Introduction to the Holographic Transformation

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1 Background for Kronecker Products

Definitin 1.1. *Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, the* ***Kronecker Product*** *of A and B is defined as*

$$A \otimes B \triangleq \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}$$

Fact 1.1. *Some basic properties:*

- $A \otimes (B + C) = A \otimes B + A \otimes C$
- $(B + C) \otimes A = B \otimes A + C \otimes A$
- $(kA) \otimes B = A \otimes (kB) = k(A \otimes B)$
- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- $A \otimes 0 = 0 \otimes A = 0$

For convenience, sometimes people use the following notation.

Definitin 1.2. *Let $A \in \mathbb{R}^{m\times n}$. For any $k \in \mathbb{Z}^+$, let $A^{\otimes k} \triangleq \underbrace{A \otimes A \otimes \cdots \otimes A}_{k} \in \mathbb{R}^{m^k\times n^k}$.*

Fact 1.2. $(A \otimes B)^T = A^T \otimes B^T$.

Fact 1.3. *Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times s}$, $C \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{s\times t}$, then*

$$(A \otimes B)(C \otimes D) = AC \otimes BD (\in \mathbb{R}^{mr\times pt})$$

Proof. One could verify this by:

$$\begin{aligned}(A \otimes B)(C \otimes D) &= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \begin{bmatrix} c_{11}D & \cdots & c_{1p}D \\ \vdots & \ddots & \vdots \\ c_{n1}D & \cdots & c_{np}D \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^n a_{1k}c_{k1}BD & \cdots & \sum_{k=1}^n a_{1k}c_{kp}BD \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n a_{mk}c_{k1}BD & \cdots & \sum_{k=1}^n a_{mk}c_{kp}BD \end{bmatrix} \\ &= AC \otimes BD. \end{aligned}$$

□

Corollary 1.1. *If A and B are non-singular, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.*

Proof. $(A \otimes B)(A^{-1} \otimes B^{-1}) = I \otimes I = I$. □

Corollary 1.2. *Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$ and $k \in \mathbb{Z}^+$ be any number, then $A^{\otimes k}B^{\otimes k} = (AB)^{\otimes k}$.*

Proof. When $k = 1$, $AB = AB$ holds trivially. When $k > 1$, we could use the induction:

$$\begin{aligned} A^{\otimes k}B^{\otimes k} &= (A \otimes A^{\otimes k-1})(B \otimes B^{\otimes k-1}) \\ &= AB \otimes A^{\otimes k-1}B^{\otimes k-1}. \end{aligned}$$

□

Corollary 1.3. *If A is non-singular, then for any $k \in \mathbb{Z}^+$, we have $\left(A^{\otimes k}\right)^{-1} = \left(A^{-1}\right)^{\otimes k}$.*

2 Holant Problem and Holographic Transformation

Let $G = (V, E)$ be any graph. The incident graph $B = ((V, E), H)$ of G is a bipartite graph, where for any $v \in V, e \in E$, there is an edge (in H) between them iff v is an end-point of e in G. The Holant problem is defined on the incident graph B.

Definitin 2.1 (Holant Problem)**.** *Let $B = ((V, E), H)$ be a bipartite graph. For any vertex $x \in V \cup E$, there is a function $f_x : \{0,1\}^{d(x)} \to \mathbb{R}$, where $d(x)$ is the degree of x. Then, the Holant problem is to calculate the following formula:*

$$\begin{aligned}\mathrm{Holant}_B &\triangleq \left\langle \bigoplus_{v\in V} f_v, \bigoplus_{e\in E} f_e \right\rangle_{\sigma\in\{0,1\}^H} \\ &= \sum_{\sigma\in\{0,1\}^H} \left(\prod_{v\in V} f_v(\sigma|_{H(v)})\right)\left(\prod_{e\in E} f_e(\sigma|_{H(e)})\right),\end{aligned}$$

where $\sigma|_{H(v)}$ and $\sigma|_{H(e)}$ are generated by restricting the configuration σ into $H(v)$ and $H(e)$, respectively. Here, we note that $H(v)$ and $H(e)$ are the edges incident to v and e in H, respectively.

Remark 2.1. *In the bipartite graph B, any edge $h \in H$ is actually a half edge in the original graph G. And obviously, for any $u, v \in V$, we have $H(u) \cap H(v) = \emptyset$. Similarlly, for any $e, f \in E$, we have $H(e) \cap H(f) = \emptyset$. Finally, using the fact that $\bigcup_{v\in V} H(v) = H$ and $\bigcup_{e\in E} H(e) = H$, we know that $\bigotimes_{v\in V} f_v$ and $\bigotimes_{e\in E} f_e$ are functions defined on $\{0,1\}^H$.*

Here is an example of holographic transformation. Let $M \in \mathbb{R}^{2\times 2}$ be a non-singular matrix. Then

$$\begin{aligned}\mathrm{Holant}_B &= \left(\bigotimes_{v\in V} f_v\right)^T \left(\bigotimes_{e\in E} f_e\right) \\ &= \left(\bigotimes_{v\in V} f_v^T\right) M^{\otimes|H|} (M^{-1})^{\otimes|H|} \left(\bigotimes_{e\in E} f_e\right) \\ &= \left(\bigotimes_{v\in V} f_v^T M^{\otimes d(v)}\right)\left(\bigotimes_{e\in E} (M^{-1})^{\otimes d(e)} f_e\right)\end{aligned}$$

So, the Holant problem $\langle \bigotimes_{v\in V} f_v, \bigotimes_{e\in E} f_e \rangle$ is equivalent to the Holant problem $\left\langle \bigotimes_{v\in V} (M^T)^{\otimes d(v)} f_v, \bigotimes_{e\in E} (M^{-1})^{\otimes d(e)} f_e \right\rangle$. This kind of transformation is called holographic transformation, they really do nothing but create equivalent problems.