

# High Dimensional Random Walk: Inner Products And Operators

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## Abstract

A very interesting point to note is that if we write the down-up walk in the operator form, we will gain a lot of benefits in the analysis. In this note, we collect useful properties of operators from [CGM19] and [AL20]. And we could use these properties to reimplement the result in [AL20] by using the “level-by-level decay approach” discovered by [CGM19].

## 1 The Operators

We will use the definition appears in [CGM19] first. And we will show that they are equivalent to the definition appears in [AL20].

**Definition 1.1.**

$$[X_k \leftrightarrow X_{k+1}](I, J) \triangleq \begin{cases} 0 & I \not\subset J \\ 1 & I \subset J \end{cases}$$

$$[X_{k+1} \leftrightarrow X_k] \triangleq [X_k \leftrightarrow X_{k+1}]^T$$

**Definition 1.2** (Operators).

$$[\pi_k \leftrightarrow \pi_{k+1}] \triangleq \frac{1}{k+1} \text{diag}(\pi_k)^{-1} [X_k \leftrightarrow X_{k+1}] \text{diag}(\pi_{k+1})$$

$$[\pi_{k+1} \leftrightarrow \pi_k] \triangleq \frac{1}{k+1} [X_{k+1} \leftrightarrow X_k]$$

Note that  $[\pi_k \leftrightarrow \pi_{k+1}]$  and  $[\pi_{k+1} \leftrightarrow \pi_k]$  are denoted as  $P_k^\uparrow$  and  $P_{k+1}^\downarrow$  in [CGM19], respectively.

**Fact 1.1** (Equivalence to [AL20]). For any function  $f : X(k+1) \rightarrow \mathbb{R}$  and  $g : X(k) \rightarrow \mathbb{R}$ , we have

$$[\pi_k \leftrightarrow \pi_{k+1}] \mathbf{f}(\alpha) = \sum_{\beta \supset \alpha} \frac{\pi_{k+1}(\beta)}{(k+1)\pi_k(\alpha)} \cdot f(\beta) = \mathbb{E}_{\beta \supset \alpha} f(\beta)$$

, and

$$[\pi_{k+1} \leftrightarrow \pi_k] \mathbf{g}(\beta) = \sum_{\alpha \subset \beta} \frac{1}{k+1} \cdot g(\alpha) = \mathbb{E}_{\alpha \subset \beta} g(\alpha)$$

. Note that  $[\pi_k \leftrightarrow \pi_{k+1}]$  and  $[\pi_{k+1} \leftrightarrow \pi_k]$  are also denoted as  $D_{k+1}$  and  $U_k$  in [AL20], respectively.

Using these operators, we could define the down-up walk and up-down walk on level  $k$ .

**Definition 1.3** (down-up walk, up-down walk). *Down-up walk:*

$$P_{\pi_k}^{\triangle} \triangleq [\pi_k \leftrightarrow \pi_{k-1}] [\pi_{k-1} \leftrightarrow \pi_k]$$

*Up-down walk:*

$$P_{\pi_k}^{\nabla} \triangleq [\pi_k \leftrightarrow \pi_{k+1}] [\pi_{k+1} \leftrightarrow \pi_k]$$

And, we denote the non-lazy version of  $P_{\pi_k}^{\triangle}, P_{\pi_k}^{\nabla}$  as  $P_{\pi_k}^{\wedge}, P_{\pi_k}^{\vee}$ , respectively.

**Definition 1.4** (Inner Product on  $\pi_k$ ). For  $f, g : X(k) \rightarrow \mathbb{R}$ , let

$$\langle f, g \rangle_{\pi_k} \triangleq \sum_{\alpha \in X(k)} \pi(\alpha) f(\alpha) g(\alpha)$$

**Theorem 1.1** (Adjointness of Operators, [AL20]). For  $f : X(k) \rightarrow \mathbb{R}$  and  $g : X(k+1) \rightarrow \mathbb{R}$ , then we have

$$\langle g, [\pi_{k+1} \leftrightarrow \pi_k] f \rangle_{\pi_{k+1}} = \langle [\pi_k \leftrightarrow \pi_{k+1}] g, f \rangle_{\pi_k}$$

*Proof.* Here, we give a high level proof of this. You could verify it by brute force.

$$\begin{aligned} \langle g, [\pi_{k+1} \leftrightarrow \pi_k] f \rangle_{\pi_{k+1}} &= \mathbb{E}_{\beta \sim \pi_{k+1}} [\mathbf{g}(\beta) \cdot [\pi_{k+1} \leftrightarrow \pi_k] f(\beta)] \\ &= \mathbb{E}_{\beta \sim \pi_{k+1}} [\mathbf{g}(\beta) \mathbb{E}_{\alpha \subset \beta} [f(\alpha)]] \\ &= \mathbb{E}_{\substack{(\beta, \alpha) \sim (\pi_{k+1}, \pi_k) \\ \alpha \subset \beta}} [g(\beta) f(\alpha)] \end{aligned}$$

$$\begin{aligned} \langle [\pi_k \leftrightarrow \pi_{k+1}] g, f \rangle_{\pi_k} &= \mathbb{E}_{\alpha \sim \pi_k} [[\pi_k \leftrightarrow \pi_{k+1}] g(\alpha) f(\alpha)] \\ &= \mathbb{E}_{\alpha \sim \pi_k} [\mathbb{E}_{\beta \supset \alpha} [g(\beta)] f(\alpha)] \\ &= \mathbb{E}_{\substack{(\beta, \alpha) \sim (\pi_{k+1}, \pi_k) \\ \alpha \subset \beta}} [g(\beta) f(\alpha)] \end{aligned}$$

So, we can conclude that  $\langle g, [\pi_{k+1} \leftrightarrow \pi_k] f \rangle_{\pi_{k+1}}$  and  $\langle [\pi_k \leftrightarrow \pi_{k+1}] g, f \rangle_{\pi_k}$  gives us the same joint distribution  $(\pi_{k+1}, \pi_k)$  and thus they are equal.  $\square$

**Definition 1.5** (map  $f$  to  $f^1$ ). Suppose we have a distribution  $\pi$  and a function  $f : \text{supp}(\pi) \rightarrow \mathbb{R}$ , then we define:  $J_{\pi} f \triangleq f^1 = \langle f, \mathbf{1} \rangle_{\pi} \cdot \mathbf{1}$ . It turns out that  $J_{\pi} = \mathbf{1} \pi^T$ . Note that  $f = f^1 + f^{\perp 1}$ .

## 2 Reimplement [AL20]

See [Notes for level-by-level-decay approach](#).

**Definition 2.1.**

$$a_k = \mathbb{E}_{\pi_k} \left[ \left( \frac{\mu_k}{\pi_k} \right)^2 \right]$$

First, we have the following fact.

**Fact 2.1.**

$$\begin{aligned} a_{k+1} &= \sum_{\gamma \in X(k-1)} \pi_{k-1}(\gamma) \left( \frac{\mu_{k-1}(\gamma)}{\pi_{k-1}(\gamma)} \right)^2 \cdot \sum_{\{x,y\} \in X_\gamma(2)} \pi_2^\gamma(\{x,y\}) \left( \frac{\mu_2^\gamma(\{x,y\})}{\pi_2^\gamma(\{x,y\})} \right)^2 \\ a_k &= \sum_{\gamma \in X(k-1)} \pi_{k-1}(\gamma) \left( \frac{\mu_{k-1}(\gamma)}{\pi_{k-1}(\gamma)} \right)^2 \cdot \sum_{\{x\} \in X_\gamma(1)} \pi_1^\gamma(\{x\}) \left( \frac{\mu_1^\gamma(\{x\})}{\pi_1^\gamma(\{x\})} \right)^2 \\ a_k &= \sum_{\gamma \in X(k-1)} \pi_{k-1}(\gamma) \left( \frac{\mu_{k-1}(\gamma)}{\pi_{k-1}(\gamma)} \right)^2 \cdot 1 \end{aligned}$$

**Definition 2.2.**

$$\begin{aligned} b_{k+1} &= \sum_{\{x,y\} \in X_\gamma(2)} \pi_2^\gamma(\{x,y\}) \left( \frac{\mu_2^\gamma(\{x,y\})}{\pi_2^\gamma(\{x,y\})} \right)^2 = \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma} \\ b_k &= \sum_{\{x\} \in X_\gamma(1)} \pi_1^\gamma(\{x\}) \left( \frac{\mu_1^\gamma(\{x\})}{\pi_1^\gamma(\{x\})} \right)^2 = \left\langle \frac{\mu_1^\gamma}{\pi_1^\gamma}, \frac{\mu_1^\gamma}{\pi_1^\gamma} \right\rangle_{\pi_1^\gamma} \\ b_{k-1} &= 1 \end{aligned}$$

**Fact 2.2.**

$$b_{k-1} = \left\langle \frac{\mu_1^\gamma}{\pi_1^\gamma}, J_{\pi_1^\gamma} \frac{\mu_1^\gamma}{\pi_1^\gamma} \right\rangle_{\pi_1^\gamma} = \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, J_{\pi_2^\gamma} \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma}$$

**Fact 2.3.**

$$\frac{\mu_1^\gamma}{\pi_1^\gamma} = [\pi_1^\gamma \leftrightarrow \pi_2^\gamma] \frac{\mu_2^\gamma}{\pi_2^\gamma}$$

*Proof.*

$$\begin{aligned} [\pi_1^\gamma \leftrightarrow \pi_2^\gamma] \frac{\mu_2^\gamma}{\pi_2^\gamma}(x) &= \sum_{\{x,y\} \in X_2^\gamma} \frac{\pi_2^\gamma(\{x,y\})}{2\pi_1^\gamma(\{x\})} \cdot \frac{\mu_2^\gamma(\{x,y\})}{\pi_2^\gamma(\{x,y\})} \\ &= \sum_{\{x,y\} \in X_2^\gamma} \frac{\mu_2^\gamma(\{x,y\})}{2\pi_1^\gamma(\{x\})} \\ &= \frac{\mu_1^\gamma}{\pi_1^\gamma}(\{x\}) \end{aligned}$$

□

**Fact 2.4.**

$$\begin{aligned}
b_k &= \left\langle [\pi_1^\gamma \leftrightarrow \pi_2^\gamma] \frac{\mu_2^\gamma}{\pi_2^\gamma}, [\pi_1^\gamma \leftrightarrow \pi_2^\gamma] \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_1^\gamma} \\
&= \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, [\pi_2^\gamma \leftrightarrow \pi_1^\gamma] [\pi_1^\gamma \leftrightarrow \pi_2^\gamma] \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma} \\
&= \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, P_{\pi_2^\gamma}^\nabla \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma}
\end{aligned}$$

Having these facts in hand, we could have the following argument (similar to [AL20]).

$$\begin{aligned}
b_k - b_{k-1} &= \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, (P_{\pi_2^\gamma}^\nabla - J_{\pi_2^\gamma}) \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma} \\
&= \left\langle \left( \frac{\mu_2^\gamma}{\pi_2^\gamma} \right)^{\perp 1}, (P_{\pi_2^\gamma}^\nabla - J_{\pi_2^\gamma}) \left( \frac{\mu_2^\gamma}{\pi_2^\gamma} \right)^{\perp 1} \right\rangle_{\pi_2^\gamma} \\
&\leq \lambda_2(P_{\pi_2^\gamma}^\nabla) \left\langle \left( \frac{\mu_2^\gamma}{\pi_2^\gamma} \right)^{\perp 1}, \left( \frac{\mu_2^\gamma}{\pi_2^\gamma} \right)^{\perp 1} \right\rangle_{\pi_2^\gamma} \\
&= \lambda_2(P_{\pi_2^\gamma}^\nabla) \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, (I - J_{\pi_2^\gamma}) \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma} \\
&= \lambda_2(P_{\pi_2^\gamma}^\nabla) (b_{k+1} - b_{k-1}) \\
&= \lambda_2(P_{\pi_1^\gamma}^\Delta) (b_{k+1} - b_{k-1}) \\
&= \frac{1}{2} (\lambda_2(P_{\pi_1^\gamma}^\Delta) + 1) (b_{k+1} - b_{k-1}) \\
&\leq \frac{1}{2} (\gamma_{k-1} + 1) (b_{k+1} - b_{k-1})
\end{aligned}$$

Note that the  $\gamma_{k-1}$  we use here is defined in [AL20] and should not be confused with  $\gamma$ .

So, we also have

$$a_k - a_{k-1} \leq \frac{1}{2} (\gamma_{k-1} + 1) (a_{k+1} - a_{k-1})$$

## References

- [AL20] Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1198–1211, 2020.
- [CGM19] Mary Cryan, Heng Guo, and Giorgos Mousa. Modified log-sobolev inequalities for strongly log-concave distributions. In *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1358–1370. IEEE, 2019.