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ARRAY DESIGN FOR MEASURING SOURCE DEPTH IN A HORIZONTAL WAVEGUIDE*

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Abstract. The signal-to-noise ratio of received radiation is enhanced by properly combining the outputs from a collection of sensors which are connected as an array. In an infinite homogeneous medium, such as a deep ocean layer, the array is matched to the incoming signal by adjusting the phases or time delays of the sensor outputs so that the signals may be added coherently. Beamforming, however, is inappropriate if the waveguide boundaries significantly effect the radiation field. The maximum information array design for estimating source depth in an infinite homogeneous horizontal waveguide is shown to be horizontal, and thus a horizontal array can be used to obtain source depth as well as bearing in a shallow waveguide.

Introduction. The signal-to-noise ratio of received radiation is enhanced by properly combining the outputs from a collection of sensors which are connected as an array. In an infinite homogeneous medium, such as a deep ocean layer, the array is matched to the incoming signal by adjusting the phases or time delays of the sensor outputs so that the signals may be added coherently. This technique, called *beamforming*, can be performed in either the time-space or the frequency-wave number domains in order to estimate the bearing of the source [1], [2].

Beamforming is inappropriate if the waveguide boundaries effect the received radiation, as for example when the source or receiver are located in a canyon, or if the problem is the measurement of the depth of a source in a relatively shallow waveguide.

The maximum-likelihood estimator of source depth using a vertical array immersed in a horizontal waveguide in which the ambient noise field is Gaussian is given by Hinich [3] based on the seminal work of Clay [4]. This paper presents a result which implies that a *horizontal* rather than a vertical array design maximizes Fisher information [5] for estimating the source depth in a stationary homogeneous horizontal waveguide. Since a horizontal array configuration is best for estimating the source bearing, the same horizontal array can be processed to estimate depth. The array depth, however, for bearing estimation is generally different from the best array depth for source depth measurement, and both "optimal" depths vary over time when the parameters of the medium are nonstationary.

1. Estimating source depth in an infinite waveguide. Suppose that the signal has narrowband energy at frequency ω. Let $y(z)$ represent the output from a sensor located at depth z, filtered in a narrowband about ω. If the signal is broadband, the results in this paper apply to each frequency component of the signal. The *horizontal dimensions are suppressed* using this notation but it should be understood that two or more sensors can lie along a constant depth horizontal plane.

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Suppose that the source is located at depth z_0, $0 < z_0 < D$. In the ocean and for most waveguides thare are many noise sources which superimpose on the source radiation. Thus

$$y(z) = p(z|z_0) + \varepsilon(z)$$

where $\varepsilon(z)$ is the noise field at z in a narrowband about ω, and $p(z|z_0)$ is the signal component received from the source. The signal satisfies the wave equation with associated boundary conditions. By the horizontal homogeneity of the waveguide, $p(z_0|z) = p(z|z_0)$.

Assume that for an array geometry $(z_1, \cdots, z_n)$, the noise terms $\varepsilon(z_1), \cdots, \varepsilon(z_n)$ have a multivariate Gaussian distribution with zero mean and variance-covariance matrix $\mathbf{\Sigma}_z$. Moreover assume that if two or more sensors are at the same depth, they are placed sufficiently apart such that the *noise field varies from sensor to sensor*, i.e. $\mathbf{\Sigma}_z$ is nonsingular even if all the sensors are in a horizontal plane. This assumption is valid for the ocean.

Let $f(y_1, \cdots, y_n|z_0)$ denote the joint density of the y_i given the source depth z_0. The maximum likelihood estimator, denoted $\hat{z}_0$, is one of the solutions to the equation

$$\text{(1)} \qquad \frac{\partial}{\partial z_0} \log f(y_1, \cdots, y_n|z_0) = \left[\frac{\partial}{\partial z_0}\mathbf{p}_z(z_0)\right]^T \mathbf{\Sigma}_z^{-1}[\mathbf{y} - \mathbf{p}_z(z_0)] = 0$$

where

$$\mathbf{p}_z(z_0) = [p(z_1|z_0), \cdots, p(z_n|z_0)]^T \quad \text{and} \quad \frac{\partial}{\partial z_0}\mathbf{p} = \left[\frac{\partial}{\partial z_0} p(z_1|z_0), \cdots, \frac{\partial}{\partial z_0} f(z_n|z_0)\right]^T.$$

For large n the mean squared error of the maximum likelihood estimator is

$$\text{(2)} \qquad \text{mse } \hat{z}_0 = I_z^{-1}(z_0) + O(n^{-2})$$

where I_z denotes the *Fisher information* about z_0 in the array data. The Fisher information is defined by

$$\text{(3)} \qquad I_z(z_0) = -E\frac{\partial^2}{\partial z_0^2} \log f(y_1, \cdots, y_n|z_0)$$

where E denotes the expected value with respect to the density f. Applying (1) to (3) we then have

$$\text{(4)} \qquad I_z(z_0) = \left(\frac{\partial}{\partial z_o}\mathbf{p}\right)^T \mathbf{\Sigma}_z^{-1} \frac{\partial}{\partial z_0}\mathbf{p},$$

which is of the order $O(n)$ since $\mathbf{\Sigma}_z$ is nonsingular, and consequently mse $\hat{z}_0 = O(n^{-1})$.

Holding the sensor depths $(z_1, \cdots, z_n)$ fixed, it is easy to show that the information is maximized, and thus mse $\hat{z}_0$ is minimized, if the array can be structured such that $\mathbf{\Sigma}_z$ is an $n \times n$ diagonal matrix. As mentioned above, this can be always achieved by horizontally shifting the sensors until the cross-correlation between the noise field at any two sensors is zero, but the engineering costs for such a design might be prohibitive.

When Σ_z is diagonal, the Fisher information has the simple form

$$I_z(z_0)=\sum_{i=1}^{n}\left[\sigma_i^{-1}\frac{\partial}{\partial z_0}p(z_i|z_0)\right]^2, \tag{5}$$

where σ_i^2 denotes the variance of the noise $\varepsilon(z_i)$. Let d denote the depth which maximizes the absolute value of $\partial p/\partial z_0$, i.e.

$$\left|\frac{\partial}{\partial z_0}p(z|z_0)\right|\leqq\left|\frac{\partial}{\partial z_0}p(d|z_0)\right| \quad \text{for all } 0\leqq z\leqq D.$$

It is clear from (5) that $I_z(z_0)$ is maximized by setting $z_1=z_2=\cdots=z_n=d$. In other words, the optimal design for spatially incoherent noise is a *horizontal array* placed at a depth where a *small vertical move of the source produces the maximum change in the radiation received by the array*. The mean squared error for the maximum likelihood estimator $\hat{z}_0$, for $\bar{\sigma}=[n^{-1}\sum_{i=1}^{n}\sigma_i^2]^{1/2}$, is

$$\text{mse } \hat{z}_0=\left[n^{1/2}\bar{\sigma}\frac{\partial}{\partial z_0}p(d|z_0)\right]^{-2}+O(n^{-2}), \tag{6}$$

and the likelihood equation is simply

$$p(d|\hat{z}_0)=n^{-1}\sum_{i=1}^{n}y_i, \tag{7}$$

i.e. the estimator $\hat{z}_0$ is the *depth* which *equates the average received signal with the predicted signal*. The signal $p(z|z_0)$ can be derived from ray tracing methods as well as normal mode theory [6], or it can be experimentally determined [7], but the result is of no practical use if medium nonstationarity makes the relationship critically dependent on time. An error analysis when the propagation parameters vary is given in § 3 using normal mode theory.

It is instructive to compare this design with the design which maximizes the received signal-to-noise ratio. Clearly the signal-to-noise ratio is maximized by placing a horizontal array at depth which maximizes the signal amplitude $|p(z|z_0)|$. In general this depth will be different than d, which maximizes $|(\partial/\partial z_0)p(z|z_0)|$. Of course, both d and the signal amplitude maximizing depth depend on z_0, which is unknown, but the functional relationships can be used to position the array to measure the parameters of a source in a well behaved medium. The maxima under discussion are discrete since $p(z_0|z)=p(z|z_0)$ satisfies the wave equation and thus $(\partial^2/\partial z_0^2)p(z|z_0)\neq 0$ except for a discrete set of z's. For most waveguides $d(z_0)$ is unique [8].

These results generalize when the ambient noise is spatially coherent.

2. Coherent noise. Even when the noise is horizontally incoherent, it may not be possible to place the sensor so that the correlation between sensors is zero. Continuing with the horizontal waveguide model, assume that the noise is isotropic, but the array geometry is constrained so that the minimum correlation between two sensors is $\rho\neq 0$. Assuming that the sensors are horizontally shifted so that the minimum correlation is achieved, it follows that the i, jth element of Σ_z is $\rho^{|i-j|}$, and thus

$$(1-\rho^2)\Sigma_z^{-1}=\begin{pmatrix} 1 & -\rho & 0 & \cdots & & & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & & & \\ 0 & -\rho & 1+\rho^2 & -\rho & 0 & & \\ \vdots & 0 & -\rho & 1+\rho^2 & -\rho & \ddots & 0 \\ & & \ddots & \ddots & 1+\rho^2 & \ddots & -\rho \\ 0 & \cdots & & 0 & -\rho & & 1 \end{pmatrix}. \tag{8}$$

Applying this matrix to (4), the array geometry which maximizes the Fisher information is as follows:

THEOREM 1. *Let $\bar{d}$ denote the depth which* maximizes $(\partial/\partial z_0)p(z|z_0)$ *and $\underset{\sim}{d}$ denote the depth which* minimizes $(\partial/\partial z_0)p(z|z_0)$. *Assuming $d=\bar{d}$, the information maximizing array design is $z_1=z_2=\cdots=\bar{d}$, if*

$$\rho \leqq \frac{2-(4-\alpha^2)^{1/2}}{\alpha} \tag{9}$$

where

$$\alpha = 1+\left(\frac{\partial}{\partial z_0}p(\underset{\sim}{d}|z_0)\right)\Big/\left(\frac{\partial}{\partial z_0}p(\bar{d}|z_0)\right).$$

If, however, $\rho>\alpha^{-1}[2-(4-\alpha^2)^{1/2}]$ then the optimal design is $z_1=z_3=z_5=\cdots=\bar{d}$ and $z_2=z_4=z_6=\cdots=\underset{\sim}{d}$, i.e. the array consists of two horizontal arrays; and array at depth $\bar{d}$ containing the odd numbered sensors, and the other at depth $\underset{\sim}{d}$ containing the even numbered sensors. When n is even, the n-th sensor is at $\underset{\sim}{d}$ if and only if $\rho>\frac{1}{2}\alpha$.

Proof. Since Σ_z is positive definite, the quadratic form in (4) is a convex function of its arguments $g_i=(\partial/\partial z_0)p(z_i|z_0)$, and thus it attains its maximum on the boundary of its domain. With $\bar{g}=(\partial/\partial z_0)p(\bar{d}|z_0)$ and $\underset{\sim}{g}=(\partial/\partial z_0)p(\underset{\sim}{d}|z_0)$, this means that the quadratic form is maximized for $g_i=\bar{g}$ or $g_i=\underset{\sim}{g}$ for each i.

When n is odd, a straightforward comparison of the alternatives shows that $(\bar{g}, \bar{g},\cdots)$ maximizes the quadratic form if

$$(1+\underset{\sim}{g}/\bar{g})(1+\rho^2)\geqq 4\rho, \tag{10}$$

where $\bar{g}>\underset{\sim}{g}$. If this inequality is reversed, then $(\bar{g}, \underset{\sim}{g}, \bar{g}, \underset{\sim}{g},\cdots)$ maximizes the quadratic form. Since $\alpha=1+\underset{\sim}{g}/\bar{g}$, inequality (10) implies inequality (9).

When n is even $(\bar{g}, \bar{g},\cdots)$ is still optimal when (10) holds. Otherwise $(\bar{g}, \underset{\sim}{g}, \bar{g}, \underset{\sim}{g},\cdots, \bar{g}, \underset{\sim}{g})$ is optimal when $2\rho>\alpha$ and $(\bar{g}, \underset{\sim}{g}, \bar{g}, \underset{\sim}{g},\cdots, \bar{g}, \bar{g})$ is optimal when $2\rho<\alpha$. Note that $2\rho>\alpha$ implies $\alpha(1+\rho^2)<4\rho$ since $1+\rho^2\leqq 2$, and $\alpha(1+\rho^2)>4\rho$ implies $2\rho<\alpha$. The reader is encouraged to compute the alternatives for $n=2$ and 3.

3. Perturbations of the acoustic field. In a horizontally stratified waveguide the solution of the wave equation and associated boundary conditions can be expressed by the sum of normal modes. To be more explicit for acoustic propagation, the sound pressure field received at *depth* z and time t due to a *distant* continuous simple harmonic source at depth z_0 is the real part of the expression

$$p(z, t|z_0) = \exp\left(i\left(\omega t - \frac{\pi}{4}\right)\right) \sum_{m=1}^{M} A_m(r)\phi_m(z_0)\phi_m(z) \tag{11}$$

where

$A_m(r) = r^{-1/2} p_m \exp(-i\varkappa_m r - \delta_m r)$,
r is the horizontal distance between source and receiver,
ω is the source frequency,
p_m is the mth mode excitation,
δ_m is the mth mode attenuation,
$\varkappa_m$ is the horizontal component of wave number.

The mth eigenfunction $\phi_m(z)$ satisfies the equation

$$\frac{d^2\phi}{dz^2} + \gamma_m^2 \phi = 0,$$

where γ_m is the vertical component of wave number, and the associated boundary conditions which imply a discrete eigenvalue spectrum. The horizontal and vertical components of the mth mode wave number are related to ω and z by the dispersion equation

$$\varkappa_m^2 + \gamma_m^2 = [\omega/c(z)]^2,$$

where $c(z)$ is the sound velocity at z, and by Snell's law $\varkappa_m$ is independent of depth (Tolstoy and Clay [6]).

As is discussed by Tolstoy and Clay, the water layer in shallow coastal areas is frequently homogeneous. When the wavelengths are of the order of magnitude of the water depth, i.e. $\varkappa_m D \leqq 10$, it is a reasonably accurate approximation to make c a constant for $0 < z < D$. The simplest working model is the perfect acoustic waveguide with a free surface and a rigid bottom. In this case,

$$\phi_m(z) = \sqrt{2} \sin(\gamma_m z), \qquad \gamma_m = (m - \tfrac{1}{2})\frac{\pi}{D} \tag{12}$$

for $m = 1, 2, \cdots$. The mth mode low frequency cutoff for this waveguide is $c\gamma_m$, and thus M is the greatest integer less than or equal to $(\omega/\pi)(D/c) - \frac{1}{2}$.

Given an observation period of discrete samples, the signal filtered in a narrowband about ω is

$$p(z|z_0) = \frac{1}{T} \sum_{t=1}^{T} p(z, t|z_0) \exp(-i(\omega t - \pi/4)). \tag{13}$$

The $\pi/4$ phase term is included in (13) so that from (11), the gradient of p as a function of z_0 is

$$g(z) = \sum_{m=1}^{M} A_m(r)\phi_m'(z_0)\phi_m(z). \tag{14}$$

In general the propagation characteristics of the medium vary over time. In keeping with the statistical model for time fluctuations used in Tolstoy and Clay [6, Chap. 6], let the $A_m(r)$ term in (14) be given by

$$A_m(r)=\bar{A}_m(r)(1+u_m), \tag{15}$$

where $\bar{A}_m(r)$ is the ensemble average and u_m is a stochastic zero mean perturbation term. Since A_m can continuously vary over the period, the variance

$$\sigma_m^2=E(u_m^2)$$

can be dependent on the sampling period T. In the following propagation of error analysis, assume that T and the variation in the parameters are small enough such that σ_m^2 is small for each m.

Let d_u denote the depth which *maximizes* $g(z)$ when A_m is given by expression (15). Using the linear approximation of $\phi'_m(d_u)$ in $g'(d_u)=0$ for $d_u=d+\delta$, we then have

$$\delta\approx\frac{-\sum_{m=1}^M \bar{A}_m(r)u_m\phi'_m(z_0)\phi'_m(d)}{\sum_{m=1}^M \bar{A}_m(r)\phi'_m(z_0)\phi''_m(d)}, \tag{16}$$

for small δ. This expression holds in the limit when $\sigma_m^2\to 0$ for each m. The mean $|\delta|^2$ from (16) is

$$\frac{\sum_{m=1}^M |\bar{A}_m(r)|^2\sigma_m^2[\phi'_m(z_0)\phi'_m(d)]^2}{|\sum_{m=1}^M \bar{A}_m(r)\phi'_m(z_0)\phi''_m(d)|^2} \tag{17}$$

For the perfect waveguide, (16) become

$$\delta\approx\frac{D}{\pi}\frac{\sum_{m=1}^M \bar{A}_m(r)u_m\gamma_m^2\cos(\gamma_m z_0)\cos(\gamma_m d)}{\sum_{m=1}^M \bar{A}_m(r)\gamma_m^3\cos(\gamma_m z_0)\sin(\gamma_m d)}, \tag{18}$$

where d satisfies the equation

$$\sum_{m=1}^M \bar{A}_m(r)\gamma_m^2\cos(\gamma_m z_0)\cos(\gamma_m d)=0. \tag{19}$$

If several modes are received from the source, the relationship between z and z_0 as given by the normal mode expression can possibly be used to estimate source depth. The physics of the waveguide determines the resolvability of the depth, not the statistical model.

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