Exercise of Chapter 6

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1 Exercise 6.3

Show that the uniform distribution on K is an invariant measure for the Metropolis version of the ball walk.

1.1 solution

The most interesting thing here is that the transition distribution $P(x, \cdot)$ of the Matropolis version of the ball walk is not continuous on its range $B(x, \delta) \cap K$. More specificly, for $y \in B(x, \delta) \cap K$, we have:

$$P(x, \mathrm{d}y) = \begin{cases} \frac{\mathrm{d}y}{\mathrm{Vol}_n B(x,\delta)}, & y \neq x \\ \frac{\mathrm{Vol}_n (B(x,\delta) \setminus K)}{\mathrm{Vol}_n B(x,\delta)}, & y = x \end{cases}$$

So, for any $x, y \in K$, we have:

$$P(x, \mathrm{d}y) = P(y, \mathrm{d}x)$$

Suppose X_0 has the uniform distribution μ and let μ_1 be the distribution of X_1, then:

$$\begin{aligned}
\mu_1(A) &= \int_A \mu_1(dy) \\
&= \int_A \int_K \mu(\mathrm{d}x) P(x, \mathrm{d}y) \\
&= \int_A \int_{B(y,\delta)\cap K} \mu(\mathrm{d}x) P(x, \mathrm{d}y), \quad \text{since if } x \notin B(y,\delta) \cap K \text{ then } P(x, \mathrm{d}y) = 0 \\
&= \int_A \int_{B(y,\delta)\cap K} \mu(\mathrm{d}x) P(y, \mathrm{d}x), \quad \text{since } P(x, \mathrm{d}y) = P(y, \mathrm{d}x) \\
&= \int_A \int_{B(y,\delta)\cap K} \mu(\mathrm{d}y) P(y, \mathrm{d}x), \quad \text{since } \mu(\mathrm{d}x) = \mu(\mathrm{d}y) \text{ when we let } \mathrm{Vol}_n(\mathrm{d}x) = \mathrm{Vol}_n(\mathrm{d}y) \\
&= \int_A \mu(\mathrm{d}y) \int_{B(y,\delta)\cap K} P(y, \mathrm{d}x), \\
&= \int_A \mu(\mathrm{d}y), \quad \text{since } P(y, \cdot) \text{ is a distribution} \\
&= \mu(A)
\end{aligned}$$

2 Exercise 6.25

Flesh out the details of the above proof sketch. (Prove the Poincare inequality with out the curvatre condition. This proof will loose a factor n in λ.)

2.1 Solution

2.1.1 Flesh out the proof of Lemma 6.21.(i)

Recall It Here:

$$l(x)^{1/n} \text{ is concave over } K$$

Where the $l(x)$ is defined as

$$\frac{\mathrm{Vol}_n(B(x,\delta)\cap K)}{\mathrm{Vol}_n B(x,\delta)}$$

Proof. Suppose there are two points x and y in K. From (6.32) we know that

$$(\lambda x+(1-\lambda)y+B(0,\delta))\cap K \supseteq \lambda((x+B(0,\delta))\cap K)+(1-\lambda)((y+B(0,\delta))\cap K)$$

Then, by Brunn-Minkowski theorem, we have

$$\begin{aligned}&\mathrm{Vol}_n[(\lambda x+(1-\lambda)y+B(0,\delta))\cap K]^{1/n}\\ &\qquad \geq \lambda \mathrm{Vol}_n[(x+B(0,\delta))\cap K]^{1/n}+(1-\lambda)\mathrm{Vol}_n[(y+B(0,\delta))\cap K]^{1/n}\end{aligned}$$

Which means:

$$l(\lambda x+(1-\lambda)y)^{1/n} \geq \lambda l(x)^{1/n}+(1-\lambda)l(y)^{1/n}$$

So, $l(x)^{1/n}$ is concave on K. □

2.1.2 Fix the upper bound of $a_{i,j}$

From Lemma 6.22, we know that the sequence of

$$\frac{1}{\mu(S_0)}, \frac{1}{\mu(S_1)}, \cdots, \frac{1}{\mu(S_{m-1})}$$

is convex. So we have

$$\begin{aligned}
a_{i,j} &\le 2w_iw_j\sum_{k=i}^{j}\frac{1}{w_k} \\
&= w_iw_j\sum_{k=i}^{j}(\frac{1}{w_k}+\frac{1}{w_{i+j-k}}) \\
&\le w_iw_j\sum_{k=1}^{j}(\frac{j-k}{j-i}\frac{1}{w_i}+\frac{k-i}{j-i}\frac{1}{w_j})+(\frac{k-i}{j-i}\frac{1}{w_i}+\frac{j-k}{j-i}\frac{1}{w_j}) \\
&= w_iw_j\sum_{k=i}^{j}(\frac{1}{w_i}+\frac{1}{w_j}) \\
&= w_iw_j(j-i+1)(\frac{1}{w_i}+\frac{1}{w_j}) \\
&= (j-i+1)(w_i+w_j)
\end{aligned}$$

This establishes Claim 6.16 in the absencse of the curvature condition.

2.1.3 Establish (6.29)

Let $\eta = c_3\delta/n$, then by Lemma 6.21.(ii) we have

$$\begin{aligned}
|\ln l(x)-\ln l(y)| &\le \frac{n}{\delta}||x-y|| \\
&\le \frac{n}{\delta}(c_3\delta/n) \\
&= c_3
\end{aligned}$$

and hence

$$\begin{aligned}
l(x)/l(y) &\le e^{c_3} \\
l(x) &\le l(y)e^{c_3}
\end{aligned}$$

According to symmetry

$$l(y) \le l(x)e^{c_3}$$

And since we could set the value of δ as we want, we may cliam that:

$$\mathrm{Vol}_nB(x,\delta') \le s_1\mathrm{Vol}_nB(y,\delta')$$

where s_1 is some constant. So we have a lower bound for all $B(y,\delta')$, i.e.

$$\frac{1}{s_1}\mathrm{Vol}_nB(x,\delta') \le \mathrm{Vol}_nB(y,\delta')$$

So, from Lemma 6.24. we have

$$\begin{aligned}
&\mathrm{Vol}_n(B(x,\delta') \cap B(y,\delta') \cap K) \\
&\qquad \geq \frac{1}{1+e} \min\{\mathrm{Vol}_n(B(x,\delta') \cap K), \mathrm{Vol}_n(B(y,\delta') \cap K)\}
\end{aligned}$$

and hence

$$\mathrm{Vol}_n I \geq \frac{s_2}{1+e} \mathrm{Vol}_n(B(x,\delta') \cap K)$$

where $s_2 = \min\{1, \frac{1}{s_1}\}$ is some constant.

Observe that $\delta' = \delta - \varepsilon\sqrt{n}$, so if we choose ε properly, we could make $\delta' = s_3\delta$, where $0 < s_3 \leq 1$ is a constant.

After that, we have

$$\frac{\mathrm{Vol}_n B(x,\delta')}{\mathrm{Vol}_n B(x,\delta)} = \frac{1}{s_3^n}$$

When $\frac{1}{s_3^n} = \frac{1}{2}$, we have

$$\frac{\mathrm{Vol}_n(B(x,\delta') \cap K)}{\mathrm{Vol}_n(B(x,\delta) \cap K)} \geq \frac{1}{2}$$

Some Note

Although this seems right here, and some material online says it is trivial. I still can not prove it.

After that, we have

$$\begin{aligned}
\int_{K_0} h \mathrm{d}u &= \frac{1}{2} \int_{K_0} \int_{K_0} \mu(\mathrm{d}x) \frac{\mathrm{d}y}{\mathrm{Vol}_n(B(x,\delta) \cap K)} (f(x) - f(y))^2 \\
&\geq \frac{1}{2} \int_{K_0} \int_I \mu(\mathrm{d}x) \frac{\mathrm{d}y}{\mathrm{Vol}_n(B(x,\delta) \cap K)} (f(x) - f(y))^2 \\
&\geq \frac{1}{2\mathrm{Vol}_n(B(x,\delta) \cap K)} \int_I \mathrm{d}y \int_{K_0} \mu(\mathrm{d}x)(f(x) - f(y))^2 \\
&\geq \frac{1}{2\mathrm{Vol}_n(B(x,\delta) \cap K)} \int_I \mathrm{d}y \int_{K_0} \mathrm{d}\mu(f - \overline{f})^2 \\
&\geq \frac{\mathrm{Vol}_n I}{2\mathrm{Vol}_n(B(x,\delta) \cap K)} \int_{K_0} \mathrm{d}\mu(f - \overline{f})^2 \\
&\geq \frac{s_2}{4(e+1)} \int_{K_0} (f - \overline{f})^2 \mathrm{d}\mu
\end{aligned}$$

2.1.4 The Order of λ

From (6.26) we know that

$$\begin{aligned}\lambda \in \mathcal{O}(\frac{1}{m^2}) &= \mathcal{O}(\frac{c_3\delta}{Dn}) \\ &= \mathcal{O}(\frac{c_3^2\delta^2}{D^2n^2})\end{aligned}$$

So, we draw a contradiction here, and fix the original proof.