Beyond Truthful Reporting: Robust Strategies for Worst-Case Payoff Maximization in Large Markets

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We study the bidding problem faced by a participant in market-clearing mechanisms that are not incentive-compatible, focusing on settings where the bidder has limited or no information about rivals' types or strategies. Using a robust optimization framework, we model this uncertainty through an ambiguity set encompassing all possible realizations of rivals' bids. The bidder maximizes its worst-case payoff over this set, yielding robust bidding strategies independent of distributional assumptions. In particular, we demonstrate the value of this approach in the context of generalized first-price, generalized second-price, and core-selecting combinatorial auctions. In the latter case, we leverage the minimax inequality to identify an optimal robust bidding strategy for single-minded and double-minded bidders, and establish that these easy-to-implement strategies outperform truthful bidding. Compared with expected-payoff-maximizing strategies, robust bidding reduces allocation risk and yields higher payoffs under adversarial realizations of rivals' bids.

Key words: market design, auctions, incentive compatibility, robust optimization

1. Introduction

Many contemporary markets, such as those in digital advertising, energy, and telecommunications, are large-scale in terms of the number and heterogeneity of items transacted. Their effective functioning relies on algorithmic market clearing and carefully designed information systems. This, in turn, creates a significant challenge for participants, as devising even approximately optimal bidding strategies is non-trivial given the complexity and opacity of how outcomes are determined. To alleviate this challenge, market operators often provide automated support. For example, major digital advertising platforms deploy auto-bidding agents that translate bidder-reported valuations into bids. These agents are designed to promote truthful reporting, which, if followed by participants, eliminates the strategic burden of bidding.[1]

This leaves participants with a choice: report their true values as suggested, or devise a strategic policy to maximize their payoffs. For large bidders, it may be worthwhile to invest in sophisticated

[1] Although market efficiency cannot be achieved in general, the complexity of the market-clearing mechanism makes it difficult for participants to identify profitable misreporting strategies.

algorithms that incorporate distributional assumptions about rivals' valuations and equilibrium behavior. For example, in digital advertising markets, demand-side platforms offer automated bidding tools to strategize on behalf of their clients, even when market operators provide their own auto-bidding systems. In contrast, many smaller bidders lack the resources for such tools and instead pursue simpler, narrowly focused objectives, such as securing a single slot, route, or license. These bidders typically follow the operator's recommendation of truthful reporting. Our central message is that even such bidders can benefit from simple deviation strategies that improve payoffs without requiring advanced modeling capabilities.

To demonstrate this, we use a robust optimization framework to analyze three widely studied auction formats: generalized first-price (GFP), generalized second-price (GSP), and core-selecting combinatorial auctions. These formats are central in practice: GFP and GSP underpin digital advertising markets, while combinatorial auctions are used to clear high-stakes markets such as spectrum sales. In all three settings, small bidders with limited demand and simple valuation structures are commonly present. For instance, in digital advertising markets, many advertisers only care about winning a top slot for a particular keyword or page. These bidders can be modeled as single-minded, with a clear target bundle. Similarly, in logistics or telecom spectrum auctions, bidders often target a specific route or frequency block, again aligning with the single-minded structure. Our approach provides a practical alternative to traditional expected-utility maximization, yielding bidding policies that are easy to compute and do not rely on distributional or behavioral assumptions about rivals. Instead, a bidder's limited knowledge is captured by an *uncertainty set* of plausible rival bids, and the objective is to maximize the *worst-case payoff* over this set. While the robust bidding problem is trivial in incentive-compatible mechanisms, where truth-telling is optimal, it becomes challenging in the non-incentive-compatible mechanisms common in practice. In these settings, our framework provides actionable guidance for smaller bidders who would otherwise default to suboptimal truthful reporting.

In Section 3, we derive optimal robust bidding policies for GFP and GSP auctions, which are common in digital advertising markets. Section 4 then focuses on core-selecting combinatorial auctions. We first analyze the benchmark case with perfect information, and then establish robust policies for both single-minded (i.e., one with a positive valuation for only one specific bundle) and double-minded (i.e., one with distinct valuations for two inclusive bundles) bidders using minimax arguments. In these settings, a robust bidding policy is to bid the minimum amount necessary to secure the "target" bundle, regardless of the realization of rivals' bids within the uncertainty set. This holds if the bidder's valuation for her target bundle is high enough so that true bidding still results in winning the bundle, regardless of rivals' bids.

Section 5 presents numerical results based on our robust bidding policies. In Section 5.1, we evaluate the performance of the robust bidding policy against various benchmarks, including expected-payoff maximization and scenarios with misspecified uncertainty sets. The results demonstrate that even without distributional knowledge, the robust policy performs well relative to all benchmarks. In Section 5.2, we extend the analysis to a more complex environment with eight bidders of different valuation structures competing for three heterogeneous items. Across 10,000 simulations, the robust policy consistently outperforms truth-telling. Importantly, even in cases we do not analytically characterize, the robust policy suggests a clear pattern of misreporting—shading bids on the target bundle while inflating bids on the global bundle—that improves bidders' expected payoffs.

In summary, our paper utilizes a robust optimization approach to model a bidder with limited information about rivals' behavior. This framework enables a distribution-free analysis that contrasts with classic expected-payoff maximization. We establish the optimality of non-truthful bidding policies in GFP, GSP, and core-selecting auctions for bidders with simple valuation structures (single- and double-minded). Our results show that even such bidders can identify and implement strategies that consistently outperform the default of truthful reporting without requiring complex modeling or computational resources.

The rest of the paper proceeds as follows. Section 1.1 reviews related literature. Section 2 introduces notation and describes the auction models and assumptions. Section 3 presents optimal robust bidding policies for GFP and GSP auctions. Section 4 analyzes robust bidding for single- and double-minded bidders in core-selecting auctions. Section 5 evaluates the performance of robust bidding through numerical simulations, with detailed results presented in Appendix A. Section 6 concludes. Further results and examples for the GSP auctions are provided in Appendix C. Proofs are relegated to Appendix D.

1.1. Related literature

Incentive compatibility and market design. Incentive compatibility has been one of the central properties in classical mechanism design literature in economics. Green and Laffont (1977) established that the VCG mechanism (Vickrey 1961, Clarke 1971, Groves 1973) is the unique market-clearing procedure that maximizes total welfare (efficiency), while ensuring that participation (individual rationality) and truthful reporting (incentive compatibility) is a dominant strategy for every market participant. However, it is well documented that despite its theoretical properties, the VCG mechanism has several undesirable features that limit its practical use (e.g., Rothkopf et al. 1990, Ausubel and Milgrom 2006). Thus, market-clearing procedures that do not guarantee incentive compatibility are prevalent in practice, despite complicating agents' participation, reporting, and strategic behavior. Understanding agents' optimal policies and equilibrium behavior in such markets is crucial for designing sustainable markets that do not unravel and for evaluating overall performance in terms of efficiency, revenue, and stability. Therefore, addressing the challenges posed by the lack of incentive compatibility remains a central theme in market design research (e.g., Roth 2015, Milgrom 2017, 2019, Bichler 2017, Bichler et al. 2010, etc.). In fact, incentive compatibility issues and associated bidding policies have been the focus of many important practical applications, e.g., in digital advertising (Edelman et al. 2007), energy (Cramton 2017), telecommunication (Bichler and Goeree 2017), transportation and logistics (Karaenke et al. 2019), etc.

GFP and GSP auctions. Both GFP and GSP are widely used in digital advertising markets today. Although GFP was initially employed, GSP has historically dominated because of several advantages noted in the literature (Edelman and Ostrovsky 2007). Since 2002, most major platforms have adopted the GSP auction, in which each bidder pays the next-highest bid. While not incentive compatible in dominant strategies, GSP has been extensively analyzed both theoretically and empirically and shown to admit equilibria with desirable efficiency and fairness properties (e.g., Edelman et al. 2007, Varian 2007, Ashlagi et al. 2010, Lucier and Paes Leme 2011, Kannan et al. 2023).

However, in recent years, GFP auctions, where winners pay their own bids, have regained prominence on digital advertising platforms, offering pricing transparency to all participants. This resurgence has also drawn renewed attention in the theoretical literature; for example, Ostrovsky and Skrzypacz (2022) show that the revenue in a pure-strategy Nash equilibrium of a GFP auction can exceed that of a GSP auction.

Core-selecting auctions. Even in simple auction settings, optimal bidding policies and equilibria are often non-trivial (e.g., Krishna 2009). At the same time, the complexities of simultaneous market-clearing of multiple heterogeneous items cannot be captured by standard auction models and often require a combinatorial auction framework, introducing new challenges in analyzing market-clearing procedures in large markets: the winner determination problem is computationally intractable in general (e.g., Rothkopf et al. 1998), and the communication complexity of any bidder's reporting may be exponential in the number of items traded in the market (Nisan and Segal 2006). Yet, despite these challenges, the versatility of the combinatorial auction model allows it to effectively address complex and important market-clearing problems (see Palacios-Huerta et al. 2024, for a survey on the deployment of various combinatorial auction designs in practice). Perhaps the most successful combinatorial auction implementation format is the combinatorial clock auction (Ausubel and Baranov 2014, 2017, Levin and Skrzypacz 2016). Variants of this auction have been implemented in markets worldwide, facilitating transactions worth billions, e.g., Janssen and

Kasberger (2019). This auction format evolved from Ausubel and Milgrom (2002) and Ausubel et al. (2006). However, a crucial aspect of the implementation is its final phase, which reduces to a one-shot combinatorial auction that still lacks an incentive compatibility guarantee. A key factor behind its success is the concept of core-selecting auctions, introduced and analyzed by Day and Raghavan (2007), Day and Milgrom (2008). As the name suggests, core-selecting auctions yield an outcome within the core of the market-clearing game, potentially mitigating the lack of incentive compatibility. Due to their significance in understanding some of the most successful combinatorial market-clearing implementations, core-selecting auctions have been extensively studied in recent years, from proposing different core pricing choices (e.g., Erdil and Klemperer 2010, Day and Cramton 2012, Bünz et al. 2015, Goetzendorff et al. 2015, Niazadeh et al. 2022, Bosshard et al. 2022, Bünz et al. 2022, Emadikhiav and Day 2025), to providing game-theoretic analysis (Hafalir and Yektaş 2015, Ausubel and Baranov 2020), characterizing equilibria Guler et al. (2016), and establishing impossibility results (Goeree and Lien 2016). In the first game-theoretic analysis of core-selecting auctions, Day and Milgrom (2008) already addressed the lack of incentive compatibility and provided an optimal bid-shading policy under perfect information. Additionally, Beck and Ott (2013) demonstrated that bidders in core-selecting auctions may have an incentive to overbid. The paper goes further by providing not only a distribution-free optimal bidding policy, crucial for settings where distributional assumptions may be unrealistic, for a worst-case profit-maximizing bidder, but also the best response bidding policy when rivals bid truthfully. Heczko et al. (2018) experimentally analyzed core-selecting auctions in a similar setting and found that they outperformed the Vickrey auction.

Applying robust optimization in auction analysis. Our modeling approach falls within the robust optimization framework, which has been developed as a tool to deal with decision-making under uncertain input parameters (e.g., Ben-Tal and Nemirovski 2002, Bertsimas and Sim 2004). Under this framework, the decision maker does not know the exact distribution of the uncertain parameters. Since the mechanism design problem can be interpreted as a linear optimization problem, the robust optimization approach readily applies. Bandi and Bertsimas (2014) study the optimal mechanism design problem for multi-item auctions, in the spirit of a celebrated work by Myerson (1981). Koçyiğit et al. (2018, 2020) extend this analysis from the mechanism designer's perspective. Pınar and Kızılkale (2017) consider the robust screening problem with moment information when the support is discrete. Chen et al. (2024) propose a geometric approach to solve the robust screening problem. Anunrojwong et al. (2025a,b), Wang et al. (2024) investigate the robust selling mechanism under the performance ratio criteria. Wang (2025) studies the performance of simple selling mechanisms with a finite number of menus.

The issue of robustness has also been widely explored in the economics literature (e.g., Carroll 2017, Che and Zhong 2024, Du 2018). Carroll (2019) reviews the work on robust mechanism design from the designer's perspective. In particular, robustness to strategic behavior and distributional uncertainty, where the principal has limited information about agents' strategies and type distributions, is related to our work. For example, the modeling of uncertainty through uncertainty sets in Bergemann and Schlag (2008) is similar to our approach; however, the objectives differ. They study a mechanism designer seeking to minimize regret—the difference between realized profit and the profit achievable with full knowledge of the buyer's true value—whereas we focus on a bidder aiming to maximize the minimum profit. The key distinction of our work from the aforementioned literature is that we employ robust optimization to analyze a bidder's decision-making in the competitive auction setting, rather than focusing on the mechanism designer's problem.

This focus on the bidder's problem rather than on the mechanism designer's problem has recently also been considered in Kasberger and Schlag (2024). They explore the buyer's bidding strategies when facing an uncertainty set consisting of "conceivable" distributions of rivals' bids, which comes from sequentially eliminating "unreasonable" distributions according to certain assumptions. In their model, the buyer aims to minimize the highest possible loss due to a lack of information on the true distribution. In our work, we do not impose limitations on the structure of the rivals' bid distributions and focus on the bidder who wants to maximize the profit within the standard maximin robust optimization framework.

Utilizing a robust optimization approach to establish and analyze optimal bidding policies allows us to bypass classical game-theoretic equilibrium analysis and avoid imposing any distributional assumptions while providing actionable prescriptive decision support to bidders. Gilboa and Schmeidler (1989) proposed a related preference model in which an agent maximizes her worst-case expected payoff with respect to a family of priors. However, such an approach still relies on some probabilistic assessment of rivals' bids, whereas our approach is completely distribution-free. As we demonstrate in later sections, the analysis sometimes requires establishing minimax results in a nonstandard manner.

2. Model

We consider auctions that allocate m indivisible items from the set $M = \{1, 2, \ldots, m\}$ among n bidders in $N = \{1, 2, \ldots, n\}$. The monopolistic seller is indexed by 0. Items in M can be homogeneous (i.e., identical) or heterogeneous (i.e., distinct). A *bundle* is a set of items. A bidder is said to have *unit demand* if she has a positive valuation for only one item. Similarly, a bidder is said to have *multiple demand* if she has a positive valuation for one or more bundles of items. For each bidder $j \in N$, we use $v_j(S)$ and $b_j(S)$ to denote her non-negative truthful and reported valuation for a bundle $S \subseteq M$, respectively. When bidder j has unit demand and the items are homogeneous, we abuse the notation and simply use v_j and b_j to denote her unique valuation and bid. The auctioneer uses an *allocation rule* to determine the set of items S_j to be allocated to bidder j (S_j is an empty set if bidder j does not obtain any item). Similarly, the auctioneer uses a *payment rule* to determine the amount p_j that bidder j has to pay. We assume that bidders have quasi-linear preferences, i.e., bidder j's payoff is $\pi_j = v_j(S_j) - p_j$ for $j \in N$. For convenience, we use $b_{-j} = (b_1, b_2, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n)$ to denote the profile of bidder j's rivals' bids, where b_j represents the entire bidding profile submitted by bidder j. When all b_j are scalar, the k^{th} highest bid in b_{-j} is denoted by $b_{-j}^{(k)}$. We choose to analyze the decision problem of bidder 1. It turns out that only $b_{-1}^{(k)}$ are relevant to our analysis. Thus, for simplicity, we omit the subscript and write $b^{(k)}$ to denote the k^{th} highest bid in b_{-1}. We use $\mathbb{1}_A$ to denote the indicator function of the expression A, i.e., it has value one if A is true and zero otherwise. For any $x \in \mathbb{R}$, we denote $x^+ = \max(x, 0)$.

Next, we introduce the following formats of auctions in large markets.

GFP auctions. GFP auctions are commonly used in online advertising markets, where advertisers bid for ad slots on a webpage or search engine results page. Each slot has a different visibility level (often modeled as a click-through rate), and advertisers compete for higher positions to maximize exposure.

We consider a setting with m heterogeneous items, often called "slots," where each slot k has a quality level represented by a click-through rate, α_k. Bidders have unit demand and submit a single scalar bid, b_j. The m highest bidders are allocated the m items in descending order of quality.

Under the first-price rule, the bidder who wins the k-th item pays their own submitted bid. For convenience, we assume that bidder 1 is favored in the event of a tie. If bidder 1's bid b_1 is high enough to secure item k (i.e., $b^{(k)} \le b_1 < b^{(k-1)}$), she pays b_1. Her payoff is therefore the value of winning that slot, v_1, net of her payment b_1, scaled by the item's quality α_k. Bidder 1's overall payoff function is given by

$$\pi_1(b_1, b_{-1}) = \sum_{k=1}^{m} \alpha_k \cdot (v_1 - b_1) \cdot \mathbb{1}_{b^{(k)} \le b_1 < b^{(k-1)}}. \tag{1}$$

Here, we adopt the convention that $b^{(0)} = \infty$. If bidder 1's bid is not high enough to win any of the m items (i.e., $b_1 < b^{(m)}$), her payoff is zero.

GSP auctions. GSP auctions are another commonly used auction format in digital markets, particularly in sponsored search and online advertising. They share the same allocation rule as GFP auctions described above. Bidders submit scalar bids, and the m highest bidders are allocated the m items in descending order of quality, represented by click-through rates α_k.

However, the payment rule differs. The bidder who wins the k-th item pays the bid of the $(k+1)$-th highest bidder. This is equivalent to paying $b^{(k)}$, the k-th highest bid among the rivals. If bidder 1's bid b_1 secures item k, her payoff is her value v_1 net of her payment $b^{(k)}$, scaled by the item's quality α_k. Bidder 1's overall payoff function is therefore given by

$$\pi_1(b_1, b_{-1}) = \sum_{k=1}^{m} \alpha_k \cdot (v_1 - b^{(k)}) \cdot \mathbb{1}_{b^{(k)} \leq b_1 < b^{(k-1)}}. \tag{2}$$

As before, $b^{(0)} = \infty$, and the payoff is zero if $b_1 < b^{(m)}$.

Core-selecting combinatorial auctions. In core-selecting auctions, items are heterogeneous. Each bidder has multiple demand and submits bids $b_j(S)$ for bundles $S \subseteq M$ of her interest. When $S = M$, we refer to it as the *global bundle*. The auctioneer decides the allocation outcome by solving the *winner determination problem* (WDP):

$$\begin{aligned}
\max_x \quad & \sum_{j \in N} \sum_{S \subseteq M} b_j(S) \cdot x_j(S) && (\mathscr{W}) \\
\text{s.t.} \quad & \sum_{S \supseteq \{i\}} \sum_{j \in N} x_j(S) \leq 1, \quad \forall i \in M, \\
& \sum_{S \subseteq M} x_j(S) \leq 1, \quad \forall j \in N, \\
& x_j(S) \in \{0,1\}, \quad \forall (S,j) \text{ s.t. bid } b_j(S) \text{ was submitted.}
\end{aligned}$$

Let $w_b(N, M)$ denote the optimal value of the objective function for the winner determination problem ($\mathscr{W}$) given the bid profile b. In that problem, the auctioneer maximizes the total reported value of bundles. The first constraint ensures that each item is allocated at most once. The second constraint implies that the auctioneer only accepts at most one submitted bid from a bidder. This bidding rule is commonly referred to as the "XOR" bidding language. Thus, each binary variable $x_j(S)$ equals to one if and only if bidder j is awarded bundle $S \subseteq M$. It is possible that the problem ($\mathscr{W}$) has multiple optimal solutions. In such cases, we assume a tie-breaking rule in favor of bidder 1 winning a pre-specified bundle. For an arbitrary set of bidders $C \subseteq N$ and an arbitrary set of items $S \subseteq M$, the function $w_b(C, S)$ is defined as the maximum surplus generated by allocating items in S among bidders in C given the reported bids b. We refer to the function w_b as the *coalition value function*. When $S = M$, we abuse the notation and write $w_b(C)$ instead of $w_b(C, M)$.

Next, we define *the core* to be the set of non-negative payoff vectors $\{\pi_j\}_{j \in N \cup \{0\}}$ satisfying the *core constraints*:

$$\sum_{j \in C \cup \{0\}} \pi_j \geq w_b(C), \quad \forall C \subseteq N. \tag{3}$$

The right-hand side of (3) is the maximum surplus generated by allocating the items among members in the coalition C (hereafter referred to as a *blocking coalition*). Thus, constraints (3) ensure that no group of participants can achieve a better payoff for themselves by excluding others, i.e., there is no incentive for any subset of bidders to form a blocking coalition in the auction. The core constraints (3) can be written succinctly as linear constraints on the payment vector $\{p_j\}_{j\in N}$ (see detailed descriptions in Appendix D.1). A *core-selecting payment rule* is a payment rule that selects a vector $\{p_j\}_{j\in N}$ satisfying the core constraints (3) and the individual rationality constraint $p \leq b$. For specificity, we adopt the quadratic core-selecting payment rule proposed by Day and Cramton (2012), which selects a payment vector in the core that minimizes the Euclidean distance to a reference payment vector. A related payment rule is the *VCG rule* Vickrey (1961), Clarke (1971), Groves (1973). Under this payment rule, each winner j pays the opportunity cost that she imposes on other bidders:

$$p_j^{VCG} = w_{b_{-j}}(N \setminus j, M) - w_{b_{-j}}(N \setminus j, M \setminus S_j). \tag{4}$$

The *VCG payoff* of bidder j is then simply $\pi_j^{VCG} = v_j(S_j) - p_j^{VCG}$. It is known that the VCG payment rule satisfies incentive compatibility but may result in low seller revenue and is vulnerable to collusive bidding (Ausubel and Milgrom 2002). In contrast, core-selecting payment rules are robust to collusion but may still have incentive compatibility issues. When the quadratic rule is applied with the reference payment being the vector of VCG payments, we refer to such a rule as the *nearest-VCG rule*.

Robust optimization formulation. In each of the aforementioned auction formats, we analyze the decision problem of a particular bidder, chosen as bidder 1 without loss of generality. We assume that bidder 1 believes her rivals' bids belong to an *uncertainty set* U_{-1}. In practice, such uncertainty sets can be constructed based on historical bid data, e.g., as box-type sets from observed bidding intervals, or as polyhedral sets encoding expert knowledge or known correlations. A rich literature in robust optimization provides systematic methods for this construction (e.g., Ben-Tal et al. 2009, Delage and Ye 2010, Bandi and Bertsimas 2014, Bertsimas et al. 2018), ranging from classical polyhedral and ellipsoidal sets that protect against bounded perturbations, to more advanced data-driven methods that use statistical guarantees to create less conservative sets from finite samples. For illustration, we use the core-selecting combinatorial auction to introduce the concept of an uncertainty set. In this setting, we have $U_{-1} \subset \mathbb{R}_+^{(n-1)2^m}$. For simplicity, we assume that the uncertainty set U_{-1} is a convex polytope, i.e., it can be specified by some linear constraints on b_{-1}:

$$U_{-1} = \{b_{-1} \mid Pb_{-1} \leq q\}, \tag{5}$$

where P and q are a matrix and a vector, respectively, of appropriate dimensions. For example, U_{-1} can be a box set where the bid for each specific bundle is constrained to lie within an interval

$$U_{-1} = \{b_{-1} \mid \underline{b}_j(S) \leq b_j(S) \leq \bar{b}_j(S), \forall j \neq 1, \forall S \subseteq M\}. \tag{6}$$

This represents a scenario where the auctioneer has independent range estimates for each potential bid a rival might place. Although the exact dimensionality and structure of the uncertainty set differ across auction formats (e.g., GFP and GSP involve scalar bids rather than bundle bids), the same robust optimization framework applies in all cases.

Given an uncertainty set U_{-1}, bidder 1's objective is to maximize her worst-case payoff with respect to this uncertainty set. In other words, she needs to solve the following robust optimization problem:

$$\pi_1^{MAXMIN} = \sup_{b_1 \in U_1} \inf_{b_{-1} \in U_{-1}} \pi_1(b_1, b_{-1}). \tag{$\mathscr{P}$}$$

In the above problem, we denote bidder 1's feasible policy space as U_1. We impose no restrictions except for the non-negativity condition, so we have $U_1 = \mathbb{R}_+^{2^m}$ for core-selecting auctions. We call bidding policy b_1 a *robust policy* if it is an optimal solution to ($\mathscr{P}$).

From a computational perspective, deriving robust policies in GFP auctions is straightforward. Because bidders submit scalar bids and allocations reduce to order statistics, computing robust policies mainly requires evaluating worst-case bid ranks within the uncertainty set. In GSP auctions, the analysis is somewhat more involved: for each slot, one must compute a worst-case payoff function (see Section 3.2 and Appendix C for details) and then take the lower envelope of these functions. Even so, the resulting optimization remains a one-dimensional search problem and is computationally efficient in practice.

By contrast, in combinatorial auctions, the robust optimization problem ($\mathscr{P}$) is computationally challenging. The winner determination problem (WDP) is NP-hard, the nearest-VCG payment rule requires solving a quadratic program, where the number of core constraints can be exponential in the number of bidders (n), and the inner minimization $\inf_{b_{-1} \in U_{-1}} \pi_1(b_1, b_{-1})$ is typically non-concave and discontinuous, making the outer maximization a difficult global optimization task.

Despite these computational hurdles, a key contribution of our framework is that it provides a path to identifying simple and profitable deviations from truth-telling without requiring the bidder to solve the full robust optimization problem. By exploiting the structure of the worst-case payoff and applying the minimax inequality (see Appendix B), as we demonstrate for single- and double-minded bidders, we can derive easy-to-implement policies that consistently outperform truthful reporting. This underscores the practical value of the robust approach: it yields actionable strategic guidance even in computationally demanding environments.

3. Robust Bidding Policies in GFP and GSP Auctions

In this section, we study bidder 1's robust bidding policy in GFP and GSP auctions. It is worth noting that in both auction formats, truthful reporting of valuations may be suboptimal.

3.1. GFP auction

We begin by analyzing the Generalized First-Price (GFP) auction. We first present a general result for the case where items are heterogeneous, distinguished by their quality or click-through rates, and then show how this simplifies for homogeneous items.

Before presenting the main result, we introduce the following notation for convenience. Let $u(k)$ denote the maximum value of $b^{(k)}$ over the uncertainty set U_{-1}, i.e.,

$$u(k) \equiv \max_{b_{-1} \in U_{-1}} b^{(k)}. \tag{7}$$

The following proposition establishes bidder 1's optimal robust policy in the general, heterogeneous-item setting. The policy involves a discrete choice: identify the single most profitable slot to target under worst-case prices, and then bid just enough to secure that specific slot.

Proposition 1. *In a GFP auction with m heterogeneous items with click-through rates α_k, a robust policy for bidder 1 is to first identify the most profitable target slot k^* by solving*

$$k^* = \arg\max_{k \in \{1,\ldots,m\}} \alpha_k \left(v_1 - u(k)\right).$$

Let the resulting maximum potential payoff be $\pi^* = \alpha_{k^*}(v_1 - u(k^*))$. An optimal robust policy is

$$b_1^{RO} = \begin{cases} u(k^*), & \text{if } \pi^* > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The optimal worst-case payoff is $\pi_1^{MAXMIN} = \max(0, \pi^*)$.

A common special case of the GFP auction is the discriminatory auction, where all items are homogeneous. In this setting, the click-through rates are identical ($\alpha_k = 1$ for all k), and a bidder is indifferent between winning any of the m available slots. The robust policy then simplifies to targeting the cheapest possible slot to win, which is the last one, slot m.

Corollary 1. *In a GFP auction with homogeneous items, a robust policy for bidder 1 is $b_1^{RO} = u(m)\mathbb{1}_{u(m) \le v_1}$. The optimal payoff is $\pi_1^{MAXMIN} = (v_1 - u(m))^+$.*

While the preceding results prescribe a strategy for a single player, we can also analyze the game-theoretic outcome when all bidders adopt the worst-case maximization objective. This shifts the focus from an individual bidder's decision problem to the strategic interaction among all participants. The following proposition reveals a perhaps surprising result: when all bidders are worst-case maximizers in the homogeneous good setting, truth-telling constitutes a stable equilibrium.

Proposition 2. *In a GFP auction with n bidders and m homogeneous items, where $n \ge m$, if all bidders are worst-case payoff maximizers over a common knowledge valuation range $[\underline{v}, \bar{v}]$, then the strategy profile where every bidder bids their true valuation, i.e., $b_j(v_j) = v_j$, constitutes a Nash equilibrium.*

The intuition behind this equilibrium is that the worst-case for any bidder is to face rivals with the maximum possible valuation, $\bar{v}$. In this scenario, a bidder cannot guarantee a positive payoff. Since truth-telling results in a worst-case payoff of zero, and no deviation can improve upon this, no player has an incentive to unilaterally change their strategy.

3.2. GSP auction

Next, we analyze the Generalized Second-Price (GSP) auction. The GSP auction shares the same allocation rule as the GFP auction, but its second-price payment rule makes the analysis more complex.

In the simplified case where all items are homogeneous (i.e., they share a uniform click-through rate, $\alpha_k = \alpha$), a bidder's objective is to secure any slot at the lowest possible cost. The robust policy is therefore identical to the one in the homogeneous GFP auction: the bidder targets the m-th slot, as it is the cheapest to win under the worst-case scenario, and the robust bidding policy $b_1^{RO} = u(m)\mathbb{1}_{u(m)\le v_1}$.

Next, we consider the case where click-through rates are not identical. We begin by analyzing the setting with $n = 3$ bidders and $m = 2$ items, and then discuss how the analysis extends to more general settings. Before presenting the main result, we first introduce the following notations.

Let $l(k)$ represent the minimum value of $b^{(k)}$ over the uncertainty set U_{-1}, i.e.,

$$l(k) \equiv \min_{b_{-1}\in U_{-1}} b^{(k)}. \tag{8}$$

In addition, let $u(k_1 \mid k_2, x)$ denote the supremum of $b^{(k_1)}$ conditional on $b^{(k_2)} \le x$, i.e.,

$$\begin{aligned} u(k_1 \mid k_2, x) \equiv \sup_{b_{-1}\in U_{-1}} \quad & b^{(k_1)} \\ \text{s.t.} \quad & x \le b^{(k_2)}. \end{aligned} \tag{9}$$

We define $u(k_1 \mid k_2)$ as a shorthand for $u\left(k_1 \mid k_2, u(k_2)\right)$, where $u(\cdot)$ is defined by (7).

Proposition 3. *Consider a GSP auction with $n \ge 3$ bidders and $m = 2$ items, where the click-through rates satisfy $\alpha_2 < \alpha_1$. A robust policy for bidder 1 is given by*

$$b_1^{RO} = \begin{cases} u(2)\mathbb{1}_{u(2)\le v_1}, & \textit{if } \alpha_1(v_1 - u(1\mid 2)) \le \alpha_2(v_1 - u(2)), \\ u(1)\mathbb{1}_{u(1)\le v_1}, & \textit{if } \alpha_2(v_1 - u(2\mid 1)) \le \alpha_1(v_1 - u(1)), \\ x^*\mathbb{1}_{x^*\le v_1}, & \textit{otherwise}, \end{cases} \tag{10}$$

where x^* is the unique solution of the following equation

$$\alpha_1(v_1 - x) = \alpha_2(v_1 - u(2 \mid 1, x)). \tag{11}$$

The worst-case payoff is given by

$$\pi_1^{MAXMIN} = \begin{cases} \alpha_2(v_1 - u(2))^+, & \textit{if } \alpha_1(v_1 - u(1\mid 2)) \le \alpha_2(v_1 - u(2)), \\ \alpha_1(v_1 - u(1))^+, & \textit{if } \alpha_2(v_1 - u(2\mid 1)) \le \alpha_1(v_1 - u(1)), \\ \alpha_1(v_1 - x^*)^+, & \textit{otherwise}. \end{cases} \tag{12}$$

Deriving robust policies in more general settings with $n \geq 4$ bidders, $m \geq 3$ items, and heterogeneous click-through rates is more involved, as it requires a careful analysis of the interactions among the $m \geq 3$ worst-case payoff functions, which may involve numerous cases. Nevertheless, robust policies can be constructed in a manner similar to that outlined in Proposition 3. Specifically, for each $k \in M$, we compute the worst-case payoff function $f_k(x)$ that bidder 1 obtains when bidding x to win item k (see Lemma 3), together with its corresponding domain. We then form the overall worst-case payoff function by taking the pointwise minimum across all $f_k(x)$. Finally, the robust bidding policy b_1^{RO} is obtained by choosing the bid that maximizes this aggregated worst-case payoff. This procedure is computationally efficient, as each $f_k(x)$ is piecewise-defined over a bounded interval, allowing the overall worst-case payoff to be optimized via a simple one-dimensional search.

Similar to the GFP auction, we can also establish an equilibrium result when all symmetric bidders are worst-case payoff maximizers in the homogeneous item setting. The logic is analogous: the worst-case scenario for any bidder is to face rivals with the maximum possible valuation. In a GSP auction, this means the price for any winning slot would be the maximum valuation, making a positive payoff impossible to guarantee. As no strategy can achieve a better worst-case payoff than the zero payoff guaranteed by truth-telling, bidding one's true valuation is a Nash equilibrium.

We also compare the performance of the robust bidding policy against both truth-telling (Example 3) and expected-payoff maximization (Example 4) in Appendix C. Specifically, in the setting of Example 3, the robust bidding policy consistently outperforms truthful bidding for every realization $b_{-1} \in U_{-1}$.

In general, the robust bidding policy in GFP is simple, and the underlying auction rules are more transparent and easier to interpret relative to GSP. This combination of strategic simplicity and reduced cognitive burden for participants may provide a rationale for the observed shift of many digital advertising platforms from GSP to GFP.

4. Robust Bidding Policies in Core-selecting Auctions

In this section, we study bidder 1's robust bidding policy in combinatorial auctions. Section 4.1 begins with a benchmark case in which bidder 1 has perfect information about her rivals' bids b_{-1}. This benchmark illustrates the maximum payoff achievable under full information and serves as a useful reference point. We then turn to the more relevant case in which bidder 1 faces uncertainty about rivals' bids. In this setting, minimax arguments provide the key tool for characterizing robust policies. We apply this approach in Section 4.2 to the single-minded case and extend it in Section 4.3 to the double-minded case.

4.1. Optimal bidding policies under perfect information

If bidder 1 has perfect information about her rivals' bids b_{-1}, then her VCG payoff coincides with the maximum payoff she can achieve (see Day and Milgrom 2008). There may exist multiple bidding policies that allow bidder 1 to attain this payoff. In the next proposition, we present one such policy, which extends naturally to settings with imperfect information. For convenience, we assume a tie-breaking rule that favors bidder 1 winning bundle S_1, the bundle she would obtain under truthful bidding (hereafter referred to as the *truthful bundle*).

PROPOSITION 4. *Bidder 1's optimal bidding policy is given as follows:*

$$b_1^{PI}(S) = \begin{cases} p_1^{VCG}, & \text{if } S_1 \subseteq S \subsetneq M, \\ w_{b_{-1}}(N \setminus 1), & \text{if } S = M, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

REMARK 1. Policy (13) is also optimal for bidder 1 even if she uses multiple identities, also known as *shills*.

Under policy (13), bidder 1 shades (underbids) her valuation on the truthful bundle S_1 and on any other bundles that contain it, except for the global bundle M. In addition, she misreports her valuation for the global bundle and bids $w_{b_{-1}}(N \setminus 1)$ for it. This bidding policy has two main effects. First, it guarantees that bidder 1 still wins bundle S_1 under the assumed tie-breaking rule. Second, by bidding high on the global bundle, bidder 1 effectively inflates her demand for other bundles. As a result, other bidders are forced to pay for the high opportunity cost they impose on bidder 1. A bidder-optimal payment rule selects the payment vector from the core that minimizes the total payment from all winners, meaning that the sum of the winners' payments is fixed at the minimum possible level required for stability. Consequently, any increase in the payments made by other winners must be offset by a corresponding decrease in bidder 1's payment, thus reducing it. Notice that policy (13) can be adjusted to accommodate different tie-breaking rules. Specifically, if bidder 1 is not favored to win bundle S_1, she can modify her bid for S where $S_1 \subseteq S \subsetneq M$ to $b_1(S) = v_1(S_1) - \pi_1^{VCG} + \epsilon$ for some arbitrarily small $\epsilon > 0$. In this case, bidder 1 still wins S_1 and attains a payoff arbitrarily close to her VCG payoff.

4.2. Single-Minded Bidder

In this subsection, we analyze the robust bidding problem ($\mathscr{P}$) in the case where bidder 1 is *single-minded*, i.e., she has strictly positive valuation for a particular bundle S_1 and for any of its supersets, and zero valuation for all other bundles:

$$v_1(S) = \begin{cases} a > 0, \text{ if } S \supseteq S_1, \\ 0, \qquad \text{if } S \not\supseteq S_1. \end{cases} \tag{14}$$

Here we assume free disposal, so bidder 1 values any winning bundle $S \supsetneq S_1$ the same as the bundle S_1 itself. To characterize robust policies in this setting, we rely on minimax arguments. Specifically, we show that for a single-minded bidder, the maximum worst-case payoff is equal to the minimum ex post maximum payoff, thereby establishing the optimality of the robust policy.

Let $\bar{p}^{VCG}$ denote the maximum VCG payment bidder 1 might have to pay over the uncertainty set U_{-1} if she wins S_1, i.e.,

$$\bar{p}^{VCG} = \max_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M) - w_{b_{-1}}(N \setminus 1, M \setminus S_1). \tag{15}$$

If $v_1(S_1) \le \bar{p}^{VCG}$, then there exists $b^*_{-1} \in U_{-1}$ such that bidder 1 does not win S_1 when bidding truthfully, yielding a payoff of zero. By Proposition 4, any alternative bidding policy leads to the same allocation outcome for bidder 1. Therefore, her worst-case payoff is bounded above by zero, and bidding zero is a trivial robust policy. Consequently, without loss of generality, we assume for the remainder of this section that $v_1(S_1) > \bar{p}^{VCG}$; that is, bidder 1's valuation for bundle S_1 is sufficiently high that she always wins S_1 by bidding truthfully, regardless of her rivals' bids $b_{-1} \in U_{-1}$. For convenience, we also assume a tie-breaking rule that favors bidder 1 winning S_1. This assumption does not affect our results, just as in the perfect-information case. The following proposition presents a robust policy for bidder 1.

PROPOSITION 5. *If bidder 1 is single-minded and $v_1(S_1) > \bar{p}^{VCG}$, then a robust policy for bidder 1 is:*

$$\begin{aligned} b_1^{RO}(S) &= \bar{p}^{VCG}, \quad \text{if } S_1 \subseteq S \subsetneq M, \\ b_1^{RO}(M) &= \bar{p}^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \backslash 1, M \backslash S_1), \\ b_1^{RO}(S) &= 0, \quad \text{otherwise}. \end{aligned} \tag{16}$$

The optimal worst-case payoff is $\pi_1^{MAXMIN} = v_1(S_1) - \bar{p}^{VCG}$.

REMARK 2. Policy (16) is not uniquely optimal. For example, if $b_1(M)$ takes any value within the interval $[0, \bar{p}^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M \setminus S_1)]$, then the resulting policy is also optimal. However, bidding a higher $b_1(M)$ weakly increases bidder 1's payoff for any realization of $b_{-1} \in U_{-1}$, as it raises the payment of other bidders, thereby reducing bidder 1's own payment.

A key challenge with our VCG-based bidding policy is the difficulty of computing payments. Nevertheless, the robust bidding policy provides a simple deviation strategy: bidders can shade their bids on bundles of interest while inflating their bid on the global bundle, thereby reducing their final payment. This creates a straightforward and predictable path for strategic behavior.

L\L\G valuation structure. We analyze the performance of the robust policy given by Proposition 5 in a setting with $n = 3$ bidders and $m = 2$ identical items. In particular, we consider the Local-Local-Global (L\L\G) valuation structure in which bidder 1 is a *local* bidder interested in only one item while bidders 2 and 3 are *local* and *global* bidders who are interested in one and two items, respectively, following a similar valuation structure as in Goeree and Lien (2016). Table 4.2 summarizes the notation: a, b, and c represent problem parameters, and x and y are decision variables. We slightly abuse notation by using b to denote bidder 2's bid on one and two items, rather than the bid profile of all bidders. Throughout this example, we adopt the nearest-VCG payment rule.

Table 1 Bidder valuations under the L\L\G valuation structure

# items	v_1	b_1	b_2	b_3
1	a	x	b	0
2	a	y	b	c

We consider the simple box-type uncertainty set

$$U_{-1} = \{(b, c) \mid \bar{b} - \epsilon_b \leq b \leq \bar{b} + \epsilon_b, \bar{c} - \epsilon_c \leq c \leq \bar{c} + \epsilon_c\},$$

where $\epsilon_b < \bar{b}$ and $\epsilon_c < \bar{c}$. Notice that, in this setting, the coalition value function $w_{v_1,b_{-1}}$ is not necessarily bidder-submodular.[2] Bidder 1's worst-case VCG payment, as defined in (15), simplifies to $\bar{p}^{VCG} = \max_{b_{-1} \in U_{-1}} (c - b)^+$ in this case. If $\bar{p}^{VCG} < a$, according to Proposition 5, a robust bidding policy for bidder 1 is

$$(x^*, y^*) = (\bar{p}^{VCG}, \bar{p}^{VCG} + \bar{b} - \epsilon_b). \tag{17}$$

Remark 3. Under the nearest-VCG payment rule, bidder 1's payoff with the robust policy (17) exceeds her payoff under truthful bidding for any realization of b_{-1} in U_{-1} (see D.8 for details).

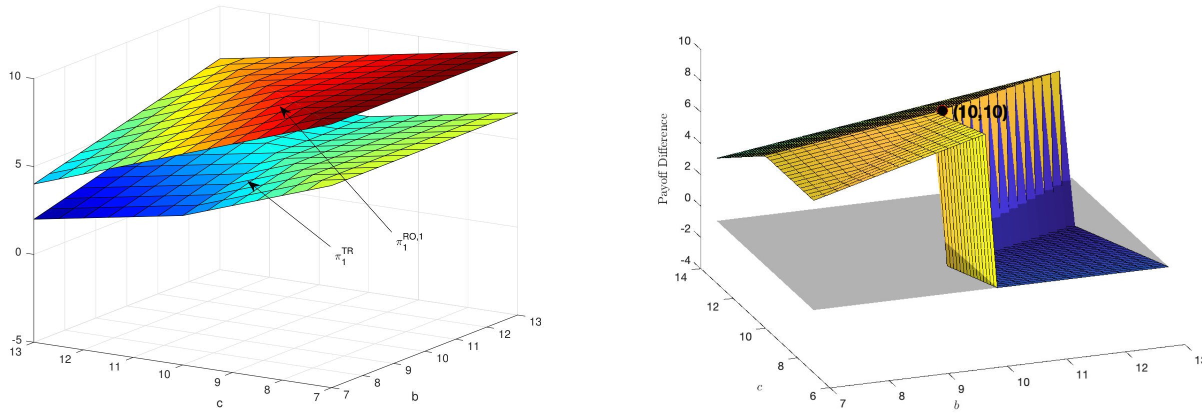
The following numerical example illustrates the performance of the robust bidding policy in the L\L\G structure.

Example 1. Consider a core-selecting auction with $n = 3$ bidders and $m = 2$ items, where bidders have an L\L\G valuation structure (see Table 4.2). Let $a = 10$ and $U_{-1} = \{(b, c) \mid 7 \leq b \leq 13, 7 \leq c \leq 13\}$. By Proposition 5, a robust bidding policy is $b_1^{RO,1} = (6, 13)$. However, as noted in

[2] A coalition value function is bidder-submodular if the marginal value of adding a new bidder to a coalition decreases (or stays the same) as the coalition gets larger. Suppose $x = 6$, $y = 10$, $b = 10$, and $c = 11$. We have

$$w(\{1,3\}) - w(\{3\}) = 0 < 5 = w(\{1,2,3\}) - w(\{2,3\}),$$

violating bidder-submodularity.



(a) Bidder 1's payoffs under $b_1^{RO,1}$ and b_1^{TR}

(b) Difference between bidder 1's payoffs under $b_1^{RO,1}$ and b_1^{PI}

Figure 1 **Illustration of Example 1: comparison of bidder 1's payoffs under the robust policy $b_1^{RO,1} = (6, 13)$, the truthful policy $b_1^{TR} = (10, 10)$, and the (potentially misspecified) perfect-information policy $b_1^{PI} = (0, 10)$.**

Remark 2, any bid y for the global bundle within $[0, 13]$ yields the same worst-case payoff, so a policy such as $b_1^{RO,2} = (6, 6)$ is also robust.

We first numerically confirm the observation from Remark 3. Figure 1a compares bidder 1's payoffs under the robust policy $b_1^{RO,1} = (6, 13)$ and truthful bidding $b_1^{TR} = (10, 10)$. For any realization of $b_{-1} \in U_{-1}$, bidder 1 attains a higher payoff with the robust policy than by reporting her true valuation.

We next compare the performance of the robust policy $b_1^{RO,1}$ with bidding under (potentially misspecified) perfect information. For concreteness, we assume that the rivals' bids are $b = c = 10$, which yields the perfect-information bid $b_1^{PI} = (0, 10)$ as defined in (13). If this information is accurate, the payoff from b_1^{PI} exceeds that of any alternative policy, including the robust policy $b_1^{RO,1}$. However, if the information about rivals' bids is misspecified, the payoff from b_1^{PI} may be lower. In this context, the uncertainty set U_{-1} represents possible misspecifications of the rivals' bids, allowing deviations of up to ϵ_b and ϵ_c in (b, c) with $\epsilon_b, \epsilon_c \leq 3$. We compare the payoffs of the robust policy $b_1^{RO,1}$ with those of the (potentially misspecified) perfect-information policy b_1^{PI}. In the latter case, bidder 1 behaves as if $b = c = 10$, even though the true realization of (b, c) may lie anywhere in U_{-1}.

Under b_1^{PI}, if bidders 2 and 3 bid such that $b \geq c$ and $b \geq 10$, then bidders 1 and 2 each win an item. Since bidder 1 bids zero for one item, bidder 2 bears the entire payment, making bidder 1 a free rider. In this case, bidder 1 pays nothing and receives her full valuation as payoff. However, by bidding b_1^{PI}, bidder 1 also faces the risk of either winning an unnecessary extra item or not winning any item at all. Specifically, if bidders 2 and 3 bid $b < c$ with $c \geq 10$, then bidder 3 wins both items and bidder 1's payoff is zero. Alternatively, if $b < 10$ and $c < 10$, then bidder 1 wins both items and pays a high price of $p_1 = \max(b, c)$. In both cases, bidder 1's payoff is substantially reduced. The robust policy $b_1^{RO,1}$ mitigates these risks by guaranteeing that bidder 1 always wins exactly one item, regardless of the bids of bidders 2 and 3.

Let $\pi(b_1^{RO,1}, b, c)$ and $\pi(b_1^{PI}, b, c)$ denote bidder 1's payoffs under $b_1^{RO,1}$ and b_1^{PI}, respectively. Figure 1b illustrates the difference $\pi(b_1^{RO,1}, b, c) - \pi(b_1^{PI}, b, c)$ for all $(b, c) \in U_{-1}$. As indicated by the positive values in every ϵ-box around $(b, c) = (10, 10)$, robust bidding may outperform (potentially misspecified) perfect-information bidding even under minimal misspecification. Moreover, for any point within the ϵ-box around $(b, c) = (10, 10)$ where the payoff of b_1^{PI} exceeds that of $b_1^{RO,1}$, there exists a corresponding "symmetric" point where $b_1^{RO,1}$ yields an even higher payoff than b_1^{PI}. More precisely, for any $(\epsilon_1, \epsilon_2) \in (0, 3] \times [-3, 3]$ with $\epsilon_1 \geq \epsilon_2$, the payoff advantage of b_1^{PI} over $b_1^{RO,1}$ is

$$\pi(b_1^{PI}, 10 + \epsilon_1, 10 + \epsilon_2) - \pi(b_1^{RO,1}, 10 + \epsilon_1, 10 + \epsilon_2) = 10 - (8.5 - 0.5\epsilon_2) = 1.5 + 0.5\epsilon_2 > 0,$$

which is strictly smaller than the payoff advantage of $b_1^{RO,1}$ at the symmetric point $(10 + \epsilon_2, , 10 + \epsilon_1)$:

$$\pi(b_1^{RO,1}, 10 + \epsilon_2, 10 + \epsilon_1) - \pi(b_1^{PI}, 10 + \epsilon_2, 10 + \epsilon_1) = (8.5 - \epsilon_1 + 0.5\epsilon_2) - 0 = 8.5 - \epsilon_1 + 0.5\epsilon_2 > 0.$$

Further, when $\epsilon_1, \epsilon_2 < 0$, the policy $b_1^{RO,1}$ always outperforms b_1^{PI}, i.e.,

$$\pi(b_1^{RO,1}, 10 + \epsilon_1, 10 + \epsilon_2) - \pi(b_1^{PI}, 10 + \epsilon_1, 10 + \epsilon_2) = 8.5 - 0.5\epsilon_2 + 0.5\min(\epsilon_1 - \epsilon_2, 0) + \max(\epsilon_1, \epsilon_2) > 0.$$

Thus, under symmetric misspecification around $(10, 10)$, the policy b_1^{PI} performs better than $b_1^{RO,1}$ only within the trapezoidal region below the diagonal. Even in that region, however, the magnitude of this advantage is smaller than the payoff advantage of $b_1^{RO,1}$ at the corresponding symmetric points. $\square$

REMARK 4. In Example 1, if bidder 1 applies the perfect-information optimal policy (13) using the worst-case rivals' bids $b = 7$ and $c = 13$, she obtains the robust policy $b_1^{RO,1} = (6, 13)$. This result follows from the minimax equality (26), which holds in the single-minded setting.

REMARK 5. In Example 1, if the two local bidders bid only for a single item and all three bidders' valuations are uniformly distributed on $[7, 13]$, Goeree and Lien (2016) show that the bidding strategies $b_1(v) = b_2(v) = v - 10/3$ and $b_3(v) = v$ constitute a Bayesian Nash equilibrium. Bidder 1's expected payoff in this equilibrium and under the robust policy $b_1^{RO,1} = (6, 13)$ are nearly identical (approximately 6.1) in this setting. Importantly, the robust policy requires substantially less information, since it does not depend on the distribution of others' valuations or their equilibrium bidding strategies. This illustrates that, although equilibrium bidding strategies may be difficult to characterize for bidder 1 in a two-dimensional bidding space, robust bidding provides a simple and tractable alternative.

4.3. Double-minded bidder

We now extend our analysis to the case where bidder 1 is *double-minded*. Similar to the single-minded case, the characterization of the robust policy is based on a minimax argument: we show that the maximum worst-case payoff coincides with the minimum ex post maximum payoff. This result establishes the optimality of the robust bidding strategy in the double-minded setting as well.

Specifically, suppose bidder 1 is interested in two nested bundles S_1 and S_2, where $\emptyset \subsetneq S_1 \subsetneq S_2 \subseteq M$:

$$v_1(S) = \begin{cases} a > 0, & \text{if } S_1 \subseteq S \subsetneq S_2, \\ a' \geq a, & \text{if } S_2 \subseteq S, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Let

$$\bar{p}_1^{VCG} = \max_{b_{-1} \in U_{-1}} \big(w_{b_{-1}}(N \setminus 1, M) - w_{b_{-1}}(N \setminus 1, M \setminus S_1) \big) \text{ and} \tag{19}$$

$$\bar{p}_2^{VCG} = \max_{b_{-1} \in U_{-1}} \big(w_{b_{-1}}(N \setminus 1, M) - w_{b_{-1}}(N \setminus 1, M \setminus S_2) \big) \tag{20}$$

be the maximum values of bidder 1's VCG payment over the uncertainty set U_{-1} when she wins bundle S_1 and S_2, respectively. As in the single-minded case, if $v_1(S_1) \leq \bar{p}_1^{VCG}$, bidder 1's worst-case payoff is bounded above by zero, so bidding zero is a robust policy. Therefore, without loss of generality, we assume $v_1(S_1) > \bar{p}_1^{VCG}$, i.e., bidder 1's valuation for S_1 is sufficiently high that she always wins either S_1 or S_2 under truthful reporting.

A bundle is called the *unique truthful allocation* for bidder 1 if, under truthful bidding, she always wins that bundle regardless of her rivals' bids within the uncertainty set. The following proposition characterizes robust policies for bidder 1 in the case where either S_1 or S_2 is her unique truthful allocation.

PROPOSITION 6. *Suppose bidder 1 is double-minded with $0 < \bar{p}_1^{VCG} < v_1(S_1) \leq v_1(S_2)$.*

1. *If S_1 is bidder 1's unique truthful allocation, then a robust policy is:*

$$\begin{aligned} b_1^{RO}(S) &= \bar{p}_1^{VCG} \quad \textit{if } S_1 \subseteq S \subsetneq M, \\ b_1^{RO}(M) &= \bar{p}_1^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M \setminus S_1), \\ b_1^{RO}(S) &= 0 \quad \textit{otherwise.} \end{aligned} \tag{21}$$

   The optimal worst-case payoff is $\pi_1^{MAXMIN} = v_1(S_1) - \bar{p}_1^{VCG}$.

2. *If S_2 is bidder 1's unique truthful allocation, then a robust policy is:*

$$\begin{aligned} b_1^{RO}(S) &= \bar{p}_2^{VCG} \quad \textit{if } S_2 \subseteq S \subsetneq M, \\ b_1^{RO}(M) &= \bar{p}_2^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M \setminus S_2), \\ b_1^{RO}(S) &= 0 \quad \textit{otherwise.} \end{aligned} \tag{22}$$

   The optimal worst-case payoff is $\pi_1^{MAXMIN} = v_1(S_2) - \bar{p}_2^{VCG}$.

REMARK 6. Note that the robust policies (21) and (22) parallel (16) in the single-minded case. In fact, Proposition 6 extends naturally to the case where bidder 1 has positive valuation for a collection of bundles $\{S_k\}_{k=1}^K$ satisfying $\emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_K$. In this setting, if S_k is the unique truthful allocation for bidder 1, then a robust policy is:

$$\begin{aligned} b_1^{RO}(S) &= \bar{p}_k^{VCG} \quad \text{if } S_k \subseteq S \subsetneq M, \\ b_1^{RO}(M) &= \bar{p}_k^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M \setminus S_k), \\ b_1^{RO}(S) &= 0 \quad \text{otherwise.} \end{aligned}$$

We now consider the case where neither S_1 nor S_2 is the unique truthful allocation for bidder 1. For analytical tractability, we study this scenario in an auction environment analogous to that of the single-minded case.

LG\L\G valuation structure. Consider a core-selecting auction with $n = 3$ bidders and $m = 2$ homogeneous items. We extend the setting of Section 4.2 by making bidder 1 double-minded; that is, she values winning either one or two items. We refer to this valuation structure as the LG\L\G structure. In this case, bidder 1's valuation for both items is $a' \geq a$, whereas in the L\L\G case it was only a. Table 2 summarizes the notation for this setting.

Table 2 Bidder valuations under the LG\L\G valuation structure

# items	v_1	b_1	b_2	b_3
1	a	x	b	0
2	a'	y	b	c

The values $\bar{p}_1^{VCG}$ and $\bar{p}_2^{VCG}$, as defined in (19) and (20), simplify in this case to

$$\bar{p}_1^{VCG} = \max_{b_{-1} \in U_{-1}} (c - b)^+, \quad \bar{p}_2^{VCG} = \max_{b_{-1} \in U_{-1}} \max(b, c).$$

We assume that $\bar{p}_1^{VCG} < a$, so bidder 1 wins either one or two items under truthful bidding. By Proposition 6, if $a' \leq a + \bar{b} - \epsilon_b$, then winning one item is bidder 1's unique truthful allocation, and her robust bidding policy is $b_1^{RO} = (\bar{p}_1^{VCG}, \bar{p}_1^{VCG} + \bar{b} - \epsilon_b)$. Similarly, if $a + \bar{b} + \epsilon_b < a'$, then winning both items is bidder 1's unique truthful allocation, and her robust bidding policy is $b_1^{RO} = (0, \bar{p}_2^{VCG})$. The optimal worst-case payoff is therefore

$$\pi_1^{MAXMIN} = \begin{cases} a - \bar{p}_1^{VCG} & \text{if } a' \leq a + \bar{b} - \epsilon_b, \\ a' - \bar{p}_2^{VCG} & \text{if } a + \bar{b} + \epsilon_b < a'. \end{cases}$$

Now consider the case where $a' \in (a + \bar{b} - \epsilon_b, a + \bar{b} + \epsilon_b]$. In this scenario, bidder 1 may win either one or both items under truthful bidding. To derive a robust bidding policy, we restrict attention to the reduced policy space $U_1' = \{(x, y) \in \mathbb{R}_+^2 \mid x = \bar{p}_1^{VCG}, y \geq x\}$. We show that by choosing the optimal value of y within U_1', bidder 1 achieves the maximum worst-case payoff characterized by the minimax inequality. Consequently, this policy is also optimal in the original policy space $U_1 = \mathbb{R}_+^2$.

For any $(x,y) \in U_1'$, let $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$ denote bidder 1's worst-case payoff functions when bidding (x,y) and winning one or two items, respectively. We have

$$\inf_{b_{-1} \in U_{-1}'} \pi_1(b_1, b_{-1}) = \min\left(\pi_1^{WO,1}(y), \pi_1^{WO,2}(y)\right).$$

The following lemma establishes key properties of $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$.

Lemma 1. *The worst-case payoff functions $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$ have the following properties:*

1. *$\pi_1^{WO,1}(y)$ is piecewise linear and increasing in y. Moreover,*

$$\pi_1^{WO,1}(y) = a \quad \textit{if} \quad y \geq \bar{p}_1^{VCG} + \bar{c} + \epsilon_c.$$

2. *$\pi_1^{WO,2}(y)$ is piecewise linear and decreasing in y. Moreover,*

$$\pi_1^{WO,2}(y) = a' - \bar{c} - \epsilon_c \quad \textit{if} \quad y \leq \bar{p}_1^{VCG} + \bar{c} + \epsilon_c.$$

The following lemma establishes the relationship between the worst-case payoff functions and the bidders' valuations.

Lemma 2. *The worst-case payoff functions satisfy the following properties:*

1. *$\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) \leq \pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b)$ if and only if $a' \leq a + \bar{b} - \epsilon_b$;*
2. *$\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) \leq \pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b)$ if and only if $a' \leq a + \bar{b} + \epsilon_b$.*

Figure 2 illustrates the worst-case payoff functions under different valuation scenarios. When either winning one item or winning two items is bidder 1's unique truthful allocation, the corresponding worst-case payoff function dominates the other (Figures 2(a) and (b)). Otherwise, the two functions intersect (Figures 2(c) and (d)).

The following proposition characterizes the robust bidding policy in the LG\L\G setting when $a' \in (a + \bar{b} - \epsilon_b, a + \bar{b} + \epsilon_b]$.

Proposition 7. *In the LG\L\G setting, if $a' \in (a + \bar{b} - \epsilon_b, a + \bar{b} + \epsilon_b]$, then a robust bidding policy for bidder 1 is $b_1^{RO} = (x^*, y^*)$, where $x^* = \bar{p}_1^{VCG}$ and y^* is the unique solution to $\pi_1^{WO,1}(y) = \pi_1^{WO,2}(y)$. The corresponding worst-case payoff is $\pi_1^{MAXMIN} = \min(a, a' - \bar{c} - \epsilon_c)$.*

Remark 7. In the LG\L\G setting, truthful bidding is also a robust policy provided that there exists $b_{-1} \in U_{-1}$ such that bidder 1 wins the global bundle under truthful bidding; that is, when $a + \bar{b} - \epsilon_b < a'$ (see Appendix D.15). This conclusion, however, does not necessarily extend to cases where S_2 is not the global bundle.

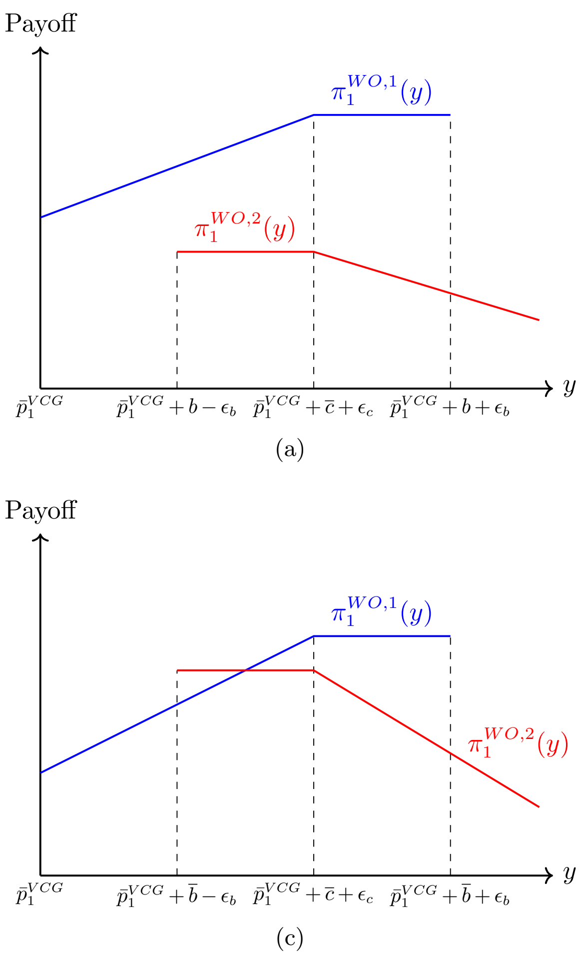


Figure 2 **Worst-case payoff functions in different cases under LG\L\G setting: (a)** $a' \leq a + \bar{b} - \epsilon_b$**, winning one item always yields better worst-case payoff; (b)** $a + \bar{b} + \epsilon_b < a'$**, winning two items always yields better worst-case payoff; (c) and (d)** $a + \bar{b} - \epsilon_b < a' \leq a + \bar{b} + \epsilon_b$**, two worst-case payoff functions intersect.**

5. Numerical Results

In this section, we present numerical results evaluating the performance of the robust bidding policies derived in Section 4. Section 5.1 compares the robust bidding policy with other common strategies, including truth-telling, expected-payoff maximization, and misspecified perfect-information bidding, in the L\L\G setting. Section 5.2 then validates the effectiveness of the robust policy in a more complex combinatorial auction with multiple bidders and heterogeneous items, demonstrating its consistent outperformance relative to truthful bidding.

5.1. Comparison of Robust and Benchmark Bidding Strategies in the L\L\G Setting

In this subsection, we numerically evaluate the performance of the robust bidding policy against benchmarks, including expectation maximization (EM), truth-telling (TR), and potentially misspecified perfect information (PI). Assuming various distributions of rivals' bidding profiles, we compute bidder 1's expected payoff under these policies. We begin with the following example.

Example 2. Consider a core-selecting auction with settings similar to Example 1, but with bidder 1's valuation varying. We compare bidder 1's payoff under the robust policy (17) to her payoff under an expected-payoff maximizing policy. For the latter, we assume bidder 1 believes her rivals' bids b and c are independent, with the following probability density functions:

$$f_b(b) = \begin{cases} (b-7)/18, & \text{if } 7 \le b \le 13, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_c(c) = \begin{cases} (c-7)/18, & \text{if } 7 \le c \le 13, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

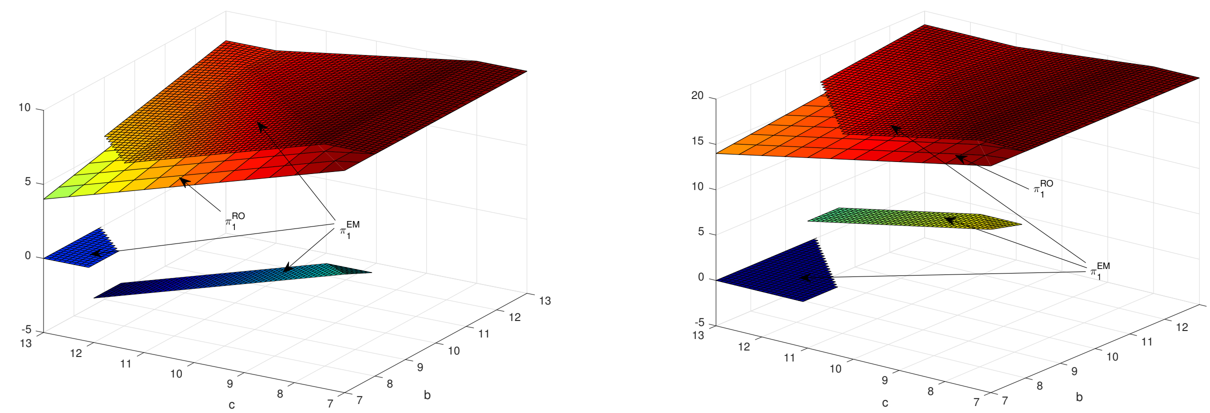
Note that if b and c are uniformly distributed, bidder 1's expected-payoff maximizing policy coincides with her robust policy (17), making the comparison trivial. Therefore, for illustration purposes, we instead consider the distributions f_b and f_c described above. Our observations remain qualitatively similar for other choices of f_b and f_c.

When $v_1 = (4,4)$, bidder 1's robust and expected-payoff maximizing policies are $b_1^{RO} = b_1^{EM} = (0,0)$. Thus, bidder 1's payoff is the same under both policies. When $v_1 = (10,10)$, we have $b_1^{RO} = (6,13)$ and $b_1^{EM} = (4,12)$ (see Appendix D.10 for details). Notice that f_b is increasing in b for $b \in [7,13]$ and f_c is decreasing in c for $c \in [7,13]$. Under these distributional assumptions, bidder 2 is more likely to bid high, and bidder 3 is more likely to bid low, relative to the support ranges given by U_{-1}. Consequently, under b_1^{EM}, bidder 1 bids low for one item, anticipating winning one item with high probability and paying less due to her low bid. However, this policy exposes her to the risk of either not winning any item or winning both items.

Figure 3a compares bidder 1's payoff under b_1^{RO} and b_1^{EM} when $v_1 = (10,10)$. As shown, bidder 1's payoff under b_1^{RO} is significantly higher than under b_1^{EM} for realizations of rivals' bids in U_{-1} where bidder 1 wins both items or wins no items by bidding b_1^{EM}. Figure 3b compares the payoffs when $v_1 = (20,20)$, where similar observations hold. Note that b_1^{RO} remains the same as in the case of $v_1 = (10,10)$ since the policy depends only on the parameters of U_{-1} as long as $a > \bar{p}^{VCG}$. On the other hand, $b_1^{EM} = (3,11)$ when $v_1 = (20,20)$, meaning bidder 1 shades her bids even further compared to the case of $v_1 = (10,10)$. □

In the rest of this subsection, we extend the analysis from Example 2 to evaluate the performance of the robust bidding policy against multiple benchmarks under various distributions.

As described earlier, the robust, truth-telling, and (potentially misspecified) perfect-information bidding policies are independent of the distribution of the uncertainty set. Therefore, given a specific distribution of b and c, the expected payoff under these three policies can be computed directly. It remains to determine the bidding policy that maximizes bidder 1's expected payoff. Due to the complexity of the payoff function, it is challenging to analytically establish a closed-form expression for the expected payoff. Therefore, we compute the expectation numerically. Let (x^*_{EM}, y^*_{EM}) denote the bidding policy that maximizes the bidder's expected payoff.



(a) Bidder 1's payoffs under $b_1^{RO,1}$ and $b_1^{EM} = (4, 12)$ when $v_1 = (10, 10)$

(b) Bidder 1's payoffs under $b_1^{RO,1}$ and $b_1^{EM} = (3, 11)$ when $v_1 = (20, 20)$

Figure 3 **Illustration of Example 2: comparison of bidder 1's payoffs under the robust policy $b_1^{RO,1} = (6, 13)$ and the expected-payoff maximizing policy b_1^{EM}.**

Under the TR policy, the bidder bids its true valuation, which is $(x_{TR}^*, y_{TR}^*) = (10, 10)$ in this setting.

Under the PI policy, the bidder ignores the uncertainty and assumes the realization of (b, c) to be $(\bar{b}, \bar{c})$. In this case, the bidding policy is given by $(x_{PI}^*, y_{PI}^*) = (\bar{c} - \bar{b}, \bar{c})$.

We consider one more bidding strategy as a benchmark, where the bidder misspecifies the values of ϵ_b and ϵ_c, and maximizes its expected payoff using this imprecise uncertainty set

$$U_{-1}' = \left\{(b, c) \mid \bar{b} - \epsilon_b' \leq b \leq \bar{b} + \epsilon_b',\ \bar{c} - \epsilon_c' \leq c \leq \bar{c} + \epsilon_c'\right\}.$$

The corresponding bidding policy, denoted as PI', is computed numerically in the same way as under the EM policy.

To account for the potential impact of the uncertainty set's distribution, we consider the following seven specific distributions: one with evenly distributed masses (D1), four biased towards the corners (D2–D5), and two biased towards the center (D6–D7). In all cases, to maintain tractability, the bids submitted by bidders 2 and 3 (b, c) are independent on $\left[\bar{b} - \epsilon_b, \bar{b} + \epsilon_b\right] \times [\bar{c} - \epsilon_c, \bar{c} + \epsilon_c]$.

The distributions we assume are as follows: given the values for $\bar{b}$, $\bar{c}$, ϵ_b, and ϵ_c, for any $(b, c) \in \left[\bar{b} - \epsilon_b, \bar{b} + \epsilon_b\right] \times [\bar{c} - \epsilon_c, \bar{c} + \epsilon_c]$:

D1: Both b and c are uniformly distributed: $f(b) = f(c) = \frac{1}{6}$.

D2: $f(b)$ and $f(c)$ are linear and both increasing: $f(b, c) = \frac{1}{324}(b - (\bar{b} - 3))(c - (\bar{c} - 3))$.

D3: $f(b)$ and $f(c)$ are linear and both decreasing: $f(b, c) = \frac{1}{324}((\bar{b} + 3) - b)((\bar{c} + 3) - c)$.

D4: $f(b)$ and $f(c)$ are linearly increasing and decreasing: $f(b, c) = \frac{1}{324}(b - (\bar{b} - 3))((\bar{c} + 3) - c)$.

D5: $f(b)$ and $f(c)$ are linearly decreasing and increasing: $f(b,c) = \frac{1}{324}((\bar{b}+3) - b)(c - (\bar{c}-3))$.

D6: b and c are (truncated) normally distributed with means $\bar{b}$ and $\bar{c}$, and variances of 1.

D7: Both b and c follow a triangular distribution, where the probability density function is maximized at 10: $f(b,c) = \frac{1}{81}(3 - |\bar{b} - b|)(3 - |\bar{c} - c|)$.

For D1, the variances of b and c are 3; for D2-D5, the variances are 2; for D6, the variances are approximately 0.9733; and for D7, the variances are 1.5.

In addition to the impact of the uncertainty set's distribution, we also consider the ratio of $\bar{b}$ to $\bar{c}$. Specifically, we analyze the following three settings, where $\bar{b}/\bar{c}$ decreases: (1) $\bar{b} = \bar{c} = 10$, i.e., $\bar{b}/\bar{c} = 1$; (2) $\bar{b} = 10$ and $\bar{c} = 13.5$, i.e., $\bar{b}/\bar{c} = 0.74$; and (3) $\bar{b} = 8$ and $\bar{c} = 11.5$, i.e., $\bar{b}/\bar{c} = 0.70$. It is straightforward to verify that under these three settings, the assumption $v_a(S_1) > \bar{p}_{VCG}$ always holds.

Tables 4–6 (see Appendix A) present the expected payoffs under different bidding policies, given the distribution of the uncertainty set. The percentage values in parentheses indicate the relative performance of RO, TR, PI, and PI' compared to the EM policy. As shown in the tables, the robust bidding policy slightly underperforms the EM policy, with a relative difference of less than 5% in most cases, while significantly outperforming the TR, PI, and PI' ($\epsilon'_b = \epsilon'_c = 1$) policies. While our policy performs well against a range of benchmarks, it is unlikely that a worst-case theoretical performance bound can be established. For instance, under a degenerate distribution concentrated at a single point, the robust policy can perform significantly worse than the expectation-maximization policy, which effectively represents the optimal solution in the absence of asymmetric information.

To summarize, we have the following observations. In general, the relative differences between the expected payoffs generated by EM and RO policies are less than 10% (except under distribution D2 with $\bar{b} = 8$ and $\bar{c} = 11.5$). The other two simple bidding policies, TR and PI, perform much worse. Policy TR achieves approximately 60% of the maximum expected payoff, while the performance of PI heavily depends on the distribution. For instance, PI yields roughly 80% of the maximum under distribution D4 but only 20% under D5.

Under distributions D1 (uniform), D3, and D5 ($f(b)$ strictly decreasing), the RO policy achieves at least 99% of the expected payoff generated by the EM policy. Intuitively, the "worst" cases for bidder 1 occur when b is small. Under the RO policy, bidder 1 must submit a higher bid for a single item to ensure winning one of the two items, resulting in a lower payoff. When $f(b)$ is decreasing, such scenarios are more likely and significantly influence the design of the EM policy, leading to near-equivalence in performance. Note that the RO policy also performs well under a uniform distribution. Even under extreme circumstances where the mass is heavily biased, RO performs slightly worse but still achieves approximately 90% of the maximum payoff.

As $\bar{b}/\bar{c}$ decreases, the difference between RO and EM policies slightly increases for a similar reason: the "worst" cases occur when b is small. Furthermore, compared to the triangular distribution, RO performs slightly worse under the truncated normal distribution, even though both are concentrated around the median. This is because the truncated normal distribution has heavier tails, increasing the likelihood of extreme cases.

5.2. Validation in a Complex Combinatorial Auction Environment

To further evaluate our theoretical findings in a more challenging environment, we simulate a combinatorial auction with three heterogeneous items and eight bidders exhibiting diverse valuation structures, providing a more realistic representation of practical markets. Such markets often contain a mix of large, sophisticated bidders and smaller participants with simpler objectives. For example, in digital advertising, many small bidders are primarily concerned with securing the top ad slot and thus submit a single bid, effectively behaving as the single-minded bidders modeled here. This heterogeneity in bidder valuations offers a rigorous test of the practical effectiveness of our robust policies.

Experimental setup. The auction consists of three heterogeneous items, $M \equiv \{1,2,3\}$, and eight bidders whose valuation structures are summarized in Table 3. The bidders include single-minded bidders (1 and 5-7), cardinality bidders (2-4), and a double-minded bidder (8). We assume a box uncertainty set for the rivals' bids, as detailed in each numerical simulation.

Table 3 **Bidders' valuation structure for the 8-bidder simulation.**

Bundle	1	2	3	4	5	6	7	8
$\{1\}$	v_1	v_2	0	0	0	0	0	v_8'
$\{2\}$	0	v_2	0	0	0	0	0	0
$\{3\}$	0	v_2	0	0	0	0	0	0
$\{1,2\}$	v_1	v_2	v_3	0	v_5	0	0	v_8''
$\{1,3\}$	v_1	v_2	v_3	0	0	v_6	0	v_8'
$\{2,3\}$	0	v_2	v_3	0	0	0	v_7	0
$\{1,2,3\}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8''

Single-minded bidder (bidder 1). We first analyze the decision problem for Bidder 1, a single-minded bidder with a true valuation of $v_1 = 16$ for item 1. The uncertainty set is given by

$$U_{-1} \equiv \left\{ \{v_i\}_{i=2}^{8} : (v_2, v_8') \in [7,10]^2, (v_3, v_5, v_6, v_8'') \in [10,18]^4, v_4 \in [15,28], v_7 \in [15,20] \right\}.$$

This serves as an empirical validation of the robust policy derived in Proposition 5.

The first step is to calculate the maximum VCG payment, $\bar{p}^{VCG}$, over the rivals' uncertainty set. As shown in Appendix ..., this value is $\bar{p}^{VCG} = 13$. Following Proposition 5, the optimal robust policy for Bidder 1 is to bid 13 for any bundle containing item 1 and 28 for the grand bundle.

We simulated 10,000 auctions, drawing rivals' bids uniformly from the uncertainty set. The robust policy yielded an average payoff of 6.30, compared to 5.59 for the truthful policy. Figure 4 in Appendix A shows the empirical cumulative distribution functions (CDFs) of the payoffs. The robust strategy's CDF stochastically dominates that of the truthful strategy. This confirms the robust policy is not just better on average, but also consistently less risky.

Single-minded bidder for a multi-item bundle (bidder 7). We also consider bidder 7, who is interested in the bundle $\{2,3\}$, to demonstrate the applicability of our framework to bidders who are not single- or double-minded. We assume $v_7 = 20$ and the uncertainty set is given by

$$U_{-7} \equiv \left\{ (\{v_i\}_{i=1}^{6}, v_8) : v_1 \in [10,18], (v_2, v_8') \in [7,10]^2, (v_3, v_5, v_6, v_8'') \in [10,18]^4, v_4 \in [15,28] \right\}.$$

The maximum VCG payment for bidder 7 is $\bar{p}^{VCG} = 18$, as shown in Appendix .... Consequently, bidder 7's optimal robust policy is to bid 18 for any bundle containing items {2,3} and 28 for the grand bundle.

Based on 10,000 simulations, the average payoffs of the robust and truth-telling policies are 4.89 and 4.54, respectively. Figure 5 in Appendix A again illustrates that the robust policy's empirical CDF stochastically dominates that of the truthful policy.

Double-minded bidder (bidder 8). Next, we validate the robust policy for a double-minded bidder (Proposition 6) by analyzing bidder 8, who values bundle {1} at $v_8' = 15$ and bundle {1,2} at $v_8'' = 18$. We assume that the uncertainty set bidder 8 faces is given by

$$U_{-8} \equiv \left\{ \{v_i\}_{i=1}^{7} : (v_1, v_2) \in [7,10]^2, (v_3, v_5, v_6) \in [10,18]^3, v_4 \in [15,28], v_7 \in [15,20] \right\}.$$

In this setting, truthful bidding guarantees that bidder 8 wins at least bundle $\{1\}$, and the maximal VCG payment for this bundle is $\bar{p}^{VCG} = 13$. The robust policy is therefore to bid 13 for any bundle containing item 1 and 28 for the grand bundle.

The simulation results again show a clear advantage for the robust strategy, with an average payoff of 5.39 versus 4.84 for truthful bidding. The CDF plot in Figure 6 in Appendix A further demonstrates that the robust policy consistently outperforms the truthful one, reducing risk and improving returns.

Cardinality bidder (bidder 3). Although we do not derive the analytical robust bidding policy for a cardinality bidder, we can still use a numerical example to demonstrate that a simple deviation from truth-telling can be beneficial. Here, we analyze bidder 3, whose valuation depends on the number of items won. Specifically, let $v_3 = 17$, and define the uncertainty set as

$$U_{-3} \equiv \left\{ \{v_i\}_{i \neq 3} : (v_1, v_8') \in [7,10]^2, v_2 \in [10,13], v_4 \in [15,25], (v_5, v_6, v_7, v_8'') \in [10,15]^4 \right\}.$$

We test a simple strategic policy where bidder 3 bids 15 on all two-item bundles, while truthfully reporting its valuation for winning three items.

Based on 10,000 simulations, the average payoffs of this strategic policy and the truth-telling policy are 2.8962 and 2.8610, respectively. As shown in Figure 7 in Appendix A, even this simple deviation yields a modest improvement over truthful bidding, indicating that the principle of strategic bidding, such as shading bids on bundles of interest, remains valuable for bidders with complex valuation structures for which an optimal policy is not analytically tractable.

6. Concluding Remarks

In this paper, we address the strategic challenges faced by market participants who have limited information about their rivals' actions and simple valuation structures, particularly in markets offering automated bidding services. We formalize this problem using a robust optimization approach to identify worst-case optimal bidding policies across three prevalent auction formats: GFP, GSP, and core-selecting combinatorial auctions. By modeling the bidder's uncertainty via a set of possible rival bids, our framework enables a distribution-free analysis of bidding strategies.

A key finding of our approach is that simple, robust bidding policies can consistently outperform the default strategy of truth-telling. For instance, we establish that for a single-minded bidder in a core-selecting auction, the robust policy yields a higher guaranteed payoff and performs well against other standard benchmarks. This demonstrates that bidders with simple preferences can profitably deviate without complex distributional or equilibrium assumptions.

One of the main challenges with robust bidding is computational tractability, as the problem generally involves maximizing a non-concave, discontinuous worst-case payoff function, particularly when the strategy space is multi-dimensional. Nevertheless, our analysis provides insights that extend to other non-incentive-compatible market-clearing settings. Specifically, our framework provides a useful heuristic: when a bidder has a unique truthful allocation that is consistently favorable, the robust strategy is to bid the minimum needed to secure it. In more complex environments, the search for optimal bids can be simplified by focusing on the intersections of the allocation-specific worst-case payoff functions, as these points are the candidates for the optimal robust policy.

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Appendix

This appendix contains numerical results, proofs, and additional discussion of the GSP auction.

Appendix A: Numerical Results

Table 4 Expected payoff of different bidding policies when $\bar{b} = \bar{c} = 10$.

Dist.	EM	RO	TR	PI	PI ($\epsilon'_b = \epsilon'_c = 1$)	PI ($\epsilon'_b = \epsilon'_c = 1.5$)	PI ($\epsilon'_b = \epsilon'_c = 2$)
D1	7.9963	7.9963	4.8761	3.9973	5.9410	6.6419	7.1956
		(100%)	(60.98%)	(49.99%)	(74.30%)	(83.06%)	(89.99%)
D2	7.9092	7.6010	4.7253	4.7284	7.1695	7.6453	7.8315
		(96.10%)	(59.74%)	(59.78%)	(90.65%)	(96.66%)	(99.02%)
D3	8.5904	8.5904	5.2267	2.8548	4.7150	5.6994	6.6821
		(100%)	(60.84%)	(33.23%)	(54.89%)	(66.35%)	(77.79%)
D4	9.1212	8.9027	5.5274	7.1266	8.5371	8.8532	9.007
		(97.60%)	(60.60%)	(78.13%)	(93.60%)	(97.06%)	(98.75%)
D5	6.8877	6.8877	4.0236	1.2946	3.1104	4.1164	5.0845
		(100%)	(58.42%)	(18.80%)	(45.16%)	(59.76%)	(73.82%)
D6	8.5508	8.2207	4.9186	3.8648	7.6708	8.3991	8.5491
		(96.14%)	(57.52%)	(45.20%)	(89.71%)	(98.23%)	(99.98%)
D7	8.3014	8.1500	4.9000	3.8986	6.8839	7.6702	8.0826
		(98.18%)	(59.03%)	(46.96%)	(82.92%)	(92.40%)	(97.36%)

Table 5 Expected payoff of different bidding policies when $\bar{b} = 10$ and $\bar{c} = 13.5$.

Dist.	EM	RO	TR	PI	PI ($\epsilon'_b = \epsilon'_c = 1$)	PI ($\epsilon'_b = \epsilon'_c = 1.5$)	PI ($\epsilon'_b = \epsilon'_c = 2$)
D1	5.0112	4.9651	3.2138	2.5228	3.6165	4.0111	4.4317
		(99.08%)	(64.13%)	(50.34%)	(72.17%)	(80.04%)	(88.44%)
D2	4.9728	4.4871	3.2362	3.3582	4.6680	4.8681	4.9411
		(90.23%)	(65.08%)	(67.53%)	(93.87%)	(97.89%)	(99.36%)
D3	5.4876	5.4870	3.2361	0.7910	2.0499	2.8268	3.6691
		(99.99%)	(58.97%)	(14.41%)	(37.36%)	(51.51%)	(66.86%)
D4	6.7200	6.3850	4.1342	5.5605	6.4263	6.5740	6.6690
		(95.01%)	(61.52%)	(82.75%)	(95.63%)	(97.83%)	(99.24%)
D5	3.5130	3.4996	2.2487	0.3855	1.1891	1.6592	2.1539
		(99.62%)	(64.01%)	(10.97%)	(33.85%)	(47.23%)	(61.31%)
D6	5.3573	4.9989	3.2489	2.0920	4.6624	5.2244	5.3573
		(93.31%)	(60.64%)	(39.05%)	(87.03%)	(97.52%)	(100%)
D7	5.1933	4.9950	3.2450	2.2286	4.1309	4.6990	5.0201
		(96.18%)	(62.48%)	(42.91%)	(79.54%)	(90.48%)	(96.66%)

Table 6 **Expected payoff of different bidding policies when $\bar{b} = 8$ and $\bar{c} = 11.5$.**

Dist.	EM	RO	TR	PI	PI ($\epsilon_b' = \epsilon_c' = 1$)	PI ($\epsilon_b' = \epsilon_c' = 1.5$)	PI ($\epsilon_b' = \epsilon_c' = 2$)
D1	5.0113	4.9651	3.3079	3.0280	4.0667	4.3866	4.6198
		(99.08%)	(66.01%)	(60.42%)	(81.15%)	(87.53%)	(92.19%)
D2	5.0007	4.4871	3.2519	3.4864	4.7772	4.9248	4.9752
		(89.73%)	(65.03%)	(69.72%)	(95.53%)	(98.48%)	(99.49%)
D3	5.4876	5.4870	3.4085	1.9229	3.0465	3.6470	4.2632
		(99.99%)	(62.11%)	(35.04%)	(55.52%)	(66.46%)	(77.69%)
D4	6.7803	6.3850	4.3066	5.9418	6.6319	6.7281	6.7629
		(94.17%)	(63.52%)	(87.63%)	(97.81%)	(99.23%)	(99.74%)
D5	3.5131	3.4996	2.2644	0.7634	1.6885	2.1524	2.5791
		(99.62%)	(64.46%)	(21.73%)	(48.06%)	(61.27%)	(73.41%)
D6	5.4023	4.9989	3.2624	2.6021	4.9365	5.3465	5.4022
		(92.53%)	(60.39%)	(48.17%)	(91.38%)	(98.97%)	(100%)
D7	5.2268	4.9950	3.2762	2.7370	4.4327	4.8567	5.1073
		(95.57%)	(62.68%)	(52.36%)	(84.81%)	(92.92%)	(97.71%)

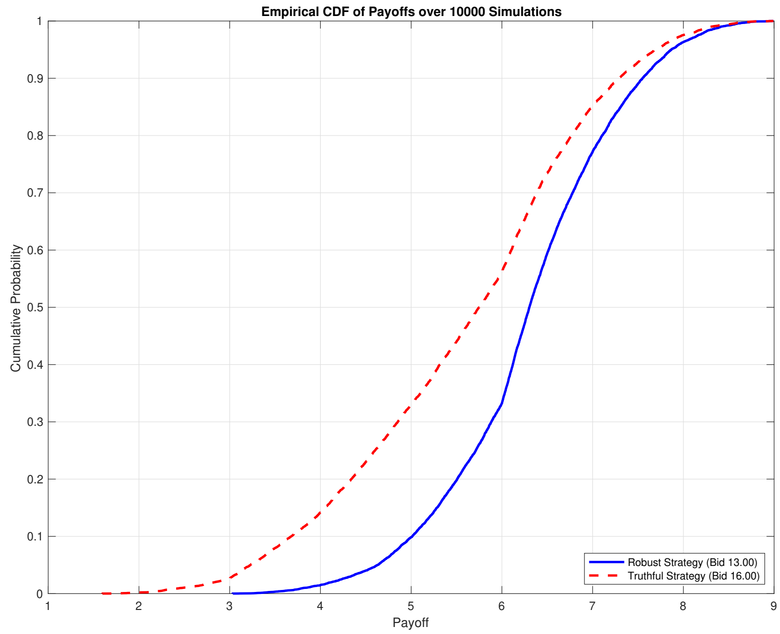

Figure 4 **Empirical CDF of Bidder 1's payoffs over 10,000 simulations.**

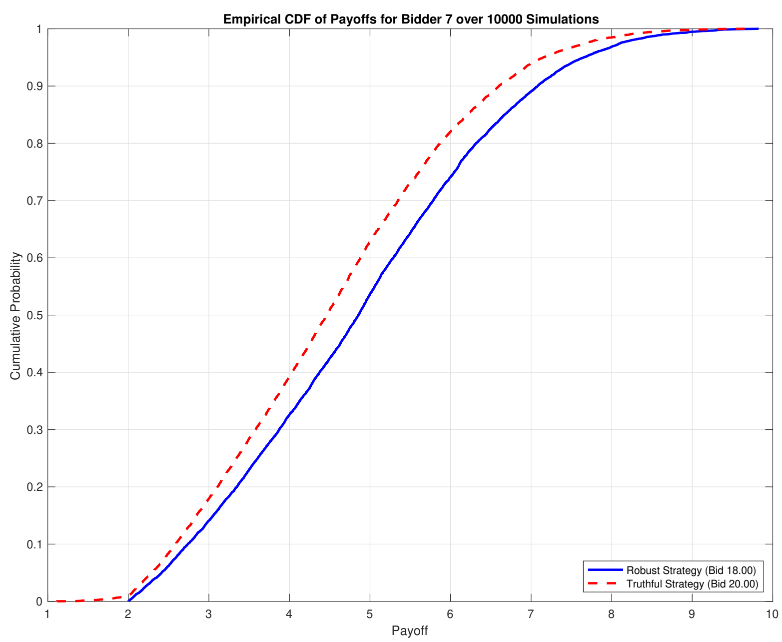

Figure 5 **Empirical cdf of bidder 7's payoffs based on 10,000 simulations.**

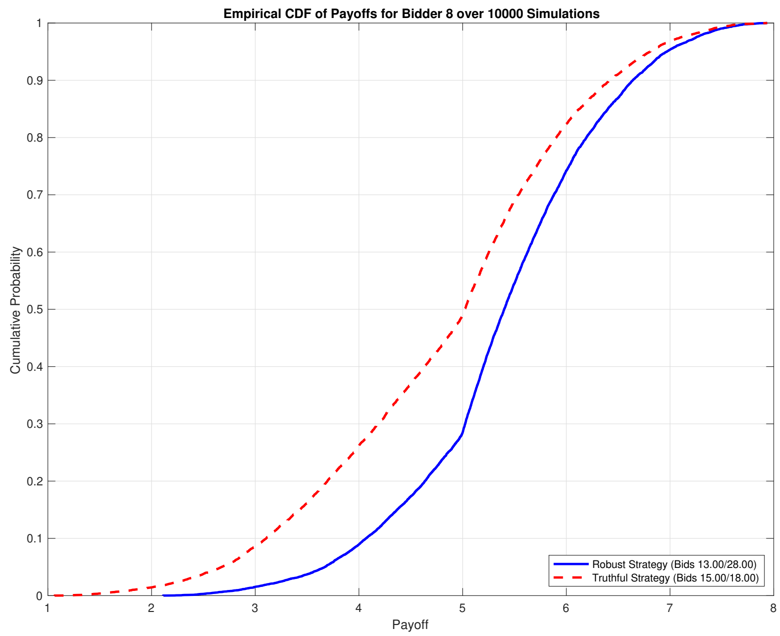

Figure 6 **Empirical CDF of bidder 8's payoffs over 10,000 simulations.**

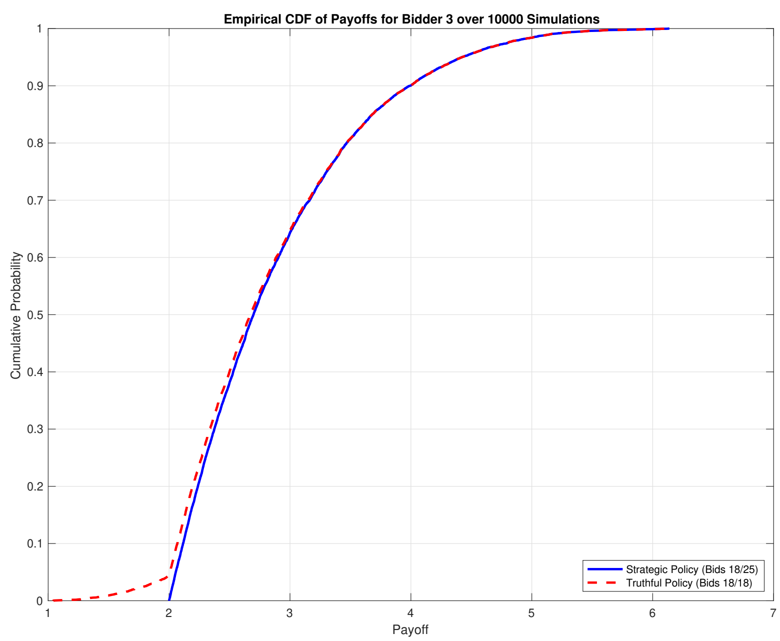

Figure 7 **Empirical cdf of bidder 3's payoffs based on 10,000 simulations.**

Appendix B: The minimax theorem

Let π_1^{MINMAX} denote the minimum value of bidder 1's maximum ex post payoff over U_{-1}, i.e., the payoff that she can achieve using an ex post optimal policy:

$$\pi_1^{MINMAX} = \inf_{b_{-1} \in U_{-1}} \sup_{b_1 \in U_1} \pi_1(b_1, b_{-1}). \tag{24}$$

By the minimax inequality (e.g. Boyd and Vandenberghe 2004), we have

$$\pi_1^{MAXMIN} \leq \pi_1^{MINMAX}. \tag{25}$$

Therefore, π_1^{MINMAX} provides an upper bound for the optimal objective of the robust optimization problem ($\mathscr{P}$). If (25) holds with equality, i.e.,

$$\pi_1^{MAXMIN} = \pi_1^{MINMAX}, \tag{26}$$

then there are two implications. First, the robust policy can be viewed as an ex post optimal policy applied to a specific worst-case bid $b_{-1} \in U_{-1}$. Second, if a policy $b_1 \in U_1$ results in bidder 1's worst-case payoff being equal to π_1^{MINMAX}, then this policy must be a robust policy. The latter observation is particularly useful for proving the optimality of bidding policies in core-selecting auctions, as we will see in Section 4.

For completeness, we recall Sion's minimax theorem (Sion 1958), which provides sufficient conditions for equality in the minimax inequality. We do not verify these conditions in our analysis, but note them here as a general reference.

Proposition 8 **(Sion (1958))**. *Let X be a convex subset of a linear topological space, Y a compact convex subset of a linear topological space, and $f : X \times Y \to \mathbb{R}$ be upper semi-continuous in x and lower semi-continuous in y. If for any $\lambda \in \mathbb{R}$, the sets*

$$LE(x, \lambda) = \{y \in Y : f(x, y) \leq \lambda\} \quad \textit{and} \quad GE(\lambda, y) = \{x \in X : f(x, y) \geq \lambda\}$$

are convex, then

$$\min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y). \tag{27}$$

Appendix C: Additional Results on Generalized Second-Price Auction

In this section, we present additional results on GSP auctions, including graphical illustrations and examples.

Figure 8 shows the worst-case payoff function of bidder 1 in a GSP auction with $n = 3$ bidders and $m = 2$ items with identical click-through rates. As we can see, it is optimal for bidder 1 to bid $b_1^{RO} = u(2)\mathbb{1}_{u(2) \leq v_1}$. If bidder 1 bids below $u(2)$, it is possible that she does not obtain any item, resulting in a worst-case payoff of zero. On the other hand, by bidding $b_1 > u(2)$, bidder 1's worst-case payoff is given by $f_1(b_1)$, her payoff when winning item 1. Since this payoff is decreasing in b_1 (by Lemma 3), bidding more than $u(2)$ leads to a lower worst-case payoff than bidding b_1^{RO}.

Figures 9–11 illustrate bidder 1's worst-case payoff function under different configurations of click-through rates α_1 and α_2, as described in (10). If $\alpha_1(v_1 - u(1 \mid 2)) \leq \alpha_2(v_1 - u(2))$, as shown in Figure 9, the robust bidding strategy $b_1^{RO} = u(2)\mathbb{1}_{u(2) \leq v_1}$ remains optimal, similar to the case with identical click-through rates. If $\alpha_2(v_1 - u(2 \mid 1)) \leq \alpha_1(v_1 - u(1))$, then the robust bidding strategy is $b_1^{RO} = u(1)\mathbb{1}_{u(1) \leq v_1}$. Note that if

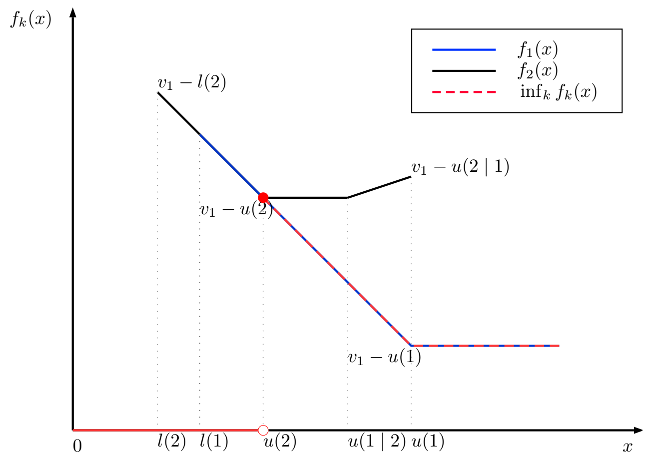

Figure 8 **Worst-case payoff function in a GSP auction with $n = 3$ bidders and $m = 2$ items which have identical click-through rates**

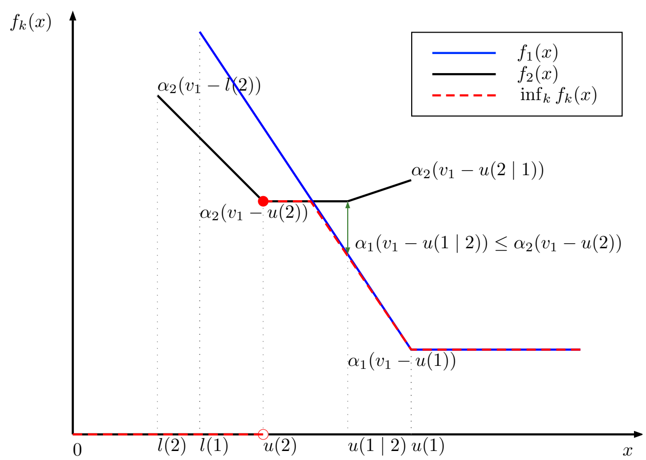

Figure 9 **Worst-case payoff function in a GSP auction with $n = 3$ bidders and $m = 2$ items which have different click-through rates satisfying $\alpha_1(v_1 - u(1 \mid 2)) \leq \alpha_2(v_1 - u(2))$**

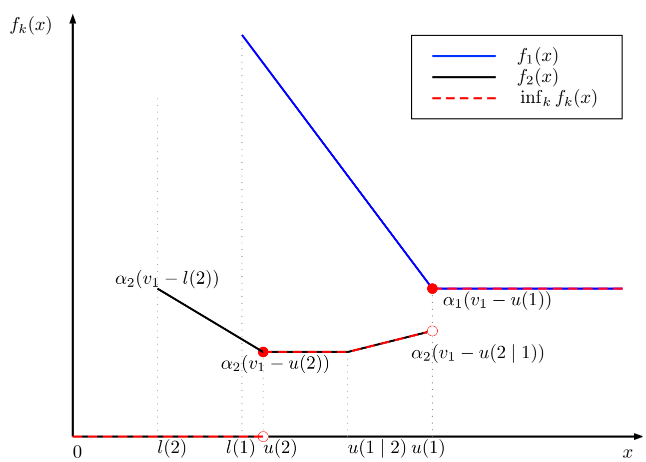

Figure 10 **Worst-case payoff function in a GSP auction with $n = 3$ bidders and $m = 2$ items which have different click-through rates satisfying $\alpha_2(v_1 - u(2 \mid 1)) \leq \alpha_1(v_1 - u(1))$**

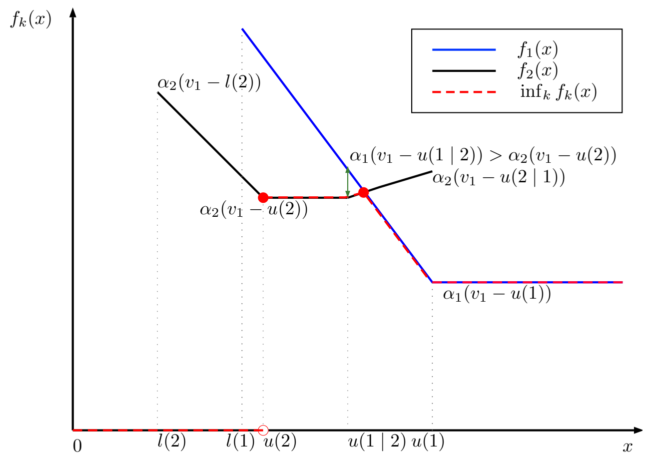

Figure 11 **Worst-case payoff function in a GSP auction with $n = 3$ bidders and $m = 2$ items which have different click-through rates satisfying $\alpha_2(v_1 - u(2)) < \alpha_1(v_1 - u(1 \mid 2))$ and $\alpha_1(v_1 - u(1)) < \alpha_2(v_1 - u(2 \mid 1))$**

$v_1 > u(1)$ in this case, bidder 1 would not want to bid below $u(1)$, as doing so risks either winning item 2 or not winning any item at all, both of which result in a lower worst-case payoff. Finally, Figure 11 illustrates the worst-case payoff function when $\alpha_2(v_1 - u(2)) < \alpha_1(v_1 - u(1 \mid 2))$ and $\alpha_1(v_1 - u(1)) < \alpha_2(v_1 - u(2 \mid 1))$. In this setting, the optimal robust bidding strategy is to bid at the point where the two worst-case payoff functions, $f_1(x)$ and $f_2(x)$, intersect.

Next, in the following examples, we compare the robust bidding policy with truthful bidding and the expected-payoff-maximizing policy.

Example 3. Consider a GSP auction with $n = 3$ bidders and $m = 2$ items. Let $\alpha = (1, 0.7)$, $v_1 = 10$, and $U_{-1} = \{(b^{(1)}, b^{(2)}) \mid 4 \le b^{(2)} \le 5, 6 \le b^{(1)} \le 7\}$. Since $\alpha_1(v_1 - u(1 \mid 2)) = 3 < 3.5 = \alpha_2(v_1 - u(2))$, according to Proposition 3, a robust policy for bidder 1 is $b_1^{RO} = 5$. The worst-case payoff under this policy is $\pi_1^{MAXMIN} = 3.5$, while the worst-case payoff under truthful bidding is $\pi_1^{TR} = 3 < 3.5 = \pi_1^{MAXMIN}$. One can also verify that, in this case, bidder 1's payoff under the robust bidding policy $b_1^{RO} = 5$ is greater than her payoff under truthful bidding for every realization of $b_{-1} \in U_{-1}$. □

Example 4. Consider a GSP auction with $n = 3$ bidders and $m = 2$ items. The settings for click-through rates and uncertainty set are similar to those in Example 3. However, in this case, we allow v_1 to vary and examine the performance of the robust policy b_1^{RO} and the expected-payoff-maximizing policy b_1^{EM}, based on the distributional assumption that $b^{(1)}$ and $b^{(2)}$ are independent and uniformly distributed on $[6, 7]$ and $[4, 5]$, respectively. From Proposition 3, we have

$$b_1^{RO} = \begin{cases} 0, \text{ if } v_1 < 5, \\ 5, \text{ if } 5 \le v_1 < \frac{35}{3}, \\ 7, \text{ otherwise.} \end{cases} \tag{28}$$

Given that b_2 and b_3 are independent and uniform, we have

$$\mathbb{E}[\pi_1(b_1)] = \begin{cases} 0, & \text{if } b_1 < 4, \\ 0.7(8 - \frac{1}{2}b_1^2 - 4v_1 + b_1 v_1), & \text{if } 4 \le b_1 < 5, \\ 0.7(v_1 - \frac{9}{2}), & \text{if } 5 \le b_1 < 6, \\ 18 - \frac{1}{2}b_1^2 - 6v_1 + b_1 v_1, & \text{if } 6 < b_1 \le 7, \\ v_1 - \frac{13}{2}, & \text{if } 7 < b_1. \end{cases} \tag{29}$$

Therefore, the expected-payoff maximizing policy (i.e., the maximizer of $\mathbb{E}[\pi_1(b_1)]$) is given by

$$b_1^{EM} = \begin{cases} 0, & \text{if } v_1 < 4, \\ v_1, & \text{if } 4 \le v_1 < 5, \\ 5, & \text{if } 5 \le v_1 < \frac{201}{18}, \\ 7, & \text{otherwise.} \end{cases} \tag{30}$$

Note that b_1^{RO} and b_1^{EM} coincide except when $v_1 \in [4,5)$ and $v_1 \in \left[\frac{201}{18}, \frac{35}{3}\right)$. Figures 12 and 13 show bidder 1's payoff functions under these policies for $v_1 = 4.5$ and $v_1 = 11.5$, respectively. When $v_1 = 4.5$, we have $b_1^{RO} = 0$, so bidder 1 receives a payoff of $\pi_1^{RO} = 0$ for all realizations of $b_{-1} \in U_{-1}$. In contrast, the expected-payoff-maximizing bid is $b_1^{EM} = 4.5$, which allows bidder 1 to win the second item and receive a positive payoff whenever $b^{(2)} < 4.5$. For $v_1 = 11.5$, we have $b_1^{RO} = 5$ and $b_1^{EM} = 7$. As shown in Figure 13, bidder 1's worst-case payoff under b_1^{RO} occurs when $b^{(2)}$ is at its maximum, while under b_1^{EM} it occurs when $b^{(1)}$ is highest. In this case, b_1^{RO} yields a slightly higher worst-case payoff than b_1^{EM}, though the latter may outperform the former under other realizations of $b_{-1} \in U_{-1}$. □

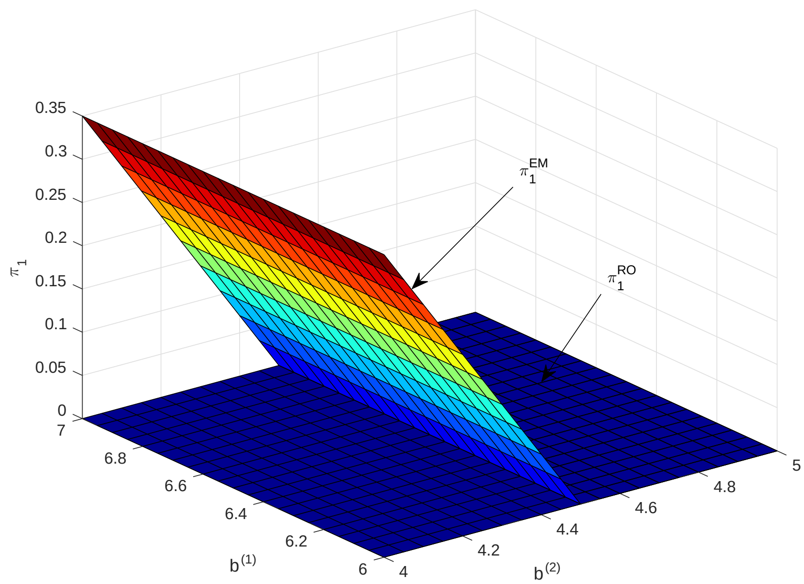

Figure 12 Illustration of Example 4 – Bidder 1's payoff function under b_1^{RO} and b_1^{EM} when $v_1 = 4.5$

Appendix D: Proofs

D.1. Quadratic core-selecting payment rule

We can rewrite (3) to be constraints on payments as follows. First, recall that S_j is the allocated bundle for bidder j. By substituting $\pi_0 = \sum_{j \in N} p_j$ and $\pi_j = b_j(S_j) - p_j$, we get

$$\sum_{j \in W} p_j \ge w_b(C) - \sum_{j \in C \cap W} (b_j(S_j) - p_j), \quad \forall C \subseteq N, \tag{31}$$

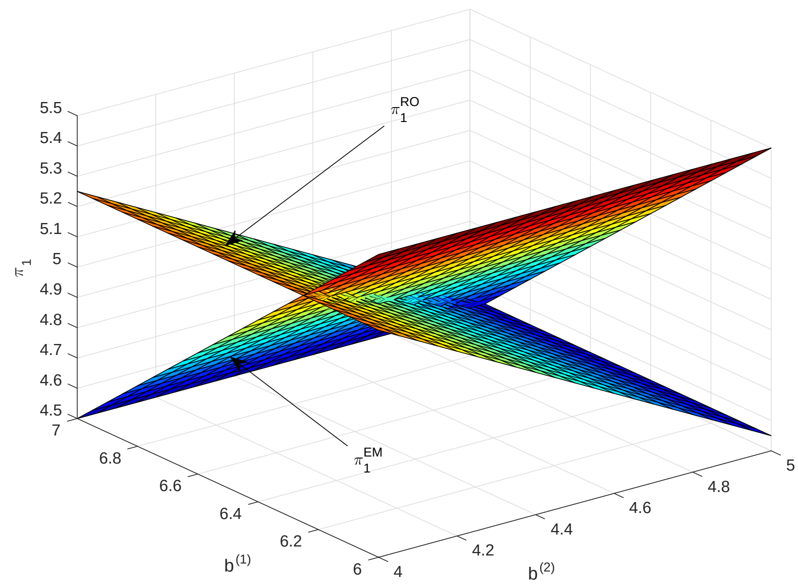

Figure 13 Illustration of Example 4 – Bidder 1's payoff function under b_1^{RO} and b_1^{EM} when $v_1 = 11.5$

where W is the set of bidders who receive nonempty bundles. After rearranging, the above constraints become:

$$\sum_{j \in W \setminus C} p_j \geq w_b(C) - \sum_{j \in C \cap W} b_j(S_j), \quad \forall C \subseteq N. \tag{32}$$

Let $\beta_C = w_b(C) - \sum_{j \in C \cap W} b_j(S_j)$ and $\beta \in \mathbb{R}^{2^n}$ be the vector of all β_C's. Also, let A be an $n \times 2^n$ matrix composed of columns a_C, each indexed by a coalition C. The jth entry of column a_C is 0 if bidder $j \in C$, and 1 otherwise. The constraints (32) are then of the form

$$pA \geq \beta.$$

Let p^0 be a reference payment vector. Under this quadratic core-selecting payment rule, the payment vector p is the optimal solution of the following quadratic program:

$$\begin{aligned} \min_p \quad & (p - p^0)(p - p^0)^T \\ \text{s.t.} \quad & pA \geq \beta, \quad p \leq b, \quad p1 = \mu, \end{aligned} \tag{$\mathscr{Q}$}$$

where μ is defined as

$$\begin{aligned} \mu \equiv \min_p \quad & p1 \\ \text{s.t.} \quad & pA \geq \beta, \quad p \leq b. \end{aligned} \tag{33}$$

The payment vector p determined by ($\mathscr{Q}$) minimizes the Euclidean distance from the reference payment vector p^0 to the core. The quantity μ is the minimum value of the total payment from bidders. Thus, the constraint $p1 = \mu$ guarantees that the payment rule is bidder-optimal, i.e., the total payment from bidders is minimized. This has the effect of minimizing the bidders' total incentive to deviate, as shown by Day and Milgrom (2008).

D.2. Remark 1

Let $C \subseteq N$ be the set of bidders corresponding to bidder 1's shills. Also, given a bid profile b, let S_j be the bundle allocated to bidder j. We have

$$\begin{aligned}\pi_0 + \sum_{j\in N\setminus C} \pi_i &= \sum_{j\in N} p_j + \sum_{j\in N\setminus C} (b_j(S_j) - p_j) \\ &= \sum_{j\in C} p_j + \sum_{j\in N\setminus C} b_j(S_j) \\ &\le \sum_{j\in C} p_j + w_b(N\setminus C, M\setminus \cup_{j\in C} S_j),\end{aligned}$$

where the last inequality follows from the definition of w_b. From the core constraints, we have $w_b(N\setminus S) \le \pi_0 + \sum_{j\in N\setminus C}\pi_i$. Hence,

$$p_1^{VCG} = w_b(N\setminus C) - w_b(N\setminus C, M\setminus \cup_{j\in C} S_j) \le \sum_{j\in C} p_j.$$

Thus, bidder 1 pays at least p_1^{VCG} even when she uses shills, which implies that her payoff is at most π_1^{VCG}. Since bidding policy (13) gives bidder 1 this VCG payoff, it is also optimal in the extended policy space in which bidder 1 uses shills.

D.3. Proposition 1

If $b_1 < u(m)$, there exists $b_{-1} \in U_{-1}$ such that $b_1 < b^{(m)}$ so that bidder 1 loses the auction and receives zero payoff. Thus, bidder 1's worst-case payoff is no more than zero if she bids $b_1 < u(m)$. On the other hand, if $b_1 \ge u(m)$, then bidder 1 always wins an item and receives a payoff $\pi_1(b_1, b_{-1}) = v_1 - b_1$. Hence, her optimal policy is to bid exactly $u(m)$ if $u(m) \le v_1$ and bid zero if $v_1 < u(m)$. In other words, a solution to ($\mathscr{P}$) is $b_1^{RO} = u(m)\mathbb{1}_{u(m)\le v_1}$. The optimal worst-case payoff is thus $\pi_1^{MAXMIN} = (v_1 - u(m))^+$.

D.4. Proposition 2

Assume all bidders $j \neq 1$ adopt the truth-telling strategy, i.e., their bid b_j is equal to their private valuation v_j. Suppose bidder 1's private valuation is $v_1 \in [\underline{v}, \bar{v}]$. Her objective is to choose a bid b_1 to solve

$$\max_{b_1\le v_1} \min_{v_{-1}\in[\underline{v},\bar{v}]^{n-1}} \pi_1(b_1, b_{-1}(v_{-1}), v_1).$$

From bidder 1's perspective, the worst-case scenario that minimizes her payoff occurs when her rivals' bids are maximized. Since rivals are truth-telling, this corresponds to their valuations being maximized. The worst-case is therefore when at least m of her rivals have a valuation of $\bar{v}$ and consequently submit bids of $b_j = \bar{v}$.

We evaluate bidder 1's payoff from the proposed equilibrium strategy, $b_1 = v_1$, in this worst-case scenario.

- If $v_1 < \bar{v}$, her bid is lower than the m rival bids of $\bar{v}$. She does not win an item, and her payoff is 0.
- If $v_1 = \bar{v}$, her bid is tied with at least m rivals. Her payment would be her bid, c, resulting in a payoff of $v_1 - b_1 = \bar{v} - \bar{v} = 0$.

In either case, her worst-case payoff from truth-telling is 0.

Now, we check if any deviation could yield a strictly positive worst-case payoff. To guarantee winning an item in the worst-case scenario, bidder 1 must submit a bid $b_1' > \bar{v}$. However, since her valuation is $v_1 \le \bar{v}$, any such winning bid would result in a negative payoff ($v_1 - b_1' < 0$). Bidding above one's valuation is a dominated strategy. Therefore, no deviation can guarantee a worst-case payoff greater than 0.

Since the truth-telling strategy yields the maximum possible worst-case payoff (0), there is no profitable deviation for bidder 1. By symmetry, this logic holds for all bidders. Thus, the strategy profile where all bidders bid their true valuation is a Nash Equilibrium.

D.5. Proposition 3

For each $k \in M$, let $f_k(x)$ denote the worst-case payoff that bidder 1 obtains when she wins item k by bidding x. Clearly, for a fixed k, the function $f_k(x)$ is defined only on the interval $x \in [l(k), u(k-1))$. The following lemma characterizes the structure of $f_k(\cdot)$.

Lemma 3. *For any $k \in \{1, 2, \dots, m\}$, we have*

$$f_k(x) = \begin{cases} \alpha_k(v_1 - x), & \text{if } l(k) \le x < u(k), \\ \alpha_k(v_1 - u(k)), & \text{if } u(k) \le x < u(k-1 \mid k), \\ \alpha_k(v_1 - u(k \mid k-1, x)), & \text{if } u(k-1 \mid k) \le x < u(k-1). \end{cases}$$

Proof of Lemma 3. By definition, we have

$$\begin{aligned} f_k(x) = \inf_{b_{-1} \in U_{-1}} \quad & \alpha_k(v_1 - b^{(k)}) \\ \text{s.t.} \quad & b^{(k)} \le x < b^{(k-1)}. \end{aligned} \tag{34}$$

When $l(k) \le x < u(k)$, the constraint $b^{(k)} \le x$ is always binding, and hence we have $f_k(x) = \alpha_k(v_1 - x)$. If $u(k) \le x < u(k-1 \mid k)$, then by bidding x, bidder 1 is guaranteed to win item k. In this case, it is possible for the rivals to bid in a way that makes bidder 1 pay the maximum possible amount, $u(k)$. Therefore, the worst-case payoff is given by $f_k(x) = \alpha_k(v_1 - u(k))$. Finally, if $u(k-1 \mid k) \le x < u(k-1)$, then the constraint $b^{(k)} \le x$ is never binding, and the maximal amount bidder 1 may pay is $u(k \mid k-1, x)$. Therefore, we have $f_k(x) = \alpha_k(v_1 - u(k \mid k-1, x))$. □

From Lemma 3, we observe that $f_k(x)$ is decreasing on $[l(k), u(k))$, constant on $[u(k), u(k-1 \mid k))$, and weakly increasing on $[u(k-1 \mid k), u(k-1))$.

Proof of Proposition 3. When $\alpha_1(v_1 - u(1 \mid 2)) \le \alpha_2(v_1 - u(2))$, according to Lemma 3, bidder 1's worst-case payoff is at most $\alpha_2(v_1 - u(2))$ (see Figure 9). Therefore, a robust policy is $b_1^{RO} = u(2)\mathbb{1}_{u(2) \le v_1}$, and the corresponding optimal worst-case payoff is $\pi_1^{MAXMIN} = \alpha_2(v_1 - u(2))^+$. If $\alpha_2(v_1 - u(2 \mid 1)) \le \alpha_1(v_1 - u(1))$, as shown in Figure 10, bidder 1 is always better off, in terms of worst-case payoff, by winning item 1 rather than item 2. Thus, the robust policy is given by $b_1^{RO} = u(1)\mathbb{1}_{u(1) \le v_1}$, and the corresponding worst-case payoff is $\pi_1^{MAXMIN} = \alpha_1(v_1 - u(1))^+$. Finally, when $\alpha_2(v_1 - u(2)) < \alpha_1(v_1 - u(1 \mid 2))$ and $\alpha_1(v_1 - u(1)) < \alpha_2(v_1 - u(2 \mid 1))$, bidder 1's worst-case payoff function on $x \in [u(2), \infty)$ is maximized at the intersect of $y = \alpha_1(v_1 - x)$ and $y = \alpha_2(v_1 - u(2 \mid 1, x))$(see Figure 11). Thus, we have $b_1^{RO} = x^* \mathbb{1}_{x* \le v_1}$, and the optimal worst-case payoff in this case is $\pi_1^{MAXMIN} = \alpha_1(v_1 - x^*)^+$.

The result extends directly to the case of $n \ge 4$ and $m = 2$, since the robust bidding problem depends only on bidder 1's belief about $b^{(1)}$ and $b^{(2)}$. □

D.6. Proposition 4

Let S_1 be the bundle that bidder 1 wins when she bids truthfully. By definition, we have

$$w_{v_1,b_{-1}}(N) = v_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1).$$

We observe that by using policy (13), bidder 1 also wins S_1. In fact, the maximum reported valuation generated by allocating S_1 to bidder 1 and $M \setminus S_1$ to the remaining bidders is

$$v_1(S_1) - \pi_1^{VCG} + w_{b_{-1}}(N \setminus 1, M \setminus S_1) = w_{v_1,b_{-1}}(N) - (w_{v_1,b_{-1}}(N) - w_{b_{-1}}(N \setminus 1)) = w_{b_{-1}}(N \setminus 1),$$

which is equal to the maximum reported valuation generated by allocating items in M to the remaining bidders. Under the assumed tie-breaking rule, S_1 is her allocation outcome. The VCG payoffs with respect to the reported valuation profile (b_1, b_{-1}) is

$$\pi_j^{VCG} = w_b(N) - w_b(N \setminus j) = w_{b_{-1}}(N \setminus 1) - w_{b_{-1}}(N \setminus 1) = 0,$$

for $j \in N$, and

$$\pi_0^{VCG} = w_b(N) - \sum_{j \in N} \pi_j^{VCG} = w_{b_{-1}}(N \setminus 1).$$

Next, we verify that this VCG profile is in the core with respect to the reported bids b. If $1 \notin C$, the core constraint (3) boils down to

$$w_{b_{-1}}(N \setminus 1) \geq w_{b_{-1}}(C),$$

which follows from the monotonicity of w directly. Instead, if $1 \in C$, the desired constraint becomes

$$w_{b_{-1}}(N \setminus 1) \geq w_{b_{-1}}(N \setminus 1),$$

which holds automatically. Therefore, bidder 1 is charged exactly her VCG payment p_1^{VCG}, which is a function of b_{-1} only. As a result, bidder 1 gets

$$\pi_1 = v_1(S_1) - p_1^{VCG} = v_1(S_1) - (v_1(S_1) - \pi_1^{VCG}) = \pi_1^{VCG},$$

which is her VCG payoff with respect to the reported valuation (v_1, b_{-1}). Since π_1^{VCG} is the maximum payoff that bidder 1 can get, policy (13) is optimal.

D.7. Proposition 5

We first show that bidder 1's worst-case payoff under policy (16) is at least $v_1(S_1) - \bar{p}^{VCG}$. By construction, we have that

$$b_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1) \geq b_1(M) \quad \forall b_{-1} \in U_{-1}. \tag{35}$$

Furthermore, from the definition of $\bar{p}^{VCG}$, we have

$$b_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1) \geq w_{b_{-1}}(N \setminus 1, M) \quad \forall b_{-1} \in U_{-1}. \tag{36}$$

Since $b_1(S) = 0$ for all $S \not\supseteq S_1$ and $w_{b_{-1}}(N \setminus 1, M \setminus S) \leq w_{b_{-1}}(N \setminus 1, M)$, the above inequality also implies that

$$b_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1) > b_1(S) + w_{b_{-1}}(N \setminus 1, M \setminus S), \tag{37}$$

for all $S \not\supseteq S_1$ and $b_{-1} \in U_{-1}$. The inequalities (35), (36) and (37) jointly show that it is always optimal for the seller to allocate bundle S_1 to bidder 1 and the rest of the items to other bidders. Since a bidder's payment can never exceed her bid, bidder 1's payoff under policy (16) is at least $v_1(S_1) - \bar{p}^{VCG}$. Hence, the optimal worst-case payoff under robust bidding satisfies $\pi_1^{MAXMIN} \geq v_1(S_1) - \bar{p}^{VCG}$. On the other hand, recall that we have the minimax inequality (25), so $\pi_1^{MAXMIN} \leq \pi_1^{MINMAX}$. For any realization of $b_{-1} \in U_{-1}$, bidder 1 can respond optimally by bidding according to a perfect-information optimal policy, e.g., policy (13). Under such a policy, bidder 1 receives her VCG payoff, so we have

$$\begin{aligned}
\pi_1^{MINMAX} &= \min_{b_{-1} \in U_{-1}} \left(w_b(N, M) - w_b(N \setminus 1, M)\right) \\
&= \min_{b_{-1} \in U_{-1}} \left(v_1(S_1) + w_b(N \setminus 1, M \setminus S_1) - w_b(N \setminus 1, M)\right) \\
&= v_1(S_1) - \max_{b_{-1} \in U_{-1}} \left(w_b(N \setminus 1, M) - w_b(N \setminus 1, M \setminus S_1)\right) \\
&= v_1(S_1) - \bar{p}^{VCG}.
\end{aligned}$$

Since $v_1(S_1) - \bar{p}^{VCG} \leq \pi_1^{MAXMIN} \leq \pi_1^{MINMAX} = v_1(S_1) - \bar{p}^{VCG}$, we must have that $\pi_1^{MAXMIN} = \pi_1^{MINMAX} = v_1(S_1) - \bar{p}^{VCG}$ and the bidding policy (16) is optimal.

D.8. Remark 3

When bidders 1 and 2 win exactly one item each, their VCG payments are $p_1^{VCG} = \max(0, c - b)$ and $p_2^{VCG} = \max(y - x, c - x)$, and the core constraints are $p_1 \leq x$, $p_2 \leq b$, and $p_1 + p_2 \geq c$. The projection of VCG on the core gives us bidder 1's payment:

$$p_1 = \left(x - \frac{1}{2}\min(x, x + b - c) + \frac{1}{2}\min(c - y, 0)\right)^+ . \tag{38}$$

The right-hand side of (38) corresponds to projecting the VCG payment (p_1^{VCG}, p_2^{VCG}) onto the blocking constraint $p_1 + p_2 \geq c$ created by bidder 3's bid on the global bundle. This term can also be written as

$$\max(0, c - b) + \frac{1}{2}(c - \max(0, c - b) - \max(y - x, c - x)),$$

which is bidder 1's VCG payment plus an extra amount that is half of the total surcharge that bidder 1 and 2 together have to pay to overcome the blocking global bidder. However, such a projection does not always yield a non-negative payment, so it is truncated at zero as in (38). By substituting $(x, y) = (\bar{p}^{VCG}, \bar{p}^{VCG} + \bar{b} - \epsilon_b)$ into (38), we get bidder 1's payment under the robust policy:

$$p_1^{RO} = \left(\bar{p}^{VCG} - \frac{1}{2}\min(\bar{p}^{VCG}, \bar{p}^{VCG} + b - c) + \frac{1}{2}\min(c - \bar{p}^{VCG} - \bar{b} + \epsilon_b, 0)\right)^+ . \tag{39}$$

Bidder 1's payoff under the robust policy is thus given by

$$\pi_1^{RO} = \min\left(a + \frac{1}{2}\min(0, b - c) - \frac{1}{2}\min(c - \bar{b} + \epsilon_b, \bar{p}^{VCG}), a\right). \tag{40}$$

On the other hand, by bidding truthfully bidder 1 gets

$$\pi_1^{TR} = \min\left(a + \frac{1}{2}\min(0, b - c) - \frac{1}{2}\min(c, a), a\right). \tag{41}$$

Since $\min(c - \bar{b} + \epsilon_b, \bar{p}^{VCG}) = c - \bar{b} + \epsilon_b \leq \min(c, a)$, we have $\pi_1^{RO} \geq \pi_1^{TR}$ for any realization of $b_{-1} \in U_{-1}$.

D.9. Remark 5

The global bidder's payment under the bidder-optimal, core-selecting, and Vickrey-nearest (BCV) auction does not depend on its own bid. Therefore, it is optimal for the global bidder to bid truthfully.

Now suppose local bidder 1 with valuation v_1 submits a bid $b_1 \geq 0$. Consider a deviation to $b_1 + \varepsilon$ with $\varepsilon > 0$. The expected gain from this deviation arises only when it changes bidder 1's status from losing to winning—namely, when the global bidder's valuation B lies between $b_1 + b_2(v_2)$ and $b_1 + b_2(v_2) + \varepsilon$. In that case, bidder 1's BCV payment is simply b_1, and her payoff is $v_1 - b_1$. Since B is uniformly distributed on $[7, 13]$, the probability that B falls in this interval is $\varepsilon/6$. Therefore, the expected gain from the deviation is $\frac{\varepsilon}{6} \cdot (v_1 - b_1)$.

To compute the expected cost of the deviation, observe that the only term in local bidder 1's payment affected by an increase in her bid is $\max(0, B - b_1)/2$. Thus, if bidder 1 raises her bid by ε, her payment increases by $\varepsilon/2$ if and only if $b_1 \leq B \leq b_1 + b_2(v_2)$. Since B is uniformly distributed on $[7, 13]$, the probability of this event is proportional to the length of the interval, and the expected cost of deviation is

$$\frac{\varepsilon}{2} \cdot \mathbb{E}\left[\int_{b_1}^{b_1+b_2(v_2)} \frac{1}{6}\, \mathrm{d}B\right] = \frac{\varepsilon}{12} \cdot \mathbb{E}[b_2(v_2)].$$

Combining the discussion above, we obtain that when $v_1 > \mathbb{E}[b_2(v_2)]/2$, bidder 1's optimal bid is

$$b_1^* = v_1 - \frac{1}{2}\,\mathbb{E}[b_2(v_2)].$$

If instead $v_1 \leq \mathbb{E}[b_2(v_2)]/2$, then any positive bid would make deviation profitable, so the optimal bid is $b_1^* = 0$. Hence, the unique optimal bidding strategy of a local bidder i is

$$b_i(v_i) = \max\left(v_i - \frac{1}{2}\,\mathbb{E}[b_{-i}(v_{-i})], 0\right).$$

Let $\alpha_i \equiv \mathbb{E}[b_{-i}(v_{-i})]$. Then we have

$$\alpha_2 = \mathbb{E}[b_1(v_1)] = \int_7^{13} \max\left(v_1 - \frac{1}{2}\,\alpha_1, 0\right) \cdot \frac{1}{6}\, \mathrm{d}v_1 = \int_7^{13} \left(v_1 - \frac{1}{2}\,\alpha_1\right) \cdot \frac{1}{6}\, \mathrm{d}v_1 = 10 - \frac{1}{2}\,\alpha_1.$$

By symmetry, we know that $\alpha_1 = \alpha_2 = 20/3$. Hence, in the Bayesian Nash equilibrium, for both $i = 1, 2$,

$$b_i(v_i) = v_i - \frac{10}{3}.$$

D.10. Example 2

Bidder 1's payoff function is:

$$\pi_1(x, y, b, c) = \begin{cases} 0 & \text{if } x + b < c \text{ and } y < c \\ \min\left(a, a - x + \frac{1}{2}\min(x, x + b - c) - \frac{1}{2}\min(c - y, 0)\right) & \\ & \text{if } c \leq x + b \text{ and } y \leq x + b \\ a - \max(b, c) & \text{if } x + b < y \text{ and } c \leq y \end{cases} \tag{42}$$

Since b and c are independent and distributed according to f_b and f_c given by (23), we have:

$$\begin{aligned} &\mathbb{E}[\pi_1(x, y, b, c)] \\ &\quad = \frac{1}{324} \int_{\max(y-x, \bar{b}-\epsilon_b)}^{\bar{b}+\epsilon_b} \int_{\bar{c}-\epsilon_c}^{\min(x+b, \bar{c}+\epsilon_c)} \pi_1^1(x, y, b, c)(b-7)(13-c)\; dc\; db \\ &\quad + \frac{1}{324} \int_{\bar{b}-\epsilon_b}^{\min(y-x, \bar{b}+\epsilon_b)} \int_{\bar{c}-\epsilon_c}^{\min(y, \bar{c}+\epsilon_c)} \pi_1^2(b, c)(b-7)(13-c)\; dc\; db, \end{aligned}$$

where $\pi_1^1(x, y, b, c) = \left(a, a - x + \frac{1}{2}\min(x, x + b - c) - \frac{1}{2}\min(c - y, 0)\right)$ and $\pi_1^2(b, c) = a - \max(b, c)$. The expected-payoff maximizing policy b_1^{EM} can then be obtained by choosing (x, y) that maximizes $\mathbb{E}[\pi_1(x, y, b, c)]$ over $\mathbb{R}_+^2$.

D.11. Proposition 6

(a) Bidder 1 has unique truthful allocation S_1 if and only if, for all $b_{-1} \in U_{-1}$:

$$v_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1) \geq v_1(S_2) + w_{b_{-1}}(N \setminus 1, M \setminus S_2), \tag{43}$$

and

$$v_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1) \geq w_{b_{-1}}(N \setminus 1, M), \quad \forall b_{-1} \in U_{-1}. \tag{44}$$

We will prove that, similar to the single-minded case, the worst-case payoff of bidder 1 under the robust policy b_1^{RO} is at least as large as the upper bound provided by minimax inequality. Therefore, the robust policy b_1^{RO} must be optimal. First, given policy b_1^{RO}, we can partition the uncertainty set U_{-1} into two disjoint subsets $U_{-1}^{(1)}$ and $U_{-1}^{(2)}$ such that bidder 1 always wins S_i on the set $U_{-1}^{(i)}$ for $i \in \{1,2\}$. The worst-case payoff under policy b_1^{RO} is given by

$$\inf_{b_{-1} \in U_{-1}} \pi_1(b_1^{RO}, b_{-1}) = \min\left(\inf_{b_{-1} \in U_{-1}^{(1)}} (v_1(S_1) - p_1^{(1)}), \inf_{b_{-1} \in U_{-1}^{(2)}} (v_1(S_2) - p_1^{(2)}) \right),$$

where $p_1^{(1)}$ and $p_1^{(2)}$ denote the corresponding payments made by bidder 1. Note that by individual rationality, we have $p_1^{(1)} \leq \bar{p}_1^{VCG}$ and $p_1^{(2)} \leq \bar{p}_1^{VCG}$. In addition, because $v_1(S_1) \leq v_1(S_2)$, we have

$$\inf_{b_{-1} \in U_{-1}} \pi_1(b_1^{RO}, b_{-1}) \geq v_1(S_1) - \bar{p}_1^{VCG}.$$

Since $\pi_1^{MAXMIN} \geq \inf_{b_{-1} \in U_{-1}} \pi_1(b_1^{RO}, b_{-1})$, we have $\pi_1^{MAXMIN} \geq v_1(S_1) - \bar{p}_1^{VCG}$. On the other hand, if we let $p_1^{VCG}(S_1)$ and $p_1^{VCG}(S_2)$ be bidder 1's VCG payment when she wins S_1 and S_2, respectively, then the minimax payoff is

$$\begin{aligned} \pi_1^{MINMAX} &= \inf_{b_{-1} \in U_{-1}} \max\left(v_1(S_1) - p_1^{VCG}(S_1), v_1(S_2) - p_1^{VCG}(S_2)\right) \\ &= v_1(S_1) - \bar{p}_1^{VCG}, \end{aligned}$$

where the last equality follows directly from the condition (43) that guarantees bidder 1 always win bundle S_1 under truthful bidding. Finally, by minimax inequality, we have $\pi_1^{MAXMIN} \leq \pi_1^{MINMAX}$. As a result, the minimax equality (26) must hold and b_1^{RO} is the optimal solution to ($\mathscr{P}$).

(b) We observe that under policy b_1^{RO} defined in (22), bidder 1 always wins S_2. As a result, the optimality follows directly from the minimax inequality argument analogous to the proof of Proposition 5.

D.12. Lemma 1

When bidder 1 wins one item, her payment is given by (38). Recall that when $(x,y) \in U_1'$, we have $x = \bar{p}_1^{VCG}$. As a result, $\pi_1^{WO,1}(y)$ is given by

$$\begin{aligned} \pi_1^{WO,1}(y) = \inf \quad & \min\left\{a, a - \bar{p}_1^{VCG} + \frac{1}{2}\min(\bar{p}_1^{VCG}, \bar{p}_1^{VCG} + b - c) + \frac{1}{2}(y-c)^+\right\} \\ \text{s.t.} \quad & b_{-1} \in U_{-1} \\ & y \leq \bar{p}_1^{VCG} + b. \end{aligned} \tag{45}$$

The objective function on the right-hand side is increasing in b and decreasing in c. Therefore, the worst-case scenario corresponds to $b^* = \max(\bar{b} - \epsilon_b, y - \bar{p}_1^{VCG})$ and $c^* = \bar{c} + \epsilon_c$. Substituting these values into (45), we have

$$\pi_1^{WO,1}(y) = \min\left\{a, a - \bar{p}_1^{VCG} + \frac{1}{2}\min\left(\bar{p}_1^{VCG}, (y - \bar{c} - \epsilon_c)^+\right) + \frac{1}{2}(y - \bar{c} - \epsilon_c)^+\right\}. \tag{46}$$

Similarly, when bidder 1 wins both items, her payment is $\max(b,c)$, so $\pi_1^{WO,2}(y)$ is given by

$$\begin{aligned}
\pi_1^{WO,2}(y) = \inf \quad & a' - \max(b,c) \\
\text{s.t.} \quad & b_{-1} \in U_{-1} \\
& \bar{p}_1^{VCG} + b < y.
\end{aligned} \tag{47}$$

In this case, the worst-case bids are $b^* = \min(\bar{b} + \epsilon_b, y - \bar{p}_1^{VCG})$, $c^* = \bar{c} + \epsilon_c$. Therefore, we have

$$\pi_1^{WO,2}(y) = a' - \max\left\{\min(\bar{b} + \epsilon_b, y - \bar{p}_1^{VCG}), \bar{c} + \epsilon_c\right\}. \tag{48}$$

We can see that $\pi_1^{WO,1}(y)$ is a piecewise linear increasing function in y while $\pi_1^{WO,2}(y)$ is a piecewise linear decreasing function in y. Furthermore, one can directly verify that $\pi_1^{WO,1}(y) = a$ for $y \geq \bar{p}_1^{VCG} + \bar{c} + \epsilon_c$ and $\pi_1^{WO,2}(y) = a' - \bar{c} - \epsilon_c$ for $y \leq \bar{p}_1^{VCG} + \bar{c} + \epsilon_c$.

D.13. Lemma 2

There are three cases to consider:

1. $\bar{b} + \epsilon_b \leq \bar{c} + \epsilon_c$:

   We have

$$\begin{aligned}
\pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) &= a - (\bar{c} - \bar{b} + \epsilon_b + \epsilon_c), \\
\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) &= a' - \bar{c} - \epsilon_c, \\
\pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) &= a - (\bar{c} - \bar{b} - \epsilon_b + \epsilon_c), \\
\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) &= a' - \bar{c} - \epsilon_c.
\end{aligned}$$

2. $\bar{b} - \epsilon_b \leq \bar{c} + \epsilon_c < \bar{b} + \epsilon_b$:

   We have

$$\begin{aligned}
\pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) &= a - (\bar{c} - \bar{b} + \epsilon_b + \epsilon_c), \\
\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) &= a' - \bar{c} - \epsilon_c, \\
\pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) &= a, \\
\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) &= a' - \bar{b} - \epsilon_b.
\end{aligned}$$

3. $\bar{c} + \epsilon_c < \bar{b} - \epsilon_b$:

   We have

$$\begin{aligned}
\pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) &= a, \\
\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) &= a' - \bar{b} + \epsilon_b \\
\pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) &= a \\
\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) &= a' - \bar{b} - \epsilon_b
\end{aligned}$$

In all three cases, one can verify that $\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) \leq \pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b)$ is equivalent to $a' \leq a + \bar{b} - \epsilon_b$ and $\pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) \leq \pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b)$ is equivalent to $a' \leq a + \bar{b} + \epsilon_b$.

D.14. Proposition 7

We first consider the case when bidder 1 bids in the restricted policy space $U_1' = \{(x,y) \in \mathbb{R}_+^2 \mid x = \bar{p}_1^{VCG}, y \geq x\}$. When $a' \in (a+\bar{b}-\epsilon_b, a+\bar{b}+\epsilon_b]$, by Lemma 2, we have $\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) < \pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b)$ and $\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) \geq \pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b)$. Due to the monotonicity of $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$ (Lemma 1), the optimal choice of y (with respect to the restricted space U_1') is at the intersection of $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$. To determine the optimal worst-case payoff with respect to the restricted space U_1', recall that $\pi_1^{WO,1}(y)$ is constant for $y \geq \bar{p}_1^{VCG}+\bar{c}+\epsilon_c$ and $\pi_1^{WO,2}(y)$ is constant for $y \leq \bar{p}_1^{VCG}+\bar{c}+\epsilon_c$. Therefore, at the intersection point of these two payoff functions, bidder 1's payoff is equal to $\min\{\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{c}+\epsilon_c), \pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{c}+\epsilon_c)\} = \min(a, a'-\bar{c}-\epsilon_c)$ (see Figure 2c and 2d). Since this is a lower bound for the optimal worst-case payoff on the original policy space $U_1 = \mathbb{R}_+^2$, we have

$$\min(a, a'-\bar{c}-\epsilon_c) \leq \pi_1^{MAXMIN}. \tag{49}$$

We now show that the above lower bound is the same as the upper bound on π_1^{MAXMIN} provided by the minimax inequality. For any $b_{-1} \in U_{-1}$, bidder 1's best response payoff is her VCG payoff. Therefore, we have $\sup_{b_1 \in U_1} \pi_1(b_1, b_{-1}) = \max(a+b, a') - \max(b,c)$, implying that

$$\begin{aligned}
\pi_1^{MINMAX} &= \inf_{b_{-1}\in U_{-1}} \sup_{b_1 \in U_1} \pi_1(b_1, b_{-1}) \\
&= \inf_{b_{-1}\in U_{-1}} (\max(a+b, a') - \max(b,c)) \\
&= \min\begin{pmatrix} \inf & a' - \max(b,c), & \inf & a-(c-b)^+ \\ \text{s.t.} & b < a'-a & \text{s.t.} & a'-a \leq b \\ & b_{-1} \in U_{-1} & & b_{-1}\in U_{-1} \end{pmatrix} \\
&= \min\{a' - \max(a'-a, \bar{c}+\epsilon_c), a - (\bar{c}+\epsilon_c - a'+a)^+\} \\
&= \min(a, a'-\bar{c}-\epsilon_c),
\end{aligned}$$

where the second last equality follows by substituting the minimizing values for b and c. By minimax inequality, we have $\pi_1^{MAXMIN} \leq \pi_1^{MINMAX} = \min(a, a'-\bar{c}-\epsilon_c)$. Together with (49), we have $\pi_1^{MAXMIN} = \min(a, a'-\bar{c}-\epsilon_c)$, and the policy (x^*, y^*) is optimal.

D.15. Remark 7

Let $\pi_1^{WO,TR}$ be her worst-case payoff under truthful bidding. We want to show that if $a+\bar{b}-\epsilon_b < a'$ then this truthful worst-case payoff is the same as the optimal worst-case payoff from robust bidding, i.e.,

$$\pi_1^{WO,TR} = \pi_1^{MAXMIN}. \tag{50}$$

First, let us consider the case when $a+\bar{b}+\epsilon_b < a'$. By reporting truthfully, bidder 1 always wins two items, so her worst-case payoff is

$$\pi_1^{WO,TR} = a' - \max(\bar{b}+\epsilon_b, \bar{c}+\epsilon_c),$$

On the other hand, according to Proposition 6, we have $\pi_1^{MAXMIN} = a' - \max(\bar{b}+\epsilon_b, \bar{c}+\epsilon_c)$. Hence, $\pi_1^{WO,TR} = \pi_1^{MAXMIN}$.

Now let us assume that $a+\bar{b}-\epsilon_b < a' \leq a+\bar{b}+\epsilon_b$. By Proposition 7, $\pi_1^{MAXMIN} = \min(a, a'-\bar{c}-\epsilon_c)$. There are three possible cases to be considered:

1. $\bar{b} + \epsilon_b \leq \bar{c} + \epsilon_c$:

   By reporting truthfully, bidder 1's worst-case payoffs when winning one and two items are respectively:

$$\begin{aligned}\pi_1^{WO1,TR} &= \min(a, \frac{1}{2}\min(a, a' - \bar{c} - \epsilon_c) + \frac{1}{2}(a' - \bar{c} - \epsilon_c)) \\ &= \min(a, a' - \bar{c} - \epsilon_c), \\ \pi_1^{WO2,TR} &= a' - \max(\min(a' - a, \bar{b} + \epsilon_b), \bar{c} + \epsilon_c) \\ &= a' - \bar{c} - \epsilon_c.\end{aligned}$$

   Hence, bidder 1's worst-case payoff is

$$\pi_1^{WO,TR} = \min(\pi_1^{WO1,TR}, \pi_2^{WO2,TR}) = \min(a, a' - \bar{c} - \epsilon_c)$$

   which is the same with π_1^{MAXMIN}.

2. $\bar{b} - \epsilon_b \leq \bar{c} + \epsilon_c < \bar{b} + \epsilon_b$:

   Similar to the previous case, by reporting truthfully, bidder 1's worst-case payoffs when winning one and two items are respectively:

$$\begin{aligned}\pi_1^{WO1,TR} &= \min(a, a' - \bar{c} - \epsilon_c), \\ \pi_1^{WO2,TR} &= a' - \max(\min(a' - a, \bar{b} + \epsilon_b), \bar{c} + \epsilon_c).\end{aligned}$$

   Now we prove that $\min(\pi_1^{WO1,TR}, \pi_2^{WO2,TR}) = \min(a, a' - \bar{c} - \epsilon_c)$. If $a' - \bar{c} - \epsilon_c < a$ then $\pi_1^{WO1,TR} = \pi_1^{WO1,TR} = a' - \bar{c} - \epsilon_c$ so we indeed have $\min(\pi_1^{WO1,TR}, \pi_2^{WO2,TR}) = \min(a, a' - \bar{c} - \epsilon_c)$. On the other hand, if $a \leq a' - \bar{c} - \epsilon_c$ then $\pi_1^{WO1,TR} = a$ and $\pi_1^{WO2,TR} = a' - \min(a' - a, b + \epsilon_b) \geq a$, so $\min(\pi_1^{WO1,TR}, \pi_2^{WO2,TR}) = a = \min(a, a' - \bar{c} - \epsilon_c)$. Therefore, $\pi_1^{WO,TR} = \pi_1^{MAXMIN}$.

3. $\bar{c} + \epsilon_c < \bar{b} - \epsilon_b$:

   In this case, the worst-case payoffs of bidder 1 when winning one and two items are $\pi_1^{WO1,TR} = \pi_1^{WO2,TR} = a = \min(a, a' - \bar{c} - \epsilon_c)$. Thus, we also have $\pi_1^{WO,TR} = \pi_1^{MAXMIN}$, which completes our proof.