High Dimensional Random Walk: Inner Products And Operators

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Abstract

A very intresting point to note is that if we write the down-up walk in the operator form, we will gain a lot of benefits in the analysis. In this note, we collect useful properties of operators from [CGM19] and [AL20]. And we could use these properies to reimplement the result in [AL20] by using the "level-by-level decay approach" discovered by [CGM19].

1 The Operators

We will use the definition appears in [CGM19] first. And we will show that they are equivalent to the definition appears in [AL20].

Definition 1.1.

$$[X_k \leftrightarrow X_{k+1}]\,(I,J) \triangleq \begin{cases} 0 & I \not\subset J \\ 1 & I \subset J \end{cases}$$

$$[X_{k+1} \leftrightarrow X_k] \triangleq [X_k \leftrightarrow X_{k+1}]^T$$

Definition 1.2 (Operators)**.**

$$[\pi_k \leftrightarrow \pi_{k+1}] \triangleq \frac{1}{k+1}\mathrm{diag}(\pi_k)^{-1}\,[X_k \leftrightarrow X_{k+1}]\,\mathrm{diag}(\pi_{k+1})$$

$$[\pi_{k+1} \leftrightarrow \pi_k] \triangleq \frac{1}{k+1}\,[X_{k+1} \leftrightarrow X_k]$$

Note that $[\pi_k \leftrightarrow \pi_{k+1}]$ and $[\pi_{k+1} \leftrightarrow \pi_k]$ are denoted as $P_k^{\uparrow}$ and $P_{k+1}^{\downarrow}$ in [CGM19], respectively.

Fact 1.1 (Equivalence to [AL20])**.** *For any function $f : X(k+1) \to \mathbb{R}$ and $g : X(k) \to \mathbb{R}$, we have*

$$[\pi_k \leftrightarrow \pi_{k+1}]\,\mathbf{f}\;(\alpha) = \sum_{\beta \supset \alpha} \frac{\pi_{k+1}(\beta)}{(k+1)\pi_k(\alpha)} \cdot f(\beta) = \mathop{\mathbb{E}}_{\beta \supset \alpha} f(\beta)$$

, and

$$[\pi_{k+1} \leftrightarrow \pi_k]\,\mathbf{g}\;(\beta) = \sum_{\alpha \subset \beta} \frac{1}{k+1} \cdot g(\alpha) = \mathop{\mathbb{E}}_{\alpha \subset \beta} g(\alpha)$$

. Note that $[\pi_k \leftrightarrow \pi_{k+1}]$ and $[\pi_{k+1} \leftrightarrow \pi_k]$ are also denoted as D_{k+1} and U_k in [AL20], respectively.

Using these operators, we could define the down-up walk and up-down walk on level k.

Definition 1.3 (down-up walk, up-down walk)**.** *Down-up walk:*

$$P^{\triangle}_{\pi_k} \triangleq [\pi_k \leftrightarrow \pi_{k-1}]\,[\pi_{k-1} \leftrightarrow \pi_k]$$

Up-down walk:

$$P^{\triangledown}_{\pi_k} \triangleq [\pi_k \leftrightarrow \pi_{k+1}]\,[\pi_{k+1} \leftrightarrow \pi_k]$$

And, we denote the non-lazy version of $P^{\triangle}_{\pi_k}, P^{\triangledown}_{\pi_k}$ as $P^{\wedge}_{\pi_k}, P^{\vee}_{\pi_k}$, respectively.

Definition 1.4 (Inner Product on π_k)**.** *For $f, g : X(k) \to \mathbb{R}$, let*

$$\langle f, g \rangle_{\pi_k} \triangleq \sum_{\alpha \in X(k)} \pi(\alpha) f(\alpha) g(\alpha)$$

Theorem 1.1 (Adjointness of Operators, [AL20])**.** *For $f : X(k) \to \mathbb{R}$ and $g : X(k+1) \to \mathbb{R}$, then we have*

$$\langle g, [\pi_{k+1} \leftrightarrow \pi_k]\, f \rangle_{\pi_{k+1}} = \langle [\pi_k \leftrightarrow \pi_{k+1}]\, g, f \rangle_{\pi_k}$$

Proof. Here, we give a high level proof of this. You could verify it by brute force.

$$\begin{aligned}
\langle g, [\pi_{k+1} \leftrightarrow \pi_k]\, f \rangle_{\pi_{k+1}} &= \mathop{\mathbb{E}}_{\beta \sim \pi_{k+1}} [\mathbf{g}(\beta) \cdot [\pi_{k+1} \leftrightarrow \pi_k]\, f(\beta)] \\
&= \mathop{\mathbb{E}}_{\beta \sim \pi_{k+1}} [\mathbf{g}(\beta) \mathop{\mathbb{E}}_{\alpha \subset \beta} [f(\alpha)]] \\
&= \mathop{\mathbb{E}}_{\substack{(\beta, \alpha) \sim (\pi_{k+1}, \pi_k) \\ \alpha \subset \beta}} [g(\beta) f(\alpha)]
\end{aligned}$$

$$\begin{aligned}
\langle [\pi_k \leftrightarrow \pi_{k+1}]\, g, f \rangle &= \mathop{\mathbb{E}}_{\alpha \sim \pi_k} [[\pi_k \leftrightarrow \pi_{k+1}]\, g(\alpha) f(\alpha)] \\
&= \mathop{\mathbb{E}}_{\alpha \sim \pi_k} [\mathop{\mathbb{E}}_{\beta \supset \alpha} [g(\beta)] f(\alpha)] \\
&= \mathop{\mathbb{E}}_{\substack{(\beta, \alpha) \sim (\pi_{k+1}, \pi_k) \\ \alpha \subset \beta}} [g(\beta) f(\alpha)]
\end{aligned}$$

So, we can conclude that $\langle g, [\pi_{k+1} \leftrightarrow \pi_k]\, f \rangle_{\pi_{k+1}}$ and $\langle [\pi_k \leftrightarrow \pi_{k+1}]\, g, f \rangle_{\pi_k}$ gives us the same joint distribution (π_{k+1}, π_k) and thus they are equal. ☐

Definition 1.5 (map f to $f^{\mathbf{1}}$)**.** *Suppose we have a distribution π and a function $f : \mathrm{supp}(\pi) \to \mathbb{R}$, then we define: $J_\pi f \triangleq f^{\mathbf{1}} = \langle f, \mathbf{1} \rangle_\pi \cdot \mathbf{1}$. It turns out that $J_\pi = \mathbf{1}\pi^T$. Note that $f = f^{\mathbf{1}} + f^{\perp \mathbf{1}}$.*

2 Reimplement [AL20]

See Notes for level-by-level-decay approach.

Definition 2.1.

$$a_k = \mathop{\mathbb{E}}_{\pi_k}\left[\left(\frac{\mu_k}{\pi_k}\right)^2\right]$$

First, we have the following fact.

Fact 2.1.

$$a_{k+1} = \sum_{\gamma\in X(k-1)} \pi_{k-1}(\gamma)(\frac{\mu_{k-1}(\gamma)}{\pi_{k-1}(\gamma)})^2 \cdot \sum_{\{x,y\}\in X_\gamma(2)} \pi_2^\gamma(\{x,y\})(\frac{\mu_2^\gamma(\{x,y\})}{\pi_2^\gamma(\{x,y\})})^2$$

$$a_k = \sum_{\gamma\in X(k-1)} \pi_{k-1}(\gamma)(\frac{\mu_{k-1}(\gamma)}{\pi_{k-1}(\gamma)})^2 \cdot \sum_{\{x\}\in X_\gamma(1)} \pi_1^\gamma(\{x\})(\frac{\mu_1^\gamma(\{x\})}{\pi_1^\gamma(\{x\})})^2$$

$$a_k = \sum_{\gamma\in X(k-1)} \pi_{k-1}(\gamma)(\frac{\mu_{k-1}(\gamma)}{\pi_{k-1}(\gamma)})^2 \cdot 1$$

Definition 2.2.

$$b_{k+1} = \sum_{\{x,y\}\in X_\gamma(2)} \pi_2^\gamma(\{x,y\})(\frac{\mu_2^\gamma(\{x,y\})}{\pi_2^\gamma(\{x,y\})})^2 = \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma}$$

$$b_k = \sum_{\{x\}\in X_\gamma(1)} \pi_1^\gamma(\{x\})(\frac{\mu_1^\gamma(\{x\})}{\pi_1^\gamma(\{x\})})^2 = \left\langle \frac{\mu_1^\gamma}{\pi_1^\gamma}, \frac{\mu_1^\gamma}{\pi_1^\gamma} \right\rangle_{\pi_1^\gamma}$$

$$b_{k-1} = 1$$

Fact 2.2.

$$b_{k-1} = \left\langle \frac{\mu_1^\gamma}{\pi_1^\gamma}, J_{\pi_1^\gamma} \frac{\mu_1^\gamma}{\pi_1^\gamma} \right\rangle_{\pi_1^\gamma} = \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, J_{\pi_2^\gamma} \frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma}$$

Fact 2.3.

$$\frac{\mu_1^\gamma}{\pi_1^\gamma} = [\pi_1^\gamma \leftrightarrow \pi_2^\gamma]\, \frac{\mu_2^\gamma}{\pi_2^\gamma}$$

Proof.

$$\begin{aligned}
[\pi_1^\gamma \leftrightarrow \pi_2^\gamma]\, \frac{\mu_2^\gamma}{\pi_2^\gamma}(x) &= \sum_{\{x,y\}\in X_2^\gamma} \frac{\pi_2^\gamma(\{x,y\})}{2\pi_1^\gamma(\{x\})} \cdot \frac{\mu_2^\gamma(\{x,y\})}{\pi_2^\gamma(\{x,y\})} \\
&= \sum_{\{x,y\}\in X_2^\gamma} \frac{\mu_2^\gamma(\{x,y\})}{2\pi_1^\gamma(\{x\})} \\
&= \frac{\mu_1^\gamma}{\pi_1^\gamma}(\{x\})
\end{aligned}$$

□

Fact 2.4.

$$
\begin{aligned}
b_k &= \left\langle [\pi_1^\gamma \leftrightarrow \pi_2^\gamma]\,\frac{\mu_2^\gamma}{\pi_2^\gamma}, [\pi_1^\gamma \leftrightarrow \pi_2^\gamma]\,\frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_1^\gamma} \\
&= \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, [\pi_2^\gamma \leftrightarrow \pi_1^\gamma]\,[\pi_1^\gamma \leftrightarrow \pi_2^\gamma]\,\frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma} \\
&= \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, P^{\triangledown}_{\pi_2^\gamma}\,\frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma}
\end{aligned}
$$

Having these facts in hand, we could have the following argument (similar to [AL20]).

$$
\begin{aligned}
b_k - b_{k-1} &= \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, (P^{\triangledown}_{\pi_2^\gamma} - J_{\pi_2^\gamma})\frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma} \\
&= \left\langle (\frac{\mu_2^\gamma}{\pi_2^\gamma})^{\perp \mathbf{1}}, (P^{\triangledown}_{\pi_2^\gamma} - J_{\pi_2^\gamma})(\frac{\mu_2^\gamma}{\pi_2^\gamma})^{\perp \mathbf{1}} \right\rangle_{\pi_2^\gamma} \\
&\le \lambda_2(P^{\triangledown}_{\pi_2}) \left\langle (\frac{\mu_2^\gamma}{\pi_2^\gamma})^{\perp \mathbf{1}}, (\frac{\mu_2^\gamma}{\pi_2^\gamma})^{\perp \mathbf{1}} \right\rangle_{\pi_2^\gamma} \\
&= \lambda_2(P^{\triangledown}_{\pi_2}) \left\langle \frac{\mu_2^\gamma}{\pi_2^\gamma}, (I - J_{\pi_2^\gamma})\frac{\mu_2^\gamma}{\pi_2^\gamma} \right\rangle_{\pi_2^\gamma} \\
&= \lambda_2(P^{\triangledown}_{\pi_2})(b_{k+1} - b_{k-1}) \\
&= \lambda_2(P^{\triangle}_{\pi_1})(b_{k+1} - b_{k-1}) \\
&= \frac{1}{2}(\lambda_2(P^{\wedge}_{\pi_1}) + 1)(b_{k+1} - b_{k-1}) \\
&\le \frac{1}{2}(\gamma_{k-1} + 1)(b_{k+1} - b_{k-1})
\end{aligned}
$$

Note that the γ_{k-1} we use here is defined in [AL20] and should not be confused with γ.

So, we also have

$$
\begin{aligned}
a_k - a_{k-1} &\le \frac{1}{2}(\gamma_{k-1} + 1)(a_{k+1} - a_{k-1}) \\
\frac{1-\gamma_{k-1}}{1+\gamma_{k-1}}(a_k - a_{k-1}) &\le a_{k+1} - a_k \\
\frac{1-\gamma_{k-1}}{1+\gamma_{k-1}} A_k &\le A_{k+1}
\end{aligned}
$$

So, in the worst case, we have

$$
\begin{aligned}
A_n &\ge \frac{1-\gamma_{n-2}}{1+\gamma_{n-2}} A_{n-1} \\
&\ge \prod_{i=0}^{n-2}\left(\frac{1-\gamma_i}{1+\gamma_i}\right) A_1 \\
A_n &\ge \frac{1}{n}\prod_{i=0}^{n-2}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)\left(\sum_{i=1}^{n} A_i\right)
\end{aligned}
$$

So, we have

$$\mathrm{Var}_{\pi_n}\left[\left(\frac{P_n^{\triangledown}\mu_n}{\pi_n}\right)^2\right] \le \left(1-\frac{1}{n}\prod_{i=1}^{n-2}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)\right)\mathrm{Var}_{\pi_n}\left[\left(\frac{\mu_n}{\pi_n}\right)^2\right]$$

All in all, this reimplementation gives us a deeper understanding of down-up walk on simplicial complex:

There is a level-by-level decay of f-divergence on the simplicial complex and thus the random walk convergences rapidly. So, we think that the spectral method on the random walk P is actually equivalent to analysis the decay of f-divergence.

References

[AL20] Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1198–1211, 2020.

[CGM19] Mary Cryan, Heng Guo, and Giorgos Mousa. Modified log-sobolev inequalities for strongly log-concave distributions. In *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1358–1370. IEEE, 2019.