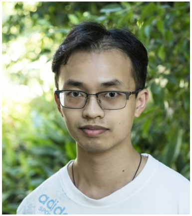
# Uniqueness and Rapid Mixing in the Bipartite Hardcore Model

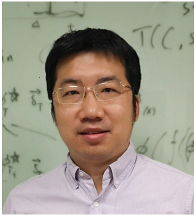
Xiaoyu Chen

Nanjing University

based on joint work with



Jingcheng Liu



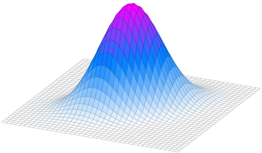
Yitong Yin

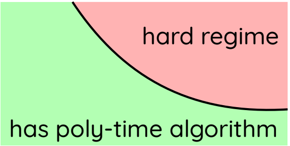
Sampling problem:

Draw (approximate) random samples from a distribution

Gibbs distribuiton:

- high-dimensional joint distribution
- described by few parameters and local interactions





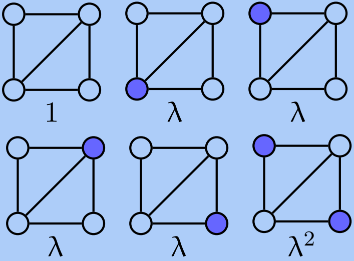

Computational phase transition:
computational complexity of sampling problem changes sharply around certain parameter values

# Hardcore model

- $G = ([n], E)$ with $n$ vertices and max degree $\Delta$.
- Fugacity $\lambda > 0$ is a real number.
- $\mathrm{Ind}(G) = \{S \subseteq [n] \mid S \text{ is an independent set}\}$.
- Gibbs distribution

$$\forall S \in \mathrm{Ind}(G), \quad \mu(S) := \frac{\lambda^{|S|}}{Z}, \quad \text{where } Z_G(\lambda) := \sum_{I \in \mathrm{Ind}(G)} \lambda^{|I|}.$$

**an example**




Partition function:

$$Z = 1 + 4\lambda + \lambda^2$$

This model is self-reducible

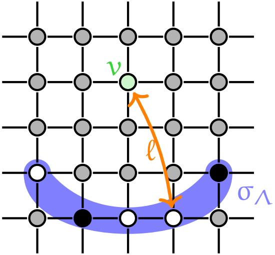
# Computational phase transition

On general graph with maximum degree $\Delta$:

uniqueness $\lambda_c(\Delta)$ non-uniqueness

$\lambda : 0$ ⊢———○———→ $\infty$

tree uniqueness threshold: $\lambda_c(\Delta) := (\Delta - 1)^{(\Delta-1)}/(\Delta - 2)^{\Delta} \approx \frac{e}{\Delta}$




**Spatial mixing (SM)**

$\forall G, v, \mathbf{Pr}_{S\sim\mu}[v \in S \mid \sigma_\Lambda]$ does not depend on $\sigma_\Lambda$ as $\ell \to +\infty$

SM $\iff \lambda \leq \lambda_c(\Delta)$

$\sigma_\Lambda$ : fixed configuration in $\Lambda$

easy $\lambda_c(\Delta)$ hard

$\lambda : 0$ ⊢———○———→ $\infty$

Computational phase transition:

- $\lambda < \lambda_c$: poly-time algorithm for approx. sampling [Wei06]
- $\lambda > \lambda_c$: no poly-time algorithm unless **NP = RP** [Sly10]

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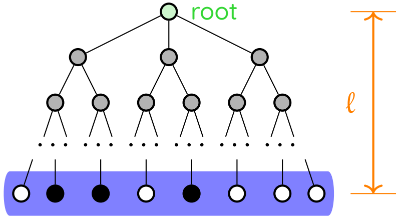


$\sigma$ : boundary condition on level $\ell$

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$\mathbf{Pr}_{S\sim\mu}$[root $\in S \mid \sigma$] does not depend on $\sigma$ when $\ell \to \infty$

SM $\Longleftrightarrow \lambda \leq \lambda_c(\Delta)$

$\Delta$-regular tree is the worst case [Wei06]

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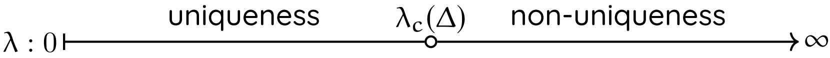


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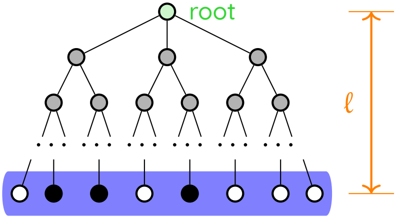


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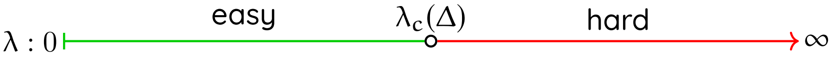


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# Hardcore model on bipartite graph (weighted #BIS)

**It is easy**: there is a poly-time algorithm to find a matximum independent set in the bipartite graph (Kőnig's theorem[1]).

**It is hard**: many important problems are proved to be #BIS-equivalent or #BIS-hard under AP-reductions.

**Selected examples**

- stable matchings (counting)
- ferro. Potts model (parti. func.)
- ferro. Ising with mixed external fields (parti. func.)

[ DGGJ04, GJ07, DGJ10, CGM12 DGJR12, GJ12a, BDG+13, LLZ14, GJ15, CGG+16, GŠVY16, GGY21, ]

**Conjecture**[DGGJ04]:
#BIS represents an intermediate complexity class:

- it has no FPRAS in general
- it is easier than #SAT

[1] In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

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# Previous algorithmic results

**Non-uniqueness regime:**

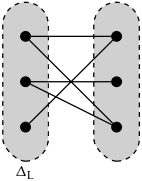
- $\alpha$-expander bipartite graph:
  - $\lambda \geq (C_0\Delta)^{4/\alpha}$, an $n^{O(\log \Delta)}$ time sampler [JKP20]
  - $\lambda \geq (C_1\Delta)^{6/\alpha}$, an $O(n \log n)$ time sampler [CGG+21]
  - $\lambda \geq (C_2\Delta)^{2/\alpha}$, an $n^{O(\log \Delta)}$ time sampler [FGKP23]
- $\Delta$-regular $\alpha$-expander bipartite graph:
  - $\lambda \geq \frac{f(\alpha)\log\Delta}{\Delta^{1/4}}$, an $n^{O(\Delta)}$ time sampler [JPP22]
- random $\Delta$-regular bipartite graph:
  - $\Delta \geq \Delta_0, \lambda \geq \frac{\log^4 \Delta}{\Delta}$, an $n^{O(1)}$ time sampler [LLLM19]
  - $\Delta \geq \Delta_1, \lambda \geq \frac{50\log^2 \Delta}{\Delta}$, an $n^{1+O(\frac{\log^2(\Delta)}{\Delta})}$ time sampler [JKP20]
  - $\Delta \geq \Delta_2, \lambda \geq \frac{100\log \Delta}{\Delta}$, an $O(n \log n)$ time sampler [CGŠV22]
- unbalanced bipartite graph:
  - $6\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $n^{O(\log(\Delta_L\Delta_R))}$ time sampler [CP20]
  - $3.4\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $n^{O(\log(\Delta_L\Delta_R))}$ time sampler [FGKP23]
  - $(1+e)\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $O(n \log n)$ time sampler [BCP22]

# Previous algorithmic results

**Uniqueness regime:**

- general graph: if $\lambda < \lambda_c(\Delta)$, there is an $O(n \log n)$ time sampler
- bipartite graph: if $\lambda = 1, \Delta_L \leq 5$, an $O(n^2)$ time sampler [LL15]
  $(\lambda = 1 \wedge \lambda < \lambda_c(\Delta) \Leftrightarrow \Delta \leq 5)$

$$\lambda_c(\Delta) = \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^{\Delta}}$$




**Uniqueness regime when $\Delta_L$ is bounded:**




**Spatial mixing (SM)**

**bipartite graph** $G$ with **degree bound** $\Delta_L$ **on left side**, $v \in G$, $\Pr_{S \sim \mu}[v \in S \mid \sigma_\Lambda]$ does not depend on $\sigma_\Lambda$ as $\ell \to +\infty$

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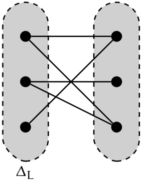


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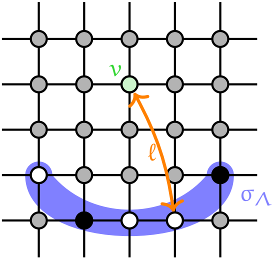


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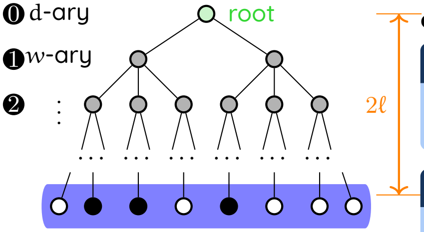


**odd level**: right **even level**: left

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**Tree recursion of $(d, w)$-ary tree**

$$F(x) := \lambda(1 + \lambda(1 + x)^{-w})^{-d}$$

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**marginal ratio** of $v$, $R_v := \mathbf{Pr}_{S\sim\mu}[v \in S] / \mathbf{Pr}_{S\sim\mu}[v \notin S]$



diverge ($> 1$ fixed point)



converge ($= 1$ fixed point)

## Definition

Let $\delta \in [0, 1)$ be a real number. The pair $(\lambda, d) \in \mathbb{R}^2_{>0}$ is $\delta$-**unique** if for any $w \in \mathbb{R}_{>0}$, all fixpoints $\hat{x} = F(\hat{x})$ of $F$ satisfy $F'(\hat{x}) \leq 1 - \delta$.

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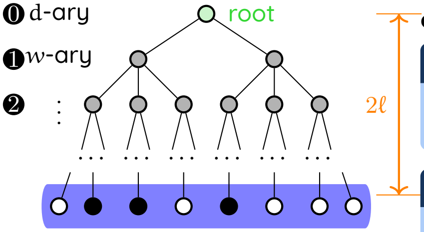


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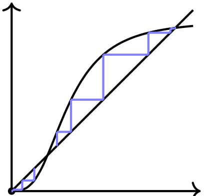

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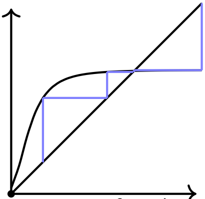

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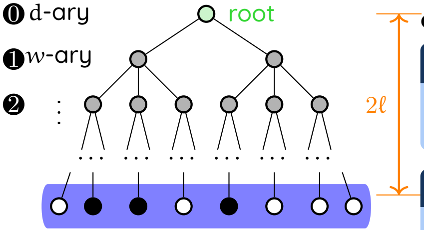


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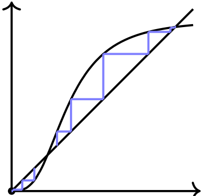

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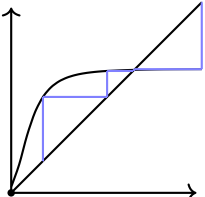

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## Theorem

Fix any $\Delta = d+1 \ge 3$ and any $\delta \in [0,1)$, the pair $(\lambda, d)$ is $\frac{\delta}{10}$-unique if

$$\lambda \le (1-\delta)\lambda_c(\Delta) = (1-\delta)\frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}.$$

## Theorem

For bipartite graph $G = (L \cup R, E)$ with maximum degree $\Delta_L = d+1 \ge 2$ on $L$ and fugacity $\lambda > 0$, let $n = |L|$, then for any $\delta \in (0,1)$, if $(\lambda, d)$ is $\delta$-unique, then we have a sampler for this hardcore model that runs in time

$$n\left(\frac{\Delta_L \log n}{\lambda}\right)^{O(C/\delta)}, \text{where } \begin{cases} C = O(1), & \Delta_L \ge 3 \\ C = (1+\lambda)^{10}, & \Delta_L = 2. \end{cases}$$

- When $\Delta_L = 1$, $G$ is a forest.
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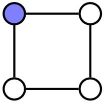
Glauber dynamics for Hardcore model:

start from an arbitrary independent set $X_0$;
**for** $t$ from $1$ to $T$ **do**:

- pick a vertex $v \in V$ uniformly at random;
- with prob. $\frac{\lambda}{1+\lambda}$, let $S = X_{t-1} \cup \{v\}$;
  with prob. $\frac{1}{1+\lambda}$, let $S = X_{t-1} \setminus \{v\}$;
- **if** $S \in \mathrm{Ind}(G)$ **then** $X_t = S$ **else** $X_t = X_{t-1}$;



irreducible + aperiodic + reversible $\Longrightarrow X_t \sim \mu$ as $t \to \infty$

mixing time: essential running time of Glauber dynamics

$$T_{\mathrm{mix}} := \max_{X_0} \min\{t \mid D_{\mathrm{TV}}(X_t \parallel \mu) \le 1/100\}$$

total variation distance: conanical distance between distributions

$$D_{\mathrm{TV}}(X_t \parallel \mu) := \frac{1}{2} \sum_{S \in \mathrm{Ind}(G)} |\mathbf{Pr}[X_t = S] - \mu(S)|$$

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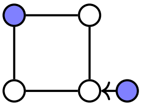

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# Our results

Glauber dynamics for Hardcore model:

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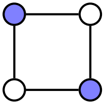

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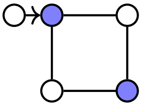

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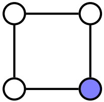

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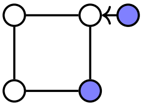

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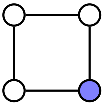

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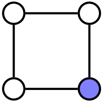

## Theorem

For bipartite graph $G = (L \cup R, E)$ with maximum degree $\Delta_L = d + 1 \geq 3$ on $L$, $\delta \in (0, 1)$, and fugacity $\lambda \in (0, (1 - \delta)\lambda_c(\Delta))$. Then the mixing time of the Glauber dynamics is bounded as

$$T_{\mathrm{mix}} \leq \left(\frac{\Delta \log n}{\lambda}\right)^{O(C/\delta)} \cdot n^3 \cdot \log \frac{1+\lambda}{\min\{1, \lambda\}}.$$

- When $\Delta_L \geq 3$, then $C = O(1)$.
- When $\Delta_L = 2$, $(\lambda, d)$ is $\delta$-unique, the bound holds with $C = (1 + \lambda)^{10}$.

## Background

## Proof outline

- Fast sampler
- Mixing of Glauber dynamics on $L \cup R$
- Spectral independence
- $\delta$-uniqueness

# Background

Let $\nu$ be a distribution over $\Omega = \{-1,+1\}^n$. $\forall \sigma \in \Omega, \quad \|\sigma\|_+ = |\{i \mid \sigma_i = 1\}|$

**impose external field $\theta > 0$**

$\theta * \nu$: a distribution on $\Omega$:

$$\forall \sigma, \quad (\theta * \nu)(\sigma) \propto \nu(\sigma) \cdot \theta^{\|\sigma\|_+}$$

**flip the distribution**

$\overline{\nu}$: a distribution on $\Omega$:

$$\forall \sigma, \quad \overline{\nu}(\sigma) = \nu(-\sigma)$$

- hardcore model: $\mu$ (fugacity $\lambda$) $\Longrightarrow$ $\theta * \mu$ (fugacity $\theta\lambda$)

For $0 < \theta \neq 1$, Field dynamics $P^{\mathrm{FD}}_{\theta,\nu}$: Markov chain $(X_t)_{t \geq 0}$ on $\Omega$:

$X_0$ is an arbitrary vector in $\Omega$ and let $s \in \{-1,+1\}$ so that $\theta^s < 1$;
for each $t > 0$:

1. **generate** $R \subseteq [n]$: **for** $i \in [n]$ with $X_{t-1}(i) = s$ add $i$ to $R$ with prob. $1 - \theta^s$
2. **let** $X_t = \sigma$ with prob. $\Pr_{\sigma \sim \theta * \nu}[\sigma \mid \sigma_R = s]$

irreducible + aperiodic + reversible [CFYZ21] $\Longrightarrow$ $X_t \sim \nu \quad$ as $t \to \infty$

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# Background

### Theorem ([CFYZ21, AJKPV22, CFYZ22, CE22])

Let $0 < \theta \neq 1$ and $\nu$ be a distribution over $\{-1,+1\}^n$ that

1. $\lambda * \nu$ is K-**marginally stable** for all $\lambda$ between $\theta$, $1$,
2. $\lambda * \nu$ is $\eta$-**spectrally independent** for all $\lambda$ between $\theta$, $1$,
3. the Glauber dynamics on $\theta * \nu$ mixes in time $\widetilde{O}(n)$,

then

$$1 \wedge 2 \;\Rightarrow T_{\text{mix}}(P^{\text{FD}}_{\theta,\nu}) \approx \max\{\theta, 1/\theta\}^{\eta\cdot\text{poly}(K)}.$$

$$1 \wedge 2 \wedge 3 \;\Rightarrow \text{sampler for } \nu \text{ in time } \widetilde{O}(n)\cdot\max\{\theta, 1/\theta\}^{\eta\cdot\text{poly}(K)}$$

$$1 \wedge 2 \wedge 3 \;\overset{\text{Var}}{\Rightarrow} T_{\text{mix}}(P^{\text{GD}}_{\nu}) \approx \widetilde{O}(n)\cdot\underbrace{n\cdot\max\{\theta, 1/\theta\}^{\eta\cdot\text{poly}(K)}}_{\text{relaxation time}}$$

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Let $\nu$ be a distribution over $\{-1,+1\}^n$ and $X \sim \nu$ be a random vector.

**influence matrix $\Psi_\nu \in \mathbb{R}^{n\times n}$**

$$\Psi_\nu(i,j) := \begin{cases} 0, & \text{if } \mathbf{Pr}_\nu[i] \in \{0,1\} \\ \mathbf{Pr}_\nu[j \mid i] - \mathbf{Pr}_\nu\left[j \mid \bar{i}\right] \end{cases}$$

$$i = \{X_i = +1\}, \bar{i} = \{X_i = -1\}$$

**$\mathrm{Corr}(X) \in \mathbb{R}^{n\times n}$**

$$\mathrm{Corr}(X)_{ij} = \frac{\mathrm{Cov}(X_i, X_j)}{\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)}}$$

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- $\Psi_\nu$ is similar to $\mathrm{Corr}(X)$

**$\eta$-spectral independence (in self-reducible models)**

$$\lambda_{\max}(\Psi_\nu) \le \eta \Longleftarrow \|\Psi_\nu\|_\infty \le \eta$$

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there is $\rho \in \{\nu, \bar{\nu}\}$ that for $i \in [n]$, $S \subseteq \Lambda \subseteq [n] \setminus \{i\}$, $\tau \in \Omega(\rho_\Lambda)$,

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# Proof outline

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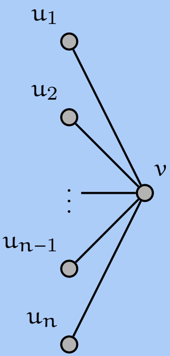


Let $\lambda = 1$ be the fugacity
$\mu$: Gibbs distribution of the hardcore model

**$\|\Psi_\mu\|_\infty$ is unbounded**

- $\forall i, \quad |\Psi_\mu(v, u_i)| = \frac{\lambda}{\lambda+1} = \frac{1}{2}$
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What could we do? 🤔

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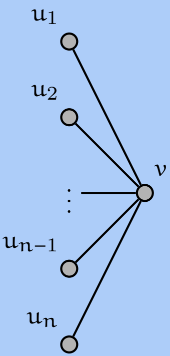


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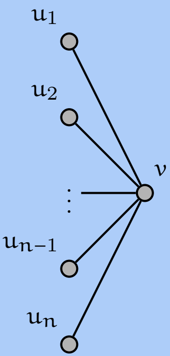


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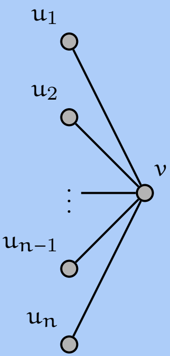


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$\mu$ is the Gibbs distribution of the hardcore model and $\nu$ is $\mu_L$

$$\nu \xrightarrow[\text{BHC}(\lambda,\lambda) \qquad P^{\mathrm{FD}}_{\theta,\nu} \text{ with } \theta = \Theta(\frac{\Delta \log n}{\lambda}) > 1 \qquad \text{BHC}(\theta\lambda,\lambda)]{O(1/\delta)\text{-spectrally independent} \qquad O(1)\text{-marginally stable}} \theta * \nu$$

Glauber dynamics mixes in $\widetilde{O}(n)$

- fast sampler for $\nu$ in time $n \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$ ($\Rightarrow$ fast sampler for $\mu$)
- Glauber dynamics on $\nu$ mixes in time $n^2 \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$

For $\nu = \mu_L$ on BHC$(\lambda, \alpha)$: $\delta$-uniqueness $\Longrightarrow$ $O(1/\delta)$-spectral independence




$\Delta_L = d + 1$

$F(x) = \lambda(1 + \alpha(1 + x)^{-w})^{-d}$

$\forall w > 0, \forall \hat{x}, F'(\hat{x}) \le 1$




**uniqueniss regime**

Fix $d \ge 1$, the pair $(\lambda, d, \alpha)$ is unique if the point $(\lambda, \alpha)$ is on above of the following parametric curve for $w > d^{-1}$:

$$\begin{cases} \alpha(w) = \dfrac{d\,w(w+1)^{w+1}}{(d\,w-1)^{w+1}} \\ \lambda(w) = \dfrac{w\,d(d+1)^{d+1}}{(d\,w-1)^{d+1}} \end{cases}$$

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BHC$(\lambda, \lambda)$ $\qquad$ $P^{FD}_{\theta,\nu}$ with $\theta = \Theta(\frac{\Delta \log n}{\lambda}) > 1$ $\qquad$ BHC$(\theta\lambda, \lambda)$

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$F(x) = \lambda(1 + \alpha(1 + x)^{-w})^{-d}$

$\forall w > 0, \forall \hat{x}, F'(\hat{x}) \le 1$




**uniqueniss regime**

Fix $d \ge 1$, the pair $(\lambda, d, \alpha)$ is unique if the point $(\lambda, \alpha)$ is on above of the following parametric curve for $w > d^{-1}$:

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# Proof outline: fast sampler for $\mu$

$\mu$ is the Gibbs distribution of the hardcore model and $\nu$ is $\mu_L$

$$\nu \xrightarrow[\text{BHC}(\lambda,\lambda) \quad P^{\text{FD}}_{\theta,\nu} \text{ with } \theta = \Theta(\frac{\Delta \log n}{\lambda}) > 1]{O(1/\delta)\text{-spectrally independent} \qquad O(1)\text{-marginally stable ✅}} \theta * \nu \quad \text{BHC}(\theta\lambda, \lambda)$$

Glauber dynamics mixes in $\widetilde{O}(n)$✅

- fast sampler for $\nu$ in time $n \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$ ($\Rightarrow$ fast sampler for $\mu$)
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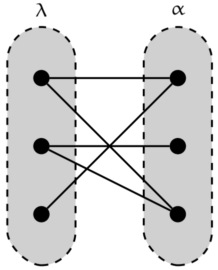


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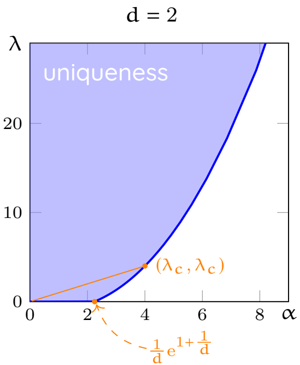


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Background

Proof outline

Fast sampler

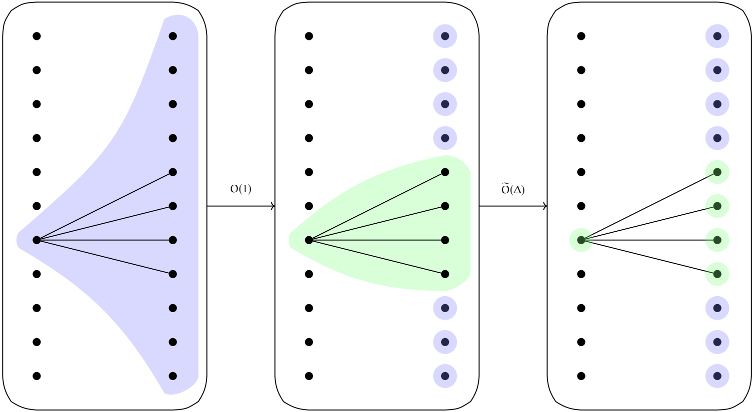
Mixing of Glauber dynamics on $L \cup R$

Spectral independence

$\delta$-uniqueness

# Proof outline: mixing of GD on $\mu$

Glauber dynamics on $\nu$ mixes in time $n^2 \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$
To get an algorithm that only updates each site seperately:




- This algorithm also runs in $n^2 \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$ round
- A vertex $u \in R$ is updated with rate $1$ in each round
- The Glauber dynamics on $\mu$ mixes in time $n^3 \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$

Could be implemented by block factorization [CMT15, CP20, CLV21].

Background

Proof outline

- Fast sampler
- Mixing of Glauber dynamics on $L \cup R$
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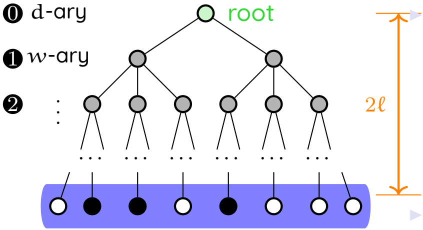
# Proof outline: spectral independence




- $\nu = \mu_L$ for BHC($\lambda$, $\alpha$) that is $\delta$-unique
- reduce the general case to the $(d+1, w+1)$-regular tree via the SAW tree [CLV21] and a special potential function [LL15]

$$\Phi(x) := \log(\log(1+x))$$

$$\Phi'(x) =: \phi(x) = \frac{1}{(1+x)\log(1+x)}$$

- recursion on $(d, w)$-ary tree
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the total influence is bounded by

$$\sum_{\ell=1}^{+\infty}\left(\sum_{v \in L_r(2\ell)} \left|\Psi_\mu(\text{root}, v)\right|\right) \overset{\text{[CLV21]}}{\leq} O(1) \cdot \sum_{\ell=1}^{+\infty}\left\{\sup_R (\Phi \circ F \circ \Phi^{-1})'(\Phi(R))\right\}^\ell$$

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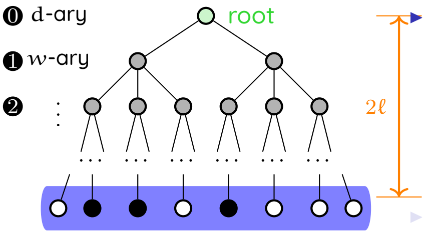


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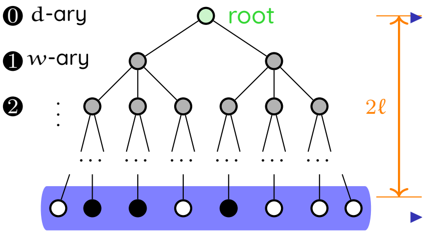


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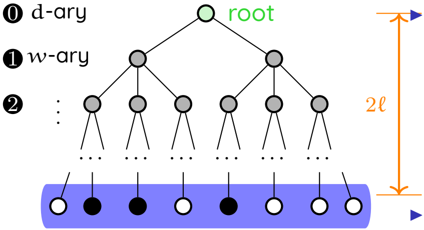


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# Proof outline: contraction ($\delta = 0$)

**$\delta$-uniqueness**

The tuple $(\lambda, d, \alpha)$ is $\delta$-unique if $\forall w \in \mathbb{R}_{>0}$, and $\forall \hat{x}$, $F'(\hat{x}) \le 1 - \delta$.

**contraction**

If $(\lambda, d, \alpha)$ is $\delta$-unique, $d \ge 1$, $\sup_{x \ge 0} H(x) := \sup_{x \ge 0} \frac{\phi(F(x))}{\phi(x)} F'(x) \le 1 - \delta$.

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2. There are function $c_1(x) > 0, c_2(x) > 0$ (when $x > 0$) that
$$H'(x) = c_1(x) \cdot (1 - H(x)) + c_2(x) \cdot (\alpha d - (x+1)^w (F(x) + 1))$$

3. Uniqueness regime




**uniqueniss boundary**

parametric curve:
$$\begin{cases} \alpha(w) = \frac{d^w (w+1)^{w+1}}{(dw-1)^{w+1}} \\ \lambda(w) = \frac{w^d (d+1)^{d+1}}{(dw-1)^{d+1}} \end{cases}$$

On the boundary $(\alpha, \lambda)$:
$\exists$ unique $w = w_c$:
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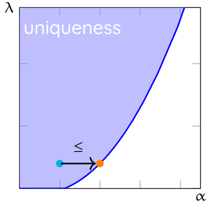


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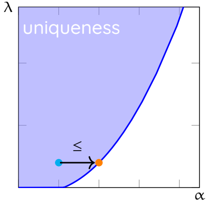


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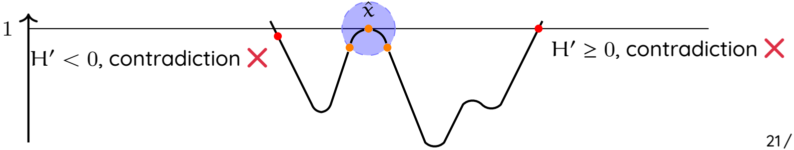
where $\hat{x}$ is the unique fixpoint.

4. Move $(\lambda, \alpha)$ to the boundary and let $w = w_c$ in $H(x) = \frac{\phi(F(x))}{\phi(x)} F'(x)$.

$$H(\hat{x}) = 1, H'(\hat{x}) = 0, H''(\hat{x}) < 0$$



# Proof outline: contraction ($\delta > 0$)

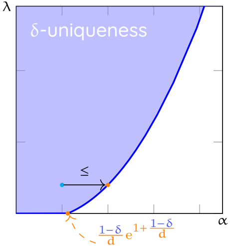
For simplicity, we assume $\delta = 0$. The $\delta > 0$ case could be handled by a similar high level idea.

1. $w$ could be eliminated $\cdots$
2. There are function $c_1(x) > 0, c_2(x) > 0$ (when $x > 0$) that

$$H'(x) = c_1(x) \cdot ((1-\delta) - H(x)) + c_2(x) \cdot B_\delta(x), \text{ where}$$

$$B_\delta(x) = w \log(x+1)\left(\alpha d \cdot \frac{x+1}{F(x)+1} - (x+1)^{w+1}\right) + \delta(x+1)(\alpha + (x+1)^w)$$

3. Uniqueness regime




No parametric equation avaliable

On the boundary $(\alpha, \lambda)$: $\exists$ unique $w = w_c$:

$$\begin{cases} (1-\delta)(\hat{x}+1)(\alpha + (1+\hat{x})^w) - \alpha d w \hat{x} = 0 \\ w \log(1+\hat{x})(\alpha d - (1+\hat{x})^{w+1}) + \delta(\hat{x}+1)(\alpha + (\hat{x}+1)^w) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} F'(\hat{x}) = 1-\delta \\ w \log(1+\hat{x})(\alpha d - (1+\hat{x})^{w+1}) + \delta(\hat{x}+1)(\alpha + (\hat{x}+1)^w) = 0 \end{cases},$$

where $\hat{x} = F(\hat{x})$ is the unique fixpoint.

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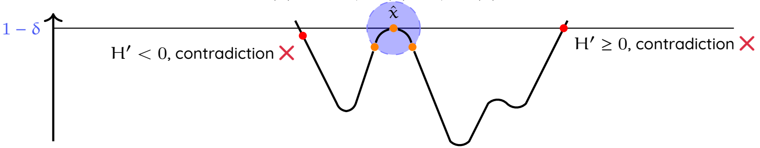

# Proof outline: $\delta$-uniqueness

$F(x) = \lambda(1+\alpha(1+x)^{-w})^{-d}$

**$\delta$-uniqueness**

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- The requiremnt on fixpoint is not easy to use 🤔.
- $(\hat{x}, d, \alpha, w)$ determines a unique $\lambda$: $\lambda(\hat{x}) = \hat{x}(1+\alpha(1+\hat{x})^{-w})^{d}$
- Change the coordinates: $(\lambda, d, \alpha, w) \leftrightarrow (\hat{x}, d, \alpha, w)$
- $F'(\hat{x}) \leq 1-\delta \Leftrightarrow$ $\underbrace{(1-\delta)(\hat{x}+1)(\alpha+(1+\hat{x})^{w}) - \alpha d w \hat{x}}_{=:T_\delta(\hat{x})} \geq 0$




has more than one fixed points

$\cup$ $F'(\hat{x}) > 1-\delta$ at some fixed point $\hat{x}$

Fix $d, \alpha, w$, a typical case is




- as $w \to +\infty$, we have $\lambda^\delta_i \to 0$
- $\lambda \geq \lambda^\delta_{2,c}$ implies $\delta$-uniqueness
- fix $d, \alpha$, critical $\hat{x}, w$ arise when $\begin{cases} T_\delta(x) = 0 \\ \partial_w \lambda^\delta_2(w) = 0 \end{cases} \overset{\delta=0}{\Longrightarrow}$ parametric curve

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The tuple $(\lambda, d, \alpha)$ is $\delta$-unique if $\forall w \in \mathbb{R}_{>0}$, and $\forall \hat{x}$, $F'(\hat{x}) \leq 1-\delta$.

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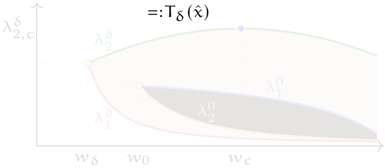
- $F'(\hat{x}) \leq 1-\delta \Leftrightarrow$

$$\underbrace{(1-\delta)(\hat{x}+1)(\alpha+(1+\hat{x})^{w}) - \alpha d w \hat{x}}_{=:T_\delta(\hat{x})} \geq 0$$




has more than one fixed points

$\cup$ $F'(\hat{x}) > 1-\delta$ at some fixed point $\hat{x}$

Fix $d, \alpha, w$, a typical case is




- as $w \to +\infty$, we have $\lambda^\delta_i \to 0$
- $\lambda \geq \lambda^\delta_{2,c}$ implies $\delta$-uniqueness
- fix $d$, $\alpha$, critical $\hat{x}, w$ arise when $\begin{cases} T_\delta(x) = 0 \\ \partial_w \lambda^\delta_2(w) = 0 \end{cases} \overset{\delta=0}{\Longrightarrow}$ parametric curve

# Proof outline: $\delta$-uniqueness

$F(x) = \lambda(1+\alpha(1+x)^{-w})^{-d}$

**$\delta$-uniqueness**

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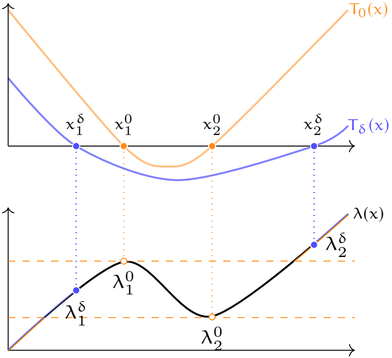


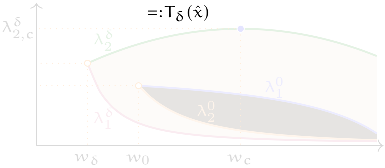


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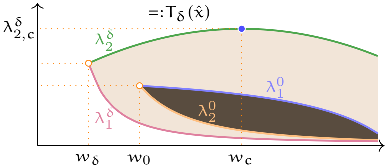


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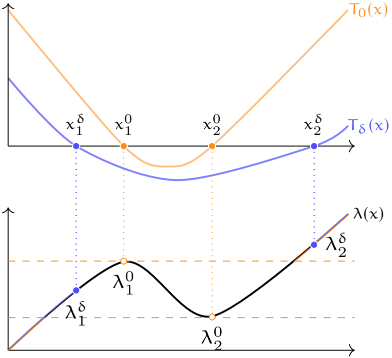


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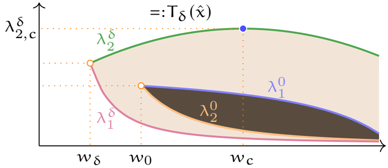


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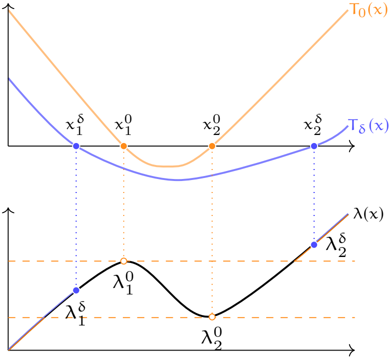


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# Thank you

arXiv:2305.00186

**Summary**

For $\delta \in (0,1)$, $\Delta_L \geq 3$, if $\lambda \leq (1-\delta)\lambda_c(\Delta_L)$, then

- the system is in the uniqueness regime
- there is a sampler that runs in time
$$T := n\left(\frac{\Delta_L \log n}{\lambda}\right)^{O(1/\delta)}$$
- the mixing time of Glauber dynamics is bounded by $O(n^2) \cdot T$

**Open problems**

- Remove the depedency on $\Delta_L$ in the running time of the sampler.
- Better mixing time for the Glauber dynamics.
- Bipartite hardcore model for negative $\lambda$.