

Adjoint Walks on Disjoint State Spaces

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Let Ω_1 and Ω_2 be two disjoint state spaces. Let π_1 and π_2 be two distribution on Ω_1 and Ω_2 , respectively. Let $P_{12} \in \mathbb{R}^{\Omega_1 \times \Omega_2}$ and $P_{21} \in \mathbb{R}^{\Omega_2 \times \Omega_1}$ be two stochastic matrices.

Definition 1. If for any $x \in \Omega_1$ and $y \in \Omega_2$, we have $\pi_1(x)P_{12}(x, y) = \pi_2(y)P_{21}(y, x)$, then we say P_{12} and P_{21} are adjoint random walk over π_1 and π_2 .

Fact 1. P_{12} and P_{21} are adjoint random walk over π_1 and π_2 iff

$$\forall f : \Omega_1 \rightarrow \mathbb{R}, \forall g : \Omega_2 \rightarrow \mathbb{R}, \langle f, P_{12}g \rangle_{\pi_1} = \langle P_{21}f, g \rangle_{\pi_2}.$$

Proof. \Rightarrow :

$$\begin{aligned} \langle f, P_{12}g \rangle_{\pi_1} &= \sum_{x \in \Omega_1} \pi_1(x) f(x) [P_{12}g](x) = \sum_{x \in \Omega_1} \sum_{y \in \Omega_2} \pi_1(x) P_{12}(x, y) f(x) g(y) \\ &\parallel \\ \langle P_{21}f, g \rangle_{\pi_2} &= \sum_{y \in \Omega_2} \pi_2(y) [P_{21}f](y) g(y) = \sum_{x \in \Omega_1} \sum_{y \in \Omega_2} \pi_2(y) P_{21}(y, x) f(x) g(y) \end{aligned}$$

\Leftarrow : Let $x \in \Omega_1$ and $y \in \Omega_2$ be arbitrary elements in Ω_1 and Ω_2 , respectively. Let $\delta_x : \Omega_1 \rightarrow \{0, 1\}$ and $\delta_y : \Omega_2 \rightarrow \{0, 1\}$, be the indicator function of x and y , respectively. Then,

$$\langle \delta_x, P_{12}\delta_y \rangle_{\pi_1} = \langle P_{21}\delta_x, \delta_y \rangle_{\pi_2} \Rightarrow \pi_1(x)P_{12}(x, y) = \pi_2(y)P_{21}(y, x) \quad \square$$

Fact 2. If P_{12} and P_{21} are adjoint random walk over π_1 and π_2 , then

$$\pi_1 P_{12} = \pi_2 \text{ and } \pi_2 P_{21} = \pi_1$$

Fact 3. Let P_{12} and P_{21} be adjoint random walk over π_1 and π_2 . Then for any distribution ν_1 over Ω_1 , we have,

$$\frac{\nu_1 P_{12}}{\pi_1 P_{12}} = P_{21} \frac{\nu_1}{\pi_1}. \quad (1)$$

Symmetrically, for any distribution ν_2 over Ω_2 , we have,

$$\frac{\nu_2 P_{21}}{\pi_2 P_{21}} = P_{12} \frac{\nu_2}{\pi_2}. \quad (2)$$

Proof. We only proof (1). For any $y \in \Omega_2$, we have

$$\frac{\nu_1 P_{12}}{\pi_1 P_{12}}(y) = \frac{[\nu_1 P_{12}](y)}{\pi_2(y)}$$

On the other hand, we have

$$\begin{aligned}
P_{21} \frac{\nu_1}{\pi_1}(y) &= \sum_{x \in \Omega_1} P_{21}(y, x) \frac{\nu_1(x)}{\pi_1(x)} \\
&= \sum_{x \in \Omega_1} \pi_2(y) P_{21}(y, x) \frac{\nu_1(x)}{\pi_2(y) \pi_1(x)} \\
&= \sum_{x \in \Omega_1} \pi_1(x) P_{12}(x, y) \frac{\nu_1(x)}{\pi_2(y) \pi_1(x)} \\
&= \sum_{x \in \Omega_1} \pi_1(x) P_{12}(x, y) \frac{\nu_1(x)}{\pi_2(y) \pi_1(x)} \\
&= \frac{[\nu_1 P_{12}](y)}{\pi_2(y)}
\end{aligned}$$

□