2010 IEEE 51st Annual Symposium on Foundations of Computer Science

# Subexponential Algorithms for Unique Games and Related Problems

Sanjeev Arora
*Computer Science Dept and CCI*
*Princeton University*
*Princeton, NJ USA*
*email: arora@princeton.edu*

Boaz Barak*
*Microsoft Research and*
*Princeton University*
*email: boaz@microsoft.com*

David Steurer*
*Microsoft Research*
*Cambridge, MA USA*
*email: dsteurer@cs.princeton.edu*

***Abstract*—We give a subexponential time approximation algorithm for the Unique Games problem. The algorithms run in time that is exponential in an arbitrarily small polynomial of the input size, $n^{\varepsilon}$. The approximation guarantee depends on $\varepsilon$, but not on the alphabet size or the number of variables. We also obtain a subexponential algorithms with improved approximations for Small-Set Expansion and Multicut. For Max Cut, Sparsest Cut, and Vertex Cover, we give subexponential algorithms with improved approximations on some interesting subclasses of instances.**

**Khot's Unique Games Conjecture (UGC) states that it is NP-hard to achieve approximation guarantees such as ours for the Unique Games. While our results stop short of refuting the UGC, they do suggest that Unique Games is significantly easier than NP-hard problems such as Max 3Sat, Max 3Lin, Label Cover and more, that are believed not to have a subexponential algorithm achieving a non-trivial approximation ratio.**

**The main component in our algorithms is a new result on graph decomposition that may have other applications. Namely we show that for every $\varepsilon > 0$ and every regular $n$-vertex graph $G$, by changing at most $\varepsilon$ fraction of $G$'s edges, one can break $G$ into disjoint parts so that the stochastic adjacency matrix of the induced graph on each part has at most $n^{\varepsilon}$ eigenvalues larger than $1 - \eta$, where $\eta$ depends polynomially on $\varepsilon$.**

***Keywords*-Approximation Algorithms, Unique Games, Subexponential Algorithms, Spectral Methods, Eigenvalues, Graph Decompositions, Constraint Satisfaction Problems.**

## I. Introduction

Among the important open questions of computational complexity, Khot's Unique Games Conjecture (UGC) [20] is one of the very few that looks like it could "go either way." The conjecture states that for a certain constraint satisfaction problem, called Unique Games, it is NP-hard to distinguish between instances that are *almost satisfiable* — at least $1 - \varepsilon$ of the constraints can be satisfied — and *almost completely unsatisfiable* — at most $\varepsilon$ of the constraints can be satisfied. (See Section V for a formal definition.)

A sequence of works have shown that this conjecture has several important implications (see Khot [21] for a comprehensive survey and bibliography) in particular showing that for many important computational problems, the currently approximation algorithms have *optimal* approximation guarantees. Perhaps most strikingly, Raghavendra [32] showed that the UGC, if true, implies that every constraint satisfaction problem (CSP) has an associated *sharp approximation threshold* $\tau$: for every $\varepsilon > 0$ one can achieve a $\tau - \varepsilon$ approximation in polynomial (and in fact even quasilinear [36]) time, but obtaining a $\tau + \varepsilon$ approximation is NP-hard. But of course, these profound implications by themselves need not be any evidence for truth of the Unique Games Conjecture. The deeper reason for belief in UGC is that in trying to design algorithms for it using current techniques, such as semidefinite programs (SDPs), one seems to run into the same bottlenecks as for all the other problems alluded to above, and indeed there are results showing limitations of SDPs in solving Unique Games [23], [33], [22]. Moreover, recently it was shown that solving Unique Games is at least as hard as some other hard-looking problem — the Small-Set Expansion problem [34]. Another reason one might believe the Unique Games problem is hard is that it shares a superficial similarity with Label Cover, a problem known to be NP-hard to approximate. Our work shows that the two problems have in fact quite different complexity.

We give a subexponential algorithm for Unique Games as well as Small-Set Expansion, as explained in the next two theorems. (Sometimes "subexponential" is meant to refer to $\exp(n^{o(1)})$ time, which we do not obtain when $\varepsilon$ is a fixed constant. If we did, that would disprove the UGC under the ETH assumption explained below.)

**Theorem I.1** (See Theorem V.1)**.** *There is some absolute constant $\alpha > 0$ and an $\exp(kn^{\varepsilon^{\alpha}})$-time algorithm that given a $(1 - \varepsilon)$-satisfiable unique game of alphabet size*[1] *$k$, outputs an assignment satisfying $1 - \varepsilon^{\alpha}$ fraction of the constraints.*

**Theorem I.2** (See Theorem II.1)**.** *There is some absolute constant $\alpha > 0$ and an $\exp(n^{\varepsilon^{\alpha}}/\delta)$-time algorithm that given $\varepsilon, \delta > 0$ and a graph that has a set of measure at most $\delta$ and edge expansion at most $\varepsilon$, finds a set of measure at most $\delta$ and edge expansion at most $\varepsilon^{\alpha}$.*

[*] Research done while at the Department of Computer Science and Center for Computational Intractability, Princeton University.

[1] The *alphabet size* of a unique game is the number of symbols that each variable can be assigned. In the context of the UGC one can think of $\varepsilon$ as some arbitrarily small constant, and $k$ as some arbitrarily large constant depending on $\varepsilon$.

0272-5428/10 $26.00 © 2010 IEEE
DOI 10.1109/FOCS.2010.59

IEEE computer society

(Our results for small set expansion are slightly better quantitatively; see Theorem II.1 for more details.) In fact, our algorithm for Unique Games is obtained by extending the algorithm for Small-Set Expansion, thus giving more evidence for the connection between these two problems.

What do these results imply for the status of the UGC? In a nutshell, they still don't rule out the UGC, but imply that (1) unique-game hardness results cannot be used to establish full exponential hardness of a computational problem regardless of the truth of the UGC, and (2) even if the UGC is true then (assuming 3-Sat has fully exponential complexity) the corresponding reduction from 3-Sat to Unique Games would have to run in $n^{1/\varepsilon^{0.01}}$ time, where $\varepsilon$ is the completeness parameter of the Unique Games instance; in particular the UGC cannot be proved via a gadget reduction from Label Cover of the type pioneered by Håstad [18].

Thus Unique Games is qualitatively different from many NP-complete problems, which seem to require fully exponential time, as pointed out by Stearns and Hunt [35] and Impagliazzo, Paturi and Zane [19]. The latter paper formulated the *Exponential Time Hypothesis* (ETH) — there is no $\exp(o(n))$ algorithm for solving $n$-variable 3-Sat — and showed that it implies that many computational problems such as Clique, $k$-Colorability, and Vertex Cover require $2^{\Omega(n)}$ time as well. (Here, $n$ refers not to the input size but the size of the solution when represented as a string.)

In fact, there are very few problems whose complexity is known to be subexponential but believed not to be polynomial — two famous examples are Factoring and Graph Isomorphism problems, which can be solved in time roughly $\exp(n^{1/3})$ [27] and $\exp(\sqrt{n}\log n)$ [29] respectively.[2] Because of this paucity of counterexamples, researchers' intuition has been that "natural" problems exhibit a dichotomy — they are either in P or require fully exponential time (i.e., have essentially no nontrivial algorithms). For example the *algebraic dichotomy conjecture* of Bulatov, Jeavons and Krokhin [6] (see also [26]) says that under the ETH *every* constraint satisfaction problem is either in P or requires $2^{\Omega(n)}$ time. *Fixed parameter intractability* also tries to formalize the same phenomenon in another way.

Accumulating evidence in recent years suggested that a similar dichotomy might hold for *approximation*. To give an example, it is now known (due to efficient-PCP constructions, the last one by Moshkovitz and Raz [30]) that ETH implies that achieving $7/8+\varepsilon$-approximation to Max 3-Sat requires $2^{n^{1-o(1)}}$ time for every fixed $\varepsilon > 0$, and similar statements are true for Max-3Lin, and Label Cover. Thus it would be natural to interpret the spate of recent UGC-hardness results, especially Raghavendra's result for all CSPs, as suggesting that the same is true for many natural classes of NP-hard optimization problems such as CSPs: there are no approximation algorithms for these problems that run in subexponential time but achieve a better approximation ratio than current polynomial time algorithms. Our results show that this interpretation would be incorrect and in fact is inconsistent with the UGC since Unique Games itself — an important example of a constraint satisfaction problem in Raghavendra's class — has a subexponential time approximation algorithm that beats the polynomial time algorithms if the UGC is true. Similarly our result also refutes the NP-hardness of variants of the UGC, such as those considered by Chawla et al. [8], where the completeness parameter $\varepsilon$ is a function tending to 0 with the input length. (Curiously, our subexponential algorithm really depends on completeness parameter being close to 1; the result of Feige and Reichman [13] mentioned above rules out under the ETH a subexponential approximation algorithm for games with completeness bounded away from 1.)

While (with the exception of the Multi Cut problem) our ideas do not yet apply to problems "downstream of Unique Games" (e.g., Max Cut), we do indicate in the full version of this paper how to get better algorithms on subfamilies of interesting instances for some of these problems.

### A. Comparison with prior algorithms for Unique Games

Several works have given algorithms for approximating Unique Games. Most of these can be broadly divided into two categories: (1) polynomial-time algorithms giving relaxed notions of approximation (i.e., deteriorating as the alphabet size grows) for *all* instances [20], [38], [16], [7], [11] and (2) polynomial-time algorithms for *certain families* of instances such games whose constraint graphs are random graphs, graphs with expansion properties, and random geometric graphs [4], [25], [5], [3], [24]. An outlier to these categories is the recent work of Arora, Impagliazzo, Matthews, and Steurer [3] that gave an $\exp(2^{-\Omega(1/\varepsilon)}n)$ algorithm for unique games that are $1-\varepsilon$ satisfiable.

Compared to papers from the first category, our algorithms run in subexponential as opposed to polynomial time, but give an approximation guarantee that is independent of the alphabet size. At the moment the constant $\alpha$ in Theorem I.1 is about $1/6$, and although it could perhaps be optimized further, there are some obstacles in making it smaller than $1/2$, which means that for very small alphabet size our approximation guarantee will be worse than that of [7], that gave an algorithm that on input a $k$-alphabet $(1-\varepsilon)$-satisfiable unique game, outputs a solution satisfying $1-O(\sqrt{\varepsilon\log k})$ fraction of the constraints.

### B. Overview: Threshold rank and decompositions

Our basic approach for the Unique Games algorithm is divide and conquer (similarly to [38] and [3]): Partition the

[2] Of course there also exist NP-hard problems with subexponential algorithms (e.g. finding an $n^{\varepsilon}$-clique / $n^{1-\varepsilon}$ approximation for clique). There are also examples without brute force search such as subset sum on $n$ integers of $\sqrt{n}$ bits, and $\log^{1.99} n$ approximation to the group Steiner tree problem (these were pointed out to us by Russell Impagliazzo and Anupam Gupta respectively; see [10] as reference for the group Steiner tree problem).

constraint graph of the Unique Games instance into disjoint blocks, throw out all constraints corresponding to edges that go between blocks, and solve independently within each block. However, the underlying "divide" step involves a new notion of graph decomposition, which we now explain.

Consider the adjacency matrix of an undirected regular graph $G$, whose every row/column is scaled to 1. (In other words, a *symmetric stochastic matrix.*) Our algorithm will use the fact that graphs with only a few eigenvalues close to 1 are "simple" because exhaustive enumeration in the subspace spanned by the corresponding eigenvalues will quickly give a good-enough solution, as explained below. Thus "complex" graphs are those with many eigenvalues close to 1. The core of our result is a new way of partitioning *every* graph into parts that are "simple." This decomposition result seems different from existing notions of partitions such as *Szemerédi's regularity lemma* [37], *low-width cut decomposition of matrices* [14], *low-diameter decompositions* [28] and *padded decompositions* [17]. The first two of the above notions really only apply to dense or pseudo-dense graphs, not all graphs. The latter two apply to all graphs but involve a "penalty term" of $O(\log n)$ that is too expensive in our setting (see discussion following Theorem I.4).

For $\tau \in [0, 1)$ let the *$\tau$-threshold rank* of $G$, denoted $\mathrm{rank}_\tau(G)$, be the number (with multiplicities) of eigenvalues $\lambda$ of $G$ satisfying $|\lambda| > \tau$. Thus $\mathrm{rank}_0(G)$ coincides with the usual rank of the matrix $G$, i.e., number of non-zero eigenvalues. We will usually be interested in $\tau$ close to 1, say 0.9. The higher the parameter $\mathrm{rank}_\tau(G)$, the "more complex" $G$ is for us. Unlike many existing notions of "rank" or "complexity", $\mathrm{rank}_\tau(G)$ is small –actually, 1 — for a random graph, and more generally is 1 for any good expander. This should not be viewed as a bug in the definition: after all, expander graphs and random graphs are easy instances for problems such as Unique Games [4] and Small-Set Expansion. In fact, very recently Kolla [24], extended [25]'s proof of this result to give an algorithm for unique games that runs in time exponential in the threshold rank of the corresponding constraint graph (assuming a certain bound on the $\ell_\infty$ norm of the eigenvectors).[3] Our algorithm uses a very simple version of the key step in [24], [25], see Section II-A.

*Relating threshold rank and small-set expansion:* The basic result underlying our graph decomposition algorithm is the following inequality that relates the threshold rank and small set expansion:

**Theorem I.3** (Rank/expansion tradeoff, see Theorem II.3)**.** *If $G$ is an $n$ vertex regular graph in which every set $S$ of at most $s$ vertices has edge expansion at least* 0.1 *(i.e., at least* 0.1 *fraction of $S$'s edges go to $[n] \setminus S$), then*

$$\mathrm{rank}_{1-\varepsilon}(G) \cdot s \leqslant n^{1+O(\varepsilon)} .$$

*Furthermore, there is a polynomial-time algorithm that given any graph $G$ will recover a set of size $n^{1+O(\varepsilon)}/\mathrm{rank}_{1-\varepsilon}(G)$ with edge expansion at most* 0.1.

This result can be seen as a generalization of Cheeger's inequality [9], [1]. The usual Cheeger's inequality would yield a non-expanding set in the graph if there is even a single eigenvalue close to 1, but this set might be as large as half the vertices, while we obtain a set that (up to $n^{O(\varepsilon)}$ slackness) of measure inversely proportional to the number of large eigenvalues. Theorem I.3 directly implies a simple "win-win" algorithm for the small set expansion problem. Either the $(1 - \varepsilon)$-threshold rank is larger than $n^{c\varepsilon}$ for some large constant $c$, in which case we can find a very small (say of size less than $n^{1-\varepsilon}$) non-expanding set in polynomial time. Or, in the spirit of [25], [24], we can do in $\exp(n^{O(\varepsilon)}$-time a brute force enumeration over the span of the eigenvectors with eigenvalues larger than $1 - \varepsilon$, and we are guaranteed to find if some non-expanding set $S$ exists in the graph then we will recover $S$ (up to a small error) via this enumeration, see Theorem II.2.

*Threshold-rank decomposition:* By applying Theorem I.3 repeatedly and recursively, we obtain our decomposition result:

**Theorem I.4** (Threshold-rank decomposition theorem, see Theorem IV.1)**.** *There is a constant $c$ and a polynomial-time algorithm that given an $n$ vertex regular graph $G$ and $\varepsilon > 0$, partitions the vertices of $G$ to sets $A_1, \ldots, A_q$ such that the induced*[4] *graph $G_i$ on $A_i$ satisfies $\mathrm{rank}_{1-\varepsilon^c}(G_i) \leqslant n^\varepsilon$ and at most a $\varepsilon$ fraction of $G$'s edges have their endpoints in different sets of this partition.*

Key to this decomposition is the advantage Theorem I.3 has over Cheeger's inequality. Since every application of Cheeger's Inequality might leave us with half the vertices, one generally needs $\Omega(\log n)$ recursion depth to get a partition where each block has, say, size $\sqrt{n}$. This could end up removing all the edges unless $\varepsilon = O(1/\log n)$. In contrast, using Theorem I.3 (or rather its more precise variant Theorem II.3) we can get to such a partition with using a constant (depending on $\varepsilon$) depth of recursion.

Our Unique Games algorithm is obtained from Theorem I.4 as follows. Given a Unique Games instance, we apply Theorem I.4 to partition it (after removing a small fraction

[3] Specifically, Kolla and Tulsiani [25] gave an algorithm that finds a satisfying assignment in time exponential in the threshold rank of the *label extended graph* of the unique game (see Section V, [25, § 6], [24, Thm 5]) and used it to obtain a polynomial time algorithm for unique games on expanders. [24] showed that in certain cases bounds on the threshold rank of the constraint graph translate into bounds on the threshold rank of the label extended graph, hence allowing to use the [25] algorithm in these cases as well. In this work we observe a more general, though quantitatively worse, relation between the threshold ranks of the label-extended and constraint graphs, see Corollary V.3.

[4] The notion of "induced graph" we use involves "regularizing" the graph via self-loops, see Section IV for the precise definition.

of the constraints) into disjoint parts each having small rank. We then look at the *label extended graph* corresponding to every part. (This is the graph that contains a "cloud" of $k$ vertices for every variable of a $k$-alphabet unique game, and there is a matching between pairs of clouds according to the corresponding permutation, see Section V.) We use the previously known observation that a satisfying assignment corresponds to a non-expanding set in the label extended graph, and combine it with a new observation (Lemma V.2) relating the threshold rank of the label extended graph and the corresponding constraint graph. The result then follows by enumerating the span of the top eigenvectors (suitably discretized) to recover (up to some small noise) the satisfying assignment in every part.

*Proof of the rank/expansion relation:* Now we give some intuition behind Theorem I.3, which underlies all this. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the graph's eigenvalues. Let us pick $\tau = 1 - \eta$ for a small enough $\eta$ and suppose $m = \mathrm{rank}_\tau(G)$. Then (the $2k$-th power of) the *Schatten* $2k$*-norm* $\mathrm{Tr}(G^{2k}) = \sum_{i \leqslant n} |\lambda_i|^{2k}$ is at least $m(1-\eta)^{2k}$. On the other hand, $\mathrm{Tr}(G^{2k})$ is also equal to $\sum_{i \leqslant n} \|G^k e_i\|_2^2$, where $e_i$ is the unit vector whose only nonzero coordinate is the $i$-th. But $\|G^k e_i\|_2^2$ simply expresses the collision probability of a $k$-step random walk that starts from $i$. Then we can use a "local" version of Cheeger's inequality (in this form due to Dimitriou and Impagliazzo [12]), which shows that if all small sets expand a lot, then the collision probability of the $k$-step random walk decays very fast with $k$. We conclude that if all small sets expand a lot, then the expression in $\|G^k e_i\|_2^2$ must be small, which yields an upper bound on $m(1-\eta)^{2k}$, and hence on the threshold rank $m$.

A related bound (in the other direction) connecting Schatten norms to small-set expansion was shown by Naor [31], who used the Schatten norm to certify small-set expansion of Abelian Cayley graphs (or more generally, graphs with $\ell_\infty$ bounded eigenvectors).

### C. Organization of the paper

The main ideas of this work appear in the simplest form in the subexpontial algorithm for Small-Set Expansion that is described in Section II. The main component used is Theorem II.3, showing that small set expanders must have low threshold rank. This theorem is proven in Section III.

Section IV contains our decomposition theorem, which is used in our algorithm for Unique Games appearing in Section V. Complete proofs of lemmas can be found in a full version of this paper found on ECCC. The full version also contains some partial results for other computational problems such as Multi Cut, and for interesting subcases of Max Cut and Vertex Cover. We show that *hypercontractive graphs*, that appear in many of the known integrality gap examples, have much smaller threshold rank than the bound guaranteed by Theorem II.3, and use this to show a quasipolynomial time algorithm for efficiently certifying that a hypercontractive graph is a small-set expander.

### D. Notation

Throughout this paper we restrict our attention to *regular* undirected graphs only, though we allow self loops and weighted edges (as long as the sum of weights on edges touching every vertex is the same). In our context this is without loss of generality, because the computational problems we are interested reduce easily to the regular case (as explained in the complete version).[5] We consider graphs with vertex set $V = [n] = \{1, \ldots, n\}$, and use $G$ to denote both the graph and its stochastic walk matrix. We use $(i, j) \sim G$ to denote the distribution obtained by choosing a random edge of $G$ (i.e., obtaining $(i, j)$ with probability $G_{i,j}/n$). We define the measure of a subset $S \subseteq [n]$, denoted $\mu(S)$, to be $|S|/n$. For $S, T \subseteq [n]$ we denote $G(S,T) := \frac{1}{n} \sum_{i \in S, j \in T} G_{i,j} = \mathbb{P}_{(i,j) \sim G}[i \in S, j \in T]$. The *expansion* of a subset $S$ of the vertices of a graph $G$, denoted $\Phi_G(S)$, is defined as $G(S, \overline{S})/\mu(S) = \mathbb{P}_{(i,j) \in G}[j \in \overline{S} | i \in S]$, where $\overline{S} = [n] \setminus S$. We will often drop the subscript $G$ and use only $\Phi(S)$ when the graph is clear from the context.

For $x, y \in \mathbb{R}^n$ we let $\langle x, y \rangle = \mathbb{E}_{i \in [n]}[x_i y_i]$. We define the corresponding 2-norm and 1-norm as $\|x\| = \sqrt{\langle x, x \rangle}$ and $\|x\|_1 = \mathbb{E}_{i \in [n]}[|x_i|]$. Note that $\Phi(S) = 1 - \langle \chi_S, G\chi_S \rangle$ where $\chi_S$ is the normalized characteristic vector of $S$, that is $\chi_S(i) = \sqrt{n/|S|}$ if $i \in S$ and $\chi_S(i) = 0$ otherwise. Indeed, $\langle \chi_S, G\chi_S \rangle = G(S,S)/\mu(S)$.

We say that $f(n) = \exp(g(n))$ if there is some constant $c$ such that $f(n) \leqslant 2^{c \cdot g(n)}$ for every sufficiently large $n$. Throughout this paper, the implicit constants used in $O(\cdot)$ notation are absolute constants, independent of any other parameters.

## II. An algorithm for Small-Set Expansion

In this section we give a subexponential algorithm for Small-Set Expansion. Specifically we prove the following theorem.

**Theorem II.1** (Subexponential algorithm for Small-Set Expansion)**.** *For every $\beta \in (0,1)$, $\varepsilon > 0$, and $\delta > 0$, there is an $\exp(n^{O(\varepsilon^{1-\beta})})\,\mathrm{poly}(n)$-time algorithm that on input a regular graph $G$ with $n$ vertices that has a set $S$ of at most $\delta n$ vertices satisfying $\Phi(S) \leqslant \varepsilon$, finds a set $S'$ of at most $\delta n$ vertices satisfying $\Phi(S) \leqslant O(\varepsilon^{\beta/3})$.*

Note that by setting $\beta = O(1/\log(1/\varepsilon))$ we can get an $\exp(n^{O(\varepsilon)})$-time algorithm that given a graph with a small set of expansion at most $\varepsilon$, finds a small set of expansion at most, say, 0.01.

[5] For simplicity, we *define* the Small-Set Expansion problem only for regular graphs, although one can easily generalize the definition and our results to non-regular graphs, assuming the measure of each vertex is weighted by its degree.

We prove Theorem II.1 by combining two methods. First we show that if the input graph has at most $m$ large eigenvalues then one can find the non expanding set (if it exists) in time $\exp(m)$. Then we show that if a graph has many eigenvalues that are fairly large then it *must* contain a small set with poor expansion, and in fact there is an efficient way to find such a set. The algorithm is obtained by applying one of these methods to the input graph depending on the number of eigenvalues larger than $1 - \eta$ (for some suitably chosen threshold $\eta$).

### *A. Enumerating non-expanding sets in low-rank graphs*

We start by showing that the search for a non-expanding set in a graph can be greatly sped up if it has only few large eigenvalues. The following result is an analog of [25], [24]'s algorithms for unique games whose label-extended graphs have few large eigenvalues (see also Footnote 3).

**Theorem II.2** (Subspace enumeration)**.** *There is an* $\exp(\operatorname{rank}_{1-\eta}(G))\operatorname{poly}(n)$*-time algorithm that given* $\varepsilon > 0$ *and a graph* $G$ *containing a set* $S$ *with* $\Phi(S) \leqslant \varepsilon$*, outputs a sequence of sets, one of which has symmetric difference at most* $8(\varepsilon/\eta)|S|$ *with the set* $S$*.*

In particular for $\varepsilon < 0.01$ and $\eta = 1/2$ the algorithm will output a list of sets containing a set $S'$ such that $|S'| \leqslant 1.1|S|$ and $\Phi(S') \leqslant 13\varepsilon$.

*Proof:* Let $\delta = \mu(S) = |S|/n$, and let $\chi_S$ be the normalized indicator vector of $S$, that is $\chi_S(i) = 1/\sqrt{\delta}$ if $i \in S$ and $\chi_S(i) = 0$ otherwise. Let $U \subseteq \mathbb{R}^V$ be the span of the eigenvectors with eigenvalue greater than $1-\eta$. The dimension of $U$ is equal to $m = \operatorname{rank}_{1-\eta}(G)$. Suppose $\chi_S = u + u^{\perp}$, where $u \in U$ and $u^{\perp}$ is orthogonal to $U$ (and hence $u^{\perp}$ is in the span of the eigenvectors with eigenvalue at most $1-\eta$). Since $\Phi(S) \leqslant \varepsilon$, we have

$$\varepsilon \geqslant 1 - \langle \chi_S, G\chi_S \rangle = 1 - \langle u, Gu \rangle - \langle u^{\perp}, Gu^{\perp} \rangle \geqslant \eta \|u^{\perp}\|^2 ,$$

where the last step uses $\langle u, Gu \rangle \leqslant \|u\|^2$ and $\langle u^{\perp}, Gu^{\perp} \rangle \leqslant (1-\eta)\|u^{\perp}\|^2$, as well as $\|u\|^2 + \|u^{\perp}\|^2 = 1$. Hence, the distance of $\chi_S$ to the subspace $U$ is bounded $\|\chi_S - u\|^2 = \|u^{\perp}\|^2 \leqslant \varepsilon/\eta$. If we enumerate over a $\sqrt{\varepsilon/100\eta}$-net in the unit ball of the $m$-dimensional subspace $U$, then we will find a vector $v$ satisfying $\|v - \chi_S\|_2^2 \leqslant 2\varepsilon/\eta$. (The size of such a net in $U$ is at most $\exp(m\log(1/\varepsilon))$.) Thus, at most $8\delta\varepsilon/\eta$ fraction of the coordinates of $v$ differ from $\chi_S$ by more than $1/(2\sqrt{\delta})$. Let $S' = S'(v)$ be the set defined by setting $i \in S'$ if $v_i \geqslant 1/(2\sqrt{\delta})$ and $i \notin S'$ otherwise. Every coordinate $i$ in the symmetric difference between $S$ and $S'$ corresponds to a coordinate in which $v$ and $\chi_S$ differ by at least $1/(2\sqrt{\delta})$ and so the symmetric difference of $S$ and $S'$ has measure at most $8\varepsilon\delta/\eta$. ■

*Remark:* While the above proof is basically a variant of the "easy direction" of Cheeger's inequality (showing that if $\lambda_2 \leqslant 1-\varepsilon$ then every set $S$ of size at most $n/2$ satisfies $\Phi(S) \geqslant \varepsilon/2$), it is actually a competitor of the "hard direction", which produces a set $S$ with $\Phi(S) \leqslant O(\sqrt{\varepsilon})$ in a graph where $\lambda_2 \geqslant 1-\varepsilon$. Our algorithm finds a set with $\Phi(S) \leqslant O(\varepsilon)$ albeit via brute force enumeration.

### *B. Finding small non-expanding sets in high-rank graphs*

Our next step is to show that every graph with high threshold-rank contains a small non-expanding vertex set.

**Theorem II.3** (Rank bound for small-set expanders)**.** *Let* $G$ *be a regular graph on* $n$ *vertices such that* $\operatorname{rank}_{1-\eta}(G) \geqslant n^{100\eta/\gamma}$*. Then there exists a vertex set* $S$ *of size at most* $n^{1-\eta/\gamma}$ *that satisfies* $\Phi(S) \leqslant \sqrt{\gamma}$*. Moreover,* $S$ *is a level set of a column of* $(\frac{1}{2}\cdot I + \frac{1}{2}\cdot G)^j$ *for some* $j \leqslant O(\log n)$*, where a* level set *of a vector* $x \in \mathbb{R}^V$ *is a set of the form* $\{i \in V \mid x_i \geqslant \tau\}$ *for some threshold* $\tau \in \mathbb{R}$*.*

One can think of Theorem II.3 as a generalization of the "difficult direction" of Cheeger's Inequality. The latter says that if $\operatorname{rank}_{1-\eta}(G) > 1$ then there exists a set $S$ with $\mu(S) \leqslant 1/2$ and $\Phi(S) \leqslant O(\sqrt{\eta})$. Theorem II.3 gives the same guarantee, but in addition the measure of the set $S$ is inversely proportional to the threshold rank (i.e., number of large eigenvalues), assuming this rank is larger than $n^{\Omega(\eta)}$. We note that we have made no attempt to optimize the constants of this theorem, and in fact do not know if the constant 100 above cannot be replaced with $1 + o(1)$ (though such a strong bound, if true, will require a different proof).

We now combine Theorem II.2 and Theorem II.3 to obtain our subexponential algorithm for Small-Set Expansion, namely Theorem II.1.

*Proof of Theorem II.1:* Set $\eta = \varepsilon^{1-\beta/3}$ and $\gamma = \varepsilon^{2\beta/3}$. If $\operatorname{rank}_{1-\eta}(G) \geqslant n^{100\eta/\gamma}$, then we can compute in polynomial time a set of size at most $n^{1-\eta/\gamma} \ll \delta n$ and expansion at most $10\sqrt{\gamma} = O(\varepsilon^{\beta/3})$ by Theorem II.3. Otherwise, if $\operatorname{rank}_{1-\eta}(G) < n^{100\eta/\gamma}$, we can compute in time $\exp\left(n^{O(\eta/\gamma)}\log(1/\varepsilon)\right) = \exp\left(n^{O(\varepsilon^{1-\beta})}\right)$ a set with measure at most $(1 + O(\varepsilon/\eta))\delta \leqslant 2\delta$ and expansion at most $O(\varepsilon/\eta) = O(\varepsilon^{\beta/3})$. ■

## III. Threshold-rank bounds for small-set expanders

As mentioned, the proof of Theorem II.3 goes by looking at $\operatorname{Tr}(G^k)$. Define the *k-Schatten norm*, denoted $S_k(M)$, of a symmetric matrix $M$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ to be $\lambda_1^k + \ldots + \lambda_n^k = \operatorname{Tr}(M^k)$ by the trace formula. We say a graph $G$ is *lazy* if $G = 1/2 \cdot G' + 1/2 \cdot I$ for some regular graph $G'$. (In other words, $G$ is lazy if $G \geqslant 1/2 \cdot I$ entry-wise.) For technical reasons, we will prove a Schatten norm bound only for lazy graphs. (This bound will suffice to prove Theorem II.3 also for non-lazy regular graphs.)

**Theorem III.1** (Schatten norm bound)**.** *Let* $G$ *be a lazy regular graph on* $n$ *vertices. Suppose every vertex set* $S$ *with* $\mu(S) \leqslant \delta$ *satisfies* $\Phi(S) \geqslant \varepsilon$*. Then, for all even* $k \geqslant 2$*, the* $k$*-Schatten norm of* $G$ *satisfies*

$$S_k(G)^k \leqslant \max\left\{ n \cdot \left(1 - \varepsilon^2/32\right)^k, \tfrac{4}{\delta} \right\} .$$

*Moreover, for any graph that does not satisfy the above bound, we can compute in polynomial time a vertex subset $S$ with $\mu(S) \leqslant \delta$ and $\Phi(S) \leqslant \varepsilon$, where $S$ is a level set of a column of $G^j$ for some $j \leqslant k$.*

Before proving Theorem III.1, lets see how it implies Theorem II.3

*Proof of Theorem II.3 from Theorem III.1:* Let $G, \eta, \gamma$ be as in the theorem, and let $m = \operatorname{rank}_{1-\eta}(G)$. (Note that we can assume $\eta < 100\gamma$ as otherwise the statement is trivial.) Set $G' = {}^1\!/\!{}_2 I + {}^1\!/\!{}_2 G$ to be the "lazy version" of $G$ and note that (1) for every set $S$, $\Phi_{G'}(S) = \Phi(S)/2$ and (2) since every eigenvalue $\lambda$ in $G$ translates to an eigenvalue ${}^1\!/\!{}_2 + {}^1\!/\!{}_2\lambda$ in $G'$, $m = \operatorname{rank}_{1-\eta/2}(G')$. Now set $k$ to be such that $(1-\gamma/64)^k = 1/n$ and $\delta = n^{-\eta/\gamma}$ and apply Theorem III.1 to $G', k$ with $\varepsilon = \sqrt{\gamma}/2$. We get that if $\Phi_{G'}(S) \leqslant \sqrt{\gamma}/2$ for every $S$ of measure at most $\delta$, then

$$m(1-\eta/2)^k \leqslant S_k(G')^k \leqslant 4/\delta = 4n^{\eta/\gamma}\,.$$

(Since the first term in the max expression is 1.) Now use $(1-\eta/2) \sim (1-\gamma/64)^{64\eta/(2\gamma)} \geqslant (1-\gamma/64)^{65\eta/\gamma}$ (in the range we care about) to argue that $m(1-\gamma/64)^{k65\eta/\gamma} \leqslant 4n^{\eta/\gamma}$, but by our choice of $k$, we get $mn^{-65\eta/\gamma} \leqslant 4n^{\eta/\gamma}$, or (assuming $n^{\eta/\gamma} \gg 4$) $m \leqslant n^{100\eta/\gamma}$. Moreover, if $G'$ violates the last inequality, we can find efficiently a level set $S$ of a column $G'^j$ that will satisfy $\mu(S) \leqslant \delta$ and $\Phi_{G'}(S) = \Phi_G(S)/2 \leqslant \sqrt{\gamma}/2$. ■

*A trace bound:* The proof of Theorem II.3 actually achieves a somewhat stronger statement. Define the $(1-\eta)$ *trace threshold rank* of $G$, denoted $\operatorname{rank}^*_{1-\eta}(G)$, to be the infimum over $k \in \mathbb{N}$ of $\operatorname{Tr}(G^{2k})/(1-\eta)^{2k}$. Clearly $\operatorname{rank}_{1-\eta}(G) \leqslant \operatorname{rank}^*_{1-\eta}(G)$, since $\operatorname{Tr}(G^{2k}) \geqslant \operatorname{rank}_{1-\eta}(G)(1-\eta)^{2k}$. Because our proof bounds the rank of $G$ via the trace of the lazy graph ${}^1\!/\!{}_2 I + {}^1\!/\!{}_2 G$, it actually achieves the following statement:

**Theorem III.2** (Trace rank bound for small-set expanders)**.** *Let $G$ be a regular lazy graph on $n$ vertices such that $\operatorname{rank}^*_{1-\eta}(G) \geqslant n^{100\eta/\gamma}$. Then there exists a vertex set $S$ of size at most $n^{1-\eta/\gamma}$ that satisfies $\Phi(S) \leqslant \sqrt{\gamma}$. Moreover, $S$ is a level set of a column of $G^j$ for some $j \leqslant O(\log n)$.*

One can also show that the trace rank bound is not too far from the threshold rank, in the range of parameters of interest in this work:

**Lemma III.3.** *For every $\delta, \eta \in (0,1)$, $\operatorname{rank}^*_{1-\delta\eta}(G) \leqslant \operatorname{rank}_{1-\eta}(G) n^{5\delta}$.*

*Proof:* For every $k$, one can see by the definition of $\operatorname{rank}^*$ and the formula $\operatorname{Tr}(G^{2k}) = \sum_{i=1}^n |\lambda_i|^{2k}$ that

$$\operatorname{rank}^*_{1-\delta\eta}(G)(1-\eta\delta)^{2k} \leqslant \operatorname{Tr}(G^{2k}) \leqslant \operatorname{rank}_{1-\eta}(G) + n(1-\eta)^{2k}$$

plugging in $k = \log n/\eta$ we get that $\operatorname{rank}^*_{1-\delta\eta}(G) n^{-4\delta} \leqslant \operatorname{rank}_{1-\eta}(G) + 1$. ■

### A. Proof of Theorem III.1

In the following, we let $G$ be a fixed lazy graph with vertex set $V = [n]$. Recall that we identify $G$ with its stochastic adjacency matrix. The proof of Theorem III.1 is based on the relation of the following parameter to Schatten norms and the expansion of small sets,

$$\Lambda(\delta) \overset{\text{def}}{=} \max_{x \in \Omega_\delta} \frac{\|Gx\|}{\|x\|}\,.$$

Here, the set $\Omega_\delta \subseteq \mathbb{R}^V$ is defined as

$$\Omega_\delta \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^V \mid 0 < \|x\|_1^2 \leqslant \delta \cdot \|x\|^2 \right\}.$$

By Cauchy–Schwarz, every vector with support of measure at most $\delta$ is contained in $\Omega_\delta$.

Since the spectral radius of $G$ is at most 1, the parameter $\Lambda(\delta)$ is upper bounded by 1 for all $\delta > 0$. The following lemma shows that if $G$ is an expander for sets of measure at most $\delta$, then $\Lambda(\delta)$ is bounded away from 1. (In fact, small-set expansion is equivalent to $\Lambda(\delta)$ being bounded away from 1. However, we only need one direction of this equivalence for the proof.)

**Lemma III.4.** *Suppose $\Phi(S) \geqslant \varepsilon$ for all sets $S$ of measure at most $\delta$. Then, $\Lambda(\delta/4) \leqslant 1 - \varepsilon^2/32$. Moreover, if $x \in \Omega_{\delta/4}$ is a unit vector such that $\|Gx\| > 1 - \varepsilon^2/32$, then there exists a level set $S$ of $x$ such that $\mu(S) \leqslant \delta$ and $\Phi(S) < \varepsilon$.*

The proof of Lemma III.4 combines a few standard techniques (Cheeger's inequality with Dirichlet boundary conditions and a truncation argument; see for example [15], [2]). A variant of this lemma that is strong enough for our purposes was given by Dimitriou and Impagliazzo [12].

Next, we obtain a bound on Schatten norms in terms of the parameter $\Lambda(\delta)$. We need the following simple technical lemma, which almost follows immediately from the definition of $\Lambda(\delta)$.

**Lemma III.5.** *For every $j \in \mathbb{N}$, $x \in \mathbb{R}^V$, and $\delta > 0$,*

$$\|G^j x\| \leqslant \max\left\{ \Lambda(\delta)^j \cdot \|x\|\,,\ \tfrac{1}{\sqrt{\delta}} \cdot \|x\|_1 \right\}. \tag{1}$$

*Proof:* Indeed, suppose that $\|G^j x\| \geqslant \|x\|_1/\sqrt{\delta}$. Then, since $G$ is stochastic and hence $\|Gy\|_2 \leqslant \|y\|_2$ and $\|Gy\|_1 \leqslant \|y\|_1$ for all $y$,

$$\begin{aligned}\|x\|_2 \geqslant \|Gx\|_2 \geqslant \cdots \geqslant \|G^{j-1}x\|_2 \geqslant \|G^j x\|_2 \geqslant \|x\|_1/\sqrt{\delta} \\ \geqslant \|Gx\|_1/\sqrt{\delta} \geqslant \cdots \geqslant \|G^{j-1}x\|_1/\sqrt{\delta} \geqslant \|G^j x\|_1/\sqrt{\delta}\,.\end{aligned}$$

Therefore, we see that $G^i x \in \Omega_\delta$ for all $i \in \{0, \ldots, j\}$, which implies that

$$\|G^j x\| \leqslant \Lambda(\delta)\|G^{j-1}x\| \leqslant \Lambda(\delta)^2\|G^{j-1}x\| \leqslant \ldots \leqslant \Lambda(\delta)^j\|x\|\,.$$

■

With this lemma, we can prove the following bound on Schatten norms in terms of the parameter $\Lambda(\delta)$.

**Lemma III.6.** *For every even integer $k \geqslant 2$,*

$$S_k(G)^k \leqslant \max\left\{n \cdot \Lambda(\delta)^k, \tfrac{1}{\delta}\right\}.$$

*Proof:* Let $e_1, \ldots, e_n$ be the normalized standard basis vectors, that is, the $j$-th coordinate of $e_i$ is equal to $\sqrt{n}$ if $i = j$ and equal to 0 otherwise. Note that $\|e_i\| = 1$ and $\|e_i\|_1 = 1/\sqrt{n}$. Using the identity $S_k(G)^k = \mathrm{Tr}(G^k)$, we obtain

$$S_k(G)^k = \mathrm{Tr}(G^k) = \sum_{i=1}^{n} \langle e_i, G^k e_i\rangle = \sum_{i=1}^{n} \langle G^{k/2} e_i, G^{k/2} e_i\rangle$$

$$= \sum_{i=1}^{n} \|G^{k/2} e_i\|^2 \leqslant n \cdot \max\left\{\Lambda(\delta)^k, \tfrac{1}{n\delta}\right\},$$

where the last inequality uses Lemma III.5. ■

Lemma III.4 and Lemma III.6 immediately imply Theorem III.1 by noting that under the condition of the theorem, $\Lambda(\delta/4) \geqslant 1 - \varepsilon^2/32$, hence implying that $S_k(G)^k \leqslant \max\{n(1 - \varepsilon^2/32)^k, 4/\delta\}$. Moreover, following the proof we see that if the condition is violated then we can get a set $S$ with $|S| \leqslant \delta n$ and $\Phi(S) \leqslant \varepsilon$ by looking at a level set of the vector $G^j e_i$ for some $j \leqslant k$ and standard basis vector $e_i$.

## IV. Low threshold-rank decomposition of graphs

Our main technical tool for extending our Small-Set Expansion algorithm to Unique Games is an algorithm to decompose a graph into parts with low threshold rank.

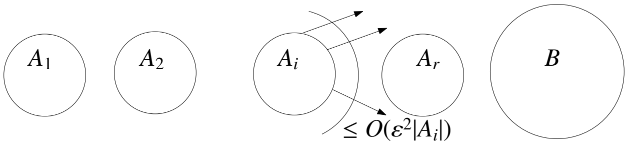
We will use the following notation. For a graph $G$, a *partition* of the vertices $V = V(G)$ is a function $\chi : V \to \mathbb{N}$. We will not care about the numerical values of $\chi$ and so identify $\chi$ with the family of disjoint sets $\{\chi^{-1}(i)\}_{i \in \mathrm{Image}(\chi)}$. The size of the partition $\chi$, denoted by $\mathrm{size}(\chi)$ is the number of sets/colors it contains. We define the *expansion* or *cost* of the partition, denoted $\Phi(\chi)$, to be the fraction of edges $i, j$ for which $\chi(i) \neq \chi(j)$. If $G$ is a $d$-regular graph and $U \subseteq V(G)$, we let $G[U]$ be the induced graph on $U$ that is "regularized" by adding to every vertex sufficient number of weight half self loops to achieve degree $d$. Our decomposition result is the following:

**Theorem IV.1** (Low threshold rank decomposition theorem)**.** *There is a polynomial time algorithm that on input a graph $G$ and $\varepsilon > 0$, outputs a partition $\chi = (A_1, \ldots, A_q)$ of $V(G)$ such that $\Phi(\chi) \leqslant O(\varepsilon \log(1/\varepsilon))$ and for every $i \in [q]$, $\mathrm{rank}_{1-\varepsilon^5}(G[A_i]) \leqslant n^{100\varepsilon}$.*

### *A. Proof of the decomposition theorem (Theorem IV.1)*

The proofs of lemmas below can be found in the complete version of this paper.

We start with some notation. Throughout the proof we'll fix the graph $G$ (which we also think of as a stochastic matrix) on the set $V = [n]$ of vertices. If $U \subseteq V$, and $S \subseteq U$, then the *relative* expansion of $S$ with respect to $U$, denoted by $\Phi_U(S)$, is defined as $\mathbb{P}_{(i,j)\sim G}\{j \in U \setminus S \mid i \in S\}$. Note that $\Phi_U(S)$ is equal to the expansion of $S$ in the graph $G[U]$. (Recall that $G[U]$ is regularized by adding self-loops.) If $\chi = (S_1, \ldots, S_q)$ is a partition of $V$, then we define the relative expansion of $\chi$ with respect to $U$, denoted as $\Phi_U(\chi)$, to be $\sum_{i=1}^{q} \Phi_U(S_i \cap U)$. We say that $\chi$ *refines* a partition $\tau$ if $\chi(x) = \chi(y) \Rightarrow \tau(x) = \tau(y)$. We define the relative cost of $\chi$ with respect to $\tau$, denoted $\Phi_\tau(\chi)$ to be the fraction of edges that are cut in $\chi$ but not in $\tau$. That is, $\Phi_\tau(\chi)$ is equal to $1/n$ times the sum of $G_{i,j}$ over all $i, j$ such that $\chi(i) \neq \chi(j)$ but $\tau(i) = \tau(j)$. Note that if $\tau = (S_1, \ldots, S_q)$ then $\Phi_\tau(\chi) = \sum_{i=1}^{q} \mu(S_i)\Phi_{S_i}(\chi)$. It's not hard to verify that $\Phi(\chi) \leqslant \Phi(\tau) + \Phi_\chi(\tau)$ with equality if $\chi$ is a refinement of $\tau$.



We prove Lemma IV.3 by repeatedly using Lemma IV.2 to remove from the graph sets $A_1, \ldots, A_r$ that are "somewhat small" (size $\leqslant n^{1-\varepsilon}$) until the remaining part $B$ has $1 - \varepsilon^5$ threshold rank at most $n^{100\varepsilon}$. To count the expansion of the partition we orient all edges to the right in the figure and charge each edge $\overrightarrow{u\,v}$ crossing the partition to the set $A_i$ containing $u$. Lemma IV.2 guarantees that the weight of directed edges from $A_i$ to $A_{i+1} \cup \cdots \cup B$ is at most $O(\varepsilon^2|A_i|)$. Theorem IV.1 is an immediate implication of Lemma IV.4, and the latter is proven by recursively applying Lemma IV.3 to each of the sets $A_i$ up to $\varepsilon^{-1}\log(1/\varepsilon)$ times until the only non-expander parts remaining are "very small" (size $\leqslant n^{\varepsilon}$). The overall cost of the partition is $O(\varepsilon^2\varepsilon^{-1}\log(1/\varepsilon)) = O(\varepsilon\log(1/\varepsilon))$.

Figure 1. Overview of proof of Theorem IV.1.

The proof will be obtained by a sequence of three lemmas, see Figure 1 for an overview. We start with the following instantiation of our small set expander algorithm:

**Lemma IV.2.** *Let $G$ be an $n$ vertex graph, and $\varepsilon > 0$. If $\mathrm{rank}_{1-\varepsilon^5}(G) \leqslant n^{100\varepsilon}$ then we can find in polynomial time a set $S \subseteq V(G)$ with $|S| \leqslant n^{1-\varepsilon}$ and $\Phi(S) \leqslant O(\varepsilon^2)$.*

*Proof:* Instantiate Theorem II.3 with $\gamma = \varepsilon^4$ and $\eta = \varepsilon^5$. ■

The precise powers of $\varepsilon$ in Lemma IV.2 are not important. The main point is that the set $S$ in the conclusion of the lemma satisfies $\log_n(1/\mu(S)) \gg \Phi(S)$ (which is the case because $\log_n(1/\mu(S)) = \varepsilon$ and $\phi(S) = O(\varepsilon^2)$).

Next, we apply Lemma IV.2 repeatedly to obtain a *partition* of the graph into sets that are either somewhat small, or are small set expanders.

**Lemma IV.3.** *There is a polynomial-time algorithm that given an $n$ vertex graph $G$ and $\varepsilon > 0$, outputs a partition $\chi = (A_1, \ldots, A_r, B)$ of $V(G)$, such that $\Phi(\chi) \leqslant O(\varepsilon^2)$, $|A_i| \leqslant n^{1-\varepsilon}$ for all $i \in [r]$, and $\mathrm{rank}_{1-\varepsilon^5}(G[B]) \leqslant n^{100\varepsilon}$.*

*Proof:* We start with an empty partition $\chi$ and will repeatedly add sets to $\chi$ until we cover $V = V(G)$. Suppose we have already obtained the sets $A_1, \ldots, A_{i-1}$. Let $U_i = V \setminus (A_1 \cup \cdots \cup A_{i-1})$. We run the algorithm of Lemma IV.2 on

$G[U_i]$. If it fails to return anything we add the set $B = U_i$ to the partition and halt. (In this case, Lemma IV.2 guarantees that $\mathrm{rank}_{1-\varepsilon^5}(G[B]) \leqslant n^{100\varepsilon}$.) Otherwise, the algorithm returns a set $A_i \subseteq U_i$ with $|A_i| \leqslant |U_i|^{1-\varepsilon} \leqslant n^{1-\varepsilon}$ and $\Phi_{U_i}(A) \leqslant O(\varepsilon^2)$. We continue in this way until we have exhausted all of $V$. Let $\chi = (A_1, \dots, A_r, B)$ be the partition that we obtained in this way. Note that

$$\Phi(\chi) = 2\sum_{i=1}^{r} G(A_i, A_{i+1} \cup \cdots \cup A_r \cup B) = 2\sum_{i=1}^{r} G(A_i, U_i \setminus A_i) .$$

But since $G(A_i, U_i \setminus A_i) = \Phi'_U(A_i)\mu(A_i) \leqslant O(\varepsilon^2\mu(A_i))$, we can upper bound the cost of $\chi$ as desired $\Phi(\chi) \leqslant O(\varepsilon^4 \sum_{i=1}^r \mu(A_i)) \leqslant O(\varepsilon^2)$ (using $\sum_{i=1}^r \mu(A_i) = 1$). ■

The idea for the next lemma is to apply Lemma IV.3 recursively until we obtain a partition of the vertices into sets $A_i$ are very small ($|A_i| \ll n^\varepsilon$) and sets $B_i$ that are small-set expanders. To achieve the bound on the size of the sets, it is enough to recurse up to depth $O(\log(1/\varepsilon)/\varepsilon)$. In each level of the recursion, we cut at most an $O(\varepsilon^2)$ fraction of edges. Hence, the total fraction of edges that we cut across all levels is at most $O(\varepsilon\log(1/\varepsilon))$. (For this argument, it was important that the algorithm of Lemma IV.2 outputs set with $\log_n(1/\mu(S)) \gg \Phi(S)$.)

Note that the next lemma immediately implies Theorem IV.1 (since the threshold rank of a subgraph is at most the number of its vertices).

**Lemma IV.4.** *There is an algorithm that given an $n$ vertex graph $G$, and $\varepsilon > 0$, outputs a partition $\chi = (A_1, \dots, A_r, B_1, \cdots, B_{r'})$ of $[n]$, such that $\Phi(\chi) \leqslant O(\varepsilon\log(1/\varepsilon))$, $|A_i| \leqslant n^\varepsilon$ for all $i \in [r]$, and $\mathrm{rank}_{1-\varepsilon^5}(G[B_j]) \leqslant n^{100\varepsilon}$ all $j \in [r']$.*

*Proof:* We let $\chi_0$ be the trivial partition of one set (with $\Phi(\chi_0) = 0$) and will continually refine the partition using Lemma IV.3 until we reach the desired form. Now for $i = 0, 1, \dots, 10\log(1/\varepsilon)/\varepsilon$ we repeat the following steps. As long as $\chi_i$ does not satisfy the above form, then for every set $A$ of $\chi_i$ that satisfies $\mathrm{rank}_{1-\varepsilon^5}(G[A]) \geqslant n^{100\varepsilon} \geqslant |A|^{100\varepsilon}$ we run Lemma IV.3 to obtain a partition $\chi_A$ of $A$ with $\Phi_A(\chi_A) \leqslant O(\varepsilon^2)$. We then let $\chi_{i+1}$ be the partition obtained by refining every such set $A$ in $\chi_i$ according to $\chi_A$. Note that we maintain the invariant that in $\chi_i$, every set $A$ such that $\mathrm{rank}_{1-\varepsilon^5}(G[A]) \geqslant n^{100\varepsilon}$ has size at most $n^{(1-\varepsilon)^i}$. Thus, after $10\log(1/\varepsilon)/\varepsilon$ iterations every such set will have size at most $n^\varepsilon$. At the end we output the final partition $\chi = \chi_{10\log(1/\varepsilon)/\varepsilon}$. It just remains to bound $\Phi(\chi)$. To do that it suffices to prove that $\Phi(\chi_{i+1}) \leqslant \Phi(\chi_i) + O(\varepsilon^2)$, since this implies $\Phi(\chi) \leqslant O(\varepsilon^2 \cdot \log(1/\varepsilon)/\varepsilon) = O(\varepsilon\log(1/\varepsilon))$. So we need to prove $\Phi_{\chi_i}(\chi_{i+1}) \leqslant O(\varepsilon^2)$. But indeed, if we let $A_1, \dots, A_r$ be the sets in $\chi_i$ that $\chi_{i+1}$ refines, then one can see that

$$\Phi_{\chi_i}(\chi_{i+1}) = \textstyle\sum_{j=1}^{b} \mu(A_j)\Phi_{A_j}(\chi_{A_j}) \leqslant O(\varepsilon^2) ,$$

where the last inequality follows from $\sum \mu(A_j) \leqslant 1$ and the guarantee $\Phi_A(\chi_{A_j}) \leqslant O(\varepsilon^2)$ provided by Lemma IV.3. ■

*Trace rank bound:* Note that by using Theorem III.2 instead of Theorem II.3, if we assume the original graph is lazy, then we can get a partition of small trace threshold-rank instead of threshold rank. (One just needs to note that if $G$ is lazy then $G[A]$ is lazy as well for every subset $A$ of $G$'s vertices.) Thus our proof actually yields the following theorem as well:

**Theorem IV.5** (Low trace threshold rank decomposition theorem)**.** *There is a polynomial time algorithm that on input a graph $G$ and $\varepsilon > 0$, outputs a partition $\chi = (A_1, \dots, A_q)$ of $V(G)$ such that $\Phi(\chi) \leqslant O(\varepsilon\log(1/\varepsilon))$ and for every $i \in [q]$, $\mathrm{rank}^*_{1-\varepsilon^5}(G[A_i]) \leqslant n^{100\varepsilon}$.*

## V. A subexponential algorithm for Unique Games

In this section we give a subexponential algorithm for Unique Games. A *unique game* of $n$ variables and alphabet $k$ is an $n$ vertex graph $G$ whose edges are labeled with permutations on the set $[k]$, where the edge $(i, j)$ is labeled with $\pi$ iff the edge $(j, i)$ is labeled with $\pi^{-1}$. An *assignment* to the game is a string $y = (y_1, \dots, y_n) \in [k]^n$, and the *value* of $y$ is the fraction of edges $(i, j)$ for which $y_j = \pi(y_i)$, where $\pi$ is the label of $(i, j)$. The *value* of the game $G$ is the maximum value of $y$ over all $y \in [k]^n$.

**Theorem V.1** (Subexponential algorithm for Unique Games)**.** *There is an $\exp(kn^{O(\varepsilon)})\,\mathrm{poly}(n)$-time algorithm that on input a unique game $G$ on $n$ vertices and alphabet size $k$ that has an assignment satisfying $1 - \varepsilon^6$ of its constraints outputs an assignment satisfying $1 - O(\varepsilon\log(1/\varepsilon))$ of the constraints.*

### A. Proof of Theorem V.1.

We assume the unique game constraint graph is $d$-regular for some $d$ — this is without loss of generality. For a unique game $G$, the *label extended graph* of $G$, denoted $\hat{G}$, is a graph on $nk$ vertices, where for $i, j \in [n]$ and $a, b \in [k]$ we place an edge between $(i, a)$ and $(j, b)$ iff there is an edge $(i, j)$ in $G$ labeled with a permutation $\pi$ such that $\pi(a) = b$. That is, every vertex $i \in V(G)$ corresponds to the "cloud" $C_i := \{(i, 1), \dots, (i, k)\}$ in $V(\hat{G})$. We say that $S \subseteq V(\hat{G})$ is *conflict free* if $S$ intersects each cloud in at most one vertex. Note that a conflict free set $S$ in $\hat{G}$ corresponds to a partial assignment $f = f_S$ for the game $G$ (i.e., a partial function from $V(G)$ to $[k]$). We define the *value* of a partial assignment $f$, denoted $\mathrm{val}(f)$, to be $2/(nd)$ times the number of labeled edges $(i, j, \pi)$ such that both $f(i)$ and $f(j)$ are defined, and $\pi(f(i)) = f(j)$.

We say that a unique game is *lazy* if each vertex has half of its constraints as self loops with the identity permutation. We use the following simple lemma, whose proof consists of observing that if there is a length $t$ walk in $\hat{G}$ from the vertex $(i, a)$ back to itself then there must be a corresponding length $t$ walk from $i$ back to itself in $G$ (and in fact one where composing the corresponding permutation yields a permutation that has $a$ as a fixed point).

**Lemma V.2.** *Suppose that $G$ is lazy. Then* $\mathrm{rank}^*_{1-\eta}(\hat{G}) \leqslant k \cdot \mathrm{rank}^*_{1-\eta}(G)$.

Combining this with Lemma III.3 we get the following corollary:

**Corollary V.3.** *For every $\delta, \eta$ and $n$ vertex constraint graph $G$ on alphabet $k$,* $\mathrm{rank}_{1-\delta\eta}(\hat{G}) \leqslant kn^{\eta}\mathrm{rank}_{1-\eta}(G)$.

Because of Lemma V.2, we will find it convenient to use the *trace* threshold rank partitioning algorithm of Theorem IV.5. We note that we could have instead used Corollary V.3 instead, at some quantitative loss to the parameters. The main idea is that any assignment that satisfies $(1-\varepsilon)$ fraction of the unique game constraints corresponds in the label extended graph to a set of about $1/k$ of all the vertices, and expansion $\varepsilon$. Our earlier subspace enumeration algorithm outputs *all* non-expanding subsets, so it will also find this one (or one close to it, which yields almost as good an assignment). Now we give the algorithm:

**Input:** Unique game $G$ on $n$ variables of alphabet $k$ that has value at least $1-\varepsilon^6$.

1) Make $G$ lazy by adding to every vertex self loops accounting to half the weight labeled with the identity permutation.
2) Run the partition algorithm of Theorem IV.5 to obtain a partition $\chi = \{A_1, \dots, A_q\}$ of the graph $G$ with $\Phi(\chi) \leqslant O(\varepsilon \log(1/\varepsilon))$ such that for every $i$, $\mathrm{rank}^*_{1-\varepsilon^5}(G[A_i]) \leqslant n^{100\varepsilon}$.
3) Let $\hat{A}_1, \dots, \hat{A}_q$ be the corresponding partition of the label-extended graph $\hat{G}$. Note that for all $t \in [q]$, $\hat{G}[A_t] = G[\hat{A}_t]$ and hence by Lemma V.2 $\mathrm{rank}_{1-\varepsilon^5}(\hat{G}[A_i]) \leqslant \mathrm{rank}^*_{1-\varepsilon^5}(\hat{G}[A_i]) \leqslant kn^{100\varepsilon}$.
4) For every $t = 1 \dots q$ do the following:
    i) Run the $\exp(\mathrm{rank}_{1-\varepsilon^5}(\hat{G}[A_t])$-time enumeration algorithm of Theorem II.2 on the graph $\hat{G}[\hat{A}_t]$ to obtain a sequence of sets $\mathcal{S}_t$.
    ii) For every set $S \in \mathcal{S}_t$, we compute an assignment $f_S$ to the vertices in $A_t$ as follows: For every $i \in A_t$, if $C_i \cap S = \emptyset$, then $f_S$ assigns an arbitrary label to the vertex $i$, if $|C_i \cap S| > 0$, then $f_S$ assigns one of the labels in $C \cap S$ to the vertex $i$. Let $f_t$ be the assignment of maximum value, and assign the variables corresponding to vertices in $A_t$ according to $f_t$. (Note that since the sets $A_1, \dots, A_q$ are disjoint, every variable will be assigned exactly one label.)

We now turn to analyze the algorithm. We assume the game has an assignment $f_{\mathrm{opt}}$ satisfying $1-\varepsilon^6$ of the constraints. Note that $f_{\mathrm{opt}}$ still has the same value, and in fact even somewhat better — $1-\varepsilon^6/2$ — after we make the graph lazy. Let $\chi = (A_1, \dots, A_t)$ be the partition obtained by the algorithm in Step 2. Since $\Phi(\chi) \leqslant 1/2$, the assignment $f_{\mathrm{opt}}$ satisfies at least $1-2(\varepsilon^6/2) = 1-\varepsilon^6$ of the constraints that are not cut by $\chi$. Let $\mu_t$ be the measure of $A_t$ (also equalling the measure of $\hat{A}_t$), and let $\varepsilon_t$ be the fraction of constraints in $A_t$ that are violated by $f_{\mathrm{opt}}$. We know that $\sum_{t=1}^{q} \mu_t\varepsilon_t \leqslant 2\varepsilon^6$.

The following lemma implies that the algorithm will output an assignment satisfying at least $1-O(\varepsilon)$ fraction of the constraints:

**Lemma V.4.** *Every partial assignment $f_t$ satisfies all but a $20\varepsilon_t/\eta$ fraction of the constraints in $A_t$.*

*Proof:* Let $S_{\mathrm{opt}}$ be the subset of $\hat{A}_t$ corresponding to the assignment $f_{\mathrm{opt}}$. Note that $|S_{\mathrm{opt}}| = |A_t|$ and $\Phi_{\hat{A}_t}(S_{\mathrm{opt}}) \leqslant \varepsilon_t$. Thus, the sequence $\mathcal{S}_t$ contains a set $S$ that has symmetric difference with $S_{\mathrm{opt}}$ at most $8(\varepsilon_t/\eta)|A_t|$ (Theorem II.2). Let $S'$ be the subset of $\hat{A}_t$ corresponding to the assignment $f_S$. The construction of $f_S$ (and thus $S'$) ensures that the symmetric difference between $S'$ and $S$ is at most the symmetric difference between $S$ and $S_{\mathrm{opt}}$. (In fact, the symmetric difference of $S$ and $S'$ is equal to $\sum_{i \in A_t} ||S \cap C_i| - 1|$.) Hence, $S'$ has symmetric difference with $S_{\mathrm{opt}}$ at most $16(\varepsilon_t/\eta)|A_t|$. In other words, $f_S$ agrees with $f_{\mathrm{opt}}$ on all but a $16\varepsilon_t/\eta$ fraction of the vertices in $A_t$. Thus $f_S$ violates at most $\varepsilon_t + 16\varepsilon_t/\eta \leqslant 20\varepsilon_t/\eta$ of the constraints in $A_t$. The lemma follows because we choose $f_t$ as the best assignment among all assignments $f_S$ for $S \in \mathcal{S}_t$. ■

Lemma V.4 implies that among the constraints not cut by $\chi$, the assignment we output satisfies all but a

$$\sum_t \mu_t \cdot 20\varepsilon_t/\eta = (20/\eta)\sum_u \mu_i\varepsilon_t = O(\varepsilon^6/\varepsilon^5) = O(\varepsilon)$$

fraction of constraints.

Since $\chi$ cuts at most $O(\varepsilon\log(1/\varepsilon))$ fraction of the constraints, the correctness of the algorithm and thus Theorem V.1 follow. (One also has to note that any solution satisfying $1-\gamma$ fraction of the lazy game's constraints satisfies at least $1-2\gamma$ fraction of the original game's constraints.) □

## VI. Conclusions and Open Questions

The obvious open question is whether our methods can be extended to yield an $\exp(n^{o(1)})$-time algorithms for Unique Games, hence refuting the Unique Games Conjecture. More generally one can ask what is the true complexity of the Unique Games and Small-Set Expansion problems? Any quantitative improvement to the bounds of Theorem II.3 would translate to an improvement in our algorithm for the Small-Set Expansion problem, and so will result in refuting the stronger variant of the UGC proposed in [34]. Another open question is whether our techniques can yield subexponential algorithms with better approximation guarantees for unique-games hard problems such as Vertex Cover, Max Cut, Sparsest Cut on every instance.

## Acknowledgements

This work stemmed from a suggestion of Assaf Naor that eigenvalue distribution, and in particular Schatten norms, could be related to small set expansion, and his related manuscript [31]. We thank Alexandra Kolla for giving us an early copy of the manuscript [24]. The authors also had a number of very fruitful conversations on this topic with several people including Moritz Hardt, Thomas Holenstein, Russell Impagliazzo, Guy Kindler, William Matthews, Prasad Raghavendra, and Prasad Tetali. This work was financially supported by NSF, Packard and Sloan foundations.

## References

[1] N. Alon and V. D. Milman, "$\lambda_1$, isoperimetric inequalities for graphs, and superconcentrators," *J. Combin. Theory Ser. B*, vol. 38, no. 1, pp. 73–88, 1985. [Online]. Available: http://dx.doi.org/10.1016/0095-8956(85)90092-9

[2] N. Alon and B. Klartag, "Economical toric spines via Cheeger's inequality," *J. Topol. Anal.*, vol. 1, no. 2, pp. 101–111, 2009. [Online]. Available: http://dx.doi.org/10.1142/S1793525309000096

[3] S. Arora, R. Impagliazzo, W. Matthews, and D. Steurer, "Improved algorithms for unique games via divide and conquer," ECCC, Tech. Rep. ECCC TR10-041, 2010.

[4] S. Arora, S. Khot, A. Kolla, D. Steurer, M. Tulsiani, and N. Vishnoi, "Unique games on expanding constraints graphs are easy," in *Proc. 40th STOC*. ACM, 2008.

[5] B. Barak, M. Hardt, T. Holenstein, and D. Steurer, "Subsampling mathematical relaxations and average-case complexity," 2010, manuscript.

[6] A. Bulatov, P. Jeavons, and A. Krokhin, "Classifying the complexity of constraints using finite algebras," *SIAM J. Comput*, vol. 34, no. 3, pp. 720–742, 2005, prelim version in ICALP '00.

[7] M. Charikar, K. Makarychev, and Y. Makarychev, "Near-optimal algorithms for unique games," in *STOC*, 2006, pp. 205–214.

[8] S. Chawla, R. Krauthgamer, R. Kumar, Y. Rabani, and D. Sivakumar, "On the hardness of approximating multicut and sparsest-cut," *Computational Complexity*, vol. 15, no. 2, pp. 94–114, 2006, prelim version in CCC 2005.

[9] J. Cheeger, "A lower bound for the smallest eigenvalue of the Laplacian," in *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*. Princeton Univ. Press, 1970, pp. 195–199.

[10] C. Chekuri and M. Pal, "A recursive greedy algorithm for walks in directed graphs," in *Proc. FOCS*, 2005, pp. 245–253.

[11] E. Chlamtac, K. Makarychev, and Y. Makarychev, "How to play unique games using embeddings," in *Proc. 47th FOCS*. Citeseer, 2006, pp. 687–696.

[12] T. Dimitriou and R. Impagliazzo, "Go with the winners for graph bisection," in *Proc 9th SODA*, 1998, pp. 510–520.

[13] U. Feige and D. Reichman, "On systems of linear equations with two variables per equation," *Proc. of RANDOM-APPROX*, pp. 117–127, 2004.

[14] A. M. Frieze and R. Kannan, "Quick approximation to matrices and applications," *Combinatorica*, vol. 19, no. 2, pp. 175–220, 1999.

[15] S. Goel, R. Montenegro, and P. Tetali, "Mixing time bounds via the spectral profile," *Electron. J. Probab.*, vol. 11, pp. no. 1, 1–26 (electronic), 2006.

[16] A. Gupta and K. Talwar, "Approximating unique games," in *Proc. 17th SODA*. ACM, 2006, p. 106.

[17] A. Gupta, R. Krauthgamer, and J. R. Lee, "Bounded geometries, fractals, and low-distortion embeddings," in *FOCS*, 2003, pp. 534–543.

[18] J. Håstad, "Some optimal inapproximability results," *J. ACM*, vol. 48, no. 4, pp. 798–859, 2001, prelim version STOC '97.

[19] R. Impagliazzo, R. Paturi, and F. Zane, "Which problems have strongly exponential complexity?" *Journal of Computer and System Sciences*, vol. 63, no. 4, pp. 512–530, 2001.

[20] S. Khot, "On the power of unique 2-prover 1-round games," in *Proceedings of 34th STOC*. New York: ACM Press, 2002, pp. 767–775.

[21] ——, "On the unique games conjecture (invited survey)," *Computational Complexity, Annual IEEE Conference on*, vol. 0, pp. 99–121, 2010.

[22] S. Khot and R. Saket, "SDP integrality gaps with local $\ell_1$-embeddability," in *FOCS*, 2009, pp. 565–574.

[23] S. Khot and N. K. Vishnoi, "The unique games conjecture, integrality gap for cut problems and embeddability of negative type metrics into $\ell_1$," in *FOCS*, 2005, pp. 53–62.

[24] A. Kolla, "Spectral algorithms for unique games," in *Proc. CCC*, 2010, to appear.

[25] A. Kolla and M. Tulsiani, "Playing random and expanding unique games," 2007, unpublished manuscript available from the authors' webpages, to appear in journal version of [4].

[26] G. Kun and M. Szegedy, "A new line of attack on the dichotomy conjecture," in *Proceedings of the 41st annual ACM symposium on Symposium on theory of computing*. ACM New York, NY, USA, 2009, pp. 725–734.

[27] A. K. Lenstra and H. W. J. Lenstra, *The Development of the Number Field Sieve*. Springer-Verlag, 1993.

[28] N. Linial and M. E. Saks, "Low diameter graph decompositions," *Combinatorica*, vol. 13, no. 4, pp. 441–454, 1993.

[29] E. M. Luks, "Isomorphism of graphs of bounded valence can be tested in polynomial time," *J. Comput. Syst. Sci.*, vol. 25, no. 1, pp. 42–65, 1982.

[30] D. Moshkovitz and R. Raz, "Two query PCP with sub-constant error," in *Proc. 49th FOCS*, 2008, pp. 314–323.

[31] A. Naor, "On the Banach space valued Azuma inequality and small set isoperimetry in Alon-Roichman graphs," 2004, unpublished manuscript.

[32] P. Raghavendra, "Optimal algorithms and inapproximability results for every csp?" in *Proc. 40th STOC*. ACM, 2008, pp. 245–254.

[33] P. Raghavendra and D. Steurer, "Integrality gaps for strong SDP relaxations of Unique Games," in *FOCS*, 2009, pp. 575–585.

[34] ——, "Graph expansion and the unique games conjecture," in *STOC*, 2010, to appear.

[35] R. Stearns and H. Hunt III, "Power indices and easier hard problems," *Math. Syst. Theory*, vol. 23, no. 4, pp. 209–225, 1990.

[36] D. Steurer, "Fast SDP algorithms for constraint satisfaction problems," in *SODA*, 2010, to appear.

[37] E. Szemerédi, "Regular partitions of graphs," *Problèmes combinatoires et théorie des graphes, Orsay*, 1976.

[38] L. Trevisan, "Approximation algorithms for unique games," in *FOCS*, 2005, pp. 197–205.