Refined Understanding: Local to Global Argument

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1 Basic Notations

Definition 1.1. *Ground set* $[n]$.

Definition 1.2. *A simplcial complex* $X \subset 2^{[n]}$ *is a downclose set family, i.e.*

$$\beta \in X \wedge \alpha \subset \beta \Rightarrow \alpha \in X$$

Definition 1.3. X *could be partited in* $n+1$ *disjoint parts* $X_0, X_1, X_2, \cdots, X_n$*, such that* $X_i = \{\alpha \in X \mid |\alpha| = i\}$

Definition 1.4. *There is a distribution* $\pi = \pi_n$ *that we are interested in. Its support set is* X_n*.*

Definition 1.5. *The distribution on* X_n *could imply nature distributions on* X_k *where* $k < n$*, that is:*

$$\forall \alpha \in X_k \,.\, \pi_k(\alpha) \propto \sum_{\substack{\beta \in X_n \\ \beta \supset \alpha}} \pi_n(\beta)$$

One can easily see that we could normalize the summation using the factor $1/\binom{n}{k}$*.*

Fact 1.1.

$$\forall \alpha \in X_k, k \le \ell \le n \,.\, \pi_k(\alpha) \propto \sum_{\substack{\beta \in X_\ell \\ \beta \supset \alpha}} \pi_\ell(\beta)$$

Fact 1.2. *For any* $\gamma \in X$*, the link of* γ *is another simplcial complex defined as*

$$X^\gamma \triangleq \{\beta \setminus \gamma \mid \beta \in X, \beta \supset \gamma\}$$

Definition 1.6. *From the distribuiton on* X*, there is a nature distribution on* X^γ *which is defined as*

$$\forall \alpha \in X_k^\gamma \,.\, \pi_k^\gamma(\alpha) \propto \pi_{|\gamma|+k}(\alpha \cup \gamma)$$

It is easy to see that the normalize factor of this distribution is

$$\sum_{\substack{\beta \in X_{|\gamma|+k} \\ \beta \supset \gamma}} \pi_{|\gamma|+k}(\beta) = \pi_{|\gamma|}(\gamma) \cdot \binom{|\gamma|+k}{|\gamma|}$$

And it easy to notice that

$$\pi_k^\gamma(\underbrace{\beta \setminus \gamma}_{\alpha}) = \Pr[\beta \sim \pi_{|\gamma|+k} \mid \beta \supset \gamma]$$

Definition 1.7. *The Down-Operator and Up-Operator is defined as follows*

- $\mathbb{R}^{X_k \times X_{k+1}}$, $[\pi_k \leftrightarrow \pi_{k+1}](\alpha, \beta) \triangleq \Pr[\beta \sim \pi_{k+1} \mid \beta \supset \alpha] = \pi_1^\alpha(\beta \setminus \alpha)$
- $\mathbb{R}^{X_k \times X_\ell}, k < \ell$, $[\pi_k \leftrightarrow \pi_\ell](\alpha, \beta) \triangleq \Pr[\beta \sim \pi_\ell \mid \beta \supset \alpha] = \pi_{\ell-k}^\alpha(\beta \setminus \alpha)$
- $\mathbb{R}^{X_{k+1} \times X_k}$,$[\pi_{k+1} \leftrightarrow \pi_k](\beta, \alpha) \triangleq \frac{1}{k+1}\mathbb{1}[\beta \supset \alpha]$

- $\mathbb{R}^{X_\ell \times X_k}, k < \ell$, $[\pi_\ell \leftrightarrow \pi_k](\beta, \alpha) \triangleq \frac{1}{\binom{\ell}{k}} \mathbb{1}[\beta \supset \alpha]$

Fact 1.3. *For any $k \neq \ell$, it holds that $[\pi_k \leftrightarrow \pi_\ell]$ and $[\pi_\ell \leftrightarrow \pi_k]$ are adjoint operators w.r.t. distribution π_k and π_ℓ, that is for all $\alpha \in X_k, \beta \in X_\ell$, it holds that*

$$\pi_k(\alpha) [\pi_k \leftrightarrow \pi_\ell](\alpha, \beta) = \pi_\ell(\beta) [\pi_\ell \leftrightarrow \pi_k](\beta, \alpha),$$

and thus, it holds that

$$\pi_k [\pi_k \leftrightarrow \pi_\ell] = \pi_\ell \quad \text{and} \quad \pi_\ell [\pi_\ell \leftrightarrow \pi_k] = \pi_k.$$

Remark 1.1. *When $k < \ell$, $\pi_k(\alpha) [\pi_k \leftrightarrow \pi_\ell](\alpha, \beta) = \pi_\ell(\beta) [\pi_\ell \leftrightarrow \pi_k](\beta, \alpha)$, also implies*

$$\begin{aligned} \pi_k(\alpha) \pi_{\ell-k}^{\alpha}(\beta \setminus \alpha) &= \pi_k(\alpha) [\pi_k \leftrightarrow \pi_\ell](\alpha, \beta) \\ &= \pi_\ell(\beta) [\pi_\ell \leftrightarrow \pi_k](\beta, \alpha) = \pi_\ell(\beta) \cdot \frac{1}{\binom{\ell}{k}}. \end{aligned}$$

Fact 1.4. *For $k < \ell$, we have*

$$[\pi_k \leftrightarrow \pi_\ell] = [\pi_k \leftrightarrow \pi_{k+1}] [\pi_{k+1} \leftrightarrow \pi_{k+2}] \cdots [\pi_{\ell-1} \leftrightarrow \pi_\ell]$$

Proof. Intuition:

$$\pi_k [\pi_k \leftrightarrow \pi_{k+1}] [\pi_{k+1} \leftrightarrow \pi_{k+2}] \cdots [\pi_{\ell-1} \leftrightarrow \pi_\ell] = \pi_\ell$$

Verification: Note that we have

$$[\pi_k \leftrightarrow \pi_{k+1}] = \mathrm{diag}^{-1}(\pi_k) A_k \mathrm{diag}(\pi_{k+1}) \frac{1}{k+1}$$

where $A_k \in \mathbb{R}^{X_k \times X_{k+1}}$ and $A_k(\alpha, \beta) = \begin{cases} 0 & \alpha \not\subset \beta \\ 1 & \alpha \subset \beta \end{cases}$ So, we have

$$\begin{aligned} \prod_{i=k}^{\ell-1} [\pi_i \leftrightarrow \pi_{i+1}] &= \prod_{i=k}^{\ell-1} \mathrm{diag}^{-1}(\pi_i) A_i \mathrm{diag}(\pi_{i+1}) \frac{1}{i+1} \\ &= \prod_{i=k}^{\ell-1} \frac{1}{i+1} \cdot \mathrm{diag}^{-1}(\pi_k) A_{k,\ell} \mathrm{diag}(\pi_\ell) \end{aligned}$$

Where,

$$A_{k,\ell}(\alpha, \beta) = \begin{cases} 0 & \alpha \not\subset \beta \\ (\ell - k)! & \alpha \subset \beta \end{cases}$$

So,

$$\prod_{i=k}^{\ell-1} [\pi_i \leftrightarrow \pi_{i+1}] = \frac{1}{\binom{\ell}{k}} \mathrm{diag}^{-1}(\pi_k) A'_{k,\ell} \mathrm{diag}(\pi_\ell) = [\pi_k \leftrightarrow \pi_\ell]$$

□

Definition 1.8. *Suppose there is a function $f^{(\ell)} = f : X_\ell \to \mathbb{R}$ that we are interested in. Then it will naturally imply function $f^{(k)}$ on X_k for $k < \ell$ such that*

$$f^{(k)} \triangleq [\pi_k \leftrightarrow \pi_\ell] f^{(\ell)}$$

Definition 1.9. *For $f^{(\ell)}$ and $\gamma \in X_k$, we have $f_\gamma^{(\ell-k)}(\alpha) = f^{(\ell)}(\gamma \cup \alpha)$*

Fact 1.5. $f^{(k)}(\alpha) = \sum_\beta [\pi_k \leftrightarrow \pi_\ell](\alpha, \beta) \cdot f^{(\ell)}(\beta) = \sum_\beta \pi_{\ell-k}^{\alpha}(\beta \setminus \alpha) \cdot f^{(\ell)}(\beta) = \mathbb{E}_{\pi_{\ell-k}^{\alpha}}[f_\alpha^{(\ell-k)}]$

Fact 1.6. $f^{(|\gamma|+k)} = [\pi_{|\gamma|+k} \leftrightarrow \pi_{|\gamma|+\ell}] f^{(|\gamma|+\ell)} \Rightarrow f_\gamma^{(k)} = [\pi_k^\gamma \leftrightarrow \pi_\ell^\gamma] f_\gamma^{(\ell)}$

Proof.

$$[\pi_{|\gamma|+k} \leftrightarrow \pi_{|\gamma|+\ell}](\alpha \cup \gamma, \beta \cup \gamma) \propto \frac{\pi_{|\gamma|+\ell}(\beta \cup \gamma)}{\pi_{|\gamma|+k}(\alpha \cup \gamma)} \propto [\pi_k^\gamma \leftrightarrow \pi_\ell^\gamma](\alpha, \beta)$$

□

Fact 1.7. *For any function $f : X_{k+1} \to \mathbb{R}, g : X_k \to \mathbb{R}$, we have*

$$\langle g, [\pi_k \leftrightarrow \pi_{k+1}] f \rangle_{\pi_k} = \langle [\pi_{k+1} \leftrightarrow \pi_k] g, f \rangle_{\pi_{k+1}}$$

Proof.

$$\begin{aligned} \text{LHS} &= \sum_{\alpha \in X_k} \pi_k(\alpha) g(\alpha) \sum_{\substack{\beta \in X_{k+1} \\ \beta \supset \alpha}} \frac{\pi_{k+1}(\beta)}{(k+1)\pi_k(\alpha)} f(\beta) \\ &= \sum_{\alpha \in X_k} \sum_{\substack{\beta \in X_{k+1} \\ \beta \supset \alpha}} \frac{1}{k+1} \pi_{k+1}(\beta) \cdot g(\alpha) f(\beta) \end{aligned}$$

$$\text{RHS} = \sum_{\beta \in X_{k+1}} \pi_{k+1}(\beta) f(\beta) \sum_{\substack{\alpha \in X_k \\ \alpha \subset \beta}} g(\alpha) \frac{1}{k+1}$$

□

Definition 1.10 (map f to $f^{\mathbf{1}}$)**.** *Suppose we have a distribution π and a function $f : \mathrm{supp}(\pi) \to \mathbb{R}$, then we define: $J_\pi f \triangleq f^{\mathbf{1}} = \langle f, \mathbf{1} \rangle_\pi \cdot \mathbf{1}$. It turns out that $J_\pi = \mathbf{1}\pi^T$. Note that $f = f^{\mathbf{1}} + f^{\perp \mathbf{1}}$.*

2 Variance Decay (Reimplement [AL20])

See Notes for level-by-level-decay approach.

Definition 2.1.

$$a_k = \mathop{\mathbb{E}}_{\pi_k} \left[\left(f^{(k)} \right)^2 \right]$$

First, we have the following fact.

Fact 2.1.

$$\begin{aligned} a_{k+1} &= \sum_{\gamma \in X_{k-1}} \pi_{k-1}(\gamma) \sum_{\{x,y\} \in X_2^\gamma} \pi_2^\gamma(\{x,y\}) \left(f_\gamma^{(2)}(\{x,y\}) \right)^2 \\ a_k &= \sum_{\gamma \in X_{k-1}} \pi_{k-1}(\gamma) \sum_{\{x\} \in X_1^\gamma} \pi_1^\gamma(\{x\}) \left(f_\gamma^{(1)}(\{x\}) \right)^2 \\ a_{k-1} &= \sum_{\gamma \in X_{k-1}} \pi_{k-1}(\gamma) \left(f_\gamma^{(0)} \right)^2 \end{aligned}$$

Definition 2.2.

$$\begin{aligned} b_{k+1} &= \sum_{\{x,y\} \in X_\gamma(2)} \pi_2^\gamma(\{x,y\}) (f_\gamma^{(2)})^2 = \left\langle f_\gamma^{(2)}, f_\gamma^{(2)} \right\rangle_{\pi_2^\gamma} \\ b_k &= \sum_{\{x\} \in X_\gamma(1)} \pi_1^\gamma(\{x\}) (f_\gamma^{(1)})^2 = \left\langle f_\gamma^{(1)}, f_\gamma^{(1)} \right\rangle_{\pi_1^\gamma} \\ b_{k-1} &= (f_\gamma^{(0)})^2 \end{aligned}$$

Fact 2.2.

$$b_{k-1} = \left\langle f_\gamma^{(1)}, J_{\pi_1^\gamma} f_\gamma^{(1)} \right\rangle_{\pi_1^\gamma} = \left\langle f_\gamma^{(2)}, J_{\pi_2^\gamma} f_\gamma^{(2)} \right\rangle_{\pi_2^\gamma}$$

Fact 2.3 (see Fact 1.6)**.**

$$f_\gamma^{(1)} = [\pi_1^\gamma \leftrightarrow \pi_2^\gamma] f_\gamma^{(2)}$$

Fact 2.4.

$$\begin{aligned} b_k &= \left\langle [\pi_1^\gamma \leftrightarrow \pi_2^\gamma] f_\gamma^{(2)}, [\pi_1^\gamma \leftrightarrow \pi_2^\gamma] f_\gamma^{(2)} \right\rangle_{\pi_1^\gamma} \\ &= \left\langle f_\gamma^{(2)}, [\pi_2^\gamma \leftrightarrow \pi_1^\gamma][\pi_1^\gamma \leftrightarrow \pi_2^\gamma] f_\gamma^{(2)} \right\rangle_{\pi_2^\gamma} \\ &= \left\langle f_\gamma^{(2)}, P^{\triangledown}_{\pi_2^\gamma} f_\gamma^{(2)} \right\rangle_{\pi_2^\gamma} \end{aligned}$$

Having these facts in hand, we could have the following argument (similar to [AL20]).

$$\begin{aligned} b_k - b_{k-1} &= \left\langle f_\gamma^{(2)}, (P^{\triangledown}_{\pi_2^\gamma} - J_{\pi_2^\gamma}) f_\gamma^{(2)} \right\rangle_{\pi_2^\gamma} \\ &= \left\langle (f_\gamma^{(2)})^{\perp \mathbf{1}}, (P^{\triangledown}_{\pi_2^\gamma} - J_{\pi_2^\gamma}) (f_\gamma^{(2)})^{\perp \mathbf{1}} \right\rangle_{\pi_2^\gamma} \\ &\leq \lambda_2(P^{\triangledown}_{\pi_2}) \left\langle (f_\gamma^{(2)})^{\perp \mathbf{1}}, (f_\gamma^{(2)})^{\perp \mathbf{1}} \right\rangle_{\pi_2^\gamma} \\ &= \lambda_2(P^{\triangledown}_{\pi_2}) \left\langle f_\gamma^{(2)}, (I - J_{\pi_2^\gamma}) f_\gamma^{(2)} \right\rangle_{\pi_2^\gamma} \\ &= \lambda_2(P^{\triangledown}_{\pi_2})(b_{k+1} - b_{k-1}) \\ &= \lambda_2(P^{\triangle}_{\pi_1})(b_{k+1} - b_{k-1}) \\ &= \frac{1}{2}(\lambda_2(P^{\wedge}_{\pi_1}) + 1)(b_{k+1} - b_{k-1}) \\ &\leq \frac{1}{2}(\gamma_{k-1} + 1)(b_{k+1} - b_{k-1}) \end{aligned}$$

Note that the γ_{k-1} we use here is defined in [AL20] and should not be confused with γ.

So, we also have

$$\begin{aligned} a_k - a_{k-1} &\leq \frac{1}{2}(\gamma_{k-1} + 1)(a_{k+1} - a_{k-1}) \\ \frac{1 - \gamma_{k-1}}{1 + \gamma_{k-1}}(a_k - a_{k-1}) &\leq a_{k+1} - a_k \\ \frac{1 - \gamma_{k-1}}{1 + \gamma_{k-1}} A_k &\leq A_{k+1} \end{aligned}$$

To analysis block dynamics let $\beta_{k-1} = \frac{1-\gamma_{k-1}}{1+\gamma_{k-1}}$. We have

$$\forall k \,.\, a_{k+1} \geq (1 + \beta_{k-1}) a_k - \beta_{k-1} a_{k-1}$$

Fact 2.5 ([CLV20], Theorem 5.4)**.**

$$a_{k+1} \geq \frac{\sum_{i=0}^{k} \Gamma_i}{\sum_{i=0}^{k-1} \Gamma_i} a_k$$

Where $\Gamma_i = \prod_{j=0}^{i-1} \beta_j$ *and* $\Gamma_0 = 1$

Proof. (Base case): $a_2 \geq (1 - \beta_0) a_1$

(Induction): Assume that we have $a_{k+1} \geq \frac{\sum_{i=0}^{k} \Gamma_i}{\sum_{i=0}^{k-1} \Gamma_i} a_k$, then

$$\begin{aligned}
a_{k+2} &\geq (1+\beta_k)a_{k+1} - \beta_k a_k \\
&\geq a_{k+1} + \beta_k \left(1 - \frac{\sum_{i=0}^{k-1} \Gamma_i}{\sum_{i=0}^{k} \Gamma_i}\right) a_{k+1} \\
&= a_{k+1} + \beta_k \left(\frac{\Gamma_k}{\sum_{i=0}^{k} \Gamma_i}\right) a_{k+1} \\
&= \left(1 + \frac{\Gamma_{k+1}}{\sum_{i=0}^{k} \Gamma_i}\right) a_{k+1} \\
&= \frac{\sum_{i=0}^{k+1} \Gamma_i}{\sum_{i=0}^{k} \Gamma_i} a_{k+1}
\end{aligned}$$

□

3 Analysis of The Block Dynamics [CLV20]

By Fact 3.5, we know that

$$a_n \geq \frac{\sum_{i=0}^{n-1} \Gamma_i}{\sum_{i=0}^{n-2} \Gamma_i} a_{n-1} \geq \frac{\sum_{i=0}^{n-1} \Gamma_i}{\sum_{i=0}^{n-2} \Gamma_i} \frac{\sum_{i=0}^{n-2} \Gamma_i}{\sum_{i=0}^{n-3} \Gamma_i} a_{n-2} \geq \frac{\sum_{i=0}^{n-1} \Gamma_i}{\sum_{i=0}^{n-\ell-1} \Gamma_i} a_{n-\ell}$$

So, we have

$$a_{n-\ell} \leq \frac{\sum_{i=0}^{n-\ell-1} \Gamma_i}{\sum_{i=0}^{n-1} \Gamma_i} a_n$$

Then,

$$a_n - a_{n-\ell} \geq \left(1 - \frac{\sum_{i=0}^{n-\ell-1} \Gamma_i}{\sum_{i=0}^{n-1} \Gamma_i}\right) a_n = \frac{\sum_{i=n-\ell}^{n-1} \Gamma_i}{\sum_{i=0}^{n-1} \Gamma_i} a_n \triangleq \kappa a_n$$

Then, we give a lowerbound for $\frac{\sum_{i=n-\ell}^{n-1} \Gamma_i}{\sum_{i=0}^{n-1} \Gamma_i}$.

First, recall that $\beta_k = \frac{1-\gamma_k}{1+\gamma_k}$. And we assume that $\gamma_k \leq \frac{\eta}{n-k-1}$. So, we have

$$\beta_k \geq \frac{(n-k-1)-\eta}{(n-k-1)+\eta} = 1 - \frac{2\eta}{(n-k-1)+\eta} \geq 1 - \frac{\lceil 2\eta \rceil}{n-k-1} \triangleq \frac{n-k-1-R}{n-k-1}$$

Fact 3.1.

$$\forall k \,.\, \frac{\partial \kappa}{\partial \beta_k} \geq 0$$

Proof. The numerator of $\frac{\partial \kappa}{\partial \beta_k}$ is

$$\begin{aligned}
&\frac{\partial(\sum_{i=n-\ell}^{n-1} \Gamma_i)}{\partial \beta_k} \left(\sum_{i=0}^{n-1} \Gamma_i\right) - \left(\sum_{i=n-\ell}^{n-1} \Gamma_i\right) \frac{\partial(\sum_{i=0}^{n-1} \Gamma_i)}{\partial \beta_k} \\
=& \frac{\sum_{i=\max\{n-\ell,k+1\}}^{n-1} \Gamma_i}{\beta_k} \left(\sum_{i=0}^{n-1} \Gamma_i\right) - \left(\sum_{i=n-\ell}^{n-1} \Gamma_i\right) \frac{\sum_{i=k+1}^{n-1} \Gamma_i}{\beta_k}
\end{aligned}$$

And we ened the proof by

$$(\sum_{i=\max\{n-\ell,k+1\}}^{n-1}\Gamma_i)(\sum_{i=0}^{n-1}\Gamma_i)\geq(\sum_{i=n-\ell}^{n-1}\Gamma_i)(\sum_{i=k+1}^{n-1}\Gamma_i) \qquad \square$$

So, to give a lowerbound for κ, we could let β_k meets its lowerbound $\max\{\frac{n-k-1-R}{n-k-1},0\}$, that is, we assume $\beta_k=\max\{\frac{n-k-1-2\eta}{n-k-1},0\}$ in the rest of the article. Then we have the following for $n-k-R>0$:

$$\Gamma_k=\prod_{i=0}^{k-1}\beta_i=\prod_{i=0}^{k-1}\frac{n-i-1-R}{n-k-1}$$

Note that the numerator occupys the interval $[n-k-R,n-1-R]$, and the donominator occupys the interval $[n-k,n-1]$. So,

$$\Gamma_k=\frac{\prod_{i=n-k-R}^{n-k-1}i}{\prod_{j=n-R}^{n-1}j}$$

Note that when $n-k-R\leq 0$, we have $\Gamma_k=0$. So, the above fomula also works for the case $n-k-R\leq 0$. This formula is good, since its denominator does not contains k.

Then, we know that

$$\begin{aligned}
\kappa &\geq \frac{\sum_{i=n-\ell}^{n-1}\Gamma_i}{\sum_{i=0}^{n-1}\Gamma_i}=\frac{\sum_{i=n-\ell}^{n-1}(n-i-R)(n-i-R+1)\cdots(n-i-1)}{\sum_{i=0}^{n-1}(n-i-R)(n-i-R+1)\cdots(n-i-1)}\\
&=\frac{\sum_{i=0}^{\ell-1}i(i-1)\cdots(i-R+1)}{\sum_{i=0}^{n-1}i(i-1)\cdots(i-R+1)}, \qquad i\leftarrow n-i-1\\
&=\frac{\binom{R}{R}+\binom{R+1}{R}+\binom{R+2}{R}+\cdots+\binom{\ell-1}{R}}{\binom{R}{R}+\binom{R+1}{R}+\binom{R+2}{R}+\cdots+\binom{n-1}{R}}, \qquad \text{divid } R! \text{ in numerator and denominator}\\
&=\frac{\binom{R+1}{R+1}+\binom{R+1}{R}+\binom{R+2}{R}+\cdots+\binom{\ell-1}{R}}{\binom{R+1}{R+1}+\binom{R+1}{R}+\binom{R+2}{R}+\cdots+\binom{n-1}{R}}\\
&=\frac{\binom{\ell}{R+1}}{\binom{n}{R+1}}\\
&=\frac{\ell(\ell-1)\cdots(\ell-R)}{n(n-1)\cdots(n-R)}
\end{aligned}$$

3.1 Mixing time of block dynamics

Note that $2R\leq\ell\Rightarrow\frac{2Rn}{n+R}\leq\ell\Rightarrow\frac{n-R}{\ell-R}\leq\frac{2n}{\ell}$.

$$\begin{aligned}
\frac{1}{\kappa}&\leq\frac{n(n-1)\cdots(n-R)}{\ell(\ell-1)\cdots(\ell-R)}\\
&\leq(\frac{n-R}{\ell-R})^{R+1}\\
&\leq(\frac{2n}{\ell})^{R+1} \qquad \text{let } 2R\leq\ell\\
&\leq(\frac{2}{\theta})^{\lceil 2\eta\rceil+1}
\end{aligned}$$

By $2R\leq\ell$, we have $\lceil n\theta\rceil\geq 2\lceil 2\eta\rceil\Rightarrow\theta\geq\frac{4\eta+2}{n}$

4 Block Factorization

Let $P^{\nabla}_{n,n-\ell} = [\pi_n \leftrightarrow \pi_{n-\ell}][\pi_{n-\ell} \leftrightarrow \pi_n]$ be the nature block dynamics where we choose a set S of size ℓ u.a.r. and then we resample the configuration in S by the correct conditional distribution. [CLV20] use a suitable representation of dirichlet form.

$$
\begin{aligned}
\left\langle f, (I - P^{\nabla}_{n,n-\ell})f \right\rangle_{\mu} &= \frac{1}{2} \sum_{\sigma,\tau \in \Omega} \mu(\sigma) P^{\nabla}_{n,n-\ell}(\sigma,\tau) \left(f(\sigma) - f(\tau)\right)^2 \\
&= \frac{1}{2} \sum_{\sigma \in \Omega} \mu(\sigma) \left(\sum_{S \in \binom{V}{\ell}} \sum_{\substack{\tau_S \in \Omega_u^{\sigma_{V\setminus S}} \\ \tau_{V\setminus S} = \sigma_{V\setminus S}}} \frac{1}{\binom{n}{\ell}} \mu_S^{\sigma_{V\setminus S}}(\tau_S) \right) \left(f(\sigma) - f(\tau)\right)^2 \\
&= \frac{1}{\binom{n}{\ell}} \sum_{S \in \binom{V}{\ell}} \sum_{\gamma \in \Omega_{V\setminus S}} \mu_{V\setminus S}(\gamma) \cdot \frac{1}{2} \sum_{\alpha \in \Omega_S^{\gamma}} \mu_S^{\gamma}(\alpha) \sum_{\beta \in \Omega_S^{\gamma}} \mu_S^{\gamma}(\beta) (f_\gamma(\alpha) - f_\gamma(\beta))^2 \\
&= \frac{1}{\binom{n}{\ell}} \sum_{S \in \binom{V}{\ell}} \sum_{\gamma \in \Omega_{V\setminus S}} \mu_{V\setminus S}(\gamma) \cdot \mathrm{Var}_{\mu_S^{\gamma}}(f_\gamma) \\
&= \frac{1}{\binom{n}{\ell}} \sum_{S \in \binom{V}{\ell}} \mu\left[\mathrm{Var}_S(f)\right]
\end{aligned}
$$

Fact 4.1 (Equivalence for Block Dynamics)**.** *Recall that the poincaré inequality for the block dynamics is*

$$\forall f, (1-\lambda_2)\mathrm{Var}_{\mu} f \le \left\langle f, (I - P^{\nabla}_{n,n-\ell})f \right\rangle_{\mu}$$

And we could restate it as follows

$$\forall f, (1-\lambda_2)\mathrm{Var}_{\mu} f \le \frac{1}{\binom{n}{\ell}} \sum_{S \in \binom{V}{\ell}} \mu[\mathrm{Var}_S(f)]$$

Moreover, we could establish the connection between the notation of **block fractorization** and the **decay of down-up walk**.

$$
\begin{aligned}
a_n &= \sum_{\gamma \in X_{n-\ell}} \pi_{n-\ell}(\gamma) \sum_{\alpha \in X_\ell^{\gamma}} \pi_\ell^{\gamma}(\alpha) (f_\gamma^{(\ell)}(\alpha))^2 \\
a_{n-\ell} &= \sum_{\gamma \in X_{n-\ell}} \pi_{n-\ell}(\gamma) (f^{(n-\ell)}(\gamma))^2
\end{aligned}
$$

It is easy to see that $f^{(n-\ell)}(\gamma) = \mathbb{E}_{\pi_\ell^{\gamma}}[f_\gamma^{(\ell)}(\alpha)]$

So, we have

$$
\begin{aligned}
a_n - a_{n-\ell} &= \sum_{\gamma \in X_{n-\ell}} \pi_{n-\ell}(\gamma) \mathrm{Var}_{\pi_\ell^\gamma}(f_\gamma^{(\ell)}) \\
&= \sum_{(U,\sigma) \in X_{n-\ell}} \pi_{n-\ell}(U,\sigma) \mathrm{Var}_{\pi_\ell^{(U,\sigma)}}(f_\gamma^{(\ell)}) \\
&= \sum_{U \in \binom{V}{n-\ell}} \sum_{\sigma \in \Omega_U} \pi_{n-\ell}(U,\sigma) \mathrm{Var}_{\pi_\ell^{(U,\sigma)}}(f_\gamma^{(\ell)}) \\
&= \sum_{U \in \binom{V}{n-\ell}} \sum_{\sigma_U \in \Omega_U} \frac{1}{\binom{n}{\ell}} \mu_U(\sigma) \mathrm{Var}_{\mu_{V \setminus U}^\sigma}(f_\sigma^{(\ell)}) \\
&= \frac{1}{\binom{n}{\ell}} \sum_{S \in \binom{V}{\ell}} \sum_{\sigma \in \Omega_{V \setminus S}} \mu_{V \setminus S}(\sigma) \mathrm{Var}_{\mu_S^\sigma}(f_\sigma^{(\ell)}) \\
&= \frac{1}{\binom{n}{\ell}} \sum_{S \in \binom{V}{\ell}} \mu[\mathrm{Var}_S(f)]
\end{aligned}
$$

5 Break Blocks into Single Vertices

Now, we want to build the connection between

$$
\frac{1}{\binom{n}{\ell}} \sum_{S \in \binom{V}{\ell}} \mu[\mathrm{Var}_S(f)] \text{ and } \frac{1}{n} \sum_{v \in V} \mu[\mathrm{Var}_v(f)]
$$

Fact 5.1. *For $S \subset V$, let $\mathcal{P}(S)$ be any partition of S, and if*

$$
\forall \sigma \in \Omega_{V \setminus S} \,.\, \mathrm{Var}_S^\sigma(f) \le C \cdot \sum_{U \in \mathcal{P}(S)} \mu_S^\sigma[\mathrm{Var}_U(f)]
$$

for some constant C (does not effect by σ). Then we have for $S \subset V$:

$$
\mu[\mathrm{Var}_S(f)] \le C \cdot \sum_{U \in \mathcal{P}(S)} \mu[\mathrm{Var}_U(f)]
$$

Proof.

$$
\begin{aligned}
\mu[\mathrm{Var}_S(f)] &= \sum_{\sigma \in \Omega_{V \setminus S}} \mu_{V \setminus S}(\sigma) \mathrm{Var}_{\mu_S^\sigma}(f) \\
&\le \sum_{\sigma \in \Omega_{V \setminus S}} \mu_{V \setminus S}(\sigma)\, C \cdot \sum_{U \in \mathcal{P}(S)} \mu_S^\sigma[\mathrm{Var}_U(f)] \\
&\le C \cdot \sum_{\sigma \in \Omega_{V \setminus S}} \mu_{V \setminus S}(\sigma) \sum_{U \in \mathcal{P}(S)} \sum_{\gamma \in \Omega_{S \setminus U}^\sigma} \mu_{S \setminus U}^\sigma(\gamma) \mathrm{Var}_{\mu_U^{\sigma \cup \gamma}}(f) \\
&= C \cdot \sum_{U \in \mathcal{P}(S)} \sum_{\sigma \cup \gamma \in \Omega_{S \setminus U}} \mu_{V \setminus U}(\sigma \cup \gamma) \mathrm{Var}_{\mu_U^{\sigma \cup \gamma}}(f) \\
&= C \cdot \sum_{U \in \mathcal{P}(S)} \mu[\mathrm{Var}_U(f)]
\end{aligned}
$$

□

Fact 5.2 ([CLV20])**.** *Let $\mathcal{C}(S)$ be the set of all disconnected parts of S, then we have*

$$
\forall \sigma \in \Omega_{V \setminus S} \,.\, \mathrm{Var}_S^\sigma(f) \le \sum_{U \in \mathcal{C}(S)} \mu_S^\sigma[\mathrm{Var}_U(f)]
$$

Fact 5.3 ([CLV20])**.** *For S of size k be a connect subgraph, we have*

$$\mathrm{Var}_S^{\gamma}(f) \leq \frac{k}{2b^{2k+2}} \sum_{v \in V} \mu_S^{\gamma}[\mathrm{Var}_v(f)]$$

Proof Sketch. Fix any configuration γ on $V \setminus S$. Let σ and τ be two configrations on S.

$$\mu^{\gamma}(\sigma) P^{\gamma}(\sigma, \tau) \geq b^k \cdot \frac{b}{k}$$

So, $\Phi \geq \frac{2b^{k+1}}{k}$. And recall that we have $1 - \lambda_2 \geq \frac{\Phi^2}{2} = \frac{2b^{2k+2}}{k^2}$.

So we have

$$\frac{2b^{2k+2}}{k^2} \mathrm{Var}_S^{\gamma}(f) \leq (1 - \lambda_2) \mathrm{Var}_S^{\gamma}(f) \leq \frac{1}{k} \sum_{v \in V} \mu_S^{\gamma}[\mathrm{Var}_v(f)]$$

$$\mathrm{Var}_S^{\gamma}(f) \leq \frac{k}{2b^{2k+2}} \sum_{v \in V} \mu_S^{\gamma}[\mathrm{Var}_v(f)]$$

□

Fact 5.4 ([CLV20])**.** *Let $G = (V, E)$ be an n-vertex graph of maximum degree at most Δ. Then for every $K \in \mathbb{N}^+$ we have*

$$\mathbb{P}_S(|S_v| = k) \leq \frac{\ell}{n} \cdot (2e\Delta\theta)^{k-1}$$

where the probability $\mathbb{P}$ is taken over a uniform random subset $S \subset V$ of size $\ell = \lceil \theta n \rceil$. ($S_v$ a the connected component in S which contains v)

Then, we have

$$
\begin{aligned}
a_n &\leq \frac{1}{\kappa}(a_n - a_{n-\ell}) \\
\mathrm{Var}(f) &\leq \frac{1}{\kappa}\frac{1}{\binom{n}{\ell}} \sum_{S\in\binom{V}{\ell}} \mu[\mathrm{Var}_S(f)] \\
&\leq \frac{1}{\kappa}\frac{1}{\binom{n}{\ell}} \sum_{S\in\binom{V}{\ell}} \sum_{U\in\mathcal{C}(S)} \mu[\mathrm{Var}_U(f)] \\
&\leq \frac{1}{\kappa}\frac{1}{\binom{n}{\ell}} \sum_{S\in\binom{V}{\ell}} \sum_{U\in\mathcal{C}(S)} \frac{|U|}{2b^{2|U|+2}} \sum_{v\in U} \mu[\mathrm{Var}_v(f)] \\
&= \frac{1}{\kappa} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \sum_{k=1}^{\ell} \mathbb{P}_S(|S_v| = k) \cdot \frac{k}{2b^{2k+2}} \\
&\leq \frac{1}{\kappa} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \sum_{k=1}^{\ell} \frac{\ell}{n} \cdot (2e\Delta\theta)^{k-1} \cdot \frac{k}{2b^{2k+2}} \\
&= \frac{1}{\kappa}\frac{\ell}{n}\frac{1}{2b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \sum_{k=1}^{\ell} \cdot (\frac{2e\Delta\theta}{b^2})^{k-1} \cdot k \\
&\leq \frac{1}{\kappa}\frac{\ell}{n}\frac{1}{2b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \sum_{k=1}^{\ell} \cdot (1/2)^{k-1} \cdot k, \qquad \text{let } \theta < \tfrac{b^2}{4e\Delta} \\
&\leq \frac{1}{\kappa}\frac{\ell}{n}\frac{1}{2b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \cdot 12, \qquad \text{see [CLV20]} \\
&\leq \frac{1}{\kappa}\frac{\ell}{n}\frac{6}{b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \\
&\leq \frac{n(n-1)\cdots(n-R)}{\ell(\ell-1)\cdots(\ell-R)} \cdot \frac{\ell}{n}\frac{6}{b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \\
&\leq (\frac{n-R}{\ell-R})^R \cdot \frac{6}{b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \\
&\leq (\frac{n-\lceil 2\eta\rceil}{\ell-\lceil 2\eta\rceil})^{\lceil 2\eta\rceil} \cdot \frac{6}{b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)] \\
&\quad (\text{Since } 2R \leq \ell \Rightarrow \tfrac{2Rn}{n+R} \leq \ell \Rightarrow \tfrac{n-R}{\ell-R} \leq \tfrac{2n}{\ell}) \\
&\leq (\frac{2n}{\ell})^R \cdot \frac{6}{b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)], \qquad \text{let } \ell = \lceil \theta n\rceil \geq 2R \\
&\leq (\frac{2}{\theta})^{2\eta+1} \cdot \frac{6}{b^4} \sum_{v\in V} \mu[\mathrm{Var}_v(f)]
\end{aligned}
$$

Moreover, note that θ should satisfies

$$\lceil n\theta\rceil \geq 2\lceil 2\eta\rceil \text{ and } \theta < \frac{b^2}{4e\Delta}$$

,which equivalants to

$$\frac{4\eta+2}{n} \leq \theta < \frac{b^2}{4e\Delta}$$

. Finally, note that as $n \to \infty$, the lowerbound could be omitted. And it is easy to give a lowerbound for b using the tree recursion (see [CLV20])

References

[AL20] Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1198–1211, 2020.

[CLV20] Zongchen Chen, Kuikui Liu, and Eric Vigoda. *Optimal Mixing of Glauber Dynamics: Entropy Factorization via High-Dimensional Expansion*. CORR, 2020. https://arxiv.org/abs/2011.02075.