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FULL LENGTH PAPER

On second-order conditions in unconstrained optimization

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Abstract The main purpose of this paper is to establish the second-order nonsmooth sufficient unconstrained optimality condition for so called ℓ-stable at some point functions and in this way to generalize some previous results in this direction. We provide the comparisons with other results by examples.

Keywords Locally Lipschitz function · Regular function · $C^{1,1}$ function · Peano derivative · Stable function · Isolated minimizer of order k · Dini derivative

Mathematics Subject Classification (2000) 49K10 · 26B05

1 Introduction and preliminaries

Second-order optimality conditions play a crucial role in optimization theory. In particular, they are very useful for the study of sensitivity analysis of optimal solutions and convergence analysis of optimal algorithms. For more details

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concerning wide range of applications, see e.g. [5,7,11,12,14,18–21,25–27,29–31] and references therein.

Various generalized second-order derivatives have been introduced to obtain optimization results. In this paper, we will focus on the Peano type of directional derivatives (deeply studied e.g. in [6,13–17,24,29]) and the Dini type of directional derivatives [3,4,29]. The previous two types of derivatives are compared (and their advantages are shown) with some other types of generalized derivatives for instance in [3,4,22,23,29–31]. At this point we should say that the terminology and notations of the previous types of derivatives are not unified, maybe because it has been made a big development since 1980s of twentieth century.

The main aim of this paper is to generalize early obtained sufficient second-order optimality conditions which were introduced in terms of the previous types of derivatives for the class of $C^{1,1}$ functions or for the class of stable functions. We will introduce a new optimality condition expressed by means of a certain directional derivative of the Peano type which suits the purpose (see Theorem 6).

$\mathbb{R}^N$ means N dimensional arithmetical space equipped with the Euclidean norm $\|\cdot\|$. A symbol $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^N$. We also always identify the space $\mathbb{R}^N$ with its dual space. S denotes the unit sphere $\{x \in \mathbb{R}^N : \|x\| = 1\}$, and $B(x,r) = \{y \in \mathbb{R}^N : \|y-x\| \leq r\}$ for $x \in \mathbb{R}^N$ and $r > 0$. The symbols $[a,b]$, $(a,b]$, $[a,b)$ and (a,b) are reserved for intervals in $\mathbb{R}^N$ with endpoints a and b (e.g., $[a,b) = \{ta + (1-t)b : 0 < t \leq 1\}$). In the case of $N = 1$, we assume $a < b$.

We say that $f : \mathbb{R}^N \to \mathbb{R}$ is *Fréchet differentiable at* $x \in \mathbb{R}^N$ if there is $f'(x) \in \mathbb{R}^N$ such that

$$\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} = \langle f'(x), h\rangle, \quad \forall h \in S,$$

and, moreover, this limit is uniform for $h \in S$. By the *strict differentiability* at x we mean that for some $L \in \mathbb{R}^N$ it holds

$$\langle L, h\rangle = \lim_{y\to x, t\downarrow 0} \frac{f(y+th) - f(y)}{t}, \quad \forall h \in \mathbb{R}^N,$$

and the limit is uniform with respect to $h \in S$. It is easy to show that the strict differentiability implies the Fréchet differentiability and $L = f'(x)$.

For a function $f : \mathbb{R}^N \to \mathbb{R}$, we define the first-order directional derivative of f at $x \in \mathbb{R}^N$ in the direction $h \in \mathbb{R}^N$ by

$$f'(x;h) = \lim_{t\downarrow 0} \frac{f(x+th) - f(x)}{t},$$

and the lower and upper first-order directional derivative of f at x in the direction h by, respectively,

$$f^{\ell}(x;h) = \liminf_{t\downarrow 0} \frac{f(x+th)-f(x)}{t},$$
$$f^{u}(x;h) = \limsup_{t\downarrow 0} \frac{f(x+th)-f(x)}{t}.$$

If f is a function of a one real variable, we will use $f'(x), f^{\ell}(x)$ and $f^{u}(x)$ instead of $f'(x;1), f^{\ell}(x;1)$ and $f^{u}(x;1)$, respectively. In this case, it will be clear from the context whether by $f'(x)$ we mean directional derivative $f'(x;1)$ or an element of the dual of $\mathbb{R}$.

Let us recall that a locally Lipschitz function $f:\mathbb{R}^N\to\mathbb{R}$ is said to be *regular at* $x\in\mathbb{R}^N$ provided that $f'(x;h)=f^{\circ}(x;h)$ for every $h\in\mathbb{R}^N$, where $f^{\circ}(x;h)$ denotes the upper Clarke directional derivative of f at x in the direction h defined in [10] as

$$f^{\circ}(x;h) = \limsup_{y\to x, t\downarrow 0} \frac{f(y+th)-f(y)}{t}.$$

The Clarke subdifferential of a locally Lipschitz function $f:\mathbb{R}^N\to\mathbb{R}$ at $x\in\mathbb{R}^N$ is defined by

$$\partial_c f(x) = \{x^*\in\mathbb{R}^N : \langle x^*, h\rangle \le f^{\circ}(x;h) \quad \forall h\in\mathbb{R}^N\}.$$

We say that $f:\mathbb{R}^N\to\mathbb{R}$ is a $C^{1,1}$ *function* near $x\in\mathbb{R}^N$ if it is Fréchet differentiable on some neighbourhood of x and its derivative $f'(\cdot)$ is Lipschitz there.

Analogously, we will say that a function $f:\mathbb{R}^N\to\mathbb{R}$ satisfies a p-property near $x\in\mathbb{R}^N$ if that p-property holds on some neighbourhood of x.

A function $f:\mathbb{R}^N\to\mathbb{R}$, for which there exist a neighbourhood U of $x\in\mathbb{R}^N$ and $K>0$ such that for all $y\in U$ there exists the Fréchet derivative $f'(y)$ and

$$\|f'(y)-f'(x)\| \le K\|y-x\|, \quad \forall y\in U,$$

is called *stable at* x. Notice that if $f:\mathbb{R}^N\to\mathbb{R}$ is a $C^{1,1}$ function near $x\in\mathbb{R}^N$, then f is stable at x.

The rest of this section is devoted to the generalized second-order directional derivatives.

Peano's second-order directional derivative of $f:\mathbb{R}^N\to\mathbb{R}$ at $x\in\mathbb{R}^N$ in the direction $h\in\mathbb{R}^N$ is given as

$$f''_P(x;h) = \lim_{t\downarrow 0} \frac{f(x+th)-f(x)-tf'(x;h)}{t^2/2}.$$

We note that $f''_P(x;h)$ coincides with a classical second-order directional derivative for functions which are twice Fréchet differentiable at x.

In this paper, we also use the following second-order directional derivatives of the Peano type of f at x in the direction h:

$$f'^{\ell}_P(x;h) = \liminf_{t\downarrow 0} \frac{f(x+th) - f(x) - tf'(x;h)}{t^2/2},$$
$$\underline{f}'^{\ell}_P(x;h) = \liminf_{t\downarrow 0} \frac{f(x+th) - f(x) - tf^{\ell}(x;h)}{t^2/2},$$

For the following theorem, see [6, Theorem 3.2].

Theorem 1 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be Fréchet differentiable near $x \in \mathbb{R}^N$ and let f be stable at x. If $f'(x) = 0$ and*

$$f''_P(x;h) > 0, \quad \forall h \in S,$$

then x is a strict local minimizer for f.

Recall that $x \in \mathbb{R}^N$ is an *isolated minimizer of order k* ($k \in \mathbb{N}$) for a function $f : \mathbb{R}^N \to \mathbb{R}$ if there are a neighbourhood U of x and $A > 0$ satisfying $f(y) \geq f(x) + A\|y - x\|^k$ for every $y \in U$. It is easy to verify that each isolated minimizer is a strict local minimizer. The notion of an isolated minimizer was studied e.g. by Auslender [1].

Ginchev et al. [13] presented the problem for what class $\mathcal{F}$ of functions $f : \mathbb{R}^N \to \mathbb{R}$ the following theorem holds:

Theorem P (problem what $\mathcal{F}$) *Let $f : \mathbb{R}^N \to \mathbb{R}$ be of class $\mathcal{F}$, and let $x \in \mathbb{R}^N$. If for each $h \in S$ one of the following conditions holds:*

(i) $f^{\ell}(x;h) > 0$,
(ii) $f^{\ell}(x;h) = 0$ *and* $\underline{f}'^{\ell}_P(x;h) > 0$,

then x is a strict local minimizer for f. Conversely, if x is an isolated minimizer of order 2, then the previous conditions hold.

They showed that the class of $C^{1,1}$ functions solves this problem:

Theorem 2 ([13, Theorem 2]). *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a $C^{1,1}$ function near $x \in \mathbb{R}^N$. If $f'(x) = 0$ and*

$$\underline{f}'^{\ell}_P(x;h) > 0, \quad \forall h \in S,$$

then x is an isolated minimizer of order 2 for f. Conversely, each isolated minimizer of order 2 satisfies these sufficient conditions.

We turn now to some definitions and notations which are mostly taken from Chaney's papers [8,9].

Definition 1 Let u be a nonzero vector in $\mathbb{R}^N$. Suppose that the sequence $\{x_k\}_{k=1}^{+\infty}$ in $\mathbb{R}^N$ converges to x. We say that $\{x_k\}_{k=1}^{+\infty}$ *converges to x in the direction u*, denoted by $x_k \to_u x$, if $x_k \neq x$ for every $k \in \mathbb{N}$ and the sequence $\{(x_k - x)/\|x_k - x\|)\}_{k=1}^{+\infty}$ converges to $u/\|u\|$.

Definition 2 Let $f : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz function, $x \in \mathbb{R}^N$, and let u be a nonzero vector in $\mathbb{R}^N$. We define the subset $\partial_u f(x)$ of $\mathbb{R}^N$ by

$$\begin{aligned}\partial_u f(x) := \{x^* \in \mathbb{R}^N;\ &\text{there exist sequences } \{x_k\}_{k=1}^{+\infty} \text{ and } \{x_k^*\}_{k=1}^{+\infty}\\ &\text{such that } x_k^* \in \partial_c f(x_k), x_k \to_u x, \text{ and } x_k^* \to x^* \text{ in norm}\}.\end{aligned}$$

Definition 3 Let $f : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz function, $x \in \mathbb{R}^N$, and let u be a nonzero vector in $\mathbb{R}^N$. Suppose that $x^* \in \partial_u f(x)$. Then $f''_-(x, x^*, u)$ is defined to be the infimum of all numbers

$$\liminf_{k\to+\infty} \frac{f(x_k) - f(x) - \langle x^*, x_k - x\rangle}{t_k^2/2},$$

taken over all triples of sequences $\{x_k\}_{k=1}^{+\infty}$, $\{x_k^*\}_{k=1}^{+\infty}$ and $\{t_k\}_{k=1}^{+\infty}$ for which

(a) $t_k > 0$ for each $k \in \mathbb{N}$ and $x_k \to x$,
(b) $t_k \downarrow 0$ and $\{(x_k - x)/t_k\}_{k=1}^{+\infty}$ converges to u,
(c) $x_k^* \in \partial_c f(x_k)$ and $x_k^* \to x^*$.

L.R. Huang and K.F. Ng introduced the second-order sufficient condition by means of $f''_-(x, 0, u)$.

Theorem 3 ([16, Theorem 2.9]) *Suppose that $f : \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz function and $x \in \mathbb{R}^N$. Suppose that $f^\ell(x; h) \geq 0$ for every $h \in S$. If $f''_-(x, 0, h) > 0$ for all $h \in S$ for which $f^\ell(x; h) = 0$, then x is a strict local minimizer for f.*

Theorem 4 uses a certain directional derivative of the Dini type.

Theorem 4 ([4, Theorem 3.2]) *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a $C^{1,1}$ function near $x \in \mathbb{R}^N$. If $f'(x) = 0$ and*

$$f_D'^{\ell}(x; h) := \liminf_{t\downarrow 0} \frac{f'(x + th; h) - f'(x; h)}{t} > 0, \quad \forall h \in S,$$

then x is a strict local minimizer for f.

Theorem 4 covers also the following result due to Cominetti and Correa (see [11] and also [7, Proposition 6.2], [30, Theorem 5.1(ii)], [31, Theorem 4.2(ii)]):

Theorem 5 ([11, Proposition 5.2]) *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a $C^{1,1}$ function near $x \in \mathbb{R}^N$. If $f'(x) = 0$, and*

$$f_\infty'^{\ell}(x; h) := \liminf_{y\to x, t\downarrow 0} \frac{\langle \nabla f(y + th) - \nabla f(y), h\rangle}{t} > 0, \quad \forall h \in S,$$

then x is a strict local minimizer for f.

The main purpose of this paper is to establish the generalization of Theorems 1, 2, 4 and 5.

2 Main result

A stability property reminded in Sect. 1 can be weakened by the following way.

Definition 4 We say that a function $f : \mathbb{R}^N \to \mathbb{R}$ is ℓ-stable at $x \in \mathbb{R}^N$ if there exist a neighbourhood U of x and $K > 0$ such that

$$|f^\ell(y;h) - f^\ell(x;h)| \le K\|y-x\|, \quad \forall y \in U, \ \forall h \in S.$$

Since $C^{1,1}$ functions appear in, e.g. the augmented Lagrange method, the penalty function method and the proximal point method, and since the property to be $C^{1,1}$ function near some point requires stronger assumptions than the property to be ℓ-stable function at considered point, it seems to be useful to study the class of ℓ-stable functions. We will show in this section, among the others, some properties of the functions which are ℓ-stable at some point.

In order to generalize Theorems 1, 2, 4 and 5, we first derive the generalization of Lagrange's mean value theorem (see Lemma 4). The following lemma is an easy consequence of [28, p. 135].

Lemma 1 *Let $a, b \in \mathbb{R}$, and let $f : [a,b] \to \mathbb{R}$ be a continuous function. If $f^u(x) > 0$ for every $x \in (a,b)$, then f is increasing.*

Next lemma is the generalization of classical Cauchy's theorem.

Lemma 2 *Let $a, b \in \mathbb{R}$, let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous functions, and let $g'(x) > 0$ for every $x \in (a,b)$. Then there are $\xi_1, \xi_2 \in (a,b)$ satisfying*

$$\frac{f^\ell(\xi_1)}{g'(\xi_1)} \le \frac{f(b)-f(a)}{g(b)-g(a)} \le \frac{f^\ell(\xi_2)}{g'(\xi_2)}.$$

Proof We consider the function $F : [a,b] \to \mathbb{R}$ such that

$$F(x) = (f(b)-f(a))(g(x)-g(a)) - (f(x)-f(a))(g(b)-g(a)), \quad \forall x \in [a,b].$$

Since F is continuous and $F(a) = F(b)$, by Lemma 1 there is $\xi_2 \in (a,b)$ with the property

$$F^u(\xi_2) = (f(b)-f(a))g'(\xi_2) - f^\ell(\xi_2)(g(b)-g(a)) \le 0.$$

Indeed, supposing $F^u(\xi) > 0$ for every $\xi \in (a,b)$, we obtain by Lemma 1 that $F(b) > F(a)$, a contradiction. Note that due to Lemma 1, it holds $g(b) > g(a)$.

Therefore

$$\frac{f(b)-f(a)}{g(b)-g(a)} \le \frac{f^\ell(\xi_2)}{g'(\xi_2)},$$

which is the second inequality. The first inequality will be obtained by transition to the function $-f$ and by using the obvious inequality $f^\ell \le f^u$. □

Setting $g(x) = x$ in Lemma 2, we obtain the following lemma immediately.

Lemma 3 *Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $\xi_1, \xi_2 \in (a, b)$ such that*

$$(b-a)f^\ell(\xi_1) \le f(b) - f(a) \le (b-a)f^\ell(\xi_2).$$

Lemma 4 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function, and let $a, b \in \mathbb{R}^N$. Then there exist $\xi_1, \xi_2 \in (a, b)$ such that*

$$f^\ell(\xi_1; b-a) \le f(b) - f(a) \le f^\ell(\xi_2; b-a). \tag{1}$$

Proof Considering the function $g : [0, 1] \to \mathbb{R}$, $g(t) = f(a + t(b-a))$, due to Lemma 3 we can find $t_1, t_2 \in (0, 1)$ satisfying

$$g^\ell(t_1) \le g(1) - g(0) \le g^\ell(t_2). \tag{2}$$

For every $t \in (0, 1)$, we have

$$\begin{aligned} g^\ell(t) &= \liminf_{s \downarrow 0} \frac{g(t+s) - g(t)}{s} \\ &= \liminf_{s \downarrow 0} \frac{f(a + (t+s)(b-a)) - f(a + t(b-a))}{s} \\ &= f^\ell(a + t(b-a); b-a). \end{aligned} \tag{3}$$

Setting $\xi_i = a + t_i(b-a)$ for $i = 1, 2$, formulas (2) and (3) imply formula (1). □

It follows immediately from Definition 4 that if f is ℓ-stable, then $f^\ell(y; h)$ is finite for every y sufficiently near x and for every $h \in \mathbb{R}^N$. In fact, we can say more.

Lemma 5 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function near $x \in \mathbb{R}^N$ and let f be ℓ-stable at x. Then there exists a neighbourhood V of x such that*

$$\sup_{h \in S,\, y \in V} |f^\ell(y; h)| < +\infty. \tag{4}$$

Proof Suppose for a contradiction that there are sequences $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, $\{h_n\}_{n=1}^{\infty} \subset S$ such that $y_n \to x$ as $n \to +\infty$ and

$$\lim_{n\to\infty} |f^{\ell}(y_n; h_n)| = +\infty.$$

Without any loss of generality we can assume that either

$$\lim_{n\to+\infty} f^{\ell}(y_n, h_n) = -\infty$$

or

$$\lim_{n\to+\infty} f^{\ell}(y_n, h_n) = +\infty.$$

We suppose that the first case occurs (the second case can be treated by an analogous way).

Next we can assume that for certain $\gamma > 0$ the condition in Definition 4 of ℓ-stability is fulfilled on $B(x, \gamma)$, and moreover f is continuous and bounded on $B(x, \gamma)$. Let $\delta > 0$ denotes a constant such that for each sufficiently large $n \in \mathbb{N}$ we have : $y_n + \delta h_n \in B(x, \gamma)$.

Now, if we combine ℓ-stability and Lemma 4, for each sufficiently large $n \in \mathbb{N}$ we get $\xi_n \in (y_n, y_n + \delta h_n)$ such that the following holds :

$$\begin{aligned} f(y_n + \delta h_n) &\le f(y_n) + \delta f^{\ell}(\xi_n; h_n) \\ &= f(y_n) + \delta[f^{\ell}(\xi_n; h_n) - f^{\ell}(x; h_n) + f^{\ell}(x; h_n) \\ &\quad - f^{\ell}(y_n; h_n) + f^{\ell}(y_n; h_n)] \\ &\le f(y_n) + 2K\delta\gamma + \delta f^{\ell}(y_n; h_n). \end{aligned}$$

Since f is bounded on $B(x, \gamma)$ and $f^{\ell}(y_n; h_n) \to -\infty$, the previous inequality does not hold for every $n \in \mathbb{N}$, a contradiction. □

Proposition 1 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function near $x \in \mathbb{R}^N$ and let f be ℓ-stable at x. Then f is Lipschitz near x.*

Proof Thanks to Lemma 5 there is some ball $B(x, \delta)$ on which f is continuous and

$$L := \sup_{y\in B(x,\delta), h\in S} |f^{\ell}(y; h)| < +\infty.$$

Next by Lemma 4, for any pair of points $a, b \in B(x, \delta)$ there exists $\xi \in (a, b) \subset B(x, \delta)$ such that

$$\begin{aligned} |f(b) - f(a)| &\le |f^{\ell}(\xi; (b - a)/\|b - a\|)|\|b - a\| \\ &\le L\|b - a\|. \end{aligned}$$ □

Proposition 2 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function near $x \in \mathbb{R}^N$ and let f be ℓ-stable at x. Then f is strictly differentiable at x.*

Proof By Proposition 1 there is a neighbourhood U of x on which f is Lipschitz. Now, due to the Rademarcher theorem, we can find a sequence $\{x_n\}_{n=1}^{\infty} \subset U$ such that $f'(x_n)$ exists for each $n \in \mathbb{N}$, and $x_n \to x$ as $n \to +\infty$. We claim that for any fixed $h \in \mathbb{R}^N$, the sequence $\{f'(x_n)h\}_{n=1}^{\infty}$ is Cauchy and hence converges. Indeed, using ℓ-stability at x, we get for any $m, n \in \mathbb{N}$:

$$\begin{aligned}
0 &\le |f'(x_n)h - f'(x_m)h| \\
&= |f'(x_n)h - f^{\ell}(x;h) + f^{\ell}(x;h) - f'(x_m)h| \\
&\le |f'(x_n)h - f^{\ell}(x;h)| + |f^{\ell}(x;h) - f'(x_m)h| \\
&\le K(\|x_n - x\| + \|x - x_m\|)\|h\|.
\end{aligned}$$

If we here pass to the limit where $m, n \to +\infty$, we see that

$$|f'(x_n)h - f'(x_m)h| \to 0, \quad \text{as } m, n \to +\infty.$$

Thus, for any $h \in \mathbb{R}^N$, we can put $L(h) := \lim_{n\to\infty} f'(x_n)h$. Clearly $L \in \mathbb{R}^N$. We claim that for any $h \in \mathbb{R}^N$, $L(h) = f^{\ell}(x;h)$. Indeed, for fixed $h \in \mathbb{R}^N$, it holds

$$\begin{aligned}
0 &\le |L(h) - f^{\ell}(x;h)| = |L(h) - f'(x_n)h + f'(x_n)h - f^{\ell}(x;h)| \\
&\le |L(h) - f'(x_n)h| + |f'(x_n)h - f^{\ell}(x;h)| \\
&\le |L(h) - f'(x_n)h| + K\|x_n - x\|\|h\|.
\end{aligned}$$

In view of definition of $L(h)$ and the fact that $x_n \to x$, we infer $|L(h) - f^{\ell}(x;h)| \le 0$, i.e. $L(h) = f^{\ell}(x;h)$ for each $h \in \mathbb{R}^N$. This equality, Lemma 4, and ℓ-stability now imply that for all y sufficiently close to x, all $t > 0$ sufficiently small, and all $h \in S$, there is $\xi \in (y, y + th)$ such that it holds

$$|[f(y + th) - f(y)]/t - L(h)| \le |f^{\ell}(\xi;h) - f^{\ell}(x;h)| \le K\|\xi - x\|$$

This immediately yields that $[f(y+th)-f(y)]/t \to L(h)$ as $y \to x, t \downarrow 0$ uniformly with respect to $h \in S$. □

Now, we are able to show that the class of the functions which are ℓ-stable at some point and continuous near this point solves the problem given before Theorem P.

Theorem 6 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be continuous near $x \in \mathbb{R}^N$ and let f be ℓ-stable at x. If $f^{\ell}(x;h) = 0$ for every $h \in S$, and*

$$\underline{f}'^{\ell}_{P}(x;h) > 0, \quad \forall h \in S, \tag{5}$$

then x is an isolated minimizer of order 2 for f. Conversely, each isolated minimizer of order 2 satisfies these sufficient conditions.

Proof We can assume, without any loss of generality, that $x = 0$ and $f(0) = 0$.

Suppose for a contradiction that 0 is not an isolated minimizer of order 2 for f. Then there exists a sequence $\{x_n\}_{n=1}^{+\infty} \subset \mathbb{R}^N \setminus \{0\}$ satisfying $\lim_{n\to+\infty} x_n = 0$ and

$$f(x_n) < \frac{1}{n}\|x_n\|^2, \quad \forall n \in \mathbb{N}. \tag{6}$$

Setting

$$h_n = \frac{x_n}{\|x_n\|}, \quad t_n = \|x_n\|, \quad \forall n \in \mathbb{N},$$

we have $h_n \in S$ for every $n \in \mathbb{N}$. Because of the compactness of S, we can find, without any loss of generality, $\tilde{h} \in S$ with the property

$$\lim_{n\to+\infty} h_n = \tilde{h}. \tag{7}$$

Formula (5) and $f^\ell(0;\tilde{h}) = 0$ imply the existence of $c > 0$ and $\delta > 0$ such that

$$\frac{f(t\tilde{h})}{t^2} > c, \quad \forall 0 < t \le \delta. \tag{8}$$

Since f is ℓ-stable at 0, there are an open neighbourhood U of 0 and $K > 0$ satisfying

$$|f^\ell(y;h)| \le K\|y\|\|h\|, \quad \forall y \in U, \ \forall h \in \mathbb{R}^N. \tag{9}$$

Using the formula (6) and Lemma 4, respectively, it holds for every sufficiently large $n \in \mathbb{N}$ that

$$\frac{f(t_n\tilde{h}) - \frac{1}{n}\|x_n\|^2}{t_n^2} < \frac{f(t_n\tilde{h}) - f(t_nh_n)}{t_n^2} \le \frac{f^\ell(a_n; t_n(\tilde{h} - h_n))}{t_n^2}, \tag{10}$$

where $a_n \in (x_n, t_n\tilde{h})$.

Due to the inequalities (8), (9) and (10), we obtain for every sufficiently large $n \in \mathbb{N}$ that

$$\begin{aligned} 0 < \frac{c}{2} \le \frac{f(t_n\tilde{h}) - \frac{1}{n}\|x_n\|^2}{t_n^2} \le \frac{f^\ell(a_n; t_n(\tilde{h} - h_n))}{t_n^2} \\ \le \frac{K\|a_n\|t_n\|\tilde{h} - h_n\|}{t_n^2}. \end{aligned}$$

Because of $\|a_n\| \leq t_n$ for every $n \in \mathbb{N}$ considered before, it follows from the previous inequality that

$$\frac{c}{2K} \leq \lim_{n\to+\infty} \|\tilde{h} - h_n\|,$$

what is a contradiction with (7).

Conversely, if x is an isolated minimizer of order 2 for a function $f : \mathbb{R}^N \to \mathbb{R}$ which is continuous near $x \in \mathbb{R}^N$ and ℓ-stable at x, then f is differentiable at x by Proposition 2, $f'(x) = 0$ and $\underline{f}'^{\ell}_{P}(x;h) > 0$ for every $h \in S$. □

3 Comparisons and examples

For a function $f : \mathbb{R}^N \to \mathbb{R}$ which satisfies the $C^{1,1}$ property near $x \in \mathbb{R}^N$, and $h \in \mathbb{R}^N$, we can easily derive from [29, Theorem 4], the relation

$$f'^{\ell}_{D}(x;h) \leq f'^{\ell}_{P}(x;h).$$

Theorem 6 generalizes Theorems 1, 2, 4 and 5 now, as one can see immediately from the definitions and from the previous relation.

Example 1 Let us consider a function

$$f(x) = \begin{cases} \int_0^{|x|} t\left(\frac{19}{20} + \sin\ln t\right) dt & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

f is $C^{1,1}$ function since $f'(x)h = x((19/20) + \sin(\ln|x|))h$ for $x \neq 0, h \in \mathbb{R}$, and $f'(0) = 0$. Integration per partes gives

$$\int t\sin(\ln\, t)dt = (1/5)t^2(2\sin(\ln\, t) - \cos(\ln\, t)).$$

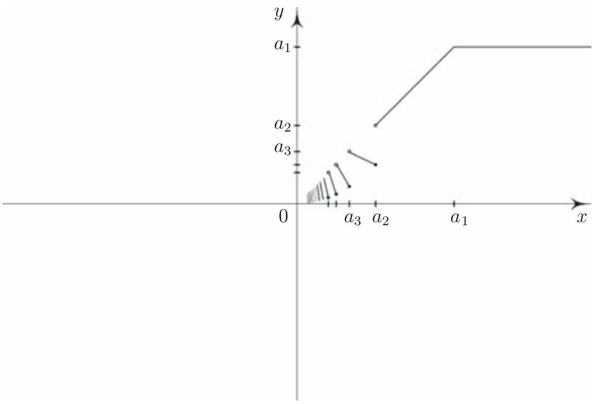
As a consequence, we have :

$$\int\limits_0^{|x|} t\sin(\ln\, t)dt = \frac{1}{5}x^2(2\sin(\ln\, |x|) - \cos(\ln\, |x|)).$$

Then

$$\begin{aligned} \underline{f}'^{\ell}_{P}(0;1) = \underline{f}'^{\ell}_{P}(0;-1) &= \liminf_{x\downarrow 0} \frac{\frac{19}{20}\frac{x^2}{2} + \frac{1}{5}x^2(2\sin(\ln\, x) - \cos(\ln\, x))}{x^2/2} \\ &= \frac{19}{20} + \frac{2}{5}\liminf_{x\downarrow 0}(2\sin(\ln\, x) - \cos(\ln\, x)) = \frac{19}{20} + \frac{2}{5}(-\sqrt{5}) > 0. \end{aligned}$$

Fig. 1 Function φ



Consequently, according to Theorem 6 $x = 0$ is a strict local minimizer for f. Further calculation shows : $f_D^{\prime\ell}(0;1) = (19/20) - 1 < 0, f_\infty^{\prime\ell}(0;1) = (19/20) - \sqrt{2} < 0$ (for this calculation, see [2]), and $f_P''(0;1)$ does not exist. In contrast to Theorems 2 and 6, we can see that Theorems 1, 4 and 5 are not applicable in this context.

The following example illustrates that Theorem 6 also overcomes Theorem 2.

Example 2 Consider a sequence $a_n = 1/n, n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} \frac{a_{n+1} + a_n^2}{a_{n+1} + a_n} = \frac{1}{2} > 0.$$

Let us define a function $\varphi : [0, \infty) \to \mathbb{R}$ as follows (see Fig. 1 for the construction of φ).

$$\varphi(u) = \begin{cases} a_1 & \text{if } u > a_1, \\ \frac{a_n^2 - a_{n+1}}{a_n - a_{n+1}}(u - a_{n+1}) + a_{n+1} & \text{if } u \in (a_{n+1}, a_n], \\ 0 & \text{if } u = 0. \end{cases}$$

Next, we will define a function $f : \mathbb{R} \to \mathbb{R}$ via the Riemann integral :

$$f(x) := \int_0^{|x|} \varphi(u)du, \quad x \in \mathbb{R}.$$

Since φ is a piecewise affine function, the integral exists. Because of $f'(a_n; 1) = a_n$ and $f'(a_n; -1) = a_n^2$ for every $n > 1$, the function f is not differentiable on any neighbourhood of 0. Thus, we cannot use Theorems 1, 2, 4 and 5.

On the other hand, f is Lipschitz on $\mathbb{R}$ with with Lipschitz constant equal to a_1. Let $x, y \in \mathbb{R}$ be arbitrary points such that $|x| \leq |y|$. We have:

$$|f(x) - f(y)| = \left| \int_0^{|x|} \varphi(u)du - \int_0^{|y|} \varphi(u)du \right| = \left| \int_{|x|}^{|y|} \varphi(u)du \right|$$
$$\leq \int_{|x|}^{|y|} |\varphi(u)|du \leq a_1(|y| - |x|) \leq a_1|x - y|.$$

It is easy to show that $f'(0) = 0$ and that f is ℓ-stable at $x = 0$. Now we claim that $f_P^{'\ell}(0; \pm 1) > 0$. So it suffices to show that

$$\liminf_{t \downarrow 0} \frac{f(t)}{t^2/2} > 0.$$

Note that there is $\epsilon > 0$ such that for each $n \in \mathbb{N}$ it holds:

$$\frac{a_{n+1} + a_n^2}{a_{n+1} + a_n} \geq \epsilon > 0. \tag{11}$$

Now consider $t \in [a_{j+1}, a_j)$ for some $j \in \mathbb{N}$ and fix $k \in \mathbb{N}, k \geq j + 2$. Let S_n denotes an area of a trapezoid over the interval $(a_{n+1}, a_n), n = j + 1, \ldots, k$, bounded by a graph of φ. Let R denotes an area of a trapezoid over the interval (a_{j+1}, t) bounded by the graph of φ. Now we can write down the formula for the integral:

$$\int_{a_k}^{t} \varphi(u)du = \left(\sum_{n=j+1}^{k} S_n \right) + R. \tag{12}$$

Further $\tilde{S}_n$ stands for an area of a trapezoid over the interval $(a_{n+1}, a_n), n = j + 1, \ldots, k$ bounded by the linear function $y = x$, and $\tilde{R}$ stands for an area of a trapezoid over the interval (a_{j+1}, t) bounded also by the function $y = x$. Now it can be shown that

$$\int_{a_k}^{t} \varphi(u)du = \left(\sum_{n=j+1}^{k} S_n \right) + R \geq \epsilon \sum_{n=j+1}^{k} \tilde{S}_n + \epsilon \tilde{R}$$
$$= \epsilon \left(\sum_{n=j+1}^{k} \tilde{S}_n + \tilde{R} \right).$$

Letting $k \to +\infty$, we will get:

$$f(t) = \int_0^t \varphi(u)dt \geq \epsilon \left(\sum_{n=j+1}^{\infty} \tilde{S}_n + \tilde{R} \right)$$
$$= \epsilon \frac{t^2}{2}.$$

Hence $2f(t)/t^2 \geq \epsilon > 0$, where $t \in [a_{j+1}, a_j)$. Since this holds for almost any $j \in \mathbb{N}$ and for all $t \in [a_{j+1}, a_j)$, we have for any $\delta > 0$ sufficiently small,

$$\inf \left\{ 2\frac{f(t)}{t^2} : t \in (0, \delta) \right\} \geq \epsilon > 0.$$

Hence $\liminf_{t \downarrow 0} 2f(t)/t^2 \geq \epsilon > 0$. Now, we can conclude that f satisfies the assumptions of Theorem 6 and therefore $x = 0$ is a strict local minimizer for f.

Finishing this section, we compare Theorems 6 and 3. We will need the following proposition which can be obtained as a special case of [16, Theorem 4.1].

Proposition 3 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz function, and $x, u \in \mathbb{R}^N$ with $\partial_u f(x) = \{0\}$. Then*

$$f''_{-}(x, 0, h) \leq \underline{f}'^{\ell}_{P}(x; h).$$

Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. Since f is not differentiable at 0, f is not ℓ-stable at 0 by Proposition 2 and we cannot use Theorem 6 to verify that 0 is a strict local minimizer for f in contrast to Theorem 3.

Moreover, it is not possible to find an example of function f which satisfies the assumptions of Theorem 6 but doesn't satisfy the assumptions of Theorem 3. Indeed, if $f : \mathbb{R}^N \to \mathbb{R}$ is ℓ-stable at $x \in \mathbb{R}^N$, continuous near x, $f^{\ell}(x; h) = 0$ and $\underline{f}'^{\ell}_{P}(x; h) > 0$ for every $h \in S$,then from Proposition 2 and the well known fact that the Clarke subdifferential of strictly differentiable function f at x is $\{f'(x)\}$, we have that $\partial_u f(x) = \{0\}$ for every $h \in S$. Because x is a local minimizer of order 2 for f by Theorem 6, it follows from the definition that $f''_{-}(x, 0, h) > 0$ for every $h \in S$.

On the other hand, for this special situation (i.e. f is continuous near x and ℓ-stable at x) Theorem 6 offers a sharper conclusion (x is a strict local minimizer of order 2) and it seems that the calculus with $\underline{f}'^{\ell}_{P}(x; h)$ is more comfortable than with $f''_{-}(x, 0, h)$. Notice also that $f'(x) = 0$ implies by Propositions 2 and 3 that $f''_{-}(x, 0, h) \leq \underline{f}'^{\ell}_{P}(x; h)$.

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