Tutorial 2

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Overview

1. Review
2. Exercises
3. Matrix Operations in R
4. Importing Data to R

Matrix

Eigenvalue λ with corresponding eigenvector $\mathbf{x} \neq 0$ if

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Ordinarily, x is normalized so that it has length unity; that is $x'x = 1$

Positive Definite Matrices

Spectral decomposition for symmetric matrices

$$\mathbf{A}_{(k\times k)} = \lambda_1 \mathbf{e}_1 \mathbf{e}_1' + \lambda_2 \mathbf{e}_2 \mathbf{e}_2' + \cdots + \lambda_k \mathbf{e}_k \mathbf{e}_k'$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$ are the associated normalized $k \times 1$ eigenvectors. $\mathbf{e}_i'\mathbf{e}_i = 1$ for $i = 1, 2, \ldots, k$ and $\mathbf{e}_i'\mathbf{e}_j = 0$ for $i \neq j$

A is positive definite if

$$0 < \mathbf{x}'\mathbf{A}\mathbf{x}$$

for all vectors $\mathbf{x} \neq 0$

Random Vectors and Matrices

- The expected value of a random matrix

$$\mathrm{E}(\boldsymbol{X}) = \begin{bmatrix} \mathrm{E}(X_{11}) & \mathrm{E}(X_{12}) & \cdots & \mathrm{E}(X_{1p}) \\ \mathrm{E}(X_{21}) & \mathrm{E}(X_{22}) & \cdots & \mathrm{E}(X_{2p}) \\ & & \cdots & \mathrm{E}(X_{np}) \\ \mathrm{E}(X_{n1}) & \mathrm{E}(X_{n2}) & \cdots & \mathrm{E}(X_{np}) \end{bmatrix}$$

•$\mathrm{E}(\boldsymbol{X} + \boldsymbol{Y}) = \mathrm{E}(\boldsymbol{X}) + \mathrm{E}(\boldsymbol{Y})$
•$\mathrm{E}(\mathbf{A}\boldsymbol{X}\mathbf{B}) = \mathbf{A}\mathrm{E}(\boldsymbol{X})\mathbf{B}$

The Mean Vector and Covariance Matrix for Linear Combinations of Random Variables

The linear combination $\mathbf{c}'\boldsymbol{X} = c_1X_1 + \cdots + c_pX_p$ has

$$\begin{aligned} \text{mean} &= \mathrm{E}\left(\mathbf{c}'\boldsymbol{X}\right) = \mathbf{c}'\boldsymbol{\mu} \\ \text{variance } &= \mathrm{Var}\left(\mathbf{c}'\boldsymbol{X}\right) = \mathbf{c}'\Sigma\mathbf{c} \end{aligned}$$

where $\boldsymbol{\mu} = \mathrm{E}(\boldsymbol{X})$ and $\Sigma = \mathrm{Cov}(\boldsymbol{X})$
Let C be a matrix, then the linear combinations of $\mathbf{Z} = \mathbf{C}X$ have

$$\begin{aligned} \mu_{\mathrm{Z}} &= \mathrm{E}(\mathrm{Z}) = \mathrm{E}(\mathrm{CX}) = \mathrm{C}\mu_{\mathrm{x}} \\ \Sigma_{\mathrm{Z}} &= \mathrm{Cov}(\mathrm{Z}) = \mathrm{Cov}(\mathrm{CX}) = \mathrm{C}\Sigma_{\mathrm{X}}\mathrm{C}' \end{aligned}$$

Example 1

$\boldsymbol{A}$ is a positive definite matrix. Show that $\boldsymbol{A}^{-1}$is also a positive definite matrix.

Solutions: Let $\boldsymbol{x} \neq \mathbf{0}$. Then

$$\boldsymbol{x}'\boldsymbol{A}^{-1}\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{A}^{-1}\boldsymbol{x} = \left(\boldsymbol{A}^{-1}\boldsymbol{x}\right)'\boldsymbol{A}\left(\boldsymbol{A}^{-1}\boldsymbol{x}\right) > 0$$

since $\boldsymbol{A}^{-1}\boldsymbol{x} \neq \mathbf{0}$ if $\boldsymbol{x} \neq \mathbf{0}$. So, $\boldsymbol{A}^{-1}$is also a positive definite matrix.

Example 2

If λ is a non-zero eigenvalue of $\mathbf{M}'\mathbf{M}$ with eigenvector $\mathbf{u}$, show that λ is an eigenvalue of $\mathbf{M}\mathbf{M}'$ with eigenvector $\mathbf{Mu}$

Solutions: $\mathbf{M}^T\mathbf{Mu} = \lambda\mathbf{u} \neq \mathbf{0}$, so certainly $\mathbf{Mu} \neq \mathbf{0}$. It is an eigenvector of $\mathbf{M}\mathbf{M}'$ with eigenvalue λ, since

$$\mathbf{M}\mathbf{M}'(\mathbf{Mu}) = \mathbf{M}\left(\mathbf{M}'\mathbf{Mu}\right) = \mathbf{M}(\lambda\mathbf{u}) = \lambda(\mathbf{Mu})$$

Example 3

Suppose $\boldsymbol{X} = [X_1, X_2, X_3]$ is a 3×1 random vector with mean $\boldsymbol{\mu}' = [2, -3, 1]$ and covariance

$$\Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

Let $\boldsymbol{Y} = (3X_1 + X_3, X_1 + X_2)$,find $\mathrm{E}(\boldsymbol{Y})$ and $\mathrm{Cov}(\boldsymbol{Y})$.

Note that

$$\mathbf{Y} = \mathbf{C}\boldsymbol{X}$$

where $\boldsymbol{C} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Then

$$\mathrm{E}(\mathbf{Y}) = \mathrm{E}(\mathbf{C}\boldsymbol{X}) = \mathbf{C}\boldsymbol{\mu} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\mathrm{Cov}(\mathbf{Y}) = \mathrm{Cov}(\mathbf{CX}) = \mathbf{C}\Sigma\mathbf{C}' = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 9 \\ 9 & 6 \end{bmatrix}$$

Example 4

Suppose $\boldsymbol{X} = [X_1, X_2]$ is a 2×1 random vector with $\boldsymbol{\mu}' = [2, -3]$ and covariance

$$\Sigma = \left[\begin{array}{cc} 1 & 1 \\ 1 & 3 \end{array} \right]$$

Find a 2×1 vector $\boldsymbol{a}$ such that $\mathrm{Cov}(X_1, Z) = 0$, where $Z = X_1 - \mathbf{a}' \left[\begin{array}{c} X_1 \\ X_2 \end{array} \right]$.

Soluitons:

Let $\boldsymbol{a}' = [a_1, a_2]$, Z and X_1 can be written as linear combination of $\boldsymbol{X}$

$$Z = [1 - a_1, -a_2] \left[\begin{array}{c} X_1 \\ X_2 \end{array} \right] = \boldsymbol{c'X}$$

$$X_1 = [1, 0] \left[\begin{array}{c} X_1 \\ X_2 \end{array} \right] = \boldsymbol{d'X}$$

Then $\mathrm{E}Z = \mathrm{E}\left(\mathbf{c}'\boldsymbol{X}\right) = \mathbf{c}'\boldsymbol{\mu}$, and $\mathrm{E}X_1 = \mathrm{E}\left(\mathbf{d}'\boldsymbol{X}\right) = \mathbf{d}'\boldsymbol{\mu}$

$$\begin{aligned} \mathrm{Cov}(X_1, Z) &= \mathrm{E}\left(Z - \mathrm{E}(Z)\right)\left(X_1 - \mathrm{E}(X_1)\right) \\ &= \mathrm{E}\left[\boldsymbol{c'(X - \mu)(X - \mu)'d}\right] \\ &= \boldsymbol{c'}\mathrm{E}\left[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})'\right]\boldsymbol{d} \\ &= \boldsymbol{c'}\Sigma\boldsymbol{d} = 1 - a_1 - a_2 = 0 \end{aligned}$$

Thus $a_1 + a_2 = 1$

Example 5

Suppose $\boldsymbol{X} = [X_1, X_2, \ldots, X_k]$ is a $k \times 1$ random vectors with the mean vector $\boldsymbol{\mu}$ and the covariance matrix Σ . Spectral decomposition for symmetric matrix Σ

$$\Sigma = \lambda_1 \mathbf{e}_1 \mathbf{e}_1' + \lambda_2 \mathbf{e}_2 \mathbf{e}_2' + \cdots + \lambda_k \mathbf{e}_k \mathbf{e}_k'$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$ are the associated normalized $k \times 1$ eigenvectors. $\mathbf{e}_i' \mathbf{e}_i = 1$ for $i = 1, 2, \ldots, k$ and $\mathbf{e}_i' \mathbf{e}_j = 0$ for $i \neq j$.
Show $\mathrm{Cov}(Y_i, Y_j) = 0$ for $i \neq j \in \{1, \ldots, k\}$ and
$\mathrm{Var}\,(Y_i) = \lambda_i, i = 1, \ldots, k$ where $Y_i = \mathbf{e}_i' \mathbf{X} = e_{i1} X_1 + e_{i2} X_2 + \cdots + e_{ip} X_k$.

Proof: $\mathrm{E}Y_i = \mathrm{E}(\mathbf{e}_i'\mathbf{X}) = \mathbf{e}_i'\boldsymbol{\mu}$ Then

$$\begin{aligned}
\mathrm{Cov}(Y_i, Y_j) &= \mathrm{E}\left[(Y_i - \mathrm{E}Y_i)(Y_j - \mathrm{E}Y_j)\right] \\
&= \mathrm{E}\left[\boldsymbol{e_i'(X - \mu)(X - \mu)'e_j}\right] \\
&= \boldsymbol{e_i'}\mathrm{E}\left[(\boldsymbol{X} - \mu)(\boldsymbol{X} - \mu)'\right]\boldsymbol{e_j} \\
&= \boldsymbol{e_i'}\Sigma\boldsymbol{e_j} \\
&= \boldsymbol{e_i'}\left(\lambda_1\mathbf{e}_1\mathbf{e}_1' + \lambda_1\mathbf{e}_2\mathbf{e}_2' + \cdots + \lambda_1\mathbf{e}_k\mathbf{e}_k'\right)\boldsymbol{e_j} \\
&= \boldsymbol{e_i'}(\lambda_j\boldsymbol{e_j}) = 0
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}(Y_i) &= \mathrm{E}\left[(Y_i - \mathrm{E}Y_i)^2\right] \\
&= \mathrm{E}\left[\boldsymbol{e_i'(X-\mu)(X-\mu)'e_i}\right] \\
&= \boldsymbol{e_i'}\mathrm{E}\left[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})'\right]\boldsymbol{e_i} \\
&= \boldsymbol{e_i'}\Sigma\boldsymbol{e_i} \\
&= \boldsymbol{e_i'}\left(\lambda_1\mathbf{e}_1\mathbf{e}_1' + \lambda_1\mathbf{e}_2\mathbf{e}_2' + \cdots + \lambda_1\mathbf{e}_k\mathbf{e}_k'\right)\boldsymbol{e_i} \\
&= \boldsymbol{e_i'}(\lambda_i\boldsymbol{e_i}) = \lambda_i
\end{aligned}
$$

Example 6

Suppose $\boldsymbol{X}$ is a 2×1 random vector with mean vector $\boldsymbol{\mu}_X' = [2, -3]$ and covariance matrix

$$\Sigma_X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$\boldsymbol{Y}$ is a 2×1 random vector with mean vector $\boldsymbol{\mu}_Y' = [2, 1]$ and covariance matrix

$$\Sigma_Y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Supppose $\mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{0}$ and $\boldsymbol{a}' = [1, -1]$
Find the mean $\mathrm{E}(2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y})$ variance $\mathrm{Var}(2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y})$

Soluitons

Let $U = (2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y})$
$\mathrm{E}U = \mathrm{E}(2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y}) = 2\mathrm{E}(\boldsymbol{a}'\boldsymbol{X}) + 3\mathrm{E}(\boldsymbol{a}'\boldsymbol{Y}) = 2\boldsymbol{a}'\boldsymbol{\mu}_X + 3\boldsymbol{a}'\boldsymbol{\mu}_Y = 13.$

$$
\begin{aligned}
&\mathrm{Var}(2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y}) \\
&= \mathrm{E}((2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y}) - \mathrm{E}(2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y}))^2 \\
&= \mathrm{E}\left(2\boldsymbol{a}'(\boldsymbol{X} - \boldsymbol{\mu}_X) + 3\boldsymbol{a}'(\boldsymbol{Y} - \boldsymbol{\mu}_Y)\right)^2 \\
&= 4\boldsymbol{a}'\mathrm{E}(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{X} - \boldsymbol{\mu}_X)'\boldsymbol{a} + 12\boldsymbol{a}'\mathrm{E}(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{Y} - \boldsymbol{\mu}_Y)'\boldsymbol{a} \\
&+ 9\boldsymbol{a}'\mathrm{E}(\boldsymbol{Y} - \boldsymbol{\mu}_Y)(\boldsymbol{Y} - \boldsymbol{\mu}_Y)'\boldsymbol{a} \\
&= 4\mathbf{a}'\Sigma_X\mathbf{a} + \mathbf{0} + 9\mathbf{a}'\Sigma_Y\mathbf{a} = 17
\end{aligned}
$$

Let $\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{X} \\ \cdots \\ \boldsymbol{Y} \end{bmatrix}$, then $\mathrm{E}\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{\mu}_X \\ \cdots \\ \boldsymbol{\mu}_Y \end{bmatrix}$, and

$$\mathrm{Cov}\boldsymbol{Z} = \begin{bmatrix} \Sigma_X & \mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Y}) \\ \mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Y}) & \Sigma_Y \end{bmatrix} = \begin{bmatrix} \Sigma_X & \mathbf{0} \\ \mathbf{0} & \Sigma_Y \end{bmatrix}$$

$U = 2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y} = \boldsymbol{c}'\boldsymbol{Z}$, where $\boldsymbol{c} = \begin{bmatrix} 2\boldsymbol{a} \\ \cdots \\ 3\boldsymbol{a} \end{bmatrix}$

Thus

$$\mathrm{E}(2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y}) = \mathrm{E}\left(\mathbf{c}'\boldsymbol{Z}\right) = \mathbf{c}'\,\mathrm{E}\left(\boldsymbol{Z}\right) = 13$$

$$\mathrm{Var}(2\boldsymbol{a}'\boldsymbol{X} + 3\boldsymbol{a}'\boldsymbol{Y}) = \mathrm{Var}\left(\mathbf{c}'\boldsymbol{Z}\right) = \mathbf{c}'\mathrm{Cov}\boldsymbol{Z}\mathbf{c} = 4\mathbf{a}'\Sigma_X\mathbf{a} + 9\mathbf{a}'\Sigma_Y\mathbf{a} = 17$$

Multivariate Statistical Techniques: Matrix Operations in R

- Multiplication by a Scalar:

$$c * A$$

- Matrix Addition and Subtraction:

$$A + B, A - B$$

- Element-wise multiplication:

$$A * B$$

- Matrix Multiplication:

$$A\% * \%B$$

Multiplies two matrices, if they are conformable. If one argument is a vector, it will be promoted to either a row or column matrix to make the two arguments conformable. If both are vectors of the same length, it will return the inner product (as a matrix).

- Computing Column & Row Means

$$colMeans(A), rowMeans((A), mean(A)$$

- Computing Column & Row Sums

$$colSums(A), rowSums(A), sum(A)$$

- Horizontal Concatenation

$$cbind(A, B)$$

- Vertical Concatenation

$$rbind(A, B)$$

- Transpose of a Matrix:

$$t(A)$$

- Spectral Decomposition of a Matrix:

$$eigen(A)$$

- Inverse of a Matrix:

$$solve(A)$$

Importing Data to R

Importing common data format that we often encounter, such as Excel, or Text data.

- From Excel to R
  Most of the data are saved in MS Excel, and the best way to import this is to save this as in CSV format.

$$\text{MyRData} <- \text{read.csv}(\text{"Data.csv"}, \text{header} = \text{TRUE})$$

The argument header = TRUE tells R that the first row of the data are the labels of every column. If set to FALSE, means the first row of the data are not the labels, but are considered as data points.

- Text data
  In some cases, data are saved in Text (.txt) format. And to import this, we use the read.table function. The data is saved as Data1.txt
  To import this to R, simply run

```
MyRData2 <- read.table("Data1.txt", header = TRUE)
```

Working directory: Where the files are stored with R and how to manipulate those files? Every R session has a default location on your operating system's file structure called the working directory. Unless you specify it otherwise, all files will be read and saved into the working directory.
How to get and setup your working directory.

```
#Get the working directory:
getwd()
#Setup the working directory:
setwd("C:/Documents and Settings/Folder name")
```

There are times, however, when the Text data are saved in the internet, here is an example. To import this to R, of course, make sure of the internet connection first. Next, copy the URL of the data and assign this to any variable name, then apply read.table. Try the codes below,

```
library(RCurl)
web <- "https://raw.githubusercontent.com/alstat/Analysis-with-Programming/master/2013/R/How%20to%20Enter%20Your%20Data/Data.dat"
x <- getURL(web)
y <- read.table(text = x, header = TRUE)
```