1 basic

Fact 1. *If we have a matroid $\mathcal{B}(M)$, and an element $e \in M$. Then any basis in $\mathcal{B}_e$ has at least one neighbor in $\mathcal{B}_{\overline{e}}$ and vice versa.*

Proof. Actually, this is a corollary of strong base exchange theorem, which is quite difficult to prove. $\square$

2 deleting

Fact 2. *If C is a cycle in $\mathcal{B}(M \setminus e)$ and $e \notin C$, then C is a cycle in $\mathcal{B}(M)$.*

Proof. Say we have a cycle C of $\mathcal{B}(M \setminus e)$. More specificly,

$$\begin{aligned} &B_0 \in \mathcal{B}(M \setminus e) \\ &B_1 \in \mathcal{B}(M) \\ &B_1 = B_0 \cup \{e\} \setminus \{g\}, \qquad g \neq e \end{aligned}$$

By the definition of C, we know that C could not extend to B_0, which means there is an element $f \in C$ and $f \notin B_0$. Then, by the definition of B_1, we know that $f \notin B_1$, so C could not extend to B_1. $\square$

Fact 3. *If D is a cut in $\mathcal{B}(M \setminus e)$, then D is a cut in $\mathcal{B}(M)$ or $D \cup \{e\}$ is a cut in $\mathcal{B}(M)$.*

Proof. Now, we have

$$\begin{aligned} &B_0 \in \mathcal{B}(M \setminus e) \\ &B_1 \in \mathcal{B}(M) \\ &B_1 = B_0 \cup \{e\} \setminus \{g\}, \qquad g \neq e \end{aligned}$$

By the definition of D we know that B_0 is not contained in the complement of D. So there is an element $f \in B_0$ which is not contained by the complement of D. So if $f \neq e$, then B_1 is not contained by the complement of D. Or if $f = e$, then B_1 is not contained by the complement of $D \cup \{e\}$. $\square$

Theorem 1. *$\mathcal{B}(M \setminus e)$ is an orientable matroid.*

Proof. If $\mathcal{B}(M)$ is an orientable matroid, then we have

$$\begin{aligned} \gamma(C, g) \neq 0 &\text{ iff } g \in C \\ \delta(D, g) \neq 0 &\text{ iff } g \in D \\ \sum_{g \in E} \gamma(C, g)\delta(D, g) = 0&, \qquad E \text{ is the groud set} \end{aligned}$$

Now, consider $\gamma' : \mathcal{C} \times E \setminus \{e\} \to \{-1, 0, 1\}$, and $\delta' : \mathcal{D} \times E \setminus \{e\} \to \{-1, 0, 1\}$ for $\mathcal{B}(M \setminus e)$. Set their values by the following rules:

$$\begin{aligned}
\gamma'(C, g) &= \gamma(C, g) \\
\delta'(D, g) &= \delta(D, g), \text{ if } D \text{ is a cut in } \mathcal{B}(M) \\
\text{or } \delta'(D, g) &= \delta(D \cup \{e\}, g), \text{ if } D \cup \{e\} \text{ is a cut in } \mathcal{B}(M)
\end{aligned}$$

According to Fact 1 and Fact 2, we could claim that γ' and δ' are well defined. Then we have:

$$\begin{aligned}
\sum_{g \in E \setminus \{e\}} \gamma'(C, g)\delta'(D, g) &= \sum_{g \in E \setminus \{e\}} \gamma'(C, g)\delta'(D, g) + 0 \\
&= \sum_{g \in E \setminus \{e\}} \gamma'(C, g)\delta'(D, g) + \gamma'(C, e)\delta'(D, g), \qquad \text{since } e \notin C \\
&= \begin{cases} \sum_{g \in E} \gamma(C, g)\delta(D, g), \text{ if } D \text{ is a cut in } \mathcal{B}(M) \\ \sum_{g \in E} \gamma(C, g)\delta(D \cup \{e\}, g), \text{ if } D \cup \{e\} \text{ is a cut in } \mathcal{B}(M) \end{cases} \\
&= 0
\end{aligned}$$

So, we know that $\mathcal{B}(M \setminus \{e\})$ is an orientable matroid. □

3 contracting

Fact 4. *If D is a cut in $\mathcal{B}(M/e)$ and $e \notin D$, then D is a cut in $\mathcal{B}(M)$.*

Proof. Say we have a cut D of $\mathcal{B}(M/e)$. More specificly,

$$\begin{aligned}
&B_0 \in \mathcal{B}(M/e) \\
&B_1 \in \mathcal{B} \\
&B_1 = B_0 \cup \{g\}, \qquad g \neq e
\end{aligned}$$

By the definition of D, we know that D's complement does not contain B_0. So, there is an element $f \in B_0$, which does not contained by D's complement. Note that this element is also in B_1, so D's complement also does not contain B_1. □

Fact 5. *If C is a cycle in $\mathcal{B}(M/e)$ and $e \notin C$, then C is a cycle in $\mathcal{B}(M)$ or $C \cup \{e\}$ is a cycle in $\mathcal{B}(M)$.*

Proof. Now we have

$$\begin{aligned}
&B_0 \in \mathcal{B}(M/e) \\
&B_1 \in \mathcal{B} \\
&B_1 = B_0 \cup \{g\}, \qquad g \neq e
\end{aligned}$$

By the definition of C, we know that C could not extend to B_0. So there is an element $f \in C$ and $f \notin B_0$. If $f \neq g$, then C could not extend to B_1. Or if $f = g$, then $C \cup \{e\}$ could not extend to B_1. □

Theorem 2. *$\mathcal{B}(M/e)$ is an orientable matroid.*

The proof of this theorem is similar to Theorem 1.