CS-E4850 Computer Vision
Exercise Round #6
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Exercise 1. Least-squares fitting for affine transformations.
a) **Solution**

$$
\begin{aligned}
E &= \sum_{i=1}^{n} \left\| \mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t} \right\|^2 \\
&= \sum_{i=1}^{n} \left\| \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \right\|^2 \\
&= \sum_{i=1}^{n} \left\| \begin{bmatrix} x_i' - m_1 x_i - m_2 y_i - t_1 \\ y_i' - m_3 x_i - m_4 y_i - t_2 \end{bmatrix} \right\|^2 \\
&= \sum_{i=1}^{n} ((m_1 x_i + m_2 y_i + t_1 - x_i')^2 + (m_3 x_i + m_4 y_i + t_2 - y_i')^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial m_1} &= \frac{\partial}{\partial m_1} \sum_{i=1}^{n} ((m_1 x_i + m_2 y_i + t_1 - x_i')^2 + (m_3 x_i + m_4 y_i + t_2 - y_i')^2) \\
&= 2 \sum_{i=1}^{n} x_i (m_1 x_i + m_2 y_i + t_1 - x_i') \\
\frac{\partial E}{\partial m_2} &= 2 \sum_{i=1}^{n} y_i (m_1 x_i + m_2 y_i + t_1 - x_i') \\
\frac{\partial E}{\partial m_3} &= 2 \sum_{i=1}^{n} x_i (m_3 x_i + m_4 y_i + t_2 - y_i') \\
\frac{\partial E}{\partial m_4} &= 2 \sum_{i=1}^{n} y_i (m_3 x_i + m_4 y_i + t_2 - y_i') \\
\frac{\partial E}{\partial t_1} &= 2 \sum_{i=1}^{n} (m_1 x_i + m_2 y_i + t_1 - x_i') \\
\frac{\partial E}{\partial t_2} &= 2 \sum_{i=1}^{n} (m_3 x_i + m_4 y_i + t_2 - y_i')
\end{aligned}
$$

b)

Proof. By setting above gradients to zero, one can get:

$$\begin{bmatrix} & & \dots & & & \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \\ & & \dots & & & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \dots \\ x'_i \\ y'_i \\ \dots \end{bmatrix}$$

And this is corresponding to the equation:

$$\mathbf{Sh} = \mathbf{u}$$

□

c)
Solution
From b)

$$\mathbf{h} = \mathbf{S}^{-1}\mathbf{u} \tag{1}$$

Transformation from the following points correspondences $\{(0,0) \to (1,2)\}, \{(1,0) \to (3,2)\}, \{(0,1) \to (1,4)\}$. Equation(**??**) becomes:

$$\begin{aligned} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} &= \begin{bmatrix} x_1 & y_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_1 & y_1 & 0 & 1 \\ x_2 & y_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 1 \\ x_3 & y_3 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_3 & y_3 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial E}{\partial \mathbf{M}} &= \frac{\partial}{\partial \mathbf{M}} \sum_{i=1}^{n} ||\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}||^2 \\
&= \frac{\partial}{\partial \mathbf{M}} \sum_{i=1}^{n} (\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t})^\top (\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) \\
&= \frac{\partial}{\partial \mathbf{M}} \sum_{i=1}^{n} (\mathbf{x}_i'^\top - \mathbf{x}_i^\top \mathbf{M}^\top - \mathbf{t}^\top)(\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{M}} (\mathbf{x}_i'^\top - \mathbf{x}_i^\top \mathbf{M}^\top - \mathbf{t}^\top)(\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{M}} (\mathbf{x}_i^\top \mathbf{M}^\top \mathbf{M}\mathbf{x}_i - (\mathbf{x}_i'^\top - \mathbf{t}^\top)\mathbf{M}\mathbf{x}_i - \mathbf{x}_i^\top \mathbf{M}^\top (\mathbf{x}_i' - \mathbf{t})) \\
&= \sum_{i=1}^{n} 2(\mathbf{x}_i^\top \mathbf{M}^\top \mathbf{x}_i - (\mathbf{x}_i'^\top - \mathbf{t}^\top)\mathbf{x}_i) \\
&= 2\sum_{i=1}^{n} (\mathbf{x}_i^\top \mathbf{M}^\top \mathbf{x}_i - \mathbf{x}_i'^\top \mathbf{x}_i + \mathbf{t}^\top \mathbf{x}_i)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial \mathbf{t}} &= \frac{\partial}{\partial \mathbf{t}} \sum_{i=1}^{n} ||\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}||^2 \\
&= \frac{\partial}{\partial \mathbf{t}} \sum_{i=1}^{n} (\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t})^\top (\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) \\
&= \frac{\partial}{\partial \mathbf{t}} \sum_{i=1}^{n} (\mathbf{x}_i'^\top - \mathbf{x}_i^\top \mathbf{M}^\top - \mathbf{t}^\top)(\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{t}} (\mathbf{x}_i'^\top - \mathbf{x}_i^\top \mathbf{M}^\top - \mathbf{t}^\top)(\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{t}} (\mathbf{t}^\top \mathbf{t} - (\mathbf{x}_i'^\top - \mathbf{x}_i^\top \mathbf{M}^\top)\mathbf{t} - \mathbf{t}^\top (\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i)) \\
&= \sum_{i=1}^{n} (2\mathbf{t}^\top - 2(\mathbf{x}_i'^\top - \mathbf{x}_i^\top \mathbf{M}^\top)) \\
&= 2\sum_{i=1}^{n} (\mathbf{t}^\top - \mathbf{x}_i'^\top + \mathbf{x}_i^\top \mathbf{M}^\top)
\end{aligned}
$$

Exercise 2. Similarity transformation from two point correspondences.
a)

Solution

$$\begin{aligned}\mathbf{v}' &= \mathbf{x}_2' - \mathbf{x}_1' \\ &= s\mathbf{R}\mathbf{x}_2 + \mathbf{t} - s\mathbf{R}\mathbf{x}_1 - \mathbf{t} \\ &= s\mathbf{R}(\mathbf{x}_2 - \mathbf{x}_1) \qquad (2)\end{aligned}$$

$$\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1 \qquad (3)$$

$$\therefore \mathbf{v}' = s\mathbf{R}\mathbf{v} \qquad (4)$$

$$\begin{bmatrix} v_x' \\ v_y' \end{bmatrix} = s \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

$$v_x' = s(cos\theta v_x - sin\theta v_y) \qquad (5)$$

$$v_y' = s(sin\theta v_x + cos\theta v_y) \qquad (6)$$

From (**??**)(**??**)

$$v_x' v_x + v_y' v_y = s(v_x^2 + v_y^2) \qquad (7)$$

$$(v_x'^2 + v_y'^2) = s^2(v_x^2 + v_y^2) \qquad (8)$$

$$\mathbf{v}' \cdot \mathbf{v} = s|\mathbf{v}|^2 cos\theta \qquad (9)$$

$$|\mathbf{v}'| = s|\mathbf{v}| \qquad (10)$$

$$\therefore cos\theta = \frac{\mathbf{v}' \cdot \mathbf{v}}{|\mathbf{v}'||\mathbf{v}|}$$

$$\theta = arccos(\frac{\mathbf{v}' \cdot \mathbf{v}}{|\mathbf{v}'||\mathbf{v}|}) \qquad (11)$$

b)

Proof. From (**??**) above in a):

$$\begin{aligned}|\mathbf{v}'| &= s|\mathbf{v}| \\ s &= \frac{|\mathbf{v}'|}{|\mathbf{v}|}\end{aligned}$$

□

c) **Solution**

$$\begin{aligned}\mathbf{t} &= \mathbf{x}' - s\mathbf{R}\mathbf{x} \\ \begin{bmatrix} t_x \\ t_y \end{bmatrix} &= \begin{bmatrix} x' \\ y' \end{bmatrix} - s \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x' - s(xcos\theta - ysin\theta) \\ y' - s(xsin\theta + ycos\theta) \end{bmatrix}\end{aligned}$$

d)

Solution

Transformation from the following points correspondences $\{(1/2, 0) \rightarrow (0, 0)\}, \{(0, 1/2) \rightarrow (-1, -1)\}$.

$$\mathbf{v'} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$\theta = arccos(\frac{\mathbf{v'} \cdot \mathbf{v}}{|\mathbf{v'}||\mathbf{v}|}) = arccos(0)$$

$$\theta = \pi/2$$

$$\therefore \mathbf{R} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

And

$$s = \frac{|\mathbf{v'}|}{|\mathbf{v}|} = 2$$

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x' - s(xcos\theta - ysin\theta) \\ y' - s(xsin\theta + ycos\theta) \end{bmatrix}$$

$$= \begin{bmatrix} x' + sy \\ y' - sx \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 2 * 0 \\ 0 - 2 * 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$