Optimal Push, Pull, and Failure Funding for Global Health

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Malaria caused over 600,000 deaths in 2021, yet commercial incentives are weak for drug and vaccine development for malaria and other tropical diseases. Governments and nonprofits address these market failures through push (e.g., grants) and pull (e.g., prizes). We propose a third mechanism in which the funder only pays when the firm fails. That is, the funder reimburses the firm a share of its testing costs, which we refer to as "insurance." Insurance is optimal when the commercial market is large enough to reward success, but too small to induce investment. The optimal mechanism addresses adverse selection and moral hazard. For most tropical diseases, including malaria, we recommend pull funding with supplementary push support. Insurance is optimal for tuberculosis if testing costs are less than a billion dollars. These findings challenge current practices dominated by push funding and extend to funding innovations in other sectors.

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1. Introduction

Malaria and tuberculosis each caused over 600,000 deaths in 2021 (World Health Organization 2024). Tropical diseases like these also cause significant non-fatal suffering. However, commercial incentives for developing new drugs for tropical diseases are weak because they primarily affect people in low-income countries. Also, commercial markets fail to fully internalize the societal benefits of both research and reduced disease transmission.

Governments and foundations address these market failures by funding drug development. For example, in 2022, the Gates Foundation, with assets exceeding $70 billion, invested $624 million in global health research and development (R&D), including over $100 million each for HIV/AIDS, malaria, and tuberculosis (Impact Global Health 2023). Similarly, the Wellcome Trust, with assets exceeding $30 billion, invested $106 million in global health R&D (Impact Global Health 2023).

Funding new product development involves incentive challenges. Funders have less information than firms about the drug's likelihood of success and the firm's capabilities. Also, funders cannot fully monitor effort, leading to moral hazard. Firms may overstate a drug's potential to secure funding and then shirk, knowing the drug is likely to fail regardless. In extreme cases, perverse incentives

can lead to fraud, as in the U.S. Agency for International Development's malaria vaccine program, where staff overstated a product's prospects and embezzled funds (Desowitz 1991, Kremer 2002).

Governments and nonprofits use push and pull mechanisms to address these challenges. Push mechanisms, such as grants and tax credits for firms, reduce development costs, but provide funds upfront, before the drug succeeds. Pull mechanisms, including prizes, advance market commitments, and priority review vouchers, reward success and promote accountability. However, pull mechanisms require firms to secure initial funding independently, often from investors.

We recommend a third mechanism: insurance, where funders pay firms for failure. If the firm fails, then a portion of its testing cost is reimbursed plus interest.[1] If the insurance is not too generous, and the commercial market potential is moderately high, then the firm will still be motivated to exert effort toward success. Insurance complements a moderate commercial market and substitutes for push funding. Insurance can be less costly ex-ante than push funding, because it is only paid upon failure.

This is the first study to model and calibrate the optimal mix of push, pull, and insurance mechanisms under adverse selection and moral hazard constraints. Furthermore, it is the first to show that rewarding only failure, and not success, is optimal when the drug has medium commercial value.

Using a principal-agent framework, we model the funder's problem of mitigating adverse selection and moral hazard while maximizing social welfare. The model assumes private information on the drug's ex-ante probability of success, and the firm's effort in the clinical trials. The model assumes known testing costs for clinical trials, based on the experience of funders like the Gates Foundation that support tropical disease product development, as well as data from the Drugs for Neglected Diseases initiative (2019). We formulate the problem as a linear optimization model with incentive constraints and analytically characterize the optimal solutions for all combinations of model parameters. We rely on (infinite-dimensional) linear optimization and its duality to theoretically establish the optimal solutions, which further reveal insights into the funding mechanisms for different settings.

We demonstrate a novel reason that regulators should sometimes reward failure when managing moral hazard and adverse selection: paying for failure is better than push funding, because then the regulator avoids unnecessarily funding drugs that succeed ex-post. A key difference between our study and previous studies is that an exogenous commercial profit can reward success. If the commercial profit from success is high but not too high, then rewarding failure (and not rewarding success) is optimal. The commercial profit must already exceed what is needed to incentivize the agent towards success, but not be so high that the market can fully take over without external funding. Furthermore,

[1] We will sometimes refer to insurance as "payment for failure" but the optics might be better if the funder frames it as "partial reimbursement". The present value of payment upon failure is capped at a fraction of the firm's sunk costs, making it a risk-sharing instrument.

this strategy mitigates the agent's perverse incentive to seek more push funding up front by under-reporting the chance of success. We show that funders should not combine rewards for success and failure. That is paying regardless of success or failure, which can be done more cheaply through push funding due to firms' higher discount rates.

Our analysis reveals that diseases fall into distinct funding categories based on consumer welfare and profit potential. First, for most tropical diseases, including dengue, malaria, onchocerciasis, and schistosomiasis, funders should combine push and pull mechanisms. Second, funders should occasionally use push funding alone, as in the case of hepatitis B. Third, for diseases like trichomoniasis with moderate profit potential, funders should combine push funding with insurance. Fourth, for diseases like tuberculosis, insurance alone is optimal due to moderate profit potential and high consumer welfare. Some drugs, such as those for leprosy, should not be funded, because testing costs exceed potential welfare gains. Also, diseases like HIV/AIDS require no public intervention in clinical testing due to high profitability.[2]

Our calibrated model challenges prevailing funding norms in global health. Whereas donors now rely almost entirely on push mechanisms, we find that high-burden tropical diseases, including dengue, malaria, onchocerciasis, and schistosomiasis, require a pull incentive of roughly $500 million, supplemented by smaller push funding. Malaria alone justifies a pull commitment of about $2 billion, with lesser push funding at late stages of development. These findings reinforce earlier proposals for an advance market commitment for malaria (Berndt et al. 2007) but do not support analogous mechanisms for tuberculosis or HIV/AIDS. Although malaria, tuberculosis, and HIV impose comparable disease burdens, their divergent commercial prospects warrant distinct funding approaches.

While focused on global health, our framework extends to other innovative sectors, such as renewable energy. In general, our model provides a tool that takes parameters as inputs to generate a corresponding funding strategy.

The paper proceeds as follows. In the remainder of this section, we review the literature. We then introduce the model in Section 2 and establish the main theoretical results on optimal mechanisms in Section 3. In Section 4, we present benchmark cases in which at least one of the two incentive issues is absent. Section 5 provides a calibration study of communicable diseases and associated drugs and vaccines. Section 6 describes the practical implications of our analysis. All the proofs are in Appendices A-D.

[2] If testing costs were below $49 million, then it would be optimal to fund leprosy testing. Furthermore, if the profit from HIV/AIDS were lower, say for heat-stable formulations for low-income countries, then it would be optimal to fund HIV/AIDS clinical testing.

1.1. Literature on Rewarding Failure

One reason a principal might reward failure is to incentivize the agent to reveal poor product performance early, rather than prolonging investment in failing projects. Simester and Zhang (2010) provides a rational explanation for why many firms continue investing in bad products. The paper explores a principal-agent model where the principal motivates the agent to exert effort and truthfully disclose demand information, even when demand is low. While rewarding success motivates effort, it may also encourage continued investment in low-demand products. Similarly, Mayer (2022) examines a dynamic contracting problem in which an agent exerts unobservable effort to increase the probability of a project's success. Because failure may not be immediately observable, the optimal contract includes rewarding failure to prevent the agent from hiding negative outcomes to prolong funding. In both studies, rewarding failure helps reveal truthful information.

Another rationale for rewarding failure is to encourage experimentation. Bolton et al. (2024) examines moral hazard in experimental design under pay-for-performance contracts, demonstrating that agents often prefer experiments with a higher chance of success. They argue that rewarding failure motivates the firm to design experiments that reveal failure, thereby facilitating more informative trials. Manso (2011) investigates a two-period, two-armed bandit problem and shows that rewarding failure in the first period can provide the agent with an incentive to explore alternative strategies in the second period.

We introduce a third rationale for rewarding failure: the principal rewards failure but not success because success yields a moderate commercial profit. This profit is sufficient to incentivize effort but not so large that the market alone can support development without external funding. Rewarding only failure also mitigates the agent's incentive to exaggerate the project's risk in order to secure more push funding. Consistent with this logic, we show that when it is optimal to reward failure, there should be no reward for success. Prior models do not exhibit this feature, as they do not account for the exogenous commercial reward associated with success in our setting.

1.2. Literature on the Principal-Agent Models for Innovation

A seminal paper in the literature on optimal funding mechanisms for stimulating innovation using a principal-agent framework is by Aghion and Tirole (1994), who models moral hazard in product development. The agent's effort and the principal's investment affect the chance of success. Because of this success uncertainty, we interpret these models as concerning innovation.

Our work relates to Rietzke and Chen (2020), which examines a setting where the agent privately knows the probability of success, makes contractible investments, and exerts non-contractible effort. Our analysis differs in several ways. First, we allow rewards for failure in the model as mentioned before. Second, we account for differing discount rates between the principal and agent, such as the

Gates Foundation being more patient than a firm. Third, while our model permits payments from the funder to the firm, we exclude payments in the reverse direction, reflecting the absence of such arrangements in global health contexts. Fourth, we allow testing costs to exceed social benefits, so the optimal action is sometimes doing nothing. Fifth, we calibrate the model.

Our work is also related to Lach et al. (2021), which studies the design of an optimal government loan program for risky R&D projects with a positive societal value under both adverse selection and moral hazard. The agent privately knows the probability of success, and cannot fully self-fund the effort costs. A key difference between their setting and ours is that the loan program may require the agent to pay the principal if the project is successful. This effectively allows the principal to mitigate the information rent due to adverse selection. In our model, the funder does not have the option to reclaim funds after the innovation outcome is finalized, which complicates the optimal incentive mechanism.

Socorro (2007) also considers both moral hazard and adverse selection, but in a two-agent setting. Both agents need to exert effort to guarantee a positive probability of success. The agents' private information is the value of success, not the probability of success, as in our setting.

Managing innovation has been a topic of study in the economics and management literature. Crama et al. (2008) studies a principal-agent problem in which a licensor, as the principal, maximizes the expected monetary transfer from a licensee plus a royalty percentage of the sales. The agent pays the principal because the agent is getting a profit (for example, paying taxes or march-in rights), but we primarily study products that are unprofitable but socially valuable. Bergmann and Friedl (2008) examines the interaction between a risk-neutral principal and a risk-averse R&D agent, who privately knows the project's profitability and exerts effort that is not perfectly observable. They derive the optimal compensation scheme under the linear contracts, exponential utility, and normally distributed uncertainty (LEN) framework (Holmstrom and Milgrom 1987). Hellmann and Thiele (2011) studies an incentive design problem in which employees choose between assigned standard tasks governed by a performance measure and privately observed innovation opportunities that require *ex-post* renegotiation. Xiao and Xu (2012) studies the impact of royalty revisions in a two-stage alliance with a marketer and an innovator. Bhattacharya et al. (2015) examines challenges associated with double-sided moral hazard, risk aversion, and holdup in R&D partnerships while allowing renegotiation. Schlapp et al. (2015) studies a resource allocation problem among two agents with dual asymmetric information, where the principal induces the agent to exert effort to acquire information about the project's potential and report the findings truthfully. Crama et al. (2017) considers a contracting problem between an innovator (the principal) and a marketer (the agent). The contract involves payments, control rights, buy-back options, and other components.

1.3. Literature on Drug Development

The probability of approval by the U.S. Food and Drug Administration (FDA) for an infectious disease drug is 23% for those entering Phase II testing and 59% for those entering Phase III. Approval probabilities differ for other disease areas, with drugs targeting higher-profit diseases typically having lower average success rates. For example, the probability of ultimate approval for a drug entering Phase II testing is 17% for non-cancer drugs and 11% for cancer drugs. Similarly, approval probabilities are 25% for rare diseases and 13% for high-prevalence diseases (Biotechnology Innovation Organization 2021).

These probabilities inform our model and calibration in two ways. First, our model predicts that drugs with higher profit potential, such as cancer drugs or those for high-prevalence diseases, have lower testing thresholds and, consequently, lower average success probabilities. Second, because these probabilities are averages, the testing thresholds in our calibration exercise should be even lower.

Drug development's low probability of success makes it a risky endeavor, particularly for firms with small portfolios. To mitigate this risk, investors can pool resources into megafunds, diversifying across multiple drug development projects (Lo and Thakor 2022). Additionally, securitization techniques, such as issuing bonds backed by projected revenues of successful drugs, can provide investors with steady returns while channeling funds into early-stage research (Lo and Thakor 2022).

Clinical trial success depends not only on product characteristics, but also on actions by the firm. Firms minimize the cost of retaining participants by exerting effort to reduce participants' inconvenience costs and offering financial incentives to participants (Song et al. 2023).

The U.S. government spends an order of magnitude more on R&D for HIV/AIDS than for malaria (Kyle et al. 2017, Impact Global Health 2023). In contrast, Brazil allocates less funding to HIV/AIDS than to malaria, dengue, and leishmaniasis, despite HIV/AIDS imposing a larger burden of disease in Brazil. One possible explanation is that Brazilian health authorities are reacting to U.S. funding for HIV/AIDS research. Indeed, a 10% increase in U.S. government funding for infectious diseases correlates with a 2–3% reduction in funding by other governments (Kyle et al. 2017). Also, perhaps Brazil's leaders believe that HIV/AIDS needs less government R&D funding due to its profit potential.

In high-stakes scenarios like the COVID-19 pandemic, funders must invest heavily and attract even high-cost suppliers (Snyder et al. 2023). In such crises, costs are often unknown due to limited time for due diligence, as well as the lack of precedent for the cost of rapid scaling. Similarly, Lopomo et al. (2024) assumes unknown testing costs for drug regulators.

In contrast, we consider specialized funders like the Gates Foundation or Wellcome Trust that have domain-specific expertise and access to cost data. For example, the Drugs for Neglected Diseases Initiative reports costs for new chemical entities, which vary based on study length and participant numbers (Drugs for Neglected Diseases initiative 2019). These funders are well-positioned to leverage the data, especially when directly funding or reimbursing development costs.

1.4. Literature on Pull Mechanisms for Drug Development

We examine the importance of pull funding for promoting drug development for tropical diseases. Among pull mechanisms, advance market commitments are theoretically robust and practically impactful for incentivizing R&D (Berndt et al. 2007, Kremer et al. 2022). A funder would commit to purchase a specified quantity of a product at a pre-agreed price. The commitment is not to one firm, but to purchase from any firm meeting the criteria. Hence, the commitment rewards firms only when their products are used, promoting access as well as innovation.

Advance market commitments inspired practical applications, including a $1.5 billion pneumococcal vaccine program, which facilitated affordable vaccine distribution in low-income countries (Kremer et al. 2022). The U.S. government employed a similar approach in Operation Warp Speed to accelerate COVID-19 vaccine development (Kremer et al. 2022).

Berndt et al. (2007) recommended $3 billion advance market commitments for HIV/AIDS, malaria, and tuberculosis. They estimated the optimal amount based on returns for commercial products. In contrast, our framework tailors funding to disease-specific characteristics. For malaria, our optimal policy aligns with Berndt et al. (2007). However, for tuberculosis and HIV/AIDS, we propose different strategies, reflecting their varying commercial potentials.

Priority review vouchers are another pull mechanism. Proposed by Ridley et al. (2006) and enacted in the United States in 2007, the program grants a transferable FDA voucher to a firm securing approval for a drug that treats a designated tropical disease. The voucher entitles its holder to an expedited review of a second, typically more lucrative, product. Hence, there are two drugs for each voucher. Because the voucher can be sold, the drug that earns the voucher and the drug that redeems it may belong to different firms, enabling small developers to participate and broadening the scope of innovation (Gans and Ridley 2013). Vouchers have sold for $100 million or more (Ridley et al. 2021), lowering the pull funding required by governments and foundations.

While priority review vouchers drive product development and regulatory approval, they do not directly address access. Funders may need to subsidize drug purchases to ensure widespread availability. For instance, the President's Emergency Plan for AIDS Relief (PEPFAR) has purchased antiretroviral drugs for low- and middle-income countries. Additionally, patent pools enhance access. In such pools, patent holders voluntarily license intellectual property to a centralized entity, allowing generic manufacturers to produce and sell drugs in low-income markets (Wang 2022).

2. Model

A principal determines whether, when, and how much to fund an agent. The principal in this context refers to a funder such as the Gates Foundation or the Wellcome Trust.

Our model represents a drug-developing firm and its investors as a single agent. We omit inter-firm competition because neglected diseases seldom attract many firms.[3]

The firm has a candidate drug for a possible clinical trial followed by submission to a regulator, such as the FDA. We call the clinical trial and the regulatory approval process a "test," the result of which is uncertain.

Before the test, the firm privately knows the probability of success, based on earlier testing of the drug and the firm's capabilities. In order for the test to be successful, the firm needs to exert costly effort by carefully designing the study and recruiting appropriate providers and patients. The effort is not directly observable by the funder. Therefore, our setting involves both adverse selection and moral hazard. Specifically, we model the firm's effort cost as c and the probability of the test being successful given the firm exerts effort as θ, which is the firm's private information (adverse selection). The funder believes that θ is drawn from a distribution with support $\Theta \equiv [\theta_L, \theta_H]$, probability density function f, and cumulative distribution function F (common prior distribution).

If the firm shirks instead, the test will definitely fail, and the corresponding cost to the firm is $\alpha\, c$ for some $\alpha \in [0, 1]$. The value α measures the severity of moral hazard: if $\alpha = 0$, the firm incurs no cost from shirking, and moral hazard is the most severe. If $\alpha = 1$, on the other hand, shirking costs the same as exerting effort, and moral hazard is eliminated. One can interpret a strictly positive α as capturing the fixed cost associated with the test, including, for example, filing paperwork to get permission, opening clinical sites to make the trial accessible, and compensating investigators and participants.

The funder can decide how much to fund the firm before and after the testing result is realized. The funding provided after the test can be conditioned on the test result. Because we analyze a single late-stage trial, the model includes only one signal: success or failure.

Following the revelation principle (Myerson 1981), we study incentive-compatible direct mechanisms without loss of generality. That is, the funder publicly commits to a funding policy in the beginning based on the agent's reported value $\theta \in \Theta$, which specifies: (1) whether to induce testing $q(\theta) \in \{0, 1\}$; (2) an amount of upfront payment $\hat{m}_0(\theta)$ (*push*); and (3) amounts of subsequent payments conditional on test results. In particular, the payment after a successful test is denoted as $\hat{m}_S(\theta)$ (*pull*), and after a failed test as $\hat{m}_F(\theta)$ (*insurance*).

A successful drug brings the funder a value ω, representing consumer welfare. Therefore, when a firm with success probability θ reports truthfully and exerts effort in the test, the funder's expected

[3] Neglected diseases, by definition, have few drugs in the pipeline or marketed. Resistance in bacteria and viruses creates demand for multiple treatments, yet limited commercial returns discourage entry. We include certain non-neglected diseases, such as HIV, but the model predicts they merit no subsidy.

payoff $u_F(\theta)$ is the potential consumer welfare $\theta\,\omega\,q(\theta)$ minus its expected payments in both periods, i.e.,

$$u_F(\theta) \equiv \left[\theta\,\omega - \hat{m}_0(\theta) - \theta\,\hat{m}_S(\theta) - (1-\theta)\,\hat{m}_F(\theta)\right] q(\theta). \tag{1}$$

We assume that the funder does not discount future payments because it is not cash-constrained and prioritizes long-term societal benefits.

If the regulator approves the drug, the firm receives a profit of π from the market. The present value of the profit is $\delta\,\pi$, where $\delta \in [0,1]$ is the firm's discount factor, reflecting the fact that the firm discounts future payments due to cash constraints, the length of testing, and the risks of product failure.

Let $u_1(\theta,\hat{\theta})$ denote the firm's expected payoff when its true type is θ, but it reports $\hat{\theta}$ and exerts effort in the test. With probability $q(\hat{\theta})$, its expected payoff is composed of three components: (1) in period 1, the firm bears the testing cost c and receives an upfront payment of $\hat{m}_0(\hat{\theta})$; (2) the test succeeds with probability θ, and therefore the firm receives a commercial profit π along with a subsequent payment of $\hat{m}_S(\hat{\theta})$; and (3) the trial fails with the remaining probability $1-\theta$, and the firm receives $\hat{m}_F(\hat{\theta})$ as specified by the mechanism. To summarize, we have

$$u_1(\theta,\hat{\theta}) \equiv \Big[-c + \hat{m}_0(\hat{\theta}) + \delta\,\theta\,\Big(\hat{m}_S(\hat{\theta}) + \pi\Big) + \delta\,(1-\theta)\,\hat{m}_F(\hat{\theta})\Big]\, q(\hat{\theta}). \tag{2}$$

If the firm of type θ misreports as $\hat{\theta}$ and shirks, with probability $q(\hat{\theta})$, it receives an upfront payment $\hat{m}_0(\hat{\theta})$ and bears the partial testing cost $\alpha\,c$, followed by a subsequent payment $\hat{m}_F(\hat{\theta})$ because the test will failure. Its total payoff boils down to

$$u_0(\hat{\theta}) \equiv \Big[-\alpha\,c + \hat{m}_0(\hat{\theta}) + \delta\,\hat{m}_F(\hat{\theta})\Big]\, q(\hat{\theta}). \tag{3}$$

A feasible mechanism must satisfy the incentive compatibility constraint, meaning that a firm with type θ should not benefit from misreporting its type as $\hat{\theta}$ or shirking, i.e.,

$$u_1(\theta,\theta) \geq \max\Big\{u_0(\hat{\theta}), u_1(\theta,\hat{\theta})\Big\}, \quad \forall(\theta,\hat{\theta}) \in \Theta^2, \tag{IC}$$

where $u_1(\theta,\theta)$ represents the type θ firm's expected payoff when it reports truthfully and exerts effort.

Furthermore, assuming the firm's outside option has a value of zero, the following individual rationality constraint must hold to guarantee the firm's participation

$$u_1(\theta,\theta) \geq 0, \quad \forall\theta \in \Theta. \tag{IR}$$

Finally, we recall the constraints on q,

$$q(\theta) \in \{0,1\}, \quad \forall\theta \in \Theta, \tag{4}$$

and impose the following limited liability constraints, restricting all monetary transfers from the funder to the firm

$$\hat{m}_0(\theta),\ \hat{m}_S(\theta),\ \hat{m}_F(\theta) \geq 0, \quad \forall \theta \in \Theta. \tag{5}$$

The funder's problem can be stated as follows:

$$u^* \equiv \max_{\{q(\cdot),\hat{m}_0(\cdot),\hat{m}_S(\cdot),\hat{m}_F(\cdot)\} \in \hat{\Omega}} \int_{\theta_L}^{\theta_H} u_F(\theta)\, \mathrm{d}F(\theta), \tag{6}$$

where the feasible set $\hat{\Omega}$ is determined by constraints (IC), (IR), (4), and (5).

To implement the optimal solution of (6), the funder does not need the firm to report the probability of success θ, and does not dictate the firm's participation. Instead, the funder extends a take-it-or-leave-it contract, specifying payment levels for push, pull, and insurance. The firm can either accept or reject the offer. If the firm accepts, it effectively reveals that its probability of success is above the threshold for $q(\theta)$ to be 1. If the common prior distribution is uniform, we show that such an indirect mechanism is indeed optimal. Under general distributions, such a take-it-or-leave-it offer performs well. Specifically, it consistently achieves at least 99% of the optimal value under the Beta distributions in Figure 3(a).

Large-scale Phase II or III testing typically takes about two years per phase (DiMasi et al. 2016). Assuming four years of testing and a 16% discount rate – the average for a large sample of firms (Gormsen and Huber 2023) – yields a discount factor greater than 0.5. Therefore, we maintain the following assumption throughout the paper.

Assumption 1. *The firm's discount factor is no less than* $1/2$*, i.e.,* $\delta \geq 1/2$*.*

In order to transform the non-linear optimization problem (6) into a linear one, we introduce the following alternative decision variables,

$$m_0(\theta) \equiv q(\theta)\, \hat{m}_0(\theta), \quad m_S(\theta) \equiv q(\theta)\, \hat{m}_S(\theta), \quad \text{and} \quad m_F(\theta) \equiv q(\theta)\, \hat{m}_F(\theta), \quad \forall \theta \in \Theta, \tag{7}$$

which represent the *expected* upfront payment, payment upon success, and payment upon failure, respectively. We also relax the binary constraint on q such that

$$0 \leq q(\theta) \leq 1, \quad \forall \theta \in \Theta. \tag{8}$$

We can rewrite both parties' payoff functions in (1), (2), and (3) as

$$u_F(\theta) = \theta\, \omega\, q(\theta) - m_0(\theta) - \theta\, m_S(\theta) - (1-\theta)\, m_F(\theta), \tag{9}$$

$$u_1(\theta, \hat{\theta}) = -c\, q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\, \theta \left[m_S(\hat{\theta}) + \pi\, q(\hat{\theta}) \right] + \delta\, (1-\theta)\, m_F(\hat{\theta}), \text{ and} \tag{10}$$

$$u_0(\hat{\theta}) = -\alpha\, c\, q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\, m_F(\hat{\theta}), \tag{11}$$

which are all linear functions of the decision variables.

We impose the following feasibility conditions on m's:

$$m_0(\theta),\ m_S(\theta),\ m_F(\theta) \geq 0, \quad \forall \theta \in \Theta. \tag{12}$$

This is a relaxation of the optimization problem (6) because we ignore the constraint that enforces $m_0(\theta) = m_S(\theta) = m_F(\theta) = 0$ if $q(\theta) = 0$. The optimal solution to this relaxed problem satisfies the ignored constraint automatically, and hence it remains optimal for the original problem.

Define the funder's expected utility as

$$U_F(q, m_0, m_S, m_F) \equiv \int_{\theta_L}^{\theta_H} u_F(\theta)\, \mathrm{d}F(\theta), \tag{13}$$

where u_F is specified in (9). The relaxed funder's problem can be formulated as follows:

$$\max_{\{q(\cdot), m_0(\cdot), m_S(\cdot), m_F(\cdot)\} \in \Omega} U_F(q, m_0, m_S, m_F), \tag{14}$$

where the feasible set Ω is determined by (IC), (IR), (8), and (12).

The following result implies that under the optimal mechanism, the funder never needs to reward success and failure simultaneously.

Lemma 1. *For any mechanism $\{q, m_0, m_S, m_F\} \in \Omega$, there exists a mechanism $\{q, \tilde{m}_0, \tilde{m}_S, \tilde{m}_F\} \in \Omega$ such that $\tilde{m}_S(\theta) \cdot \tilde{m}_F(\theta) = 0$ for any $\theta \in \Theta$, and $U_F\left(q, \tilde{m}_0, \tilde{m}_S, \tilde{m}_F\right) \geq U_F\left(q, m_0, m_S, m_F\right)$.*

That is, any feasible mechanism that offers pull and insurance together is weakly dominated by another one that does not. Intuitively, when both rewards m_S and m_F are strictly positive, conditional on being induced to test ($q = 1$), the funder can increase m_0 by $\delta \cdot \min\{m_S, m_F\}$ while decreasing both m_S and m_F by $\min\{m_S, m_F\}$. This change does not affect the firm's incentive while saving the funder $(1 - \delta) \cdot \min\{m_S, m_F\}$ due to the firm's time discounting.

3. Optimal Mechanism

In this section, we characterize the optimal mechanism that solves the funder's problem (14). First, we show the optimal solution under the following assumption on the common prior distribution of the agent's private information in Section 3.1.

Assumption 2. *The common prior of the agent's private information θ follows a uniform distribution on $[0, 1]$.*

The results obtained in this section under Assumption 2 are intuitive and easy to implement. Then, in Section 3.2, we study the problem for general distributions.

Before going into details, we provide a concise overview of the optimal mechanisms. The optimal solution to the funder's problem (14) follows a simple threshold policy under Assumption 2. That

is, the funder incentivizes the firm to test if and only if the success rate θ is above a threshold. Furthermore, the optimal compensations $m_0(\theta)$, $m_S(\theta)$, and $m_F(\theta)$ are constants that do not change with θ as long as it is above the threshold. Thus, the funder can implement this optimal mechanism as a simple *take-it-or-leave-it offer*: the firm will accept the offer if and only if its privately known success probability exceeds the threshold.

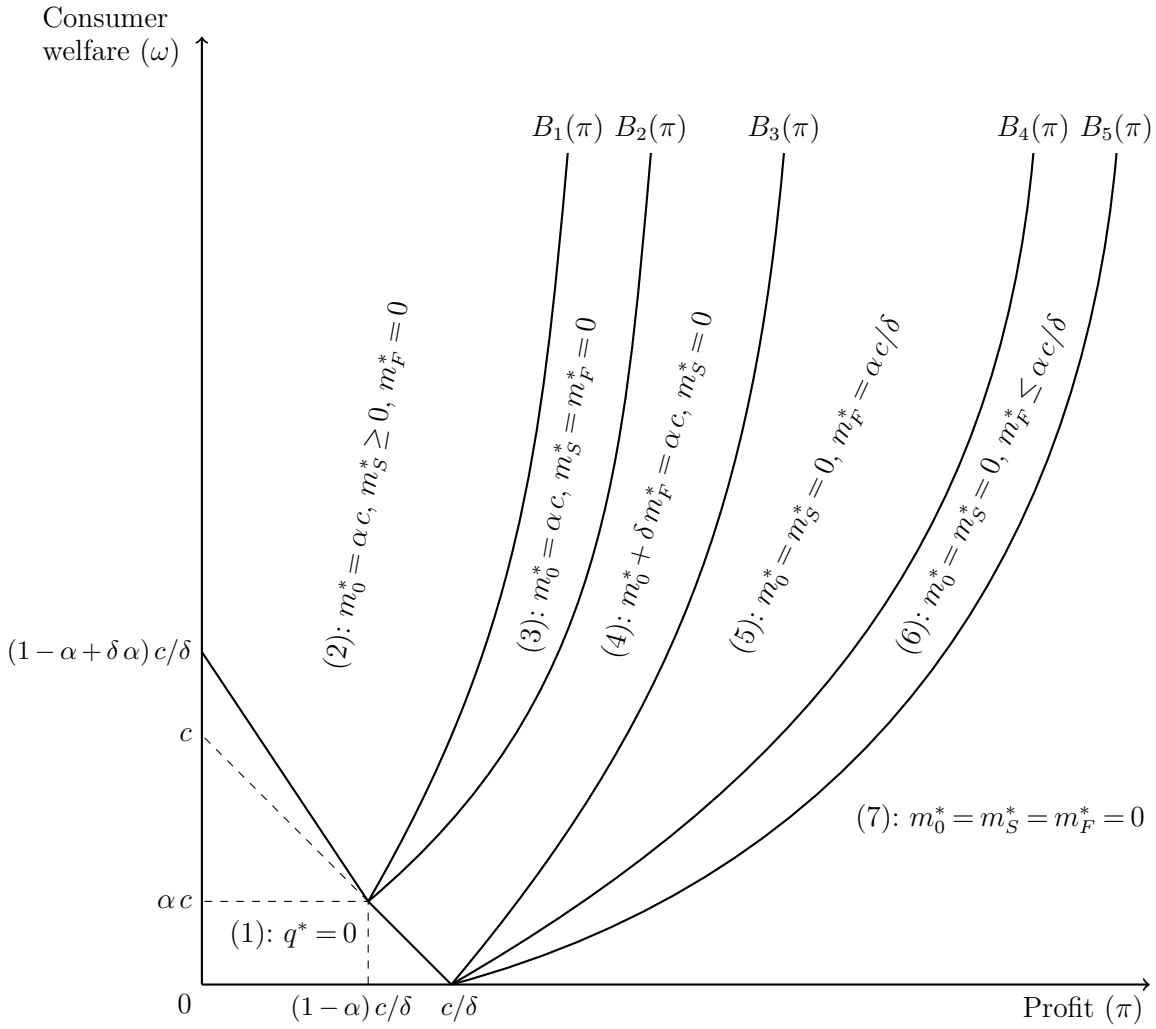


Figure 1 **Optimal mechanisms under given profit π and consumer welfare ω.**

The main results of this section are summarized in Figure 1. There are seven regions in the (π, ω) space when holding constant the testing cost (1), shirking cost (α), and time discount factor (δ). Below, we list each region and the associated optimal policy.

Region (1): Do nothing. Here, both the welfare ω and profit π are low relative to the testing cost. The funder's optimal policy is not to invest.

Region (2): Pull and push funding. The welfare ω is high, but the profit π is low. The funder should offer pull $m_S^* \geq 0$ and push $m_0^* = \alpha\, c$ funding. By rewarding success, pull funding provides accountability and thus mitigates the moral hazard problem. The up-front payment cannot be greater than $\alpha\, c$. Otherwise, the firm with a very low probability of success would have the incentive to over-report its drug's promise to be funded and then shirk.

Region (3): Push funding. The profit potential is higher than in Region (2), so pull funding is unnecessary, though push funding continues at $\alpha\, c$.

Region (4): Push and insurance. The profit potential is higher than in Region (3), so less push funding is needed. Some of the push funding m_0^* is replaced with insurance m_F^*. Insurance has an advantage over push funding in mitigating the adverse selection problem. Without insurance, firms may under-report the drug's ex-ante chance of success to attract more push funding to cover the testing cost. Compared with push funding, insurance saves the funder money because it does not need to pay when the firm succeeds, so the funder is better off in expectation. However, should the drug fail, the funder would be worse off due to the firm's relatively high discount rate. Using insurance, the funder can encourage the firm to invest in less-promising drugs that would not otherwise have been tested.

Region (5): Insurance only. The profit potential is higher than in Region (4), so the funder no longer needs to provide push funding ($m_0^* = 0$) and only provides insurance $m_F^* = \alpha\, c\,/\,\delta$. Within this region, total funding remains constant. As profit π increases, drugs with a lower probability of success participate in testing. The testing threshold on the probability of success keeps decreasing throughout Regions (2) to (5), but may not be decreasing in the next region, Region (6).

Region (6): Less insurance. Within this region, as the profit π rises, the insurance falls. The funder is trading off between bringing more drugs into testing and providing more insurance, and hence the testing threshold may or may not decrease in π.

Region (7): Do nothing. The profit potential π is so high that the funder is not needed to encourage investment.

Figure 2 displays the optimal mechanism for increasing π values under a fixed ω. The partition of the x-axis represents the seven cases described above. Figure 2(a) depicts the present value of the maximum payments, and (b) illustrates the testing threshold. As the profit π potential increases, the testing threshold is non-increasing for Regions (2) to (5), as well as in Region (7).

In Appendix E, we quantify the benefit of allowing the insurance funding to reward failure. Our numerical tests show that not using insurance funding can imply a 20% suboptimality (Figure EC.1).

Next, we describe the optimal solution structures in detail in Section 3.1. In the following subsection, we relax the assumption of a uniformly distributed θ and demonstrate robustness of the result.

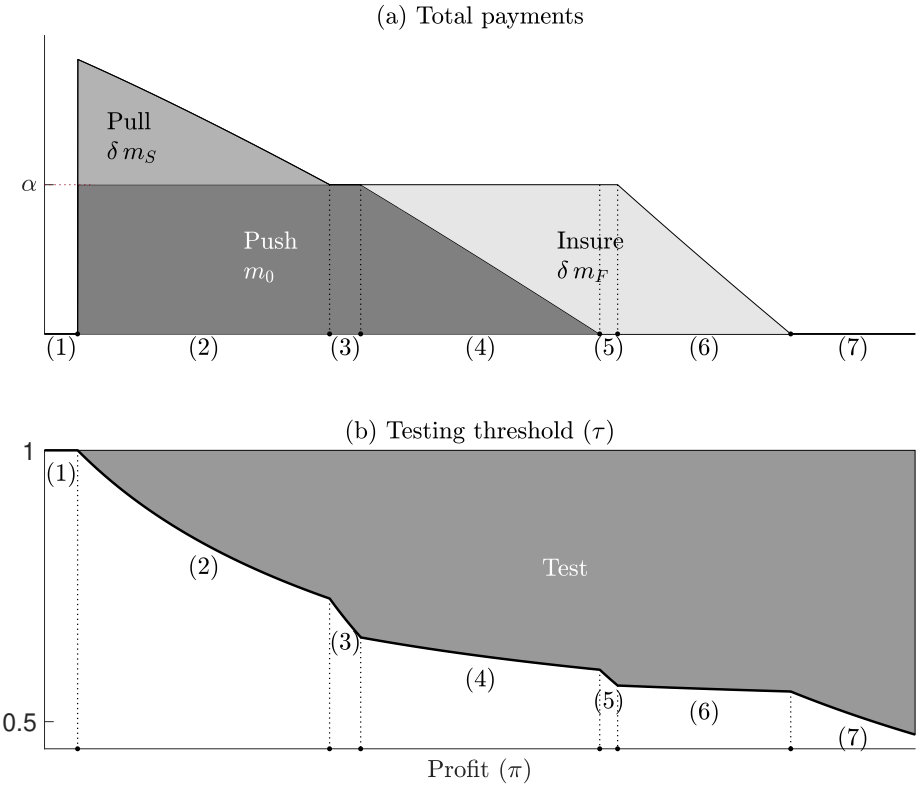


Figure 2 The total payment is the sum between the upfront payment m_0 and either payment upon success $\delta\, m_S$ or payment upon failure $\delta\, m_F$, which captures the maximum present value of payments. The testing threshold (τ) under the optimal mechanism is non-increasing as profit potential rises for these parameter values ($\alpha = 0.5$, $c = 1$, $\delta = 0.6$, and $\omega = 1.2$).

3.1. Optimal Solutions

Proposition 1 considers the case in which both the welfare ω and the profit π are relatively low. In this scenario, the optimal mechanism is not to develop the drug. This result corresponds to Region (1) in Figure 1.[4]

PROPOSITION 1. *If $\omega + \delta\,\pi \leq c$ or $\omega + \pi \leq (1 - \alpha + \delta\,\alpha)\, c/\delta$, the optimal solution to (14) is*

$$q^*(\theta) = m_0^*(\theta) = m_S^*(\theta) = m_F^*(\theta) = 0,\ \forall \theta \in [0,1].$$

In order to show this result, we need to verify optimality in two separate cases: (1) $\omega + \delta\,\pi \leq c$; or (2) $c < \omega + \delta\,\pi \leq (1 - \alpha + \delta\,\alpha)\, c + (1 - \delta)\,\omega$. In case (1), the social surplus $\omega + \delta\,\pi$ is smaller than the testing cost, implying that it is socially inefficient to induce testing. Hence, the optimal strategy for

[4] It is without loss of generality to normalize one of the following three parameters, c, π, or ω, to 1. In the Appendix, we set $c = 1$ for simplicity. In this section, we retain all of them to more explicitly express how various quantities depend on these parameters.

a centralized social planner is to induce no testing and provide no funding. In case (2), even though the social surplus exceeds the testing cost, due to the firm's discounting of future income and the relatively small profit π, the compensation to induce the firm's honest effort would be too high to be worth doing. It is worth noting that if the firm does not discount future income, i.e., $\delta = 1$, case (2) is reduced to $c < \omega + \delta\,\pi \leq c$ and hence vanishes.

To better express the following results, we define the self-funding threshold

$$\bar{\tau} \equiv \frac{c}{\delta\,\pi}, \tag{15}$$

such that for a success rate $\theta > \bar{\tau}$, the expected profit $\delta\,\pi\,\theta$ exceeds the testing cost c. Such a firm always has an incentive to test even without support from the funder.

Next, Proposition 2 focuses on the scenario where the consumer welfare ω is relatively large compared to the firm's profit π. In this case, the optimal decision is to provide an upfront payment $\alpha\,c$ and a subsequent one conditional on success, which reflects the principal's intuitive desire to hold the firm accountable. This result corresponds to Region (2) in Figure 1.

PROPOSITION 2. *For any $(1-\alpha)\,\bar{\tau} \leq 1$, define function*

$$B_1(\pi) \equiv \pi\,\frac{1-(1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\bar{\tau}^2}{2\,(1-\alpha)^2\,\bar{\tau}^2}. \tag{16}$$

If $\omega + \pi \geq (1-\alpha+\delta\,\alpha)\,c/\delta$ and $\omega \geq B_1(\pi)$, the following solution is optimal to (14)*:*

$$q^*(\theta) = \begin{cases} 0, \ \forall\theta \in [0,\tau), \\ 1, \ \forall\theta \in [\tau,1], \end{cases} \qquad m_0^*(\theta) = \begin{cases} 0, \quad \forall\theta \in [0,\tau), \\ \alpha\,c, \ \forall\theta \in [\tau,1], \end{cases}$$

and

$$m_S^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ \pi\,\left[(1-\alpha)\,\bar{\tau}/\tau - 1\right], & \forall\theta \in [\tau,1], \end{cases} \qquad m_F^*(\theta) = 0, \quad \forall\theta \in [0,1],$$

where $\tau \in [0,(1-\alpha)\,\bar{\tau}]$ is the unique solution to

$$2\,\delta\,(\omega+\pi)\,\tau^3 + c\,(\alpha - 2\,\delta\,\alpha - 1)\,\tau^2 - c\,(1-\alpha) = 0.$$

In the (π,ω) parametric space, the policy introduced in Proposition 2 is optimal when the pair is not too close to $(0,0)$ and the welfare ω is above the boundary $B_1(\pi)$. Equivalently, the profit π is not excessively high compared to ω. In such scenarios, even a firm with a high success rate requires compensation to participate. The upfront transfer is set at size $\alpha\,c$ due to moral hazard: a larger upfront payment would incentivize a firm with a low success rate to misreport as a high type and shirk, resulting in a strictly positive expected payoff. The testing threshold τ is determined by optimality conditions, and the subsequent payment upon success, m_S^*, is designed to ensure that the threshold type is indifferent between participating and not. Furthermore, as shown in Proposition 2 and Figure 2(a), the total payment $m_0^* + \delta\,m_S^*$ decreases with increasing profit π.

As the profit π continues increasing, successful firms have already received enough incentives from the commercial market. In this case, the optimal policy now only involves an upfront payment of αc (push) and no after-test payment (pull). The conditions are formally established in Proposition 3, and demonstrated in Region (3) of Figure 1.

PROPOSITION 3. *For any* $(1-\alpha)\bar{\tau} \leq 1$*, define function*

$$B_2(\pi) \equiv B_1(\pi) - \pi \frac{(1-\delta)[1-(1-\alpha)\bar{\tau}]}{(1-\alpha)^2\bar{\tau}^2} \leq B_1(\pi). \tag{17}$$

If $B_2(\pi) \leq \omega < B_1(\pi)$*, the following solution is optimal to* (14)*:*

$$q^*(\theta) = \begin{cases} 0, \forall \theta \in [0,\tau), \\ 1, \forall \theta \in [\tau,1], \end{cases} \quad m_0^*(\theta) = \begin{cases} 0, & \forall \theta \in [0,\tau), \\ \alpha c, & \forall \theta \in [\tau,1], \end{cases} \quad m_S^*(\theta) = m_F^*(\theta) = 0, \quad \forall \theta \in [0,1],$$

where $\tau \equiv (1-\alpha)\bar{\tau}$*.*

The two boundaries $B_1(\pi)$ and $B_2(\pi)$ intersects at $\pi = (1-\alpha)c/\delta$, where $\omega = \alpha c$. The condition $B_2(\pi) \leq \omega < B_1(\pi)$ in Proposition 3 can also be equivalently described as the profit π falls into an interval for any given $\omega \geq \alpha c$. It is worth mentioning that when the firm does not discount future cash flow, i.e., $\delta = 1$, we have $B_1(\pi) = B_2(\pi)$, and Region (3) in Figure (1) disappears.

As we increase the profit π further, it becomes sufficient to induce the firm's effort. However, for a firm with a relatively low success rate to be willing to test, it becomes optimal to "reward failure." We formally present the result in Proposition 4, which captures Region (4) of Figure 1.

PROPOSITION 4. *For any* $(1-\alpha)\bar{\tau} \leq 1$*, define*

$$B_3(\pi) \equiv B_2(\pi) - \pi \frac{\alpha\left[3(2\delta-1) + (2+\alpha-4\delta\alpha-2\delta)\bar{\tau} + (2\delta\alpha-2\alpha+1)\bar{\tau}^2\right]}{2(1-\alpha)^2\bar{\tau}} \leq B_2(\pi).$$

If $\omega + \delta\pi > c$ *and* $B_3(\pi) \leq \omega < B_2(\pi)$*, the following solution is optimal to* (14)*:*

$$q^*(\theta) = \begin{cases} 0, \forall \theta \in [0,\tau), \\ 1, \forall \theta \in [\tau,1], \end{cases} \quad m_0^*(\theta) = \begin{cases} 0, & \forall \theta \in [0,\tau), \\ \delta\pi\left[\alpha\bar{\tau} + (1-\alpha)\bar{\tau}/\tau - 1\right], & \forall \theta \in [\tau,1], \end{cases}$$

and

$$m_F^*(\theta) = \begin{cases} 0, & \forall \theta \in [0,\tau), \\ \pi\left[1-(1-\alpha)\bar{\tau}/\tau\right], & \forall \theta \in [\tau,1], \end{cases} \quad m_S^*(\theta) = 0, \ \forall \theta \in [0,1],$$

where the threshold $\tau \in [0,1]$ *is uniquely determined by*

$$2\delta(\omega+\pi)\tau^3 - \left[2\delta\pi(1-\delta) + (1-\alpha+2\delta\alpha)c\right]\tau^2 - (1-\alpha)(2\delta-1)c = 0.$$

As illustrated in Figure 2(a), the present value of the total payment that the firm receives is at least $m_0^* + \delta m_F^* = \alpha c$ in Region (4). Therefore, even if the drug fails, the firm is partially compensated, which encourages its participation. In other words, rewarding failure provides insurance for the firm. The threshold τ in Proposition 4 guarantees that if the firm's success probability θ is above this

threshold, the present value from success ($\delta\,\pi\,\theta$) is sufficiently high to motivate the firm's participation and exerting effort.

The expression of $m_0^*(\theta) = \delta\,\pi\,[\alpha\,\bar{\tau} + (1-\alpha)\,\bar{\tau}/\tau - 1]$ in Proposition 4 eventually becomes zero for sufficiently high profit π. If we increase π further, m_0^* will stay at zero, while the threshold τ decreases, allowing drugs with an even lower prior chance of success to be tested. In this case, the firm only receives a payment from the funder if the test fails. The detailed result is presented in Proposition 5, which is represented by Region (5) of Figure 1.

PROPOSITION 5. *When $\bar{\tau} \leq 1$, define*

$$B_4(\pi) \equiv B_3(\pi) - \pi\,\frac{(2\,\delta - 1)\,(1-\bar{\tau})\,(1-\alpha\,\bar{\tau})^2}{2\,(1-\alpha)^2\,\bar{\tau}^2} \leq B_3(\pi).$$

If $B_4(\pi) \leq \omega < B_3(\pi)$, the following solution is optimal to (14)*:*

$$q^*(\theta) = \begin{cases} 0, \forall\theta \in [0,\tau), \\ 1, \forall\theta \in [\tau,1], \end{cases} \quad m_F^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ \alpha\,c/\delta, & \forall\theta \in [\tau,1], \end{cases} \quad m_0^*(\theta) = m_S^*(\theta) = 0, \quad \forall\theta \in [0,1],$$

where $\tau = (1-\alpha)\,\bar{\tau}/(1-\alpha\,\bar{\tau})$.

If the profit π grows even higher, the funder can start lowering its support level and rely more and more on the commercial market. The corresponding optimality conditions are characterized in Proposition 6, depicting Region (6) of Figure 1.

PROPOSITION 6. *For any $\bar{\tau} \leq 1$, define*

$$B_5(\pi) \equiv B_4(\pi) - \pi\,\frac{\alpha\,(1-\bar{\tau}^2)}{2\,(1-\alpha)\,\bar{\tau}} \leq B_4(\pi).$$

If $B_5(\pi) \leq \omega < B_4(\pi)$, the following solution is optimal to (14)*: $m_0^*(\theta) = m_S^*(\theta) = 0,\ \forall\theta \in [0,1]$,*

$$q^*(\theta) = \begin{cases} 0, \forall\theta \in [0,\tau), \\ 1, \forall\theta \in [\tau,1], \end{cases} \quad m_F^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ \pi\,(\bar{\tau}-\tau)/(1-\tau), & \forall\theta \in [\tau,1], \end{cases}$$

where $\tau \equiv (1+\bar{\tau})/[2\,(\omega/\pi+1)]$.

Starting from this region, the funder no longer guarantees to cover the cost of shirking $\alpha\,c$ for the participating firm. The reason is that the profit is high enough so that the firm does not need to use the highest insurance level ($\alpha\,c/\delta$) to lure a drug with a relatively low chance of success to be tested. In fact, the testing threshold $\tau \equiv (1+\bar{\tau})/[2\,(\omega/\pi+1)]$ is increasing in π when $\omega > \bar{\tau}\,\pi$. That is, it can be optimal to exclude more types of firms as the profit π increases. The optimality conditions drive this feature: the benefit of tightening the testing threshold, which saves money for the funder, dominates giving a drug with a relatively low *ex ante* efficacy level a chance to be tested.

Finally, when the profit π is high enough to incentivize testing, the funder does not need to interfere. This is shown in Region (7) of Figure 1 and summarized in Proposition 7 below.

PROPOSITION 7. *If $\omega < B_5(\pi)$, the following solution is optimal to* (14)*:*

$$q^*(\theta) = \begin{cases} 0, \forall \theta \in [0, \tau), \\ 1, \forall \theta \in [\tau, 1], \end{cases} \quad m_0^*(\theta) = m_S^*(\theta) = m_F^*(\theta) = 0, \quad \forall \theta \in [0, 1],$$

where $\tau = \bar{\tau}$.

We prove the results in this section using duality for the (infinite-dimensional) linear optimization problem (14). In particular, we first follow the standard step on the (IC) constraint from Myerson (1981) to eliminate the decision $m_0(\theta)$. The objective function then involves three virtual valuation functions, corresponding to coefficients of $q(\theta)$, $m_S(\theta)$, and $m_F(\theta)$, respectively. We then construct a dual feasible solution for each of the seven cases. The corresponding dual objective values are upper bounds for the optimal objective values in (14). However, these upper bounds can be achieved using the primal solutions proposed in the aforementioned seven propositions, which validates optimality. The detailed proofs are provided in Appendix B.

The following corollary further states that as the profit level π keeps increasing, the principal generally funds less promising drugs for testing, except in Region (6) of Figure 1; the same monotonicity consistently holds across all regions as the consumer welfare ω increases.

COROLLARY 1. *Fixing other model parameters,*

(a) the optimal testing threshold τ according to Propositions 1-5 and 7 is non-increasing in π; and

(b) the optimal testing threshold τ according to Propositions 1-7 is non-increasing in ω.

Figure 2(b) illustrates the non-increasing testing threshold τ with profit π in Regions (1)-(5) and (7). In this particular parameter setting, the threshold is decreasing in Region (6) as well.

3.2. General common-prior distributions

Now we study the model without Assumption 2. That is, we allow the common prior for drug efficacy to take general distributions. The corresponding optimal solution to (14) may be much more complex than the one illustrated in Section 3.1. In particular, it may involve a multi-menu contract that is hard to implement and therefore less practically relevant. Therefore, in this section, we explore the optimal take-it-or-leave-it mechanisms. For any given testing threshold, we characterize the optimal funding levels in closed form. Our result indicates that under the optimal threshold, the funding levels demonstrate very similar characteristics as the solution in Section 3.1. Interestingly, there is one more possible funding scenario under general distributions compared with the uniform $[0, 1]$ case: the funder may only push with a level less than $\alpha\, c$.

Recall the definition of self-funding threshold $\bar{\tau}$ in (15). If a drug's success rate is higher than $\bar{\tau}$, the firm has sufficient incentive to test. Therefore, the funder only needs to evaluate any potential threshold in $\breve{\Theta} \equiv [\theta_L,\ \min\{\bar{\tau}, \theta_H\}]$.

The optimization problem (14) with the following additional constraints computes the best take-it-or-leave-it offer for a given testing threshold $\check{\tau} \in \check{\Theta}$:

$$\begin{aligned} &q(\theta) = m_0(\theta) = m_S(\theta) = m_F(\theta) = 0, \ \forall \theta \in [\theta_L, \check{\tau}), \text{ and} \\ &q(\theta) = 1, \ m_0(\theta) = \check{m}_0, \ m_S(\theta) = \check{m}_S, \ m_F(\theta) = \check{m}_F, \ \forall \theta \in [\check{\tau}, \theta_H], \end{aligned} \tag{18}$$

for some decision variables $\check{m}_0$, $\check{m}_S$, $\check{m}_F$. More precisely, we consider the following optimization problem,

$$u^{\mathrm{TH}}(\check{\tau}) \equiv \max_{\{\check{m}_0, \check{m}_S, \check{m}_F\} \in \bar{\Omega}(\check{\tau})} \int_{\theta_L}^{\theta_H} u_F(\theta) \, \mathrm{d}F(\theta), \tag{19}$$

where u_F follows (9), and the feasible set $\bar{\Omega}(\check{\tau})$ is determined by (IC), (IR), (8), (12), and (18).

Lemma 2. *For any $\check{\tau} \in \check{\Theta}$, we have*

$$u^{\mathrm{TH}}(\check{\tau}) = \max_{\{\check{m}_0, \check{m}_S, \check{m}_F\} \in \bar{\Omega}'(\check{\tau})} \int_{\check{\tau}}^{\theta_H} [\theta \, \omega - \check{m}_0 - \theta \, \check{m}_S - (1-\theta) \, \check{m}_F] \, \mathrm{d}F(\theta), \tag{20}$$

in which the feasible set $\bar{\Omega}'(\check{\tau})$ is defined by the following constraints:

$$\check{m}_S - \check{m}_F + \pi \geq 0, \tag{21}$$

$$\check{m}_0 + \delta \, \check{\tau} \, \check{m}_S + \delta \, (1 - \check{\tau}) \, \check{m}_F = c - \delta \, \pi \, \check{\tau}, \tag{22}$$

$$\check{m}_0 + \delta \, \check{m}_F \leq \alpha \, c, \ \textit{and} \tag{23}$$

$$\check{m}_0, \ \check{m}_S, \ \check{m}_F \geq 0. \tag{24}$$

The following proposition identifies the optimal solution to the linear optimization problem (20) in closed form.

Proposition 8. *Define functions*

$$\mathcal{G}(\check{\tau}) \equiv \alpha + [(1-\alpha)/\check{\tau} - 1/\bar{\tau}] \quad \textit{and} \quad \mathcal{H}(\check{\tau}) \equiv \delta \, (1 - \check{\tau}) \, [1 - F(\check{\tau})] - \int_{\check{\tau}}^{\theta_H} (1-\theta) \, \mathrm{d}F(\theta).$$

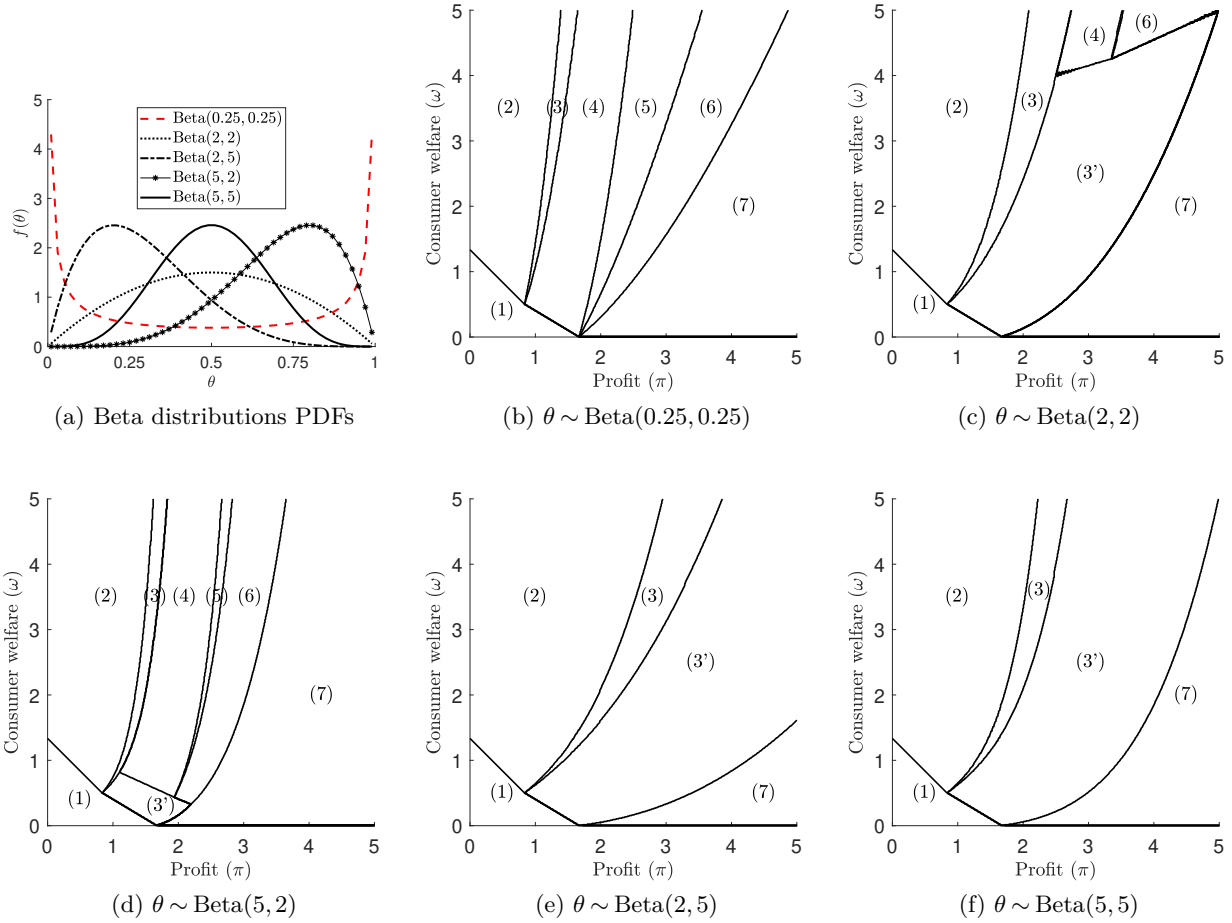
The following defined $(\check{m}_0^*, \check{m}_S^*, \check{m}_F^*)$ is optimal to (19) as well as (20) for a given $\check{\tau} \in \check{\Theta}$:

1. *when $\mathcal{G}(\check{\tau}) \geq \alpha$, we have $\check{m}_0^* = \alpha \, c$, $\check{m}_S^* = \pi \, [\mathcal{G}(\check{\tau}) - \alpha] \, \bar{\tau}$, and $\check{m}_F^* = 0$;*
2. *when $0 \leq \mathcal{G}(\check{\tau}) \leq \alpha$, and $\mathcal{H}(\check{\tau}) \geq 0$, we have $\check{m}_0^* = \mathcal{G}(\check{\tau}) \, c$, $\check{m}_S^* = 0$, and $\check{m}_F^* = \pi \, [\alpha - \mathcal{G}(\check{\tau})] \, \bar{\tau}$;*
3. *when $\mathcal{G}(\check{\tau}) < \alpha$ and $\mathcal{H}(\check{\tau}) \leq 0$, we have $\check{m}_0^* = \delta \, \pi \, (\bar{\tau} - \check{\tau})$ and $\check{m}_S^* = \check{m}_F^* = 0$; and*
4. *when $\mathcal{G}(\check{\tau}) < 0$ and $\mathcal{H}(\check{\tau}) \geq 0$, we have $\check{m}_0^* = \check{m}_S^* = 0$ and $\check{m}_F^* = \pi \, (\bar{\tau} - \check{\tau})/(1 - \check{\tau})$.*

Note that under Assumption 2 (θ is uniformly distributed on $[0, 1]$), we have

$$\mathcal{H}(\check{\tau}) = \left(\delta + \frac{1}{2}\right) (1 - \check{\tau})^2 > 0, \ \forall \check{\tau} < 1.$$

This implies that Case 3 in Proposition 8, which requires $\mathcal{H}(\check{\tau}) \leq 0$, does not arise in the optimal solution under Assumption 2. In fact, the condition $\mathcal{G}(\check{\tau}) < \alpha$ in Case 3 directly implies that the



(a) Beta distributions PDFs (b) $\theta \sim \text{Beta}(0.25, 0.25)$ (c) $\theta \sim \text{Beta}(2, 2)$

(d) $\theta \sim \text{Beta}(5, 2)$ (e) $\theta \sim \text{Beta}(2, 5)$ (f) $\theta \sim \text{Beta}(5, 5)$

Figure 3 Optimal threshold policies when θ is not uniformly distributed ($\alpha = 0.5$, $c = 1$, and $\delta = 0.6$).

corresponding $\check{m}_0^* < \alpha\, c$. That is, the funder only uses push funding at a level strictly less than $\alpha\, c$. This corresponds to none of the seven cases in Section 3.1.

In order to obtain the best take-it-or-leave-it offer, we use the optimal solution described in Proposition 8 together with a line search for the optimal threshold value $\check{\tau}$. Figure 3 displays the optimal threshold policy when the firm's private information follows the Beta distributions as depicted in Figure 3(a). Regions (1)–(7) in panels (b)-(f) correspond to the optimal solutions defined in Propositions 1–7 (and hence regions (1)-(7) in Figure 1), respectively. We use (3') to capture the parameter region corresponding to Case 3 in Proposition 8, which involves only an upfront payment less than $\alpha\, c$. As discussed earlier, this solution does not arise when θ is uniformly distributed.

As we can see from the figure, the overall structure of the parameter regions in which different optimal solutions arise largely mirrors that of the uniform distribution case. The main difference is the new Region (3'). For a better illustration of it, we present Figure 4, which resembles Figure 2 for the uniform distribution case.

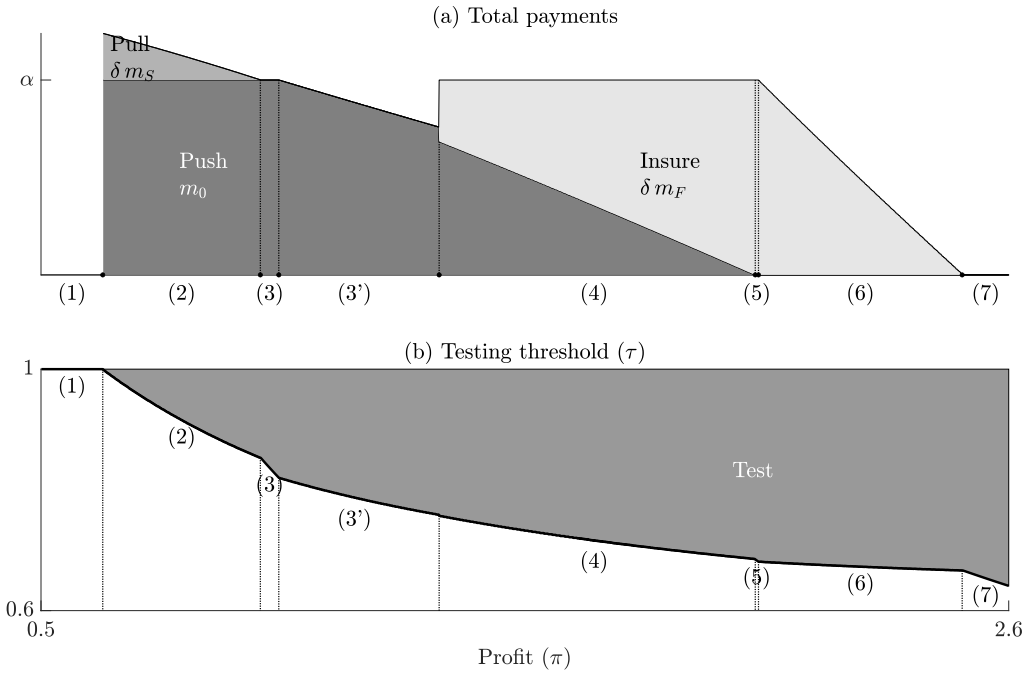


Figure 4 The total payment is the sum between the upfront payment m_0 and either payment upon success $\delta\, m_S$ or payment upon failure $\delta\, m_F$, which captures the maximum present value of payments ($\theta \sim$ Beta$(5,2)$, $\alpha = 0.5$, $c = 1$, $\delta = 0.6$, and $\omega = 0.7$).

Figure 4 illustrates that as the commercial profit π increases, policy structures (1) to (7) become optimal in sequence, consistent with the uniform distribution case. Between (3) and (4), however, it may be optimal to provide only push funding less than $\alpha\, c$, as illustrated in (3'). The optimal testing threshold τ appears to decrease in π in Panel (b).

Overall, it appears that the general structure of the optimal solution established in Section 3.1 largely remains the same under general common prior distributions.

4. Benchmark comparisons

To illustrate the roles of adverse selection and moral hazard, we examine counterfactual scenarios in which each element is absent. In Sections 4.1 and 4.2, we characterize the optimal mechanism when either moral hazard or adverse selection is removed. In Section 4.3, we examine the benchmark with neither adverse selection nor moral hazard.

4.1. No Moral Hazard

Here we consider a second-best benchmark with only adverse selection but no moral hazard, i.e., $\alpha = 1$. The optimal solutions introduced in Propositions 1–7 still hold, although some of the regions disappear. We summarize the optimal mechanisms in the following corollary.

Corollary 2. *Assuming $\alpha = 1$, the following four cases summarize the optimal solution to* (14)*:*

(a) If $\omega + \delta\,\pi \le c$, the solution described in Proposition 1 is optimal. That is,

$$q^*(\theta) = m_0^*(\theta) = m_S^*(\theta) = m_F^*(\theta) = 0,\ \forall\theta \in [0,1],$$

which implies no drug will be developed.

(b) If $\delta\,\pi \le c < \omega + \delta\,\pi$, the solution described in Proposition 4 is optimal. That is, for a threshold $\tau = 1 - \dfrac{\omega + \delta\,\pi - c}{\omega + \pi} \in (0,1)$, we have $q^*(\theta) = m_0^*(\theta) = m_F^*(\theta) = 0$ for $\theta \in [0,\tau)$,

$$q^*(\theta) = 1,\ m_0^*(\theta) = c - \delta\,\pi,\ m_F^*(\theta) = \pi,\ \forall\theta \in [\tau,1],\ \text{and } m_S^*(\theta) = 0,\ \forall\theta \in [0,1].$$

(c) If $\dfrac{\delta\,\pi}{1 + 2\,\omega/\pi} < c < \delta\,\pi$, the solution described in Proposition 6 is optimal. That is, $m_0^*(\theta) = m_S^*(\theta) = 0,\ \forall\theta \in [0,1]$; for $\theta \in [0,\tau)$, $q^*(\theta) = m_F^*(\theta) = 0$, and for $\theta \in [\tau,1]$,

$$q^*(\theta) = 1,\ m_F^*(\theta) = \frac{2\,\omega\,c - \pi\,(\delta\,\pi - c)}{2\,\delta\,\omega + \delta\,\pi - c},\ \text{where } \tau = \frac{\delta\,\pi + c}{2\,\delta\,(\omega + \pi)}.$$

(d) If $0 \le c \le \dfrac{\delta\,\pi}{1 + 2\,\omega/\pi}$, the solution described in Proposition 7 is optimal. That is, $m_0^*(\theta) = m_S^*(\theta) = m_F^*(\theta) = 0,\ \forall\theta \in [0,1]$ and

$$q^*(\theta) = \begin{cases} 0,\ \forall\theta \in [0,\tau), \\ 1,\ \forall\theta \in [\tau,1], \end{cases} \quad \text{where } \tau = \bar{\tau} = \frac{c}{\delta\,\pi},$$

which implies letting the commercial market take over.

According to Corollary 2, Regions (2), (3), and (5) in Figures 1 and 2 vanish when there is no moral hazard. In the remaining regions, Cases (b) and (c) of Corollary 2 require funding for drug development. Both cases involve rewarding failure ($m_F > 0$). This implies that rewarding failure is a robust strategy even if the funder is not concerned with moral hazard.

More specifically, in Case (b), the reward for failure is exactly the profit π. Therefore, as long as θ is above the threshold τ, the firm's expected payoff does not depend on the test outcome. This highlights the fact that the reward for failure serves as "insurance." Furthermore, without moral hazard, there is no need for pull funding. In fact, the present value of the total reward exactly equals the development cost c. That is, the funder extracts all the surplus, and the firm receives no information rent.

The situation is different for Case (c). For the drug with boundary type $\theta = \tau$, one can verify that $\delta\,[\tau\,\pi + (1-\tau)\,m_F^*(\theta)] = c$. That is, the firm's expected net payoff equals zero. Any drug with θ above the threshold τ receives a positive information rent.

Overall, the different cases, especially Case (b), imply that the model, even without moral hazard, does not reduce to the standard screening problem studied in Myerson (1981).

Without moral hazard, pull funding is unnecessary because failure results solely from chance rather than effort. The funder does not need to differentiate between rewarding success and failure. Instead,

the funder offers a combination of push funding and insurance. As profit potential increases, optimal insurance and push funding both decline. In Case (b), push funding decreases and reaches zero when π increases to c/δ. In Case (c), insurance also decreases and reaches zero when π increases into Case (d).

4.2. No Adverse Selection

Here, we consider a second-best benchmark with moral hazard, but without adverse selection. The success probability θ is therefore public information.

This analysis highlights the value of information. Comparing outcomes in this setting to the base case from the previous section quantifies the value to the funder of learning about whether the firm is likely to succeed with the drug.

In this second-best problem with only moral hazard and no adverse selection, the value θ is fixed, and hence the decision variables q, m_0, m_S, and m_F are scalars instead of functions. The incentive constraint (IC) is reduced to

$$-c\,q+\delta\,\theta\,(m_S+\pi\,q)+\delta\,(1-\theta)\,m_F\geq-\alpha\,c\,q+\delta\,m_F. \tag{25}$$

The corresponding funder's problem is formulated as follows:

$$\max_{\{q,m_0,m_S,m_F\}\in\Omega_{\mathrm{SB}}}\ \theta\,\omega\,q-m_0-\theta\,m_S-(1-\theta)\,m_F, \tag{26}$$

where the feasible set Ω_{SB} is determined by (25) and the following linear inequality constraints:

$$-c\,q+m_0+\delta\,\theta\,(m_S+\pi\,q)+\delta\,(1-\theta)\,m_F\geq 0, \tag{27}$$

$$0\leq q\leq 1,\ \text{and} \tag{28}$$

$$m_0,\ m_S,\ m_F\geq 0. \tag{29}$$

The next proposition characterizes the funder's optimal mechanism according to (26).

Proposition 9. *The following four cases summarize the optimal solution to the funder's problem without adverse selection* (26)*:*

(a) if $\theta\in[0,\ \max\{c/(\omega+\delta\,\pi),(\delta\,\alpha-\alpha+1)\,c/(\delta\,(\omega+\pi))\})$*, it is optimal to not induce testing and provide no funding, i.e.,*

$$q=m_0=m_S=m_F=0;$$

(b) if $\theta\in[(\delta\,\alpha-\alpha+1)\,c/(\delta\,(\omega+\pi)),\ (1-\alpha)\,\bar{\tau})$*, it is optimal to induce testing and provide a combination of push and pull funding with an expected present value of* $c-\delta\,\pi\,\theta$*, i.e.,*

$$q=1,\quad m_0=\alpha\,c,\quad m_S=\frac{(1-\alpha)\,c}{\delta\,\theta}-\pi,\quad m_F=0;$$

(c) if $\theta \in [\max\{c/(\omega+\delta\pi), (1-\alpha)\bar{\tau}\}, \bar{\tau})$*, it is optimal to induce testing and provide a push funding of amount* $c-\delta\pi\theta$*, i.e.,*

$$q=1, \quad m_0=c-\delta\pi\theta, \quad m_S=m_F=0; \text{ and}$$

(d) if $\theta \in [\bar{\tau}, 1]$*, the firm is willing to test without the funder, i.e.,*

$$q=1, \quad m_0=m_S=m_F=0.$$

Note that the condition $c/(\omega+\delta\pi) > (\delta\alpha-\alpha+1)\,c/(\delta(\omega+\pi))$ in Case (a) is equivalent to $c/(\omega+\delta\pi) > (1-\alpha)\bar{\tau}$ in Case (c), which further implies that $(\delta\alpha-\alpha+1)\,c/(\delta(\omega+\pi)) < (1-\alpha)\bar{\tau}$, or, the disappearance of Case (b). Therefore, the intervals for θ in the cases presented in Proposition 9 are indeed mutually exclusive and collectively exhaustive.

To summarize, when the success rate θ is public information, the funder should not offer any funding if the drug is ex-ante very ineffective (Case (a)), and let the commercial market take over if the drug is sufficiently promising (Case (d)). For intermediate levels of θ, the funder should rely on either push only (Case (c)) or pull plus push (Case (b)).

It is also worth noting that in both Cases (b) and (c), the present value of the firm's expected value is $m_0+\delta\theta(m_S+\pi)=c$, which equals the testing cost. In Case (b), the drug is less promising ex-ante, which implies that the expected profit from potential success is not high enough for the firm to exert effort. Therefore, the funder needs to provide some pull ($m_S>0$) to guarantee that the incentive constraint (25) is satisfied. Indeed, this constraint binds at the optimum. In Case (c), on the other hand, the probability of success θ is higher, and the expected profit from success is sufficiently high so that the funder no longer needs to provide additional reward for success. In this case, the constraint (25) is not even binding at optimality. In both cases, the push funding ($m_0>0$) is to guarantee the firm's participation.

Without adverse selection, the funder has no reason to provide insurance ($m_F=0$). Insurance is used to counteract the firm's incentive to under-report the drug's probability of success, but in this case, the funder already knows the true probability.

4.3. No Moral Hazard and No Adverse Selection

For completeness, we present the first-best optimal solution with neither adverse selection nor moral hazard. The funder's problem boils down to

$$\max_{\{q,m_0,m_S,m_F\}\in\Omega_{\mathrm{FB}}} \theta\omega q - m_0 - \theta m_S - (1-\theta)m_F, \tag{30}$$

where the feasible set Ω_{FB} is determined by (27), (28), and (29). We present the optimal funding mechanism in the following proposition.

PROPOSITION 10. *The following three cases summarize the optimal solution to the funder's problem without asymmetric information* (30)*:*

(a) if $\theta \in [0, c/(\omega + \delta\pi))$*, it is optimal to not induce testing, i.e.,*

$$q = m_0 = m_S = m_F = 0;$$

(b) if $\theta \in [c/(\omega + \delta\pi), \bar{\tau}]$*, it is optimal to induce testing with a push amount of* $c - \delta\theta\pi$*, i.e.,*

$$q = 1, \quad m_0 = c - \delta\theta\pi, \quad m_S = m_F = 0;$$

(c) if $\theta \in [\bar{\tau}, 1]$*, the firm is willing to test without the funder, i.e.,*

$$q = 1, \quad m_0 = m_S = m_F = 0.$$

When there are no incentive issues, i.e., both the probability of success and the agent's effort are publicly observable, the result is intuitive. The funder induces testing if and only if the expected societal value is higher than the cost, that is, $\theta(\omega + \delta\pi) \geq c$, corresponding to Cases (b) and (c). In this case, if the firm's expected payoff is insufficient to cover the testing cost (Case (b)), the funder pays the shortfall using push funding to ensure the firm's participation.

5. Calibration

To estimate optimal policies, we need four values: the testing cost c, the relative cost of shirking α, the consumer welfare ω, and the profit π. We assume values for these to illustrate potential applications of the model. For simplicity, we hold some model parameters constant across diseases, but our analytical framework allows for disease-specific variation.

First, we assume that the cost of late-stage testing for a new drug is either \$100 million or \$400 million. The assumption of \$100 million is based on a report by the Drugs for Neglected Diseases Initiative, which estimates testing costs for a new chemical entity to be €60 to 190 million (Drugs for Neglected Diseases initiative 2019). We also consider testing costs of \$400 million based on a study of out-of-pocket testing costs for Phase III infectious disease drugs, ranging from \$126 million to \$628 million (Wouters et al. 2020). We also report optimal policy solutions as a function of a range of testing costs.

Second, we assume that the firm bears 70% of the testing cost if it shirks ($\alpha = 0.7$) (Table 1). The firm must exert more than half effort, as it still needs to enroll a substantial number of patients in clinical trials. However, it will not give full effort; it will not dedicate its best staff or full focus if it does not have a sufficient reward for success.

Third, we make assumptions about the consumer welfare gains from a given drug for a disease based on its associated burden. For select communicable diseases, the World Health Organization reports

disability-adjusted life years (DALYs) lost in 2021 by income group (World Health Organization 2024). We denote DALYs from high-income countries as DALY_h and aggregate DALYs from low- and middle-income countries as DALY_l. The World Bank defines high-income countries as those with a gross national income per capita of $14,000 or more. We also indicate whether the WHO considers each disease to be neglected.

We assume that each product, if successful, reduces DALYs per disease by 10%, reflecting the limited market share and product effectiveness. For example, for anti-malarials, the burden alleviated was 16% for Artemether-Lumefantrine, 12% for Artesunate-Amodiaquine, and 1% for Artesunate-Sulfadoxine-Primaquine in 2015 (Hassoun 2015).

Notation	Description	Value
α	Relative cost of shirking	70%
θ	Probability of technical success	$\mathcal{U}_{[0,1]}$
π	Profit	Equation (31)
ω	Consumer welfare	Equation (32)
c	Testing cost	$100 or $400 million
	Time until profit due to clinical trial	4 years
$\hat{\delta}$	Annual discount rate for firm	16%
δ	Discount factor for firm $1/(1+\hat{\delta})^4$	0.55
share	Share of profit and DALYs per product	10%
$\omega_\text{l}/\text{DALY}_\text{l}$	Consumer welfare per DALY for low and middle-income countries	$2,000
$\pi_\text{l}/\text{DALY}_\text{l}$	Profit per DALY for low and middle-income countries	0
$\omega_\text{h}/\text{DALY}_\text{h}$	Consumer welfare per DALY for high-income countries	$12,000
$\pi_\text{h}/\text{DALY}_\text{h}$	Profit per DALY for high-income countries	$24,000

Table 1 Model parameters and their assumed values.

For lower-income countries, we assume that consumer welfare is $2,000 per DALY saved ($\omega_\text{l}/\text{DALY}_\text{l}$). It is standard practice to assume that the value of a drug in a country depends on the country's income per capita (Kazibwe et al. 2022). Gross national income per capita in sub-Saharan Africa is $1,700 and in South Asia is $2,400 (World Bank 2025). These are two places where the burden of tropical diseases is high.

For high-income countries, we assume that consumer welfare is $12,000 per DALY saved ($\omega_\text{h}/\text{DALY}_\text{h}$). This factor is based on national income, but not as high as national income, because

more of the surplus is captured by profit. We assume that, in high-income countries, a DALY saved generates consumer welfare equal to a quarter of per capita income, with the rest captured in profit and variable cost (Appendix F). The average gross national income per capita in high-income countries is $48,000 (World Bank 2025).

Fourth, we estimate profit. Regarding profit in lower-income countries, (π_{l}/DALY$_{\text{l}}$), we assume it is zero, because the drug is sold at cost. Indeed, malaria drug Coartem and tuberculosis drug Sirturo are both sold for less than $1 per day in low-income countries. Gilead licenses its Hepatitis C drug for free to generic makers in India. Merck charges nothing for its onchocerciasis drug ivermectin in lower-income countries.

Regarding profit in high-income countries, we assume that profit per DALY is half of income per capita (see Appendix F for a detailed explanation). To summarize, the profit would be $24,000 per DALY ($\pi_{\text{h}}$/DALY$_{\text{h}}$). Consider a high-income country like the United Kingdom, where the gross national income per capita of $48,000 is at the mean for high-income countries (World Bank 2025). Profit of $24,000 per life year gained is consistent with the well-known threshold used by health authorities in the United Kingdom.[5]

We assume 4 years of no profit due to testing, followed by 10 years of profit accrued at the beginning of each year. We use an annual discount rate of 16%, reflecting the mean for 2,500 firms based on earnings calls (Gormsen and Huber 2023).[6] We calculate the present value of profit per drug, π, at the beginning of the 10-year profit period as follows:

$$\pi = \sum_{t=1}^{10} \frac{\text{share} \times \text{DALY}_{\text{h}} \times (\pi_{\text{h}}/\text{DALY}_{\text{h}})}{(1+\hat{\delta})^{t-1}}. \tag{31}$$

We calculate the present value of consumer welfare by summing up ten years with no discount:

$$\omega = 10 \times \text{share} \times [\text{DALY}_{\text{l}} \times (\omega_{\text{l}}/\text{DALY}_{\text{l}}) + \text{DALY}_{\text{h}} \times (\omega_{\text{h}}/\text{DALY}_{\text{h}}))]. \tag{32}$$

Based on these parameters, we determine the optimal funding mechanism according to Propositions 1-7.

5.1. Calibration Results

Malaria and tuberculosis each caused a loss of more than 5 million DALYs in low- and middle-income countries in 2021. However, malaria and tuberculosis have different burdens in high-income countries, differing by two orders of magnitude (DALY$_{\text{h}}$=3,000 versus DALY$_{\text{h}}$=716,000) (Table 2).

[5] Health authorities in England will pay no more than £30,000 ($39,000) per quality-adjusted life year gained. If about a third of that is variable costs, then the remaining two-thirds ($26,000) is profit, which is consistent with our assumption of $24,000 in profit. For more on the threshold in England, see `https://www.gov.uk/guidance/cost-utility-analysis-health-economic-studies`.

[6] With a four-year delay before the firm receives payment, the resulting discount factor is slightly above 0.5, satisfying Assumption 1.

Disease	Type	Neglected	Annual DALYs (000) by income group High ($DALY_h$)	Low+Middle ($DALY_l$)
Acute hepatitis B	Viral	0	82	1969
Ascariasis	Parasitic	1	0	681
Chagas disease	Parasitic	1	8	170
Cysticercosis	Parasitic	1	43	1063
Dengue	Viral	1	7	1742
Food-borne trematodes	Parasitic	1	33	861
Gonorrhoea	Bacterial	0	5	68
HIV/AIDS	Viral	0	2419	37749
Hookworm disease	Parasitic	1	0	534
Leishmaniasis	Parasitic	1	13	857
Leprosy	Bacterial	1	0	28
Lymphatic filariasis	Parasitic	1	3	1320
Malaria	Parasitic	0	3	52046
Onchocerciasis	Parasitic	1	0	1307
Rabies	Viral	1	9	2588
Schistosomiasis	Parasitic	1	4	1851
Syphilis	Bacterial	0	30	6111
Trachoma	Bacterial	1	0	134
Trichomoniasis	Parasitic	0	36	236
Trichuriasis	Parasitic	1	0	188
Trypanosomiasis	Parasitic	1	0	69
Tuberculosis	Bacterial	0	716	59867

Table 2 The harm from communicable diseases can be measured as disability-adjusted life years lost (DALYs). Source: authors' calculations using 2021 data on DALYs from the World Health Organization available at `https://www.who.int/data/gho/data/`.

Because malaria and tuberculosis have such different disease burdens in high-income countries, they have different potentials for commercial success and thus need different incentives. For malaria, the optimal policy is mainly pull with some push, regardless of the testing cost (Table 3).

Like malaria, Chagas disease is best addressed by pull with some push, for typical values of testing costs. For Chagas, if testing costs are below about $500 million, the funder should primarily pull with some push. If testing costs were greater, funding would not be optimal. Chagas disease is representative of many of the tropical diseases we study. For most tropical diseases, the optimal mechanism is mostly pull, with some complementary push if testing costs are sufficiently low, and no funding otherwise (Figure 5).

Within a given policy, there is variation in the magnitude of funding. For example, malaria and Chagas disease have the same optimal policy: mostly pull funding with push funding for partial testing costs. However, they have different magnitudes of pull funding. Malaria merits much bigger pull funding because of its much bigger disease burden. For a malaria product that reduces the disease

burden by 10% and has testing costs of $400 million, pull funding should be more than two billion. For a testing cost of $100 million, the pull reward should be about $800 million.

Tuberculosis merits a different approach than malaria, due to its higher prevalence in high-income countries (Table 2), which increases profit potential. Here, the funder should offer insurance, covering a share of the losses in failure rather than rewarding success, because the commercial market will reward success, assuming that testing costs are below about half a billion dollars. If the testing costs exceed one billion dollars, then partial push funding is needed.

HIV/AIDS is an extreme case in which the large commercial market is likely sufficient to incentivize testing of promising products. We estimate that the present value of profit is about $18 billion for an HIV/AIDS drug or vaccine that alleviates 10% of the disease burden in high-income countries. Our assumption about the commercial potential for an HIV drug is consistent with the actual market. Indeed, sales of Biktarvy were nearly $13 billion in 2023.[7] Hence, the commercial market is sufficient if testing costs are below $2.4 billion. If testing costs are more than two billion, then the funder should provide insurance, partially reimbursing the firm when the firm fails (Table 3).

	Trypanosomiasis	**Malaria**	**Tuberculosis**	**HIV**
(1) Nothing	111 to 5,000	-	-	-
(2) Pull+push	1 to 111	1 to 5,000	3,540 to 5,000	-
(3) Push	-	-	1,504 to 3,540	-
(4) Push+insure	-	-	1,179 to 1,504	-
(5) Insure	-	-	635 to 1,179	-
(6) Insure less	-	-	192 to 635	2,422 to 5,000
(7) Commercial	-	-	1 to 192	1 to 2,422

Table 3 Critical values of testing costs (millions of US dollars). Testing costs are assumed to be at least $1 million and at most $5 billion. Source: authors' calculations.

5.2. Calibration Robustness Check

The calibration results are robust to changes in the parameter assumptions.

First, recall that the results are robust to different assumptions about testing cost. For malaria, the optimal policy is mainly pull with a complementary push for any testing cost within a reasonable range (Table 3). Funding for most tropical diseases, including malaria, is a mix of pull and push, unless the testing cost is prohibitively high. For HIV, the optimal policy is to leave it to the commercial market unless testing costs are incredibly high, in which case insurance is justified.

Second, the results are robust to changes in discount rates and testing periods. While we assume a 16% discount rate, reflecting the average for a large sample of firms (Gormsen and Huber 2023), we could instead assume 10.5%, a reasonable assumption for large firms (DiMasi et al. 2016). Assuming

[7] Sales data are based on the authors' calculation using IQVIA MIDAS data.

	Testing cost 400				Testing cost 100			
	Policy	**Push**	**Pull**	**Insure**	**Policy**	**Push**	**Pull**	**Insure**
Leprosy		0	0	0	(1) Nothing	0	0	0
Gonorrhoea		0	0	0		70	21	0
Trachoma	(1) Nothing	0	0	0		70	84	0
Trichuriasis		0	0	0		70	103	0
Trypanosomiasis		0	0	0		70	58	0
Ascariasis		280	393	0		70	180	0
Chagas disease		280	121	0		70	13	0
Dengue		280	529	0		70	174	0
Hookworm disease		280	347	0	(2) Pull+push	70	165	0
Leishmaniasis		280	311	0		70	44	0
Lymphatic filariasis		280	510	0		70	201	0
Malaria	(2) Pull+push	280	2082	0		70	812	0
Onchocerciasis		280	538	0		70	235	0
Rabies		280	607	0		70	184	0
Schistosomiasis		280	573	0		70	211	0
Syphilis		280	615	0		70	14	0
Food-borne trematodes		280	85	0	(4) Push+insure	10	0	108
Cysticercosis		280	7	0	(5) Insure	0	0	127
Acute hepatitis B	(3) Push	280	0	0	(6) Insure less	0	0	71
Trichomoniasis	(4) Push+insure	265	0	27		0	0	85
Tuberculosis	(6) Insure less	0	0	377	(7) Commercial	0	0	0
HIV/AIDS	(7) Commercial	0	0	0		0	0	0

Table 4 Optimal policies depend on disease characteristics. Units are millions of dollars. We round the present value of profits for drugs to $1 million if they are less than that amount. Source: authors' calculations using 2021 data on DALYs from the World Health Organization available at `https://www.who.int/data/gho/data/`.

a 10.5% discount rate and a five-year testing period yields a discount factor of approximately 0.6, satisfying Assumption 1 and leaving the results unchanged. For testing longer than 5 years and an interest rate of 12% or more, the discount factor would fall below 0.5. While we do not provide proofs for the optimal policies when the discount factor is below 0.5, we can numerically compute the policies in a calibration exercise.

Third, the results are robust to reducing the profit potential for bacterial diseases by half. We check the effects of lower profit for antibiotics, because healthcare providers often reserve new antibacterials for emergencies. We study five bacterial diseases: gonorrhoea, leprosy, syphilis, trachoma, and tuberculosis. The optimal policy categories for four of the bacterial diseases are unchanged. For tuberculosis, the optimal policy changes from less insurance to more insurance. It is only if the profit is reduced by more than 60%, the optimal policy shifts from insurance to push funding.

Fourth, without moral hazard, the funder offers a mix of push funding and insurance. For products with limited profit potential, the funder favors push funding. We report the results in Table 5.

Fifth, without adverse selection, the funder offers a mix of push and pull funding. We assume the probability of success θ is fixed at 0.45 (Aitken et al. 2024) and report the results in Table 5.[8]

[8] Without adverse selection, the model sets the success–probability threshold so high that the optimal late-stage subsidy for tuberculosis is driven to zero. Given tuberculosis' high disease burden, a lower threshold and greater funding would be warranted.

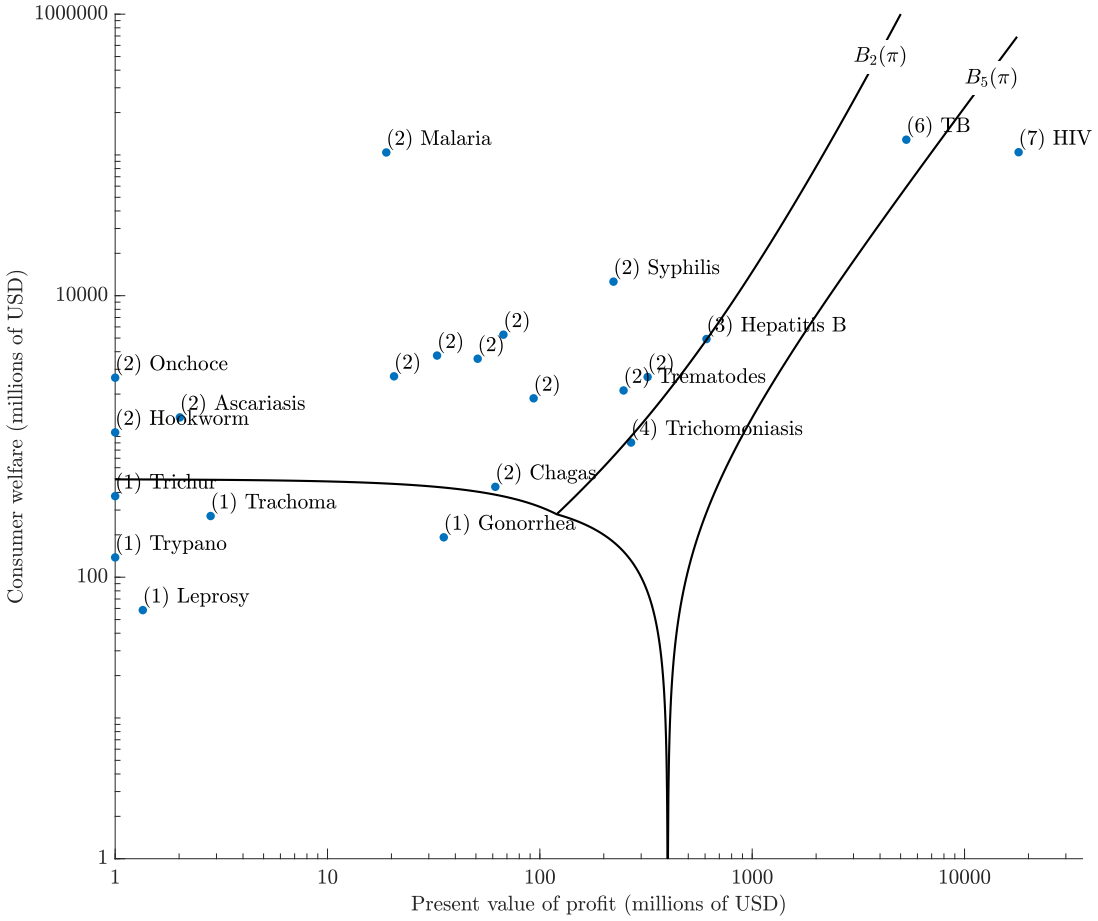


Figure 5 Optimal funding mechanisms for selected communicable diseases. The testing cost is $400 million, and the number preceding each disease name indicates the corresponding policy. Source: authors' calculations using 2021 data on DALYs from the World Health Organization at `https://www.who.int/data/gho/data/`.

Sixth, the results remain robust when accounting for competition in the HIV/AIDS market, because the optimal policy for HIV/AIDS remains no funder contribution for product development. For HIV/AIDS, profit potential is sufficient to motivate development, but HIV/AIDS is exceptional among the diseases we study. For the other diseases, we assume no competition. After all, the World Health Organization considers most tropical diseases to be neglected, meaning that few firms develop new products.

Finally, if the funder assumes a value greater than $2,000 per DALY in low- and middle-income countries, higher funding levels would be justified. This increased funding would encourage participation by more firms, including those with lower probabilities of success.

	No moral hazard			No adverse selection		
	Push	**Pull**	**Insure**	**Push**	**Pull**	**Insure**
Gonorrhoea	-	-	-	-	-	-
Leprosy	-	-	-	-	-	-
Trachoma	-	-	-	-	-	-
Trichuriasis	-	-	-	-	-	-
Trypanosomiasis	-	-	-	-	-	-
Ascariasis	398	-	4	280	479	-
Chagas disease	338	-	112	-	-	-
Cysticercosis	78	-	583	255	-	-
Dengue	349	-	92	280	391	-
Food-borne trematodes	152	-	449	280	34	-
Hookworm disease	399	-	1	-	-	-
Leishmaniasis	307	-	169	280	314	-
Lymphatic filariasis	379	-	37	280	446	-
Malaria	381	-	34	280	449	-
Onchocerciasis	400	-	-	280	483	-
Rabies	333	-	122	280	361	-
Schistosomiasis	367	-	59	280	423	-
Syphilis	178	-	402	280	81	-
Trichomoniasis	131	-	487	279	-	-
Acute hepatitis B	-	-	657	126	-	-
Tuberculosis	-	-	377	-	-	-
HIV/AIDS	-	-	-	-	-	-

Table 5 Optimal policies without moral hazard or adverse selection. Without loss of adverse selection, we assume that the probability of success θ is 0.45. Units are millions of dollars. Source: authors' calculations assuming testing costs of $400 million and using 2021 data on DALYs from the World Health Organization available at `https://www.who.int/data/gho/data/`.

6. Conclusion

We analyze optimal push, pull, and insurance mechanisms in a context where expected profits are nearly always negative, yet the funder has a strong incentive to invest. We present a model of optimal funding that addresses both moral hazard and adverse selection. Using linear optimization and duality theory, we derive analytical expressions for optimal funding mechanisms under various parameters.

Our analysis prescribes how funders should optimize three tools: insurance, pull funding, and push funding. Insurance mitigates adverse selection by reducing firms' incentives to under-report a product's potential to secure more push funding. Push funding is often necessary because firms discount the future more heavily than funders. However, contrary to current practice, push funding should represent only a small share of total funding. Push funding and insurance are subject to moral hazard, so funders should cap these payments at the cost of shirking. Pull funding, by tying rewards to success, mitigates moral hazard and should be used more extensively.

Funders should use up to two of the three tools for each disease. However, funders should not combine insurance with pull funding, as this is equivalent to push funding but more costly due to firms' higher discount rates.

Our findings challenge prevailing global health practices in three ways. First, pull funding is underused for most tropical diseases. We estimate that a $500 million pull mechanism is optimal for high-burden tropical diseases, including dengue, lymphatic filariasis, onchocerciasis, and schistosomiasis. Priority review vouchers currently offer only about $100 million per product (Ridley et al. 2021), so additional pull funding is needed. Achieving the optimal value requires implementing an advance market commitment (Kremer et al. 2022), reforming the voucher program, and/or establishing a European voucher program (Ridley et al. 2024). For malaria, an advance market commitment of more than a billion dollars is justified by its massive disease burden.

Second, funders should use a different approach for malaria than for tuberculosis. In 2022, the Gates Foundation invested $152 million in R&D for malaria and $155 million for tuberculosis (Impact Global Health 2023). We show that funders should offer large pull funding for malaria, and use insurance for tuberculosis, given its commercial potential. However, if the commercial potential for a tuberculosis drug is lower than we estimate, pull funding for tuberculosis would be justified (Table 3).

Third, due to strong commercial incentives, HIV/AIDS product development does not require as much funding to motivate late-stage clinical testing as malaria and other diseases. Yet governments and foundations spent more in 2023 on R&D for HIV/AIDS than on any other infectious or parasitic disease (Impact Global Health 2024). Governments and foundations should shift product development funding away from HIV/AIDS to product development for other diseases, to basic science, and to purchasing existing HIV/AIDS drugs for low-income people. In general, funders should focus on beneficial but unprofitable activities.

While this study focuses on global health, our framework could extend to other sectors, such as renewable energy. By strategically combining push, pull, and insurance mechanisms, governments and nonprofits can align incentives, foster innovation, and save lives.

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Appendix

Appendix A: Proofs of Section 2

Proof of Lemma 1. For any feasible mechanism $\Xi \equiv \{q(\cdot), m_0(\cdot), m_S(\cdot), m_F(\cdot)\} \in \Omega$, we propose the following mechanism $\tilde{\Xi} \equiv \{\tilde{q}(\cdot), \tilde{m}_0(\cdot), \tilde{m}_S(\cdot), \tilde{m}_F(\cdot)\}$, where

$$\begin{aligned}
&\tilde{q}(\theta) = q(\theta), && \forall \theta \in \Theta,\\
&\tilde{m}_0(\theta) = m_0(\theta) + \delta \min\{m_S(\theta), m_F(\theta)\}, && \forall \theta \in \Theta,\\
&\tilde{m}_S(\theta) = m_S(\theta) - \min\{m_S(\theta), m_F(\theta)\}, && \forall \theta \in \Theta,\\
&\tilde{m}_F(\theta) = m_F(\theta) - \min\{m_S(\theta), m_F(\theta)\}, && \forall \theta \in \Theta.
\end{aligned}$$

With a slight abuse of notation, we use $\hat{u}_1(\theta, \hat{\theta}; \Xi)$ and $\hat{u}_0(\theta; \Xi)$ to denote the firm's expected payoff when its true type is θ and it reports $\hat{\theta}$, and when it reports θ and shirks, respectively, both under mechanism Ξ. It is straightforward to verify that the firm's payoff functions remain the same, i.e.,

$$\begin{aligned}
\hat{u}_1(\theta, \hat{\theta}; \tilde{\Xi}) =& -c\,\tilde{q}(\hat{\theta}) + \tilde{m}_0(\hat{\theta}) + \delta\,\theta\left[\tilde{m}_S(\hat{\theta}) + \pi\,\tilde{q}(\hat{\theta})\right] + \delta\,(1-\theta)\,\tilde{m}_F(\hat{\theta})\\
=& -c\,q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\,\theta\left[m_S(\hat{\theta}) + \pi\,q(\hat{\theta})\right] + \delta\,(1-\theta)\,m_F(\hat{\theta})\\
=& \hat{u}_1(\theta, \hat{\theta}; \Xi)
\end{aligned}$$

and

$$\hat{u}_0(\hat{\theta}; \tilde{\Xi}) = -\alpha\,c\,\tilde{q}(\hat{\theta}) + \tilde{m}_0(\hat{\theta}) + \delta\,\tilde{m}_F(\hat{\theta}) = -\alpha\,c\,q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\,m_F(\hat{\theta}) = \hat{u}_0(\hat{\theta}; \Xi).$$

By construction, we know that $\tilde{m}_0(\cdot)$, $\tilde{m}_S(\cdot)$, and $\tilde{m}_F(\cdot)$ are all nonnegative. Therefore, the proposed mechanism $\tilde{\Xi}$ is feasible.

It remains to verify that the funder's expected payoff under truth-telling is weakly higher, which follows from

$$\begin{aligned}
u_F(\theta; \tilde{\Xi}) &= \theta\,\omega\,\tilde{q}(\theta) - \tilde{m}_0(\theta) - \theta\,\tilde{m}_S(\theta) - (1-\theta)\,\tilde{m}_F(\theta)\\
&= \theta\,\omega\,q(\theta) - m_0(\theta) - \theta\,m_S(\theta) - (1-\theta)\,m_F(\theta) + (1-\delta)\min\{m_S(\theta), m_F(\theta)\}\\
&\geq u_F(\theta; \Xi).
\end{aligned}$$

This completes the proof. □

Appendix B: Proofs of Section 3

By dividing the objective function and both sides of constraints (IC) and (IR) by c, we can, without loss of generality, normalize the model by setting $c = 1$ and replacing the parameters (c, ω, π) with $(1, \omega/c, \pi/c)$.

For notational simplicity, we set $c = 1$ for the remainder of the appendix. Accordingly, the optimality conditions and solutions presented in the main text apply with c replaced by 1. For instance, the boundary B_1 defined in equation (16) simplifies to

$$B_1(\pi) \equiv \frac{\delta^2\,\pi^3 - (1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi}{2\,(1-\alpha)^2}.$$

We omit other resulting simplified expressions for brevity.

Reformulation of the funder's Problem (14). We use $u(\theta)$ to denote the firm's expected payoff when it reports its type truthfully and exerts effort, i.e.,

$$u(\theta) \equiv u_1(\theta, \theta), \quad \forall \theta \in \Theta.$$

The (IC) constraint can be rewritten as two linear inequalities, i.e.,

$$u(\theta) \geq u_1(\theta, \hat{\theta}), \quad \forall (\theta, \hat{\theta}) \in \Theta^2 \tag{EC.1}$$

and

$$u(\theta) \geq u_0(\hat{\theta}), \quad \forall (\theta, \hat{\theta}) \in \Theta^2. \tag{EC.2}$$

The following lemma is standard in mechanism design and allows us to eliminate the decision variable m_0 (See Krishna (2009), Chapter 5).

LEMMA EC.1. *The incentive constraint* (EC.1) *is equivalent to*

$$m_S(\cdot) - m_F(\cdot) + \pi\, q(\cdot) \text{ nondecreasing} \tag{EC.3}$$

and

$$m_0(\theta) = u(\theta_L) + \delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y + q(\theta) - \delta\, \theta\, [m_S(\theta) + \pi\, q(\theta)] - \delta\, (1-\theta)\, m_F(\theta). \tag{EC.4}$$

Proof of Lemma EC.1 First, we prove constraint (EC.1) implies (EC.3) and (EC.4). Constraint (EC.1) can be rewritten as

$$u(\theta) \geq u(\hat{\theta}) + \delta\, (\theta - \hat{\theta})\, \left[m_S(\hat{\theta}) - m_F(\hat{\theta}) + \pi\, q(\hat{\theta}) \right].$$

Flipping the role of θ and $\hat{\theta}$ yields

$$u(\hat{\theta}) \geq u(\theta) + \delta\, (\hat{\theta} - \theta)\, [m_S(\theta) - m_F(\theta) + \pi\, q(\theta)].$$

Combining the two inequalities above yields

$$\delta\, (\theta - \hat{\theta})\, \left[m_S(\hat{\theta}) - m_F(\hat{\theta}) + \pi\, q(\hat{\theta}) \right] \leq u(\theta) - u(\hat{\theta}) \leq \delta\, (\theta - \hat{\theta})\, [m_S(\theta) - m_F(\theta) + \pi\, q(\theta)],$$

implying the monotonicity condition (EC.3).

Invoking envelope theorem yields

$$u'(\theta) = \delta\, [m_S(\theta) - m_F(\theta) + \pi\, q(\theta)],$$

which implies that

$$u(\theta) = u(\theta_L) + \delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y.$$

Solving for m_0 yields (EC.4).

To prove the other direction, without loss of generality, fix $0 \leq \hat{\theta} < \theta \leq 1$, and we have

$$\begin{aligned}
u(\theta) = &\ u(\theta_L) + \delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y \\
= &\ u(\theta_L) + \delta \int_{\theta_L}^{\hat{\theta}} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y + \delta \int_{\hat{\theta}}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y \\
= &\ u(\hat{\theta}) + \delta \int_{\hat{\theta}}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y \\
\geq &\ u(\hat{\theta}) + \delta\, (\theta - \hat{\theta})\, \left[m_S(\hat{\theta}) - m_F(\hat{\theta}) + \pi\, q(\hat{\theta}) \right],
\end{aligned}$$

where the last inequality follows from the monotonicity condition (EC.3).

This completes the proof. □

The "envelope" condition (EC.4) allows us to eliminate m_0 and rewrite the funder's problem in terms of the "virtual valuation" functions defined by

$$\begin{aligned} w_q(\theta) &\equiv \theta\,\omega + \delta\,\pi\,\theta - 1 - \delta\,\pi\,\frac{1-F(\theta)}{f(\theta)}, \\ w_S(\theta) &\equiv -(1-\delta)\,\theta - \delta\,\frac{1-F(\theta)}{f(\theta)}, \text{ and} \\ w_F(\theta) &\equiv -(1-\delta)\,(1-\theta) + \delta\,\frac{1-F(\theta)}{f(\theta)}, \end{aligned}$$

The funder's problem (14) can now be reformulated as follows:

$$\max_{\{q(\cdot),m_S(\cdot),m_0(\cdot),u(\theta_L)\}\in\Omega_1} -u(\theta_L) + \int_{\theta_L}^{\theta_H} \left[w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)\right] f(\theta)\,\mathrm{d}\theta, \tag{EC.5}$$

where the feasible set Ω_1 is defined by (8), (EC.3), and the following linear inequality constraints: for any $(\theta,\hat{\theta})\in\Theta^2$,

$$\begin{aligned} &\delta \int_{\theta_L}^{\theta} \left[m_S(y) - m_F(y) + \pi\,q(y)\right]\mathrm{d}y \\ &\quad\geq \delta \int_{\theta_L}^{\hat{\theta}} \left[m_S(y) - m_F(y) + \pi\,q(y)\right]\mathrm{d}y + (1-\alpha)\,q(\hat{\theta}) - \delta\,\hat{\theta}\,\left[m_S(\hat{\theta}) + \pi\,q(\hat{\theta})\right] + \delta\,\hat{\theta}\,m_F(\hat{\theta}), \end{aligned} \tag{EC.6}$$

$$u(\theta_L) + \delta \int_{\theta_L}^{\theta} \left[m_S(y) - m_F(y) + \pi\,q(y)\right]\mathrm{d}y \geq 0, \tag{EC.7}$$

$$u(\theta_L) + \delta \int_{\theta_L}^{\theta} \left[m_S(y) - m_F(y) + \pi\,q(y)\right]\mathrm{d}y + q(\theta) - \delta\,\theta\,\left[m_S(\theta) + \pi\,q(\theta)\right] - \delta\,(1-\theta)\,m_F(\theta) \geq 0, \text{ and} \tag{EC.8}$$

$$m_S(\theta),\ m_F(\theta) \geq 0. \tag{EC.9}$$

The inequalities (EC.6), (EC.7), and (EC.8) are the moral hazard constraint, the individual rationality constraint, and the feasibility constraint on m_0, respectively.

Proof of Proposition 1. Notice that our candidate primal solution is always feasible.

The remaining proof is partitioned into two parts: (1) $\omega + \delta\,\pi \leq 1$; and (2) $1 < \omega + \delta\,\pi \leq (1-\alpha+\delta\,\alpha) + (1-\delta)\,\omega$.

For part (1), define

$$\begin{aligned} &\chi_L \equiv 1 \\ &\eta(\theta) \equiv \frac{1}{2\,\delta}\left(\frac{2\,\delta-1}{\theta^2}+1\right) - 1, && \forall\theta\in(0,1], \\ &\xi(\theta) \equiv -\left[w_S(\theta) + w_F(\theta)\right] f(\theta), && \forall\theta\in[0,1], \text{ and} \\ &\beta(\theta) \equiv (1-\alpha-\delta\,\pi\,\theta)\,\eta(\theta) + \delta\,\pi\int_{\theta}^{1}\eta(y)\,\mathrm{d}y - w_q(\theta)\,f(\theta), && \forall\theta\in(0,1]. \end{aligned}$$

Because

$$w_S(\theta) + w_F(\theta) = -1 + \delta \leq 0,$$

the nonnegativity of ξ holds automatically. By Assumption 1, we know that η is decreasing, implying that for any $\theta\in[0,1]$,

$$\eta(\theta) \geq \eta(1) = 0.$$

Differentiating β yields

$$\beta'(\theta) = -2\,\delta\,\pi\,\eta(\theta) + (1-\alpha-\delta\,\pi\,\theta)\;\eta'(\theta) - \left[w_q'(\theta)\,f(\theta) + w_q(\theta)\,f'(\theta)\right] = -\frac{(1-\alpha)\,(2\,\delta-1)}{\delta\,\theta^3} - (\omega+\pi) \leq 0,$$

and hence, the nonnegativity of β boils down to

$$\beta(1) = 1-\omega-\delta\,\pi \geq 0,$$

which is implied by the optimality condition.

We have

$$\begin{aligned}
&\int_0^1 \eta(\theta)\,\left[\delta\int_0^\theta \left[m_S(y) - m_F(y) + \pi\,q(y)\right]\mathrm{d}y + (1-\alpha)\,q(\theta) - \delta\,\theta\,\left[m_S(\theta) + \pi\,q(\theta)\right] + \delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta \\
&+ \chi_L\,\left[-u(0)\right] + \int_0^1 \beta(\theta)\,\left[-q(\theta)\right]\mathrm{d}\theta + \int_0^1 \xi(\theta)\,\left[-m_S(\theta)\right]\mathrm{d}\theta \\
=& -u(0) + \int_0^1 \left[w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)\right]\,f(\theta)\,\mathrm{d}\theta.
\end{aligned}$$

Therefore, the objective function is bounded by 0. Substituting our conjectured primal solution into the objective function also yields 0, establishing the optimality.

For part (2), let

$$\begin{aligned}
\Psi(\theta) \equiv\; & 2\,\delta^2\,\pi\,(\omega+\pi)\,\theta^3 \\
& - \delta\,\left[2\,\delta\,\pi^2\,(1-\delta) + (1-\alpha)\,(3\,\omega+4\,\pi) + 2\,\delta\,\alpha\,\pi\right]\,\theta^2 \\
& + 2\,(1-\alpha)\,\left[2\,\delta\,\alpha + 2\,\delta\,\pi\,(1-\delta) + (1-\alpha)\right]\,\theta \\
& + \delta\,(1-\alpha)\,\omega - 2\,\delta\,\alpha\,(1-\alpha) - 2\,(1-\delta)\,(1-\alpha)^2
\end{aligned}$$

Define

$$\begin{aligned}
&\chi_L \equiv 1, \\
&\eta(\theta) \equiv \begin{cases} \dfrac{1}{2\,\delta}\,\left(\dfrac{C_2}{\theta^2}+1\right) - 1, & \forall\theta\in(0,\tau), \\ \dfrac{C_1}{2\,\delta\,\pi\,\left(\alpha-1+\delta\,\pi\,\theta\right)^2} - \dfrac{\omega}{2\,\delta\,\pi} - 1, & \forall\theta\in[\tau,1], \end{cases} \\
&\xi(\theta) \equiv \begin{cases} -\left[w_S(\theta)+w_F(\theta)\right]\,f(\theta), & \forall\theta\in[0,\tau), \\ -\delta\,\theta\,\eta(\theta) + \delta\int_\theta^1 \eta(y)\,\mathrm{d}y - w_S(\theta)\,f(\theta), & \forall\theta\in[\tau,1], \end{cases} \\
&\beta(\theta) \equiv (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) + \delta\,\pi\int_\theta^1 \eta(y)\,\mathrm{d}y - w_q(\theta)\,f(\theta), \qquad \forall\theta\in(0,\tau], \text{ and} \\
&\nu(\theta) \equiv \delta\,\theta\,\eta(\theta) - \delta\int_\theta^1 \eta(y)\,\mathrm{d}y - w_F(\theta)\,f(\theta), \qquad \forall\theta\in[\tau,1],
\end{aligned}$$

where

$$\begin{aligned}
&\tau = \sup\left\{\theta\in\Theta : \Psi(\theta) = 0\right\}, \\
&C_1 = -\left(\delta\,\pi - 1 + \alpha\right)\,\left(\omega + \delta\,\pi\,\omega - \alpha\,\omega - 2\,\delta\,\alpha\,\pi\right), \text{ and} \\
&C_2 = \frac{\tau\,\left[\delta\,\left(\omega+\pi\right)\,\tau^2 + \left(\alpha - 2\,\delta\,\pi - 1 + 2\,\delta^2\,\pi - 2\,\alpha\,\delta\right)\,\tau + \left(2 - \delta\,\omega - 2\,\alpha - 2\,\delta + 4\,\alpha\,\delta\right)\right]}{\alpha - 1 + \delta\,\pi\,\tau}.
\end{aligned}$$

The threshold τ is chosen such that η is continuous, i.e.,

$$\lim_{\theta\uparrow\tau}\eta(\theta)=\lim_{\theta\downarrow\tau}\eta(\theta).$$

First, we prove that $C_1\geq 0$. We have

$$\delta\,\pi-1+\alpha\leq\delta\cdot\frac{1-\alpha}{\delta}-1+\alpha=0.$$

Fix π, and the second term $(\delta\,\pi+1-\alpha)\,\omega-2\,\delta\,\alpha\,\pi$ is increasing in ω. Hence, its minimum is attained at $\omega=1-\delta\,\pi$. We have

$$(\delta\,\pi+1-\alpha)\,\omega-2\,\delta\,\alpha\,\pi\geq h(\omega)\equiv(-\omega+2-\alpha)\,\omega-2\,\alpha\,(1-\omega)=-\omega^2+(\alpha+2)\,\omega-2\,\alpha.$$

The minimum on $\left[\alpha,\frac{1}{\delta}\,(1-\alpha+\delta\,\alpha)\right]$ is attained at one of the two endpoints, and we have

$$h(\alpha)=0$$

and

$$h\left(\frac{1}{\delta}\,(1-\alpha+\delta\,\alpha)\right)=\frac{(1-\alpha)\overbrace{(\alpha+2\,\delta-\alpha\,\delta-1)}^{=\alpha\,(1-\delta)+(2\,\delta-1)\geq 0}}{\delta^2}\geq 0.$$

Therefore, the constant C_1 is nonnegative, and the multiplier η is nondecreasing on $[\tau,1]$.

Next, we verify the existence of τ on $[0,1]$. We have

$$\Psi(1)=2\,\delta\,\underbrace{(\omega+\delta\,\pi-1)}_{\geq 0}\,\underbrace{(\alpha+\delta\,\pi-1)}_{\leq 0}\leq 0$$

and

$$\Psi'(\theta)=2\,\underbrace{(\alpha+\delta\,\pi\,\theta-1)}_{\leq\alpha+\delta\,\frac{1-\alpha}{\delta}-1=0}\,\underbrace{[3\,\delta\,(\omega+\pi)\,\theta-2\,\delta\,\pi\,(1-\delta)-\alpha\,(2\,\delta-1)-1]}_{\text{increasing in }\theta}$$

Because $\Psi'(0)>0>\Psi'(1)$, where the second inequality follows from

$$\begin{aligned}3\,\delta\,(\omega+\pi)-2\,\delta\,\pi\,(1-\delta)-\alpha\,(2\,\delta-1)-1&=3\,\delta\,(\omega+\delta\,\pi)+\delta\,\pi\,(1-\delta)-\alpha\,(2\,\delta-1)-1\\&\geq 3\,\delta+\delta\,\pi\,(1-\delta)-\alpha\,(2\,\delta-1)-1\\&\geq 3\,\delta-\alpha\,(2\,\delta-1)-1\\&\geq 3\,\delta-(2\,\delta-1)-1\\&=\delta>0,\end{aligned}$$

function Ψ is increasing and decreasing on $\left(-\infty,\frac{2\,\delta\,\pi\,(1-\delta)+\alpha\,(2\,\delta-1)+1}{3\,\delta\,(\omega+\pi)}\right]$ and $\left[\frac{2\,\delta\,\pi\,(1-\delta)+\alpha\,(2\,\delta-1)+1}{3\,\delta\,(\omega+\pi)},1\right]$, respectively.

To prove the existence of τ, it suffices to show that there exists $\hat{\theta}\in(-\infty,1]$ such that $\Psi(\hat{\theta})\geq 0$:

1. if $\hat{\theta}\geq 0$, the existence of a zero point for the function Ψ on the interval $[0,1]$ follows directly; and
2. if $\hat{\theta}<0$, we know that

$$\Psi\left(\frac{2\,\delta\,\pi\,(1-\delta)+\alpha\,(2\,\delta-1)+1}{3\,\delta\,(\omega+\pi)}\right)\geq\Psi(\hat{\theta})\geq 0,$$

and hence the desired result still holds.

Because η is increasing on $[\tau,1]$, we have

$$\eta(\tau)\geq 0 \quad \Leftrightarrow \quad \tau\geq \frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_1}{\omega+2\,\delta\,\pi}}\right) \quad \Leftarrow \quad \Psi\left(\frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_1}{\omega+2\,\delta\,\pi}}\right)\right)\geq 0,$$

where the last step is implied by the monotonicity of Ψ.

Therefore, to prove the existence of τ and the nonnegativity of η, it suffices to show that

$$\Psi\left(\frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_1}{\omega+2\,\delta\,\pi}}\right)\right)\geq 0,$$

which can be rewritten as

$$\begin{aligned}
&(\omega+\pi)\sqrt{-\frac{(\alpha+\delta\,\pi-1)\,(\omega-\alpha\,\omega+\delta\,\pi\,\omega-2\,\delta\,\pi\,\alpha)}{\omega+2\,\delta\,\pi}}-(1-\alpha)\,(\omega+\pi)+\delta\,\pi\,(1+\pi-\delta\,\pi)\leq 0,\\
\Leftrightarrow\ &-\frac{(\alpha+\delta\,\pi-1)\,(\omega-\alpha\,\omega+\delta\,\pi\,\omega-2\,\delta\,\pi\,\alpha)}{\omega+2\,\delta\,\pi}\leq\left(1-\alpha+\frac{\delta\,\pi\,(\delta\,\pi-\pi-1)}{\omega+\pi}\right)^2,\\
\Leftrightarrow\ &g(\omega)\geq 0,
\end{aligned}$$

where

$$\begin{aligned}
g(\omega)\equiv\ &\delta^2\,\pi^2\,\omega^3+2\,\delta\,\pi^2\,(-2\,\alpha\,\delta+\alpha+\delta+\delta\,\pi-1)\,\omega^2\\
&+\delta\,\pi^2\left[(\delta^3-2\,\delta^2+2\,\delta)\,\pi^2+(2\,\alpha-2\,\alpha\,\delta-4\,\alpha\,\delta^2+2\,\delta^2-2)\,\pi+4\,\alpha\,\delta-3\,\delta-2\,\alpha+2\right]\omega\\
&+2\,\delta\,\pi^3\,(\delta\,\pi-1)\left[\delta\,(1-\delta)^2\,\pi+\alpha+2\,\delta-2\,\alpha\,\delta-\delta^2-1\right].
\end{aligned}$$

Fix $\pi\in\left[0,\frac{1-\alpha}{\delta}\right]$, and the second-order derivative of g is increasing, i.e.,

$$g''(\omega)\geq g''(1-\delta\,\pi)=2\,\delta\,\pi^2\,\left(2\,\alpha+5\,\delta+2\,\delta\,\pi-4\,\alpha\,\delta-3\,\delta\,\pi^2-2\right)\geq 0,$$

which implies that

$$g'(\omega)\geq g'(1-\delta\,\pi)=-2\,\delta\,\pi^2\,(2\,\delta-1)\,(\pi-\delta\,\pi+1)\,(\alpha+\delta\,\pi-1)\geq 0$$

Because $g(1-\delta\,\pi)=0\geq 0$, the desired result holds automatically.

By $\eta(\tau)\geq 0$ and

$$\delta\,[2\,\eta(\theta)+\theta\,\eta'(\theta)]=1-2\,\delta\leq 0,\quad \forall\theta\in[0,\tau],$$

we know that η is decreasing on $[0,\tau]$, guaranteeing its nonnegativity.

It remains to prove the nonnegativity of β, ξ, and ν. For any $\theta\in[0,\tau]$, we have

$$\beta'(\theta)=-2\,\delta\,\pi\,\underbrace{\eta(\theta)}_{\geq 0}+\underbrace{(1-\alpha-\delta\,\pi\,\theta)}_{\geq 0}\,\underbrace{\eta'(\theta)}_{\leq 0}-\underbrace{\left[w_q'(\theta)\,f(\theta)+w_q(\theta)\,f'(\theta)\right]}_{=2\,\delta\,\pi+\omega\geq 0}\leq 0,$$

and the desired property is guaranteed by $\beta(\tau)=0$.

Similarly, for any $\theta\in[\tau,1]$, we have

$$\xi'(\theta)=-\delta\,\left[2\,\underbrace{\eta(\theta)}_{\geq 0}+\theta\,\underbrace{\eta'(\theta)}_{\geq 0}\right]-\underbrace{\left[w_S'(\theta)\,f(\theta)+w_S(\theta)\,f'(\theta)\right]}_{\geq 0}\leq 0,$$

and the desired property boils down to

$$\xi(1)=-\delta\,\eta(1)-w_S(1)\,f(1)=\frac{\delta\,(\omega+\pi)-(1-\alpha+\delta\,\alpha)}{\alpha+\delta\,\pi-1}\geq 0,$$

which is implied by $\delta\,(\omega+\pi)\leq 1-\alpha+\delta\,\alpha$.

Finally, for any $\theta \in [\tau, 1]$, we have

$$\nu'(\theta) = \delta \left[2 \underbrace{\eta(\theta)}_{\geq 0} + \theta \underbrace{\eta'(\theta)}_{\geq 0} \right] - \underbrace{[w_F'(\theta)\, f(\theta) + w_F(\theta)\, f'(\theta)]}_{\leq 0} \geq 0,$$

and hence, the desired property boils down to $\nu(\tau) \geq 0$, which holds automatically.

We have

$$\begin{aligned}
&\int_0^1 \eta(\theta) \left[\delta \int_0^\theta [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y + (1-\alpha)\, q(\theta) - \delta\, \theta\, [m_S(\theta) + \pi\, q(\theta)] + \delta\, \theta\, m_F(\theta) \right] \mathrm{d}\theta \\
&+ \chi_L\, [-u(0)] + \int_0^\tau \beta(\theta)\, [-q(\theta)]\, \mathrm{d}\theta + \xi_L\, [-m_S(0)] + \int_0^1 \xi(\theta)\, [-m_S(\theta)]\, \mathrm{d}\theta + \int_\tau^1 \nu(\theta)\, [-m_F(\theta)]\, \mathrm{d}\theta \\
=& -u(0) + \int_0^1 [w_q(\theta)\, q(\theta) + w_S(\theta)\, m_S(\theta) + w_F(\theta)\, m_F(\theta)]\, f(\theta)\, \mathrm{d}\theta.
\end{aligned}$$

The nonnegativity of all the multipliers we define imply that, the objective function is bounded by 0. Substituting our conjectured primal solution into the objective function also yields 0, establishing the optimality.

This completes the proof. □

Proof of Proposition 2. To verify our candidate primal solution is feasible, we need to prove that

$$\frac{1-\alpha}{\delta\, \tau} - \pi \geq 0 \quad \Leftrightarrow \quad \tau \leq \frac{1-\alpha}{\delta\, \pi}.$$

Define

$$h(\theta) \equiv 2\, \delta\, (\omega + \pi)\, \theta^3 + (\alpha - 2\, \delta\, \alpha - 1)\, \theta^2 + \alpha - 1$$

Taking derivative yields

$$h'(\theta) = 6\, \delta\, (\omega + \pi)\, \theta^2 + 2\, (\alpha - 2\, \delta\, \alpha - 1)\, \theta = 2\, \theta\, \underbrace{[3\, \delta\, (\omega + \pi)\, \theta + (\alpha - 2\, \delta\, \alpha - 1)]}_{\text{increasing in } \theta},$$

and we have $\lim_{\theta \downarrow 0} h'(\theta) < 0$ and

$$h'(1) = 2\, [3\, \delta\, (\omega + \pi) + (\alpha - 2\, \delta\, \alpha - 1)] \geq 2\, [3\, (1 - \alpha + \delta\, \alpha) + (\alpha - 2\, \delta\, \alpha - 1)] > 0$$

Therefore, the derivative h' is first negative and then positive, and function h is first decreasing and then increasing on $[0, 1]$. Because

$$h(0) = -(1-\alpha) \leq 0$$

and

$$h\left(\frac{1-\alpha}{\delta\, \pi} \right) = \frac{1}{\delta^2\, \pi^3} \cdot \left[-\delta^2\, \pi^3 + (1-\alpha)\, (1 - \alpha - 2\, \delta\, \alpha)\, \pi + 2\, \omega\, (1-\alpha)^2 \right] \geq 0,$$

the unique existence of $\tau \in \left[0, \frac{1-\alpha}{\delta\, \pi}\right]$ is guaranteed.

Let

$$\begin{aligned}
\Psi(\hat{\tau}) = \; & 2\, \delta^2\, \pi\, \tau\, (\omega + \pi)\, \hat{\tau}^3 \\
& - \delta\, \tau\, [2\, \delta\, \pi^2\, (1 - \delta) + (1 - \alpha)\, (3\, \omega + 4\, \pi) + 2\, \delta\, \alpha\, \pi]\, \hat{\tau}^2 \\
& + 2\, \tau\, (1 - \alpha)\, [2\, \delta\, \alpha + 2\, \delta\, \pi\, (1 - \delta) + (1 - \alpha)]\, \hat{\tau} \\
& + \delta\, (1 - \alpha)\, (\omega + \pi)\, \tau^3 - (1 - \alpha)\, (1 - \alpha + 2\, \delta\, \alpha)\, \tau^2 \\
& - (1 - \alpha)\, [2\, (1 - \delta)\, (1 - \alpha) + \delta\, \pi]\, \tau + (1 - \alpha)^2.
\end{aligned}$$

Define

$$
\begin{aligned}
&\chi_L \equiv 1,\\
&\eta(\theta) \equiv \begin{cases} \eta_1(\theta) = \dfrac{1}{2\,\delta}\left(\dfrac{C_3}{\theta^2}+1\right)-1, & \forall \theta \in (0,\hat{\tau}),\\ \eta_2(\theta) = \dfrac{C_2}{2\,\delta\,\pi\,(\alpha-1+\delta\,\pi\,\theta)^2} - \dfrac{\omega}{2\,\delta\,\pi} - 1, & \forall \theta \in [\hat{\tau},\tau),\\ \eta_3(\theta) = \dfrac{1}{2\,\delta}\left(\dfrac{1}{\theta^2}+1\right)-1, & \forall \theta \in [\tau,1], \end{cases}\\
&\xi(\theta) \equiv \begin{cases} -\left[w_S(\theta)+w_F(\theta)\right]\,f(\theta), & \forall \theta \in [0,\hat{\tau}],\\ -\delta\,\theta\,\eta_2(\theta) + \delta\,\left[\displaystyle\int_\theta^\tau \eta_2(y)\,\mathrm{d}y + \int_\tau^1 \eta_3(y)\,\mathrm{d}y\right] - w_S(\theta)\,f(\theta), & \forall \theta \in [\hat{\tau},\tau], \end{cases}\\
&\nu(\theta) \equiv \begin{cases} \delta\,\theta\,\eta_2(\theta) - \delta\,\left[\displaystyle\int_\theta^\tau \eta_2(y)\,\mathrm{d}y + \int_\tau^1 \eta_3(y)\,\mathrm{d}y\right] - w_F(\theta)\,f(\theta), & \forall \theta \in [\hat{\tau},\tau],\\ -\left[w_S(\theta)+w_F(\theta)\right]\,f(\theta), & \forall \theta \in [\tau,1], \end{cases}\\
&\beta(\theta) \equiv (1-\alpha-\delta\,\pi\,\theta)\,\eta_1(\theta) + \delta\,\pi\int_\theta^1 \eta(y)\,\mathrm{d}y - w_q(\theta)\,f(\theta), \qquad \forall \theta \in (0,\hat{\tau}], \text{ and}\\
&\sigma(\theta) \equiv -(1-\alpha-\delta\,\pi\,\theta)\,\eta_3(\theta) - \delta\,\pi\int_\theta^1 \eta_3(y)\,\mathrm{d}y + w_q(\theta)\,f(\theta), \qquad \forall \theta \in [\tau,1],
\end{aligned}
$$

where

$$
\begin{aligned}
&\hat{\tau} = \sup\left\{\theta \in \Theta : \Psi(\theta) = 0\right\},\\
&C_2 = \frac{(\alpha-1+\delta\,\pi\,\tau)\,(\delta\,\pi^2 - \delta\,\pi^2\,\tau^2 - \omega\,\tau - \delta\,\pi\,\omega\,\tau^2 + \omega\,\alpha\,\tau + 2\,\delta\,\pi\,\alpha\,\tau)}{\tau}, \text{ and}\\
&C_3 = \frac{\begin{array}{c}\delta\,(\omega+\pi)\,\hat{\tau}\,\tau\,(\hat{\tau}^2-\tau^2) + (\alpha-2\,\delta\,\pi-1+2\,\delta^2\,\pi-2\,\alpha\,\delta)\,\hat{\tau}^2\,\tau + (1-\alpha+2\,\alpha\,\delta)\,\hat{\tau}\,\tau^2\\ +(2+\delta\,\pi-2\,\alpha-2\,\delta+2\,\alpha\,\delta)\,\hat{\tau}\,\tau + (\alpha-1)\,\hat{\tau}\end{array}}{\delta\,\pi\,\hat{\tau}\,\tau - (1-\alpha)\,\tau}.
\end{aligned}
$$

The thresholds $\hat{\tau}$ and τ are chosen such that $\beta(\hat{\tau}) = \sigma(\tau) = 0$, implying the continuity of η at $\hat{\tau}$ and τ, i.e.,

$$
\lim_{\theta\uparrow\hat{\tau}} \eta_1(\theta) = \lim_{\theta\downarrow\hat{\tau}} \eta_2(\theta) \quad \text{and} \quad \lim_{\theta\uparrow\tau} \eta_2(\theta) = \lim_{\theta\downarrow\tau} \eta_3(\theta).
$$

Next, we verify the existence of $\hat{\tau}$ on $[0,\tau]$. Notice that

$$
\begin{aligned}
&\Psi(\tau)\\
&= (\alpha-1+\delta\,\pi\,\tau)\,\left[2\,\delta\,(\omega+\pi)\,\tau^3 + (\alpha-2\,\delta\,\pi-1+2\,\delta^2\,\pi-2\,\alpha\,\delta)\,\tau^2 + 2\,(1-\delta)\,(1-\alpha)\,\tau + \alpha - 1\right]\\
&= \underbrace{(\alpha-1+\delta\,\pi\,\tau)}_{\leq 0}\left[-2\,(1-\delta)\,\tau\,\underbrace{[\delta\,\pi\,\tau-(1-\alpha)]}_{\leq 0}\right]\\
&\leq 0.
\end{aligned}
$$

Differentiating Ψ yields

$$
\Psi'(\theta) = 2\,\tau\,\underbrace{(\alpha-1+\delta\,\pi\,\theta)}_{\leq 0}\,\underbrace{[3\,\delta\,(\omega+\pi)\,\theta - [2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]]}_{\text{increasing in } \theta}, \quad \forall \theta \in [0,\tau].
$$

Because $\Psi'(0) > 0$, function Ψ is either increasing or first increasing and then decreasing on $(-\infty,\tau]$.

To prove the existence of $\hat{\tau}$, it suffices to show that there exists $\hat{\theta} \in (-\infty,\tau]$ such that $\Psi(\hat{\theta}) \geq 0$:

1. if $\hat{\theta} \geq 0$, the existence of a zero point for the function Ψ on $[0,1]$ follows directly; and
2. if $\hat{\theta} < 0$, we know that $\Psi(0) \geq \Psi(\hat{\theta}) \geq 0$, and hence the desired result still holds.

Because

$$-2\,\delta\,\pi\,\eta_2(\theta)+\underbrace{(1-\alpha-\delta\,\pi\,\theta)}_{\geq 0}\,\eta_2'(\theta)=\omega+2\,\delta\,\pi,\quad \forall\theta\in[\hat{\tau},\tau]$$

and $\eta_2(\tau)=\eta_3(\tau)>0$, we know that η_2 is increasing on $[\hat{\tau},\tau]$, and hence,

$$\eta_2(\hat{\tau})\geq 0 \quad\Leftrightarrow\quad \hat{\tau}\geq\frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_2}{\omega+2\,\delta\,\pi}}\right) \quad\Leftarrow\quad \Psi\left(\frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_2}{\omega+2\,\delta\,\pi}}\right)\right)\geq 0,$$

where the last step is implied by the monotonicity of Ψ.

Therefore, to prove the existence of $\hat{\tau}$ and the nonnegativity of η_2, it suffices to show that

$$\Psi\left(\frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_2}{\omega+2\,\delta\,\pi}}\right)\right)\geq 0,$$

which can be rewritten as

$$-(\omega+\pi)^2\,\frac{(\alpha+\delta\,\pi\,\tau-1)\,(\omega\,\tau-\delta\,\pi^2+\delta\,\pi^2\,\tau^2-\alpha\,\omega\,\tau+\delta\,\pi\,\omega\,\tau^2-2\,\delta\,\pi\,\alpha\,\tau)}{\omega+2\,\delta\,\pi}\leq\left[(1-\alpha)\,(\omega+\pi)-\delta\,\pi\,(1+\pi-\delta\,\pi)\right]^2\,\tau,$$

or equivalently,

$$\begin{aligned}&(\omega+2\,\delta\,\pi)\,\left[(1-\alpha)\,(\omega+\pi)-\delta\,\pi\,(1+\pi-\delta\,\pi)\right]^2\,\tau\\&+(\omega+\pi)^2\,(\alpha+\delta\,\pi\,\tau-1)\,(\omega\,\tau-\delta\,\pi^2+\delta\,\pi^2\,\tau^2-\alpha\,\omega\,\tau+\delta\,\pi\,\omega\,\tau^2-2\,\delta\,\pi\,\alpha\,\tau)\geq 0.\end{aligned}$$

We can use the definition of τ to eliminate the constant term, i.e.,

$$1-\alpha=2\,\delta\,(\omega+\pi)\,\tau^3+\delta\,\pi\,(\alpha-2\,\delta\,\alpha-1)\,\tau^2,$$

and the desired result boils down to $g(\omega)\geq 0$, where

$$\begin{aligned}g(\omega)\equiv\;&3\,\delta\,(\omega+\pi)^3\,\tau\,(\omega)^2-2\,(\omega+\pi)^2\,(2\,\delta\,\alpha-\alpha+1)\,\tau\,(\omega)\\&+(\omega+\pi)\,\left[-\delta\,\pi\,(\omega+\pi)+2\,(1-\alpha)\,\left[1-2\,\delta-(1-\delta)\,(\omega+2\,\delta\,\pi)\right]\right]+\delta\,(\omega+2\,\delta\,\pi)\,(1+\pi-\delta\,\pi)^2.\end{aligned}$$

Applying the implicit function theorem to the definition of τ yields

$$\tau'(\omega)=-\frac{\delta\,\tau(\omega)^2}{3\,\delta\,(\omega+\pi)\,\tau(\omega)+(\alpha-2\,\delta\,\alpha-1)}.$$

Because

$$2\,\delta\,(\omega+\pi)\,\tau^3+(\alpha-2\,\delta\,\alpha-1)\,\tau^2=1-\alpha\geq 0,$$

we know that

$$3\,\delta\,(\omega+\pi)\,\tau+(\alpha-2\,\delta\,\alpha-1)>2\,\delta\,(\omega+\pi)\,\tau+(\alpha-2\,\delta\,\alpha-1)\geq 0,$$

and hence, the threshold τ is decreasing in ω.

Next, we will prove the nonnegativity of g in three steps.

LEMMA EC.2. *Function*

$$l(\omega)\equiv 3\,\delta\,(\omega+\pi)^2\,\tau\,(\omega)^2-2\,(\omega+\pi)\,(2\,\delta\,\alpha-\alpha+1)\,\tau\,(\omega)-\delta\,\pi\,(\omega+\pi)+2\,(1-\alpha)\,\left[1-2\,\delta-(1-\delta)\,(\omega+2\,\delta\,\pi)\right]$$

is increasing.

Proof of Lemma EC.2. Taking derivative yields

$$\begin{aligned} l'(\omega) = & \, 3\,\delta\,[2\,(\omega+\pi)\,\tau(\omega)^2 + 2\,(\omega+\pi)^2\,\tau(\omega)\,\tau'(\omega)] - 2\,(2\,\delta\,\alpha - \alpha + 1)\,[\tau(\omega) + (\omega+\pi)\,\tau'(\omega)] \\ & -\delta\,\pi - 2\,(1-\alpha)\,(1-\delta) \\ = & \, 4\,\delta\,(\omega+\pi)\,\tau\,(\omega)^2 + 2\,(\alpha - 2\,\alpha\,\delta - 1)\,\tau\,(\omega) - 2\,(1-\delta)\,(1-\alpha) - \delta\,\pi \\ = & \, \frac{2\,(1-\alpha)}{\tau(\omega)} - 2\,(1-\delta)\,(1-\alpha) - \delta\,\pi, \end{aligned}$$

where the last equality follows from the definition of τ.

When $\pi \geq \frac{1-\alpha}{\delta}$, the threshold τ is bounded by $\frac{1-\alpha}{\delta\,\pi}$, and hence, we have

$$\frac{2\,(1-\alpha)}{\tau} - 2\,(1-\delta)\,(1-\alpha) - \delta\,\pi \geq \delta\,\pi - 2\,(1-\delta)\,(1-\alpha) \geq (1-\alpha)\,(2\,\delta - 1) \geq 0$$

When $0 \leq \pi \leq \frac{1-\alpha}{\delta}$, the threshold τ is bounded by 1, and hence, we have

$$\frac{2\,(1-\alpha)}{\tau} - 2\,(1-\delta)\,(1-\alpha) - \delta\,\pi \geq 2\,\delta\,(1-\alpha) - \delta\,\pi \geq (1-\alpha)\,(2\,\delta - 1) \geq 0$$

Therefore, function l is nondecreasing. □

LEMMA EC.3. *Function g is nondecreasing.*

Proof of Lemma EC.3. Function g can be rewritten in terms of l, i.e.,

$$g(\omega) = (\omega+\pi)\,l(\omega) + \delta\,(\omega + 2\,\delta\,\pi)\,(1 + \pi - \delta\,\pi)^2.$$

Taking derivative yields

$$g'(\omega) = l(\omega) + (\omega+\pi)\,l'(\omega) + \delta\,(1 + \pi - \delta\,\pi)^2.$$

Because both $l(\cdot)$ and $l'(\cdot)$ are increasing, we know that g' is increasing.

1. When $0 \leq \pi \leq \frac{1-\alpha}{\delta}$, we have

$$\begin{aligned} & g'(\omega) \\ \geq & \, g'\left(\frac{1-\alpha+\delta\,\alpha}{\delta} - \pi\right) \\ = & \, \frac{\delta^2\,(1-\delta)^2\,\pi^2 + 2\,\delta^2\,(-2\,\delta\,\alpha + 2\,\alpha + \delta - 2)\,\pi + \left(-5\,\alpha^2 + 8\,\alpha - 3\right)\,\delta^2 + \left(6\,\alpha^2 - 12\,\alpha + 6\right)\,\delta - \alpha^2 + 2\,\alpha - 1}{\delta} \\ \geq & \, \left.\frac{\delta^2\,(1-\delta)^2\,\pi^2 + 2\,\delta^2\,(-2\,\delta\,\alpha + 2\,\alpha + \delta - 2)\,\pi + \left(-5\,\alpha^2 + 8\,\alpha - 3\right)\,\delta^2 + \left(6\,\alpha^2 - 12\,\alpha + 6\right)\,\delta - \alpha^2 + 2\,\alpha - 1}{\delta}\right|_{\pi = \frac{1-\alpha}{\delta}} \\ = & \, 0, \end{aligned}$$

where the second inequality follows from the numerator decreasing in π for any $\pi \in \left[0, \frac{1-\alpha}{\delta}\right]$.

2. When $\pi > \frac{1-\alpha}{\delta}$, we have

$$g'(\omega) \geq g'\left(\frac{\delta^2\,\pi^3 - (1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi}{2\,(1-\alpha)^2}\right) \equiv \frac{k(\pi)}{4\,\delta\,(1-\alpha)^2},$$

where

$$\begin{aligned} k(\pi) \equiv & \, 3\,\delta^4\,\pi^4 + (8\,\alpha\,\delta^3 - 8\,\alpha\,\delta^4 - 8\,\delta^3 + 8\,\delta^4)\,\pi^3 \\ & + (4\,\alpha^2\,\delta^4 - 12\,\alpha^2\,\delta^3 + 6\,\alpha^2\,\delta^2 - 8\,\alpha\,\delta^4 + 20\,\alpha\,\delta^3 - 12\,\alpha\,\delta^2 + 4\,\delta^4 - 8\,\delta^3 + 6\,\delta^2)\,\pi^2 \\ & + (8\,\alpha^2\,\delta^3 - 8\,\alpha^2\,\delta^2 - 16\,\alpha\,\delta^3 + 16\,\alpha\,\delta^2 + 8\,\delta^3 - 8\,\delta^2)\,\pi \\ & + 4\,(-\alpha^4 + 6\,\alpha^3 - 12\,\alpha^2 + 10\,\alpha - 3)\,\delta^2 + 4\,(\alpha^4 - 5\,\alpha^3 + 9\,\alpha^2 - 7\,\alpha + 2)\,\delta \\ & - \alpha^4 + 4\,\alpha^3 - 6\,\alpha^2 + 4\,\alpha - 1. \end{aligned}$$

Because $k^{(3)}(\pi) \geq 0$, we have

$$k''(\pi) \geq k''\left(\frac{1-\alpha}{\delta}\right) = 8\,\delta^3\,(\alpha-1)\,(3\,\alpha-\delta+\alpha\,\delta-4) \geq 0,$$

which further implies

$$k'(\pi) \geq k'\left(\frac{1-\alpha}{\delta}\right) = 8\,\delta^3\,(1-\alpha)^2\,(2-\alpha) \geq 0.$$

Therefore, we have

$$k(\pi) \geq k\left(\frac{1-\alpha}{\delta}\right) = 0,$$

which implies the monotonicity of g. □

LEMMA EC.4. *Function g is nonnegative.*

Proof of Lemma EC.4. When $0 \leq \pi \leq \frac{1-\alpha}{\delta}$ and ω attains its minimal feasible value, we have $\tau = 1$, and hence,

$$\begin{aligned} g(\omega) &\geq g\left(\frac{1-\alpha+\delta\,\alpha}{\delta} - \pi\right) \\ &= \frac{(1-\delta)\,(\alpha+\delta\,\pi-1)^2\,[\delta\,(1-\delta)\,(2\,\delta-1)\,\pi+\alpha+3\,\delta-4\,\alpha\,\delta+3\,\alpha\,\delta^2-1]}{\delta^2} \\ &\geq \frac{(1-\delta)\,(\alpha+\delta\,\pi-1)^2\,(\alpha+3\,\delta-4\,\alpha\,\delta+3\,\alpha\,\delta^2-1)}{\delta^2} \\ &\geq 0. \end{aligned}$$

When $\omega > \frac{1-\alpha}{\delta}$, we have

$$\begin{aligned} g(\omega) &\geq g\left(\frac{\delta^2\,\pi^3-(1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi}{2\,(1-\alpha)^2}\right) \\ &\equiv \frac{\pi\,(\delta^2\,\pi^2-2\,\delta^2\,\pi\,\alpha+2\,\delta\,\pi\,\alpha+2\,\delta^2\,\pi-2\,\delta\,\pi-2\,\alpha^2\,\delta+\alpha^2+4\,\alpha\,\delta-2\,\alpha-2\,\delta+1)^2\,k(\pi)}{8\,\delta\,(1-\alpha)^4}, \end{aligned}$$

where

$$\begin{aligned} k(\pi) &= \delta^2\,\pi^2+2\,\alpha^2\,\delta-\alpha^2-6\,\alpha\,\delta+2\,\alpha+4\,\delta-1 \\ &\geq (1-\alpha)^2+2\,\alpha^2\,\delta-\alpha^2-6\,\alpha\,\delta+2\,\alpha+4\,\delta-1 \\ &= 2\,\delta\,(1-\alpha)\,(2-\alpha) \\ &\geq 0. \end{aligned}$$

□

The nonnegativity of g guarantees that $\eta(\theta) \geq 0$ for any $\theta \in [\hat{\tau}, 1]$. Because

$$\delta\,[2\,\eta_1(\theta)+\theta\,\eta_1'(\theta)] = 1-2\,\delta \leq 0,$$

we know that η_1 is decreasing on $[0, \hat{\tau}]$, guaranteeing the nonnegativity of η on its entire domain $[0, 1]$.

Differentiating $\beta(\cdot)$ yields

$$\beta'(\theta) = -2\,\delta\,\pi\,\eta_1(\theta)+(1-\alpha-\delta\,\pi\,\theta)\,\eta_1'(\theta)-\left[w_q'(\theta)\,f(\theta)+w_q(\theta)\,f'(\theta)\right] = -\frac{C_3\,(1-\alpha)+\delta\,\theta^3\,(\omega+\pi)}{\delta\,\theta^3}.$$

Because η_1 is decreasing on $[0, \hat{\tau}]$, we know that $C_3 \geq 0$, and hence, the multiplier β is nonincreasing in θ, and the nonnegativity condition is implied by $\beta(\hat{\tau}) = 0$.

Similarly, multipliers ξ, ν, and σ are decreasing, increasing, and increasing on $[\hat{\tau}, \tau]$, $[\hat{\tau}, \tau]$, and $[\tau, 1]$, respectively, and the desired properties follow from $\xi(\tau) = \nu(\hat{\tau}) = \sigma(\tau) = 0$.

We have

$$\begin{aligned}
&\chi_L\,[-u(0)] + \int_0^{\hat\tau} \beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta + \int_\tau^1 \sigma(\theta)\,q(\theta)\,\mathrm{d}\theta + \int_0^\tau \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta + \int_{\hat\tau}^1 \nu(\theta)\,[-m_F(\theta)]\,\mathrm{d}\theta \\
&+ \int_0^1 \eta(\theta)\left[\delta\int_0^\theta [m_S(y) - m_F(y) + \pi\,q(y)]\,\mathrm{d}y + (1-\alpha)\,q(\theta) - \delta\,\theta\,[m_S(\theta) + \pi\,q(\theta)] + \delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta \\
=& -u(0) + \int_0^1 [w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,f(\theta)\,\mathrm{d}\theta.
\end{aligned}$$

Hence, the objective function is bounded by

$$\int_\tau^1 \sigma(\theta)\,\mathrm{d}\theta = \int_\tau^1 \left[\frac{2\,\delta\,(\omega+\pi)\,\theta^3 + (\alpha - 2\,\delta\,\alpha - 1)\,\theta^2 + \alpha - 1}{2\,\delta\,\theta^2}\right]\mathrm{d}\theta.$$

Substituting our conjectured primal solution into the objective function yields

$$\int_\tau^1 (\theta\,\omega - m_0^* - \theta\,m_S^*)\,\mathrm{d}F(\theta),$$

achieving the upper bound above and, thereby, establishing the optimality. The desired result follows from

$$\begin{aligned}
&\int_\tau^1 \left[\frac{2\,\delta\,(\omega+\pi)\,\theta^3 + (\alpha - 2\,\delta\,\alpha - 1)\,\theta^2 + \alpha - 1}{2\,\delta\,\theta^2}\right]\mathrm{d}\theta - \int_\tau^1 (\theta\,\omega - m_0^* - \theta\,m_S^*)\,\mathrm{d}F(\theta) \\
&= \int_\tau^1 \left(\pi\,\theta + \frac{\alpha-1}{2\,\delta} + \frac{\alpha-1}{2\,\delta\,\theta^2}\right)\mathrm{d}\theta - \int_\tau^1 (-\theta\,m_S^*)\,\mathrm{d}F(\theta) \\
&= \frac{\pi}{2}\,(1-\tau^2) + \frac{\alpha-1}{2\,\delta}\,(1-\tau) + \frac{1-\alpha}{2\,\delta}\left(1 - \frac{1}{\tau}\right) + \frac{1}{2}\,(1-\tau^2)\left(\frac{1-\alpha}{\delta\,\tau} - \pi\right) \\
&= 0.
\end{aligned}$$

This completes the proof. □

Proof of Proposition 3. Primal feasibility follows directly from the condition $(1-\alpha)\,\bar\tau \le 1$.

Define

$$\begin{aligned}
&\chi_L \equiv 1, \\
&\xi(\theta) \equiv -\,[w_S(\theta) + w_F(\theta)]\,f(\theta), \quad \forall \theta \in [0,\tau) \cup [\hat\tau, 1], \\
&\nu(\theta) \equiv -\,[w_S(\theta) + w_F(\theta)]\,f(\theta), \quad \forall \theta \in [\tau, \hat\tau), \\
&\eta(\theta) \equiv \begin{cases} \eta_1(\theta) \equiv \dfrac{1}{2\,\delta}\left(\dfrac{C_3}{\theta^2} + 1\right) - 1, \ \forall \theta \in (0,\tau), \\ \eta_2(\theta) \equiv \dfrac{1}{2\,\delta}\left(\dfrac{C_2}{\theta^2} + 1\right) - 1, \ \forall \theta \in [\tau,\hat\tau), \\ \eta_3(\theta) \equiv \dfrac{1}{2\,\delta}\left(\dfrac{C_1}{\theta^2} + 1\right) - 1, \ \forall \theta \in [\hat\tau, 1], \end{cases} \\
&\sigma(\theta) \equiv \begin{cases} w_q(\theta)\,f(\theta) - (1-\alpha-\delta\,\pi\,\theta)\,\eta_2(\theta) - \delta\,\pi\left[\displaystyle\int_\theta^{\hat\tau} \eta_2(y)\,\mathrm{d}y + \int_{\hat\tau}^1 \eta_3(y)\,\mathrm{d}y\right], & \forall \theta \in [\tau,\hat\tau), \\ w_q(\theta)\,f(\theta) - (1-\alpha-\delta\,\pi\,\theta)\,\eta_3(\theta) - \delta\,\pi\displaystyle\int_\theta^1 \eta_3(y)\,\mathrm{d}y, & \forall \theta \in [\hat\tau, 1], \text{ and} \end{cases} \\
&\beta(\theta) \equiv (1-\alpha-\delta\,\pi\,\theta)\,\eta_1(\theta) + \delta\,\pi\left[\int_\theta^\tau \eta_1(y)\,\mathrm{d}y + \int_\tau^{\hat\tau} \eta_2(y)\,\mathrm{d}y + \int_{\hat\tau}^1 \eta_3(y)\,\mathrm{d}y\right] - w_q(\theta)\,f(\theta), \quad \forall \theta \in (0,\tau),
\end{aligned}$$

where

$$\begin{aligned}
&C_1 = 2\,\delta - 1, \\
&C_2 = 2\,\hat\tau\,(1-\delta) + 2\,\delta - 1 \ge 0, \\
&C_3 = 2\,(1-\delta)\,(\hat\tau - \tau) + 2\,\delta - 1 \ge 0, \\
&\hat\tau = \frac{1}{2\,\delta^2\,\pi^3\,(1-\delta)}\left[(1-\alpha)^2\,(2\,\omega + \pi) - 2\,\alpha\,\delta\,\pi\,(1-\alpha) + \delta^2\,\pi^3\,(1-2\,\delta)\right].
\end{aligned}$$

The dual feasibility conditions $\tau \le \hat{\tau} \le 1$ can be reformulated as

$$\delta^2\,(1-2\,\delta)\,\pi^3 - 2\,\delta\,(1-\delta)\,(1-\alpha)\,\pi^2 + (1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi + 2\,\omega\,(1-\alpha)^2 \ge 0$$

and

$$\frac{1}{2\,\delta^2\,\pi^3\,(1-\delta)}\,\left[(1-\alpha)^2\,(2\,\omega+\pi) - 2\,\delta\,\alpha\,\pi\,(1-\alpha) + \delta^2\,\pi^3\,(1-2\,\delta)\right] \le 1,$$

or equivalently,

$$-\delta^2\,\pi^3 + (1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi + 2\,\omega\,(1-\alpha)^2 \le 0.$$

Both conditions are guaranteed by the two inequalities given in Proposition 3.

To prove the nonnegativity of η, we have:

- Because $C_1, C_2, C_3 \ge 0$, all the three pieces η_1, η_2, and η_3 are decreasing.
- Because $\eta_3(1) = 0$, the nonnegativity of η_3 holds automatically.
- The nonnegativity condition for η_2 boils down to

$$\lim_{\theta \uparrow \hat{\tau}} \eta_2(\theta) = \frac{1}{2\,\delta}\,\left(\frac{2\,\hat{\tau}\,(1-\delta) + 2\,\delta - 1}{\hat{\tau}^2} + 1\right) - 1 \ge 0 \quad \Leftrightarrow \quad (1-2\,\delta)\,\hat{\tau}^2 + 2\,(1-\delta)\,\hat{\tau} + (2\,\delta - 1) \ge 0,$$

or equivalently,

$$\tau \le \hat{\tau} \le \underbrace{\frac{1-\delta+\sqrt{5\,\delta^2-6\,\delta+2}}{2\,\delta-1}}_{\ge 1 \text{ for any } \delta \in \left[\frac{1}{2},1\right]},$$

which is implied by $\tau \le \hat{\tau} \le 1$.

- The nonnegativity condition for η_1 boils down to

$$\lim_{\theta \uparrow \tau} \eta_1(\theta) = \frac{1}{2\,\delta}\,\left(\frac{2\,(1-\delta)\,(\hat{\tau}-\tau) + 2\,\delta - 1}{\tau^2} + 1\right) - 1 \ge 0 \quad \Leftrightarrow \quad \hat{\tau} \ge \tau + \underbrace{\frac{2\,\delta-1}{2\,(1-\delta)}\,(\tau^2-1)}_{\le 0},$$

which is implied by $\tau \le \hat{\tau}$.

After proving the nonnegativity of η's, we show that β and σ are also nonnegative. Taking derivative yields

$$\forall \theta \in [0,\tau),\ \beta'(\theta) = -2\,\delta\,\pi\,\underbrace{\eta_1(\theta)}_{\ge 0} + \underbrace{(1-\alpha-\delta\,\pi\,\theta)}_{\ge 0,\ \forall \theta \in [0,\tau]}\,\underbrace{\eta_1'(\theta)}_{\le 0} - \underbrace{\left[w_q'(\theta)\,f(\theta) + w_q(\theta)\,f'(\theta)\right]}_{=\omega + 2\,\delta\,\pi \ge 0} \le 0,$$

$$\begin{aligned}
\forall \theta \in [\tau,\hat{\tau}),\ \sigma'(\theta) &= 2\,\delta\,\pi\,\eta_2(\theta) - (1-\alpha-\delta\,\pi\,\theta)\,\eta_2'(\theta) + w_q'(\theta)\,f(\theta) + w_q(\theta)\,f'(\theta) \\
&= \frac{(1-\alpha)\,\left[2\,\hat{\tau}\,(1-\delta) + 2\,\delta - 1\right] + \delta\,\theta^3\,(\omega+\pi)}{\delta\,\theta^3} \\
&\ge 0,
\end{aligned}$$

and

$$\begin{aligned}
\forall \theta \in [\hat{\tau},1],\ \sigma'(\theta) &= 2\,\delta\,\pi\,\eta_3(\theta) - (1-\alpha-\delta\,\pi\,\theta)\,\eta_3'(\theta) + w_q'(\theta)\,f(\theta) + w_q(\theta)\,f'(\theta) \\
&= \frac{(1-\alpha)\,(2\,\delta-1) + \delta\,\theta^3\,(\omega+\pi)}{\delta\,\theta^3} \\
&\ge 0.
\end{aligned}$$

Therefore, the nonnegativity conditions boil down to

$$\beta(\tau) \ge 0, \quad \sigma(\tau) \ge 0, \quad \text{and} \quad \sigma(\hat{\tau}) \ge 0.$$

By definition of $\hat{\tau}$, we have $\beta(\tau) = \sigma(\tau) = 0$. The remaining condition $\sigma(\hat{\tau}) \ge 0$ can be rewritten as

$$2\,\delta\,(\omega+\pi)\,\hat{\tau}^3 - \left[2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha\right]\,\hat{\tau}^2 + (1-\alpha)\,(1-2\,\delta) \ge 0.$$

To prove it, define

$$h(\theta) \equiv 2\,\delta\,(\omega+\pi)\,\theta^3 - [2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha]\,\theta^2 + (1-\alpha)\,(1-2\,\delta).$$

Taking derivative yields

$$h'(\theta) = 6\,\delta\,(\omega+\pi)\,\theta^2 - 2\,[2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha]\,\theta.$$

We have

$$h'(\theta) \begin{cases} \leq 0, \text{ if } \theta \leq \dfrac{2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}, \\ \geq 0, \text{ if } \theta \geq \dfrac{2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}. \end{cases}$$

Therefore, function h is either first decreasing and then increasing or increasing on $[0,1]$. Because

$$h(0) = (1-\alpha)\,(1-2\,\delta) \leq 0 \quad \text{and} \quad h\,(\tau) \geq 0,$$

where the latter one follows from the optimality condition in the proposition, and $\hat{\tau} \geq \tau$, the desired property $h(\hat{\tau}) \geq 0$ holds for sure.

We have

$$\begin{aligned} &\chi_L\,[-u(0)] + \int_0^{\tau} \beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta + \int_{\tau}^{1} \sigma(\theta)\,q(\theta)\,\mathrm{d}\theta \\ &+ \int_0^1 \eta(\theta) \left[\delta \int_0^{\theta} [m_S(y) - m_F(y) + \pi\,q(y)]\,\mathrm{d}y + (1-\alpha)\,q(\theta) - \delta\,\theta\,[m_S(\theta) + \pi\,q(\theta)] + \delta\,\theta\,m_F(\theta)\right] \mathrm{d}\theta \\ &+ \int_0^{\tau} \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta + \int_{\hat{\tau}}^{1} \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta + \int_{\tau}^{\hat{\tau}} \nu(\theta)\,[-m_F(\theta)]\,\mathrm{d}\theta \\ =& -u(0) + \int_0^1 [w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,f(\theta)\,\mathrm{d}\theta. \end{aligned}$$

Hence, the objective function is bounded by

$$\int_{\tau}^{1} \sigma(\theta)\,\mathrm{d}\theta = -\frac{\omega\,(1-\alpha)^2}{2\,\delta^2\,\pi^2} + \frac{\alpha\,(1-\alpha)}{\delta\,\pi} + \frac{\omega}{2} - \alpha.$$

Substituting our conjectured primal solution into the objective function yields $\int_{\tau}^{1} (\theta\,\omega - \alpha)\,\mathrm{d}F(\theta)$, attaining the upper bound above and, thereby, establishing the optimality. This completes the proof. □

Proof of Proposition 4. We first verify that $B_3(\pi) \leq B_2(\pi)$ for any $\pi \geq \frac{1-\alpha}{\delta}$. When $\pi \geq \frac{1}{\delta}$, the desired condition is equivalent to

$$\phi(\pi) \equiv 3\,\delta^2\,(2\,\delta - 1)\,\pi^2 + \delta\,(2 + \alpha - 4\,\delta\,\alpha - 2\,\delta)\,\pi + 2\,\delta\,\alpha - 2\,\alpha + 1 \geq 0, \quad \forall \pi \in \left[\frac{1}{\delta}, \infty\right),$$

which follows from

$$-\frac{(2 + \alpha - 4\,\delta\,\alpha - 2\,\delta)}{6\,\delta\,(2\,\delta - 1)} - \frac{1}{\delta} = \frac{\overbrace{-\alpha - 10\,\delta + 4\,\delta\,\alpha + 4}^{\leq -10\,\delta + 4}}{6\,\delta\,(2\,\delta - 1)} \leq 0$$

and

$$\phi\left(\frac{1}{\delta}\right) = -\frac{\alpha\,\overbrace{(\alpha - 4\,\delta + 2\,\delta\,\alpha)}^{\leq 2\,\alpha - 2}}{2\,\delta\,(1-\alpha)^2} \geq 0.$$

When $\frac{1-\alpha}{\delta} \leq \pi \leq \frac{1}{\delta}$, we have $B_3(\pi) \leq 1 - \delta\,\pi \leq B_2(\pi)$, completing the proof of $B_3(\pi) \leq B_2(\pi)$.

Under the given conditions, the primal feasibility condition boils down to

$$\frac{1-\alpha}{\delta\,\pi} \leq \tau \leq \mathbb{1}_{\delta\,\pi-\alpha>0} \cdot \frac{1-\alpha}{\delta\,\pi-\alpha} + \mathbb{1}_{\delta\,\pi-\alpha\leq 0}$$

We begin by proving the unique existence of the threshold τ. Define

$$h(\theta) \equiv 2\,\delta\,(\omega+\pi)\,\theta^3 - [2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha]\;\theta^2 - (1-\alpha)\,(2\,\delta-1).$$

We have

$$h(0) = -\,(2\,\delta-1)\;(1-\alpha) \leq 0$$

and

$$h(1) = 2\,\delta\;(\omega-1+\delta\,\pi) > 0,$$

and hence, the existence of $\tau \in [0,1]$ is guaranteed.

Differentiating h yields

$$h'(\theta) = 6\,\delta\,(\omega+\pi)\,\theta^2 - 2\;[2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha]\;\theta.$$

We have

$$h'(\theta) \begin{cases} \leq 0, \text{ if } \theta \leq \dfrac{2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}, \\ \geq 0, \text{ if } \theta \geq \dfrac{2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}. \end{cases}$$

Therefore, function h is either first decreasing and then increasing or increasing on $[0,1]$. Because $h(0) < 0 < h(1)$, the threshold τ is uniquely determined.

By the monotonicity of h, to prove $\tau \geq \frac{1-\alpha}{\delta\,\pi}$, it suffices to show that

$$h\left(\frac{1-\alpha}{\delta\,\pi}\right) \leq 0,$$

which is equivalent to $\omega \leq B_2(\pi)$.

When $\pi \leq \frac{\alpha}{\delta}$, the upper bound in the primal feasibility condition for τ simplifies to $\tau \leq 1$, which holds automatically.

When $\pi > \frac{\alpha}{\delta}$, the upper bound in the primal feasibility condition for τ boils down to $\tau \leq \frac{1-\alpha}{\delta\,\pi-\alpha}$, which follows from

$$h\left(\frac{1-\alpha}{\delta\,\pi-\alpha}\right) \geq 0 \quad \Leftrightarrow \quad \omega \geq B_3(\pi).$$

Define

$$\begin{aligned}
&\chi_L \equiv 1, \\
&\xi(\theta) \equiv -\,[w_S(\theta) + w_F(\theta)]\;f(\theta), && \forall \theta \in [0,1], \\
&\eta(\theta) \equiv \frac{1}{2\,\delta}\left(\frac{2\,\delta-1}{\theta^2}+1\right) - 1, && \forall \theta \in (0,1], \\
&\beta(\theta) \equiv (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) + \delta\,\pi\int_\theta^1 \eta(y)\,\mathrm{d}y - w_q(\theta)\,f(\theta), && \forall \theta \in (0,\tau], \text{ and} \\
&\sigma(\theta) \equiv -\,(1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) - \delta\,\pi\int_\theta^1 \eta(y)\,\mathrm{d}y + w_q(\theta)\,f(\theta), && \forall \theta \in [\tau,1].
\end{aligned}$$

Because

$$w_S(\theta) + w_F(\theta) = -1 + \delta \leq 0$$

and $\eta(\theta) \geq \eta(1) = 0$, the nonnegativity of ξ and η holds automatically. Differentiating β and σ yields that these two multipliers are decreasing and increasing on $[0, \tau]$ and $[\tau, 1]$, respectively. Because $\beta(\tau) = \sigma(\tau) = 0$, they are guaranteed to be nonnegative.

We have

$$\begin{aligned}
&\int_0^1 \eta(\theta) \left[\delta \int_0^\theta [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y + (1-\alpha)\, q(\theta) - \delta\, \theta\, [m_S(\theta) + \pi\, q(\theta)] + \delta\, \theta\, m_F(\theta) \right] \mathrm{d}\theta \\
&+ \int_0^\tau \beta(\theta)\, [-q(\theta)]\, \mathrm{d}\theta + \int_\tau^1 \sigma(\theta)\, q(\theta)\, \mathrm{d}\theta + \int_0^1 \xi(\theta)\, [-m_S(\theta)]\, \mathrm{d}\theta \\
=& -u(0) + \int_0^1 [w_q(\theta)\, q(\theta) + w_S(\theta)\, m_S(\theta) + w_F(\theta)\, m_F(\theta)]\, f(\theta)\, \mathrm{d}\theta,
\end{aligned}$$

and hence, the objective function is bounded by

$$\int_\tau^1 \sigma(\theta)\, \mathrm{d}\theta = \frac{(1-\tau)\, (1 - \alpha - 2\,\delta - \tau - \delta\,\pi\,\tau + 2\,\alpha\,\delta + \delta\,\omega\,\tau + \alpha\,\tau + \delta\,\pi\,\tau^2 + 2\,\delta^2\,\pi\,\tau + \omega\,\delta\,\tau^2 - 2\,\alpha\,\delta\,\tau)}{2\,\delta\,\tau}$$

Substituting our conjectured primal solution into the objective function yields

$$\int_\tau^1 [\theta\,\omega - m_0^* - (1-\theta)\, m_F^*]\, \mathrm{d}F(\theta),$$

achieving the upper bound above and, thereby, establishing the optimality. This completes the proof. □

Proof of Proposition 5. The primal feasibility condition $\tau \in [0, 1]$ is equivalent to $\pi \geq \frac{1}{\delta}$, which is implied by $B_4(\pi) < B_3(\pi)$, as follows from their definitions.

Define

$$\begin{aligned}
\chi_L &\equiv \frac{1}{2\,\delta}, \\
\eta(\theta) &\equiv \frac{1}{2\,\delta} \left(\frac{2\,\delta - 1}{\theta^2} + 1 \right) - 1, \quad \forall \theta \in [\hat{\tau}, 1], \\
\gamma(\theta) &\equiv -\frac{1}{2\,\delta} \left[\frac{C_2}{(1-\theta)^2} + 1 \right] + 1, \quad \forall \theta \in [0, \hat{\tau}), \\
\xi(\theta) &\equiv \begin{cases} \delta\,\theta\,\gamma(\theta) + \delta \int_{\hat{\tau}}^1 \eta(y)\, \mathrm{d}y - \delta \int_\theta^{\hat{\tau}} \gamma(y)\, \mathrm{d}y - w_S(\theta)\, f(\theta), & \forall \theta \in [0, \hat{\tau}), \\ -[w_S(\theta) + w_F(\theta)]\, f(\theta), & \forall \theta \in [\hat{\tau}, 1], \end{cases} \\
\beta(\theta) &\equiv -(1 - \delta\,\pi\,\theta)\, \gamma(\theta) + \delta\,\pi \int_{\hat{\tau}}^1 \eta(y)\, \mathrm{d}y - \delta\,\pi \int_\theta^{\hat{\tau}} \gamma(y)\, \mathrm{d}y - w_q(\theta)\, f(\theta), \quad \forall \theta \in [0, \tau), \text{ and} \\
\sigma(\theta) &\equiv \begin{cases} (1 - \delta\,\pi\,\theta)\, \gamma(\theta) - \delta\,\pi \int_{\hat{\tau}}^1 \eta(y)\, \mathrm{d}y + \delta\,\pi \int_\theta^{\hat{\tau}} \gamma(y)\, \mathrm{d}y + w_q(\theta)\, f(\theta), & \forall \theta \in [\tau, \hat{\tau}), \\ -(1 - \alpha - \delta\,\pi\,\theta)\, \eta(\theta) - \delta\,\pi \int_\theta^1 \eta(y)\, \mathrm{d}y + w_q(\theta)\, f(\theta), & \forall \theta \in [\hat{\tau}, 1], \end{cases}
\end{aligned}$$

where

$$C_2 = \frac{(1 - 2\,\delta)\, (1 - \hat{\tau})^2}{\hat{\tau}} \leq 0$$

and the second threshold $\hat{\tau}$ is determined by

$$\begin{aligned}
&(1 - 2\,\delta)\, (1 - \delta\,\pi)\, \hat{\tau}^2 \\
&+ [-2\,\delta\,(\omega + \pi)\, \tau^3 + (4\,\delta\,\omega + 5\,\delta\,\pi + 1)\, \tau^2 - 2\,(\delta\,\omega + 2\,\delta\,\pi + 1)\, \tau + (-4\,\delta^2\,\pi + 3\,\delta\,\pi + 4\,\delta - 1)]\, \hat{\tau} \\
&+ (1 - 2\,\delta)\, (1 - \delta\,\pi) = 0.
\end{aligned}$$

First, we prove the unique existence of $\hat{\tau}$ on $[\tau,1]$. Define

$$\begin{aligned} h(\theta) \equiv\ & (1-2\,\delta)\,(1-\delta\,\pi)\,(\theta^2+1) \\ & +\left[-2\,\delta\,(\omega+\pi)\,\tau^3+(4\,\delta\,\omega+5\,\delta\,\pi+1)\,\tau^2-2\,(\delta\,\omega+2\,\delta\,\pi+1)\,\tau+(-4\,\delta^2\,\pi+3\,\delta\,\pi+4\,\delta-1)\right]\theta, \end{aligned}$$

and we have

$$h(1)=-\frac{(\delta\,\pi-1)^2\,\left[-\delta^2\,\pi^2+\delta\,(1-\alpha)\,\pi+\alpha+2\,\delta\,\omega\,(1-\alpha)\right]}{(\delta\,\pi-\alpha)^3}\leq 0.$$

To prove the existence of $\hat{\tau}$, it suffices to show that

$$\max_{\theta\in[\tau,1]} h(\theta)\geq 0.$$

Because $(1-2\,\delta)\,(1-\delta\,\pi)>0$, the maximum is attained at either τ or 1. Hence, the desired result boils down to $h(\tau)\geq 0$, or equivalently, $\omega\leq B_3(\pi)$. Because h is quadratic, the uniqueness of $\hat{\tau}$ holds automatically.

Differentiating β yields

$$\beta'(\theta)=2\,\delta\,\pi\,\gamma(\theta)-(1-\delta\,\pi\,\theta)\,\gamma'(\theta)-w_q'(\theta)\,f(\theta)-w_q(\theta)\,f'(\theta)=\frac{C_2\,(1-\delta\,\pi)}{\delta\,(1-\theta)^3}-(\omega+\pi),$$

which is increasing in θ. We have

$$\begin{aligned} & \beta'(\theta)\leq 0,\quad \forall\theta\in[0,\tau], \\ \Leftrightarrow\ & C_2\,(1-\delta\,\pi)\leq\delta\,(1-\theta)^3\,(\omega+\pi),\quad \forall\theta\in[0,\tau], \\ \Leftrightarrow\ & \frac{(1-2\,\delta)\,(1-\hat{\tau})^2}{\hat{\tau}}\,(1-\delta\,\pi)\leq\delta\,(1-\tau)^3\,(\omega+\pi) \\ \Leftrightarrow\ & (1-2\,\delta)\,(1-\delta\,\pi)\,(1-\hat{\tau})^2-\delta\,\hat{\tau}\,(1-\tau)^3\,(\omega+\pi)\leq 0 \\ \Leftrightarrow\ & -\hat{\tau}\,(1-\tau)^2\,\left[1+2\,\delta\,\pi+\delta\,\omega-3\,\delta\,\tau\,(\omega+\pi)\right]\leq 0 \\ \Leftrightarrow\ & 1+2\,\delta\,\pi+\delta\,\omega-3\,\delta\,\tau\,(\omega+\pi)\geq 0, \end{aligned}$$

where the second to last step follows from the definition of $\hat{\tau}$. The last inequality can be rewritten as, for any (π,ω) in the feasible region, we have

$$2\,\delta^2\,\pi^2+\delta\,(-2+\alpha)\,\pi-\alpha+\delta\,(\delta\,\pi+2\,\alpha-3)\,\omega\geq 0.$$

When $\pi\leq\frac{3-2\,\alpha}{\delta}$, fix π, and the expression above is decreasing in ω. Therefore, the minimum is attained at the upper bound of ω. If the desired result holds for any $\pi\in\left[\frac{1}{\delta},\frac{3-2\,\alpha}{\delta}\right]$, it holds automatically for any $\pi\in\left(\frac{3-2\,\alpha}{\delta},\infty\right)$, because the expression we are interested in is increasing in π on $\left[\frac{1}{\delta},\infty\right)$, which follows from

$$-\frac{\delta\,(-2+\alpha+\delta\,\omega)}{4\,\delta^2}=\frac{2-\alpha-\delta\,\omega}{4\,\delta}\leq 1\leq\frac{1}{\delta}.$$

It suffices to show that

$$2\,\delta^2\,\pi^2+\delta\,(-2+\alpha)\,\pi-\alpha+\delta\,(\delta\,\pi+2\,\alpha-3)\,B_3(\pi)=\frac{\overbrace{(1-\delta\,\pi)}^{\leq 0}\,g(\pi)}{2\,(1-\alpha)^2}\geq 0,$$

where

$$\begin{aligned} g(\pi)\equiv\ & \delta^3\,(1-2\,\delta)\,\pi^3+\delta^2\,(6\,\delta-4+\alpha)\,\pi^2+\delta\,(6\,\alpha^2\,\delta-4\,\alpha^2-12\,\alpha\,\delta+8\,\alpha-1)\,\pi \\ & +\alpha-4\,\alpha^2+2\,\alpha^3+6\,\alpha^2\,\delta-4\,\alpha^3\,\delta. \end{aligned}$$

Taking derivative yields

$$\begin{aligned} & g'(\pi)=(3\,\delta^3-6\,\delta^4)\,\pi^2+(12\,\delta^3-8\,\delta^2+2\,\alpha\,\delta^2)\,\pi+6\,\alpha^2\,\delta^2-4\,\alpha^2\,\delta-12\,\alpha\,\delta^2+8\,\alpha\,\delta-\delta, \\ & g''(\pi)=(6\,\delta^3-12\,\delta^4)\,\pi+12\,\delta^3-8\,\delta^2+2\,\alpha\,\delta^2, \\ & g^{(3)}(\pi)=-6\,\delta^3\,(2\,\delta-1)<0, \end{aligned}$$

and hence,

$$g''(\pi) \leq g''\left(\frac{1}{\delta}\right) = -2\,\delta^2\,(1-\alpha) \leq 0.$$

This further implies

$$g'(\pi) \leq g'\left(\frac{1}{\delta}\right) = 2\,\delta\,(1-\alpha)\,\underbrace{(2\,\alpha+3\,\delta-3\,\alpha\,\delta-3)}_{\leq 3-3\,\alpha+2\,\alpha-3=-\alpha\leq 0} \leq 0$$

and

$$g(\pi) \leq g\left(\frac{1}{\delta}\right) = 2\,(1-\alpha)^2\,\underbrace{(\alpha+2\,\delta-2\,\alpha\,\delta-2)}_{\leq 2-2\,\alpha+\alpha-2=-\alpha} \leq 0,$$

and hence the desired result holds.

Because β and σ are decreasing and increasing on $[0,\tau]$ and $[\tau,\hat{\tau})$, respectively, and $\beta(\tau)=\sigma(\tau)=0$, the corresponding nonnegativity conditions hold automatically.

For any $\theta\in[\hat{\tau},1]$, we have

$$\sigma'(\theta) = 2\,\delta\,\pi\,\eta(\theta) - (1-\alpha-\delta\,\pi\,\theta)\,\eta'(\theta) + w_q'(\theta)\,f(\theta) + w_q(\theta)\,f'(\theta) = \frac{(1-\alpha)\,(2\,\delta-1)}{\delta\,\theta^3} + \omega + \pi \geq 0,$$

and hence, the desired nonnegativity condition boils down to $\sigma(\hat{\tau})\geq 0$, which follows from

$$\begin{aligned}\sigma(\hat{\tau}) - \lim_{\theta\uparrow\hat{\tau}}\sigma(\theta) &= -(1-\alpha-\delta\,\pi\,\hat{\tau})\,\eta(\hat{\tau}) - (1-\delta\,\pi\,\hat{\tau})\,\gamma(\hat{\tau}) \\ &= -\frac{(2\,\delta-1)\,(\hat{\tau}+1)\,(1-\alpha-\delta\,\pi\,\hat{\tau}+\alpha\,\hat{\tau})}{2\,\delta\,\hat{\tau}^2} \\ &\geq 0.\end{aligned}$$

The last inequality follows from

$$1-\alpha-\delta\,\pi\,\hat{\tau}+\alpha\,\hat{\tau} \leq (1-\alpha) - (\delta\,\pi-\alpha)\,\tau = 0.$$

It remains to verify the nonnegativity of ξ on $[0,\hat{\tau})$. Taking derivative yields

$$\xi'(\theta) = \delta\,[2\,\gamma(\theta)+\theta\,\gamma'(\theta)] - [w_S'(\theta)\,f(\theta) + w_S(\theta)\,f'(\theta)] = \frac{(2\,\delta-1)\,(1-\hat{\tau})^2}{\hat{\tau}\,(1-\theta)^3} \geq 0,$$

and the desired property is implied by

$$\xi(0) = \frac{1}{\hat{\tau}}\left(\delta-\frac{1}{2}\right)(1-\hat{\tau})^2 + \frac{1}{2} \geq \frac{1}{2}.$$

We have

$$\begin{aligned}&\chi_L\,[-u(0)] + \int_0^{\tau}\beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta + \int_{\tau}^1\sigma(\theta)\,q(\theta)\,\mathrm{d}\theta + \int_0^1\xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta \\ &+\int_{\hat{\tau}}^1\eta(\theta)\left[\delta\int_0^{\theta}[m_S(y)-m_F(y)+\pi\,q(y)]\,\mathrm{d}y + (1-\alpha)\,q(\theta) - \delta\,\theta\,[m_S(\theta)+\pi\,q(\theta)] + \delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta \\ &-\int_0^{\hat{\tau}}\gamma(\theta)\left[u(0)+\delta\int_0^{\theta}[m_S(y)-m_F(y)+\pi q(y)]\,\mathrm{d}y + q(\theta) - \delta\theta\,[m_S(\theta)+\pi q(\theta)] - \delta(1-\theta)m_F(\theta)\right]\mathrm{d}\theta \\ =&-u(0) + \int_0^1[w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,f(\theta)\,\mathrm{d}\theta,\end{aligned}$$

and hence, the objective function is bounded by

$$\int_{\tau}^{\hat{\tau}}\sigma(\theta)\,\mathrm{d}\theta + \int_{\hat{\tau}}^1\sigma(\theta)\,\mathrm{d}\theta = \frac{(1-\tau)\,[\delta\,\omega\,(1+\tau)-\alpha\,(1-\tau)]}{2\,\delta}.$$

Substituting our conjectured primal solution into the objective function yields

$$\int_{\tau}^1\left[\theta\,\omega - (1-\theta)\cdot\frac{\alpha}{\delta}\right]\mathrm{d}F(\theta),$$

which attains the upper bound above. This completes the proof. □

Proof of Proposition 6. The primal feasibility condition boils down to

$$0 \leq \tau \leq 1 \quad \Leftrightarrow \quad \delta\,\pi - 2\,\delta\,(\omega+\pi) + 1 \leq 0$$

and

$$0 \leq \frac{1-\delta\,\pi\,\tau}{\delta\,(1-\tau)} \leq \frac{\alpha}{\delta} \quad \Leftrightarrow \quad \begin{cases} 1-\delta\,\pi\,\tau \geq 0, \\ 1-\delta\,\pi\,\tau-\alpha\,(1-\tau) \leq 0, \end{cases}$$

where the first and second conditions are for the testing threshold τ and the reward for failure, respectively. The feasibility condition for τ follows from

$$\delta\,\pi - 2\,\delta\,(\omega+\pi) + 1 = 1 - \delta\,\pi - 2\,\delta\,\omega \leq 1 - \delta\,\frac{1}{\delta} = 0.$$

The feasibility condition for m_F follows from

$$1-\delta\,\pi\,\tau = \frac{2\,(\omega+\pi) - \pi\,(\delta\,\pi+1)}{2\,(\omega+\pi)} \geq \frac{2\,[B_5(\pi)+\pi] - \pi\,(\delta\,\pi+1)}{2\,(\omega+\pi)} = 0,$$

and

$$1-\delta\,\pi\,\tau-\alpha\,(1-\tau) = \frac{-\delta^2\,\pi^2 + \delta\,(1-\alpha)\,\pi + \alpha + 2\,\delta\,\omega\,(1-\alpha)}{2\,\delta\,(\omega+\pi)} \leq 0,$$

where the last inequality follows from $\omega < B_4(\pi)$ directly.

Define

$$\begin{aligned}
&\chi_L \equiv \frac{1}{2\,\delta}, \\
&\gamma(\theta) \equiv 1 - \frac{1}{2\,\delta} \geq 0, && \forall \theta \in [0,1], \\
&\beta(\theta) \equiv -(1-\delta\,\pi\,\theta)\,\gamma(\theta) - \delta\,\pi \int_\theta^1 \gamma(y)\,\mathrm{d}y - w_q(\theta)\,f(\theta) = \frac{1}{2\,\delta}\,(\delta\,\pi+1) - \theta\,(\omega+\pi), && \forall \theta \in [0,\tau), \\
&\sigma(\theta) \equiv (1-\delta\,\pi\,\theta)\,\gamma(\theta) + \delta\,\pi \int_\theta^1 \gamma(y)\,\mathrm{d}y + w_q(\theta)\,f(\theta) = -\frac{1}{2\,\delta}\,(\delta\,\pi+1) + \theta\,(\omega+\pi), && \forall \theta \in [\tau,1], \text{ and} \\
&\xi(\theta) \equiv \delta\,\theta\,\gamma(\theta) - \delta \int_\theta^1 \gamma(y)\,\mathrm{d}y - w_S(\theta)\,f(\theta) = \frac{1}{2}, && \forall \theta \in [0,1].
\end{aligned}$$

Because β and σ are decreasing and increasing on $[0,\tau)$ and $[\tau,1]$, respectively, and

$$\lim_{\theta \uparrow \tau} \beta(\theta) = \sigma(\tau) = 0,$$

they are always nonnegative.

We have

$$\begin{aligned}
&\chi_L\,[-u(0)] + \int_0^\tau \beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta + \int_\tau^1 \sigma(\theta)\,q(\theta)\,\mathrm{d}\theta + \int_0^1 \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta \\
&- \int_0^1 \gamma(\theta) \left[u(0) + \delta \int_0^\theta [m_S(y) - m_F(y) + \pi\,q(y)]\,\mathrm{d}y + q(\theta) - \delta\,\theta\,[m_S(\theta) + \pi\,q(\theta)] - \delta\,(1-\theta)\,m_F(\theta) \right] \mathrm{d}\theta \\
&= -u(0) + \int_0^1 [w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,f(\theta)\,\mathrm{d}\theta,
\end{aligned}$$

and hence, the objective function is bounded by

$$\int_\tau^1 \sigma(\theta)\,\mathrm{d}\theta = \frac{(1-\tau)\,[\delta\,\tau\,(\omega+\pi) + \delta\,\omega - 1]}{2\,\delta}$$

Substituting our conjectured primal solution into the objective function yields

$$\int_\tau^1 \left[\theta\,\omega - (1-\theta) \cdot \frac{1}{1-\tau} \cdot \left(\frac{1}{\delta} - \tau\,\pi \right) \right] \mathrm{d}F(\theta),$$

achieving the upper bound above and, thereby, establishing the optimality. This completes the proof. □

Proof of Proposition 7. We prove the desired result in two steps: first, we show that the optimal mechanism is to provide no funding when $\omega = \frac{\pi}{2}(\delta\pi - 1)$; and then, we prove that if our candidate solution is optimal when the consumer welfare is ω_0, fixing all the remaining parameters, it remains optimal for any $0 \le \omega_1 < \omega_0$. Combining the results from both steps yields the desired conclusion.

When $\omega = \frac{\pi}{2}(\delta\pi - 1)$, by Proposition 6, the testing threshold is given by

$$\tau = \frac{\delta\pi + 1}{2\delta(\omega + \pi)} = \frac{1}{\delta\pi},$$

and the optimal mechanism boils down to

$$q(\theta) = \begin{cases} 0, \text{ if } \theta \in [0, \tau), \\ 1, \text{ if } \theta \in [\tau, 1], \end{cases} \qquad m_0(\theta) = m_S(\theta) = m_F(\theta) = 0, \quad \forall \theta \in \Theta.$$

This completes the proof of the first argument.

Lemma EC.5 states that if the optimal mechanism is to provide no funding, lowering the consumer welfare ω and keeping all the remaining parameters will not alter the optimal mechanism.

LEMMA EC.5. *If the optimal mechanism is to provide no funding when the consumer welfare is ω_0, fixing all the remaining parameters, for any $\omega \le \omega_0$, the optimal mechanism remains providing no funding.*

Proof of Lemma EC.5. Let $\tau \equiv \frac{1}{\delta\pi}$. Suppose there exists $\omega_1 < \omega_0$ such that the mechanism providing no funding is strictly dominated by

$$\hat{\Omega} \equiv \{\hat{q}(\theta), \hat{m}_0(\theta), \hat{m}_S(\theta), \hat{m}_F(\theta)\}.$$

Because the policy providing no funding is strictly dominated, we have

$$\int_0^1 [\theta\,\omega_1\,\hat{q}(\theta) - \hat{m}_0(\theta) - \theta\,\hat{m}_S(\theta) - (1-\theta)\,\hat{m}_F(\theta)]\,\mathrm{d}\theta > \omega_1 \int_\tau^1 \theta\,\mathrm{d}\theta,$$

or equivalently,

$$\omega_1 \left[\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta\right] > \int_0^1 [\hat{m}_0(\theta) + \theta\,\hat{m}_S(\theta) + (1-\theta)\,\hat{m}_F(\theta)]\,\mathrm{d}\theta. \tag{EC.10}$$

Because the feasible set of the funder's problem (14) is independent from ω, when the profit is ω_0, the mechanism $\hat{\Omega}$ remains feasible.

Since the mechanism providing no funding is optimal when the consumer welfare is ω_0, we have

$$\int_0^1 [\theta\,\omega_0\,\hat{q}(\theta) - \hat{m}_0(\theta) - \theta\,\hat{m}_S(\theta) - (1-\theta)\,\hat{m}_F(\theta)]\,\mathrm{d}F(\theta) \le \omega_0 \int_\tau^1 \theta\,\mathrm{d}\theta,$$

or equivalently,

$$\omega_0 \left[\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta\right] \le \int_0^1 [\hat{m}_0(\theta) + \theta\,\hat{m}_S(\theta) + (1-\theta)\,\hat{m}_F(\theta)]\,\mathrm{d}\theta \tag{EC.11}$$

Combining (EC.10) and (EC.11) yields

$$\omega_0 \left[\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta\right] < \omega_1 \left[\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta\right]$$

Because $\omega_0 > \omega_1 \ge 0$, we have

$$\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta < 0,$$

which implies

$$\int_0^1 [\hat{m}_0(\theta) + \theta\,\hat{m}_S(\theta) + (1-\theta)\,\hat{m}_F(\theta)]\,\mathrm{d}\theta < 0,$$

contradicting the nonnegativity of m's. This completes the proof. □

Combining the results above completes the proof. □

Proof of Corollary 1. In Region (2), the testing threshold τ is determined by the following polynomial equation:

$$2\,\delta\,(\omega+\pi)\,\tau^3+(\alpha-2\,\delta\,\alpha-1)\,\tau^2+\alpha-1=0.$$

As either ω or π increases, it is clear that τ must decrease in order to maintain equality, implying a lower testing threshold.

In Region (3), it is straightforward to see that the threshold $\tau=\frac{1-\alpha}{\delta\,\pi}$ is decreasing in π and constant in ω.

In Region (4), by the implicit function theorem, we have

$$\tau'(\pi)=-\frac{2\,\delta\,\tau^3}{6\,\delta\,(\omega+\pi)\,\tau^2-2\,[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]\,\tau}=-\frac{\delta\,\tau^2}{3\,\delta\,(\omega+\pi)\,\tau-[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]},$$

and the desired result boils down to

$$3\,\delta\,(\omega+\pi)\,\tau-[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]\geq 0\quad\Leftrightarrow\quad\tau\geq\frac{2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)},$$

which follows from

$$\begin{aligned}&2\,\delta\,(\omega+\pi)\,\tau^3-[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]\,\tau^2-(1-\alpha)\,(2\,\delta-1)\,\big|_{\tau=\frac{2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}}\\&=\frac{1}{9\,\delta^2\,(\omega+\pi)^2}\left[-\frac{1}{3}\,[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]^3-9\,\delta^2\,(\omega+\pi)^2\,(1-\alpha)\,(2\,\delta-1)\right]\\&<0.\end{aligned}$$

In the proof of Proposition 4, we have shown that function

$$2\,\delta\,(\omega+\pi)\,\theta^3-[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]\,\theta^2-(1-\alpha)\,(2\,\delta-1)$$

is either increasing or first decreasing and then increasing in θ, with the values at the two endpoints (0 and 1) being negative and positive, respectively. Therefore, the inequality above implies that $\tau\geq\frac{2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}$ and thus the desired monotonicity. The monotonicity of τ in ω follows from the fact that ω appears only in the coefficient of τ^3 in the polynomial equation.

In Regions (5) and (7), the thresholds $\frac{1-\alpha}{\delta\,\pi-\alpha}$ and $\frac{1}{\delta\,\pi}$ are both decreasing in π and constant in ω.

In Region (6), the threshold $\frac{\delta\,\pi+1}{2\,\delta\,(\omega+\pi)}$ is decreasing in ω.

This completes the proof. □

Appendix C: Proofs of Section 3.2

Proof of Lemma 2. It is straightforward to verify that the funder's expected payoff is reduced to

$$\int_{\check{\tau}}^{\theta_H}[\theta\,\omega-\check{m}_0-\theta\,\check{m}_S-(1-\theta)\,\check{m}_F]\,\mathrm{d}F(\theta)$$

under the threshold mechanism (18).

Invoking Lemma EC.1, we know that the (IC) constraint is equivalent to (EC.2), (EC.3), and (EC.4). By substituting the decision variables with the threshold mechanism defined in (18), the incentive constraints are further reduced as follows:

$$\check{m}_S-\check{m}_F+\pi\geq 0,\ \text{and}\tag{21}$$

$$\check{m}_0+\delta\,\check{\tau}\,\check{m}_S+\delta\,(1-\check{\tau})\,\check{m}_F=c-\delta\,\pi\,\check{\tau},\tag{22}$$

$$\check{m}_0+\delta\,\check{m}_F\leq\alpha\,c,\tag{23}$$

$$-c+\check{m}_0+\delta\,\theta\,(\check{m}_S+\pi)+\delta\,(1-\theta)\,\check{m}_F\geq-\alpha\,c+\check{m}_0+\delta\,\check{m}_F,\quad\forall\theta\in[\theta_L,\theta_H],\tag{EC.12}$$

where (21) and (22) come from (EC.3) and (EC.4), respectively. Constraints (23) and (EC.12) are derived from the moral hazard constraint (EC.2), ensuring that firms of types below and above $\check{\tau}$ have no incentive to (misreport and) shirk. Constraint (EC.12) is implied by (21) and (23).

The participation constraint (IR) and the feasibility condition for q hold automatically by construction. The nonnegativity condition (12) boils down to (24).

This completes the proof. □

Proof of Proposition 8. It is straightforward to verify the feasibility of our candidate solution. Given $\check{\tau}$, the term $\omega \int_{\check{\tau}}^{\theta_H} \theta \,\mathrm{d}F(\theta)$ is constant with respect to the decision variables. For notational simplicity, we omit this constant from the objective function. The simplified "objective function" is thus

$$\int_{\check{\tau}}^{\theta_H} \left[-\check{m}_0 - \theta\, \check{m}_S - (1-\theta)\, \check{m}_F\right] \mathrm{d}F(\theta).$$

In case 1, define

$$\phi = -\frac{1}{\delta\, \check{\tau}} \int_{\check{\tau}}^{\theta_H} \theta \,\mathrm{d}F(\theta), \quad \eta = -\left[1 - F(\check{\tau})\right] + \frac{1}{\delta\, \check{\tau}} \int_{\check{\tau}}^{\theta_H} \theta \,\mathrm{d}F(\theta), \text{ and } \nu = (1-\delta)\, \left[1 - F(\check{\tau})\right] \geq 0.$$

The nonnegativity of η follows from its monotonicity in $\check{\tau}$ and

$$\left[-\left[1 - F(\check{\tau})\right] + \frac{1}{\delta\, \check{\tau}} \int_{\check{\tau}}^{\theta_H} \theta \,\mathrm{d}F(\theta)\right]_{\check{\tau}=\theta_H} = 0.$$

We have

$$\phi\, \left[\check{m}_0 + \delta\, \check{\tau}\, \check{m}_S + \delta\, (1-\check{\tau})\, \check{m}_F\right] + \eta\, (\check{m}_0 + \delta\, \check{m}_F) - \nu\, \check{m}_F = (\phi + \eta)\, \check{m}_0 + \delta\, \check{\tau}\, \phi\, \check{m}_S + \left[\delta\, (1-\check{\tau})\, \phi + \delta\, \eta - \nu\right] \check{m}_F,$$

and hence, the objective function is bounded by $(1 - \delta\, \pi\, \check{\tau})\, \phi + \alpha\, \eta$. Substituting our candidate policy into the objective function attains this upper bound, establishing its optimality.

In case 2, define

$$\begin{aligned}
\phi &= \frac{1}{\delta\, \check{\tau}} \left[-\delta\, \left[1 - F(\check{\tau})\right] + \int_{\check{\tau}}^{\theta_H} (1-\theta) \,\mathrm{d}F(\theta)\right], \\
\eta &= \frac{1-\check{\tau}}{\check{\tau}}\, \left[1 - F(\check{\tau})\right] - \frac{1}{\delta\, \check{\tau}} \int_{\check{\tau}}^{\theta_H} (1-\theta) \,\mathrm{d}F(\theta), \text{ and} \\
\xi &= (1-\delta)\, \left[1 - F(\check{\tau})\right] \geq 0,
\end{aligned}$$

where the nonnegativity of η follows from $\mathcal{H}(\check{\tau}) \geq 0$. We have

$$\phi\, \left[\check{m}_0 + \delta\, \check{\tau}\, \check{m}_S + \delta\, (1-\check{\tau})\, \check{m}_F\right] + \eta\, (\check{m}_0 + \delta\, \check{m}_F) - \xi\, \check{m}_S = (\phi + \eta)\, \check{m}_0 + (\delta\, \check{\tau}\, \phi - \xi)\, \check{m}_S + \left[\delta\, (1-\check{\tau})\, \phi + \delta\, \eta\right] \check{m}_F,$$

and hence, the objective function is bounded by $(1 - \delta\, \pi\, \check{\tau})\, \phi + \alpha\, \eta$. Substituting our candidate policy into the objective function attains this upper bound, establishing its optimality.

In case 3, define

$$\begin{aligned}
\phi &= -\left[1 - F(\check{\tau})\right], \\
\xi &= -\delta\, \check{\tau}\, \left[1 - F(\check{\tau})\right] + \int_{\check{\tau}}^{\theta_H} \theta \,\mathrm{d}F(\theta) \geq 0, \text{ and} \\
\nu &= -\delta\, (1-\check{\tau})\, \left[1 - F(\check{\tau})\right] + \int_{\check{\tau}}^{\theta_H} (1-\theta) \,\mathrm{d}F(\theta),
\end{aligned}$$

where the nonnegativity of ν follows from $\mathcal{H}(\check{\tau}) \leq 0$. The nonnegativity of ξ follows from that of η in case 1. We have

$$\phi\left[\check{m}_0 + \delta\,\check{\tau}\,\check{m}_S + \delta\,(1-\check{\tau})\,\check{m}_F\right] - \xi\,\check{m}_S - \nu\,\check{m}_F = \phi\,\check{m}_0 + (\delta\,\check{\tau}\,\phi - \xi)\,\check{m}_S + \left[\delta\,(1-\check{\tau})\,\phi - \nu\right]\check{m}_F,$$

and hence, the objective function is bounded by $(1-\delta\,\pi\,\check{\tau})\,\phi$. Substituting our candidate policy into the objective function attains this upper bound, establishing its optimality.

In case 4, define

$$\begin{aligned}
\phi &= -\frac{1}{\delta\,(1-\check{\tau})}\int_{\check{\tau}}^{\theta_H}(1-\theta)\,\mathrm{d}F(\theta),\\
\gamma &= [1-F(\check{\tau})] - \frac{1}{\delta\,(1-\check{\tau})}\int_{\check{\tau}}^{\theta_H}(1-\theta)\,\mathrm{d}F(\theta), \text{ and}\\
\xi &= \frac{1}{1-\check{\tau}}\left[\int_{\check{\tau}}^{\theta_H}\theta\,\mathrm{d}F(\theta) - \check{\tau}\,[1-F(\check{\tau})]\right],
\end{aligned}$$

where the nonnegativity of γ and ξ follows from $\mathcal{H}(\check{\tau}) \geq 0$ and its monotonicity, respectively. We have

$$\phi\left[\check{m}_0 + \delta\,\check{\tau}\,\check{m}_S + \delta\,(1-\check{\tau})\,\check{m}_F\right] - \gamma\,\check{m}_0 - \xi\,\check{m}_S = (\phi-\gamma)\,\check{m}_0 + (\delta\,\check{\tau}\,\phi - \xi)\,\check{m}_S + \delta\,(1-\check{\tau})\,\phi\,\check{m}_F,$$

and hence, the objective function is bounded by $(1-\delta\,\pi\,\check{\tau})\,\phi$. Substituting our candidate policy into the objective function attains this upper bound, establishing its optimality.

This completes the proof. □

Appendix D: Proofs of Section 4

Proof of Proposition 9. We prove the desired result in the following steps.

When $\theta \geq \frac{1}{\delta\,\pi}$, ignore constraints (25) and (27), and the optimal solution to the relaxed problem is given by

$$q = 1, \quad m_0 = m_S = m_F = 0,$$

satisfying the two ignored constraints automatically.

Next, we prove the following two lemmas to simplify the problem.

LEMMA EC.6. *Without loss of generality, we can focus on mechanisms satisfying* $m_F = 0$.

Proof. Given any mechanism $(\hat{q}, \hat{m}_0, \hat{m}_S, \hat{m}_F)$ with $\hat{m}_F > 0$, we propose the following mechanism $(\tilde{q}, \tilde{m}_0, \tilde{m}_S, \tilde{m}_F)$, where

$$\begin{aligned}
\tilde{q} &= \hat{q},\\
\tilde{m}_0 &= \hat{m}_0 + \delta\,(1-\theta)\,\hat{m}_F,\\
\tilde{m}_S &= \hat{m}_S,\\
\tilde{m}_F &= 0.
\end{aligned}$$

The new mechanism is feasible and generates a (weakly) higher expected payoff for the principal. This completes the proof. □

LEMMA EC.7. *When* $\theta < \frac{1}{\delta\,\pi}$, *the agent's expected payoff is always* 0.

Proof. Suppose the agent's expected payoff is strictly positive when $\theta < \frac{1}{\delta\,\pi}$. The principal can always lower the compensation levels m's to make (IR) hold with equality:

1. first, if $m_0 > 0$, the principal can always lower m_0 without affecting the moral hazard constraint; and
2. if the agent's expected payoff is still strictly positive after setting $m_0 = 0$, it is always feasible to decrease m_S to $\tilde{m}_S$ to make (IR) hold with equality. The moral hazard constraint is automatically satisfied, because

$$-(1-\alpha)\,q + \delta\,\theta\,(\tilde{m}_S + \pi\,q) \geq -q + \delta\,\theta\,(\tilde{m}_S + \pi\,q) = 0.$$

This completes the proof.

When $\theta < \frac{1}{\delta\,\pi}$, we can use constraint (IR) to eliminate m_0, i.e.,

$$m_0 = q - \delta\,\theta\,(m_S + \pi\,q).$$

The funder's problem can be reformulated as follows:

$$\begin{aligned} \max_{q,m_S}\ & (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S, \\ \text{subject to}\ & -[\delta\,\theta\,\pi - (1-\alpha)]\,q - \delta\,\theta\,m_S \leq 0, && (\eta) \\ & -q \leq 0, && (\beta) \\ & q \leq 1, && (\sigma) \\ & (\delta\,\theta\,\pi - 1)\,q + \delta\,\theta\,m_S \leq 0, && (\gamma) \\ & -m_S \leq 0, && (\xi) \end{aligned}$$

where the Greek letters at the end of each inequality represent the shadow prices associated with this constraint.

When $\max\left\{\frac{1}{\omega+\delta\,\pi}, \frac{1-\alpha}{\delta\,\pi}\right\} \leq \theta \leq \frac{1}{\delta\,\pi}$, define

$$\sigma \equiv \theta\,\omega + \delta\,\theta\,\pi - 1, \quad \xi \equiv \theta\,(1-\delta).$$

It is straightforward to verify that both σ and ξ are nonnegative. We have

$$\sigma \cdot q - \xi \cdot m_S = (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S,$$

and hence, the objective function is bounded by

$$\sigma = \theta\,\omega + \delta\,\theta\,\pi - 1.$$

Plugging our conjectured solution into the objective function yields the upper bound, establishing the optimality.

When $\frac{\delta\,\alpha-\alpha+1}{\delta\,(\omega+\pi)} \leq \theta \leq \frac{1-\alpha}{\delta\,\pi}$,define

$$\eta \equiv \frac{1-\delta}{\delta}, \quad \sigma \equiv \frac{1}{\delta}\,[\delta\,\theta\,(\omega+\pi) - (\delta\,\alpha - \alpha + 1)]\,.$$

It is straightforward to verify that both η and σ are nonnegative. We have

$$\eta \cdot [-[\delta\,\theta\,\pi - (1-\alpha)]\,q - \delta\,\theta\,m_S] + \sigma \cdot q = (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S,$$

and hence, the objective function is bounded by

$$\sigma = \frac{1}{\delta}\left[\delta\,\theta\,(\omega+\pi) - (\delta\,\alpha - \alpha + 1)\right].$$

Plugging our conjectured solution into the objective function yields

$$(\theta\,\omega + \delta\,\theta\,\pi - 1) - \theta\,(1-\delta)\left(\frac{1-\alpha}{\delta\,\theta} - \pi\right) = \frac{1}{\delta}\left[\delta\,\theta\,(\omega+\pi) - (\delta\,\alpha - \alpha + 1)\right],$$

attaining the upper bound.

When $\theta \leq \frac{1}{\omega+\delta\,\pi}$, define

$$\beta \equiv -\theta\,\omega - \delta\,\theta\,\pi + 1, \quad \xi \equiv \theta\,(1-\delta).$$

It is straightforward to verify that both β and ξ are nonnegative. We have

$$-\beta\cdot q - \xi\cdot m_S = (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S,$$

and hence, the objective function is bounded by 0. Plugging our conjectured solution into the objective function yields the upper bound.

When $\theta \leq \frac{\delta\,\alpha-\alpha+1}{\delta\,(\omega+\pi)}$, define

$$\eta \equiv \frac{1-\delta}{\delta}, \quad \beta \equiv -\frac{1}{\delta}\left[\delta\,\theta\,(\omega+\pi) - (\delta\,\alpha - \alpha + 1)\right].$$

It is straightforward to verify that both η and β are nonnegative. We have

$$\eta\cdot\left[-\left[\delta\,\theta\,\pi - (1-\alpha)\right]q - \delta\,\theta\,m_S\right] - \beta\cdot q = (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S,$$

and hence, the objective function is bounded by 0. Plugging our conjectured solution into the objective function yields the upper bound.

This completes the proof. □

Proof of Proposition 10. If $\theta \geq \frac{1}{\delta\,\pi}$, the firm is capable of self-funding, and the optimal mechanism is to grant no external funding.

If $\theta < \frac{1}{\delta\,\pi}$, the individual rationality constraint (27) must bind; otherwise, the funder can strictly improve its own payoff by reducing the funding amount. We can solve (27) as an equality for m_0, i.e.,

$$m_0 = q - \delta\,\theta\,(m_S + \pi\,q) - \delta\,(1-\theta)\,m_F,$$

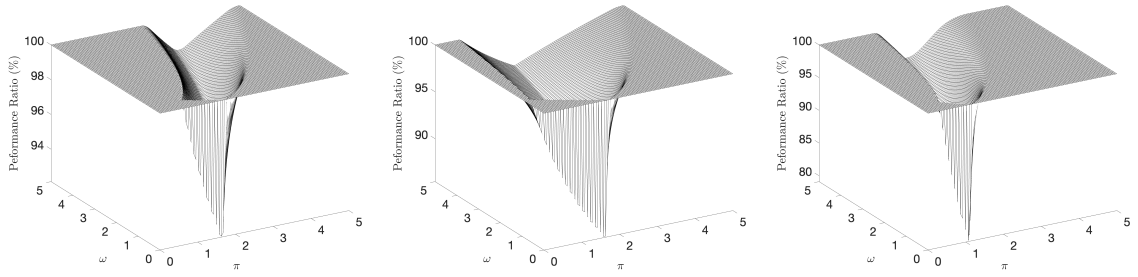
and substitute it back into the objective function, yielding

$$(\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - (1-\delta)\,\theta\,m_S - (1-\delta)\,(1-\theta)\,m_F.$$

If $\theta\,\omega + \delta\,\theta\,\pi - 1 \leq 0$, it is optimal to set $q = 0$, i.e., not to fund the test.

If $\theta\,\omega + \delta\,\theta\,\pi - 1 > 0$, ignoring the nonnegativity constraint on m_0, it is optimal to set $q = 1$, $m_S = m_F = 0$, and the resulting $m_0 = 1 - \delta\,\theta\,\pi > 0$, satisfying the previously ignored constraint.

This completes the proof. □



(a) $\alpha = 0.5$ and $\delta = 0.6$ (b) $\alpha = 0.9$ and $\delta = 0.6$ (c) $\alpha = 0.5$ and $\delta = 0.9$

Figure EC.1 Rerformance ratios (in percentage) when rewarding failure is prohibited under different values of α and δ ($c = 1$ and $\theta \sim \mathcal{U}_{[0,1]}$).

Appendix E: No Reward for Failure

In this section, we quantify the value of allowing the mechanism to reward failure under Assumption 2. That is, consider a benchmark case in which the funder is not allowed to reward failure, i.e., add the constraint $m_F(\theta) = 0$ for all $\theta \in [0,1]$ to the optimization problem (14). Denote $\hat{u}$ to be the corresponding objective value, in place of u^* in (14). Consider the performance ratio without rewarding failure, i.e.,[9]

$$\text{Performance Ratio} \equiv \frac{\hat{u}}{u^*}.$$

Figures EC.1(a) and (b) compare the performance ratios for different α values. It appears that the parameter region where the performance ratios are strictly below 100% expands as the value of α increases, and the minimal ratio can be as low as 85% in Figure EC.1(b). This highlights the importance of considering the possibility of rewarding failure in policy design.

Figures EC.1(a) and (c) compare the performance ratios under various discount factors δ. As δ increases from 0.6 to 0.9, the minimal performance ratio decreases from 92% in Figure EC.1(a) to 79% in (c). The intuition is that as the firm discounts the future less (i.e., as δ increases), future payments become more effective, making the reward for failure a more effective tool.

Appendix F: Calibration notes

In calibrating the model, we assume that the firm captures half of the drug's value. Figure EC.2 illustrates why a 50% share is plausible. With linear demand ($p(q)$), a monopoly price (p^m), and constant variable cost (v) set at 20% of the maximum willingness to pay, total surplus is split as follows: consumer welfare (ω) receives 25%, firm profit (π) 50%, and the remainder to variable costs. Similar figures appear in standard texts such as Mas-Colell et al. (1995) and Samuelson and Nordhaus (2009).

[9] When $u^* = \hat{u} = 0$, for completeness of the figures, we set the performance ratio to 100%.

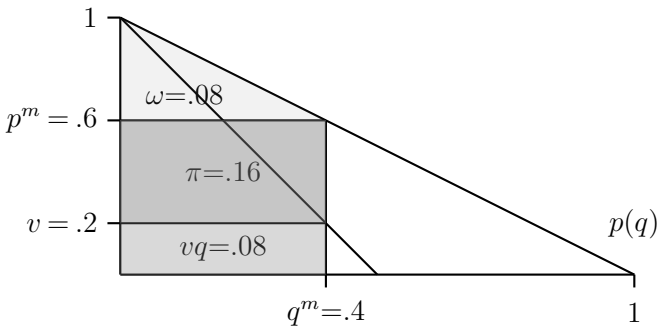


Figure EC.2 The assumed division of consumer welfare (ω), profit (π), and spending on variable cost in the calibration exercise.