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Linear Algebra: Subspaces

Tutorial Class — notes from [Poo14]

Xinye Chen xinye.chen@manchester.ac.uk

Subspaces

Definition (Subspaces)

A subspace of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that:

1. The zero vector v is in S
2. If u and v are in S, then $u + v$ is in S (S is closed under addition)
3. If u is in S and α is a scalar, then αu is in S (S is closed under scalar multiplication)

e.g., if $v_1, v_2, \dots, v_k$ are vectors in $\mathbb{R}^n$. Then span$(v_1, v_2, \dots, v_k)$ is a subspace of $\mathbb{R}^n$.

Basis

Definition (Basis)

A basis for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that

1. spans S
2. is linearly independent

Definition (Dimension)

If S is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for S is called the dimension of S, denoted $\dim S$.

The Rank Theorem

Definition (Rank)

The rank of a matrix A is the dimension of its row and column spaces and is denoted by rank(A).

Theorem (The Rank Theorem (Rank–Nullity Theorem))

If A is an $m \times n$ matrix ($m \geq n$), then

$$\textit{rank}(A) + \textit{nullity}(A) = n,$$

where nullity(A) is the dimension of the null space of a matrix A, i.e., dim Ker(A)).

Proof of the Rank Theorem

Let R be the reduced row echelon form of A, and suppose that rank$(A) = r$. Then R has r leading variables and $n - r$ free variables in the solution to $Ax = 0$. Since dim Ker$(A) = n - r$, we have

$$\text{rank}(A) + \text{nullity}(A) = r + (n - r) = n.$$

Statements of Invertible Matrices

Let A be an $n \times n$ ($m \geq n$) matrix. The following statements are equivalent:

1. A is invertible
2. $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$
3. $Ax = b$ has only the trivial solution
4. The reduced row echelon form of A is I_n
5. A is a product elementary matrices
6. rank$(A) = n$
7. nullity$(A) = 0$
8. The column vectors (resp., the row vectors) are linearly independent

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Other Theorem

Theorem

Let $A \in \mathbb{R}^{m\times n}$, then:

1. *rank(A^TA) = rank(A)*
2. *The $n \times n$ matrix A^TA is invertible if and only if rank$(A) = n$*

Proof.
For 1., It is obvious rank(A) + nullity(A) $= n =$ rank(A^TA) + nullity(A^TA). We can obtain

1. If $x \in$ dim Ker(A), then $A^TAx = A^T(Ax) = 0$
2. If $x \in$ dim Ker(A^TA), then $(Ax)^T(Ax) = xA^TAx = 0$ and hence $Ax = 0$

Therefore, nullity(A) = nullity(A^TA) and rank(A) = rank(A^TA).
For 2., A^TA is invertible if and only if rank$(A^TA) = n$ and then we naturally have A^TA is invertible if and only if rank$(A) = n$.

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Thanks for your attention!

David Poole.
Linear Algebra: A Modern Introduction.
Cengage Learning, 2014.