{U}_0 = -i\frac{\eta}{2}\sigma_z \quad (31)$$

Then we construct the transformed Hamiltonian  $\tilde{H}$ . With  $U_0^\dagger \sigma_z U_0 = \sigma_z$  and  $U_0^\dagger \sigma_x U_0 = \sigma_+ e^{i\eta t} + \sigma_- e^{-i\eta t}$ , the explicit form of  $H$  reads

$$\begin{aligned} \tilde{H} &= U_0^\dagger H U_0 - iU_0^\dagger \dot{U}_0 \\ &= \frac{\omega - \eta}{2}\sigma_z + \Omega_R (\sigma_+ e^{i\eta t} + \sigma_- e^{-i\eta t}) \cos(\nu t) \\ &= \frac{\omega - \eta}{2}\sigma_z + \frac{\Omega_R}{2} [\sigma_+ e^{i(\eta-\nu)t} + \sigma_- e^{-i(\eta-\nu)t}] + \frac{\Omega_R}{2} [\sigma_+ e^{i(\eta+\nu)t} + \sigma_- e^{-i(\eta+\nu)t}] \end{aligned} \quad (32)$$

After the *rotating wave approximation (RWA)*, we have

$$H' = \frac{\omega - \eta}{2}\sigma_z + \frac{\Omega_R}{2} [\sigma_+ e^{i(\eta-\nu)t} + \sigma_- e^{-i(\eta-\nu)t}] \quad (33)$$

Let's consider the case  $\eta = \nu$ ,

$$H' = \frac{\omega - \nu}{2}\sigma_z + \frac{\Omega_R}{2} (\sigma_+ + \sigma_-) = \frac{\omega - \nu}{2}\sigma_z + \frac{1}{2}\Omega_R \sigma_x \quad (34)$$

The associated propagator

$$\begin{aligned} \exp(-iH't) &= \mathbb{1} \cos\left(\frac{1}{2}\Omega_G t\right) - \frac{2i}{\Omega_G} H' \sin\left(\frac{1}{2}\Omega_G t\right) \\ &= \mathbb{1} \cos\left(\frac{1}{2}\Omega_G t\right) - i \left( \frac{\omega - \nu}{\Omega_G} \sigma_z + \frac{\Omega_R}{\Omega_G} \sigma_x \right) \sin\left(\frac{1}{2}\Omega_G t\right) \end{aligned} \quad (35)$$

where

$$\Omega_G = \sqrt{(\omega - \nu)^2 + \Omega_R^2} \quad (36)$$

is called *generalised Rabi frequency*. It is helpful to notice

$$(H')^2 = \frac{(\omega - \nu)^2}{4} \sigma_z^2 + \frac{1}{4} \Omega_R^2 \sigma_x^2 + \frac{1}{4} (\omega - \nu) \Omega_R \{\sigma_z, \sigma_x\} = \frac{1}{4} \Omega_G^2 \mathbb{1} \quad (37)$$

which implies that  $\mathbb{1} = \frac{4}{\Omega_G^2} (H')^2$ . Taking the time-derivative yields

$$\begin{aligned} i \frac{\partial}{\partial t} \exp(-iH't) &= -i \frac{\Omega_G}{2} \mathbb{1} \sin\left(\frac{1}{2}\Omega_G t\right) + H' \cos\left(\frac{1}{2}\Omega_G t\right) \\ &= H' \cos\left(\frac{1}{2}\Omega_G t\right) - i \frac{\Omega_G}{2} \frac{4}{\Omega_G^2} (H')^2 \sin\left(\frac{1}{2}\Omega_G t\right) \\ &= H' \left[ \mathbb{1} \cos\left(\frac{1}{2}\Omega_G t\right) - i \frac{2}{\Omega_G} H' \sin\left(\frac{1}{2}\Omega_G t\right) \right] \\ &= H' \exp(-iH't) \end{aligned} \quad (38)$$

The expression given in eqn.(35) is the correct solution of the Schrödinger equation with the Hamiltonian  $H'$ .

### 1.3.1 Resonant Driving

If the light-field is on resonance with the atomic transition, *i.e.*  $\omega - \nu = 0$ , this simplifies to

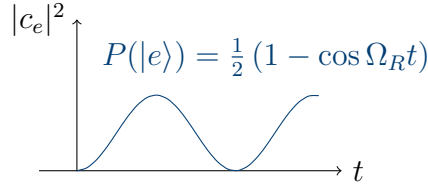
$$\exp(-iH't) = \mathbb{1} \cos\left(\frac{1}{2}\Omega_R t\right) - i \sigma_x \sin\left(\frac{1}{2}\Omega_R t\right) \quad (39)$$

Applying this to the ground state as initial state yields

$$\exp(-iH't) |g\rangle = \cos\left(\frac{1}{2}\Omega_R t\right) |g\rangle - i \sin\left(\frac{1}{2}\Omega_R t\right) |e\rangle \quad (40)$$

Together with the factor  $U_0$ , we have

$$\begin{aligned} &\exp\left(-i\frac{\omega}{2}\sigma_z t\right) \exp(-iH't) |g\rangle \\ &= \exp\left(i\frac{\omega}{2}t\right) \cos\left(\frac{\Omega_R}{2}t\right) |g\rangle - i \exp\left(-i\frac{\omega}{2}t\right) \sin\left(\frac{\Omega_R}{2}t\right) |e\rangle \\ &= \exp\left(i\frac{\omega}{2}t\right) \left[ \cos\left(\frac{\Omega_R}{2}t\right) |g\rangle - i \exp(-i\omega t) \sin\left(\frac{\Omega_R}{2}t\right) |e\rangle \right] \end{aligned} \quad (41)$$



**Figure 2:** The probability to find the atom in the excited state in Rabi oscillation.

The probability to find the atom in the excited state or ground state is given by

$$|c_e(t)|^2 = \left[ \sin \left( \frac{\Omega_R}{2} t \right) \right]^2 = \frac{1}{2} (1 - \cos \Omega_R t) \quad (42)$$

$$|c_g(t)|^2 = \left[ \cos \left( \frac{\Omega_R}{2} t \right) \right]^2 = \frac{1}{2} (1 + \cos \Omega_R t) \quad (43)$$

They are called Rabi oscillation.

### 1.3.2 Off-Resonant Driving

If the light field is far off-resonant, *i.e.*  $|\nu - \omega| \gg \Omega_R$ , the approximations

$$\frac{\omega - \nu}{\Omega_G} = \frac{\omega - \nu}{\sqrt{(\omega - \nu)^2 + \Omega_R^2}} \simeq \frac{\nu - \omega}{|\nu - \omega|} = \pm 1 \quad (44)$$

$$\frac{\Omega_R}{\Omega_G} = \frac{\Omega_R}{\sqrt{(\omega - \nu)^2 + \Omega_R^2}} \simeq \frac{\Omega_R}{|\nu - \omega|} \ll 1 \quad (45)$$

so the propagator eqn.(35) becomes

$$\exp(-iH't) = \mathbb{1} \cos \left( \frac{1}{2} \Omega_G t \right) - i \sin \left( \frac{1}{2} \Omega_G t \right) \quad (46)$$

Let  $\Omega_G$  do a Taylor expansion at  $\Omega_R = 0$

$$\Omega_G = |\nu - \omega| + \frac{\Omega_R^2}{2|\nu - \omega|} + \mathcal{O}(\Omega_R^4) \quad (47)$$

So we have

$$\Omega_G - (\omega - \nu) \simeq \frac{\Omega_R^2}{2|\delta|} \quad (48)$$

with the detuning  $\delta = \omega - \nu$ .

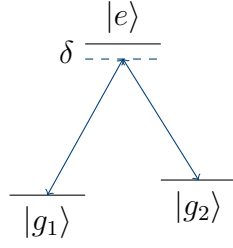
### 1.3.3 Ramsey

In the case of  $\nu = \omega$ , we found the propagator

$$U_x = \mathbb{1} \cos \left( \frac{1}{2} \Omega_R t \right) - i \sigma_x \sin \left( \frac{1}{2} \Omega_R t \right) \quad (49)$$


---





**Figure 3:** The three-level atom.

in the interaction picture. For a duration  $T = \frac{\pi}{2\Omega_R}$ , this reduce to

$$U_x(T) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_x) \quad (50)$$

Assuming the atom initially in its ground state  $|g\rangle$ , we obtain

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}(|g\rangle - i|e\rangle) \quad (51)$$

A measurement of the population of the eigenstates would yield 50% ground state and 50% excited state.

$$H_\phi = \frac{\omega}{2}\sigma_z + \Omega_R\sigma_x \cos(\nu t + \phi) \quad (52)$$

the associated propagator

$$U_\phi(T) = \frac{1}{\sqrt{2}}[\mathbb{1} - i(\cos \phi \sigma_x + \sin \phi \sigma_y)] \quad (53)$$

Applying  $U_\phi(T)$  to the state  $|\Psi(T)\rangle$

$$\begin{aligned} |\Psi(2T)\rangle &= U_\phi(T) |\Psi(T)\rangle \\ &= \frac{1}{2}[\mathbb{1} - i(\cos \phi \sigma_x + \sin \phi \sigma_y)] (|g\rangle - i|e\rangle) \\ &= -i \left( \exp\left(i\frac{\phi}{2}\right) \sin\left(\frac{\phi}{2}\right) |g\rangle + \exp\left(-i\frac{\phi}{2}\right) \cos\left(\frac{\phi}{2}\right) |e\rangle \right) \end{aligned} \quad (54)$$

The probability to find the atom in the ground state or the excited state thus oscillates with  $\phi$ .

## 1.4 The Three-Level Atom

$$H_1 = \frac{\Omega_R}{2\sqrt{2}} (|e\rangle \langle g_1| e^{i\delta t} + |g_1\rangle \langle e| e^{-i\delta t}) \quad (55)$$

$$H_2 = \frac{\Omega_R}{2\sqrt{2}} (|e\rangle \langle g_2| e^{i\delta t} + |g_2\rangle \langle e| e^{-i\delta t}) \quad (56)$$


---

$$\begin{aligned}
H = H_1 + H_2 &= \frac{\Omega_R}{2} \left( |e\rangle \frac{\langle g_1| + \langle g_2|}{\sqrt{2}} e^{i\delta t} + \frac{|g_1\rangle + |g_2\rangle}{\sqrt{2}} \langle e| e^{-i\delta t} \right) \\
&= \frac{\Omega_R}{2} (|e\rangle \langle g| e^{i\delta t} + |g\rangle \langle e| e^{-i\delta t}) \\
&= \frac{\Omega_R}{4} [(\sigma_x - i\sigma_y)(\cos \delta t + i \sin \delta t) + (\sigma_x + i\sigma_y)(\cos \delta t - i \sin \delta t)] \\
&= \frac{\Omega_R}{2} (\sigma_x \cos \delta t + \sigma_y \sin \delta t)
\end{aligned} \tag{57}$$

where  $|g\rangle = (|g_1\rangle + |g_2\rangle)/\sqrt{2}$ . The propagator

$$U(t) \simeq \mathbb{I} - i \int_0^t dt_1 H(t_1) - \int_0^t dt_1 \int_0^{t_1} dt_2 H(t_1) H(t_2) \tag{58}$$

Now we calculate  $U(t)$  at  $t = 2\pi/\delta$ . The first order

$$U^{(1)} \left( \frac{2\pi}{\delta} \right) = -i \frac{\Omega_R}{2} \int_0^{2\pi/\delta} dt_1 (\sigma_x \cos \delta t_1 + \sigma_y \sin \delta t_1) = 0 \tag{59}$$

and the second

$$U^{(2)} \left( \frac{2\pi}{\delta} \right) = -\frac{\Omega_R^2}{4} \int_0^{2\pi/\delta} dt_1 \int_0^{t_1} dt_2 (\sigma_x \cos \delta t_1 + \sigma_y \sin \delta t_1) (\sigma_x \cos \delta t_2 + \sigma_y \sin \delta t_2) \tag{60}$$

To do so, we need some integrals

$$\int_0^{\frac{2\pi}{\delta}} dt_1 \int_0^{t_1} dt_2 \cos(\delta t_1) \cos(\delta t_2) = 0 \tag{61}$$

$$\int_0^{\frac{2\pi}{\delta}} dt_1 \int_0^{t_1} dt_2 \sin(\delta t_1) \sin(\delta t_2) = 0 \tag{62}$$

$$\int_0^{\frac{2\pi}{\delta}} dt_1 \int_0^{t_1} dt_2 \cos(\delta t_1) \sin(\delta t_2) = -\frac{1}{2\delta} \frac{2\pi}{\delta} \tag{63}$$

$$\int_0^{\frac{2\pi}{\delta}} dt_1 \int_0^{t_1} dt_2 \sin(\delta t_1) \cos(\delta t_2) = \frac{1}{2\delta} \frac{2\pi}{\delta} \tag{64}$$

We thus obtain the perturbative expression

$$U \left( \frac{2\pi}{\delta} \right) \simeq \mathbb{I} + \frac{\Omega_R^2}{4} \frac{1}{2\delta} [\sigma_x, \sigma_y] \frac{2\pi}{\delta} = \mathbb{I} + i \frac{\Omega_R^2}{4\delta} \sigma_z \frac{2\pi}{\delta} \tag{65}$$

The propagator looks as if it was induced by the Hamiltonian

$$H_e = -\frac{\Omega_R^2}{4\delta} \sigma_z = \frac{\Omega_e}{2} \sigma_z \tag{66}$$

Now we can return to the explicit three levels, and obtain

$$\begin{aligned}
H_e &= \frac{\Omega_e}{2} (|e\rangle \langle e| - |g\rangle \langle g|) \\
&= \frac{\Omega_e}{2} \left( |e\rangle \langle e| - \frac{1}{2} (|g_1\rangle \langle g_1| + |g_1\rangle \langle g_2| + |g_2\rangle \langle g_1| + |g_2\rangle \langle g_2|) \right)
\end{aligned} \tag{67}$$


---

The eigenstates are  $|e\rangle$ ,  $|g\rangle$  and  $(|g_1\rangle - |g_2\rangle)/\sqrt{2}$

$$H_e |e\rangle = \frac{\Omega_e}{2} |e\rangle \quad (68)$$

$$H_e |g\rangle = -\frac{\Omega_e}{2} |g\rangle \quad (69)$$

$$H_e \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}} = 0 \quad (70)$$

with the eigenvalues  $\frac{\Omega_e}{2}$ ,  $-\frac{\Omega_e}{2}$  and 0, we can now express the propagator as

$$\exp(-iH_e t) = \exp\left(-i\frac{\Omega_e}{2}t\right) |e\rangle \langle e| + \exp\left(i\frac{\Omega_e}{2}t\right) |g\rangle \langle g| + \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}} \frac{\langle g_1| - \langle g_2|}{\sqrt{2}} \quad (71)$$

Applying this to the initial state  $|g_1\rangle$  yields

$$\begin{aligned} \exp(-iH_e t) |g_1\rangle &= \exp\left(i\frac{\Omega_e}{2}t\right) \frac{|g_1\rangle + |g_2\rangle}{2} + \frac{|g_1\rangle - |g_2\rangle}{2} \\ &= \frac{1}{2} \left( \exp\left(i\frac{\Omega_e}{2}t\right) + 1 \right) |g_1\rangle + \frac{1}{2} \left( \exp\left(i\frac{\Omega_e}{2}t\right) - 1 \right) |g_2\rangle \\ &= \exp\left(i\frac{\Omega_e}{4}t\right) \left( \cos\left(\frac{\Omega_e}{4}t\right) |g_1\rangle + i \sin\left(\frac{\Omega_e}{4}t\right) |g_2\rangle \right) \end{aligned} \quad (72)$$

## 1.5 Bloch Equations

The Bloch equations

$$\langle \Psi | \mathbb{1} | \Psi \rangle = 1 \quad (73)$$

$$\langle \Psi | \sigma_x | \Psi \rangle = S_x \quad (74)$$

$$\langle \Psi | \sigma_y | \Psi \rangle = S_y \quad (75)$$

$$\langle \Psi | \sigma_z | \Psi \rangle = S_z \quad (76)$$

that define the Bloch vector

$$\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \quad (77)$$

For the state  $|g\rangle$  one obtains

$$\langle g | \sigma_x | g \rangle = \langle g | e \rangle = 0 \quad (78)$$

$$\langle g | \sigma_y | g \rangle = -i \langle g | e \rangle = 0 \quad (79)$$

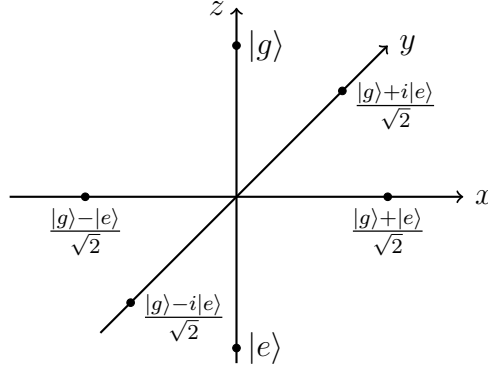
$$\langle g | \sigma_z | g \rangle = -\langle g | g \rangle = -1 \quad (80)$$

For the state  $|+\rangle$  one obtains

$$\langle + | \sigma_x | + \rangle = \langle + | + \rangle = 1 \quad (81)$$

$$\langle + | \sigma_y | + \rangle = i \langle + | - \rangle = 0 \quad (82)$$

$$\langle + | \sigma_z | + \rangle = -\langle + | - \rangle = 0 \quad (83)$$



**Figure 4:** The eigenstates of  $\hat{\sigma}_x$ ,  $\hat{\sigma}_y$  and  $\hat{\sigma}_z$  lie on the  $x$ ,  $y$  and  $z$  axis.

## 1.6 Dynamics of the Bloch Vector

Instead of the Schrödinger equation, we can describe the system dynamics in terms of an equation of motion for the Bloch vector  $\mathbf{S}$ . In addition to the Schrödinger equation

$$|\dot{\Psi}\rangle = -iH|\Psi\rangle \quad (84)$$

For the state vector  $|\Psi\rangle$

$$\begin{aligned} \dot{S}_x &= \langle \dot{\Psi} | \sigma_x | \Psi \rangle + \langle \Psi | \sigma_x | \dot{\Psi} \rangle \\ &= i \langle \Psi | H \sigma_x | \Psi \rangle - i \langle \Psi | \sigma_x H | \Psi \rangle \\ &= i \langle \Psi | [H, \sigma_x] | \Psi \rangle \end{aligned} \quad (85)$$

To simplify this, we can express the Hamiltonian in Pauli matrices

$$H = \sum_j \frac{\omega_j}{2} \sigma_j \quad (86)$$

so that

$$\dot{S}_x = i \sum_j \frac{\omega_j}{2} \langle \Psi | [\sigma_j, \sigma_x] | \Psi \rangle = \omega_y \sigma_z - \omega_z \sigma_y \quad (87)$$

Similarly

$$\dot{S}_y = \omega_z \sigma_x - \omega_x \sigma_z \quad (88)$$

$$\dot{S}_z = \omega_x \sigma_y - \omega_y \sigma_x \quad (89)$$

In terms of vector notation, we have

$$\dot{\mathbf{S}} = \boldsymbol{\omega} \times \mathbf{S} \quad (90)$$

and

$$\frac{\partial}{\partial t} |\mathbf{S}|^2 = \dot{\mathbf{S}} \mathbf{S} + \mathbf{S} \dot{\mathbf{S}} = (\boldsymbol{\omega} \times \mathbf{S}) \mathbf{S} + \mathbf{S} (\boldsymbol{\omega} \times \mathbf{S}) = 0 \quad (91)$$

## 1.7 Averages over Different States

Consider the expectation of an observable  $A$

$$\langle A \rangle = \langle \Psi | A | \Psi \rangle = \sum_j p_j \langle \Psi_j | A | \Psi_j \rangle \quad (92)$$

We can thus define the Bloch vector

$$\mathbf{S} = \sum_j p_j \mathbf{S}_j \quad (93)$$

for the ensemble average. Now we recall the Ramsey experiment

$$H_\phi = \frac{\omega}{2} \sigma_z + \Omega_R \sigma_x \cos(\nu t + \phi) \quad (94)$$

Consider the state  $|\Psi\rangle = |\pm\rangle = (|e\rangle \pm |g\rangle)/\sqrt{2}$

$$\begin{aligned} \mathbf{S} &= \frac{1}{2} \mathbf{S}_{|g\rangle} + \frac{1}{2} \mathbf{S}_{|e\rangle} = \frac{1}{2} \langle g | \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} | g \rangle + \frac{1}{2} \langle e | \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} | e \rangle \\ &= \frac{1}{2} \langle g | \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} | g \rangle + \frac{1}{2} \langle e | \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} | e \rangle = 0 \end{aligned} \quad (95)$$

and  $\dot{\mathbf{S}} = \boldsymbol{\omega} \times \mathbf{S} = 0$ . The Bloch vector for the ensemble average is thus stationary.

## 2 Harmonic Oscillator

The Hamiltonian of one dimension harmonic oscillator

$$\begin{aligned} H &= \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 \\ &= \hbar\omega \left( \frac{P^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar} X^2 \right) \\ &= \frac{1}{2}\hbar\omega (\hat{p}^2 + \hat{x}^2) \end{aligned} \quad (96)$$

in terms of the unitless operators

$$\hat{x} = \sqrt{\frac{m\omega}{\hbar}} X, \quad \hat{p} = \frac{1}{\sqrt{m\hbar\omega}} P \quad (97)$$

which satisfy the commutation relation

$$[\hat{x}, \hat{p}] = \frac{1}{\hbar}[X, P] = i \quad (98)$$

Creation and annihilation operators are defined as

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}) \quad (99)$$

with

$$[a, a^\dagger] = \frac{1}{2}[\hat{x} + i\hat{p}, \hat{x} - i\hat{p}] = \frac{1}{2}([\hat{x}, -i\hat{p}] + [i\hat{p}, \hat{x}]) = 1 \quad (100)$$

$\hat{x}$  and  $\hat{p}$  can be expressed as

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}) \quad (101)$$

The Hamiltonian becomes

$$\hat{H} = \hbar\omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad (102)$$

For the commute relation  $[a, a^\dagger] = 1$ , one obtains

$$a^\dagger a a |\mu\rangle = (a a^\dagger - 1) a |\mu\rangle = a(a^\dagger a - 1) |\mu\rangle = (\mu - 1) a |\mu\rangle \quad (103)$$

so that

$$\langle \mu | a^\dagger a |\mu\rangle = \mu \quad \Rightarrow \quad a |\mu\rangle = \sqrt{\mu} |\mu - 1\rangle \quad (104)$$

and similarly,

$$a^\dagger |\mu\rangle = \sqrt{\mu + 1} |\mu + 1\rangle \quad (105)$$

In real space representation, the annihilation operator and creation operator read

$$a \rightarrow \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right), \quad a^\dagger \rightarrow \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right) \quad (106)$$

The wave function satisfying

$$\frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right) \phi_0(x) = 0 \quad \Rightarrow \quad \phi_0(x) \sim \exp\left(-\frac{x^2}{2}\right) \quad (107)$$

$$\phi_1(x) \sim \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right) \exp\left(-\frac{x^2}{2}\right) = 2x \exp\left(-\frac{x^2}{2}\right) \quad (108)$$

### 3 Quantisation of the Electromagnetic Field

The Maxwell equations read

$$\nabla \cdot \mathbf{E} = 0 \quad (109)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (110)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (111)$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (112)$$

In terms of divergence and curl

$$\nabla \cdot \mathbf{Q} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \quad (113)$$

$$\nabla \times \mathbf{Q} = -\left(\frac{\partial Q_y}{\partial z} - \frac{\partial Q_z}{\partial y}\right) \mathbf{e}_x - \left(\frac{\partial Q_z}{\partial x} - \frac{\partial Q_x}{\partial z}\right) \mathbf{e}_y - \left(\frac{\partial Q_x}{\partial y} - \frac{\partial Q_y}{\partial x}\right) \mathbf{e}_z \quad (114)$$

Let's start with a simple ansatz

$$\mathbf{E} = f(t) \sin kz \mathbf{e}_x \quad (115)$$

$$\mathbf{B} = g(t) \cos kz \mathbf{e}_y \quad (116)$$

The Maxwell equations imply

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0 \quad (117)$$

$$\begin{aligned} \nabla \times \mathbf{E} &= \frac{\partial E_x}{\partial z} \mathbf{e}_y = f(t)k \cos kz \mathbf{e}_y \\ &= -\frac{\partial \mathbf{B}}{\partial t} = -\dot{g}(t) \cos kz \mathbf{e}_y \end{aligned} \quad (118)$$

$$\begin{aligned} \nabla \times \mathbf{B} &= -\frac{\partial B_y}{\partial z} \mathbf{e}_x = g(t)k \sin kz \mathbf{e}_x \\ &= \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2} \dot{f}(t) \sin kz \mathbf{e}_x \end{aligned} \quad (119)$$

This requires the differential equations

$$f(t)k = -\dot{g}(t), \quad g(t)k = \frac{1}{c^2} \dot{f}(t) \quad (120)$$

So we have

$$\ddot{f}(t) = c^2 k \dot{g}(t) = -c^2 k^2 f(t) = \nu_k^2 f(t) \quad (121)$$

where  $\nu_k = ck$  is called the linear dispersion. We can now quantise the electric field as

$$\mathbf{E} = \sqrt{\frac{\hbar \nu}{\varepsilon_0 V}} (\hat{a} e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t}) \sin kz \mathbf{e}_x \quad (122)$$

The creation and annihilation operators satisfy

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \quad (123)$$

We can also define creation and annihilation operators for wave packets

$$\hat{a}_\phi = \int dk \phi(k) \hat{a}_k, \quad \hat{a}_\phi^\dagger = \int dk \phi^*(k) \hat{a}_k^\dagger \quad (124)$$

and the commutator

$$[\hat{a}_\phi, \hat{a}_\psi^\dagger] = \int dk dk' \phi(k) \psi^*(k') [\hat{a}_k, \hat{a}_{k'}^\dagger] = \int dk \phi(k) \psi^*(k) \quad (125)$$



## 4 Jaynes-Cummings

we can now consider a two-level system interacting with a single mode of the quantised electro-magnetic field. The Hamiltonian reads

$$H = \frac{\omega}{2}\sigma_z + \frac{1}{2}\Omega_R\sigma_x (ae^{-i\nu t} + a^\dagger e^{i\nu t}) \quad (126)$$

1)  $U_S = \exp(i\nu a^\dagger at)$ , the Hamiltonian reads

$$H_S = U_S^\dagger H U_S - iU_S^\dagger \dot{U}_S = \frac{\omega}{2}\sigma_z + \nu a^\dagger a + \frac{1}{2}\Omega_R\sigma_x(a + a^\dagger) \quad (127)$$

2)  $U_I = \exp(-i\frac{\omega}{t}\sigma_z t)$

$$\begin{aligned} H_I &= U_I^\dagger H U_I - iU_I^\dagger \dot{U}_I \\ &= \frac{1}{2}\Omega_R (\sigma_+ e^{i\omega t} + \sigma_- e^{-i\omega t}) (ae^{-i\nu t} + a^\dagger e^{i\nu t}) \\ &= \frac{1}{2}\Omega_R [\sigma_+ a e^{i(\omega-\nu)t} + \sigma_+ a^\dagger e^{i(\omega+\nu)t} + \sigma_- a e^{-i(\omega+\nu)t} + \sigma_- a^\dagger e^{-i(\omega-\nu)t}] \\ &\approx \frac{1}{2}\Omega_R [\sigma_+ a e^{i(\omega-\nu)t} + \sigma_- a^\dagger e^{-i(\omega-\nu)t}] \end{aligned} \quad (128)$$

This Hamiltonian contains four elementary process

- $\sigma_+ a$ : atom absorbs a photon and gets excited.
- $\sigma_+ a^\dagger$ : atom emits a photon and gets excited.
- $\sigma_- a$ : atom absorbs a photon and gets de-excited.
- $\sigma_- a^\dagger$ : atom emits a photon and gets de-excited.

### 4.1 Two-Dimensional Subspaces

The Hamiltonian (in rotating wave approximation) in lab frame reads

$$H = \frac{\omega}{2}\sigma_z + \nu a^\dagger a + \frac{1}{2}\Omega_R(\sigma_+ a + \sigma_- a^\dagger) \quad (129)$$

and

$$H |g, \mu\rangle = \left(-\frac{\omega}{2} + \mu\nu\right) |g, \mu\rangle + \frac{1}{2}\Omega_R\sqrt{\mu} |e, \mu-1\rangle \quad (130)$$

$$H |e, \mu-1\rangle = \frac{1}{2}\Omega_R\sqrt{\mu} |g, \mu\rangle + \left(\frac{\omega}{2} + (\mu-1)\nu\right) |e, \mu-1\rangle \quad (131)$$

In terms of the basis  $\{|g, \mu\rangle, |e, \mu-1\rangle\}$  we can express this as the matrix

$$\begin{pmatrix} -\frac{\omega-\nu}{2} + \left(\mu - \frac{1}{2}\right)\nu & \frac{1}{2}\Omega_R\sqrt{\mu} \\ \frac{1}{2}\Omega_R\sqrt{\mu} & \frac{\omega-\nu}{2} + \left(\mu - \frac{1}{2}\right)\nu \end{pmatrix} \quad (132)$$

or, in terms of Pauli-matrices as

$$H = -\frac{\omega - \nu}{2}\sigma_z + \frac{1}{2}\Omega_R\sqrt{\mu}\sigma_x + \left(\mu - \frac{1}{2}\right)\nu\mathbb{1} \quad (133)$$

In the case of resonance between atom and light-field, this reduces to

$$H(\nu = \omega) = \frac{1}{2}\Omega_R\sqrt{\mu}\sigma_x + \left(\mu - \frac{1}{2}\right)\nu\mathbb{1} \quad (134)$$

with eigenstates

$$\frac{1}{\sqrt{2}}(|g, \mu\rangle \pm |e, \mu - 1\rangle) \quad (135)$$

## 4.2 The Lambda-System

The Hamiltonian of the Lambda-system interacting with a single-mode quantum field in rotating wave approximation reads

$$\begin{aligned} H &= \omega |e\rangle \langle e| + 0(|g_1\rangle \langle g_1| + |g_2\rangle \langle g_2|) + \nu a^\dagger a \\ &\quad + \frac{1}{2\sqrt{2}}\Omega_R (|g_1\rangle \langle e| a^\dagger + |g_2\rangle \langle e| a^\dagger + |e\rangle \langle g_1| a + |e\rangle \langle g_2| a) \\ &= \omega |e\rangle \langle e| + \frac{1}{2\sqrt{2}}\Omega_R [(|g_1\rangle \langle e| + |g_2\rangle \langle e|)a^\dagger + (|e\rangle \langle g_1| + |e\rangle \langle g_2|)a] \end{aligned} \quad (136)$$

In the interaction picture we have

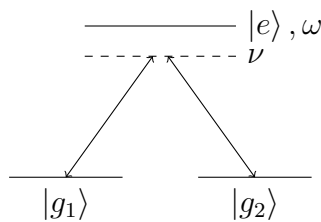
$$H_I = \frac{1}{2\sqrt{2}}\Omega_R [(|g_1\rangle \langle e| + |g_2\rangle \langle e|)a^\dagger e^{-i\delta t} + (|e\rangle \langle g_1| + |e\rangle \langle g_2|)ae^{i\delta t}] \quad (137)$$

with the detuning  $\delta = \omega - \nu$ . According to the perturbation theory

$$U = \mathbb{1} - i \int_0^t dt' H(t') - \frac{1}{2} \int_0^t dt' \int_0^{t'} dt'' H_I(t') H_I(t'') + \dots \quad (138)$$

thus we have to consider the second order

$$\begin{aligned} H_I(t') H_I(t'') &= \frac{1}{8}\Omega_R^2 [(|g_1\rangle \langle e| + |g_2\rangle \langle e|)(|e\rangle \langle g_1| + |e\rangle \langle g_2|)a^\dagger a e^{-i\delta(t'-t'')} \\ &\quad + (|e\rangle \langle g_1| + |e\rangle \langle g_2|)(|g_1\rangle \langle e| + |g_2\rangle \langle e|)aa^\dagger e^{i\delta(t'-t'')}] \\ &= \frac{1}{8}\Omega_R^2 (|g_1\rangle \langle g_1| + |g_2\rangle \langle g_2| + |g_1\rangle \langle g_2| + |g_2\rangle \langle g_1|) a^\dagger a e^{-i\delta(t'-t'')} \\ &\quad + \frac{1}{4}\Omega_R^2 |e\rangle \langle e| (a^\dagger a + 1) e^{i\delta(t'-t'')} \end{aligned} \quad (139)$$



**Figure 5:** The Lambda-system

## 5 Coherent States

For fock states  $|\mu\rangle$ , the expectation values for  $x$  and  $p$  vanish

$$\langle\mu|x|\mu\rangle = \frac{1}{\sqrt{2}}(\langle\mu|a|\mu\rangle + \langle\mu|a^\dagger|\mu\rangle) = 0 \quad (140)$$

$$\langle\mu|p|\mu\rangle = \frac{-i}{\sqrt{2}}(\langle\mu|a|\mu\rangle - \langle\mu|a^\dagger|\mu\rangle) = 0 \quad (141)$$

and the fluctuations

$$\langle\mu|x^2|\mu\rangle = \frac{1}{2}(\langle\mu|a^2|\mu\rangle + \langle\mu|aa^\dagger|\mu\rangle + \langle\mu|a^\dagger a|\mu\rangle + \langle\mu|a^\dagger a^\dagger|\mu\rangle) = \mu + \frac{1}{2} \quad (142)$$

$$\langle\mu|p^2|\mu\rangle = -\frac{1}{2}(\langle\mu|a^2|\mu\rangle - \langle\mu|aa^\dagger|\mu\rangle - \langle\mu|a^\dagger a|\mu\rangle + \langle\mu|a^\dagger a^\dagger|\mu\rangle) = \mu + \frac{1}{2} \quad (143)$$

For the ground state

$$\langle 0|x|0\rangle = \langle 0|p|0\rangle = 0 \quad (144)$$

$$\langle 0|x^2|0\rangle = \langle 0|p^2|0\rangle = \frac{1}{2} \quad (145)$$

This yields

$$\Delta x \Delta p = (\langle 0|x^2|0\rangle - (\langle 0|x|0\rangle)^2)(\langle 0|p^2|0\rangle - (\langle 0|p|0\rangle)^2) = \frac{1}{4} \quad (146)$$

which is the minimal allowed uncertainty. We can generate different states with the same uncertainty, by displacing the vacuum in phase space. The *displacement operator* is defined as

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \quad (147)$$

The coherent state

$$\begin{aligned} |\alpha\rangle &= D(\alpha)|0\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp(\alpha a^\dagger) \exp(-\alpha^* a)|0\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp(\alpha a^\dagger)|0\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\mu} \frac{\alpha^\mu (a^\dagger)^\mu}{\mu!} |0\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\mu} \frac{\alpha^\mu}{\sqrt{\mu!}} |\mu\rangle \end{aligned} \quad (148)$$

The probability to find  $\mu$  photons is thus given by the Poisson distribution.

$$P(\mu) = \exp(-|\alpha|^2) \frac{(|\alpha|^2)^\mu}{\mu!} \quad (149)$$

For the expectation value of  $x$  and  $p$  with respect to a general state  $|\Psi\rangle$  one has

$$(\langle\Psi| D^\dagger(\alpha)x(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)x D(\alpha)) |\Psi\rangle \quad (150)$$

$$(\langle\Psi| D^\dagger(\alpha)p(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)p D(\alpha)) |\Psi\rangle \quad (151)$$

then we calculate

$$D^\dagger(\alpha)x D(\alpha) = x + \frac{\alpha + \alpha^*}{\sqrt{2}} = x + x_0 \quad (152)$$

$$D^\dagger(\alpha)p D(\alpha) = p - i\frac{\alpha - \alpha^*}{\sqrt{2}} = p + p_0 \quad (153)$$

We verify the uncertainty in position and momentum of any coherent state

$$\begin{aligned} \langle\alpha| x^2 |\alpha\rangle - (\langle\alpha| x |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)x^2 D(\alpha) |0\rangle - (\langle 0| D^\dagger(\alpha)x D(\alpha) |0\rangle)^2 \\ &= \langle 0| x^2 |0\rangle - (\langle 0| x |0\rangle)^2 \end{aligned} \quad (154)$$

$$\begin{aligned} \langle\alpha| p^2 |\alpha\rangle - (\langle\alpha| p |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)p^2 D(\alpha) |0\rangle - (\langle 0| D^\dagger(\alpha)p D(\alpha) |0\rangle)^2 \\ &= \langle 0| p^2 |0\rangle - (\langle 0| p |0\rangle)^2 \end{aligned} \quad (155)$$

## 5.1 Coherent States in Real-Space Representation

The character of the displacement operator can be exemplified in the real-space representation of wave functions.

$$\Psi_\alpha(x) = \langle x|\alpha\rangle \propto \exp\left(-\frac{1}{2}(x - x_0)^2 + ip_0x - \frac{i}{2}x_0p_0\right) \quad (156)$$

with  $x_0 = (\alpha + \alpha^*)/\sqrt{2}$  and  $p_0 = (\alpha - \alpha^*)/(\sqrt{2}i)$ . It is convenient to define the vector

$$|\alpha, \tau\rangle = \exp[(\alpha a^\dagger - \alpha^* a)\tau] |0\rangle \quad (157)$$

with additional scalar parameter  $\tau$ . It satisfies the differential equation

$$\frac{\partial |\alpha, \tau\rangle}{\partial \tau} = (\alpha a^\dagger - \alpha^* a) |\alpha, \tau\rangle \quad (158)$$

The real-space representation of the operator  $(\alpha a^\dagger - \alpha^* a)$  reads

$$\frac{1}{\sqrt{2}} \left[ \alpha \left( x - \frac{\partial}{\partial x} \right) - \alpha^* \left( x + \frac{\partial}{\partial x} \right) \right] = \frac{\alpha - \alpha^*}{\sqrt{2}} x - \frac{\alpha + \alpha^*}{\sqrt{2}} \frac{\partial}{\partial x} = ip_0x - x_0 \frac{\partial}{\partial x} \quad (159)$$

We thus need to solve the equation

$$\frac{\partial \Phi}{\partial \tau} = \left( ip_0x - x_0 \frac{\partial}{\partial x} \right) \Phi \quad (160)$$

with the Ansatz

$$\Phi(\tau) = \exp\left(-\frac{1}{2}(x - f_x)^2 + if_px - i\varphi\right) \quad (161)$$

The initial conditions are  $f_x(0) = f_p(0) = \varphi(0) = 0$ . The derivatives

$$\frac{\partial \Phi(\tau)}{\partial \tau} = \left( (x - f_x) \frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} x - i \frac{\partial \varphi}{\partial \tau} \right) \Phi(\tau) \quad (162)$$

$$\frac{\partial \Phi(\tau)}{\partial x} = (-(x - f_x) + i f_p) \Phi(\tau) \quad (163)$$

This yields

$$(x - f_x) \frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} x - i \frac{\partial \varphi}{\partial \tau} = i p_0 x - x_0 (-(x - f_x) + i f_p) \quad (164)$$

Collect all terms proportional to  $x$

$$\frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} = i p_0 + x_0 \quad (165)$$

This is solved for

$$\frac{\partial f_x}{\partial \tau} = x_0 \quad \Rightarrow \quad f_x = x_0 \tau \quad (166)$$

$$\frac{\partial f_p}{\partial \tau} = p_0 \quad \Rightarrow \quad f_p = p_0 \tau \quad (167)$$

Collect all terms do not contain  $x$  yields

$$-f_x \frac{\partial f_x}{\partial \tau} - i \frac{\partial \varphi}{\partial \tau} = -x_0 f_x - i x_0 f_p \quad (168)$$

which is solved for

$$\varphi(\tau) = \frac{1}{2} x_0 p_0 \tau^2 \quad (169)$$

With  $\tau = 1$ , this gives the phase factor  $\exp(-\frac{i}{2} x_0 p_0)$ .

## 5.2 Dynamics of Coherent States

For the dynamics induced by  $U_0(t) = \exp(-i\nu a^\dagger a t)$ , one obtains

$$\begin{aligned} U_0(t) |\alpha\rangle &= U_0(t) \exp(\alpha a^\dagger - \alpha^* a) |0\rangle \\ &= U_0(t) \exp(\alpha a^\dagger - \alpha^* a) U_0^\dagger(t) U_0(t) |0\rangle \\ &= U_0(t) \exp(\alpha a^\dagger - \alpha^* a) U_0^\dagger(t) |0\rangle \\ &= \exp \left[ \alpha U_0(t) a^\dagger U_0^\dagger(t) - \alpha^* U_0(t) a U_0^\dagger(t) \right] |0\rangle \\ &= \exp(\alpha a^\dagger e^{-i\nu t} + \alpha^* a e^{i\nu t}) |0\rangle \\ &= D(\alpha e^{-i\nu t}) |0\rangle = |\alpha e^{-i\nu t}\rangle \end{aligned} \quad (170)$$

### 5.3 Light-Matter Interaction with Coherent States

Coherent states are eigenstates to the annihilation operator  $a$ .

$$\begin{aligned} a|\alpha\rangle &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\mu} \frac{\alpha^{\mu}}{\sqrt{\mu!}} a|\mu\rangle \\ &= \alpha \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\mu} \frac{\alpha^{\mu-1}}{\sqrt{(\mu-1)!}} |\mu-1\rangle = \alpha|\alpha\rangle \end{aligned} \quad (171)$$

Similarly

$$\langle\alpha|a^{\dagger} = \alpha^* \langle\alpha| \quad (172)$$

Coherent states are not orthogonal to each other

$$\begin{aligned} \langle\alpha|\beta\rangle &= \left( \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{\mu} \frac{(\alpha^*)^{\mu}}{\sqrt{\mu!}} \langle\mu| \right) \left( \exp\left(-\frac{|\beta|^2}{2}\right) \sum_{\nu} \frac{\beta^{\nu}}{\sqrt{\nu!}} |\nu\rangle \right) \\ &= \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right) \sum_{\mu,\nu} \frac{(\alpha^*)^{\mu} \beta^{\nu}}{\sqrt{\mu!} \sqrt{\nu!}} \langle\mu|\nu\rangle \\ &= \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right) \sum_{\mu} \frac{(\alpha^* \beta)^{\mu}}{\mu!} \end{aligned} \quad (173)$$

Now we want to find the eigenvector  $|\Psi\rangle$  of  $a^{\dagger}$

$$a^{\dagger}|\Psi\rangle = \lambda|\Psi\rangle = |\tilde{\Psi}\rangle \quad (174)$$

The normalised vector

$$\frac{|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}} \quad (175)$$

and

$$\left| \frac{\langle\Psi|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}} \right| = 1 \quad (176)$$

Normalising  $a^{\dagger}|\alpha\rangle$  yields

$$\frac{a^{\dagger}|\alpha\rangle}{\sqrt{\langle\alpha|aa^{\dagger}|\alpha\rangle}} = \frac{\alpha^{\dagger}|\alpha\rangle}{\sqrt{|\alpha|^2 + 1}} \quad (177)$$

and

$$\frac{\langle\alpha|a^{\dagger}|\alpha\rangle}{\sqrt{\langle\alpha|aa^{\dagger}|\alpha\rangle}} = \frac{\alpha^*}{\sqrt{|\alpha|^2 + 1}} \quad (178)$$

In the limit  $|\alpha| \rightarrow \infty$

$$\frac{\alpha^*}{\sqrt{|\alpha|^2 + 1}} \rightarrow \frac{\alpha^*}{|\alpha|} \quad (179)$$


---

with

$$\left| \frac{\alpha^*}{|\alpha|} \right| = 1 \quad (180)$$

The relation

$$a^\dagger |\alpha\rangle \simeq \alpha^* |\alpha\rangle \quad (181)$$

is thus a good approximation for  $|\alpha| \ll 1$ .

## 6 Coherent States

For fock states  $|\mu\rangle$ , the expectation values for  $x$  and  $p$  vanish

$$\langle\mu|x|\mu\rangle = \frac{1}{\sqrt{2}}(\langle\mu|a|\mu\rangle + \langle\mu|a^\dagger|\mu\rangle) = 0 \quad (182)$$

$$\langle\mu|p|\mu\rangle = \frac{-i}{\sqrt{2}}(\langle\mu|a|\mu\rangle - \langle\mu|a^\dagger|\mu\rangle) = 0 \quad (183)$$

and the fluctuations

$$\langle\mu|x^2|\mu\rangle = \frac{1}{2}(\langle\mu|a^2|\mu\rangle + \langle\mu|aa^\dagger|\mu\rangle + \langle\mu|a^\dagger a|\mu\rangle + \langle\mu|a^\dagger a^\dagger|\mu\rangle) = \mu + \frac{1}{2} \quad (184)$$

$$\langle\mu|p^2|\mu\rangle = -\frac{1}{2}(\langle\mu|a^2|\mu\rangle - \langle\mu|aa^\dagger|\mu\rangle - \langle\mu|a^\dagger a|\mu\rangle + \langle\mu|a^\dagger a^\dagger|\mu\rangle) = \mu + \frac{1}{2} \quad (185)$$

For the ground state

$$\langle 0|x|0\rangle = \langle 0|p|0\rangle = 0 \quad (186)$$

$$\langle 0|x^2|0\rangle = \langle 0|p^2|0\rangle = \frac{1}{2} \quad (187)$$

This yields

$$\Delta x \Delta p = (\langle 0|x^2|0\rangle - (\langle 0|x|0\rangle)^2)(\langle 0|p^2|0\rangle - (\langle 0|p|0\rangle)^2) = \frac{1}{4} \quad (188)$$

which is the minimal allowed uncertainty. We can generate different states with the same uncertainty, by displacing the vacuum in phase space. The *displacement operator* is defined as

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \quad (189)$$

The coherent state

$$\begin{aligned} |\alpha\rangle &= D(\alpha)|0\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp(\alpha a^\dagger) \exp(-\alpha^* a)|0\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp(\alpha a^\dagger)|0\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\mu} \frac{\alpha^\mu (a^\dagger)^\mu}{\mu!} |0\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\mu} \frac{\alpha^\mu}{\sqrt{\mu!}} |\mu\rangle \end{aligned} \quad (190)$$

The probability to find  $\mu$  photons is thus given by the Poisson distribution.

$$P(\mu) = \exp(-|\alpha|^2) \frac{(|\alpha|^2)^\mu}{\mu!} \quad (191)$$



For the expectation value of  $x$  and  $p$  with respect to a general state  $|\Psi\rangle$  one has

$$(\langle\Psi| D^\dagger(\alpha)x(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)x D(\alpha)) |\Psi\rangle \quad (192)$$

$$(\langle\Psi| D^\dagger(\alpha)p(D(\alpha)|\Psi\rangle) = \langle\Psi| (D^\dagger(\alpha)p D(\alpha)) |\Psi\rangle \quad (193)$$

then we calculate

$$D^\dagger(\alpha)x D(\alpha) = x + \frac{\alpha + \alpha^*}{\sqrt{2}} = x + x_0 \quad (194)$$

$$D^\dagger(\alpha)p D(\alpha) = p - i\frac{\alpha - \alpha^*}{\sqrt{2}} = p + p_0 \quad (195)$$

We verify the uncertainty in position and momentum of any coherent state

$$\begin{aligned} \langle\alpha| x^2 |\alpha\rangle - (\langle\alpha| x |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)x^2 D(\alpha) |0\rangle - (\langle 0| D^\dagger(\alpha)x D(\alpha) |0\rangle)^2 \\ &= \langle 0| x^2 |0\rangle - (\langle 0| x |0\rangle)^2 \end{aligned} \quad (196)$$

$$\begin{aligned} \langle\alpha| p^2 |\alpha\rangle - (\langle\alpha| p |\alpha\rangle)^2 &= \langle 0| D^\dagger(\alpha)p^2 D(\alpha) |0\rangle - (\langle 0| D^\dagger(\alpha)p D(\alpha) |0\rangle)^2 \\ &= \langle 0| p^2 |0\rangle - (\langle 0| p |0\rangle)^2 \end{aligned} \quad (197)$$

## 6.1 Coherent States in Real-Space Representation

The character of the displacement operator can be exemplified in the real-space representation of wave functions.

$$\Psi_\alpha(x) = \langle x|\alpha\rangle \propto \exp\left(-\frac{1}{2}(x - x_0)^2 + ip_0x - \frac{i}{2}x_0p_0\right) \quad (198)$$

with  $x_0 = (\alpha + \alpha^*)/\sqrt{2}$  and  $p_0 = (\alpha - \alpha^*)/(\sqrt{2}i)$ . It is convenient to define the vector

$$|\alpha, \tau\rangle = \exp[(\alpha a^\dagger - \alpha^* a)\tau] |0\rangle \quad (199)$$

with additional scalar parameter  $\tau$ . It satisfies the differential equation

$$\frac{\partial |\alpha, \tau\rangle}{\partial \tau} = (\alpha a^\dagger - \alpha^* a) |\alpha, \tau\rangle \quad (200)$$

The real-space representation of the operator  $(\alpha a^\dagger - \alpha^* a)$  reads

$$\frac{1}{\sqrt{2}} \left[ \alpha \left( x - \frac{\partial}{\partial x} \right) - \alpha^* \left( x + \frac{\partial}{\partial x} \right) \right] = \frac{\alpha - \alpha^*}{\sqrt{2}} x - \frac{\alpha + \alpha^*}{\sqrt{2}} \frac{\partial}{\partial x} = ip_0x - x_0 \frac{\partial}{\partial x} \quad (201)$$

We thus need to solve the equation

$$\frac{\partial \Phi}{\partial \tau} = \left( ip_0x - x_0 \frac{\partial}{\partial x} \right) \Phi \quad (202)$$

with the Ansatz

$$\Phi(\tau) = \exp\left(-\frac{1}{2}(x - f_x)^2 + if_px - i\varphi\right) \quad (203)$$

The initial conditions are  $f_x(0) = f_p(0) = \varphi(0) = 0$ . The derivatives

$$\frac{\partial \Phi(\tau)}{\partial \tau} = \left( (x - f_x) \frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} x - i \frac{\partial \varphi}{\partial \tau} \right) \Phi(\tau) \quad (204)$$

$$\frac{\partial \Phi(\tau)}{\partial x} = (-(x - f_x) + i f_p) \Phi(\tau) \quad (205)$$

This yields

$$(x - f_x) \frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} x - i \frac{\partial \varphi}{\partial \tau} = i p_0 x - x_0 (-(x - f_x) + i f_p) \quad (206)$$

Collect all terms proportional to  $x$

$$\frac{\partial f_x}{\partial \tau} + i \frac{\partial f_p}{\partial \tau} = i p_0 + x_0 \quad (207)$$

This is solved for

$$\frac{\partial f_x}{\partial \tau} = x_0 \quad \Rightarrow \quad f_x = x_0 \tau \quad (208)$$

$$\frac{\partial f_p}{\partial \tau} = p_0 \quad \Rightarrow \quad f_p = p_0 \tau \quad (209)$$

Collect all terms do not contain  $x$  yields

$$-f_x \frac{\partial f_x}{\partial \tau} - i \frac{\partial \varphi}{\partial \tau} = -x_0 f_x - i x_0 f_p \quad (210)$$

which is solved for

$$\varphi(\tau) = \frac{1}{2} x_0 p_0 \tau^2 \quad (211)$$

With  $\tau = 1$ , this gives the phase factor  $\exp(-\frac{i}{2} x_0 p_0)$ .

## 6.2 Dynamics of Coherent States

For the dynamics induced by  $U_0(t) = \exp(-i\nu a^\dagger a t)$ , one obtains

$$\begin{aligned} U_0(t) |\alpha\rangle &= U_0(t) \exp(\alpha a^\dagger - \alpha^* a) |0\rangle \\ &= U_0(t) \exp(\alpha a^\dagger - \alpha^* a) U_0^\dagger(t) U_0(t) |0\rangle \\ &= U_0(t) \exp(\alpha a^\dagger - \alpha^* a) U_0^\dagger(t) |0\rangle \\ &= \exp \left[ \alpha U_0(t) a^\dagger U_0^\dagger(t) - \alpha^* U_0(t) a U_0^\dagger(t) \right] |0\rangle \\ &= \exp(\alpha a^\dagger e^{-i\nu t} + \alpha^* a e^{i\nu t}) |0\rangle \\ &= D(\alpha e^{-i\nu t}) |0\rangle = |\alpha e^{-i\nu t}\rangle \end{aligned} \quad (212)$$

### 6.3 Light-Matter Interaction with Coherent States

Coherent states are eigenstates to the annihilation operator  $a$ .

$$\begin{aligned} a|\alpha\rangle &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\mu} \frac{\alpha^{\mu}}{\sqrt{\mu!}} a|\mu\rangle \\ &= \alpha \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\mu} \frac{\alpha^{\mu-1}}{\sqrt{(\mu-1)!}} |\mu-1\rangle = \alpha|\alpha\rangle \end{aligned} \quad (213)$$

Similarly

$$\langle\alpha|a^{\dagger} = \alpha^* \langle\alpha| \quad (214)$$

Coherent states are not orthogonal to each other

$$\begin{aligned} \langle\alpha|\beta\rangle &= \left( \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{\mu} \frac{(\alpha^*)^{\mu}}{\sqrt{\mu!}} \langle\mu| \right) \left( \exp\left(-\frac{|\beta|^2}{2}\right) \sum_{\nu} \frac{\beta^{\nu}}{\sqrt{\nu!}} |\nu\rangle \right) \\ &= \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right) \sum_{\mu,\nu} \frac{(\alpha^*)^{\mu} \beta^{\nu}}{\sqrt{\mu!} \sqrt{\nu!}} \langle\mu|\nu\rangle \\ &= \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right) \sum_{\mu} \frac{(\alpha^* \beta)^{\mu}}{\mu!} \end{aligned} \quad (215)$$

Now we want to find the eigenvector  $|\Psi\rangle$  of  $a^{\dagger}$

$$a^{\dagger}|\Psi\rangle = \lambda|\Psi\rangle = |\tilde{\Psi}\rangle \quad (216)$$

The normalised vector

$$\frac{|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}} \quad (217)$$

and

$$\left| \frac{\langle\Psi|\tilde{\Psi}\rangle}{\sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}} \right| = 1 \quad (218)$$

Normalising  $a^{\dagger}|\alpha\rangle$  yields

$$\frac{a^{\dagger}|\alpha\rangle}{\sqrt{\langle\alpha|aa^{\dagger}|\alpha\rangle}} = \frac{\alpha^{\dagger}|\alpha\rangle}{\sqrt{|\alpha|^2 + 1}} \quad (219)$$

and

$$\frac{\langle\alpha|a^{\dagger}|\alpha\rangle}{\sqrt{\langle\alpha|aa^{\dagger}|\alpha\rangle}} = \frac{\alpha^*}{\sqrt{|\alpha|^2 + 1}} \quad (220)$$

In the limit  $|\alpha| \rightarrow \infty$

$$\frac{\alpha^*}{\sqrt{|\alpha|^2 + 1}} \rightarrow \frac{\alpha^*}{|\alpha|} \quad (221)$$


---

with

$$\left| \frac{\alpha^*}{|\alpha|} \right| = 1 \quad (222)$$

The relation

$$a^\dagger |\alpha\rangle \simeq \alpha^* |\alpha\rangle \quad (223)$$

is thus a good approximation for  $|\alpha| \ll 1$ .