Imperial College London

NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF PHYSICS

Mathematical Methods for Physicists

Author:
Chen Huang

Email:
chen.huang23@imperial.ac.uk

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1 Vector Spaces and Tensors

1.1 vector spaces

1.1.1 Definition of a Vector Space

Definition. A real (complex) vector space is a set $\mathbb{V}$ - whose elements are called vectors - together with two operations called addition (+) and scalar multiplication such that

1. $\mathbb{V}$ is closed under **addition**: $\forall \underline{u}, \underline{v} \in \mathbb{V} \Rightarrow \underline{u} + \underline{v} \in \mathbb{V}$.
2. $\mathbb{V}$ is closed under **scalar multiplication**: $\forall \underline{u} \in \mathbb{V}$ and $\forall$ scalar $\lambda \Rightarrow \lambda \underline{u} \in \mathbb{V}$.
3. There exists a null or zero vector $\underline{0}$ such that $\underline{u} + \underline{0} = \underline{u}$.
4. Each vector $\underline{u}$ has a corresponding negative vector $-\boldsymbol{v}$ such that: $\underline{u} + (-\underline{v}) = 0$.
5. The addition operation satisfies: $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ and $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$.
6. Scalar multiplication satisfies: $a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}$, $a(b\underline{u}) = (ab)\underline{u}$

Example. 3 component real column vectors

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

2 component complex vectors

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in \mathbb{C} \right\}$$

1.1.2 Linear Independence

Definition. *A set of n non-zero vectors $\{u_1, u_2, \cdots, u_n\}$ in a vector space is linearly independent if*

$$\sum_{i=1}^{n} a_i u_i = 0 \quad \Rightarrow \quad a_i = 0 \quad \forall i$$

Otherwise we say $\{u_1, u_2, \cdots, u_n\}$ is linearly dependent.
Let N be the maximum number of linearly independent vectors in $\mathbb{V}$, then N is the dimension of $\mathbb{V}$.

Definition. *A subspace, $\mathbb{W}$, of a vector space $\mathbb{V}$ is a subset of $\mathbb{V}$ that is itself a vector space.*

1.1.3 Basis Vectors

Any set of n linearly independent vectors $\{u_i\}$ in an n-dimension vector space $\mathbb{V}$ is a *basis* for $\mathbb{V}$. Any vector v in $\mathbb{V}$ can be represented as a linear combination of the basis vectors

$$v = \sum_{i=1}^{n} a_i u_i$$

1.1.4 Inner Product

Definition. An inner product on a **real vector space** $\mathbb{V}$, is a **real** number $\langle \underline{u}, \underline{v} \rangle$ for every pair of vectors $\underline{u}$ and $\underline{v}$. The inner product has the following properties

1. $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$
2. $\langle \underline{u}, a\underline{v}_1 + b\underline{v}_2 \rangle = a\langle \underline{u}, \underline{v}_1 \rangle + b\langle \underline{u}, \underline{v}_2 \rangle$
3. $\langle \underline{v}, \underline{v} \rangle \geq 0$
4. Define $\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$. Then $\|\underline{v}\| = \underline{0} \;\Rightarrow\; \underline{v} = \underline{0}$

Definition. An inner product on a **complex space** $\mathbb{V}$, is a **real** number $\langle u, v \rangle$ for every ordered pair of vectors u and v. The inner product has the following properties

1. $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle^*$
2. $\langle \underline{u}, a\underline{v}_1 + b\underline{v}_2 \rangle = a\langle \underline{u}, \underline{v}_1 \rangle + b\langle \underline{u}, \underline{v}_2 \rangle$
   $\langle a\underline{u}_1 + b\underline{u}_2, v \rangle = a^*\langle \underline{v}, \underline{u}_1 \rangle^* + b^*\langle \underline{v}, \underline{u}_2 \rangle^* = a^*\langle \underline{u}_1, \underline{v} \rangle + b^*\langle \underline{u}_2, \underline{v} \rangle$
3. $\langle \underline{v}, \underline{v} \rangle \geq 0$
4. Define $\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$. Then $\|\underline{v}\| = 0 \;\Rightarrow\; \underline{v} = \underline{0}$

Example.

$$\mathbb{R}^3 = \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right\rangle = ad + be + cf, \qquad \mathbb{C}^2 = \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = a^*c + b^*d$$

1.1.5 Orthogonality

Two vectors are said to be **orthogonal** if their inner product is zero, i.e.

$$\langle \underline{u}, \underline{v} \rangle = 0 \tag{1}$$

A set of vectors $\{\underline{e}_1, \cdots, \underline{e}_n\}$ is **orthonormal** if

$$\langle \underline{e}_i, \underline{e}_j \rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{2}$$

where δ_{ij} is named as Kronecker delta.

1.2 Matrices

A $m \times n$ matrix is an array of numbers with with m rows and n columns.

1.2.1 Summation Convention

The expression for the elements of $C = AB$ is

$$C_{ij} = \sum_k A_{ik} B_{kj} \tag{3}$$

and this may be written as

$$C_{ij} = A_{ik} B_{kj} \tag{4}$$

where it is implicitly assumed that there is a summation over the repeated index k. This shorthand is known as the *Einstein summation convention*. In this expression, k is called a *dummy index*, and i and j are called as *free indices*.

There are three basic rules to index notation:

1. In any one term of an expression, indexes may appear only once, twice or not at all.

2. A index that appears only once on one side of an expression must also appear once on the other side. It is called a *free index*.

3. A index that appears twice is summed over. It is called a *dummy index*.

1.2.2 Recall Special Square Matrices

- **Unit matrix**.

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5}$$

- **Unitary matrix**. U is unitary if $UU^\dagger = U^\dagger U = \mathbb{I}$

- **Symmetric and anti-symmetric matrices**. S is symmetric, if $S^T = S$ or, alternatively, $S_{ij} = S_{ji}$. A is anti-symmetric if $A^T = -A$ or, alternatively, $A_{ij} = -A_{ji}$.

- **Hermitian and anti-Hermitian matrices**. These may be thought of as the complex generalisations of symmetric and anti-symmetric matrices. H is Hermitian if $H^\dagger = H$ or, alternatively, $H_{ij} = H^*_{ji}$. A is anti-Hermitian if $A^\dagger = -A$ or, alternatively, $A_{ij} = -A^*_{ji}$.

- **Orthogonal matrix**. R is orthogonal, if it satisfies

$$R^T R = R R^T = \mathbb{I} \quad \Leftrightarrow \quad R^T = R^{-1} \tag{6}$$

1.2.3 Levi-Civita Symbol (Alternating Tensor or Epsilon Tensor)

The Levi-Civita symbol has three indices, each of which can take value 1,2 or 3 and is defined as

$$\varepsilon_{ijk} = \begin{cases} 1, & (i,j,k) = (1,2,3),(2,3,1),(3,1,2) \\ -1, & (i,j,k) = (3,2,1),(2,1,3),(1,3,2) \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

The alternating tensor can be used to write 3-d Euclidean vector (cross) products in index notation:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \quad \Leftrightarrow \quad c_i = \varepsilon_{ijk} a_j b_k \tag{8}$$

A useful identity involving the contraction of two alternating tensors is

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{9}$$

Example. we can use it to prove the vector identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.

Proof.

$$\begin{aligned} [\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})]_i &= \varepsilon_{ijk} a_j (\boldsymbol{b} \times \boldsymbol{c})_k \\ &= \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m \\ &= (a_j c_j) b_i - (a_j b_j) c_i \\ &= (\boldsymbol{a} \cdot \boldsymbol{c})[\boldsymbol{b}]_i - (\boldsymbol{a} \cdot \boldsymbol{b})[\boldsymbol{c}]_i \end{aligned} \tag{10}$$

1.2.4 Eigenvalues, Eigenvectors and Diagonalization

An eigenvalue equation takes the form

$$A_{ij} x_j = \lambda x_i \tag{11}$$

where A_{ij} are the components of an $n \times n$ matrix, and x is an eigenvector with corresponding eigenvalue λ.
Form the $n \times n$ matrix M whose n columns are the vectors $\{e^{(1)}, ...e^{(n)}\}$. Then M is an orthogonal matrix and

$$M^{\dagger} A M = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \tag{12}$$

1.3 Scalars, Vectors and Tensors in 3d Space

- **Scalar** quantities have magnitude and are independent of the any direction.
- **Vector** quantities have magnitude and direction.
- **Rank-two tensor** quantities are an extension of the concept of a vector, and each has two indices.

$$J_i = \sigma_{ij} E_j \tag{13}$$

1.4 Transformations under Rotations

1.4.1 Transformation of Vectors

The two sets of components of x are related by an orthonal matrix L and $\det(L) = 1$

$$x'_i = L_{ij} x_j \tag{14}$$

Set of all such matrices form SO(3) group.

1.4.2 Transformation of Rank-Two Tensors

A rank-two tensor transforms as

$$T'_{ij}(x') = L_{ip} L_{jq} T_{pq}(x) \quad \Leftrightarrow \quad T' = LTL^T \tag{15}$$

For higher rank tensor,

$$T'_{ijk\cdots}(x') = L_{ip} L_{jq} L_{kr} \cdots T_{pqr\cdots}(x) \tag{16}$$

1.5 Tensor Calculus

1.5.1 The Gradient Operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \tag{17}$$

The definitions of grad, div and curl in Cartesian coordinates may be expressed using index notation:

$$[\nabla \phi]_i = \partial_i \phi \tag{18}$$

$$\nabla \cdot \boldsymbol{F} = \partial_i F_i \tag{19}$$

$$(\nabla \times \boldsymbol{F})_i = \varepsilon_{ijk} \partial_j F_k \tag{20}$$

where we have used the convenient shorthand $\partial_i = \frac{\partial}{\partial x_i}$.

2 Green Functions

2.1 Introduction

Green functions are an invaluable tool for the solution of inhomogeneous differential equations. Here we consider the second-order linear *ordinary differential equation (ODE)* with some boundary conditions. $\mathcal{L}$ is a linear second order differential operator, and

$$\mathcal{L}_x[y(x)] = \left[\frac{\mathrm{d}}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right] y(x) = f(x) \tag{21}$$

The range of the parameter x is $x \in [\alpha, \beta]$ where α might be finite or $-\infty$ and β might be finite or $+\infty$. $f(x)$ is a known function. If $f(x) = 0$, the ordinary is **homogeneous**; while when $f(x) \neq 0$, the equation is **inhomogeneous**.

Suppose that we know $y_1(x), y_2(x)$ are solutions of $\mathcal{L}_x[y(x)] = 0$, and they are linearly independent.

2.2 Variation of Parameters

From the assumptions above, we know that

$$y(x) = ay_1(x) + by_2(x) \tag{22}$$

is a set of $\mathcal{L}_x[y(x)] = 0$ for any constant a and b, and

$$y(x) = ay_1(x) + by_2(x) + y_0(x) \tag{23}$$

is a solution of $\mathcal{L}_x[y(x)] = f(x)$. y_0 is called particular integral, and is any solution of $\mathcal{L}_x[y(x)] = f(x)$.

Imposing the boundary conditions of a particular problem will result in equations for the numbers a and b in the general solution. These equations can be solved for a and b. Two boundary conditions will give two equations for the two unknown constants a and b.

We assume that the particular integral of ode is given by

$$y_0(x) = u(x)y_1(x) + v(x)y_2(x) \tag{24}$$

and the differential

$$y_0' = u'y_1 + uy_1' + v'y_2 + vy_2' \tag{25}$$

$$y_0'' = u''y_1 + 2u'y_1' + uy_1'' + v''y_2 + 2v'y_2' + vy_2'' \tag{26}$$

Substituting these expressions into the eqn.(21)

$$\begin{aligned}
\mathcal{L}_x[y_0(x)] =& u''y_1 + 2u'y_1' + uy_1'' + v''y_2 + 2v'y_2' + vy_2'' \\
& + p(u'y_1 + uy_1' + v'y_2 + vy_2') + q(uy_1 + vy_2) \\
=& u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) \\
& + u''y_1 + 2u'y_1' + v''y_2 + 2v'y_2' + p(u'y_1 + v'y_2) \\
=& u''y_1 + 2u'y_1' + v''y_2 + 2v'y_2' + p(u'y_1 + v'y_2)
\end{aligned} \tag{27}$$

Therefore, we will vary these parameters subject to the constraint

$$\boxed{u'y_1 + v'y_2 = 0} \tag{28}$$

and

$$u''y_1 + u'y_1' + v''y_2 + v'y_2' = 0 \tag{29}$$

Substituting these expressions into the differential equation (21), after some rearrangement, gives

$$\boxed{u'y_1' + v'y_2' = f} \tag{30}$$

So we have

$$\begin{cases} u'y_1' + v'y_2' = f \\ u'y_1 + v'y_2 = 0 \end{cases} \quad \Rightarrow \quad \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = M \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{31}$$

then

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ f \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1' \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{32}$$

where $W(x)$ is the *Wronskian*, and

$$W(x) = \det(M) = y_1 y_2' - y_2 y_1' \tag{33}$$

So the solutions are

$$u'(x) = -\frac{y_2(x)f(x)}{W(x)}, \qquad v'(x) = \frac{y_1(x)f(x)}{W(x)} \tag{34}$$

2.2.1 Homogeneous Initial Conditions

The boundary conditions $y(\alpha) = y'(\alpha) = 0$ are called *homogeneous initial conditions*. Integrating eqn.(34) gives

$$u(x) = -\int_\alpha^x \mathrm{d}\tilde{x} \frac{y_2(\tilde{x})f(\tilde{x})}{W(\tilde{x})}, \qquad v(x) = \int_\alpha^x \mathrm{d}\tilde{x} \frac{y_1(\tilde{x})f(\tilde{x})}{W(\tilde{x})} \tag{35}$$

then

$$y_0(x) = u(x)y_1(x) + v(x)y_2(x) = \int_\alpha^x \mathrm{d}\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) \tag{36}$$

satisfies $y_0(\alpha) = y_0'(\alpha) = 0$. So $y = y_0$ is a solution of the ODE with boundary conditions $y(\alpha) = y'(\alpha) = 0$.

$$\begin{aligned} y(x) &= \int_\alpha^x \mathrm{d}\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) + \int_x^\beta \mathrm{d}\tilde{x} \cdot 0 \\ &= \int_\alpha^\beta G(x, \tilde{x}) f(\tilde{x}) \mathrm{d}\tilde{x} \end{aligned} \tag{37}$$

where we have defined the *Green Function*

$$G(x, \tilde{x}) = \begin{cases} 0, & x < \tilde{x} \\ \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})}, & x > \tilde{x} \end{cases} \tag{38}$$

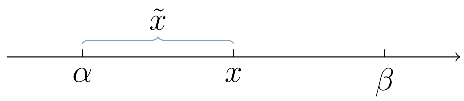

Figure 1: The range of variable x in the problem is $x \in [\alpha, \beta]$.

2.2.2 Inhomogeneous Initial Conditions

Consider more general initial conditions of the form $y(\alpha) = c_1$, $y'(\alpha) = c_2$. Choose a function $g(x)$ s.t. $g(\alpha) = c_1$ and $g'(\alpha) = c_2$. Define

$$Y(x) = y(x) - g(x) \tag{39}$$

which satisfies $Y(\alpha) = Y'(\alpha) = 0$, and $\mathcal{L}_x Y(x) = F(x)$, where

$$F(x) = f(x) - \mathcal{L}_x g(x) = f(x) - g''(x) - p(x)g'(x) - q(x)g(x) \tag{40}$$

Then we can solve for Y as before and that will give us $y(x) = Y(x) + g(x)$.

2.2.3 Homogeneous Two-Point Boundary Conditions

Consider homogeneous two-point boundary conditions $y(\alpha) = y(\beta) = 0$. A solution to eqn.(21) satisfies $y(\alpha) = 0$ is

$$y(x) = \int_\alpha^x \mathrm{d}\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) + a y_1(x) + b y_2(x) \tag{41}$$

We choose $y_1(\alpha) = y_2(\beta) = 0$. Setting $y(\alpha) = 0$ gives

$$y(\alpha) = y_0(\alpha) + a y_1(\alpha) + b y_2(\alpha) = b y_2(\alpha) = 0 \quad \Rightarrow \quad b = 0 \tag{42}$$

Similarly, setting $y(\beta) = 0$ gives

$$\begin{aligned} y(\beta) &= y_0(\beta) + a y_1(\beta) + b y_2(\beta) \\ &= -\int_\alpha^\beta \mathrm{d}\tilde{x} \frac{y_1(\beta)y_2(\tilde{x})}{W(\tilde{x})} f(\tilde{x}) + a y_1(\beta) = 0 \quad \Rightarrow \quad a = \int_\alpha^\beta \mathrm{d}\tilde{x} \frac{y_2(\tilde{x})}{W(\tilde{x})} f(\tilde{x}) \end{aligned} \tag{43}$$

which may be substituted in to the solution to give

$$\begin{aligned} y(x) &= \int_\alpha^x \mathrm{d}\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) + a y_1(x) \\ &= \int_\alpha^x \mathrm{d}\tilde{x} \frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) + \int_\alpha^\beta \mathrm{d}\tilde{x} \frac{y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) \\ &= \int_\alpha^x \mathrm{d}x \frac{y_1(\tilde{x})y_2(x)}{W(\tilde{x})} f(\tilde{x}) + \int_x^\beta \mathrm{d}x \frac{y_2(\tilde{x})y_1(x)}{W(\tilde{x})} f(\tilde{x}) \\ &= \int_\alpha^\beta \mathrm{d}\tilde{x} G(x, \tilde{x}) f(\tilde{x}) \end{aligned} \tag{44}$$

where we have defined the *Green Function*

$$G(x,\tilde{x}) = \begin{cases} \frac{y_1(\tilde{x})y_2(x)}{W(\tilde{x})}, & \alpha \le \tilde{x} < x \\ \frac{y_2(\tilde{x})y_1(x)}{W(\tilde{x})}, & x < \tilde{x} \le \beta \end{cases} \tag{45}$$

Consider $G(x,\tilde{x})$ as a function of x at a fixed value of $\tilde{x} \in [\alpha,\beta]$, which has several properties

1. When $x \neq \tilde{x}$
$$\mathcal{L}_x[G(x,\tilde{x})] = 0 \tag{46}$$

2. $G(x,\tilde{x})$ is continuous at $x = \tilde{x}$
$$\lim_{\varepsilon\to 0} [G(x,\tilde{x})]_{x=\tilde{x}-\varepsilon}^{x=\tilde{x}+\varepsilon} = \lim_{\varepsilon\to 0}\left[\frac{y_1(\tilde{x})y_2(\tilde{x}+\varepsilon)}{W(\tilde{x})} - \frac{y_2(\tilde{x})y_1(\tilde{x}-\varepsilon)}{W(\tilde{x})}\right] = 0 \tag{47}$$

3. $\frac{\partial}{\partial x}G(x,\tilde{x})$ has a unit discontinuity at $x = \tilde{x}$
$$\begin{aligned} \lim_{\varepsilon\to 0}\left[\frac{\partial G(x,\tilde{x})}{\partial x}\right]_{x=\tilde{x}-\varepsilon}^{x=\tilde{x}+\varepsilon} &= \lim_{\varepsilon\to 0}\left[\frac{y_1(\tilde{x})y_2'(\tilde{x}+\varepsilon)}{W(\tilde{x})} - \frac{y_2(\tilde{x})y_1'(\tilde{x}-\varepsilon)}{W(\tilde{x})}\right] \\ &= \frac{W(\tilde{x})}{W(\tilde{x})} = 1 \end{aligned} \tag{48}$$

2.3 Green Function More Generally

Let $G(x,\tilde{x})$ be a function that satisfies

$$\boxed{\mathcal{L}_x[G(x,\tilde{x})] = \delta(x-\tilde{x})} \tag{49}$$

$\delta(x)$ is the *Dirac delta-function* which satisfies

1. $\delta(x) = 0$ when $x \neq 0$
2. $\delta(x) = \delta(-x)$
3. $\int_a^b \delta(x-x_0)f(x)\mathrm{d}x = \begin{cases} 0, & x_0 \notin [a,b] \\ f(x_0), & x_0 \in [a,b] \end{cases}$

$G(x,\tilde{x})$ is called a *Green function* for the differential operator $\mathcal{L}_x$. If $G(x,\tilde{x})$ satisfies eqn.(49), then so does $G(x,\tilde{x}) + Y(x)$, where $\mathcal{L}_x[Y(x)] = 0$.

Now define

$$\boxed{y_0(x) = \int_\alpha^\beta \mathrm{d}\tilde{x}G(x,\tilde{x})f(\tilde{x})} \tag{50}$$

is a solution of $\mathcal{L}_x[y(x)] = f(x)$. Which can be verified by operating on both sides with $\mathcal{L}_x$, i.e.

$$\mathcal{L}_x[y_0] = \int_\alpha^\beta \mathrm{d}\tilde{x}\mathcal{L}_x[G(x,\tilde{x})]f(\tilde{x}) = \int_\alpha^\beta \mathrm{d}\tilde{x}\delta(x-\tilde{x})f(\tilde{x}) = f(x) \tag{51}$$

$f(x)$ is a "linear combination" of delta-function spikes at each $x = \tilde{x}$ with coefficient $f(\tilde{x})$. So y is a continuous linear combination of $G(x, \tilde{x})$ responses

$$y_0(x) = \int_\alpha^\beta \mathrm{d}\tilde{x} G(x, \tilde{x}) f(\tilde{x}) \tag{52}$$

This is called *linear response*.

We can now solve for a and b using the boundary conditions that y satisfies.

2.3.1 Homogeneous Initial Conditions

The boundary conditions are $y(\alpha) = y'(\alpha) = 0$. If $G(\alpha, \tilde{x}) = G'(\alpha, \tilde{x}) = 0$, then

$$y(x) = \int_\alpha^\beta \mathrm{d}\tilde{x} G(x, \tilde{x}) f(\tilde{x}) \tag{53}$$

is the solution of the ode and satisfies the boundary conditions. Now, let's look for the Green function with the right boundary conditions.

1. For $x < \tilde{x}$, $\mathcal{L}_x[G(x, \tilde{x})] = 0$. $G(x, \tilde{x}) = 0$ is a solution of the homogeneous equation that satisfies the boundary conditions that $G(\alpha, \tilde{x}) = G'(\alpha, \tilde{x}) = 0$. So for $x < \tilde{x}$
$$G(x, \tilde{x}) = 0 \tag{54}$$

2. For $x \geq \tilde{x}$, $\mathcal{L}_x[G(x, \tilde{x})] = 0$. $G(x, \tilde{x})$ equals some linear combination of $y_1(x)$ and $y_2(x)$
$$G(x, \tilde{x}) = A(\tilde{x}) y_1(x) + B(\tilde{x}) y_2(x) \tag{55}$$
We can find A and B by using the properties of G:

   (i) G is continuous at $x = \tilde{x}$
$$A(\tilde{x}) y_1(\tilde{x}) + B(\tilde{x}) y_2(\tilde{x}) = 0 \tag{56}$$

   (ii) G' has a unit discontinuity at $x = \tilde{x}$
$$A(\tilde{x}) y_1'(\tilde{x}) + B(\tilde{x}) y_2'(\tilde{x}) = 0 \tag{57}$$

   The solution is
$$A(\tilde{x}) = -\frac{y_x(\tilde{x})}{W(\tilde{x})}, \qquad B(\tilde{x}) = \frac{y_1(\tilde{x})}{W(\tilde{x})} \tag{58}$$
   where W is the Wronskian of y_1 and y_2.

So we have

$$G(x, \tilde{x}) = \begin{cases} 0, & x < \tilde{x} \\ \frac{y_1(\tilde{x}) y_2(x) - y_2(\tilde{x}) y_1(x)}{W(\tilde{x})}, & x > \tilde{x} \end{cases} \tag{59}$$

which agrees with that calculated before.

2.3.2 Homogeneous Two-Point Boundary Conditions

The boundary conditions are $y(\alpha) = y(\beta) = 0$. The Green Function should satisfies

$$G(\alpha, \tilde{x}) = G(\beta, \tilde{x}) = 0 \tag{60}$$

We assume y_1 and y_2 are linear independent solutions of homogeneous equation, and we choose $y_1(\alpha) = y_2(\beta) = 0$.

$$G(x, \tilde{x}) = \begin{cases} A(\tilde{x})y_1(x) + B(\tilde{x})y_2(x), & \alpha < x < \tilde{x} \\ C(\tilde{x})y_1(x) + D(\tilde{x})y_2(x), & \tilde{x} < x < \beta \end{cases} \tag{61}$$

1. Boundary conditions: $G(\alpha, \tilde{x}) = G(\beta, \tilde{x}) = 0$

$$A(\tilde{x})y_1(\alpha) + B(\tilde{x})y_2(\alpha) = B(\tilde{x})y_2(\alpha) = 0 \quad \Rightarrow \quad B(\tilde{x}) = 0 \tag{62}$$

$$C(\tilde{x})y_1(\beta) + D(\tilde{x})y_2(\beta) = C(\tilde{x})y_1(\beta) = 0 \quad \Rightarrow \quad C(\tilde{x}) = 0 \tag{63}$$

so we have

$$G(x, \tilde{x}) = \begin{cases} A(\tilde{x})y_1(x), & \alpha < x < \tilde{x} \\ D(\tilde{x})y_2(x), & \tilde{x} < x < \beta \end{cases} \tag{64}$$

2. Continuity of G and unit discontinuity of G' at $x = \tilde{x}$

$$A(\tilde{x})y_1(\tilde{x}) - D(\tilde{x})y_2(\tilde{x}) = 0 \tag{65}$$

$$A(\tilde{x})y_1'(\tilde{x}) - D(\tilde{x})y_2'(\tilde{x}) = 0 \tag{66}$$

so we have

$$A(\tilde{x}) = \frac{y_2(\tilde{x})}{W(\tilde{x})}, \qquad D(\tilde{x}) = \frac{y_1(\tilde{x})}{W(\tilde{x})} \tag{67}$$

The final result

$$G(x, \tilde{x}) = \begin{cases} \frac{y_2(\tilde{x})y_1(x)}{W(\tilde{x})}, & \alpha < x < \tilde{x} \\ \frac{y_1(\tilde{x})y_2(x)}{W(\tilde{x})}, & \tilde{x} < x < \beta \end{cases} \tag{68}$$

which agrees with that calculated before.

2.3.3 Higher Dimensions, More Variables

Consider a second order linear differential operator $\mathcal{L}$ on function $y(x_1, x_2, x_3)$, then

$$\mathcal{L}y = f(x_1, x_2, x_3) \tag{69}$$

and

$$\mathcal{L}G(\underline{x}, \underline{\tilde{x}}) = \delta^{(3)}(\underline{x} - \underline{\tilde{x}}) = \delta(x_1 - \tilde{x}_1)\delta(x_2 - \tilde{x}_2)\delta(x_3 - \tilde{x}_3) \tag{70}$$

Let R be a 3-d region in 3-d Euclidean space

$$\int_R \mathrm{d}\tilde{x}_1 \mathrm{d}\tilde{x}_2 \delta^{(2)}(\underline{x} - \underline{\tilde{x}}) f(\underline{\tilde{x}}) = \begin{cases} f(\underline{x}), & \underline{x} \in R \\ 0 & \underline{x} \notin R \end{cases} \tag{71}$$

Example. The most famous example is

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 \tag{72}$$

and the Green function satisfies

$$\nabla^2 G(\underline{x}, \underline{\tilde{x}}) = \delta(\underline{x} - \underline{\tilde{x}}) \tag{73}$$

Consider the Poisson equation for the scalar electric potential $\phi(\underline{x})$ in terms of the scalar charge density $\rho(\underline{x})$:

$$\nabla^2 \phi(\underline{x}) = -\frac{\rho(\underline{x})}{\varepsilon} \tag{74}$$

and

$$\phi(x) = \int \mathrm{d}\underline{\tilde{x}} G(\underline{x}, \underline{\tilde{x}}) \left[-\frac{\rho(\underline{\tilde{x}})}{\varepsilon} \right] \tag{75}$$

is a solution of Poisson's equation. The Green function for the Poisson equation that satisfying the boundary condition $G(\underline{x}, \underline{\tilde{x}}) \to 0$ as $|\underline{x}| \to \infty$ is

$$G(\underline{x}, \underline{\tilde{x}}) = \frac{1}{4\pi |\underline{x} - \underline{\tilde{x}}|} \tag{76}$$

where $|\underline{x} - \underline{\tilde{x}}| = \sqrt{(x_1 - \tilde{x}_1)^2 + (x_2 - \tilde{x}_2)^2 + (x_3 - \tilde{x}_3)^2}$.

3 Hilbert Spaces

Definition. A Hilbert space is an infinite dimensional complex vector space with inner product $\langle \cdot, \cdot \rangle$ and a infinite countable orthonormal basis $\{u_1, u_2, u_3, \cdots\}$.
The Hilbert space we will look at in this chapter will be a vector space of complex function of a real variable $x \in [a, b]$ with

1. an inner product
$$\langle f, g \rangle = \int_a^b f^*(x) g(x) \mathrm{d}x \tag{77}$$
Functions $f(x)$ and $g(x)$ are orthogonal if $\langle f, g \rangle = 0$. The *norm* of f is given by $\|f\| = \sqrt{\langle f, f \rangle}$, and $f(x)$ may be normalised in $\hat{f} = f/\|f\|$. If $\langle y_i, y_j \rangle = \delta_{ij}$, then the set of $\{y_1, y_2, y_3, \cdots\}$ is orthogonal.

2. Let $\{y_1, y_2, y_3, \cdots\}$ be an orthogonal basis, then any function $f(x) \in \mathcal{H}$ can be expanded
$$f(x) = \sum_{i=1}^{\infty} f_i y_i(x), \quad f_i \in \mathbb{C} \tag{78}$$
Then we have
$$\langle y_k, f \rangle = \langle y_k, \sum_{i=1}^{\infty} f_i y_i \rangle = \sum_{i=1}^{\infty} f_i \langle y_k, y_i \rangle = \sum_{i=1}^{\infty} f_i \delta_{ik} = f_k \tag{79}$$

3.1 Sturm-Liouville Theory

The theory of inhomogeneous differential equations of form $\mathcal{L}y(x) = f(x)$ on $x \in [a, b]$ where $\mathcal{L}$ is second order, linear and **self-adjoint**.

3.1.1 Self-Adjoint Differential Operators

Consider
$$\boxed{\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] + \sigma(x)} \tag{80}$$
and
$$\mathcal{L}y = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \sigma y = -(\rho y')' + \sigma y \tag{81}$$
where $\rho(x)$ and $\sigma(x)$ are real valued and defined on $x \in [a, b]$ and $\rho(x) > 0$ on $x \in (a, b)$. Such an operator is said to be in *self-adjoint form*[1].

Definition. A second order linear differential operator $\mathcal{D}$ is self-adjoint on Hilbert space $\mathcal{H}$ if
$$\boxed{\langle u, \mathcal{D}v \rangle = \langle v, \mathcal{D}u \rangle^*, \quad \forall u, v \in \mathcal{H}} \tag{82}$$

[1]being in self-adjoint form does not mean that a differential operator is self-adjoint, that depend on the operator and the specific Hilbert space.

c.f. the definition of a Hermitian matrix $M : M_{ij} = M_{ji}^*$.

Consider $\mathcal{L}$ as in eqn.(80),

$$\begin{aligned}
\langle u, \mathcal{L}v\rangle =& \int_a^b u^* \left[-(\rho v')' + \sigma v\right] \mathrm{d}x \\
=& -u^*\rho v'\Big|_a^b + \int_a^b \left(u^{*\prime}\rho v' + u^*\sigma v\right) \mathrm{d}x \\
=& -u^*\rho v'\Big|_a^b + u^{*\prime}\rho v\Big|_a^b + \int_a^b \left(-(u^{*\prime}\rho)'v + u^*\sigma v\right) \mathrm{d}x \\
=& (-u^*\rho v' + u^{*\prime}\rho v)\Big|_a^b + \int_a^b \left(-(u^{*\prime}\rho)' + u^*\sigma\right) v \mathrm{d}x \\
=& (-u^*\rho v' + u^{*\prime}\rho v)\Big|_a^b + \left[\int_a^b \left(-(u'\rho)' + u\sigma\right) v^* \mathrm{d}x\right]^* \\
=& (-u^*\rho v' + u^{*\prime}\rho v)\Big|_a^b + \langle v, \mathcal{L}u\rangle^*
\end{aligned} \tag{83}$$

So $\mathcal{L}$ is self-adjoint on $\mathcal{H}$ if

$$\rho(u^{*\prime}v - u^*v')\Big|_a^b = 0 \tag{84}$$

3.1.2 Boundary Conditions

1. if $\rho(a) = \rho(b) = 0$ and $u(a)u(b)$ is finite for all $u \in \mathcal{H}$, then $\mathcal{L}$ is self-adjoint.
2. if $u(a) = u(b)$ and $u'(a) = u'(b)$ for all $u \in \mathcal{H}$, and $\rho(a) = \rho(b)$, then $\mathcal{L}$ is self-adjoint. $\mathcal{H}$ is set of functions of periodic boundary conditions.
3. If $u(a) = u(b) = 0$ for all $u \in \mathcal{H}$, then $\mathcal{L}$ is self-adjoint. This is a special case of

$$\begin{cases} C_1u(a) + C_2u'(a) = 0 \\ D_1u(b) + D_2u'(b) = 0 \end{cases} \tag{85}$$

Note that these examples of boundary conditions that work are preserved under taking linear combinations

3.1.3 Weight Functions

Any second order linear differential operator can be put into self-adjoint form. Consider the most general operator

$$\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(A(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - B(x)\frac{\mathrm{d}}{\mathrm{d}x} + C(x) \tag{86}$$

where A, B, C are real and $A(x) > 0$ for $x \in [a, b]$.

Claim that there exists a function $w(x) > 0$ such that $w\tilde{\mathcal{L}}$ can be written in self-adjoint form *i.e.*

$$w(x)\left[-(Ay')' - By' + Cy\right] = -(\rho y')' + \sigma y \tag{87}$$

rearranging this

$$-w(Ay')' - Bwy' + Cwy = -(\rho y')' + \sigma y \tag{88}$$

so we have

$$\begin{cases} -(Awy')' + w'Ay' - Bwy' = -(\rho y')' \\ Cwy = \sigma y \end{cases} \tag{89}$$

then

$$\frac{w'}{w} = \frac{B}{A}, \qquad Aw = \rho, \qquad Cw = \sigma \tag{90}$$

We choose w(x) such that

$$w(x) = \exp\left[\int_a^x \frac{B(\tilde{x})}{A(\tilde{x})}\mathrm{d}\tilde{x}\right] \tag{91}$$

where $w(a) = 1$.

Definition. The inner product with weight w

$$\langle f, g\rangle_w = \langle f, wg\rangle = \int_a^b f^*(x)w(x)g(x)\mathrm{d}x = \langle wf, g\rangle \tag{92}$$

w is real.

3.1.4 Eigenfunctions and Eigenvalues

Consider the inhomogeneous eigenfunction equation

$$\tilde{\mathcal{L}}y = \lambda y \tag{93}$$

we may define an operator in self-adjoint form $\mathcal{L} = w\tilde{\mathcal{L}}$ and eigenfunction equation becomes

$$\boxed{\mathcal{L}y = \lambda wy} \tag{94}$$

A solution is called an eigenfunction of $\mathcal{L}$ with eigenvalue λ and weight $w(x)$. We claim that

1. The eigenvalues of eqn.(94) are real.
2. The eigenfunctions of eqn.(94) with distinct eigenvalues are orthogonal.

Consider two eigenfunctions, y_i and y_j of $\tilde{\mathcal{L}}$ with eigenvalues λ_i and λ_j respectively. They are also eigenfunctions of $\mathcal{L}$ with eigenvalues λ_i and λ_j and weight w. Then we have

$$\mathcal{L}y_i = \lambda_i wy_i \tag{95}$$

$$\langle y_j, \mathcal{L}y_i\rangle = \lambda_i\langle y_j, wy_i\rangle \tag{96}$$

$$\langle y_j, \mathcal{L}y_i\rangle^* = \lambda_i^*\langle y_j, wy_i\rangle^* \qquad \text{(take complex conjugate)} \tag{97}$$

$$\langle y_i, \mathcal{L}y_j\rangle = \lambda_j^*\langle y_i, wy_j\rangle = \lambda_j^*\langle y_i, y_j\rangle_w \qquad \text{(use self-adjointness)} \tag{98}$$

$$\mathcal{L}y_j = \lambda_j w y_j \tag{99}$$

$$\langle y_i, \mathcal{L}y_j\rangle = \lambda_j\langle y_i, wy_j\rangle = \lambda_j\langle y_i, y_j\rangle_w \tag{100}$$

Compare eqn.(98) and eqn.(100), we find

$$(\lambda_i^* - \lambda_j)\langle y_i, y_j\rangle_w = 0 \tag{101}$$

- For $i = j$ we have
$$(\lambda_i^* - \lambda_i)\|y_i\|_w^2 = 0 \tag{102}$$
so, if we have non-zero eigenfunctions, then $\lambda_i^* = \lambda_i$, *i.e.*, the eigenvalues are real.

- For $i \neq j$ we have
$$(\lambda_i - \lambda_j)\langle y_i, y_j\rangle_w = 0 \tag{103}$$
so, if we are considering distinct eigenvalues, then $\langle y_i, y_j\rangle_w = 0$, *i.e.*, the eigenfunctions are orthogonal with weight $w(x)$.

3.1.5 Eigenfunction Expansions

Theorem. The eigenvalues of a self-adjoint operator with w form a discrete, infinite sequence $\lambda_1, \lambda_2, \lambda_3, \cdots$ such that $|\lambda_n| \to \infty$ as $n \to \infty$, and that the corresponding eigenfunctions with weight w, $f_1, f_2, f_3 \cdots$ form a *complete orthonormal basis* for functions on $[a, b]$ in the Hilbert space. So any function $g \in \mathcal{H}$ can be expanded as

$$g(x) = \sum_n g_n f_n(x), \quad g_n \in \mathbb{C} \tag{104}$$

where

$$g_n = \langle f_n, g\rangle_\omega = \int_a^b f_n^*(x)w(x)g(x)\mathrm{d}x \tag{105}$$

Substituting into the expansion we find

$$\begin{aligned} g(x) &= \sum_n \int_a^b \mathrm{d}\tilde{x}\left[f_n^*(\tilde{x})w(\tilde{x})g(\tilde{x})\right] f_n(x) \\ &= \int_a^b \mathrm{d}\tilde{x} g(\tilde{x})\left[w(\tilde{x})\sum_n f_n(x)f_n^*(\tilde{x})\right] \\ &= \int_a^b \mathrm{d}\tilde{x}\delta(x-\tilde{x})g(\tilde{x}) \end{aligned} \tag{106}$$

where

$$\boxed{\delta(x-\tilde{x}) = w(\tilde{x})\sum_n f_n(\tilde{x})f_n^*(\tilde{x})} \tag{107}$$

Let $u \in \mathcal{H}$, consider the expression

$$\begin{aligned}\int_a^b |u|^2 \omega \mathrm{d}x =& \langle u, u\rangle_w = \langle \sum_n u_n f_n(x), \sum_m u_m f_m(x)\rangle_w \\ =& \sum_{n,m} u_n^* u_m \langle f_n, f_m\rangle_w = \sum_{n,m} u_n^* u_m \delta_{nm} = \sum_n |u_n|^2\end{aligned} \tag{108}$$

which is *Parseval's identity* in the case with a weight function $w(x)$

$$\langle u, u\rangle_w = \sum_n |u_n|^2 \tag{109}$$

3.1.6 Green Functions Revisited

If $\{y_n\}$ are a set of orthonormal eigenfunctions of self-adjoint operator $\mathcal{L}$ with weight w with corresponding eigenvalues $\{\lambda_n\}$, then the Green function for $\mathcal{L}$ is given by

$$\boxed{G(x, \tilde{x}) = \sum_n \frac{y_n(x) y_n^*(\tilde{x})}{\lambda_n}} \tag{110}$$

To prove this, we apply $\mathcal{L}$ to $G(x, \tilde{x})$

$$\begin{aligned}\mathcal{L}_x[G(x, \tilde{x})] =& \sum_n \frac{\mathcal{L}_x[y_n(x)] y_n^*(\tilde{x})}{\lambda_n} \\ =& \sum_n w(x) y_n(x) y_n^*(\tilde{x}) \\ =& \frac{\omega(x)}{\omega(\tilde{x})}\left[\omega(\tilde{x}) \sum_n y_n(x) y_n^*(\tilde{x})\right] \\ =& \delta(x - \tilde{x}) \quad \square\end{aligned} \tag{111}$$

3.1.7 Eigenfunction Expansions for Solving ODEs

As an example, consider the differential equation

$$\mathcal{L}y - \nu y = f \tag{112}$$

with some boundary conditions. $\mathcal{L}$ is a self-adjoint operator with weight function $w = 1$ and $\{y_n\}$ are eigenfunctions. Suppose $\mathcal{L}$ has eigenvalues λ_n, and corresponding eigenfunctions $\{y_n\}$, satisfying the same boundary conditions. Let

$$y(x) = \sum_n a_n y_n(x), \qquad f(x) = \sum_n f_n y_n(x) \tag{113}$$

Substituting into the original equation, we find

$$\begin{aligned} & \mathcal{L}\sum_n a_n y_n - \nu \sum_n a_n y_n = \sum_n f_n y_n \\ \Rightarrow \quad & \sum_n (a_n \lambda_n - \nu a_n) y_n = \sum_n f_n y_n \\ \Rightarrow \quad & (a_n \lambda_n - \nu a_n) = f_n\end{aligned} \tag{114}$$

So that
$$a_n = \frac{f_n}{\lambda_n - \nu}, \qquad (\lambda_n \neq \nu) \tag{115}$$
so that the solution is given by
$$y(x) = \sum_n \frac{f_n}{\lambda_n - \nu} y_n(x) \tag{116}$$

3.2 Legendre Polynomials

3.2.1 Two Examples

Example. Let
$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad x \in [0, 2\pi R] \tag{117}$$
with boundary conditions $y(0) = y(2\pi R) = 0$. Then the eigenfunction equation becomes
$$-y_n'' = \lambda_n y_n \tag{118}$$
and the eigenfunctions and the corresponding eigenvalues are
$$y_n = \sin\left(\frac{n}{2R}x\right), \qquad \lambda_n = \left(\frac{n}{2R}\right)^2, \qquad n = 1, 2, 3, \cdots \tag{119}$$

Example. Let
$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad x \in [0, 2\pi R] \tag{120}$$
with boundary conditions $y(0) = y(2\pi R)$ and $y'(0) = y'(2\pi R)$.
$$-y_m'' = \lambda_m y_m \tag{121}$$
and the eigenfunctions and the corresponding eigenvalues are
$$y_m = \exp\left(i\frac{m}{R}x\right), \qquad \lambda_m = \left(\frac{m}{2R}\right)^2, \qquad m \in \mathbb{Z} \tag{122}$$
When $m = 0$, there's the extra 'zero mode' of y_0 is a constant with eigenvalue 0.

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}y\right] = \lambda y} \tag{123}$$

Assume that eigenfunctions of eqn.(123) are polynomials

$$y_n(x) = x^{m_n} + a_{m-1}x^{m_n-1} + \cdots + a_1 x + a_0 \tag{124}$$

substituting this to the eigenfunction equation, we have

$$m_n(m_n+1) = \lambda \tag{125}$$

So eigenvalues take form

$$\lambda = l(l+1), \qquad l \in \mathbb{N} \tag{126}$$

We can label the eigenfunctions by l

- $l = 0$, $y_0(x) = 1$
- $l = 1$, $y_1(x) = x + a_0$
- $l = 2$, $y_2(x) = x^2 + a_1 x + a_0$

$$\int_{-1}^{1} y_l^*(x) y_{l'}(x) \mathrm{d}x = \delta_{ll'} \tag{127}$$

3.2.2 Legendre's Equation

Legendre's equation

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \quad \text{with} \quad x \in [-1, 1] \tag{128}$$

arises is a number of contexts in science, for example in the solution of Laplace's equation in spherical coordinates. This equation can be put into the form of a self-adjoint eigenvalue problem with $\rho = 1 - x^2$, $\sigma = 0$, $w = 1$ and $\lambda = l(l+1)$.

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)y'\right] = l(l+1)y} \tag{129}$$

or

$$\mathcal{L}y = l(l+1)y \tag{130}$$

where

$$\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right] \tag{131}$$

is self-adjoint on a Hilbert space of functions that are finite at ± 1. Assume that eigenfunctions of eqn.(129) are polynomials

$$y_n(x) = x^{m_n} + a_{m_n-1}x^{m_n-1} + \cdots + a_1 x + a_0 \tag{132}$$

Substituting the polynomial solution y_n into eqn.(129), then thinking about equation coefficients of partial of x. The highest power m_n satisfies the relation

$$m_n(m_n+1)=\lambda \tag{133}$$

So eigenvalues take form

$$\lambda = l(l+1), \qquad l\in\mathbb{N} \tag{134}$$

and can label eigenfunctions by l

- $l=0$, $y_0(x)=1$
- $l=1$, $y_1(x)=x+a_0$
- $l=2$, $y_2(x)=x^2+a_1x+a_0$

3.3 Spherical Harmonics

Laplace's equation in spherical coordinates is given by

$$\nabla^2 f(r,\theta,\phi)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{135}$$

If we take

$$f(r,\theta,\phi)=r^l\mathrm{e}^{im\phi}\Theta(\theta) \tag{136}$$

as an *ansatz*, where $l\in\mathbb{N}$ and $m\in\mathbb{Z}$, then Laplace's equation becomes

$$l(l+1)\mathrm{e}^{im\phi}\Theta(\theta)+\frac{\mathrm{e}^{im\phi}}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)-\frac{\Theta}{\sin\theta}m^2\mathrm{e}^{im\phi}=0 \tag{137}$$

Rearrange this, we have

$$\sin^2\theta l(l+1)+\frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)=m^2 \tag{138}$$

Let $u=\cos\theta$ and $\Theta(\theta)=P(u)$, where $u\in[-1,1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}=\frac{\mathrm{d}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}=-\sin\theta\frac{\mathrm{d}}{\mathrm{d}u} \tag{139}$$

Then the equation becomes

$$\underbrace{-[(1-u^2)P']'}_{\text{self-adjoint form}}+\frac{m^2}{1-u^2}P=l(l+1)P \tag{140}$$

with $\rho=1-u^2$, $\sigma=\frac{m^2}{1-u^2}$, $w=1$ and $\lambda=l(l+1)$. Now the differential operators depend on m, and there will be a different set of indefinite solutions for each m. This can show that we get non-singular solutions if $l\in\mathbb{N}$ and $m\in[-l,l]$. The solutions are called *associated Legendre polynomials* $P_l^m(u)$, which is a basis set for functions of u on $[-1,1]$.

The orthogonality

$$\int_{-1}^{1} P_l^m(u) P_{l'}^m(u) \mathrm{d}u = \frac{2(l+m)!}{(2l+1)(l-m)} \delta_{ll'} \tag{141}$$

Similarly, the equation can be expressed as

$$\underbrace{-[(1-u^2)P']'}_{\text{self-adjoint form}} - l(l+1)P = -\frac{m^2}{1-u^2} P \tag{142}$$

with $\rho = 1 - u^2$, $\sigma = -l(l+1)$ and $w = \frac{1}{1-u^2}$ This show that

$$\int_{-1}^{1} \frac{P_l^m(u) P_l^{m'}(u)}{1-u^2} \mathrm{d}u = \frac{(l+m)!}{m(l-m)} \delta_{mm'} \tag{143}$$

Finally we get

$$Y_l^m = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \mathrm{e}^{im\phi}, \quad l \in \mathbb{N}, -l \leq m \leq l \tag{144}$$

they are solutions of $\nabla^2 Y_l^m = 0$, and form an orthogonal basis of function on $\mathbf{S}^2$

$$\delta_{ll'} \delta_{mm'} = \int_0^{2\pi} \int_0^{\pi} Y_l^m(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) \sin\theta \mathrm{d}\theta \mathrm{d}\phi \tag{145}$$

So any function f can be expressed as

$$f(\theta, \phi) = \sum_l \sum_{m=-l}^{l} f_{lm} Y_l^m(\theta, \phi) \tag{146}$$

where

$$f_{lm} = \int_{\mathbf{S}^2} Y_l^{m*} f \mathrm{d}\Omega \tag{147}$$

4 Integral Transforms

4.1 Fourier Series

Consider $f(x)$ has a period of $2\pi R$, we can express $f(x)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} f_n y_n(x) \tag{148}$$

where

$$y_n(x) = \frac{1}{\sqrt{2\pi R}} \mathrm{e}^{inx/R} \tag{149}$$

and we have

$$\langle y_n, y_m \rangle = \int_0^{2\pi R} y_n^* y_m \mathrm{d}x = \delta_{nm} \tag{150}$$

We choose $x \in [-\pi R, \pi R]$, then

$$\begin{aligned} f_n &= \int_{-\pi R}^{\pi R} y_n^*(x) f(x) \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi R}} \int_{-\pi R}^{\pi R} \mathrm{e}^{-inx/R} f(x) \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi R}} \int_{-\pi R}^{\pi R} \mathrm{e}^{-ik_n x} f(x) \mathrm{d}x \end{aligned} \tag{151}$$

where $k_n = n/R$, $x \in (-\infty, \infty)$. Let $R \to \infty$ and k_n take the real continuous values from $-\infty$ to ∞, we define that

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-ikx} f(x) \mathrm{d}x \tag{152}$$

for f satisfying $\int_{-\infty}^{\infty} |f| \mathrm{d}x$ is finite. $\tilde{f}(k)$ is the *Fourier transform* of $f(x)$.

4.2 Fourier Transforms

4.2.1 Definition and Notation

Definition. Fourier transform

$$\boxed{\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-ikx} \mathrm{d}x} \tag{153}$$

The *inverse Fourier transform* is defined as

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) \mathrm{e}^{ikx} \mathrm{d}k} \tag{154}$$

In other words, this operation on $\tilde{f}(k)$ is the inverse Fourier transform and we can define

$$\mathrm{FT}^{-1}[\mathrm{FT}(f)] = f \quad \Rightarrow \quad \mathrm{FT}^{-1}\mathrm{FT} = \mathbb{1} \tag{155}$$

4.2.2 Dirac Delta-Function

$$\begin{aligned} f(x) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) \mathrm{e}^{ikx} \mathrm{d}k \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') \mathrm{e}^{-ikx'} \mathrm{d}x' \right] \mathrm{e}^{ikx} \mathrm{d}k \\ =& \int_{-\infty}^{\infty} f(x') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ik(x-x')} \mathrm{d}k \right] \mathrm{d}x' \\ =& \int_{-\infty}^{\infty} f(x') \delta(x-x') \mathrm{d}x' \end{aligned} \tag{156}$$

where we have defined the *Dirac delta-function*

$$\boxed{\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ik(x-x')} \mathrm{d}k} \tag{157}$$

4.2.3 Properties of the Fourier Transform

1. If $f(x)$ is a real function $[f(x)]^* = f(x)$

$$\tilde{f}^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-i(-k)x} f^*(x) \mathrm{d}x = \tilde{f}(-k) \tag{158}$$

- If $f(x)$ is an even function $f(-x) = f(x)$, then $\tilde{f}(x)$ is a pure real function.
  Proof. Define $y = -x$, then

$$\tilde{f}^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-iky} f(-y) \mathrm{d}(-y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-iky} f(y) \mathrm{d}y = \tilde{f}(k) \tag{159}$$

- If $f(x)$ is an off function $f(-x) = -f(x)$, then $\tilde{f}(x)$ is a pure imaging function.
  Proof. Define $y = -x$, then

$$\tilde{f}^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-iky} f(-y) \mathrm{d}(-y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-iky} f(y) \mathrm{d}y = -\tilde{f}(k) \tag{160}$$

2. Differentiation

$$\mathrm{TF}[f^{(n)}(x)] = (ik)^n \tilde{f}(k) \tag{161}$$

Proof. Consider the first order derivative

$$\begin{aligned} \mathrm{TF}[f'(x)] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{-ikx} f'(x) \\ =& \frac{1}{\sqrt{2\pi}} \left[f(x) \mathrm{e}^{-ikx} \right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x f(x) (-ik) \mathrm{e}^{-ikx} \\ =& ik \tilde{f}(k) \end{aligned} \tag{162}$$

3. Multiplication by x

$$\mathrm{FT}[xf(x)] = i\frac{\mathrm{d}}{\mathrm{d}x}\tilde{f}(k) \tag{163}$$

4. Rigid shift of coordinate

$$\mathrm{FT}[f(x-a)] = \mathrm{e}^{-ika}\tilde{f}(k) \tag{164}$$

Proof. Define $y = x - a$, then

$$\begin{aligned}\mathrm{FT}[f(x-a)] =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-ikx}f(x-a)\mathrm{d}x \\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-ika}\mathrm{e}^{-ik(x-a)}f(x-a)\mathrm{d}(x-a) \\ =& \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ika}\int_{-\infty}^{\infty}\mathrm{e}^{-iky}f(y)\mathrm{d}y = \mathrm{e}^{-ika}\tilde{f}(k)\end{aligned} \tag{165}$$

4.2.4 Parseval's Theorem

Parseval's theorem for Fourier transforms states that

$$\boxed{\int_{-\infty}^{\infty}|f(x)|^2\mathrm{d}x = \int_{-\infty}^{\infty}|\tilde{f}(k)|^2\mathrm{d}k} \tag{166}$$

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty}|f(x)|^2\mathrm{d}x =& \int_{-\infty}^{\infty}f(x)f^*(x)\mathrm{d}x \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\mathrm{d}k\int_{-\infty}^{\infty}\mathrm{d}k'\tilde{f}(k)\tilde{f}^*(k')\mathrm{e}^{i(k-k')x}\right]\mathrm{d}x \\ =& \int_{-\infty}^{\infty}\mathrm{d}k\int_{-\infty}^{\infty}\mathrm{d}k'\tilde{f}(k)\tilde{f}^*(k')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i(k-k')x}\mathrm{d}x\right] \\ =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(k)\tilde{f}^*(k')\delta(k-k')\mathrm{d}k\mathrm{d}k' \\ =& \int_{-\infty}^{\infty}\tilde{f}(k)\tilde{f}^*(k)\mathrm{d}k = \int_{-\infty}^{\infty}|\tilde{f}(k)|^2\mathrm{d}k\end{aligned} \tag{167}$$

4.2.5 Convolution Theorem

The convolution of f and g is defined as

$$\boxed{f * g = \int_{-\infty}^{\infty}f(y)g(x-y)\mathrm{d}y} \tag{168}$$

with claims

1. $f * g = g * f$

2. $f * \delta = f$

The convolution theorem can be stated in two, equivalent forms.

1.
$$\begin{aligned}\mathrm{FT}(f*g) =&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{e}^{-ikx}\int_{-\infty}^{\infty}\mathrm{d}y f(y)g(x-y)\\ =&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-iky}f(y)\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{e}^{-ik(x-y)}g(x-y)\\ =&\sqrt{2\pi}\tilde{f}(k)\tilde{g}(k)=\sqrt{2\pi}\mathrm{FT}[f]\mathrm{FT}[g]\end{aligned} \tag{169}$$

2.
$$\mathrm{FT}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}}\tilde{f}(k)*\tilde{g}(k) \tag{170}$$

4.2.6 Examples of Fourier Transform

1. Constant function $f(x) = 1$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-ikx}\mathrm{d}x = \sqrt{2\pi}\delta(k) \tag{171}$$

2. Single frequency/wavenumber mode $f(x) = \mathrm{e}^{ik_0x}$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{ik_0x}\mathrm{e}^{-ikx}\mathrm{d}x = \sqrt{2\pi}\delta(k-k_0) \tag{172}$$

3. Dirac delta-function $f(x) = \delta(x-x_0)$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-x_0)\mathrm{e}^{-ikx}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ikx_0} \tag{173}$$

4. Gaussian function $f(x) = \frac{1}{\sigma(2\pi)^{1/4}}\mathrm{e}^{-x^2/4\sigma^2}$

$$\begin{aligned}\tilde{f}(k) =&\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma(2\pi)^{1/4}}\int_{-\infty}^{\infty}\mathrm{e}^{-x^2/4\sigma^2}\mathrm{e}^{-ikx}\mathrm{d}x\\ =&\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma(2\pi)^{1/4}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{4\sigma^2}-ikx\right)\mathrm{d}x\\ =&\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma(2\pi)^{1/4}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{4\sigma^2}\left(x+ik2\sigma\right)^2-k^2\sigma^2\right]\mathrm{d}x\\ =&\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma(2\pi)^{1/4}}\mathrm{e}^{-k^2\sigma^2}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{4\sigma^2}\left(x+ik2\sigma\right)^2\right]\mathrm{d}x\\ =&\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma(2\pi)^{1/4}}\mathrm{e}^{-k^2\sigma^2}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{4\sigma^2}x'^2\right)\mathrm{d}x'\\ =&\frac{\sqrt{2\sigma}}{(2\pi)^{\frac{1}{4}}}\mathrm{e}^{-k^2\sigma^2}\end{aligned} \tag{174}$$

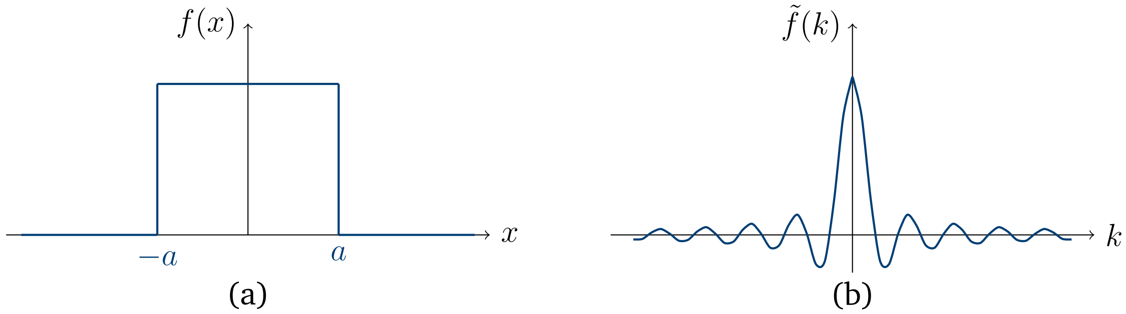

Figure 2: Top-hat function.

5. Top-hat function $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}$

$$\begin{aligned} \tilde{f}(k) =& \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{e}^{-ikx} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{ik} \mathrm{e}^{-ikx} \right]_{-a}^{a} \\ =& \sqrt{\frac{2}{\pi}} \frac{\sin(ka)}{k} = a\sqrt{\frac{2}{\pi}} \mathrm{sinc}(ak) \end{aligned} \tag{175}$$

4.3 The Applications of Fourier Transforms in Physics

4.3.1 Diffraction Through an Aperture

The geometry for Fraunhofer diffraction see Fig.3(a). For small values of θ we have $\theta \approx \sin\theta \approx \tan\theta = \frac{X}{D}$. The aperture function is given by a top-hat

$$h(x) = \begin{cases} 1 & |x| < \frac{a}{2} \\ 0 & |x| \geq \frac{a}{2} \end{cases} \tag{176}$$

so we have

$$\tilde{h}(k) = \frac{a}{\sqrt{2\pi}} \mathrm{sinc}\left(\frac{ak}{2}\right) \tag{177}$$

The intensity $I(k)$ of light observed in the diffraction pattern is the square of the Fourier transform of the aperture function $f(x)$

$$I(x = X) = I\left(k_x = \frac{2\pi X}{\lambda D}\right) = |\tilde{h}(k_x)|^2 = \frac{a^2}{2\pi} \mathrm{sinc}^2\left(\frac{a\pi X}{2\lambda D}\right) \tag{178}$$

4.3.2 Double Slit Diffraction

The aperture function is given by

$$h(x) = f(x)g(x) \tag{179}$$

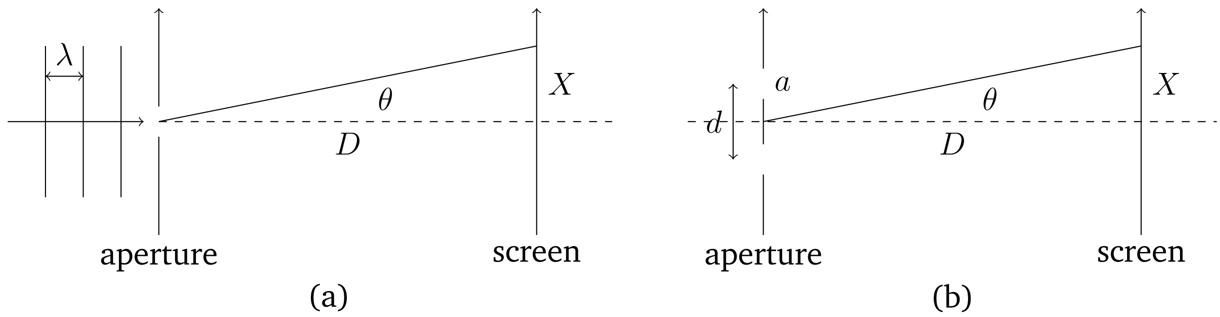

Figure 3: Geometry for Fraunhofer diffraction. (a) Diffraction through an aperture. (b) Double slit diffraction.

where

$$f(x) = \delta\left(x - \frac{d}{2}\right) + \delta\left(x + \frac{d}{2}\right) \tag{180}$$

and $g(x)$ is single aperture function. And

$$\begin{aligned} \tilde{f}(k) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{-ikx} \left[\delta\left(x - \frac{d}{2}\right) + \delta\left(x + \frac{d}{2}\right)\right] \\ =& \frac{1}{\sqrt{2\pi}} \left(\mathrm{e}^{-ikd/2} + \mathrm{e}^{ikd/2}\right) = \sqrt{\frac{2}{\pi}} \cos\left(\frac{kd}{2}\right) \end{aligned} \tag{181}$$

so we have

$$\begin{aligned} \mathrm{TF}(f * g) =& \sqrt{2\pi} \tilde{f}(k) \tilde{g}(k) \\ =& \sqrt{2\pi} \sqrt{\frac{2}{\pi}} \cos\left(\frac{kd}{2}\right) \frac{a}{\sqrt{2\pi}} \mathrm{sinc}\left(\frac{ak}{2}\right) \\ =& \sqrt{\frac{2}{\pi}} a \mathrm{sinc}\left(\frac{ak}{2}\right) \cos\left(\frac{kd}{2}\right) \end{aligned} \tag{182}$$

and the intensity on the screen is given by

$$I(k) = \frac{2a^2}{\pi} \mathrm{sinc}^2\left(\frac{ak}{2}\right) \cos^2\left(\frac{kd}{2}\right) \tag{183}$$

4.3.3 Diffusion Equation

Consider an infinite, one-dimensional conducting bar. The flow of heat is determined by the diffusion equation

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2} \tag{184}$$

where θ is the heat distribution. The boundary conditions on this problem is $\theta(\pm\infty, t = 0)$ and $\theta(x, t = 0) = \delta(x)$.

$$\frac{\partial}{\partial t} \tilde{\theta}(k, t) = D(ik)^2 \tilde{\theta}(k, t) = -Dk^2 \tilde{\theta}(k, t) \tag{185}$$

the solution is

$$\tilde{\theta}(k,t) = \tilde{\theta}(k,0)\mathrm{e}^{-Dk^2t} = \mathrm{FT}[\delta(x)]\mathrm{e}^{-Dk^2t} = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-Dk^2t} \tag{186}$$

So we have

$$\begin{aligned}\theta(x,t) =& \mathrm{FT}^{-1}[\tilde{\theta}(k,t)] \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \mathrm{e}^{ikx}\mathrm{e}^{-Dk^2t} \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \exp\left[-Dt\left(k-\frac{ix}{2Dt}\right)^2 - \frac{x^2}{4Dt}\right]\mathrm{d}k \\ =& \frac{1}{2\pi}\mathrm{e}^{-x^2/4Dt}\int_{-\infty}^{\infty} e^{-Dtq^2}\mathrm{d}q \\ =& \frac{1}{2\pi}\sqrt{\frac{\pi}{Dt}}\mathrm{e}^{-x^2/4Dt} \qquad \left(\int_{-\infty}^{\infty}\mathrm{e}^{-\alpha x^2}\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}\right)\end{aligned} \tag{187}$$

Hence the final result

$$\theta(x,t) = \frac{1}{2\sqrt{\pi Dt}}\mathrm{e}^{-x^2/4Dt} \tag{188}$$

4.4 Laplace Transforms

Laplace transforms is useful for initial value problem where $f(t)$ only exists for $t \geq 0$.

$$\mathcal{L}[f(t)] = \hat{f}(s) = \int_0^{\infty} \mathrm{d}t\mathrm{e}^{-st}f(t) \tag{189}$$

where s is a complex variable and $\mathrm{Re}(S) > 0$ is required for the convergence of the integral.

4.4.1 Properties

- $\mathcal{L}[f'(t)] = s\hat{f}(s) - f(0)$
  Proof.

$$\begin{aligned}\mathcal{L}[f'(t)] =& \int_0^{\infty}\mathrm{d}t\mathrm{e}^{-st}f'(t) \\ =& \mathrm{e}^{-st}f(t)\big|_0^{\infty} + s\int_0^{\infty}\mathrm{d}t\mathrm{e}^{-st}f(t) = s\hat{f}(s) - f(0)\end{aligned} \tag{190}$$

- $\mathcal{L}[f^{(n)}(t)] = s^n\hat{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
- $\mathcal{L}[t^n f(t)] = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}s^n}\hat{f}(s)$
  Proof.

$$\begin{aligned}(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}s^n}\hat{f}(s) =& (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}s^n}\int_0^{\infty}\mathrm{d}t\mathrm{e}^{-st}f(t) \\ =& (-1)^n\int_0^{\infty}\mathrm{d}t(-t)^n\mathrm{e}^{-st}f(t) \\ =& \int_0^{\infty}\mathrm{d}t\mathrm{e}^{-st}t^nf(t) = \mathcal{L}[t^nf(t)]\end{aligned} \tag{191}$$

4.4.2 Examples

- $\mathcal{L}[1] = \frac{1}{s}$
- $\mathcal{L}[\mathrm{e}^{-at}] = \frac{1}{s+a}$
- $\mathcal{L}[\cos\omega t] = \frac{s}{s^2+w^2}$
- $\mathcal{L}[\sin\omega t] = \frac{w}{s^2+w^2}$
- $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$
- $\mathcal{L}[t\mathrm{e}^{-at}] = \frac{1}{(s+a)^2}$
- $\mathcal{L}[\mathrm{e}^{at}f(t)] = \hat{f}(s-a)$

4.4.3 Convolution Theorem for Laplace Transforms

A convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as

$$f_1 * f_2 = \int_{-\infty}^{\infty} f_1(t')f_2(t-t')\mathrm{d}t' \tag{192}$$

If f_1 and f_2 vanish for $t < 0$, then

$$f_1 * f_2 = \int_0^t f_1(t')f_2(t-t')\mathrm{d}t' \tag{193}$$

If we apply the Laplace transform

$$\begin{aligned}\mathcal{L}[f_1 * f_2] &= \int_0^{\infty} \mathrm{d}t\mathrm{e}^{-st}\int_0^t f_1(t')f_2(t-t')\mathrm{d}t' \\ &= \int_0^{\infty} \mathrm{d}t' f_1(t')\int_{t'}^{\infty} \mathrm{d}t\mathrm{e}^{-st}f_2(t-t') \\ &= \int_0^{\infty} \mathrm{d}t'\mathrm{e}^{-st'} f_1(t')\int_{t'}^{\infty} \mathrm{d}t\mathrm{e}^{-s(t-t')}f_2(t-t') \\ &= \tilde{f}_1(s)\tilde{f}_2(s)\end{aligned} \tag{194}$$

Example. Consider the differential equation

$$f'' + 5f' + 6f = 0 \tag{195}$$

with boundary conditions $f'(0) = f(0) = 0$. Apply the Laplace transform on the equation, we have

$$s^2\hat{f} - sf(0) - f'(0) + 5[s\hat{f} - f(0)] + 6\hat{f} = \hat{f}(s^2+5s+6) = \frac{1}{s} \tag{196}$$

rearranging this, we have

$$\hat{f}(s) = \frac{1}{s(s+2)(s+3)} = \frac{1}{6s} - \frac{1}{2}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s+3} \tag{197}$$

So

$$f(t) = \frac{1}{6} - \frac{1}{2}\mathrm{e}^{-2t} + \frac{1}{3}\mathrm{e}^{-3t} \tag{198}$$

5 Complex Analysis

5.1 Complex Functions of a Complex Variable

A complex number $z = x + iy$ can be mapped to another complex number $w = f(z) = u(x, y) + iv(x, y)$. It is often useful to use the 'polar representation' of complex numbers where

$$z = r\mathrm{e}^{i\theta} \tag{199}$$

where $r = |z| = \sqrt{x^2 + y^2}$ is called the modulus of z and $\theta = \arg(z)$ is called the argument of z. $\arg(z)$ can be made unambiguous by a choice of 'branch'. We will write the principal branch as $\mathrm{Arg}(z)$, which is values $-\pi < \mathrm{Arg}(z) \leq \pi$.

Example.

1) $f(z) = |z| = \sqrt{x^2 + y^2}$

2) $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

3) $f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i\,(3x^2y - y^3)$

4) $f(z) = z^{1/3} = r^{1/3}\mathrm{e}^{(i\theta+2\pi in)/3} = \begin{cases} r^{1/3}\exp\left(\frac{i\theta}{3}\right) \\ r^{1/3}\exp\left(\frac{i\theta}{3} + \frac{2\pi i}{3}\right) \\ r^{1/3}\exp\left(\frac{i\theta}{3} + \frac{4\pi i}{3}\right) \end{cases}$

Complex functions defined as power series

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{200}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \tag{201}$$

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad (|z| < 1) \tag{202}$$

5.2 Continuity, Differentiability and Analyticity

5.2.1 Definitions

Definition. $f(z)$ is *continuous* at $z = z_0$ if $\forall\varepsilon > 0$, there exists a $\delta > 0$, such that, if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$. We also say

$$\lim_{z\to z_0} f(z) = f(z_0) \tag{203}$$

Definition. $f(z)$ is differentiable at $z = z_0$ if $\exists F \in \mathbb{C}$ such that

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = F \tag{204}$$

we say $f'(z_0) = (\mathrm{d}f/\mathrm{d}z)\big|_{z_0} = F$.

Definition. A subset $D \in \mathbb{C}$ is *open* if for every $z \in D$, there is an open disc centred at z entirely contained in D.

Definition. A function $f(z)$ is *analytic* at z_0 if $f(z)$ is differentiable everywhere in an open domain containing z_0; if $f(z)$ is NOT analytic at z_0 we say $f(z)$ is *singular* at z_0.

Example. $f(z) = z^2$ and $z = z_0 + \delta z$

$$\lim_{\delta z \to 0} \frac{(z_0 + \delta z)^2 - z_0^2}{\delta z} = 2z_0 \tag{205}$$

$f(z) = z^2$ is differentiable everywhere in $\mathbb{C}$. So we say $f(z)$ is analytic in C and $f(z)$ is entire.

Example. $f(z) = z^* = x - iy$ and $z = z_0 + \delta z$

$$\lim_{\delta z \to 0} \frac{(z_0 + \delta z)^* - z_0^*}{\delta z} = \lim_{\delta z \to 0} \frac{\delta z^*}{\delta z} = \mathrm{e}^{-2i\theta} \tag{206}$$

$f(z) = z^*$ is not differentiable anywhere so $f(z)$ is not analytic in $\mathbb{C}$.

5.2.2 The Cauchy-Riemann Conditions

In this section we ask: *Under what conditions is a complex function $f(z) = u(x,y) + iv(x,y)$ analytic in a domain D?* Let us assume that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist in D, *i,e,* $f(z)$ is analytic in D.

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}\frac{\mathrm{d}f}{\mathrm{d}z} = f', \qquad \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}\frac{\mathrm{d}f}{\mathrm{d}z} = if' \tag{207}$$

which shows

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \quad \Rightarrow \quad i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \tag{208}$$

Rearranging this, now we get the *Cauchy-Riemann equations*

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{209}$$

It is a theorem that $f(z)$ is analytic if and only id Cauchy-Riemann equations hold in D.

Example. $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + 2ixy$. In this function, $u = x^2 - y^2$ and $v = 2xy$.

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = -2y \tag{210}$$

$$\frac{\partial v}{\partial x} = 2y, \qquad \frac{\partial v}{\partial y} = 2x \tag{211}$$

satisfy the C-R equations.

Example. $f(z) = x = (z + z^*)/2$. In this function, $u = x$ and $v = 0$, so we have

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0 \tag{212}$$

C-R equations fail.

Example. $f(z) = x^2 + y^2 = zz^*$ with $u = x^2 + y^2$ and $v = 0$.

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = 2y, \qquad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \tag{213}$$

So $f(z)$ satisfies C-R equations at $x = y = 0$ but nowhere else.

Theorem. $f(z)$ is analytic at $z = z_0$ if and only if $f(z)$ has a power series expansion around $z = z_0$ that converges in an open neighborhood of z_0.

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots = \sum_{k=0}^{\infty} c_k(z - z_0)^2 \tag{214}$$

with

$$c_k = \frac{f^{(k)}(z_0)}{k!} \tag{215}$$

5.2.3 Harmonic Functions

Definition. $g(x, y)$ is harmonic if $\nabla^2 g = 0$.
Now we look at C-R equations

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \tag{216}$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0 \tag{217}$$

$u(x, y)$ is harmonic. Similarly, $v(x, y)$ is harmonic. We conclude that if $f = u + iv$ is analytic, u and v are *conjugate* harmonic functions.

Example. Consider the real function $u(x, y) = \cos x \cosh y$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0 \tag{218}$$

hence u is harmonic. Then we find the conjugate harmonic function $v(x, y)$. Using the C-R equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -\sin x \cosh y \quad \Rightarrow \quad v = -\sin x \sinh y + c_1(y) \tag{219}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\cos x \sinh y \quad \Rightarrow \quad v = -\sin x \sinh y + c_2(x) \tag{220}$$

so that $c_1 = c_2 = c$ and $v(x, y) = -\sin x \sinh y + c$, where c is a constant. Hence

$$f(z) = \cos x \cosh y - i \sin x \sinh y + \tilde{c} \tag{221}$$

is analytic by construction.

5.3 Multi-Valued Functions

Example. $f(z) = z^{1/3}$. There are three related branches of $z^{1/3}$

$$\begin{cases} F_1(z) = r^{1/3} e^{i\theta/3} \\ F_2(z) = r^{1/3} e^{i\theta/3 + 2\pi i/3} \\ F_3(z) = r^{1/3} e^{i\theta/3 + 4\pi i/3} \end{cases} \tag{222}$$

with $\theta \in [-\pi, \pi]$. Each one is single valued, but discontinuous along the negative real axis. If we glue sheets together on the branch cuts, then the three sheets form a *Riemann surface*. $f(z) = z^{1/3}$ is defined on the Riemann surface on the following way

$$f(z) = F_i(z) \quad \text{on sheet } i \tag{223}$$

$f(z)$ is single valued and continuous on the Riemann surface.

Example. $f(z) = z^{1/2}$: 2 branches and 2 Riemann sheets.

Example. $f(z) = z^{1/n}$: n branches and n Riemann sheets.

Example. $f(z) = \ln z = \ln\left(re^{i\theta}\right)$ not defined at $z = 0$.

$$f(z) = \ln r + i\theta + 2\pi i n \tag{224}$$

has one branch for each integer n.

Example. $f(z) = (z - z_0)^{1/3}$. A *branch point* is a point that cannot be encircled without moving on to a different sheet of the Riemann surface of $f(z)$.

Example. $f(z) = (z-a)^{1/2}(z-b)^{1/2}$, $a, b \in \mathbb{R}$. The function has two branch points a and b, the branch cuts must begin or end there (see Fig.4).

5.4 Integration of Complex Functions

5.4.1 Contours

We focus on *contour integrals*, $\int_C f(z)\mathrm{d}z$, along lines or paths C in the complex plane.

Example. Evaluate $\int_c z\mathrm{d}z$ along (i) $y = x^2$ and (ii) $y = x$.

$$\int_C z\mathrm{d}z = \int_C (x+iy)(\mathrm{d}x + i\mathrm{d}y) = \int_C (x\mathrm{d}x - y\mathrm{d}y) + i\int_C (y\mathrm{d}x + x\mathrm{d}y) \tag{225}$$

(i) $\int_0^1 \int (x\mathrm{d}x - 2x^3\mathrm{d}x) + i\int_0^1 (x^2\mathrm{d}x + 2x^2\mathrm{d}x) = i$

(ii) $\int_0^1 (x\mathrm{d}x - x\mathrm{d}x) + i\int_0^1 (x\mathrm{d}x + x\mathrm{d}x) = i$

Figure 4: The two possible ways to place branch cuts for $f(z) = (z-a)^{1/2}(z-b)^{1/2}$, and they form the same Riemann surface.

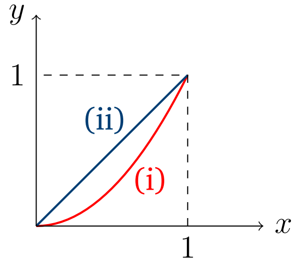

Figure 5: The two paths, (i) $y = x^2$ and (ii) $y = x$, along with the function $f(z)$ is to be integrated in the example.

5.4.2 Cauchy's Theorem

Theorem. Cauchy's theorem. If $f(z)$ is analytic everywhere on and within a closed contour C

$$\oint_C f(z)\mathrm{d}z = 0 \tag{226}$$

Theorem. Green's theorem in the plane. P and Q are functions of x and y, and C is a closed contour in the $x - y$ plane, then

$$\oint_C (P\mathrm{d}x + Q\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y \tag{227}$$

Proof. Proof of Cauchy's theorem

$$\begin{aligned}
\oint_C f(z)\mathrm{d}z &= \oint_C (u(x,y) + iv(x,y))(\mathrm{d}x + i\mathrm{d}y) \\
&= \oint_C (u\mathrm{d}x - v\mathrm{d}y) + i\oint_C (v\mathrm{d}x + u\mathrm{d}y) \\
&= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x\mathrm{d}y + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \mathrm{d}x\mathrm{d}y = 0
\end{aligned} \tag{228}$$

5.4.3 Path Independence

Theorem. Let C_1 and C_2 be two contours from z_a to z_b. If $f(z)$ is analytic on C_1 and C_2 and the region between them, then

$$\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z \tag{229}$$

Proof. Consider closed contour $C = C_1 - C_2$. By Cauchy's theorem

$$\oint_C f(z)\mathrm{d}z = \int_{C_1} f(z)\mathrm{d}z - \int_{C_2} f(z)\mathrm{d}z = 0 \tag{230}$$

5.4.4 Contour Deformation

Theorem. If C_1 and C_2 are closed contours, and C_1 can be defined into C_2 entirely in a region where $f(z)$ is analytic, then

$$\oint_{C_1} f(z)\mathrm{d}z = \oint_{C_2} f(z)\mathrm{d}z \tag{231}$$

Proof. Choose line segment AB as shown in the Fig.6. Consider $C = C_1 + \overline{BA} - C_2 + \overline{AB}$. By Cauchy's theorem

$$\begin{aligned}\oint_C f(z)\mathrm{d}z &= \left(\int_{C_1} + \int_{\overline{BA}} - \int_{C_2} + \int_{\overline{AB}}\right) f(z)\mathrm{d}z \\ &= \int_{C_1} f(z)\mathrm{d}z - \int_{C_2} f(z)\mathrm{d}z = 0\end{aligned} \tag{232}$$

Example. Evaluate $\oint_{|z=1|} \frac{1}{z}\mathrm{d}z$. Deform the contour into a small circle, radius r, centred on the origin, then

$$\oint_{|z=1|} \frac{1}{z}\mathrm{d}z = \oint_{|z=1|} \mathrm{e}^{-i\theta} i\mathrm{e}^{i\theta}\mathrm{d}\theta = 2\pi i \tag{233}$$

5.4.5 Cauchy's Integral Theorem

Theorem. If $f(z)$ is analytic within and on a closed contour C and z_0 is any point within C, then

$$\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z = 2\pi i f(z_0) \tag{234}$$

or

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z \tag{235}$$

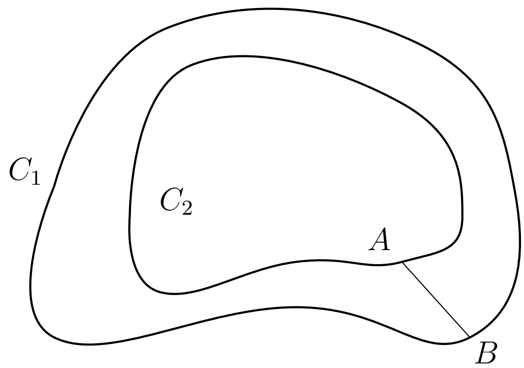

Figure 6: Caption

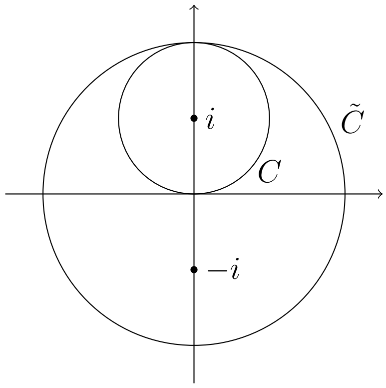

Figure 7: The contour C and $\tilde{C}$.

Proof. The integral is analytic within and on C except at $z = z_0$. Let C_r be a small circle around z_0, *i.e.* $C_r : z = z_0 + re^{i\theta}$, then

$$\begin{aligned}\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z = \oint_{C_r} \frac{f(z)}{z-z_0}\mathrm{d}z = \int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\mathrm{d}\theta \\ = i\int_0^{2\pi} f(z_0+re^{i\theta})\mathrm{d}\theta = \lim_{r\to 0} i\int_0^{2\pi} f(z_0+re^{i\theta})\mathrm{d}\theta = 2\pi i f(z_0)\end{aligned} \tag{236}$$

Example. Consider the integral

$$\oint \frac{\sin z}{z^2+1}\mathrm{d}z = \oint_C \frac{\sin z}{(z+i)(z-i)}\mathrm{d}z \tag{237}$$

and consider the closed contour C and $\tilde{C}$, which are showed in Fig.7. For the contour C, We choose

$$f(z) = \frac{\sin z}{z+i} \tag{238}$$

Then

$$\oint_C \frac{\sin z}{z^2+1}\mathrm{d}z = \oint_C \frac{f(z)}{z-i}\mathrm{d}z = 2\pi i \frac{\sin i}{2i} = \pi i \sinh 1 \tag{239}$$

$\tilde{C}$ is a circle of radius 2 centred at origin, so

$$\begin{aligned}\oint_{\tilde{C}} \frac{\sin z}{z^2+1}\mathrm{d}z = \oint_{\tilde{C}} \frac{\sin z}{(z+i)(z-i)}\mathrm{d}z = \frac{i}{2}\oint_{\tilde{C}} \left(\frac{\sin z}{z+i} - \frac{\sin z}{z-i}\right)\mathrm{d}z \\ = -\pi(\sin(-i) - \sin(i)) = 2\pi i \sinh 1\end{aligned} \tag{240}$$

5.4.6 Derivatives of Analytic Functions

Cauchy's integral theorem gives

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z \tag{241}$$

If we differentiate both sides of Cauchy's integral formula with respect to z_0, interchanging the orders of integration and differentiation, we get

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\mathrm{d}z \tag{242}$$

Similarly,

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\mathrm{d}z \tag{243}$$

$$\vdots$$

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z_0)}{(z-z_0)^{n+1}}\mathrm{d}z} \tag{244}$$

Example.

$$\oint_C \frac{1}{z^n}\mathrm{d}z \quad \text{with} \quad C: |z| = r \tag{245}$$

- $n = 1$, $\oint_C (1/z)\mathrm{d}z = 2\pi i$
- $n \geq 2$, $\oint_C (1/z)\mathrm{d}z = 0$

5.4.7 Fourier Transform of a Gaussian

We have known that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-(x'+ia)^2}\mathrm{d}x' \tag{246}$$

where a is a real number. Now we use Cauchy's theorem to prove it.

Proof.

$$I_1 = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \int_{C_1} \mathrm{e}^{-z^2}\mathrm{d}z \tag{247}$$

$$I_2 = \int_{-\infty}^{\infty} \mathrm{e}^{-(x'+ia)^2}\mathrm{d}x' = \int_{C_2} \mathrm{e}^{-z^2}\mathrm{d}z \tag{248}$$

where C_1 is the whole x-axis and C_2 is the line parallel to the x-axis at $z = x + ia$. Let's assume $a > 0$. To begin with, we construct a closed contour $C_R = C_{2R} + E_R^+ - C_{1R} + E_R^-$ (see in Fig.8).

$$\oint_{C_R} \mathrm{e}^{-z^2}\mathrm{d}z = 0 \tag{249}$$

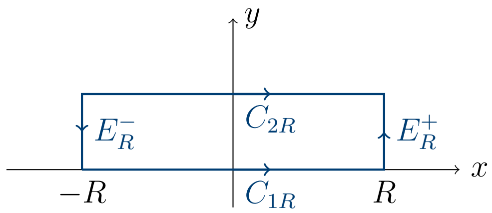

Figure 8: The contour C_R.

for any R. When $R \to \infty$, then

$$\lim_{R\to\infty} \oint_{C_R} \mathrm{e}^{-z^2} \mathrm{d}z = \lim_{R\to\infty} \left(\int_{C_1} + \int_{E_R^+} - \int_{C_2} + \int_{E_R^-} \right) \mathrm{e}^{-z^2} \mathrm{d}z = I_1 - I_2 = 0 \tag{250}$$

5.5 Power Series Representations of Complex Functions

5.5.1 Taylor Series

$f(z)$ is analytic at z_0 if it has a Taylor series in a neighbourhood of z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_n)^2 \tag{251}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z \tag{252}$$

5.5.2 Singularities

If $f(z)$ is analytic except at specific points in the complex plane, those points are called isolated singularities or *poles*.

Example.

$$f(z) = \frac{\mathrm{e}^z}{(z-5)(z+i)(z-(1+i))^2} \tag{253}$$

has isolated singularities at $z = 5, i, 1+i$.

There two types of singularities:

1. $f(z)$ has a *pole of order* $m (m \geq 1)$ at z_0 if there exists a $g(z)$ which is analytic at z_0 and $g(z_0) \neq 0$ *s.t.*

$$f(z) = \frac{g(z)}{(z - z_0)^m} \tag{254}$$

This implies $f(z)$ has a power series except around z_0

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n \tag{255}$$

Poles of order 1 are called *single poles*.

2. $f(z)$ has an essential singularity at z_0 if $f(z)$ has a power series except around $z = z_0$ with infinitely many negtive powers

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{256}$$

Example.

$$f(z) = \mathrm{e}^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z} \right)^n \tag{257}$$

5.6 Contour Integration using the Residue Theorem

5.6.1 The Residue Theorem

Definition. Let f has an isolated singularity at z_0, then the residue of f at z_0 is

$$Res_f(z_0) = \frac{1}{2\pi i}\oint_{C_{z_0}} f(z)\mathrm{d}z \tag{258}$$

where C_{z_0} is a closed contour *s.t.* z_0 is inside and $f(z)$ is analytic inside except at z_0. If $f(z)$ has a pole of order m at z_0, then

$$f(z) = \frac{g(z)}{(z-z_0)^m} \tag{259}$$

and

$$\mathrm{Res}_f(z_0) = \frac{1}{2\pi i}\oint_C \frac{g(z)}{(z-z_0)^m}\mathrm{d}z = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}g(z)}{\mathrm{d}z^{m-1}}\bigg|_{z=z_0} \tag{260}$$

Example.

(1) $f(z) = 1/(z-z_0)$

$$\mathrm{Res}_f(z_0) = 1 \tag{261}$$

(2) $f(z) = \sin z/(1+z)^2$

$$\mathrm{Res}_f(-1) = \frac{\mathrm{d}\sin z}{\mathrm{d}z}\bigg|_{z=-1} = \cos(-1) = \cos 1 \tag{262}$$

Theorem. Let C is a closed contour, $f(z)$ is a function that is analytic on C and inside C except at $z = z_1, \cdots, z_N$. Then

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{Res}_f(z_k) \tag{263}$$

Proof. By Cauchy's theorem

$$\oint_{\tilde{C}} f(z)\mathrm{d}z = \oint_C f(z)\mathrm{d}z - 2\pi i \sum_{k=1}^{N} \mathrm{Res}_f(z_k) = 0 \tag{264}$$

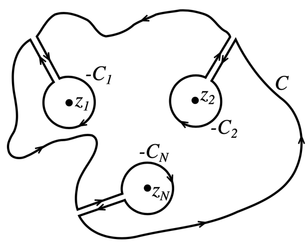

Figure 9: The contour C used in the proof of the residue theorem.

5.6.2 Contour Integration Examples

Example. 1

$$\begin{aligned} I &= \oint_{|z|=1} \mathrm{e}^{1/z}\mathrm{d}z \\ &= \oint_{|z|=1} \left[1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots\right]\mathrm{d}z = 2\pi i \end{aligned} \tag{265}$$

Example. 2

$$\begin{aligned} I &= \oint_{|z|=3} \frac{z+2}{2z^2+1}\mathrm{d}z = \oint_{|z|=3} \frac{z+2}{2(z^2+1/2)}\mathrm{d}z \\ &= \oint_{|z|=3} \frac{z+2}{2(z+\frac{i}{\sqrt{2}})(z-\frac{i}{\sqrt{2}})}\mathrm{d}z \\ &=2\pi i\left[\mathrm{Res}\left(\frac{i}{\sqrt{2}}\right) + \mathrm{Res}\left(-\frac{i}{\sqrt{2}}\right)\right] \\ &=2\pi i\left[\frac{\frac{i}{\sqrt{2}}+2}{2(\frac{i}{\sqrt{2}}+\frac{i}{\sqrt{2}})} + \frac{-\frac{i}{\sqrt{2}}+2}{2(-\frac{i}{\sqrt{2}}-\frac{i}{\sqrt{2}})}\right] = \pi i \end{aligned} \tag{266}$$

Example. 3

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)(x^2+9)} \tag{267}$$

Consider the contour $C = C_R + S_R$ (see in fig.10(a)), we have

$$\begin{aligned} I &= \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+1)(x^2+9)} \\ &= \lim_{R\to\infty} \oint_C \frac{\mathrm{d}z}{(z+i)(z-i)(z+3i)(z-3i)} - \lim_{R\to\infty} \int_{C_R} \frac{\mathrm{d}z}{(z^2+1)(z^2+9)} \\ &=2\pi i\left[\mathrm{Res}(i) + \mathrm{Res}(3i)\right] - 0 \\ &=2\pi i\left(\frac{1}{16i} + \frac{1}{-48i}\right) = \frac{\pi}{12} \end{aligned} \tag{268}$$

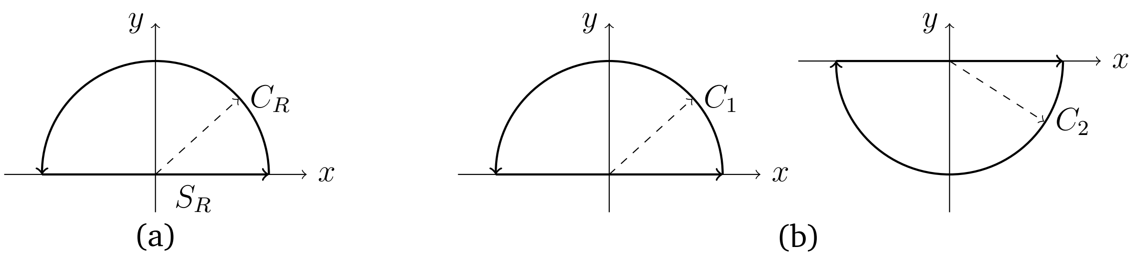

Figure 10: (a) The contour for example 3. (b) The contour for example 4.

Example. 4

$$
\begin{aligned}
I =& \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\mathrm{d}x = \int_{x-\text{axis}} \frac{\cos z}{z^2+1}\mathrm{d}z \\
=& \int_{x-\text{axis}} \frac{\mathrm{e}^{iz}}{2(z+i)(z-i)}\mathrm{d}z + \int_{x-\text{axis}} \frac{\mathrm{e}^{-iz}}{2(z+i)(z-i)}\mathrm{d}z \\
=& I_1 + I_2
\end{aligned}
\tag{269}
$$

Consider the contour C_1 and C_2 (see in fig.10(b))

$$
\begin{aligned}
I_1 =& \lim_{R\to\infty} \oint_C \frac{\mathrm{e}^{iz}}{2(z+i)(z-i)}\mathrm{d}z - \lim_{R\to\infty} \int_{C_1} \frac{\mathrm{e}^{iz}}{2(z+i)(z-i)}\mathrm{d}z \\
=& 2\pi i\,\mathrm{Res}(i) - \lim_{R\to\infty} \int_{C_1} \frac{\mathrm{e}^{ix-y}}{2(z+i)(z-i)}\mathrm{d}z \\
=& 2\pi i \frac{\mathrm{e}^{-1}}{4i} - 0 = \frac{\pi}{2}\mathrm{e}^{-1}
\end{aligned}
\tag{270}
$$

$$
\begin{aligned}
I_2 =& -\lim_{R\to\infty} \oint_C \frac{\mathrm{e}^{-iz}}{2(z+i)(z-i)}\mathrm{d}z - \lim_{R\to\infty} \int_{C_2} \frac{\mathrm{e}^{-iz}}{2(z+i)(z-i)}\mathrm{d}z \\
=& -2\pi i\,\mathrm{Res}(-i) - \lim_{R\to\infty} \int_{C_2} \frac{\mathrm{e}^{-ix+y}}{2(z+i)(z-i)}\mathrm{d}z \\
=& -2\pi i \frac{\mathrm{e}^{-1}}{-4i} - 0 = \frac{\pi}{2}\mathrm{e}^{-1}
\end{aligned}
\tag{271}
$$

So we have

$$
I = I_1 + I_2 = \pi\mathrm{e}^{-1}
\tag{272}
$$

5.6.3 Inverting Laplace Transforms

Suppose

$$
F(s) = \mathrm{LT}[f(t)] = \int_0^s f(t)\mathrm{e}^{-st}\mathrm{d}t
\tag{273}
$$

where $t > 0$, then

$$
f(t) = \mathrm{LT}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\mathrm{e}^{st}\mathrm{d}s
\tag{274}
$$

which is called *Bromwich integral*.To invert a Laplace transform $F(s)$

(i) Find the singular points $a_1, a_2, \cdots,$ of $F(s)$ and choose a real number c such that $c > \mathrm{Re}(a_i)$ for all i.

(ii) Close the Bromwich integral contour show in fig.11 with a large semicircle in the left-hand half-plane.

(iii) If the integral around the semicircle vanished as $R \to \infty$, then

$$
f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\mathrm{e}^{st}\mathrm{d}s = \sum_i \mathrm{Res}(a_i), \qquad (t > 0)
\tag{275}
$$

where $\mathrm{Res}(a_i)$ is the residues of $F(s)\mathrm{e}^{st}$.

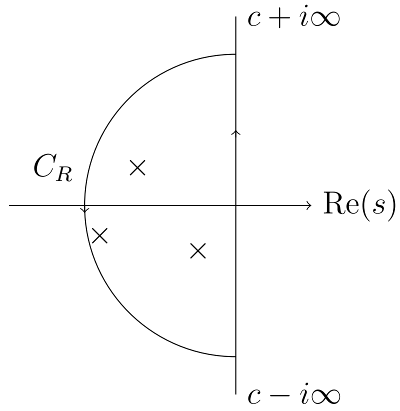

Figure 11: The contour for inverting Laplace transforms.

6 Calculus of Variations

- A **function** f maps a number, x, to another number, $f(x)$.
- A **functional** I maps a function, f, to a number $I[f]$.

Example.

(a) $I[f] = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$

(b) $T(\psi) = \int \psi^*(x)\frac{\hat{p}^2}{2m}\psi(x)\mathrm{d}x$

(c) $U(\rho) = \frac{1}{2}\iint \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}\mathrm{d}^3\boldsymbol{r}'$

(d) $S[y] = \int_a^b \sqrt{1+(\frac{\mathrm{d}y}{\mathrm{d}x})^2}\mathrm{d}x =$ length of curve from $x=a$ to $x=b$ given by $y(x)$.

(e) $S[x] = \int_{t_1}^{t_2}\left[\frac{1}{2}m\dot{x}^2 - V(x)\right]\mathrm{d}t =$ action.

6.1 The Euler-Lagrange Problem

Let y be a function of variable $y(x)$

$$I[y] = \int_{x_A}^{x_B} f(x, y(x), y'(x))\mathrm{d}x \tag{276}$$

where f is a function of 3 arguments, and $x_A, x_B, y(x_A), y(x_B)$ are fixed. *Euler-Lagrange problem* is to find $y(x)$ such that $\delta I = \mathcal{O}(\delta y^2)$ at $y(x)$, and we say y extremises $I[y]$ or $I[y]$ is stationary at y.

Consider varying $y(x)$ slightly

$$y(x) \to y(x) + \delta y(x) \tag{277}$$

then

$$\begin{aligned} I[y+\delta y] &= \int_{x_A}^{x_B} f(x, y(x)+\delta y(x), y'+\delta y'(x))\mathrm{d}x \\ &= \int_{x_A}^{x_B} \left[f(x,y,y') + \delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} + \mathcal{O}(\delta y^2) \right] \mathrm{d}x \end{aligned} \tag{278}$$

so we have

$$\begin{aligned} \delta I =& I[y+\delta y] - I[y] \\ =& \int_{x_A}^{x_B} \left(\delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} \right) \mathrm{d}x + \mathcal{O}(\delta y^2) \\ =& \int_{x_A}^{x_B} \left(\delta y \frac{\partial f}{\partial y} \right) \mathrm{d}x + \left[\delta y \frac{\partial f}{\partial y'} \right]_{x_A}^{x_B} - \int_{x_A}^{x_B} \delta y \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \mathrm{d}x + \mathcal{O}(\delta y^2) \\ =& \int_{x_A}^{x_B} \delta y \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \mathrm{d}x + \mathcal{O}(\delta y^2) \end{aligned} \tag{279}$$

$\delta I = \mathcal{O}(\delta y^2)$ if and only if

$$\boxed{\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0} \tag{280}$$

for $x_A \leq x \leq x_B$. This equation is called *Euler-Lagrange equation*.

Example.

$$f(x,y,y') = (1+x^2)y'^2 - y^4 \tag{281}$$

$I[y] = \int_{x_A}^{x_B} f(x,y,y')\mathrm{d}x$ is stationary if

$$-4y^3 - \frac{\mathrm{d}}{\mathrm{d}x} \left[(1+x^2)2y' \right] = 0 \tag{282}$$

6.1.1 Beltrami identity

Suppose $f(x,y,y') = f(y,y')$, then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial y'} \frac{\mathrm{d}y'}{\mathrm{d}x} \tag{283}$$

if y is a solution of the Euler-Lagrange equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial f}{\partial y'} \right) y' + \frac{\partial f}{\partial y'} \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} y' \right) \tag{284}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(f - \frac{\partial f}{\partial y'} y' \right) = 0 \quad \Rightarrow \quad \boxed{f - \frac{\partial f}{\partial y'} y' = \text{const}} \tag{285}$$

with the condition $\partial f / \partial x = 0$. This equation is called *Beltrami identity*, which is the first integral of Euler-Lagrange equation.

Example.

$$I[y] = \int f \mathrm{d}x \quad \text{with} \quad f(y, y') = y'^2 - y^4 \tag{286}$$

Applying the Beltrami identity

$$y'^2 - y^4 - 2y'^2 = \text{const} \tag{287}$$

6.1.2 Functional Derivatives

We know that

$$\delta I = \int_{x_A}^{x_B} \delta y \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \mathrm{d}x + \mathcal{O}(\delta y^2) \tag{288}$$

then we can define the *functional derivative of I*

$$\frac{\delta I}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \tag{289}$$

then Euler-Lagrange equation can be written as

$$\frac{\delta I}{\delta y(x)} = 0 \tag{290}$$

6.1.3 Lagrangian Mechanics

The Lagrangian of a classical particle moving in three dimensions is

$$L = T - V = \frac{1}{2} m \dot{\boldsymbol{x}}^2 + V(\boldsymbol{x}, t) \tag{291}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$. The action

$$S[\boldsymbol{x}(t)] = \int_{t_A}^{t_B} L(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) \tag{292}$$

Vary $S[\boldsymbol{x}]$ separately for x_1, x_2, x_3 and get an Euler-Lagrange equation for each

$$\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}_i} \right) = 0, \qquad i = 1, 2, 3 \tag{293}$$

i.e.

$$m \ddot{x}_i = -\nabla_i V \tag{294}$$

which is Newton's equation.

6.1.4 Euler-Lagrange Examples

(a) **Shortest Path Problem**
Method 1
Between (x, y) and $(x + \mathrm{d}x, y + \mathrm{d}y)$ along curve $y(x)$, the distance is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \tag{295}$$

so the length of $y(x)$ is

$$\int \mathrm{d}s = \int_{x_A}^{x_B} \sqrt{1 + y'^2}\mathrm{d}x \tag{296}$$

This extremised by Euler-Lagrange equation

$$0 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0 \quad \Rightarrow \quad y' = c \quad \Rightarrow \quad y = cx + d \tag{297}$$

Method 2
We write the curve in *parametrised* form

$$y = y(\lambda), \qquad x = x(\lambda) \tag{298}$$

The curve fixed at $\lambda = \lambda_A$ at (x_A, y_A) and $\lambda = \lambda_B$ at (x_B, y_B). The length of path is

$$\int \mathrm{d}s = \int_{\lambda_A}^{\lambda_B} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2}\mathrm{d}\lambda \tag{299}$$

This extremised by Euler-Lagrange equation. For x

$$0 - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{x'}{\sqrt{x'^2 + y'^2}}\right) = 0 \quad \Rightarrow \quad \frac{x'}{\sqrt{x'^2 + y'^2}} = \alpha \tag{300}$$

Similarly, for y

$$0 - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{y'}{\sqrt{x'^2 + y'^2}}\right) = 0 \quad \Rightarrow \quad \frac{y'}{\sqrt{x'^2 + y'^2}} = \beta \tag{301}$$

So we have

$$\frac{y'}{x'} = \gamma \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \gamma \quad \Rightarrow \quad y = \gamma x + c \tag{302}$$

(b) **Brachistochrone**
A particle moving from $A(0, 0)$ to $B(x_B, y_B)$ takes the time

$$T = \int_A^B \mathrm{d}t = \int_A^B \frac{\mathrm{d}s}{v} = \int_{x=0}^{x=x_B} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\mathrm{d}x \tag{303}$$

Using the Beltrami indentity

$$\frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - \frac{\frac{y'}{\sqrt{1+y'^2}}}{\sqrt{2gy}} y' = c \quad \Rightarrow \quad y(1+y'^2) = c^2 \quad \Rightarrow \quad y' = \frac{\alpha - y}{y} \tag{304}$$

where $\alpha = c^2$. The solution is a cycloid

$$x = x(\theta) = a(\theta - \sin\theta) \tag{305}$$

$$y = y(\theta) = a(1 - \cos\theta) \tag{306}$$

where $a = \alpha/2$. Then the total time

$$T = \int_0^{\theta_B} \frac{\sqrt{(\mathrm{d}x/\mathrm{d}\theta)^2 + (\mathrm{d}y/\mathrm{d}\theta)^2}}{\sqrt{2gy}} \mathrm{d}\theta = \int_0^{\theta_B} \sqrt{\frac{a}{g}} \mathrm{d}\theta = \sqrt{\frac{a}{g}} \theta_B \tag{307}$$

6.1.5 Symmetries and Conservation

- **Conservation of Energy**
  Consider a single particle in 1D space, and the potential doesn't depend explicitly on time t. The Lagrangian

$$L(x, \dot{x}) = T - L = \frac{1}{2} m\dot{x}^2 - V(x) \tag{308}$$

  We use the Beltrami identity

$$\begin{aligned} L - \frac{\partial L}{\partial \dot{x}} \dot{x} &= \text{const} \\ -\left(\frac{1}{2} m\dot{x}^2 + V\right) &= \text{const} \\ T + V &= \text{const} \end{aligned} \tag{309}$$

  so we see that the V being independent of t leads to the conservation of total energy. More generally, for any mechanical system with position variables $\boldsymbol{q} = (q_1, q_2, \cdots, q_N)$

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T - V(\boldsymbol{q}) \tag{310}$$

  which does not depend on t. If one defines

$$H = -L + \sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \tag{311}$$

  H is the classical Hamiltonian and the total energy. Then the Beltrami identity tells us that this is a constant of the motion.

- **Conservation of Momentum**
  Consider a particle in 3D space. Suppose the potential $V(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$ is independent of $\boldsymbol{x} = (x_1, x_2, x_3)$, *i. e.*

$$\frac{\partial V}{\partial x_i} = 0, \qquad i = 1, 2, 3 \tag{312}$$

The Lagrangian

$$L = T - V = \frac{1}{2}m\left(\sum_i \dot{x}_i^2\right) - V \tag{313}$$

The Euler-Lagrange equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \quad \Rightarrow \quad m\dot{x}_i = \text{const} \tag{314}$$

which is the momentum of that particle in the i direction.

- **Conservation of Angular Momentum**
  Suppose $\boldsymbol{q} = (r(t), \theta(t).\phi(t))$, then the Lagrangian for the particle is

$$L = T - V(r, \theta, \phi) \tag{315}$$

where the kinetic energy

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \tag{316}$$

We find that T doesn't depend on ϕ. If V also doesn't depend on ϕ, then the Lagrangian doesn't depend on ϕ.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad mr^2\sin^2\theta\dot{\phi} = \text{const} \tag{317}$$

is a constant of the motion. This is the angular momentum in the z-direction. If the potential V is a function of r alone, the system is spherically symmetric, then all components of the angular momentum are conserved.

6.2 Constrain Optimisation and Lagrange Multipliers

6.2.1 Constrained Optimisation of Functions

Consider the function $f(x, y)$ and we want to find the stationary points of f subject to the constraint

$$g(x, y) - C = 0 \tag{318}$$

At the stationary point P, the contour of $f(x, y)$ are parallel to the curve $g(x, y) = C$

$$\boldsymbol{\nabla} f|_P \parallel \boldsymbol{\nabla} g|_P \quad \Rightarrow \quad \boldsymbol{\nabla}\left(f(x, y) - \lambda g(x, y)\right)_P = 0 \tag{319}$$

The gradient ratio λ, is called a *Lagrange multiplier*. This equation has two components.

In d-dimension with the function $f(x_1, \cdots, x_d)$ and constraint $g(x_1, \cdots, x_d) = C$.
With more constraints

$$\begin{aligned} g_1(\boldsymbol{x}) &= C_1 \\ g_2(\boldsymbol{x}) &= C_2 \\ &\vdots \\ g_k(\boldsymbol{x}) &= C_k \end{aligned} \tag{320}$$

constraint surface is $n-k$ dimension. f is constrained on the constraint surface if

$$\nabla(f-\lambda_1 g_1-\lambda_2 g_2-\cdots-\lambda_k g_k)=0 \tag{321}$$

Example. Find the minimum distance between curves $xy=1$ and $x+2y=1$. Our task is to minimise $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, which is the same problem as minimising

$$f(x_1,x_2,y_1,y_2)=(x_2-x_1)^2+(y_2-y_1)^2 \tag{322}$$

Solve $\nabla u=0$, where

$$u(x_1,x_2,y_1,y_2)=f(x_1,x_2,y_1,y_2)+\lambda_1 x_1 y_1+\mu(x_2+2y_2) \tag{323}$$

Unconstrained minimisation of $u(x_1,x_2,y_1,y_2)$ gives

$$\frac{\partial u}{\partial x_1}=2(x_1-x_2)+\lambda_1 y_1=0 \tag{324}$$

$$\frac{\partial u}{\partial x_2}=2(x_2-x_1)+\mu_2=0 \tag{325}$$

$$\frac{\partial u}{\partial y_1}=2(y_1-y_2)+\lambda_1 x_1=0 \tag{326}$$

$$\frac{\partial u}{\partial y_2}=2(y_2-y_1)+2\mu_2=0 \tag{327}$$

The solution is

$$(x_1,y_1)=\left(\sqrt{2},\frac{\sqrt{2}}{2}\right), \qquad (x_2,y_2)=\left(\frac{1+3\sqrt{2}}{5},\frac{4-3\sqrt{2}}{10}\right) \tag{328}$$

The minimum distance is then $(2\sqrt{2}-1)/\sqrt{5}$

6.2.2 Constrained Optimisation of Functionals

Functional problem is to extremise $F[y]$ subject to $G[y]=C$. The Lagrange multiplier is to find solutions of $\delta(F-\lambda G)=0$.

Example. The catenary: Find the shape formed by a heavy rope of a chain hanging between two fixed end points $A(-a,0)$ and $B(a,0)$. Our task is to minimise the total energy. Suppose the mass density is ρ, and the mass of piece is $\mathrm{d}m=\rho\mathrm{d}s$. The total enegy

$$E=g\int_A^B y\mathrm{d}m=\rho g\int_A^B y\mathrm{d}s=\rho g\int_{-a}^{a} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \tag{329}$$

The length of the rope is fixed. So the constraint

$$L=\int_A^B \mathrm{d}s=\int_{-a}^{a}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \tag{330}$$

Then we have to extremise

$$\begin{aligned} U = E - \lambda L =& \rho g \int_{-a}^{a} y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x - \lambda \int_{-a}^{a} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x \\ =& \int_{-a}^{a} (\rho g y - \lambda) \sqrt{1 + y'^2} \mathrm{d}x = \int_{-a}^{a} f \mathrm{d}x \end{aligned} \tag{331}$$

U does not depend on x, then we can use Beltrami identity

$$f - \frac{\partial f}{\partial y'} y' = C \tag{332}$$

which is

$$(\rho g y - \lambda)\sqrt{1 + y'^2} - (\rho g y - \lambda)\frac{y'^2}{\sqrt{1 + y'^2}} = C \tag{333}$$

$$\rho g y - \lambda = C\sqrt{1 + y'^2} \tag{334}$$