Intigrate over x

$$\begin{aligned}
&\int p_{ij}\left(x_{ij}\right)\mathcal{N}\left(x_{ij};\tilde{\mu}_{ij}\left(z_i\right),w_j^2\right)dx_{ij} \\
=&\int \exp\left(-\lambda x_{ij}^2\right)\frac{1}{\sqrt{2\pi}w_j}\exp\left(-\frac{\left(x_{ij}-\tilde{\mu}_{ij}\right)^2}{2w_j^2}\right)dx_{ij} \\
=&\frac{1}{\sqrt{1+2\lambda w_j^2}}\exp\left(-\frac{\lambda\tilde{\mu}_{ij}^2}{1+2\lambda w_j^2}\right)
\end{aligned} \tag{1}$$

The conditional distribution of y is

$$\begin{aligned}
\log p\left(\mathbf{y}_i|\mathbf{z}_i\right) = &\sum_{j:y_{ij}=0}\log\left[\frac{1}{\sqrt{1+2\lambda w_j^2}}\exp\left(-\frac{\lambda\tilde{\mu}_{ij}^2}{1+2\lambda w_j^2}\right)\right] \\
&+\sum_{j:y_{ij}\neq 0}\log\left[\left(1-\exp\left(-\lambda y_{ij}^2\right)\right)\frac{1}{\sqrt{2\pi}w_j}\exp\left(-\frac{\left(y_{ij}-\tilde{\mu}_{ij}\right)^2}{2w_j^2}\right)\right].
\end{aligned} \tag{2}$$

Our target is to get $p(z_i|y_i)$, so we can use VAE.

Encoder:

$$\begin{aligned}
\boldsymbol{\mu}_i,\boldsymbol{\sigma}_i^2 &= f_\phi\left(\mathbf{y}_i\right) \\
q_\phi\left(\mathbf{z}_i|\mathbf{y}_i\right) &= \mathcal{N}\left(\boldsymbol{\mu}_i,\boldsymbol{\sigma}_i^2\right)
\end{aligned} \tag{3}$$

Decoeder:

$$p\left(\mathbf{z}_i\right)=\mathcal{N}(\mathbf{0},\mathbf{I}) \tag{4}$$

$$\tilde{\mathbf{x}}_i = f_\theta\left(\mathbf{z}_i\right) \tag{5}$$

The evidence lower bound (ELBO):

$$elbo = \mathbb{E}_{q_\phi(\mathbf{z}_i|\mathbf{y}_i)}\left[\log p_{\boldsymbol{\theta},\mathbf{w},\boldsymbol{\lambda}}\left(\mathbf{y}_i|\mathbf{z}_i\right)\right] - D_{KL}\left(q_\phi\left(\mathbf{z}_i|\mathbf{y}_i\right)\|p\left(\mathbf{z}_i\right)\right) \tag{6}$$