Preliminary

April 9, 2019

Contents

1 logic

1.1 and

$\wedge$

1.2 or

$\vee$

1.3 imply

$\rightarrow$

1.4 iff

$\iff$

1.5 not

$\neg$

1.6 any

$\forall$

1.7 exist

$\exists$

2 set and class

2.1 class

a class is a collection A of objects such that given any object x, it is possible to determine whether or not x is a member of A.

2.2 set

a class A is defined to be a set iff exists a class B and $A \in B$.

2.3 axiom of extensionality

$[x \in A \iff x \in B] \rightarrow A = B$

2.4 axiom of class formation

for any statements $P(y)$ in the first-order predicate calculus involving a variable y, there exists a class A such that $x \in A$ if and only if x is a set and the statement $P(x)$. we denote this class A by $\{x|P(x)\}$.

2.5 axiom of operation

for union, intersection, functions, relations, Cartesian products, if one of these operation is performed on a set, then the result is also a set.

2.6 power axiom

for all set A, the class $P(A)$ of all subsets of A is itself a set. $P(A)$ is called the power set of A.

2.7 subclass

A, B are classes, then $A \subset B \iff (\forall x \in A) x \in A \to x \in B$ A is a subclass of B. if B is a set, then A is a subset.

2.8 empty set

$\emptyset$

2.9 family of set

a family of sets indexed by I is a collection of sets A_i.

2.10 disjoint set

$A \cap B = \emptyset$.

3 Function

3.1 preliminary for funcion

@page[3]

3.2 f and g injective $\to$ gf is injective

proof: $x \neq y \to f(x) \neq f(y) \to g(f(x)) \neq g(f(y))$

3.3 $f : A \to B$ and $g : B \to C$ surjective $\to$ gf is surjective

proof: $f(A) = f(B) \to g(f(A)) = g(B) = C$.

3.4 gf injective $\rightarrow f$ is injective

proof: assume f is not injective, then $(\exists x)(\exists y)f(x) = f(y)$. so $g(f(x)) = g(f(y))$. so gf is not injective, contradiction.

3.5 gf surjective $\rightarrow g$ is surjective

proof: assume g is not surjective, then it is easy to see that gf is not surjective, contradiction.

4 Integer

4.1 Theorem for gcd

If $a_1, a_2, \cdots, a_n$ are integers, not all 0, then $(a_1, a_2, \cdots, a_n)$ exists. Futhermore, there are integers $k_1, k_2, \cdots, k_n$ such that:

$$(a_1, a_2, \cdots, a_n) = k_1a_1 + k_2a_2 + \cdots + k_na_n$$

.

4.1.1 proof:

Let $S = \{x_1a_1 + x_2a_2 + \cdots + x_na_n | x_i \in Z, \sum_i x_ia_i > 0\}$. It is easy to see that $S \neq \emptyset$. Let $c = \sum_i x_ia_i$ be the least number in S. We claim that:

1. $c|a_i$ for $1 \leq i \leq n$.
2. $d \in Z$ and $d|a_1$ for $1 \leq i \leq n \rightarrow d|c$.

Then c is obviously a gcd for $\{a_i\}$.

- claim 1: $c|a_i$ for $1 \leq i \leq n$. Assume $\exists o$ such that $c \nmid a_o$. Then, $\exists q, k$ such that $a_o = q\sum_i x_ia_i + k$. $a_o - q\sum_i x_ia_i = k > 0$. $(1 - qx_o)a_o + \sum_{i \neq o} x_ia_i = k > 0 \Rightarrow k \in S$. And because $k < c$, we get a contradiction. $\square$
- claim 2: $d \in Z$ and $d|a_1$ for $1 \leq i \leq n \rightarrow d|c$. $\forall d$ that devided $\{a_i\}$, we have $a_i = k_id, i = 1, 2, \cdots, n$. Then $c = \sum_i x_ia_i = \sum_i x_ik_id = d\sum_i x_ik_i$. So $d|c$. $\square$

5 Axiom of choice

5.1 axiom of choice

The product of family of nonempty sets indexed by a nonempty set is nonempty.

5.2 zorn's lemma

If A is a nonempty partially ordered set such that every chain in A has upper bound in A, then A contains a maximal element. zorn's lemma wiki

5.3 ordinal number

the number of all the ordinal's is more than the number of element in any sets.

5.3.1 Definition of the ordinal number

1. $\emptyset$
2. $\{\emptyset\}$
3. $\{\emptyset, \{\emptyset\}\}$
4. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
5. $\cdots$

according to the definition of the ordinal number. if there is a set M which contains all the ordinal numbers, then we have $M \in M$, this is a paradox for a sets. I think things like $M \in M$ may happen on proper class. here on stackexchange is a discussion for this issue.

5.4 exercise 1 (p14)

- for all subset $\{a, b\} \subset P(S)$. the g.l.b. is $a \cap b$. the l.u.b. is $a \cup b$. and the unique maximal element is S.
- $\{a \leq b, c \leq d\}$. the sub set $\{a, c\}$ do not have lower bound or upper bound.
- partially set $\{a \leq b, c \leq d\}$ has no maximal elements. partially set $\{a \leq b, c \leq d, a \leq d, c \leq b\}$ has maximal elements b, d.

5.5 exercise 2 (p15)

$\mathcal{A}$ is a complte lattice $\Rightarrow$ there is a g.l.b. and l.u.b. for $A \in \mathcal{A}$. from the antisymmetric property of $\mathcal{A}$, we know that $\mathcal{A}$ has a unique maximal element and a unique minimal element. we denote the maximal element of $\mathcal{A}$ by m. then it is easy to see that $m = f(m)$.

5.6 exercise 3 (p15)

$$\underbrace{\frac{1}{1}}_{2}, \underbrace{\frac{1}{2}, \frac{2}{1}}_{3}, \underbrace{\frac{1}{3}, \frac{2}{2}, \frac{3}{1}}_{4}, \ldots$$

5.7 exercise 4 (p15)

we need to prove that: the axiom of choice $\iff$ every set S has a choice function.

5.7.1 proof:

$\Rightarrow$: we could construct a product on $P(S) \setminus \emptyset$. from axiom of choice, the result of this product is not empty. So, $\exists f \in \prod_{A \in S} A$ and we have $f(A) \in A$. obviously, f is the choice function for S. $\Leftarrow$: Let $\{A_i | i \in I\}$ be any family, such that $(\forall i) A_i \neq \emptyset$. every set $S \neq \emptyset$ has a choice function $\Rightarrow (\forall i) A_i$ has a choice function f_i. from these choice function f_i, we could then construct another function φ, by defining $\varphi(i) = f_i(A_i)$. it is quite clear that $\varphi \in \prod_{i \in I} A_i$, which is a nonempty set.

5.8 exercise 5 (P15)

$(\forall x \in R)(x, 0)$ is the maximal element in S. thus, S has infinitely many maximal elements.

5.9 exercise 6 (P15)

from exercise 4, we know that $(\forall i) A_i$ has a choice functon f, mapping all the subsets B of A_i to an element in B. once we have the function f, we could simpliy enumerate $f(A_i)$ over all the elements in A_i ...

5.10 exercise 7 (P15)

There are only 2 cases in which one element $a \in A$ does not have an immediate successor:

1. $\{x \in A | a < x\} = \emptyset$.

2. $\{x \in A | a < x\}$ does not have a least element.

5.10.1 A is well-ordered

under this condition, only the first case could happen. assume that we have 2 elements a, b in A that has no immediate successor. however, A is well-ordered, so there must be a least element in $\{a, b\}$. assuming that the least element is a, we will find that the set $\{x \in A | a < x\}$ is not empty. which means that a has a immediate successor in A, which is a contradiction.

5.10.2 A is a linearly ordered set

$\{10, \cdots, -1, 0, 1, 2, 3, 4, 5\}$. we set that $10 \leq \cdots \leq -1 \leq 0 \leq 2 \leq 3 \leq 4 \leq 5$. then, $10, 5$ are 2 elements with no immediate successor.

6 Cardinal numbers

6.1 If A is a set and $P(A)$ its power set, then $|A| < |P(A)|$

6.1.1 proof

to finish the proof, we claim that:

1. $|A| \leq |P(A)|$

2. $|A| \neq |P(A)|$

we prove them one by one.

1. $|A| \leq |P(A)|$: define a map $f : A \to P(A)$ as $a \mapsto \{a\}$. this map is obviously a injection. so $|A| \leq |P(A)|$.

2. $|A| \neq |P(A)|$: to prove this, we only need to prove that: $[(\forall) f : A \to P(A)] \to f$is not surjective. for any function f, we define a set $B = \{a \in A | a \notin f(a)\}$. it is easy to see that by definition, $B \subset A$. so, if f is surjective, then $(\exists) x \in A \wedge x \mapsto B$. then we could get $x \in B \wedge x \notin B$, which is a contradiction.

6.2 ordered by extension (p18)

6.3 Exercise 1 (P21)

1. (a) omit
2. (b) (a) $\Rightarrow$ (b)
3. (c) omit

6.4 Exercise 2 (P21)

1. Assume that we have a infinit set $A = \{a_0, a_1, a_2, \cdots\}$. Then we could build a bijection $f : A \to A - \{a_0\}$ by setting $f(a_i) = a_{i+1}, i \in N$. It is easy to see that $A - \{a_0\} \subset A$.
2. $\Leftarrow$: could be easily got from (1). $\Rightarrow$: could be got from 1.(1).

6.5 Exercise 3 (P21)

1. we could build a bijection $f : Z \to N$ by setting:

$$f(x) = \begin{cases} 0, & x = 0 \\ -x * 2, & x < 0 \\ x * 2 + 1, & x > 0 \end{cases}$$

2. omit.

6.6 Exercise 4, 5, 6, 7, 8 (P21)

omit.

6.7 Exercise 9 (P21)