

CS-E4850 Computer Vision
Exercise Round #6
Submitted by Chen Xu, ID 000000
2024-10-17

Exercise 1. Least-squares fitting for affine transformations.

a) **Solution**

$$\begin{aligned} E &= \sum_{i=1}^n \|\mathbf{x}'_i - \mathbf{M}\mathbf{x}_i - \mathbf{t}\|^2 \\ &= \sum_{i=1}^n \left\| \begin{bmatrix} x'_i \\ y'_i \end{bmatrix} - \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \right\|^2 \\ &= \sum_{i=1}^n \left\| \begin{bmatrix} x'_i - m_1x_i - m_2y_i - t_1 \\ y'_i - m_3x_i - m_4y_i - t_2 \end{bmatrix} \right\|^2 \\ &= \sum_{i=1}^n ((m_1x_i + m_2y_i + t_1 - x'_i)^2 + (m_3x_i + m_4y_i + t_2 - y'_i)^2) \end{aligned}$$

$$\frac{\partial E}{\partial m_1} = \frac{\partial}{\partial m_1} \sum_{i=1}^n ((m_1x_i + m_2y_i + t_1 - x'_i)^2 + (m_3x_i + m_4y_i + t_2 - y'_i)^2)$$

$$= 2 \sum_{i=1}^n x_i (m_1x_i + m_2y_i + t_1 - x'_i)$$

$$\frac{\partial E}{\partial m_2} = 2 \sum_{i=1}^n y_i (m_1x_i + m_2y_i + t_1 - x'_i)$$

$$\frac{\partial E}{\partial m_3} = 2 \sum_{i=1}^n x_i (m_3x_i + m_4y_i + t_2 - y'_i)$$

$$\frac{\partial E}{\partial m_4} = 2 \sum_{i=1}^n y_i (m_3x_i + m_4y_i + t_2 - y'_i)$$

$$\frac{\partial E}{\partial t_1} = 2 \sum_{i=1}^n (m_1x_i + m_2y_i + t_1 - x'_i)$$

$$\frac{\partial E}{\partial t_2} = 2 \sum_{i=1}^n (m_3x_i + m_4y_i + t_2 - y'_i)$$

b)

Proof. By setting above gradients to zero, one can get:

$$\begin{bmatrix} \dots & & & & & \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \\ \dots & & & & & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \dots \\ x'_i \\ y'_i \\ \dots \end{bmatrix}$$

And this is corresponding to the equation:

$$\mathbf{Sh} = \mathbf{u}$$

□

c)

Solution

From b)

$$\mathbf{h} = \mathbf{S}^{-1}\mathbf{u} \tag{1}$$

Transformation from the following points correspondences $\{(0,0) \rightarrow (1,2)\}, \{(1,0) \rightarrow (3,2)\}, \{(0,1) \rightarrow (1,4)\}$. Equation(??) becomes:

$$\begin{aligned} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} &= \begin{bmatrix} x_1 & y_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_1 & y_1 & 0 & 1 \\ x_2 & y_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 1 \\ x_3 & y_3 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_3 & y_3 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Exercise 2. Similarity transformation from two point correspondences.

a)

Solution

$$\begin{aligned}
\mathbf{v}' &= \mathbf{x}'_2 - \mathbf{x}'_1 \\
&= s\mathbf{R}\mathbf{x}_2 + \mathbf{t} - s\mathbf{R}\mathbf{x}_1 - \mathbf{t} \\
&= s\mathbf{R}(\mathbf{x}_2 - \mathbf{x}_1)
\end{aligned} \tag{2}$$

$$\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1 \tag{3}$$

$$\therefore \mathbf{v}' = s\mathbf{R}\mathbf{v} \tag{4}$$

$$\begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = s \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

$$v'_x = s(\cos\theta v_x - \sin\theta v_y) \tag{5}$$

$$v'_y = s(\sin\theta v_x + \cos\theta v_y) \tag{6}$$

From (??)(??)

$$v'_x v_x + v'_y v_y = s(v_x^2 + v_y^2) \tag{7}$$

$$(v_x'^2 + v_y'^2) = s^2(v_x^2 + v_y^2) \tag{8}$$

$$\mathbf{v}' \cdot \mathbf{v} = s|\mathbf{v}|^2 \cos\theta \tag{9}$$

$$|\mathbf{v}'| = s|\mathbf{v}| \tag{10}$$

$$\therefore \cos\theta = \frac{\mathbf{v}' \cdot \mathbf{v}}{|\mathbf{v}'||\mathbf{v}|}$$

$$\theta = \arccos\left(\frac{\mathbf{v}' \cdot \mathbf{v}}{|\mathbf{v}'||\mathbf{v}|}\right) \tag{11}$$

b)

Proof. From (??) above in a):

$$|\mathbf{v}'| = s|\mathbf{v}|$$

$$s = \frac{|\mathbf{v}'|}{|\mathbf{v}|}$$

□

c) **Solution**

$$\mathbf{t} = \mathbf{x}' - s\mathbf{R}\mathbf{x}$$

$$\begin{aligned}
\begin{bmatrix} t_x \\ t_y \end{bmatrix} &= \begin{bmatrix} x' \\ y' \end{bmatrix} - s \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} x' - s(x\cos\theta - y\sin\theta) \\ y' - s(x\sin\theta + y\cos\theta) \end{bmatrix}
\end{aligned}$$

d)

Solution

Transformation from the following points correspondences $\{(1/2, 0) \rightarrow (0, 0)\}, \{(0, 1/2) \rightarrow (-1, -1)\}$.

$$\mathbf{v}' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$\theta = \arccos\left(\frac{\mathbf{v}' \cdot \mathbf{v}}{|\mathbf{v}'||\mathbf{v}|}\right) = \arccos(0)$$

$$\theta = \pi/2$$

$$\begin{aligned} \therefore \mathbf{R} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} s &= \frac{|\mathbf{v}'|}{|\mathbf{v}|} = 2 \\ \begin{bmatrix} t_x \\ t_y \end{bmatrix} &= \begin{bmatrix} x' - s(x\cos\theta - y\sin\theta) \\ y' - s(x\sin\theta + y\cos\theta) \end{bmatrix} \\ &= \begin{bmatrix} x' + sy \\ y' - sx \end{bmatrix} \\ &= \begin{bmatrix} 0 + 2 * 0 \\ 0 - 2 * 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{aligned}$$