Mixing Time of Continuous Time Markov Chain

Xiaoyu Chen

1 Background

For the background of constructing CTMC, please refer to a previous note[1] for details. The rest of this article mainly comes from [MT06].

2 Measures and Mixing Times

Definition 2.1 (ℓ^p distance)**.** *For any function $f : \Omega \to \mathbb{R}$, we have:*

$$\| f \|_{p,\pi} := \left(\sum_{x \in \Omega} \pi(x) |f(x)|^p \right)^{1/p}$$

Definition 2.2 (An Entropy-like Measure)**.**

$$\mathrm{Ent}_\pi(f) := \mathbb{E}_\pi[f \log f] - (\mathbb{E}_\pi f) \log \mathbb{E}_\pi f$$

Note that, if $\mathbb{E}_\pi f = 1$, we have

$$\mathrm{Ent}_\pi(f) = \mathbb{E}_\pi[f \log f]$$

There are many ways to measure the distance between $P^t(x, \cdot)$ and π.

Definition 2.3. *For a discrete Markov chain P, let $k_t^x(y) := P^t(x, y)/\pi(y)$.*

Since $k_t^x \to \mathbf{1}$ as $t \to \infty$, so many important measures of mixing are defined as the ℓ^p distance (norm) of $k_t^x - \mathbf{1}$. An interesting counterexample of this is $\mathrm{Ent}_\pi(k_t^x)$, it is defined as the relative entropy between $P^t(x, \cdot)$ and π.

Fact 2.1.

$$\| P^t(x, \cdot) - \pi \|_{TV} = \frac{1}{2} \| k_t^x - 1 \|_{1,\pi}$$

Fact 2.2.

$$\mathrm{Var}_\pi(k_t^x) = \| k_t^x - 1 \|_{2,\pi}$$

Fact 2.3.

$$D(P^t(x, \cdot) \parallel \pi) = \sum_{y \in \Omega} \pi(y) \frac{P^t(x, y)}{\pi(y)} \log \frac{P^t(x, y)}{\pi(y)} = \mathrm{Ent}_\pi(k_t^x)$$

[1] Notes for Continuous Time Markov Chains

Having these measures in hand, we could define their mixing time respectively.

Definition 2.4 (Some Mixing Times)**.**

$$\begin{aligned}
\tau(\varepsilon) &= \min\{n : \forall x \in \Omega, \| p^n(x,\cdot) - \pi \|_{TV} \leq \varepsilon\} \\
\tau_D(\varepsilon) &= \min\{n : \forall x \in \Omega, D(p^n(x,\cdot) \| \pi) \leq \varepsilon\} \\
\tau_2(\varepsilon) &= \min\{n : \forall x \in \Omega, \| p^n(x,\cdot) - \pi \|_{2,\pi} \leq \varepsilon\}
\end{aligned}$$

3 Continuous Time Markov Chain Mixing

See a clip[2] from [MT06] for details.

[2] [Clip] Mathematical Aspects of Mixing Times in Markov Chains.pdf

References

[MT06] Ravi R Montenegro and Prasad Tetali. *Mathematical aspects of mixing times in Markov chains.* Now Publishers Inc, 2006.