CS-E4850 Computer Vision
Exercise Round #1
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Exercise 1. Homogeneous coordinates.

Solution

a)

Proof. The homogeneous coordinates of $\mathbf{x}$ is $(x\ y\ 1)^\top$, and $\mathbf{l} = (a\ b\ c)^\top$,
therefore the left-hand side of $ax + by + c = 0$ can be written as the inner product of $\mathbf{x}$ and $\mathbf{l}$. Thus,

$$\mathbf{x}.\mathbf{l} = \mathbf{x}^\top \mathbf{l} = 0$$

$\square$

b)

Proof. $\because \mathbf{l}.(\mathbf{l} \times \mathbf{l}') = 0$
$\therefore$ the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$ is on the line $\mathbf{l}$.
Similarly, $\because \mathbf{l}'.(\mathbf{l} \times \mathbf{l})' = 0$,
$\therefore$ the point $\mathbf{x}$ is on the line $\mathbf{l}'$.
The point $\mathbf{x}$ is on both $\mathbf{l}$ and $\mathbf{l}'$, which means $\mathbf{x}$ is the intersection of $\mathbf{l}$ and $\mathbf{l}'$. $\square$

c)

Proof. $\because \mathbf{x}.(\mathbf{x} \times \mathbf{x}') = 0$
$\therefore$ the line $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$ is through the point $\mathbf{x}$.
Similarly, $\because \mathbf{x}'.(\mathbf{x} \times \mathbf{x})' = 0$,
$\therefore$ the line $\mathbf{l}$ is through the point $\mathbf{x}'$.
The point line $\mathbf{l}$ is through both points $\mathbf{x}$ and $\mathbf{x}'$. $\square$

d)

Proof. From c), the line through both $\mathbf{x}$ and $\mathbf{x}'$ can be written as $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.
$\because \mathbf{y}.\mathbf{l} = (\alpha\mathbf{x} + (1-\alpha)\mathbf{x}').(\mathbf{x} \times \mathbf{x}') = \alpha\mathbf{x}.(\mathbf{x} \times \mathbf{x}') + (1-\alpha)\mathbf{x}'.(\mathbf{x} \times \mathbf{x}') = 0$
$\therefore$ the point $\mathbf{y} = \alpha\mathbf{x} + (1-\alpha)\mathbf{x}'$ lies on the line $\mathbf{l}$, which is through both points $\mathbf{x}$ and $\mathbf{x}'$.

$\square$

Exercise 2. Transformations in 2D.
a) Use homogeneous coordinates and give the matrix representations of the following transformation groups: translation, Euclidean transformation (rotation+translation), similarity transformation (scaling+rotation+translation), affine transformation, projective transformation.

- Translation:

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Euclidean transformation (rotation+translation):

$$\begin{bmatrix} cos\theta & -sin\theta & t_x \\ sin\theta & cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Similarity transformation (scaling+rotation+translation):

$$\begin{bmatrix} s\ cos\theta & -s\ sin\theta & t_x \\ s\ sin\theta & s\ cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Affine transformation:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Projective transformation:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

b) What is the number of degrees of freedom in these transformations?

- Translation: 2
- Euclidean transformation (rotation+translation): 3
- Similarity transformation (scaling+rotation+translation): 4
- Affine transformation: 6
- Projective transformation: 8

c) Why is the number of degrees of freedom in a projective transformation less than the number of elements in a 3×3 matrix?

The matrix $\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$ has 9 elements, but only their ratio matters, so the perspective transformation can be determined by 8 elements.

Exercise 3. Planar projective transformation.

Solution

a)

$$\begin{aligned}
&\mathbf{l}^\top \mathbf{x} = 0 \\
&\Rightarrow \mathbf{l}^\top \mathbf{H}^{-1}\mathbf{H}\mathbf{x} = 0 \\
&\Rightarrow \mathbf{l}^\top \mathbf{H}^{-1}\mathbf{x}' = 0 \\
&\therefore \mathbf{l}'^\top = \mathbf{l}^\top \mathbf{H}^{-1} \\
&\Rightarrow \mathbf{l}' = (\mathbf{l}^\top \mathbf{H}^{-1})^\top = \mathbf{H}^{-\top}\mathbf{l}
\end{aligned}$$

b)

Proof. Since $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are not lying on the lines $\boldsymbol{l}_1$ or $\boldsymbol{l}_2$, Therefore:

$$\boldsymbol{l}_i^\top \boldsymbol{x}_j \neq 0 \ \forall i, j \in \{1, 2\}$$

$$\begin{aligned}
I &= \frac{(\boldsymbol{l}_1^\top \boldsymbol{x}_1)(\boldsymbol{l}_2^\top \boldsymbol{x}_2)}{(\boldsymbol{l}_1^\top \boldsymbol{x}_2)(\boldsymbol{l}_2^\top \boldsymbol{x}_1)} \\
&= \frac{(\boldsymbol{l}_1^\top \mathbf{H}^{-1}\mathbf{H}\boldsymbol{x}_1)(\boldsymbol{l}_2^\top \mathbf{H}^{-1}\mathbf{H}\boldsymbol{x}_2)}{(\boldsymbol{l}_1^\top \mathbf{H}^{-1}\mathbf{H}\boldsymbol{x}_2)(\boldsymbol{l}_2^\top \mathbf{H}^{-1}\mathbf{H}\boldsymbol{x}_1)} \\
&= \frac{(\boldsymbol{l}_1'^\top \boldsymbol{x}_1')(\boldsymbol{l}_2'^\top \boldsymbol{x}_2')}{(\boldsymbol{l}_1'^\top \boldsymbol{x}_2')(\boldsymbol{l}_2'^\top \boldsymbol{x}_1')} = I'
\end{aligned}$$

$\therefore I$ is invariant under projective transformation. □

Why similar construction does not give projective invariants with fewer number of points or lines?

Solution

If there are fewer number of lines (1 line) and I is constructed in such a way:

$$I = \frac{(\boldsymbol{l}_1^\top \boldsymbol{x}_1)}{(\boldsymbol{l}_1^\top \boldsymbol{x}_2)}$$

Consider scaling $\boldsymbol{x}_1$ by α, $\boldsymbol{x}_2$ by β, and $\boldsymbol{l}_1$ by γ.

$$\begin{aligned}\widetilde{I} &= \frac{(\gamma \boldsymbol{l}_1^\top \alpha \boldsymbol{x}_1)}{(\gamma \boldsymbol{l}_1^\top \beta \boldsymbol{x}_2)} \\ &= \frac{\alpha(\boldsymbol{l}_1^\top \boldsymbol{x}_1)}{\beta(\boldsymbol{l}_1^\top \boldsymbol{x}_2)} \neq I\end{aligned}$$

which means I is not invariant. Similarly, I won't be invariant either if there are fewer number of points (one point).