CS-E4850 Computer Vision
Exercise Round #2
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Exercise 1. Pinhole camera.

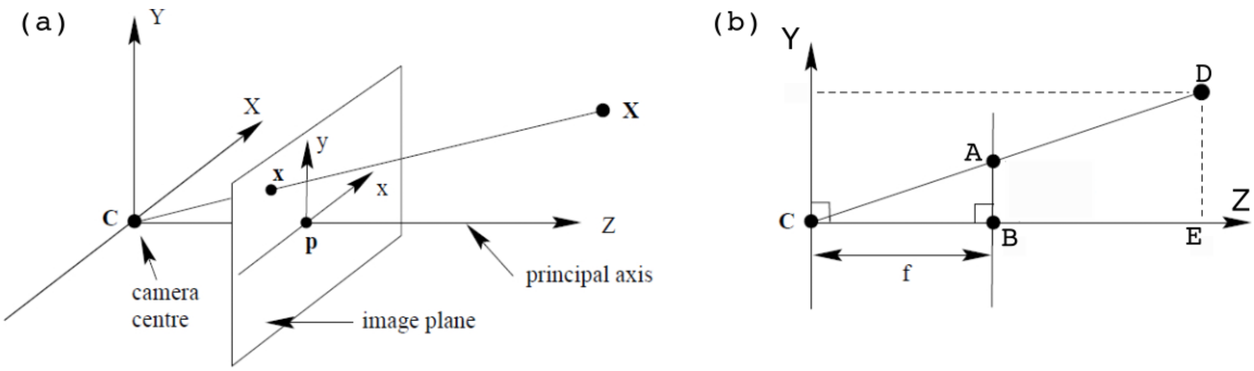

Figure 1: The pinhole model

Proof. Figure 1a shows the pinhole model. Figure 1b shows the projection if one look at the model along the x-axis. Using the rule of similar triangular, one can get:

$$\frac{y_p}{y_c} \equiv \frac{|AB|}{|DE|} = \frac{|CB|}{|CE|} \equiv \frac{f}{z_c}$$
$$\therefore y_p = f\frac{y_c}{z_c}$$

Similarly,

$$x_p = f\frac{x_c}{z_c}$$

□

Exercise 2. Pixel coordinate frame.

Solution

a)

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_x & 0 & u_0 \\ 0 & m_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_x & -m_x/tan\theta & u_0 \\ 0 & m_y/sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix}$$

Exercise 3. Intrinsic camera calibration matrix.

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_x & 0 & u_0 \\ 0 & m_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} = \begin{bmatrix} m_x & 0 & u_0 \\ 0 & m_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} m_x & 0 & u_0 \\ 0 & m_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} m_x f & 0 & u_0 \\ 0 & m_y f & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 4. Camera projection matrix.

$$\begin{aligned} \mathbf{P}_{3\times 4} = \mathbf{K}[\mathbf{R}|\mathbf{t}] &= \begin{bmatrix} m_x f & 0 & u_0 \\ 0 & m_y f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \\ &= \begin{bmatrix} r_{11} m_x f + u_0 r_{31} & r_{12} m_x f + u_0 r_{32} & r_{13} m_x f + u_0 r_{33} & t_1 m_x f + u_0 t_3 \\ r_{21} m_y f + v_0 r_{31} & r_{22} m_y f + v_0 r_{32} & r_{23} m_y f + v_0 r_{33} & t_2 m_y f + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \end{aligned}$$

Exercise 5. Rotation matrix.
a)

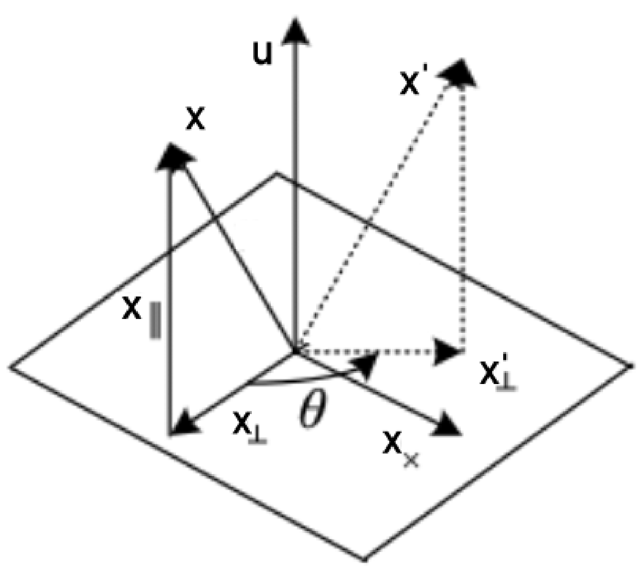

Figure 2: Rodrigues Rotation

Proof. As it is shown in Figure 2, first we project the vector $\mathbf{x}$ onto the axis $\mathbf{u}$ to obtain

$$\mathbf{x}_{\|} = \mathbf{u}(\mathbf{u} \cdot \mathbf{x}) \tag{1}$$

which is the component of $\mathbf{x}$ that is not affected by the rotation.
Next, we compute the perpendicular residual of $\mathbf{x}$ from $\mathbf{u}$,

$$\mathbf{x}_\perp = \mathbf{x} - \mathbf{x}_\parallel = \mathbf{x} - \mathbf{u}(\mathbf{u} \cdot \mathbf{x}) \tag{2}$$

This vector can rotate around $\mathbf{u}$ by 90° to get

$$\mathbf{x}_\times = \mathbf{u} - \mathbf{x}_\perp = \mathbf{u} \times \mathbf{x} \tag{3}$$

The in-plane component of the rotated vector $\mathbf{x}'$ can be calculated as

$$\begin{aligned} \mathbf{x'}_\perp &= cos\theta \mathbf{x}_\perp + sin\theta \mathbf{x}_\times \\ &= cos\theta(\mathbf{x} - \mathbf{u}(\mathbf{u} \cdot \mathbf{x})) + sin\theta(\mathbf{u} \times \mathbf{x}) \end{aligned} \tag{4}$$

Putting all these terms together, the final rotated vector is

$$\begin{aligned} \mathbf{Rx} = \mathbf{x'} &= \mathbf{x'}_\perp + \mathbf{x}_\parallel \\ &= cos\theta(\mathbf{x} - \mathbf{u}(\mathbf{u} \cdot \mathbf{x})) + sin\theta(\mathbf{u} \times \mathbf{x}) + \mathbf{u}(\mathbf{u} \cdot \mathbf{x}) \\ &= cos\theta \mathbf{x} + sin\theta(\mathbf{u} \times \mathbf{x}) + (1 - cos\theta)(\mathbf{u} \cdot \mathbf{x})\mathbf{u} \end{aligned} \tag{5}$$

$\square$

b) **Solution**
From a) above

$$\begin{aligned} \mathbf{Rx} &= cos\theta \mathbf{x} + sin\theta(\mathbf{u} \times \mathbf{x}) + (1 - cos\theta)(\mathbf{u} \cdot \mathbf{x})\mathbf{u} \\ &= sin\theta \mathbf{u} \times \mathbf{x} + cos\theta \mathbf{x} + (1 - cos\theta)(\mathbf{u} \cdot \mathbf{x})\mathbf{u} \\ &= sin\theta \mathbf{u} \times \mathbf{x} + \mathbf{x} - \mathbf{x} + cos\theta \mathbf{x} + (1 - cos\theta)(\mathbf{u} \cdot \mathbf{x})\mathbf{u} \\ &= sin\theta \mathbf{u} \times \mathbf{x} + \mathbf{x} - (1 - cos\theta)\mathbf{x} + (1 - cos\theta)(\mathbf{u} \cdot \mathbf{x})\mathbf{u} \\ &= sin\theta \mathbf{u} \times \mathbf{x} + \mathbf{x} + (1 - cos\theta)(-\mathbf{x} + (\mathbf{u} \cdot \mathbf{x})\mathbf{u}) \\ &= sin\theta \mathbf{u} \times \mathbf{x} + \mathbf{x} + (1 - cos\theta)\mathbf{u} \times (\mathbf{u} \times \mathbf{x}) \end{aligned}$$

Because $\mathbf{u}\times$ can be written as

$$\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

The above $\mathbf{Rx}$ can be written as

$$
\begin{aligned}
\mathbf{Rx} &= \mathbf{R}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= sin\theta \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&+ (1 - cos\theta) \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= sin\theta \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&+ (1 - cos\theta) \begin{bmatrix} u_1^2 - 1 & u_1 u_2 & u_1 u_3 \\ u_1 u_2 & u_2^2 - 1 & u_2 u_3 \\ u_1 u_3 & u_2 u_3 & u_3^2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} cos\theta + u_1^2(1 - cos\theta) & u_1 u_2 (1 - cos\theta) - u_3 sin\theta & u_1 u_3 (1 - cos\theta) + u_2 sin\theta \\ u_1 u_2 (1 - cos\theta) + u_3 sin\theta & cos\theta + u_2^2(1 - cos\theta) & u_2 u_3 (1 - cos\theta) - u_1 sin\theta \\ u_1 u_3 (1 - cos\theta) - u_2 sin\theta & u_2 u_3 (1 - cos\theta) + u_1 sin\theta & cos\theta + u_3^2(1 - cos\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$

Therefore,

$$
\mathbf{R} = \begin{bmatrix} cos\theta + u_1^2(1 - cos\theta) & u_1 u_2 (1 - cos\theta) - u_3 sin\theta & u_1 u_3 (1 - cos\theta) + u_2 sin\theta \\ u_1 u_2 (1 - cos\theta) + u_3 sin\theta & cos\theta + u_2^2(1 - cos\theta) & u_2 u_3 (1 - cos\theta) - u_1 sin\theta \\ u_1 u_3 (1 - cos\theta) - u_2 sin\theta & u_2 u_3 (1 - cos\theta) + u_1 sin\theta & cos\theta + u_3^2(1 - cos\theta) \end{bmatrix}
$$