Tutorial 3

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Result 3.3 If $\boldsymbol{X}$ is distributed as $N_p(\boldsymbol{\mu}, \Sigma)$, the q linear combinations

$$\mathbf{A}_{(q \times p)} \mathbf{X}_{p \times 1} = \left[\begin{array}{c} a_{11}X_1 + \cdots + a_{1p}X_p \\ a_{21}X_1 + \cdots + a_{2p}X_p \\ \vdots \\ a_{q1}X_1 + \cdots + a_{qp}X_p \end{array} \right]$$

are distributed as $N_q\left(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\Sigma\mathbf{A}'\right)$. Also $\boldsymbol{X}_{p \times 1} + \mathbf{d}_{p \times 1}$, where $\mathbf{d}$ is a vector of constants, is distributed as $N_p(\boldsymbol{\mu} + \mathbf{d}, \Sigma)$

Result 3.5 (a) If $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent, then $\text{Cov}(\boldsymbol{X}_1, \boldsymbol{X}_2) = 0$, a $q_1 \times q_2$ matrix of zeros, where $\boldsymbol{X}_1$ is $q_1 \times 1$ random vector and $\boldsymbol{X}_2$ is $q_2 \times 1$. random vector

(b) If $\begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}$ is $N_{q_1+q_2}\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$, then $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent if and only if $\Sigma_{12} = \Sigma_{21} = 0$

(c) If $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent and are distributed as $N_{q_1}(\boldsymbol{\mu}_1, \Sigma_{11})$ and $N_{q_2}(\boldsymbol{\mu}_2, \Sigma_{22})$, respectively, then $\begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}$ has the multivariate normal distribution

$$N_{q_1+q_2}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}\right)$$

Result 3.6 Let $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}$ be distributed as $N_p(\boldsymbol{\mu}, \Sigma)$ with $\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$, and $|\Sigma_{22}| > 0$. Then the conditional distribution of $\boldsymbol{X}_1$, given that $\boldsymbol{X}_2 = \mathbf{x}_2$ is normal and has

$$\begin{aligned} \text{Mean } &= \boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \\ \text{and} \qquad \text{Covariance } &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \end{aligned}$$

Note that the covariance does not depend on the value $\mathbf{x}_2$ of the conditioning variable.

Result 3.7 Let $\boldsymbol{X}$ be distributed as $N_p(\boldsymbol{\mu}, \Sigma)$ with $|\Sigma| > 0$.
Then $(\boldsymbol{X} - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X} - \boldsymbol{\mu})$ is distributed as χ^2_p, where χ^2_p denotes the chi-square distribution with p degrees of freedom.

Result 3.8 Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ be mutually independent with $\boldsymbol{X}_j$ distributed as $N_p\left(\boldsymbol{\mu}_j, \Sigma\right)$. (Note that each $\boldsymbol{X}_j$ has the same covariance matrix Σ.) Then

$$\mathbf{V}_1 = c_1\boldsymbol{X}_1 + c_2\boldsymbol{X}_2 + \cdots + c_n\boldsymbol{X}_n$$

is distributed as $N_p\left(\sum_{j=1}^n c_j\boldsymbol{\mu}_j, \left(\sum_{j=1}^n c_j^2\right)\Sigma\right)$.Moreover, $\mathbf{V}_1$ and $\mathbf{V}_2 = b_1\boldsymbol{X}_1 + b_2\boldsymbol{X}_2 + \cdots + b_n\boldsymbol{X}_n$ are jointly multivariate normal with covariance matrix

$$\begin{bmatrix} \left(\sum_{j=1}^n c_j^2\right)\Sigma & \mathbf{b}'\mathbf{c}\Sigma \\ \mathbf{b}'\mathbf{c}\Sigma & \left(\sum_{j=1}^n b_j^2\right)\Sigma \end{bmatrix}$$

Consequently, V_1 and V_2 are independent if $\mathbf{b}'\mathbf{c} = \sum_{j=1}^n c_jb_j = 0$

Result 3.13 (The central limit theorem) Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ be independent observations from any population with mean μ and finite covariance Σ. Then $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{\mu})$ has an approximate $N_p(0, \Sigma)$ distribution for large sample sizes. Here n should also be large relative to p
And

$$n(\overline{\boldsymbol{X}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\overline{\boldsymbol{X}} - \boldsymbol{\mu}) \text{ is approximately } \chi^2_p$$

for $n - p$ large.

Example 1

Suppose that $\mathbf{X}_i = (X_{1i}, X_{2i}, X_{3i}, X_{4i})'$, $i = 1, \ldots, n$ are independent, and follow the same multivariate normal distribution $N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = [1, 1, -3, 1]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.5 & 0 & 0.5 \\ 0.5 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0.5 & 0 & 1 \end{bmatrix}$$

(a)Find the mean vector and covariance matrix of the joint distribution of $(X_{1i}- \ X_{2i}, X_{1i} + X_{2i} + X_{3i} + X_{4i})'$, and the mean and variance of the conditional distribution of (X_{1i}, X_{2i}, X_{4i}) given X_{3i}.

(a)1. Let $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Then $(X_{1i} - X_{2i}, X_{1i} + X_{2i} + X_{3i} + X_{4i})' = \boldsymbol{AX}_i$ are distributed as $N_2\left(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\Sigma\mathbf{A}'\right)$,where

$$\mathbf{A}\boldsymbol{\mu} = \mathbf{0}_{2\times 1}$$

$$\mathbf{A}\Sigma\mathbf{A}' = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

(a)2. Suppose that $\mathbf{Y}_i = (X_{1i}, X_{2i}, X_{4i}, X_{3i})'$. It follows the same multivariate normal distribution $N_4(\mu_{new}, \Sigma_{new})$ with $\mu'_{new} = [1, 1, 1, -3]$and

$$\mathbf{\Sigma}_{new} = \left[\begin{array}{cccc} 1 & 0.5 & 0.5 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0.5 & 0.5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Let $\boldsymbol{Y}_i = \left[\begin{array}{c} \boldsymbol{X}_i^{(1)} \\ \boldsymbol{X}_i^{(2)} \end{array} \right]$, where $\boldsymbol{X}_i^{(2)} = X_{3i}$, $\boldsymbol{X}_i^{(1)\prime} = (X_{1i}, X_{2i}, X_{4i})$

$\boldsymbol{\mu}_{new} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$,$\Sigma_{new} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$
Note $\Sigma_{12} = \Sigma_{21} = 0$, so (X_{1i}, X_{2i}, X_{4i}) and X_{3i} are independent.
Then the conditional distribution of (X_{1i}, X_{2i}, X_{4i}) given $X_{3i} = x_3$ is normal has

Mean $= \boldsymbol{\mu}_1 + \Sigma_{12}\,(\mathrm{x}_3 + 3) = \boldsymbol{\mu}_1 = [1, 1, 1]'$
and Covariance

$$= \Sigma_{11} - \Sigma_{12}\Sigma_{21} = \Sigma_{11} = \boldsymbol{\Sigma}_{new} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}.$$

(b) Let $n = 10$, determine the distribution of

$$V_1 = \frac{1}{n} \sum_{i=1}^{n} a_i \mathbf{X}_i$$

and

$$T^2 = n[(\overline{\mathbf{X}} - \boldsymbol{\mu})]' \boldsymbol{\Sigma}^{-1} [(\overline{\mathbf{X}} - \boldsymbol{\mu})]$$

$\overline{\mathbf{X}}$ is the sample mean.
(c) Find $b_i, i = 1, \ldots, n$, such that $V_2 = \sum_{i=1}^{n} b_i \mathbf{X}_i$ and V_1 are statistically independent.

Solutions

(b)
$V_1 = \frac{1}{n}\sum_{i=1}^{n} a_i \mathbf{X}_i$ is distributed as
$N_4\left((\sum_{j=1}^{n} a_j/n)\boldsymbol{\mu}, \left(\left(\sum_{j=1}^{n} a_j^2\right)/n^2\right)\Sigma \right)$. And then $\bar{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i$ is distributed as $N_4\left(\boldsymbol{\mu}, \frac{1}{n}\Sigma\right)$. (Result 3.8)
Thus

$$T^2 = n[(\overline{\mathbf{X}} - \boldsymbol{\mu})]'\boldsymbol{\Sigma}^{-1}[(\overline{\mathbf{X}} - \boldsymbol{\mu})] = [(\overline{\mathbf{X}} - \boldsymbol{\mu})]'\left(\frac{1}{n}\boldsymbol{\Sigma}\right)^{-1}[(\overline{\mathbf{X}} - \boldsymbol{\mu})]$$

is distributed as χ_4^2, where χ_4^2 denotes the chi-square distribution with 4 degrees of freedom.(Result 3.7)

Soluitons

(c) V_1, V_2 are jointly multivariate normal with covariance matrix

$$\begin{bmatrix} \left(\sum_{j=1}^{n} a_j^2/n^2\right)\Sigma & (\mathbf{b}'\mathbf{a}/n)\Sigma \\ (\mathbf{b}'\mathbf{a}/n)\Sigma & \left(\sum_{j=1}^{n} b_j^2\right)\Sigma \end{bmatrix}$$

Consequently, V_1 and V_2 are independent if $\mathbf{b}'\mathbf{a} = \sum_{j=1}^{n} a_j b_j = 0$

Example2

Let $\mathbf{X}$ be $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = [2, -3, 1]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

(a) Find vectors a and b such that $\mathbf{a}'\mathbf{X}$ and $\mathbf{b}'\mathbf{X}$ are independent.
(b) Find the distribution of $(\mathbf{a}, \mathbf{b})'\mathbf{X}$.

Solutions

$(\mathbf{a},\mathbf{b})'\mathbf{X} = \begin{bmatrix} \mathbf{a}'\mathbf{X} \\ \mathbf{b}'\mathbf{X} \end{bmatrix}$ are distributed as $N_2((\mathbf{a},\mathbf{b})'\boldsymbol{\mu}, (\mathbf{a},\mathbf{b})'\Sigma(\mathbf{a},\mathbf{b}))$,
where $(\mathbf{a},\mathbf{b})'\boldsymbol{\mu} = \begin{bmatrix} \mathbf{a}'\boldsymbol{\mu} \\ \mathbf{b}'\boldsymbol{\mu} \end{bmatrix}$ $(\mathbf{a},\mathbf{b})'\Sigma(\mathbf{a},\mathbf{b}) = \begin{bmatrix} \mathbf{a}'\Sigma\mathbf{a} & \mathbf{a}'\Sigma\mathbf{b} \\ \mathbf{b}'\Sigma\mathbf{a} & \mathbf{b}'\Sigma\mathbf{b} \end{bmatrix}$.
Then $\mathbf{a}'\mathbf{X}$ and $\mathbf{b}'\mathbf{X}$ are independent if and only if $\mathbf{a}'\Sigma\mathbf{b} = 0$.
For example,let $\boldsymbol{a}' = (1,0,0), \boldsymbol{b}' = (1,-1,0)$. Then $\mathbf{a}'\Sigma\mathbf{b} = 0$. Thus, $\mathbf{a}'\mathbf{X}$ and $\mathbf{b}'\mathbf{X}$ are independent

Exmaple 3

Suppose that $\mathbf{X}_i = (X_{1i}, X_{2i}, X_{3i}, X_{4i})'$and
$\mathbf{Y}_i = (Y_{1i}, Y_{2i}, Y_{3i}, Y_{4i})'$, $i = 1, \ldots, n$ are independent, and $\mathbf{X}_i$ follow the same multivariate normal distribution $N_4(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$. And $\mathbf{Y}_i$ follow the same multivariate normal distribution $N_4(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$
Find the distribution of $\bar{\boldsymbol{X}} - \bar{\boldsymbol{Y}}$

Solutions

Since $\mathbf{X}_i$ and $\mathbf{Y}_i$ are independent and are distributed as $N_4(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $N_4(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$, then $\begin{bmatrix} \mathbf{X}_i \\ \mathbf{Y}_i \end{bmatrix}$ has the multivariate normal distribution

$N_8\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\right)$, $i = 1, \ldots, n$. (Result 3.5(c))

And $\begin{bmatrix} \mathbf{X}_i \\ \mathbf{Y}_i \end{bmatrix}$ are independent, $i = 1, \ldots, n$.

Then $\begin{bmatrix} \bar{\mathbf{X}} \\ \bar{\mathbf{Y}} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n} \begin{bmatrix} \mathbf{X}_i \\ \mathbf{Y}_i \end{bmatrix}$ is distributed as

$N_8\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \frac{1}{n}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\right)$. (Result 3.8)

Then $\bar{\mathbf{X}} - \bar{\mathbf{Y}} = \boldsymbol{A}\begin{bmatrix} \bar{\mathbf{X}} \\ \bar{\mathbf{Y}} \end{bmatrix}$,where $\boldsymbol{A} = [\boldsymbol{I}_4, -\boldsymbol{I}_4]$.

Thus, $\bar{\mathbf{X}} - \bar{\mathbf{Y}}$ is distributed as $N_4(\boldsymbol{A}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \frac{1}{n}\boldsymbol{A}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\boldsymbol{A}')$, that is

$N_4\left(\mu_1 - \mu_2, \frac{1}{n}(\Sigma_1 + \Sigma_2)\right)$. (Result 3.3)

Example 4

Suppose that $\mathbf{X}_i = (X_{1i}, X_{2i}, X_{3i}, X_{4i})'$and
$\mathbf{Y}_i = (Y_{1i}, Y_{2i}, Y_{3i}, Y_{4i})'$, $i = 1, \ldots, n$ are independent, and $\mathbf{X}_i$ follow the same distribution with mean and covariance $\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1$. And $\mathbf{Y}_i$ follow the same distribution with mean and covariance $\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2$.
Determine the distribution of $\sqrt{n}\left(\overline{\mathbf{X}} - \overline{\mathbf{Y}} - (\mu_1 - \mu_2)\right)$ when $n \to \infty$.

Solutions

Since $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ are independent, $Cov(\boldsymbol{X}_i, \boldsymbol{Y}_i) = 0$. Then $\begin{bmatrix} \mathbf{X}_i \\ \mathbf{Y}_i \end{bmatrix}$ are independent and has the same distribution with mean and covariance $\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$, $i = 1, \ldots, n$.

Then $\sqrt{n}\left(\begin{bmatrix} \bar{\mathbf{X}} \\ \bar{\mathbf{Y}} \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}\right)$has an approximate distribution $N_8(0, \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix})$ when $n \to \infty$. ($\begin{bmatrix} \bar{\mathbf{X}} \\ \bar{\mathbf{Y}} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n} \begin{bmatrix} \mathbf{X}_i \\ \mathbf{Y}_i \end{bmatrix}$)

Then $\sqrt{n}\left(\overline{\mathbf{X}} - \overline{\mathbf{Y}} - (\mu_1 - \mu_2)\right) = \sqrt{n}\boldsymbol{A}\left(\begin{bmatrix} \bar{\mathbf{X}} \\ \bar{\mathbf{Y}} \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}\right)$,where $\boldsymbol{A} = [\boldsymbol{I}_4, -\boldsymbol{I}_4]$. Therefore, $\sqrt{n}\left(\overline{\mathbf{X}} - \overline{\mathbf{Y}} - (\mu_1 - \mu_2)\right)$ is approximately distributed as $N_4(0, \boldsymbol{A}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\boldsymbol{A}')$, that is $N_4\left(0, (\Sigma_1 + \Sigma_2)\right)$ when $n \to \infty$.