Exercise of Chapter 4

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1 Exercise 4.1 Summary

There is a MC described below.

1. Select a vertex $v \in V$, u.a.r.
2. Select a colour $c \in Q \setminus X_0(\Gamma(v))$, u.a.r.
3. $X_1(v) \leftarrow c$ and $X_1(u) \leftarrow X_0(u)$ for all $u \neq v$.

2 Exercise 4.1(1)

Prove that the above MC is irreducible (and hence ergodic) under the (stronger) assumption $q \geq \Delta + 2$. Further prove, using Lemma 3.7, that its (unique) stationary distribution is uniform over Ω.

2.1 solution

To prove the irreducibility, we only need to design a procedure which transforms some coloring X into another coloring Y by changing the color of one vertex per step.

First, we define a measure $M(X, Y) = |\{v : X(v) \neq Y(v)\}|$.

The Procedure

1. Select a vertex v from G where $X(v) \neq Y(v)$.
2. Change the color of all the vertex in $S = \{u : u \in \Gamma(v) \wedge X(u) = Y(v)\}$ to some color other than $Y(v)$. (this could always be achieved because $q \geq \Delta + 2$)
3. $X(v) \leftarrow Y(v)$, and if $X \neq Y$ goto step 1.

For step 2 We could notice that:

$$X(u) \neq Y(u), \qquad \forall u \in S$$

Because $X(v) = Y(v)$ and $X(u) = Y(v)$, if $X(u) = Y(u)$ then we have $Y(v) = Y(u)$ which means Y is an illegal coloring. So, the measure will not increase in step 2.

And for step 3 We could easily notice that the measure will strictly decrease in step 3. So, the measure will strictly decrease in every 3 steps, which means the procedure will stop within $t \leq 3M(X,Y)$ steps. In each step, the situation we want always happens in a positive probability, so $P^t(X,Y) > 0$.

Proof for uniform distribution For $\forall X, Y \in \Omega$, if the measure $M(X,Y) \neq 1$, then $P(X,Y) = P(Y,X) = 0$. When $M(X,Y) = 1$, suppose $X(v) \neq Y(v)$, then

$$P(X,Y) = \frac{1}{n} \frac{1}{Q - |X(\Gamma(v))|}$$

and

$$P(Y,X) = \frac{1}{n} \frac{1}{Q - |Y(\Gamma(v))|}$$

. Because $M(X,Y) = 1$, we have $X(\Gamma(v)) = Y(\Gamma(v))$. So $P(X,Y) = P(Y,X)$.

3 Exercise 4.1(2)

[Alan Sokal.] Exhibit a sequence of connected graphs of increasing size, with $\Delta = 4$, such that the above MC fails to be irreducible when $q = 5$. (Hint: as a starting point, construct a "frozen" 5-colouring of the infinite square lattice, i.e., the graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ and edge set $(i,j),(i',j') : |i-i'| + |j-j'| = 1$. The adjective "frozen" applied to a state is intended to indicate that the only transition available from the state is a loop (with probability 1) to the same state.)

3.1 solution

Suppose there are 5 colors in $Q = \{0,1,2,3,4\}$ (actually, Q could be seen as addtional group $\mathbb{Z}_5$). As described below, I find a coloring which is "frozen". Note that, if we determine the color of an arbitary vertex in the graph, then we could determine the color of the rest vertices in the graph using the rule below.

$$\begin{array}{ccccc} & & i+1 & & \\ & & | & & \\ i-2 & — & i & — & i+2 \\ & & | & & \\ & & i-1 & & \end{array}$$

Clearly, for all vertex v, $Q \setminus X_0(\Gamma(v)) = X_0(v)$. So, there is a loop to the same state in this MC.

4 Exercise 4.1(3)

Design an MC on q-colourings of an arbitrary graph G of maximum degree Δ that is ergodic, provided only that $q \geq \Delta + 1$. The MC should be easily implementable, otherwise there is no challenge! (Hint: use transitions based on edge updates rather than vertex updates.)

4.1 solution

Lets consider a MC as follow:

The MC Designed by Me

1. Select an edge $e = \{u, v\} \in E$, u.a.r.
2. Select a pair of colors $(c_u, c_v) \in S_{u,v} = \{(c_u, c_v) : c_u \neq c_v \wedge c_u \notin X_0(\Gamma(u) \setminus \{v\}) \wedge c_v \notin X_0(\Gamma(v) \setminus \{u\})\}$, u.a.r.
3. $X_1(u) \leftarrow c_u$, $X_1(v) \leftarrow c_v$ and $X_1(w) \leftarrow X_0(w)$ for all $w \neq v \wedge w \neq u$.

For any two coloring X and Y, we want to show that they are reachable to each other by our operation (here, we only consider transforming X to Y). For convenience, we define $G^{\oplus} = G[\{v : X(v) \neq Y(v)\}]$ as an induced subgraph of G. It is clear that when $G^{\oplus} = \emptyset$, we have $X = Y$.

An Interesting Observation(1)

$$|Q \setminus X(\Gamma(u) \setminus \{v\})| \geq 2, \qquad \forall \{u, v\} \in G^{\oplus}$$

This is because:

$$\begin{aligned} |Q \setminus X(\Gamma(u) \setminus \{v\})| &= q - |X(\Gamma(u) \setminus \{v\})| \\ &\geq q - (|\Gamma(u)| - 1) \\ &\geq q - \Delta + 1 \\ &\geq (\Delta + 1) - \Delta + 1 = 2 \end{aligned}$$

An Interesting Observation(2)

If $\{u, v\} \in G^{\oplus}$, then:

$$S_v(u) = |\{c_v : (c_u, c_v) \in S_{u,v}\}| \geq 2$$

We could calculate S_v explicitly:

$$S_v(u) = \bigcup_{c_u \in Q \setminus X(\Gamma(u) \setminus \{v\})} Q \setminus (X(\Gamma(v) \setminus \{u\}) \cup c_u)$$

$$S_v(u) = \bigcup_{c_u \in Q \setminus X(\Gamma(u) \setminus \{v\})} (Q \setminus X(\Gamma(v) \setminus \{u\})) \cap (Q \setminus c_u)$$

$$S_v(u) = (Q \setminus X(\Gamma(v) \setminus \{u\})) \cap (\bigcup_{c_u \in Q \setminus X(\Gamma(u) \setminus \{v\})} (Q \setminus c_u))$$

$$S_v(u) = (Q \setminus X(\Gamma(v) \setminus \{u\}))$$

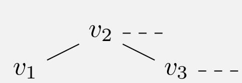
So, $|S_v(u)| \geq 2$.

Using this observation, we could design some sequences of operations which transforms X to Y. These operations may seem a little complex, because we need to make sure the edge number of $G^{\oplus}$ is strictly decrease after each operation.

Eliminate all the isolated edges in $G^{\oplus}$

Suppose $e = \{u, v\}$, then let $X(u) \leftarrow Y(u), X(v) \leftarrow Y(v)$, and we are done. Clearly, the edge number in $G^{\oplus}$ is strictly decrease in this operation.

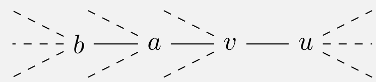
Eliminate all the 1-degree vertices in $G^{\oplus}$



Suppose $\deg(v_1) = 1$. Lets consider the edge $\{v_1, v_2\}$. If $X(v_2) = Y(v_1)$, then by $|S_{v_2}(v_3)| \geq 2$, we could change $X(v_2)$ to some color other than $Y(v_1)$ by changing $X(v_2)$ and $X(v_3)$ simutaneously. After that, we could change $X(v_1)$ to $Y(v_1)$ safely. Clearly, the edge number in $G^{\oplus}$ is strictly decrease in this operation.

Break cycles in $G^{\oplus}$

Once there is a cycle, we could remove a vertex from it.



Suppose we want to change $X(v)$ to $Y(v)$. If $Y(v) \notin X(\Gamma(v) \setminus \{u\})$, then we could use edge $\{u, v\}$ to reach this target. Else if $Y(v) \in X(\Gamma(v) \setminus \{u\})$, for example $X(a) = Y(v)$, we need to change $X(a)$ to some color other than $Y(v)$ (Note that there may be many vertices $w \in \Gamma(v)$ such that $X(w) = Y(v)$, we need to deal with them one by one). Note that for a, we could always find a vertex $b \neq v$ such that $X(b) \neq Y(b)$ because we are dealing with cycle. Because $|S_a(b)| \geq 2$, so we could change $X(a)$ to some color other than $Y(v)$ easily while making sure the edge number of $G^{\oplus}$ would not increase. After that, we could change $X(v)$ to $Y(v)$ using the edges $\{u, v\}$, this step will decrease the number of edges in $G^{\oplus}$ strictly.

We could do this operation until $G^{\oplus} = \emptyset$, which lead to $X = Y$.

5 Exercise 4.9

Suppose that $Q = \{0, 1, ..., 6\}$, $X_0(\Gamma(v)) = \{3, 6\}$ and $Y_0(\Gamma(v)) = \{4, 5, 6\}$. Thus the sets of legal colours for v in X_1 and Y_1 are $c_x \in \{0, 1, 2, 4, 5\}$ and $c_y \in \{0, 1, 2, 3\}$, respectively. Construct a joint distribution for (c_x, c_y) such that c_x is uniform on $\{0, 1, 2, 4, 5\}$, c_y is uniform on $\{0, 1, 2, 3\}$, and $Pr[c_x = c_y] = \frac{3}{5}$. Show that your construction is optimal.

5.1 solution

Construct the distribution as follow:

$Pr\{(0,0)\} = \frac{1}{5}$ $Pr\{(1,1)\} = \frac{1}{5}$ $Pr\{(2,2)\} = \frac{1}{5}$
$Pr\{(4,0)\} = \frac{1}{40}$ $Pr\{(4,1)\} = \frac{1}{40}$ $Pr\{(4,2)\} = \frac{1}{40}$
$Pr\{(5,0)\} = \frac{1}{40}$ $Pr\{(5,1)\} = \frac{1}{40}$ $Pr\{(5,2)\} = \frac{1}{40}$
$Pr\{(4,3)\} = \frac{1}{8}$ $Pr\{(5,3)\} = \frac{1}{8}$

6 Exercise 4.11

Let U be a finite set, A, B be subsets of U, and Z_a, Z_b be random variables, taking values in U. Then there is a joint distribution for Z_a and Z_b such that Z_a (respectively Z_b) is uniform and supported on A (respectively B) and, furthermore,

$$Pr[Z_a = Z_b] = \frac{|A \cap B|}{\max\{|A|, |B|\}}$$

Prove it.

6.1 solution

Assume $|A| \geq |B|$ and denote $A \cap B = C$ for convenience. First we want to show that $\frac{|C|}{|A|}$ is the best value that we could expect.

Observation

If $Pr[Z_a = Z_b] > \frac{|C|}{|A|}$, then there exists $x \in C$, such that $Pr[Z_a = x] > \frac{1}{|A|}$. So, the optimal solution would not exceed $\frac{|C|}{|A|}$.

Then we want to show that we can always reach $\frac{|C|}{|A|}$.

1. For all $x \in C$, let $Pr\{(x, x)\} = \frac{1}{|A|}$.
2. For all $x \in A \setminus C, y \in C$, let $Pr\{(x, y)\} = \frac{\frac{1}{|B|} - \frac{1}{|A|}}{|A \setminus C|}$.
3. For all $x \in A \setminus C, y \in B \setminus C$, let $Pr\{(x, y)\} = \frac{1}{|B||A \setminus C|}$.
4. For all the other pairs of x, y, let $Pr\{(x, y)\} = 0$.

First, its easy to see that $Pr\{Z_a = Z_b\} = \sum_{x \in C} Pr\{(x, x)\} = \frac{|C|}{|A|}$.

Then:

$$
\begin{aligned}
Pr\{Z_a = x \in C\} &= \frac{1}{|A|} \\
Pr\{Z_b = x \in C\} &= \frac{1}{|A|} + |A \setminus C| \times \frac{\frac{1}{|B|} - \frac{1}{|A|}}{|A \setminus C|} = \frac{1}{|B|} \\
Pr\{Z_b = x \in B \setminus C\} &= \frac{|A \setminus C|}{|B||A \setminus C|} = \frac{1}{|B|} \\
Pr\{Z_a = x \in A \setminus C\} &= |C|\frac{\frac{1}{|B|} - \frac{1}{|A|}}{|A \setminus C|} + \frac{|B \setminus C|}{|B||A \setminus C|} \\
&= (\frac{1}{|B|} - \frac{1}{|A|})\frac{|C|}{|A| - |C|} + \frac{|B| - |C|}{|B|(|A| - |C|)} \\
&= -\frac{1}{|A|}\frac{|C|}{|A| - |C|} + \frac{|B|}{|B|(|A| - |C|)} \\
&= -\frac{1}{|A|}\frac{|C|}{|A| - |C|} + \frac{|A|}{|A|(|A| - |C|)} \\
&= \frac{|A| - |C|}{|A|(|A| - |C|)} \\
&= \frac{1}{|A|}
\end{aligned}
$$

Which means Z_a and Z_b are both in uniform distribution.

7 Exercise 4.17(1)

Use Proposition 4.16 to construct an FPRAS for linear extensions of a partial order.

7.1 solution

Lets define the partial relationship as $\prec = \{\prec_1, \prec_2, \cdots, \prec_m\}$ where $\prec_i = (a \prec b)$. Then we consider the parial ordered set $O_i = (V, \{\prec_1, \prec_2, \cdots, \prec_i\})$ for $0 \le i \le m$ and $|V| = n$. The set of linear extensions of O_i could be denoted by $\Omega(O_i)$. Then the quantity $|\Omega(O_i)|$ could be expressed as follow:

$$|\Omega(O_i)| = n! \times \varrho_1 \varrho_2 \cdots \varrho_m$$

where

$$\varrho_i = \frac{|\Omega(O_i)|}{|\Omega(O_{i-1})|}$$

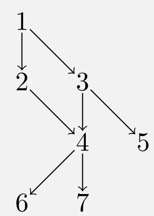
To give an appropriate lower bound on ϱ_i, we need to add some constraints on $\prec$. Intuitively, we want to sort the elements in $\prec$ by toplogical order. We could construct a DAG $G = (V, \prec)$ by seeing $\prec_i = (a \prec b)$ as an edge from b

to a. We could get a toplogical order $T = (v_1, v_2, \cdots, v_n)$ of this graph easily. And by using this toplogical order, we could define another order $\leq$:

$$d \prec b \Rightarrow (a \prec b) \leq (c \prec d)$$

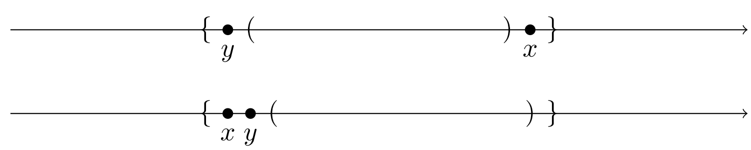
Then we could sort the elements of $\prec$ by $\leq$ easily.

Example



We could find a toplogical order $1, 2, 3, 4, 5, 6, 7$ from this graph. Then we could sort our parial relationship like this:$(3 \prec 1), (2 \prec 1), (4 \prec 2), (4 \prec 3), (5 \prec 3), (6 \prec 4), (7 \prec 4)$.

Now, we are about to find an appropiate lower bound for ϱ_i. First, it is clear that $\Omega(O_i) \subseteq \Omega(O_{i-1})$. For some element π in $\Omega(O_{i-1}) \setminus \Omega(O_i)$, it must violate $\prec_i = (x \prec y)$. Intuitively, π must look like:



We could construct a new permutation π' by moving x in front of y. And we claim that $\pi' \in \Omega(O_i)$. Let $\prec^{(i)} = \{\prec_1, \prec_2, \cdots, \prec_i\}$. If $\exists v \in V$ such that $(v \prec x) \in \prec^{(i)}$, then we have $(v \prec x)$ in front of $(x \prec y)$. But by $x \prec y$, we have $(x \prec y) \leq (v \prec x)$. Since $x \neq y$, these could not happen simutaneously. Finally, we could draw a conclusion that $\nexists v \in V$ such that $(v \prec x) \in \prec^{(i)}$. So we could move x across the inner brackets safely and hence $\pi' \in \Omega(O_i)$.

Since each $\pi' \in \Omega(O_i)$ could match at most n πs in $\Omega(O_{i-1}) \setminus \Omega(O_i)$, we could infer that:

$$\begin{aligned}
|\Omega(O_{i-1})| &= |\Omega(O_i)| + |\Omega(O_{i-1}) \setminus \Omega(O_i)| \\
&\leq |\Omega(O_i)| + n|\Omega(O_i)| \\
&= (n+1)|\Omega(O_i)|
\end{aligned}$$

which means:

$$\varrho_i = \frac{|\Omega(O_i)|}{|\Omega(O_{i-1})|} \geq \frac{1}{n+1}$$

So, by calling the sampler with $\delta = \frac{\epsilon}{3(n+1)m}$ on O_{i-1}, we could estimate ϱ_i. Suppose we got a random permutation π from our sampler. Let Z_i be the indicator variable of the event that $\pi \in \Omega(O_i)$, and set $\mu_i = \mathbb{E}(Z_i) = Pr[Z_i = 1]$. By choice of δ and the definition of the variation distance:

$$\varrho_i - \frac{\epsilon}{3(n+1)m} \leq \mu_i \leq \varrho_i + \frac{\epsilon}{3(n+1)m}$$

. Using the lower bound we get above, we could write this inequality as:

$$(1 - \frac{\epsilon}{3m})\varrho_i \leq \mu_i \leq (1 + \frac{\epsilon}{3m})\varrho_i$$

So we could find a FPRAS in time:

$$\begin{aligned} T &= c_1 n \log n + c_2 m^2 \epsilon^{-2} T_{\text{sampler}}(n, m, \frac{\epsilon}{3(n+1)m}) \\ &\leq c_1 n \log n + c_2 m^2 \epsilon^{-2} (n^3 - n)(2 \ln n + \ln(3(n+1)m/\epsilon))/6 \end{aligned}$$

Where c_1 and c_2 are some constant and $n \log n$ for sorting.

8 Exercise 4.17(2)

Reprove Proposition 4.5 using path coupling. Note the significant simplification over the direct coupling proof.

8.1 solution

Lets define the concept of adjacent: two state X and Y are adjacent if they are only have one vertex in different color. Suppose (X_0, Y_0) is a pair of states and the only difference between them is the color of one vertex u. We could define the coupling (refer to Algorithm 1).

Let $d(X, Y)$ be the number of verices in G which have different color in X and Y. Then we have these inequality: For $v \neq u \wedge v \notin \Gamma(u)$, we have:

$$\mathbb{E}[d(X_1, Y_1)|X_0, Y_0, v \neq u \wedge v \notin \Gamma(u)] = d(X_0, Y_0)$$

For $v \in \Gamma(u)$, lets assume that $Q \setminus Y_0(\Gamma(v)) \subseteq Q \setminus X_0(\Gamma(v))$. Clearly, $d(X_1, Y_1) = d(X_0, Y_0) + 1$ happens when $c_x = Y_0(u)$, which happens with probability at most $\frac{1}{|Q \setminus X_0(\Gamma(v))|}$. So, we have:

$$\begin{aligned} \mathbb{E}[d(X_1, Y_1)|X_0, Y_0, v \in \Gamma(u)] &= d(X_0, Y_0) + \frac{1}{|Q \setminus X_0(\Gamma(v))|} \\ &\leq d(X_0, Y_0) + \frac{1}{q - \deg(v)} \\ &\leq d(X_0, Y_0) + \frac{1}{q - \Delta} \end{aligned}$$

The case $u = v$ is very trivial that we could easily find that:

$$\mathbb{E}[d(X_1, Y_1)|X_0, Y_0, v = u] = d(X_0, Y_0) - 1$$

Algorithm 1: the coupling

begin
  Select $v \in V$ u.a.r.
  if $v = u$ **then**
    Select a color c from $Q \setminus X_0(\Gamma(v))$.
    $X_1(v) \leftarrow c, Y_1(v) \leftarrow c$.
  else if $Q \setminus Y_0(\Gamma(v)) \subseteq Q \setminus X_0(\Gamma(v))$ **then**
    Select a color c_x from $Q \setminus X_0(\Gamma(v))$ u.a.r.
    if $c_x \neq Y_0(u)$ **then**
      $c_y \leftarrow c_x$.
    else
      Select c_y from $Q \setminus Y_0(\Gamma(v))$ u.a.r.
    $X_1(v) \leftarrow c_x, Y_1(v) \leftarrow c_y$.
  else
    Select a color c_y from $Q \setminus Y_0(\Gamma(v))$ u.a.r.
    if $c_y \neq X_0(u)$ **then**
      $c_x \leftarrow c_y$.
    else
      Select c_x from $Q \setminus X_0(\Gamma(v))$ u.a.r.
    $X_1(v) \leftarrow c_x, Y_1(v) \leftarrow c_y$.

Combining them together, we could find that:

$$\begin{aligned}
\mathbb{E}[d(X_1, Y_1)] &\leq d(X_0, Y_0) + \frac{|\Gamma(u)|}{n}\frac{1}{q-\Delta} - \frac{1}{n} \\
&\leq d(X_0, Y_0) + \frac{\Delta}{n}\frac{1}{q-\Delta} - \frac{1}{n} \\
&= d(X_0, Y_0) - \frac{q-2\Delta}{n(q-\Delta)} \\
&\leq (1 - \frac{q-2\Delta}{n(q-\Delta)})d(X_0, Y_0)
\end{aligned}$$

By path coupling, we could then draw a conclusion that any pair of states $(X_0, Y_0) \in \Omega$, we have $\mathbb{E}[d(X_1, Y_1)] \leq (1-\alpha)d(X_0, Y_0)$, where $\alpha = \frac{q-2\Delta}{n(q-\Delta)}$. Moreover, we could find that:

$$\begin{aligned}
\Pr[d(X_t, Y_t) \neq 0] &= \mathbb{E}[d(X_t, Y_t) = 0] \\
&\leq (1-\alpha)^t \max_{(X,Y)\in\Omega^2} d(X, Y) \\
&\leq (1-\alpha)^t n \\
&\leq e^{-at} n
\end{aligned}$$

From $\Pr[d(X_t, Y_t) \neq 0] \leq e^{-at}n \leq \epsilon$, we could get:

$$\begin{aligned} t(\epsilon) &\leq \frac{1}{\alpha}(\ln n - \ln \epsilon) \\ &\leq \frac{q-\Delta}{q-2\Delta} n(\ln n - \ln \epsilon) \end{aligned}$$