Lecture 17: Adjoint, self-adjoint, and normal operators; the spectral theorems! (1)

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Goals (2)

- Adjoint operators and their properties, conjugate linearity, and dual spaces
- Self-adjoint operators, spectral theorems, and normal operators
- As time allows: corollaries.

Adjoint operators (3)

Let $T \in \mathcal{L}(V, W)$, where V and W are inner product spaces.

Definition 1. An *adjoint* operator $T^* \in \mathcal{L}(W, V)$ is one such that

$$\langle Tv, w\rangle = \langle v, T^* w\rangle, \quad \forall v \in V, w \in W. \tag{0.1}$$

Proposition 0.2. *If V is finite-dimensional, then for all $T \in \mathcal{L}(V, W)$, there exists a unique adjoint $T^* \in \mathcal{L}(W, V)$.*

Proof.

- Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V.
- Then, T^* must satisfy $\langle e_j, T^* w\rangle = \langle Te_j, w\rangle$ for all j.
- Hence, $T^* w = \sum_{j=1}^n \langle Te_j, w\rangle e_j$. So T^* must be unique.
- Define T^* in this way. Then (0.1) is satisfied for $v = e_j$.
- By linearity, (0.1) is satisfied for all $v \in V$. So T^* exists. $\square$

Conjugate-linearity (4)

Definition 2. A map $T : V \to W$ is *conjugate-linear* if $T(u+v) = T(u) + T(v)$ and $T(\lambda v) = \overline{\lambda} T(v)$.

Conjugate-linear maps still have nullspace, range, the rank-nullity theorem, etc. They also still form a vector space. We could have used this to prove $V = U \oplus U^\perp$ before: $U^\perp$ is the nullspace of the *conjugate-linear* map

$$V \to \mathcal{L}(U, \mathbf{F}), \quad v \mapsto \langle -, v \rangle.$$

The rank-nullity theorem then implies $\dim U^\perp = \dim V - \dim U$. Then, since $U \cap U^\perp = 0$, we deduce $V = U \oplus U^\perp$. (Alternatively, $U^\perp$ is the nullspace of the *linear* map $b \mapsto \langle v, - \rangle$: now $V \to \overline{\mathcal{L}}(U, \mathbf{F})$ =the vector space of *conjugate-linear* maps $U \to \mathbf{F}$.)

Properties of adjoints (5)

Let V and W be inner product spaces, and let V be finite-dimensional.

Proposition 0.3. *The map $\mathcal{L}(V, W) \to \mathcal{L}(W, V)$, $T \mapsto T^*$, is conjugate-linear.*

Proof.
- Have to check: $(T + S)^* = T^* + S^*$ and $(\lambda T)^* = \overline{\lambda} T^*$.
- These follow by additivity and homogeneity of $\langle -, - \rangle$, e.g.: $\langle v, (T + S)^* w \rangle = \langle (T + S)v, w \rangle = \langle Tv, w \rangle + \langle Sv, w \rangle = \langle v, T^* w \rangle + \langle v, S^* w \rangle = \langle v, (T^* + S^*)w \rangle$. □

Note that, when $(T^*)^*$ exists, it equals T. Hence:

Corollary 3. *If V and W are finite-dimensional, then the adjoint map is invertible, and is inverse is also the adjoint map.*

Dual space and linear functionals (6)

Definition 4. The *dual space to V* is $V^* := \mathcal{L}(V, F)$.

Elements $\varphi \in V^*$ are *linear functionals* $V \to \mathbf{F}$. Note that, when V is finite-dimensional, $V \cong V^*$ since they have the same dimension. This is *not* a "canonical" (="natural") isomorphism! When V is an inner product space, we can do better:

Corollary 5 (Theorem 6.45)**.** *Let V be a finite-dimensional inner product space. Then the adjoint map is a conjugate-linear isomorphism $V \xrightarrow{\sim} V^*$.*

Specifically, $V = \mathcal{L}(\mathbf{F}, V) \xrightarrow{\sim} \mathcal{L}(V, \mathbf{F}) = V^*$. Explicitly,

$$u \mapsto \varphi_u \in V^* \text{ s.t. } \varphi_u(v) = \langle v, u \rangle, \forall u \in V.$$

Then, for $T \in \mathcal{L}(V, W)$ and $w \in W$, $T^*(w)$ can alternatively be defined as: $T^*(w) =$ the unique $u \in V$ such that

$$\varphi_u(v) = \langle Tv, w \rangle, \text{ i.e., } \langle v, u \rangle = \langle Tv, w \rangle, \forall v \in V.$$

Further properties (7)

Let $T : V \to W$, with V, W finite-dimensional inner product spaces.

Proposition 0.4 (Proposition 6.46)**.** *(a)* $\operatorname{null} T^* = (\operatorname{range} T)^\perp$

(b) $\operatorname{range} T^* = (\operatorname{null} T)^\perp$

(c) $\operatorname{null} T = (\operatorname{range} T^*)^\perp$

(d) $\operatorname{range} T = (\operatorname{null} T^*)^\perp$

Proof. (a) $T^*w = 0 \Leftrightarrow \langle v, T^*w\rangle = 0$ for all $v \Leftrightarrow \langle Tv, w\rangle = 0$ for all $v \Leftrightarrow w \in (\operatorname{range} T)^\perp$.

(d) Take $^\perp$ of both sides of (a), using $(U^\perp)^\perp = U$.

(b)–(c) Swap T with T^*, using $(T^*)^* = T$. □

Matrix of operators and adjoints (8)

Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_m)$ be orthonormal bases of V and W.

Proposition 0.5. *Let* $A = (a_{jk}) = \mathcal{M}(T)$ *and* $B = (b_{jk}) = \mathcal{M}(T^*)$*. Then* $a_{jk} = \langle Te_k, f_j\rangle$ *and* $b_{jk} = \langle T^*f_k, e_j\rangle = \overline{a_{kj}}$*. Hence,* $B = \overline{A}^t$*.*

Proof.

- The formula for A follows because $Te_k = \sum_{j=1}^m \langle Te_k, f_j\rangle f_j$.
- The formula for B follows for the same reason (just replace T with T^*).
- Then, $b_{jk} = \overline{a_{kj}}$ is a consequence of the definition of T^* together with conjugate symmetry. □

Self-adjoint operators (9)

Definition 6. An operator $T \in \mathcal{L}(V)$ is self-adjoint if $T = T^*$.

Proposition 0.6 (Proposition 7.1)**.** *All eigenvalues of a self-adjoint operator are real.*

Proof. Let $v \in V$ be nonzero such that $Tv = \lambda v$. Then, $\lambda\langle v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\lambda}\langle v, v\rangle$. □

Spectral theorem for self-adjoint operators (10)

From now on, all our vector spaces are finite-dimensional inner product spaces.

Theorem 7 (Theorem 7.13+)**.** *T is self-adjoint iff T admits an orthonormal eigenbasis with* real *eigenvalues.*

Proof.

- Proof for $\mathbf{F} = \mathbf{C}$: we already know that $\mathcal{M}(T)$ is upper-triangular in some orthonormal basis.

- Then, $T = T^*$ iff the matrix equals its conjugate transpose, i.e., it is upper-triangular with real values on the diagonal.
- Now let $\mathbf{F} = \mathbf{R}$. In some orthonormal basis, the matrix is block upper-triangular with 1×1 and 2×2 blocks.
- Then, the matrix equals its own transpose iff it is block diagonal with real diagonal entries and symmetric 2×2 blocks.
- However, in slide (6) next lecture we show that the 2×2 blocks are anti-symmetric. So there are none. $\square$