Adjoint Walks

Xiaoyu Chen

1 Adjoint walk on two disjoint spaces

Let Ω_1 and Ω_2 be two disjoint state spaces. Let π_1 and π_2 be two distribution on Ω_1 and Ω_2, respectively. Let $P_{12} \in \mathbb{R}^{\Omega_1 \times \Omega_2}$ and $P_{21} \in \mathbb{R}^{\Omega_2 \times \Omega_1}$ be two stochastic matrices.

Definition 1. *If for any $x \in \Omega_1$ and $y \in \Omega_2$, we have $\pi_1(x)P_{12}(x,y) = \pi_2(y)P_{21}(y,x)$, then we say P_{12} and P_{21} are* adjoint random walk *over π_1 and π_2.*

Fact 1. *P_{12} and P_{21} are* adjoint random walk *over π_1 and π_2 iff*

$$\forall f : \Omega_1 \to \mathbb{R}, \forall g : \Omega_2 \to \mathbb{R} \ , \ \langle f, P_{12}g \rangle_{\pi_1} = \langle P_{21}f, g \rangle_{\pi_2}.$$

Proof. $\Rightarrow$:

$$\begin{aligned} \langle f, P_{12}g \rangle_{\pi_1} &= \sum_{x \in \Omega_1} \pi_1(x) f(x) \left[P_{12}g \right](x) = \sum_{x \in \Omega_1} \sum_{y \in \Omega_2} \pi_1(x) P_{12}(x,y) f(x) g(y) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel \\ \langle P_{21}f, g \rangle_{\pi_2} &= \sum_{y \in \Omega_2} \pi_2(y) \left[P_{21}f \right](y) g(y) = \sum_{x \in \Omega_1} \sum_{y \in \Omega_2} \pi_2(y) P_{21}(y,x) f(x) g(y) \end{aligned}$$

$\Leftarrow$: Let $x \in \Omega_1$ and $y \in \Omega_2$ be arbitrary elements in Ω_1 and Ω_2, respectively. Let $\delta_x : \Omega_1 \to \{0,1\}$ and $\delta_y : \Omega_2 \to \{0,1\}$, be the indicator function of x and y, respectively. Then,

$$\langle \delta_x, P_{12}\delta_y \rangle_{\pi_1} = \langle P_{21}\delta_x, \delta_y \rangle_{\pi_2} \Rightarrow \pi_1(x)P_{12}(x,y) = \pi_2(y)P_{21}(y,x)$$ □

Fact 2. *If P_{12} and P_{21} are* adjoint random walk *over π_1 and π_2, then*

$$\pi_1 P_{12} = \pi_2 \text{ and } \pi_2 P_{21} = \pi_1$$

Fact 3. *Let P_{12} and P_{21} be* adjoint random walk *over π_1 and π_2. Then for any distribution ν_1 over Ω_1, we have,*

$$\frac{\nu_1 P_{12}}{\pi_1 P_{12}} = P_{21} \frac{\nu_1}{\pi_1}. \tag{1}$$

Symmetrically, for any distribution ν_2 over Ω_2, we have,

$$\frac{\nu_2 P_{21}}{\pi_2 P_{21}} = P_{12} \frac{\nu_2}{\pi_2}. \tag{2}$$

Proof. We only proof (1). For any $y \in \Omega_2$, we have

$$\frac{\nu_1 P_{12}}{\pi_1 P_{12}}(y) = \frac{\left[\nu_1 P_{12} \right](y)}{\pi_2(y)}$$

On the other hand, we have

$$\begin{aligned}
P_{21}\frac{\nu_1}{\pi_1}(y) &= \sum_{x\in\Omega_1} P_{21}(y,x)\frac{\nu_1(x)}{\pi_1(x)} \\
&= \sum_{x\in\Omega_1} \pi_2(y)P_{21}(y,x)\frac{\nu_1(x)}{\pi_2(y)\pi_1(x)} \\
&= \sum_{x\in\Omega_1} \pi_1(x)P_{12}(x,y)\frac{\nu_1(x)}{\pi_2(y)\pi_1(x)} \\
&= \sum_{x\in\Omega_1} \pi_1(x)P_{12}(x,y)\frac{\nu_1(x)}{\pi_2(y)\pi_1(x)} \\
&= \frac{[\nu_1 P_{12}](y)}{\pi_2(y)}
\end{aligned}$$

□

2 Understand variance and Dirichlet form

For any distribution $\pi : \Omega \to \mathbb{R}$, and any function $f : \Omega \to \mathbb{R}$, the variance is defined as

$$\mathrm{Var}_\pi[f] \triangleq \mathbb{E}_\pi[f^2] - \mathbb{E}_\pi[f]^2.$$

We could represent the variance in a more general way:

$$\mathrm{Var}_\pi[f] = \langle f - \pi(f), f - \pi(f)\rangle_\pi,$$

where, for conveniene, we use $\pi(f)$ to stand for $\pi^T f$, which is equivalent to $\mathbb{E}_\pi[f]$.

Then, we define the projection operator as follows.

Definition 2. *We define $J_\pi \triangleq \mathbb{1}\pi^T$, which is a matrix with each row equals to π^T.*

Fact 4. *For any $f : \Omega \to \mathbb{R}$, it is easy to verify that $J_\pi f = \langle f, \mathbb{1}\rangle_\pi \mathbb{1}$. So, J_π is a projection matrix that maps f to $f^{\mathbb{1}}$.*

Then, for the variance, we have

$$\begin{aligned}
\mathrm{Var}_\pi[f] &= \langle f - J_\pi f, f - J_\pi f\rangle_\pi \\
&= \langle f^{\perp_\pi \mathbb{1}}, f^{\perp_\pi \mathbb{1}}\rangle_\pi \\
&= \langle f, (I - J_\pi)f\rangle_\pi \\
&= \mathcal{E}_{J_\pi}(f,f)
\end{aligned}$$