3 Exercise 5.18

Follow through in detail the calculations sketched above.

3.1 solution

Let $Q = P - I$ and $\widetilde{P} = e^Q$, then we have:

$$
\begin{aligned}
\mathrm{Var}_\pi(\widetilde{P}^t f) &= \sum_{x\in\Omega} \pi(x)\{[\widetilde{P}^t f](x,y)\}^2 \\
&= \sum_{x\in\Omega}\{\sum_{y\in\Omega} \widetilde{P}^t(x,y)f(y)\}^2 \\
&= \sum_{x\in\Omega}\{\sum_{y\in\Omega}[\sum_{k=0}^{\infty} \frac{t^k Q^k(x,y)}{k!}]f(y)\}^2 \\
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}_\pi &= \sum_{x\in\Omega} 2\pi(x)[\widetilde{P}^t f](x)\{[\widetilde{P}^t f](x)\}' \\
&= \sum_{x\in\Omega} 2\pi(x)[\widetilde{P}^t f](x)\{\sum_{y\in\Omega}[\sum_{k=1}^{\infty} \frac{t^{k-1} Q^k(x,y)}{(k-1)!}]f(y)\} \\
&= \sum_{x\in\Omega} 2\pi(x)[\widetilde{P}^t f](x)\{\sum_{y\in\Omega}[\sum_{k=0}^{\infty} \frac{t^k Q^{k+1}(x,y)}{k!}]f(y)\} \\
&= \sum_{x\in\Omega} 2\pi(x)[\widetilde{P}^t f](x)\{\sum_{y\in\Omega} Q\cdot e^{tQ}(x,y)f(y)\} \\
&= \sum_{x\in\Omega} 2\pi(x)[\widetilde{P}^t f](x) Q\widetilde{P}^t f(x) \\
&= \sum_{x\in\Omega} 2\pi(x)[\widetilde{P}^t f](x)\sum_{y\in\Omega} Q(x,y)[\widetilde{P}^t f](y) \\
&= 2\sum_{x\in\Omega} \pi(x)Q(x,y)[\widetilde{P}^t f](x)[\widetilde{P}^t f](y)
\end{aligned}
$$

So, when $t = 0$, we have:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}_\pi(\widetilde{P}^t f)\bigg|_{t=0} &= 2\sum_{x,y\in\Omega}\pi(x)[P(x,y)-I(x,y)]f(x)f(y) \\
&= 2\sum_{x,y\Omega}\pi(x)P(x,y)f(x)f(y) - 2\sum_{x\in\Omega}\pi(x)f^2(x) \\
&= 2\sum_{x,y\Omega}\pi(x)P(x,y)f(x)f(y) - (\sum_{x\in\Omega}\pi(x)f^2(x) + \sum_{y\in\Omega}\pi(y)f^2(y)) \\
&= 2\sum_{x,y\Omega}\pi(x)P(x,y)f(x)f(y) - (\sum_{x,y\in\Omega}\pi(x)f^2(x)P(x,y) + \sum_{x,y\in\Omega}\pi(x)P(x,y)f^2(y)) \\
&= 2\sum_{x,y\in\Omega}\pi(x)P(x,y)f(x)f(y) - (\sum_{x,y\in\Omega}\pi(x)P(x,y)(f^2(x)+f^2(y))) \\
&= -\sum_{x,y\in\Omega}\pi(x)P(x,y)(f(x)-f(y))^2 \\
&= -2\mathcal{E}_P(f,f)
\end{aligned}
$$

Actually, we could know more from above:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Val}_\pi(\widetilde{P}^t f) &= -2\mathcal{E}_P(\widetilde{P}^t f, \widetilde{P}^t f) \\
&\le -\frac{2}{\rho}\mathrm{Var}_\pi(\widetilde{P}^t f)
\end{aligned}
$$

In the worst case, we have:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Val}_\pi(\widetilde{P}^t f) = -\frac{2}{\rho}\mathrm{Var}_\pi(\widetilde{P}^t f)
$$

Let $v = \widetilde{P}^t f$, then we have:

$$
\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{2}{\rho}v \\
\frac{\mathrm{d}v}{v} &= -\frac{2}{\rho}\mathrm{d}t \\
\int\frac{\mathrm{d}v}{v} &= \int -\frac{2}{\rho}\mathrm{d}t \\
\log(v) + C_1 &= -\frac{2}{\rho}t + C_2 \\
v(t) &= \exp\{-\frac{2}{\rho}t\}\cdot C_3
\end{aligned}
$$

To determinate C_3, consider the situation where $t = 0$, then

$$
v(0) = \mathrm{Val}_\pi(f) = C_3
$$

So, we have

$$\mathrm{Val}_\pi(\widetilde{P}^t f) \leq \exp\{-\frac{2}{\rho}t\}\mathrm{Var}_\pi(f)$$