Exercies of Chapter 9

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1 Exercise 9.4

In Chapter 9, the jerrum book gives the general definition of the Dirichlet form:

$$\mathcal{E}_P(f,g) = -\mathbb{E}_\pi[fQg], \quad \text{where } Q = P - I$$

Show that

$$\frac{1}{2}\sum_{x,y}\pi(x)P(x,y)(f(y)-f(x))(g(y)-g(x))$$

is equal to $\mathcal{E}_P(f,g)$ when either $f = g$ or P is time reversible, and provide a counterexample to the equivalence in general.

1.1 Solution

Lets consider the general case first:

$$\begin{aligned}
\mathcal{E}_P(f,g) &= -\mathbb{E}_\pi[fQg] \\
&= -\sum_{x\in\Omega}\pi(x)f(x)[Qg](x) \\
&= -\sum_{x\in\Omega}\pi(x)f(x)\sum_{y\in\Omega}Q(x,y)g(y) \\
&= \sum_{x,y\in\Omega}\pi(x)f(x)(I(x,y)-P(x,y))g(y) \\
&= \sum_{x\in\Omega}\pi(x)f(x)g(x) - \sum_{x,y\in\Omega}\pi(x)P(x,y)f(x)g(y) \\
&= \frac{1}{2}\sum_{x\in\Omega}\pi(x)f(x)g(x) + \frac{1}{2}\sum_{y\in\Omega}\pi(y)f(y)g(y) - \sum_{x,y\in\Omega}\pi(x)P(x,y)f(x)g(y) \\
&= \frac{1}{2}\sum_{x\in\Omega}\pi(x)f(x)f(x)\sum_{y\in\Omega}P(x,y) + \frac{1}{2}\sum_{x,y\Omega}\pi(x)P(x,y)f(y)g(y) - \sum_{x,y\in\Omega}\pi(x)P(x,y)f(x)g(y) \\
&= \frac{1}{2}\sum_{x,y\in\Omega}\pi(x)P(x,y)f(x)f(x) + \frac{1}{2}\sum_{x,y\Omega}\pi(x)P(x,y)f(y)g(y) - \sum_{x,y\in\Omega}\pi(x)P(x,y)f(x)g(y)
\end{aligned} \tag{1}$$

At the same time, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{x,y\Omega}\pi(x)P(x,y)(f(y)-f(x))(g(y)-g(x)) \\
&\qquad\qquad = \frac{1}{2}\sum_{x,y}\pi(x)P(x,y)(f(y)g(y)+f(x)g(x)-f(x)g(y)-f(y)g(x))
\end{aligned} \tag{2}$$

By comparing Equation (1) and (2), we could know that

$$\mathcal{E}_P(f,g) = \frac{1}{2}\sum_{x,y}\pi(x)P(x,y)(f(y)-f(x))(g(y)-g(x))$$

iff

$$\sum_{x,y}\pi(x)P(x,y)f(x)g(y) = \sum_{x,y}\pi(x)P(x,y)f(y)g(x) \tag{3}$$

So, when $f = g$, its easy to verify that Equation (3) is true. And when P is time reversible, we have

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

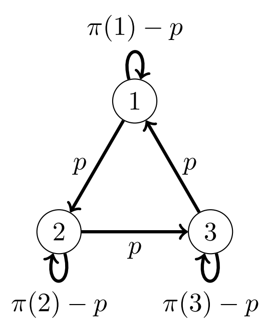
, which means:

$$\begin{aligned}
\sum_{x,y}\pi(x)P(x,y)f(x)g(y) &= \sum_{x,y}\pi(y)P(y,x)f(x)g(y) \\
&= \sum_{x,y}\pi(x)P(x,y)f(y)g(x), \qquad \text{by swapping } x \text{ and } y
\end{aligned}$$

To give a counterexample, we only need to give a MC and give two function g, f where:

$$\sum_{x\neq y}\pi(x)P(x,y)f(x)g(y) \neq \sum_{x\neq y}\pi(x)P(x,y)f(y)g(x)$$

Consider a MC like this:



The **edge measure** (i.e. $\pi(x)P(x,y)$) of each edge is marked in the figure. It is clear that this MC is not time reversible. In this MC, we have

$$\sum_{x\neq y} \pi(x)P(x,y)f(x)g(y) = p(f(1)g(2) + f(2)g(3) + f(3)g(1)) \tag{4}$$

$$\sum_{x\neq y} \pi(x)P(x,y)f(y)g(x) = p(f(2)g(1) + f(3)g(2) + f(1)g(3)) \tag{5}$$

To give a counterexample, we only need to make (4) $-$ (5) $\neq 0$, which means

$$f(1)[g(2) - g(3)] + f(2)[g(3) - g(1)] + f(3)[g(1) - g(2)] \neq 0 \tag{6}$$

Consider an assignment for f where $f = [1, 2, 1]$, and the Equation (6) will become:

$$g(3) - g(1)$$

To make $g(3) - g(1) \neq 0$, we could set g to $[0, 0, 1]$.

2 Exercise 9.8

Verify identity (9.12).

2.1 Solution

First, we could notice that

$$\begin{aligned}
&\pi(\Omega_0)\mathcal{L}_{\pi_0}(f) + \pi(\Omega_1)\mathcal{L}_{\pi_1} \\
&\qquad = \mathbb{E}_\pi[f^2 \ln f^2] - \sum_b \pi(\Omega_b)\mathbb{E}_{\pi_b}[f^2 \ln(\mathbb{E}_{\pi_b} f^2)]
\end{aligned} \tag{7}$$

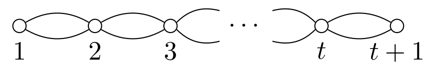
then we have

$$\begin{aligned}
\mathcal{L}_\pi(f) - (7) &= \sum_b \pi(\Omega_b)\mathbb{E}_{\pi_b}[f^2 \ln(\mathbb{E}_{\pi_b} f^2)] - \mathbb{E}_\pi[f^2 \ln(\mathbb{E}_\pi f^2)] \\
&= \sum_b \pi(\Omega_b)\mathbb{E}_{\pi_b}[f^2 \ln(\mathbb{E}_{\pi_b} f^2)] - \sum_b \pi(\Omega)_b\mathbb{E}_{\pi_b}[f^2 \ln(\mathbb{E}_\pi f^2)] \\
&= \sum_b \pi(\Omega_b)\mathbb{E}_{\pi_b}[f^2 \ln(\mathbb{E}_{\pi_b} f^2) - f^2 \ln(\mathbb{E}_\pi f^2)] \\
&= \mathcal{L}_\pi(\bar{f})
\end{aligned}$$

3 Exercise 9.14

By exhibiting a suitable graph, show that the bound in Example 9.13 is of correct order of magnitude, at least in some circumstances.

3.1 Solution



In this graph, we have $n = t + 1$ vertices with $m = 2t$ edges. Any spanning tree of this graph should be a path that connect all the vertices. And it is easy to note that the MC is now reduced to the random walk on hypercube $\{0, 1\}^t$ with $p = \frac{1}{t^2}$. Then by using the result of Exercise 8.14, we could find that the mixing time of this MC has a lower bould of $\Omega(t^2 \log t)$ which is the same order of magnitude of $\Omega(mn \log n)$.

4 Exercise 9.18

Verify that $D(\sigma||\tau)$ is non-negative, and that $D(\sigma||\tau) = 0$ implies $\sigma = \tau$.

4.1 Solution

$$
\begin{aligned}
D(\sigma||\tau) &= \sum_{x\in\Omega} \sigma(x) \ln \frac{\sigma(x)}{\pi(x)} \\
&= \sum_{x\in\Omega} \sigma(x)[- \ln \frac{\pi(x)}{\sigma(x)}] \\
&= \mathbb{E}_\sigma[- \ln \frac{\pi(x)}{\sigma(x)}], \qquad \text{by Jensen's Inequality} \\
&\geq - \ln \mathbb{E}_\sigma[\frac{\pi(x)}{\sigma(x)}] \\
&= - \ln[\sum_{x\in\Omega} \sigma(x) \frac{\pi(x)}{\sigma(x)}] \\
&= - \ln 1 = 0
\end{aligned}
$$

The Jensen's Inequality achieves equality when all the possible parameter for $-\ln$ are the same (since $-\ln$ is non-linear). So we have

$$\frac{\pi(e_1)}{\sigma(e_1)} = \frac{\pi(e_2)}{\sigma(e_2)} = \cdots = \frac{\pi(e_n)}{\sigma(e_n)} = t$$

And by

$$\sum_{e\in\Omega} \pi(e) = t \sum_{e\in\Omega} \pi(e)$$

we know that $t = 1$, which means $\sigma = \pi$.