MANCHESTER 1824
The University of Manchester

MATH19661 Mathematics 1M1 2021-22
1st Semester - Week 12

Tutorial Class

Xinye Chen xinye.chen@manchester.ac.uk

Scheme

- Tutorial problems: Additional ODEs
- Exam review

Tutorial problems: Additional ODEs

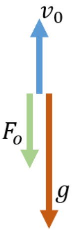
12.1. Consider an object moving through the air under the force of gravity and with a linear model for air resistance. If this object begins at the ground with a given initial velocity upwards, the goal of this problem is to determine the time when it returns to the ground.

(i) **Suppose that g is the acceleration due to gravity,** m **is the mass of the object, the magnitude of the force of air resistance is** $\gamma|v|$ **where v is the velocity upwards and** γ **is a constant, and** v_0 **is the initial velocity. Explain why**

$$\frac{dv}{dt} = -g - \frac{\gamma}{m}v, \quad v(0) = v_0$$

Tutorial problems: Additional ODEs

12.1. (i)



According to Newton's law, we have

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} \rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F}{m} \tag{1}$$

Tutorial problems: Additional ODEs

12.1. (i) The total force in the upward direction which is the opposite direction of velocity v, $F = F_o + mg$. Substitute $|F_o| = |\gamma v|$ into (1) and get $\frac{dv}{dt} = -g - \frac{\gamma}{m}v$ which subject to initial condition $v(0) = v_0$. 12.1. (ii) Solve the ODE from part (i) to determine v as a function of time.

Solution 1: Separation of variables:

$$\int \frac{1}{g + \frac{\gamma}{m}v} \mathrm{d}v = -\int \mathrm{d}t, \frac{m}{\gamma} \ln |g + \frac{\gamma}{m}v| = -t + C_0$$

Set $C = \pm e^{C_0}$, then

$$\frac{\gamma}{m}v + g = Ae^{-\gamma t/m} \Rightarrow v = Ce^{-\gamma t/m} - \frac{mg}{\gamma}$$

Tutorial problems: Additional ODEs

12.1. (ii) Solution 2: Using an integrating factor:
Rewrite the equation $\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{\gamma}{m}v$,

$$\frac{dv}{dt} + \frac{\gamma}{m}v = -g,$$

so the integrating factor is

$$I = e^{\gamma t/m}.$$

After multiplying by I, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\gamma t/m}v) = -ge^{\gamma t/m}.$$

Integrating this gives

$$v = Ce^{-\gamma t/m} - \frac{mg}{\gamma} \quad (C \text{ is the constant}).$$

Tutorial problems: Additional ODEs

12.1. (i) Both solutions give the same result, after that, we need to apply the initial condition $v(0) = v_0$ to get the concrete constant C:

$$v_0 = A - \frac{mg}{\gamma} \to A = v_0 + \frac{mg}{\gamma}.$$

Hence, the solution is

$$v = (v_0 + \frac{mg}{\gamma})e^{-\gamma t/m} - \frac{mg}{\gamma}. \tag{2}$$

The University of Manchester

Tutorial problems: Additional ODEs

12.1. (iii) **Suppose that x is the height of the object above the ground. Assuming $x(0) = 0$, find x as a function of time t.**
As we know,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v.$$

Then we substitute it into $v = (v_0 + \frac{mg}{\gamma})e^{-\gamma t/m} - \frac{mg}{\gamma}$ and have,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (v_0 + \frac{mg}{\gamma})e^{-\gamma t/m} - \frac{mg}{\gamma}.$$

Integrate and get

$$x = -\frac{m}{\gamma}(v_0 + \frac{mg}{\gamma})e^{-\gamma t/m} - \frac{mg}{\gamma}t$$

Tutorial problems: Additional ODEs

12.1. (iii) From $x(0) = 0$, accordingly, we have $C = \frac{m}{\gamma}(v_0 + \frac{mg}{\gamma})e^{-\gamma t/m}$.

Therefore,

$$x = \frac{m}{\gamma}(v_0 + \frac{mg}{\gamma})(1 - e^{-\gamma t/m}) - \frac{mg}{\gamma}t$$

Tutorial problems: Additional ODEs

12.1. (iv) **If t is the time when the object returns to the ground, explain why t satisfies the equation**

$$0 = \frac{m}{\gamma}(v_0 + \frac{mg}{\gamma})(1 - e^{-\gamma t/m}) - \frac{mg}{\gamma}t$$

Can you suggest how you could use this equation to approximate t in particular cases (i.e. when m, □and v_0 are given specific values)?

From (iii) we have

$$x(t) = \frac{m}{\gamma}(v_0 + \frac{mg}{\gamma})(1 - e^{-\gamma t/m}) - \frac{mg}{\gamma}t$$

When the object returns to the ground, denoted as time T, we have $x(t = T) = 0$. Therefore, we have $0 = \frac{m}{\gamma}(v_0 + \frac{mg}{\gamma})(1 - e^{-\gamma T/m}) - \frac{mg}{\gamma}T$.

The University of Manchester

Tutorial problems: Additional ODEs

12.1. (iv) This equation is nonlinear equation, so there is no analytical solution, we could the bisection or Newton-Raphson method to approximate T.

Exam problems: 2018

1. (d) **Use Macaulay brackets to write a formula for the function**

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & 1 < x < 3 \\ 0 & 3 < x \end{cases} .$$

Exam problems: 2018

1. (d) For simple piecewise function,

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & \text{otherwise} \end{cases} .$$

$$g(x) = \begin{cases} 0 & x < 3 \\ 1 & \text{otherwise} \end{cases} .$$

It is easy to get the formula with Macaulay brackets

$$f(x) = \langle x - 1 \rangle^0$$

$$g(x) = \langle x - 3 \rangle^0$$

The University of Manchester

Exam problems: 2018

1. (d) Further,

$$F(x) - g(x) = \begin{cases} F(x) & x < 3 \\ F(x) - 1 & \text{otherwise} \end{cases} .$$

Make $F(x) = f(x)$, then

$$f(x) - g(x) = \begin{cases} 0 & x < 1 \\ 1 & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases} .$$

$$f(x) - g(x) = \langle x - 1 \rangle^0 - \langle x - 3 \rangle^0$$

Exam problems: 2018

2. (a) **Use Simpson's rule with 4 intervals to find an approximation to the integral $\int_0^1 \frac{1}{1+x^3}\mathrm{d}x$. (work to 3 d.p.)**

Review Simpson's rule,

$$\int_a^b f(x)\mathrm{d}x \approx \frac{\delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + ... + 4y_{n-1} + y_n)$$

where $\delta x = \frac{b-a}{n}$ and n must be even.

For your convenience, we can summarize this formula as $\int_a^b f(x)\mathrm{d}x \approx \frac{\delta x}{3}[\text{Firs term} + 4(\text{Sum of Odds})) + 2(\text{Sum of Evens}) + \text{Last term}]$

Exam problems: 2018

2. (a) $x_0 = 0$, $x_1 = 1/4$, $x_3 = 3/4$, $x_4 = 1$; $h = 1/4$. Rounding to 5 d.p., we solve the corresponding values of y: $y_0 = 1$, $y_1 = 0.985$, $y_2 = 0.703$, $y_3 = 0.5$.

According to Simpson's rule, we have

$$\int_0^1 \frac{1}{1+x^3}\mathrm{d}x \approx \frac{h}{3}(y_0 + y_1 + y_2 + y_3) = 0.386$$

Exam problems: 2018

(b) Use the Newton-Raphson method to estimate $10^{1/3}$ to 3 d.p.

Exam problems: 2018

(b) Use the Newton-Raphson method to estimate $10^{1/3}$ to 3 d.p.

Let $x = 10^{1/3}$, we use Newton-Raphson method to solve the equation $x^3 - 10 = 0$. Clearly, $f(x) = x^3 - 10$. $f'(x) = 3x^2$. According to Newton-Raphson method, each iteration is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 10}{3x_k^2}. \tag{3}$$

Exam problems: 2018

The University of Manchester

(b) Use the Newton-Raphson method to estimate $10^{1/3}$ to 3 d.p.

Start with $x_0 = 2$ cause this the closest value to 10.

1. $x_1 = 2 - \frac{2^3-10}{3\times 2^2} = 2.16667.$
2. $x_2 = x_1 - \frac{x_1^3-10}{3\times x_1^2} = 2.15450.$
3. $x_3 = x_2 - \frac{x_2^3-10}{3\times x_2^2} = 2.15444.$

Stop since x_2 and x_3 share the same in the first three digits past the decimal point, so the approximation of $10^{1/3}$ is 2.154 (3 d.p.).

Exam problems: 2018

3. The line L_1 passes through the points A and B and the line L_2 passes through the points A and C. The plane P is the unique plane containing the three points A, B, and C. The coordinates of the points are

$$A(0,1,1), B(1,1,0), C(-1,0,0)$$

(a) Find a vector equation for the plane P.

We start off getting two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in the plane P. Correspondingly,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1,0,-1)^T$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-1,-1,-1)^T$$

Let (x,y,z) be the coordinates of the point P in the plane, i.e., set any vectors end with the point on the plane P by $(x,y,z)^T$.

Exam problems: 2018

(a) Find a vector equation for the plane P.

To the best of our knowledge, any three vectors in a plane are linearly dependent. $\overrightarrow{AP} = (x, y-1, z-1)$ Therefore, we have

$$\left| \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & -1 \\ x & y-1 & z-1 \end{bmatrix} \right| = 0$$

We solve this by $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \cdot \vec{x} = 1$

There are other solutions, for example, we first solve the normal vector $\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC}$ of plane P, i.e., the vector perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$. After that, we use the fact that $\vec{N} \cdot \overrightarrow{AP} = 0$ to get the solution.

Exam problems: 2018

(a) Find a vector equation for the plane P.

The norm vector $\overrightarrow{N}$ is the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

we can calculate norm vector $\overrightarrow{N}$ first by

$$\overrightarrow{N} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & -1 \\ i & j & k \end{bmatrix}$$

i, j, k correspond to the unit vector (associated with coordinates) of norm vector.

Exam problems: 2018

(b) Find parametrisations for the lines L_1 and L_2.

A parametrisation for L_1

$$\overrightarrow{OA} + t\overrightarrow{AB} = \begin{bmatrix} t \\ 1 \\ 1-t \end{bmatrix}$$

A parametrisation for L_2

$$\overrightarrow{OA} + t\overrightarrow{AC} = \begin{bmatrix} -t \\ 1-t \\ 1-t \end{bmatrix}$$

Exam problems: 2018

(c) Find the positive angle $\theta \leq \pi/2$ between lines L_1 and L_2 at the point A where they intersect.

Obviously,

$$\cos(\theta) = \frac{|\overrightarrow{AB} \cdot \overrightarrow{AC}|}{|\overrightarrow{AB}||\overrightarrow{AC}|} = 0.$$

which means $\theta = \frac{\pi}{2}$.

Exam problems: 2018

(d) Describe in words the geometric significance of the quantity

$$|\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC})|.$$

Exam problems: 2018

(d) Describe in words the geometric significance of the quantity

$$|\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC})|.$$

The magnitude of the cross product $\overrightarrow{OB} \times \overrightarrow{OC}$ is

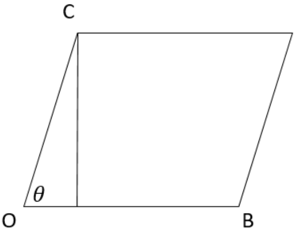
$$|\overrightarrow{OB} \times \overrightarrow{OC}| = |\overrightarrow{OB}||\overrightarrow{OC}| \sin\theta$$

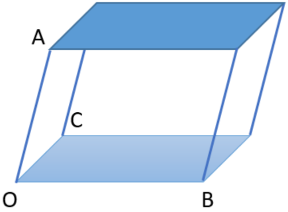
$\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC})$ is the scalar triple product, its absolute value gives the volume of the prarllelepiped with three edges given by $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$.

Exam problems: 2018

(d) Describe in words the geometric significance of the quantity

$$|\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC})|.$$





Exam problems: 2018

4. (b) Use phasors to find the amplitude and phase of the sum

$$(\frac{1-\sqrt{3}}{2})\,\sin(\omega t) + \sqrt{\frac{3}{2}}\cos(\omega t - \pi/4)$$

Recall phasors:
Define the complex number X by

$$X = Ae^{i(\omega t+\phi)} = A(\cos(\omega t+\phi)+i\sin(\omega t+\phi)) = Ae^{i\phi}e^{i\omega t} = Ze^{i\omega t}$$

where $Z = Ae^{i\phi}$.
Then, Z is the phasor for the sinusoidal function Re(X).

Exam problems: 2018

4. (b) Use phasors to find the amplitude and phase of the sum

$$(\frac{1-\sqrt{3}}{2})\,\text{sin}(\omega t) + \sqrt{\frac{3}{2}}\,\text{cos}(\omega t - \pi/4)$$

Hint:

$\frac{1-\sqrt{3}}{2}\,\text{sin}(\omega t) = \text{Re}(-\frac{1-\sqrt{3}}{2} i e^{i\omega t})$

$\sqrt{\frac{3}{2}}\,\text{cos}(\omega t - \pi/4) = \text{Re}(\sqrt{\frac{3}{2}} e^{i\omega t} e^{-i\pi/4}) = \text{Re}(\sqrt{\frac{3}{2}}(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})e^{i\omega t}))$

The phasor is

$$-i(\frac{1-\sqrt{3}}{2}) + \sqrt{\frac{3}{2}}(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}) = \frac{\sqrt{3}}{2} - i\frac{1}{2} = e^{-i\pi/6}.$$