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On the polyhedral structure of a multi–item production planning model with setup times

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Abstract. We present and study a mixed integer programming model that arises as a substructure in many industrial applications. This model generalizes a number of structured MIP models previously studied, and it provides a relaxation of various capacitated production planning problems and other fixed charge network flow problems. We analyze the polyhedral structure of the convex hull of this model, as well as of a strengthened LP relaxation. Among other results, we present valid inequalities that induce facets of the convex hull under certain conditions. We also discuss how to strengthen these inequalities by using known results for lifting valid inequalities for 0–1 continuous knapsack problems.

Key words. mixed integer programming – production planning – polyhedral combinatorics – capacitated lot–sizing – fixed charge network flow

1. Introduction

One of the most successful techniques that has been used to solve integer programming (IP) and mixed integer programming (MIP) problems in recent years is the application of polyhedral results for structured relaxations of these problems. Often facet–defining families of valid inequalities can be defined for the relaxed models. Since these inequalities are also valid for the original problem, they can be applied in a branch–and–cut algorithm.

In this paper we introduce a model that occurs as a relaxation of many structured MIP problems. This model occurs as a substructure of several production planning problems, including the multi–item capacitated lot–sizing problem with setup times (MCL) (see, e.g., Trigiero, Thomas, and McClain [1989]). MCL and many other production planning problems deal with the the interaction of both capacity constraints and demand satisfaction constraints over a finite time horizon composed of discrete time periods. These

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problems can be very difficult, both computationally and theoretically; for example, even finding a feasible solution for MCL is strongly $\mathcal{NP}$–hard.

The model that we study in this paper is a single–period relaxation of MCL; that is, we consider the interaction of demand and capacity constraints in a single period, rather than over the entire horizon. In this relaxation we consider, among other decision variables, those that represent the amount of inventory carried over from the preceding period; therefore we call this relaxation PI, for *preceding inventory*. PI can be formulated as follows:

$$\min \sum_{i=1}^{P} p^i x^i + \sum_{i=1}^{P} q^i y^i + \sum_{i=1}^{P} h^i s^i \tag{1}$$

subject to

$$x^i + s^i \geq d^i, i = 1, \dots, P, \tag{2}$$

$$\sum_{i=1}^{P} x^i + \sum_{i=1}^{P} t^i y^i \leq c, \tag{3}$$

$$x^i \leq (c - t^i) y^i, i = 1, \dots, P, \tag{4}$$

$$x^i, s^i \geq 0, i = 1, \dots, P, \tag{5}$$

$$y^i \in \{0, 1\}, i = 1, \dots, P. \tag{6}$$

In the context of multi–item lot–sizing, there are P items indexed by i. The variables x^i can be seen as the production variables, y^i as the setup variables, and s^i as the inventory variables for inventory for the *preceding* period–that is, inventory that can be used to satisfy current demand. Production, setup, and holding costs are given by p^i, q^i, and h^i, respectively. Constraints (2) ensure that the demand d^i for each item is satisfied, either from production or inventory. (We assume that $d^i > 0, i = 1, \dots, P$.) Constraint (3) enforces the capacity restriction: production time and time consumed by setups must not exceed the capacity c. (We assume that the setup times t^i are nonnegative for $i = 1, \dots, P$.) Constraints (4) ensure that a setup occurs if any quantity of an item is produced. Throughout the paper, we will let $\mathcal{P} = \{1, \dots, P\}$, and we will denote the set of points defined by (2)–(6) as X^{PI}.

During the last two decades researchers have often used strong valid inequalities for structured MIP models to solve more complicated problems. Among those who pioneered such methods were Crowder, Johnson, and Padberg [1983], who demonstrated that the use of inequalities for knapsack relaxations defined by single constraints of 0–1 IP problems can be effective in solving IP's. Today the use of valid inequalities for the knapsack polytope is a common feature in IP solvers. Padberg, Van Roy, and Wolsey [1985] derived flow cover inequalities for a relaxation of MIP's defined by a single–node fixed charge (SNFC) network flow model. This research has been continued and extended by, among others, Van Roy and Wolsey [1986], Gu, Nemhauser, and Savelsbergh [1999], and Marchand and Wolsey [1999], and the use of inequalities for SNFC to solve MIP's has become common.

If we assume that there are P inbound arcs into the single node, the basic model SNFC can be expressed as

$$\min \sum_{i=1}^{P} b^i x^i + \sum_{i=1}^{P} q^i y^i \tag{7}$$

$$\text{subject to } x^i \leq d^i y^i, i = 1, \ldots, P, \tag{8}$$

$$\sum_{i=1}^{P} x^i \leq c, \tag{9}$$

$$x^i \geq 0, i = 1, \ldots, P, \tag{10}$$

$$y^i \in \{0, 1\}, i = 1, \ldots, P. \tag{11}$$

When $t^i = 0, i = 1, \ldots, P$, PI resembles SNFC, but even in this case PI has a richer structure. To see this, consider the case of PI in which $t^i = 0, i = 1, \ldots, P$ and $p^i > 0, h^i > 0, i = 1, \ldots, P$. Because of these conditions, (2) holds at equality in every optimal solution, and therefore we can eliminate the variables s^i, letting $b^i = p^i - h^i$. Adding the simple mixed integer rounding inequalities $x^i \leq d^i y^i$ (see Gomory [1960]), we see that this special case of PI is the model SNFC. Thus, PI is a generalization of SNFC when $t^i = 0, i = 1, \ldots, P$. and allowing nonnegative setup times generalizes the model even further. In fact, PI generalizes models such as those studied by Van Roy and Wolsey [1986] and Goemans [1989], which are themselves generalizations of SNFC.

We mentioned that PI serves as a natural relaxation of MCL; there are a number of other production planning problems for which PI provides a relaxation. For example, results for PI can be applied to problems with backorders, with overtime variables, and with multiple machines (see Miller [1999]). Moreover, Pochet and Wolsey [1991] showed how to model *multi–level* capacitated lot–sizing problems (i.e., problems in which the products are organized in a bill–of–materials structure) using *echelon stocks*. Using these results, it is easy to show that PI provides a single–machine, single–period relaxation for many multi–level lot–sizing problems as well. In addition, since PI generalizes various fixed charge flow models, as discussed above, it can provide a relaxation for more general fixed charge network flow problems as well.

An outline of the rest of the paper follows. In Section 2, we analyze the polyhedral structure of PI. Since PI is $\mathcal{NP}$–hard, it is unlikely that it is possible to derive a good characterization of conv(X^{PI}) by families of linear inequalities (Karp and Papadimitrou [1982]). However, we can characterize the convex hull by extreme points and rays, and define families of facet–inducing valid inequalities. We also show which extreme points are possible optimal solutions of PI when we impose certain conditions on the objective function coefficients. We perform a similar analysis for an LP relaxation of X^{PI} that is strengthened with a family of facet–inducing valid inequalities analogous to the *path inequalities* (Barany, Van Roy, and Wolsey [1984], Van Roy and Wolsey [1987]).

In Section 3, we derive two new valid inequalities for PI. We give general conditions under which each of these classes of inequalities define facets of conv(X^{PI}), and we also show that these new inequalities cut off all possible fractional optimal solutions of the strengthened LP relaxation mentioned above.

In Section 4, we explore the relationship between the 0–1 continuous knapsack problem (CKP), studied by Marchand and Wolsey [1999], among others, and PI. In particular, we show how to use results for CKP to strengthen one of the families of inequalities defined in Section 3 through sequential lifting. We also show how to use results for CKP to derive a third new family of valid inequalities for PI.

2. Basic polyhedral results

In this section we present basic results concerning the polyhedral structure of PI. Some easy proofs are omitted. We begin by listing the trivial facets of conv(X^{PI}).

2.1. Trivial facets

Proposition 1. *If $c > t^i, i = 1, \dots, P$, conv(X^{PI}) is full–dimensional. Moreover, for $i = 1, \dots, P$, (2), (4), and the bound $x^i \geq 0$ induce facets of conv(X^{PI}).*

Proposition 2. *If for every $(i, j) \in \mathcal{P} \times \mathcal{P}, i \neq j, c > t^i + t^j$, then (3) and the bounds $y^i \leq 1, i = 1, \dots, P$ induce facets of conv(X^{PI}).*

Proposition 3. *If $c > t^i + d^i$, the valid inequality*

$$s^i + d^i y^i \geq d^i \tag{12}$$

induces a facet of conv(X^{PI}), for $i = 1, \dots, P$.

Note that (12) implies the bound $s^i \geq 0, i = 1, \dots, P$; therefore this bound never induces a facet. Also note that these inequalities can be seen as (l, S) inequalities, introduced by Barany, Van Roy, and Wolsey [1984] for the uncapacitated lot–sizing problem. Van Roy and Wolsey [1987] later generalized these to the path inequalities for more general fixed charge network flow problems.

2.2. Extreme points and rays

We will now discuss the extreme points and rays of conv(X^{PI}) and of an LP relaxation of X^{PI}. Given an extreme point $(\bar{x}, \bar{y}, \bar{s})$ of conv(X^{PI}), let $Q = \{i \in \mathcal{P} : \bar{y}^i = 1\}$. Also, let $Q_u = \{i \in Q : \bar{x}^i = d^i\}$, $Q_l = \{i \in Q : \bar{x}^i = 0\}$, and let $Q_r = \{i \in Q : \bar{x}^i > 0, \bar{x}^i \neq d^i\}$. For a given i, we define types of (x^i, y^i, s^i) triples:

$$\begin{aligned}
&\text{Type 1: } \bar{x}^i = d^i, \bar{y}^i = 1, \bar{s}^i = 0,\\
&\text{Type 2: } \bar{x}^i = 0, \bar{y}^i = 1, \bar{s}^i = d^i,\\
&\text{Type 3: } \bar{x}^i = 0, \bar{y}^i = 0, \bar{s}^i = d^i.
\end{aligned}$$

Proposition 4. (Characterization of the extreme points of conv(X^{PI})) *In every extreme point* $(\bar{x}, \bar{y}, \bar{s})$ *of conv(*X^{PI}*),* $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ *is of Type 1,* $i \in Q_u$*;* $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ *is of Type 2,* $i \in Q_l$*; and* $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ *is of Type 3,* $i \in \mathcal{P} \setminus Q$*. Moreover, either* $Q_r = \emptyset$*, or* $|Q_r| = 1$ *and*

$$\bar{x}^r = c - \sum_{i \in Q_u} (t^i + d^i) - \sum_{i \in Q_l} t^i - t^r, \bar{y}^r = 1, \bar{s}^r = (d^r - \bar{x}^r)^+, \tag{13}$$

where $Q_r = \{r\}$.

The fact that $|Q_r| \le 1$ in every extreme point of conv(X^{PI}) can be shown by contradiction. In effect, this says that there is only one $i \in \mathcal{P}$ for which the constraint (3) plays a part in determining the value $\bar{x}^i$. For the other $i \in \mathcal{P}$, $\bar{x}^i$ is determined by some interaction among (2), the variable upper bound (4), and the bound $x^i \ge 0$. If $|Q_r| > 1$ for some $(\bar{x}, \bar{y}, \bar{s}) \in X^{PI}$, then it is not difficult to construct two new points in X^{PI} such that $(\bar{x}, \bar{y}, \bar{s})$ is a convex combination of the two.

Now define X^{PI}_{LP} to be the set of feasible points to the LP relaxation of (2)–(6), (12); we can characterize its extreme points as well. Given an extreme point $(\bar{x}, \bar{y}, \bar{s})$ of X^{PI}_{LP}, let Q, Q_u, and Q_l be defined as for X^{PI}, and let $R = \{i \in \mathcal{P} : 0 < \bar{y}^i < 1\}$.

Proposition 5. (Characterization of the extreme points of X^{PI}_{LP}) *Let* $(\bar{x}, \bar{y}, \bar{s})$ *be an extreme point of* X^{PI}_{LP}*. Then* $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ *is of Type 1,* $i \in Q_u$*;* $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ *is of Type 2,* $i \in Q_l$*; and* $(\bar{x}^i, \bar{y}^i, \bar{s}^i)$ *is of Type 3,* $i \in \mathcal{P} \setminus \{Q \cup R\}$*. Moreover, if* $Q_r \cup R$ *is non–empty, it consists of a singleton* $\{r\}$*, and exactly one of the following statements is true:*

1. $(\bar{x}^r, \bar{y}^r, \bar{s}^r)$ *is defined as in (13), and thus* $(\bar{x}, \bar{y}, \bar{s})$ *is an extreme point of conv(*X^{PI}*);*
2. $$\bar{y}^r = \frac{c - \sum_{i \in Q_u}(t^i + d^i) - \sum_{i \in Q_l} t^i}{t^r}, \bar{x}^r = 0, \bar{s}^r = d^r; \tag{14}$$
3. $$\bar{y}^r = \frac{c - \sum_{i \in Q_u}(t^i + d^i) - \sum_{i \in Q_l} t^i}{t^r + d^r}, \bar{x}^r = d^r \bar{y}^r, \bar{s}^r = d^r(1 - \bar{y}^r); \tag{15}$$
4. $$\bar{y}^r = \frac{c - \sum_{i \in Q_u}(t^i + d^i) - \sum_{i \in Q_l} t^i}{c}, \bar{x}^r = (c - t^r)\bar{y}^r, \bar{s}^r = d^r(1 - \bar{y}^r). \tag{16}$$

The reasoning that $|Q_r \cup R| \le 1$ is similar to the reasoning that $|Q_r| \le 1$ in Proposition 4. For fractional extreme points defined by (14), inequality (2) and the bound $x^r \ge 0$ are both tight for r, and inequalities (4) and (12) are not. For fractional extreme points defined by (15), inequalities (2) and (12) are both tight for r, inequality (4) is not, and $x^r > 0$. For fractional extreme points defined by (16), inequalities (4) and (12) are tight, inequality (2) is not, and $x^r > 0$.

Proposition 6. *All extreme rays of both conv(*X^{PI}*) and* X^{PI}_{LP} *have the form* $s^i = 1, x^i = y^i = 0$*, for some* $i \in \mathcal{P}$*, and* $s^j = x^j = y^j = 0$*,* $j \neq i$*.*

Thus PI has a bounded optimum if and only if $h^i \geq 0, i = 1, \ldots, P$, and PI has an unbounded optimal solution if and only if $h^i \leq 0$, for some $i \in \mathcal{P}$. If $h^i \geq 0, i = 1, \ldots, P$, and there is an element i for which $h^i = 0$, then PI has an infinite number of optimal solutions. If we assume that $h^i > 0, i = 1, \ldots, P$, then every optimal solution of PI is bounded.

These results provide a characterization of both conv(X^{PI}) and conv(X^{PI}_{LP}) by their extreme points and rays. Next, we discuss which extreme points are possible optimal solutions of X^{PI} and X^{PI}_{LP} when we impose the conditions $q^i > 0, h^i > 0, i = 1, \ldots, P$. In discussing a set of points $X \subset \{(x, y, s) : x^i, s^i \geq 0, 0 \leq y^i \leq 1, i = 1, \ldots, P\}$, we will use

Definition 1. *The point $(\bar{x}, \bar{y}, \bar{s})$ is a* ***dominant solution*** *of X if there exists some cost vector (p, q, h), such that $q^i > 0, h^i > 0, i = 1, \ldots, P$, for which optimizing (1) over X yields $(\bar{x}, \bar{y}, \bar{s})$ as the* ***unique*** *optimal solution.*

This definition allows us to obtain

Proposition 7. *The point (x, y, s) is a dominant solution of X^{PI} for PI if and only if (1) (x, y, s) is an extreme point of conv(X^{PI}); and (2) $y^i = 1$ if and only if $x^i > 0, i = 1, \ldots, P$.*

Thus, the condition $q^i > 0, i = 1, \ldots, P$, ensures that a setup for an item does not take place unless production of that item also occurs. This is a reasonable condition to enforce; for example, in both lot–sizing and fixed charge problems, a fixed cost to produce/transport an item is not paid unless some positive quantity of that item is produced/transported. A similar result holds for X^{PI}_{LP}.

Proposition 8. *Given an instance of PI, no dominant solution of X^{PI}_{LP} has the form (14). Moreover, every dominant solution of X^{PI}_{LP} is either a dominant solution of X^{PI}, or else it is of one of the forms (15) or (16) with $Q_l = \emptyset$.*

3. New valid inequalities for PI

In this section we will derive two new families of valid inequalities for X^{PI} and present results concerning their strength. In defining these inequalities we use the following definition.

Definition 2. *A* cover *of PI is a set S such that $\lambda = \sum_{i \in S}(t^i + d^i) - c \geq 0$. A* reverse cover *of PI is a set $S \neq \emptyset$ such that $\mu = c - \sum_{i \in S}(t^i + d^i) > 0$.*

3.1. Cover inequalities

3.1.1. Basic cover inequalities To motivate this class, consider a cover S of PI. Recall that

$$\lambda = \sum_{i \in S}(t^i + d^i) - c \geq 0.$$

So if $y^i = 1, i \in S$,

$$\sum_{i\in S} s^i \geq \sum_{i\in S} d^i - \sum_{i\in S} x^i \geq \sum_{i\in S} d^i - (c - \sum_{i\in S} t^i) = \sum_{i\in S}(t^i + d^i) - c \geq \lambda.$$

Moreover, if $y^{i'} = 0$ for exactly one $i' \in S$, then both

$$\sum_{i\in S} s^i \geq \lambda - t^i$$

and

$$\sum_{i\in S} s^i \geq s^{i'} \geq d^{i'}$$

must hold. From such reasoning we can derive that

$$\sum_{i\in S} s^i \geq \lambda + \sum_{i\in S} \max\{-t^i, d^i - \lambda\}(1 - y^i) \tag{17}$$

is valid for PI. A more formal statement and proof follows.

Proposition 9. (Cover Inequalities, basic form) *Given a cover S of PI, (17) is valid for* X^{PI}.

Proof. Let $(\bar{x}, \bar{y}, \bar{s})$ be any point in X^{PI}. Let $S^1 = \{i \in S : \bar{y}^i = 1\}$, and let $S^0 = \{i \in S : \bar{y}^i = 0\}$. If $\{i \in S^0 : t^i + d^i > \lambda\} = \emptyset$, then

$$\begin{aligned}
\sum_{i\in S} s^i &= \sum_{i\in S^0} s^i + \sum_{i\in S^1} s^i \\
&\geq \sum_{i\in S^0} d^i + (\sum_{i\in S^1}(t^i + d^i) - c)^+ \\
&= \sum_{i\in S^0}(t^i + d^i - t^i(1 - \bar{y}^i)) + (\sum_{i\in S^1}(t^i + d^i) - c)^+ \\
&\geq \sum_{i\in S}(t^i + d^i) - c - \sum_{i\in S^0}(t^i(1 - \bar{y}^i)) \\
&= \lambda + \sum_{i\in S} \max\{-t^i, d^i - \lambda\}(1 - \bar{y}^i). \qquad (18)
\end{aligned}$$

If $\{i \in S^0 : t^i + d^i > \lambda\} \neq \emptyset$, then

$$\begin{aligned}
\sum_{i\in S} s^i &\geq \sum_{i\in S^0} s^i \qquad (19) \\
&\geq \sum_{i\in S^0} d^i \\
&\geq \lambda + \sum_{i\in S^0 : t^i + d^i > \lambda} (d^i - \lambda) + \sum_{i\in S^0 : t^i + d^i \leq \lambda} d^i \\
&\geq \lambda + \sum_{i\in S} \max\{-t^i, d^i - \lambda\}(1 - \bar{y}^i), \qquad (20)
\end{aligned}$$

and so the proposition holds. $\square$

Example 1. Consider an instance of PI with $P = 4$, with $c = 16$, and with demand and setup times given by

$$\begin{array}{ccccc} i & 1 & 2 & 3 & 4 \\ t^i & 2 & 2 & 1 & 3 \\ d^i & 5 & 4 & 5 & 10. \end{array}$$

Choose as a cover $S = \{1, 2, 3\}$; then $\lambda = 7 + 6 + 6 - 16 = 3$, and (17) yields

$$s^1 + s^2 + s^3 \geq 3 + 2(1 - y^1) + (1 - y^2) + 2(1 - y^3) = 8 - 2y^1 - y^2 - 2y^3, \quad (21)$$

a valid inequality for X^{PI}. □

The family of inequalities defined by (17) cuts off many fractional extreme points. Specifically, we have

Proposition 10. *For PI, every dominant fractional solution of X^{PI}_{LP} of the form (15) is cut off by an inequality of the form (17).*

The proof is obtained by observing that $Q_l = \emptyset$ (from Proposition 8), and taking $S = Q_u \cup \{r\}$ in order to define (17).

Under general conditions, inequalities of the form (17) define facets of conv(X^{PI}).

Proposition 11. *Given an inequality of the form (17), order the $i \in S$ such that $t^{[1]} + d^{[1]} \geq \ldots \geq t^{[|S|]} + d^{[|S|]}$, let $\mu^1 = t^{[1]} + d^{[1]} - \lambda$, and define $T = \mathcal{P} \setminus S$. If $\lambda > 0$, $t^{[2]} + d^{[2]} > \lambda$, and $t^i < \mu^1, i \in T$, then (17) induces a facet of conv(X^{PI}).*

Proof. The conditions of the Proposition imply that $c > t^i, i = 1, \ldots, P$, and so from Proposition 1 conv(X^{PI}) is full–dimensional. It therefore suffices to show that the intersection of the hyperplane defined by (33) and of conv(X^{PI}) has dimension $3P - 1$. Note that the $|T|$ rays $s^i = 1, i \in T$, are extreme rays of conv(X^{PI}) and also lie in the hyperplane defined by (33). To obtain the results, therefore, it suffices to show $3P - |T|$ linearly independent points in X^{PI} that lie in this hyperplane, and that are also linearly independent of each of these extreme rays. To simplify notation, we assume first that $d^i + d^{[1]} \geq \lambda, i \in S \setminus [1]$. Consider the $3|P| - |T|$ points

- for each $i' \in S \setminus \{[1]\}$,

$$x^{i'} = (d^{i'} - \lambda)^+, y^{i'} = 1, s^{i'} = d^{i'} - x^{i'}$$
$$x^{[1]} = d^{[1]} - (\lambda - d^{i'})^+, y^{[1]} = 1, s^{[1]} = d^{[1]} - x^{[1]},$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

  ($|S| - 1$ points)
- the point

$$x^{[1]} = (d^{[1]} - \lambda)^+, y^{[1]} = 1, s^{[1]} = d^{[1]} - x^{[1]}$$
$$x^{[2]} = d^{[2]} - (\lambda - d^{[1]})^+, y^{[2]} = 1, s^{[1]} = d^{[1]} - x^{[1]},$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

- for each $i' \in S \setminus \{[1]\}$,

$$x^{i'} = y^{i'} = 0, s^{i'} = d^{i'}$$
$$x^{[1]} = d^{[1]} - (\lambda - t^{i'} - d^{i'})^+, y^{[1]} = 1, s^{[1]} = d^{[1]} - x^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

($|S| - 1$ points)
- the point

$$x^{[1]} = 0, y^{[1]} = 0, s^{[1]} = d^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{[1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

- for each $i' \in S \setminus [1]$,

$$x^{i'} = d^{i'} + \mu^1, y^{i'} = 1, s^{i'} = 0$$
$$x^{[1]} = y^{[1]} = 0, s^{[1]} = d^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [1]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

($|S| - 1$ points)
- the point

$$x^{[1]} = d^{[1]} + t^{[2]} + d^{[2]} - \lambda, y^{[1]} = 1, s^{[1]} = 0$$
$$x^{[2]} = y^{[2]} = 0, s^{[2]} = d^{[2]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus \{i' \cup [2]\}$$
$$x^i = y^i = 0, s^i = d^i, i \notin S$$

- for each $i' \in T$,

$$x^{i'} = 0, y^{i'} = 1, s^{i'} = d^{i'}$$
$$x^{[1]} = y^{[1]} = 0, s^{[1]} = d^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus [1]$$
$$x^i = y^i = 0, s^i = d^i, i \notin S \cup i'$$

($|T|$ points)
- for each $i' \in T$,

$$x^{i'} = \mu^1 - t^{i'}, y^{i'} = 1, s^{i'} = (d^{i'} - x^{i'})^+$$
$$x^{[1]} = y^{[1]} = 0, s^{[1]} = d^{[1]}$$
$$x^i = d^i, y^i = 1, s^i = 0, i \in S \setminus [1]$$
$$x^i = y^i = 0, s^i = d^i, i \notin S \cup i'$$

($|T|$ points).

It can be checked that these points and the $|T|$ rays $s^i = 1, i \in T$, are linearly independent. When the assumption $d^i + d^{[1]} \geq \lambda, i \in S \setminus [1]$, does not hold, $3P - |T|$ linearly independent points can be chosen similarly, but either a larger number of subsets or more complex notation is required to list them. □

Example 1 (continued)

The inequality (21) satisfies the conditions of Proposition 11, and is therefore a facet of $\text{conv}(X^{PI})$. □

3.1.2. Second form of cover inequalities We now expand the class of cover inequalities by incorporating another set of variables. This allows us to cut off more (in some cases, all) fractional dominant solutions and to define more facets of $\text{conv}(X^{PI})$.

Proposition 12. (Cover Inequalities, second form) *Given a cover S of PI, order the $i \in S$ such that $t^{[1]} + d^{[1]} \geq \ldots \geq t^{[|S|]} + d^{[|S|]}$. Let $T = \mathcal{P} \setminus S$, and let (T', T'') be any partition of T. Define $\mu^1 = t^{[1]} + d^{[1]} - \lambda$; if $|S| \geq 2$ and $t^{[2]} + d^{[2]} \geq \lambda$, then*

$$\sum_{i \in S} s^i \geq \lambda + \sum_{i \in S} \max\{-t^i, d^i - \lambda\}(1 - y^i) + \frac{\lambda}{(t^{[2]} + d^{[2]})} \sum_{i \in T'} (x^i - (\mu^1 - t^i) y^i) \tag{22}$$

is valid for X^{PI}.

Proof. Let $(\bar{x}, \bar{y}, \bar{s})$ be any point in X^{PI}. Let $S^1 = \{i \in S : \bar{y}^i = 1\}$, let $S^0 = \{i \in S : \bar{y}^i = 0\}$, and let $T'^1 = \{i \in T' : \bar{y}^i = 1\}$. If $T'^1 = \emptyset$, then the validity argument is exactly the case as for Proposition 9. So let $T'^1 \neq \emptyset$; we first assume that $T'^1 = \{i'\}$, a singleton. To show that $(\bar{x}, \bar{y}, \bar{s})$ satisfies (22), it suffices to show that

$$\begin{aligned} \sum_{i \in S} \bar{s}^i &+ \sum_{i \in S} \max\{-t^i, d^i - \lambda\} \bar{y}^i \\ &\geq \lambda + \sum_{i \in S} \max\{-t^i, d^i - \lambda\} + \frac{\lambda}{(t^{[2]} + d^{[2]})} (\bar{x}^{i'} - (\mu^1 - t^{i'})). \end{aligned} \tag{23}$$

Now

$$\begin{aligned} \sum_{i \in S} \bar{s}^i &+ \sum_{i \in S} \max\{-t^i, d^i - \lambda\} \bar{y}^i \\ &\geq \min_{(x,y,s) \in X^{PI} : x^{i'} = \bar{x}^{i'}, y^{i'} = 1} \left\{ \sum_{i \in S} s^i + \sum_{i \in S} \max\{-t^i, d^i - \lambda\} y^i \right\}. \end{aligned} \tag{24}$$

This is true because the right hand side is the minimum value that the left hand side can achieve, provided that $y^{i'} = 1$ and $x^{i'} = \bar{x}^{i'}$ are both still true. Therefore we will consider the minimization problem

$$\min_{(x,y,s) \in X^{PI} : x^{i'} = \bar{x}^{i'}, y^{i'} = 1} \left\{ \sum_{i \in S} s^i + \sum_{i \in S} \max\{-t^i, d^i - \lambda\} y^i \right\}, \tag{25}$$

and show that the optimal solution to this problem has value greater than or equal to that of the right hand side of (23). To do this we will use the following lemma.

Lemma 13. *Let* $r_S = |\{i \in S : t^i + d^i > \lambda\}|$, *and define*

$$\varphi_S(u) = \begin{cases} u, & \text{if } \quad 0 \le u \le \sum_{i=r_S+1}^{|S|}(t^{[i]} + d^{[i]}) \\ \sum_{i=r_S+1}^{|S|}(t^{[i]} + d^{[i]}) + (r_S - j)\lambda, & \text{if } \sum_{i=j+1}^{|S|}(t^{[i]} + d^{[i]}) \le u \le \sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) - \lambda, \\ & \quad j = 1, \dots, r_S, \\ \sum_{i=r_S+1}^{|S|}(t^{[i]} + d^{[i]}) + (r_S - j)\lambda + & \text{if } \sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) - \lambda \le u \le \sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}), \\ \quad (u - [\sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) - \lambda]), & \quad j = 2, \dots, r_S. \end{cases} \tag{26}$$

Then

$$\min_{(x,y,s)\in X^{PI}: x^{i'} = \bar{x}^{i'}, y^{i'}=1} \left\{ \sum_{i\in S} s^i + \sum_{i \in S} \max\{-t^i, d^i - \lambda\} y^i \right\} = \sum_{i\in S} d^i - \varphi_S(c - (t^{i'} + \bar{x}^{i'})). \tag{27}$$

Proof. See Appendix A. □

So now we need to show that

$$\sum_{i\in S} d^i - \varphi_S(c - (t^{i'} + \bar{x}^{i'})) \ge \lambda + \sum_{i\in S} \max\{-t^i, d^i - \lambda\} + \frac{\lambda}{(t^{[2]} + d^{[2]})}(\bar{x}^{i'} - (\mu^1 - t^{i'})), \tag{28}$$

or, equivalently, that

$$\sum_{i\in S} d^i \ge d^1 + \sum_{i \in S\setminus[1]} \max\{-t^i, d^i - \lambda\} + \varphi_S(c - (t^{i'} + \bar{x}^{i'})) + \frac{\lambda}{(t^{[2]} + d^{[2]})}(\bar{x}^{i'} - (\mu^1 - t^{i'})). \tag{29}$$

To show this, we will use a second lemma.

Lemma 14.

$$\max_{0 \le x^i \le c - t^i} \{\varphi_S(c - (t^i + x^i)) + \frac{\lambda}{(t^{[2]} + d^{[2]})}(x^i - (\mu^1 - t^i))\} = \sum_{i\in S\setminus[1]} \min\{t^i + d^i, \lambda\}. \tag{30}$$

Proof. See Appendix B. □

Because of Lemma 14,

$$\begin{aligned}
\sum_{i\in S} d^i &= d^{[1]} + \sum_{i\in S\setminus[1]} d^i \\
&= d^{[1]} + \sum_{i\in S\setminus[1]} \max\{-t^i, d^i - \lambda\} + \sum_{i\in S\setminus[1]} \min\{t^i + d^i, \lambda\} \\
&= d^{[1]} + \sum_{i\in S\setminus[1]} \max\{-t^i, d^i - \lambda\} \\
&\quad + \max_{0\le x^{i'}\le c-t^{i'}} \{\varphi_S(c - (t^{i'} + x^{i'})) + \frac{\lambda}{(t^{[2]} + d^{[2]})}(x^{i'} - (\mu^1 - t^{i'}))\} \\
&\ge d^{[1]} + \sum_{i\in S\setminus[1]} \max\{-t^i, d^i - \lambda\} \\
&\quad + \varphi_S(c - (t^{i'} + \bar{x}^{i'})) \\
&\quad + \frac{\lambda}{(t^{[2]} + d^{[2]})}(\bar{x}^{i'} - (\mu^1 - t^{i'})),
\end{aligned} \tag{31}$$

and (29) therefore holds. This suffices to show that any point in X^{PI} with $|T'^1| = 1$ satisfies (22).

When there is more than one element in T'^1, the proof proceeds analogously. Having more than one element in T'^1 weakens (22), however, and there are no tight points for which $|T'^1| > 1$. □

Example 1 (continued)

Again choose $S = \{1, 2, 3\}$, and recall that $\lambda = 3$. Then $t^{[1]} + d^{[1]} = t^1 + d^1 = 7$, and $\mu^1 = 4$. Also note that $t^{[2]} + d^{[2]} = t^2 + d^2 = t^3 + d^3 = 6$. Take $T' = \{4\}$, then

$$\frac{\lambda}{(t^{[2]} + d^{[2]})} = \frac{3}{6} = \frac{1}{2},$$

and (22) yields

$$\begin{aligned}
s^1 + s^2 + s^3 &\ge 3 + 2(1 - y^1) + (1 - y^2) + 2(1 - y^3) + \frac{1}{2}(x^4 - y^4) \\
&= 8 - 2y^1 - y^2 - 2y^3 + \frac{1}{2}(x^4 - y^4),
\end{aligned} \tag{32}$$

a valid inequality for X^{PI}. □

In addition to the fractional dominant solutions mentioned in Proposition 10, under certain conditions, inequalities of the form (22) cut off *all* fractional dominant solutions of X^{PI}_{LP}. For example, we have

Proposition 15. *For PI, if t^i and d^i are constant for all items, i.e., if $t^i = t, d^i = d, i = 1, \dots, P$, then every dominant fractional solution of X_{LP}^{PI} of the form (16) is cut off by an inequality of the form (22).*

To prove this proposition, first observe again that $Q_l = \emptyset$. Then to define (22) let $T' = \{r\}$, and let S be any set such that $|S| = \lceil \frac{c}{t+d} \rceil$ and $S \supset Q_u$. Thus, given Proposition 8, in this case cover inequalities cut off all fractional dominant solutions of X_{LP}^{PI}. We recall at this point that X_{LP}^{PI} is an LP relaxation of X^{PI} that has already been strengthened with the facet–inducing inequalities (12).

There are facets induced by inequalities of the form (22) with $T' \neq \emptyset$. For example, we have

Proposition 16. *If $\lambda > 0$, $t^{[2]} + d^{[2]} > \lambda$, $t^i < \mu^1, i \in T$, then (22) induces a facet of conv(X^{PI}).*

Proof. As before it suffices to exhibit $3P - T$ points that are in the intersection of the hyperplane defined by (22) and of X^{PI}, and that are linearly independent of each other and of the $|T|$ rays $s^i = 1, i \in T$,

For simplicity we again first assume that $d^i + d^{[1]} \geq \lambda, i \in \{S \setminus [1]\} \cup U$. Note that if we add the qualification $x^i = y^i = 0, s^i = d^i, i \in T'$, to each of the points presented in the proof of Proposition 11, then we have $3|S| + 2|T''|$ points that are linearly independent of each other and of the extreme directions, and they all satisfy (22) at equality. Therefore, it suffices to show $2|T'|$ points in the hyperplane defined by (22) that are linearly independent of each other, of the $3|S| + 2|T''|$ points just mentioned, and of the $|T|$ extreme rays. That is, for each item $i' \in T'$, it suffices to show two points in the hyperplane defined by (22) that are linearly independent of each other and the points and rays already mentioned.

So for each $i' \in T'$, consider the points

1.
$$
\begin{gathered}
x^{i'} = (\mu^1 - t^{i'}), y^{i'} = 1, s^{i'} = (d^{i'} - x^{i'})^+, \\
x^i = y^i = 0, i \in T \setminus i' \\
x^{[1]} = y^{[1]} = 0, s^{[1]} = d^{[1]}, \\
x^{[i]} = d^{[i]}, y^{[i]} = 1, s^{[i]} = 0, i = 2, \dots, |S|;
\end{gathered}
$$

2.
$$
\begin{gathered}
x^{i'} = (\mu^1 - t^{i'}) + t^{[2]} + d^{[2]}, y^{i'} = 1, s^{i'} = (d^{i'} - x^{i'})^+, \\
x^i = y^i = 0, i \in T \setminus i', \\
x^{[i]} = y^{[i]} = 0, s^{[i]} = d^{[1]}, i = 1, 2, \\
x^{[i]} = d^{[i]}, y^{[i]} = 1, s^{[i]} = 0, i = 3, \dots, |S|.
\end{gathered}
$$

It can be checked that the $2|T'|$ points defined in this manner satisfy (22) at equality, and that they are linearly independent of each other and of the previously given points and rays.

As before, when the assumption $d^i + d^{[1]} \geq \lambda, i \in \{S \setminus [1]\} \cup U$, does not hold, the proof is similar. □

Example 1 (continued)

The inequality (32) satisfies the conditions of Proposition 16, and is therefore a facet of conv(X^{PI}). □

3.2. Reverse cover inequalities

Before formally defining this class, we provide a brief motivation. Given $(x, y, s) \in X^{PI}$, let S be a reverse cover; recall that $\mu = c - \sum_{i \in S} t^i + d^i > 0$. Let i' be some element not in S. If $x^{i'} = (c - t^{i'})y^{i'}$, either

1. $y^{i'} = 0$; or
2. $y^{i'} = 1$ and no capacity is left for $i \in S$; therefore $y^i = 0, i \in S$, and $\sum_{i \in S} s^i \geq \sum_{i \in S} d^i$.

Such reasoning yields

$$\sum_{i \in S} s^i \geq (\sum_{i \in S}(t^i + d^i))y^{i'} - \sum_{i \in S} t^i(1 - y^i) - ((c - t^{i'})y^{i'} - x^{i'}),$$

which is generalized in the following proposition.

Proposition 17. (Reverse Cover Inequalities) *Let S be a reverse cover of PI, let $T = \mathcal{P} \setminus S$, and let (T', T'') be any partition of T. Then*

$$\sum_{i \in S} s^i \geq (\sum_{i \in S}(t^i + d^i)) \sum_{i \in T'} y^i - \sum_{i \in S} t^i(1 - y^i) - \sum_{i \in T'}((c - t^i)y^i - x^i) \qquad (33)$$

is valid for X^{PI}.

Proof. Let $(\bar{x}, \bar{y}, \bar{s})$ be any point in X^{PI}. If $\bar{y}^i = 0$ (and thus $\bar{x}^i = 0$), $i \in T'$, then (33) is obviously valid. Let $T'^1 = \{j \in T' : \bar{y}_j = 1\}$; consider first the case in which $T'^1 = \{i'\}$ is a singleton. Then (33) becomes

$$\sum_{i \in S} \bar{s}^i \geq \sum_{i \in S} d^i + \sum_{i \in S} t^i \bar{y}^i - (c - t^{i'}) + \bar{x}^{i'}.$$

This inequality holds because

$$\begin{aligned} (c - t^{i'}) &\geq \textstyle\sum_{i \in S}(\bar{x}^i + t^i \bar{y}^i) + \bar{x}^{i'} && \text{(by (3))} \\ &\geq \textstyle\sum_{i \in S}(d^i - \bar{s}^i + t^i \bar{y}^i) + \bar{x}^{i'}. && \text{(by (2))} \end{aligned}$$

When $|T'^1| > 1$, the argument is similar, except that in this case the inequality is weaker and cannot ever hold at equality. □

Note that T' can be seen as a candidate set for the choice of the element $i' \notin S$ mentioned in the paragraph motivating Proposition 17.

Example 1 (continued)

Choose $S = \{2, 3\}$ as a reverse cover; thus $\mu = 16-(2+4)-(1+5) = 4$. Let $T' = \{1\}$; then (33) yields

$$\begin{aligned} s^2 + s^3 &\geq 12y^1 - 2(1 - y^2) - (1 - y^3) - (14y^1 - x^1) \\ &= -3 - 2y^1 + x^1 + 2y^2 + y^3, \end{aligned} \tag{34}$$

a valid inequality for X^{PI}. □

One property of reverse cover inequalities concerns the fractional extreme points of X^{PI}_{LP} that they cut off.

Proposition 18. *For PI, every fractional dominant solution of X^{PI}_{LP} is cut off by an inequality of the form (33).*

We can prove this proposition by considering any fractional dominant solution (as characterized by Propositions 5 and 8) and taking $S = Q_u$ and $T' = \{r\}$ to define (33). We recall here that X^{PI}_{LP} is defined by a formulation that has already been strengthened with the inequalities (12). It is also interesting to note that the family of reverse cover inequalities is stronger than the family of cover inequalities, in the sense that the former *always* cuts off all possible fractional optimal solutions to X^{PI}_{LP}.

We now state general conditions under which reverse cover inequalities define facets of conv(X^{PI}).

Proposition 19. *If $S \neq \emptyset$, $T' \neq \emptyset$, and $t^i < \mu$, $i \in T$, (33) induces a facet of conv(X^{PI}).*

Proof. Similarly to the proof of Proposition 11, it suffices to show $3P - |T|$ points and $|T|$ rays in X^{PI} that are linearly independent and that lie in the hyperplane defined by (22). Let i^* be some element in T'; consider the points

- for each $i' \in S$,

$$\begin{gathered} x^{i'} = 0, y^{i'} = 1, s^{i'} = d^{i'} \\ x^{i^*} = c - t^{i^*} - t^{i'}, y^{i^*} = 1, s^{i^*} = (d^{i^*} - x^{i^*})^+, \\ x^i = y^i = 0, s^i = d^i, i \neq i', i \neq i^*; \end{gathered}$$

  ($|S|$ points)
- for each $i' \in S$,

$$\begin{gathered} x^{i'} = d^{i'}, y^{i'} = 1, s^{i'} = 0 \\ x^{i^*} = c - t^{i^*} - t^{i'} - d^{i'}, y^{i^*} = 1, s^{i^*} = (d^{i^*} - x^{i^*})^+ \\ x^i = y^i = 0, s^i = d^i, i \neq i', i \neq i^*; \end{gathered}$$

  ($|S|$ points)

- for each $i' \in S$,

$$x^{i'} = d^{i'} + \mu, \ y^{i'} = 1, \ s^{i'} = 0$$
$$x^i = d^i, \ y^i = 1, \ s^i = 0, \ i \in S \backslash i'$$
$$x^i = y^i = 0, \ s^i = d^i, \ i \in T.$$

($|S|$ points)

- for each $i' \in T' \setminus i^*$,

$$x^{i'} = \mu - t^{i'}, \ y^{i'} = 1, \ s^{i'} = (d^{i'} - x^{i'})^+$$
$$x^i = d^i, \ y^i = 1, \ s^i = 0, \ i \in S$$
$$x^i = y^i = 0, \ s^i = d^i, \ i \in T \setminus i';$$

($|T'| - 1$ points)

- for each $i' \in T'$,

$$x^{i'} = c - t^{i'}, \ y^{i'} = 1, \ s^{i'} = (d^{i'} - x^{i'})^+$$
$$x^i = y^i = 0, \ s^i = d^i, \ i \neq i';$$

($|T'|$ points)

- for each $i' \in T''$,

$$x^{i'} = 0, \ y^{i'} = 1, \ s^{i'} = d^{i'}$$
$$x^i = d^i, \ y^i = 1, \ s^i = 0, \ i \in S$$
$$x^i = y^i = 0, \ s^i = d^i, \ i \in T \setminus i';$$

($|T''|$ points)

- for each $i' \in T''$,

$$x^{i'} = \mu - t^{i'}, \ y^{i'} = 1, \ s^{i'} = (d^{i'} - x^{i'})^+$$
$$x^i = d^i, \ y^i = 1, \ s^i = 0, \ i \in S$$
$$x^i = y^i = 0, \ s^i = d^i, \ i \in T \setminus i';$$

($|T''|$ points)

- the point

$$x^i = d^i, \ y^i = 1, \ s^i = 0, \ i \in S$$
$$x^i = y^i = 0, \ s^i = d^i, \ i \in T.$$

It can be checked that the points listed are linearly independent of each other and of the $|T|$ rays $s^i = 1, i \in T$. □

Example 1 (continued)

The inequality (34) satisfies the conditions of Proposition 19 and is therefore a facet of conv(X^{PI}). □

4. Lifting cover and reverse cover inequalities

In this section, we discuss how the sets of cover and reverse cover inequalities can be extended further by lifting. In doing so, we make frequent use of 0-1 continuous knapsack relaxations of PI. Therefore, we start by providing the necessary background information on the 0-1 continuous knapsack problem.

4.1. The 0–1 continuous knapsack problem

An instance of the 0–1 *continuous knapsack problem* (CKP) is defined by the set

$$Y = \{(y, s) \in \{0, 1\}^n \times \mathbb{R}^1_+ : \sum_{j \in N} a_j y_j \le b + s\}, \tag{35}$$

where $N = \{1, \dots, n\}$, $a_j \in \mathbb{Z}_+$, $j \in N$, and $b \in \mathbb{Z}_+$. Marchand and Wolsey [1999] studied the polyhedral structure of the convex hull of solutions of CKP and derived two classes of easily computable, facet–defining inequalities for CKP through sequential lifting. We will summarize these results here.

Given an instance of CKP, we define a (j', C, D) *cover pair* for Y as an index j and sets C and D such that

- $C \cap D = j', C \cup D = N$;
- $\lambda_C = \sum_{j \in C} a_j - b > 0$;
- $a_{j'} > \lambda_C$.

Note that these conditions imply

$$\mu_D = a_{j'} - \lambda_C = \sum_{j \in D} a_j - (\sum_{j \in N} a_j - b) > 0.$$

We first take a (j', C, D) cover pair and fix all y_j, $j \in C \setminus j$ to 1 and all y_j, $j \in D \setminus j$ to 0, thus obtaining the two–dimensional polyhedron $Y_0 = \{(y, s) \in \{0, 1\}^1 \times \mathbb{R}^1_+ : a_j y_j \le \mu_D + s\}$. The unique non–trivial facet of conv(Y_0) is given by

$$(a_j - \mu_D) y_j \le s, \tag{36}$$

or equivalently, $\lambda_C y_j \le s$. We then lift the fixed variables back into (36). Different lifting orders lead to different inequalities, which are facets for conv(Y) if the lifting is maximal. For two general lifting sequences, Marchand and Wolsey [1999] identified the lifting function and showed that it is superadditive, and thus lifting is sequence independent and can be done with negligible computation (see, e.g., Wolsey [1977] and Gu, Nemhauser, and Savelsbergh [1999] for a more elaborate discussion of sequence independent lifting). These two lifting sequences give rise to two classes of inequalities, namely *continuous cover inequalities* and *continuous reverse cover inequalities*.

Proposition 20. *(*Continuous cover inequalities–*Marchand and Wolsey [1999]) Given a (j', C, D) cover pair for Y, order the elements of C such that $a_{[1]} \geq \ldots \geq a_{[r_C]}$, where r_C is the number of elements of C with $a_j > \lambda_C$. Let $A_0 = 0$, $A_j = \sum_{p=1}^{j} a_{[p]}$, $j = 1, \ldots, r_C$, and define*

$$\phi_C(u) = \begin{cases} (j-1)\lambda_C, & \text{if } A_{j-1} \leq u \leq A_j - \lambda_C,\ j = 1, \ldots, r_C, \\ (j-1)\lambda_C + [u - (A_j - \lambda_C)], & \text{if } A_j - \lambda_C \leq u \leq A_j,\ j = 1, \ldots, r_C - 1, \\ (r_C - 1)\lambda_C + [u - (A_{r_C} - \lambda_C)], & \text{if } A_{r_C} - \lambda_C \leq u. \end{cases} \tag{37}$$

The inequality

$$\sum_{j \in C} \min\{\lambda_C, a_j\} y_j + \sum_{D \setminus j} \phi_C(a_j) y_j \leq \sum_{j \in C \setminus j} \min\{\lambda_C, a_j\} + s \tag{38}$$

defines a facet of conv(Y).

Inequalities of the form (38) are obtained from (36) by first lifting all the variables in $C \setminus j$, then all those in $D \setminus j$. Because the lifting function for each of the two sets is superadditive, the lifting order within the two sets is immaterial.

Proposition 21. *(*Continuous reverse cover inequalities–*Marchand and Wolsey [1999]) Given a (j', C, D) cover pair for Y, order the elements of D such that $a_{[1]} \geq \ldots \geq a_{[r_D]}$, where r_D is the number of elements of D with $a_j > \mu_D$. Let $A_0 = 0$, $A_j = \sum_{p=1}^{j} a_{[p]}$, $j = 1, \ldots, r_D$, and define*

$$\psi_D(u) = \begin{cases} u - j\mu_D, & \text{if } A_j \leq u \leq A_{j+1} - \mu_D,\ j = 0, \ldots, r_D - 1, \\ A_j - j\mu_D, & \text{if } A_j - \mu_D \leq u \leq A_j,\ j = 1, \ldots, r_D - 1, \\ A_{r_D} - r_D\mu_D, & \text{if } A_{r_D} - \mu_D \leq u. \end{cases} \tag{39}$$

The inequality

$$\sum_{j \in D} (a_j - \mu_D)^+ y_j + \sum_{j \in C \setminus i} \psi_D(a_j) y_j \leq \sum_{j \in C \setminus i} \psi_D(a_j) + s \tag{40}$$

defines a facet of conv(Y).

Inequalities of the form (40) are derived from (36) by lifting all variables in $D \setminus j$ first, then all variables in $C \setminus j$. As before, the coefficients do not depend on the sequence chosen within $D \setminus j$ and $C \setminus j$.

4.2. Lifting cover inequalities

In order to expand further the set of cover inequalities, we extend (22) by lifting into it variables associated with some subset $U \subset \mathcal{P} \setminus \{S \cup T'\}$. To do this, we will define a CKP relaxation of PI based on the sets S and U, where U contains the items whose variables will be lifted.

From constraints (2) and (3) we have

$$\sum_{i\in S\cup U}(t^i y^i + d^i - s^i) \le c. \tag{41}$$

Adding $\sum_{i\in S} d^i y^i$ to both sides of this inequality, we see that

$$\sum_{i\in S\cup U}(t^i + d^i)y^i \le c + \sum_{i\in S\cup U}(s^i + d^i y^i - d^i) \tag{42}$$

is also valid for PI. We know that

$$s = \sum_{i\in S\cup U}(s^i + d^i y^i - d^i) \ge 0 \tag{43}$$

because of (12), and so valid inequalities for the instance of CKP defined by

$$\sum_{i\in S\cup U}(t^i + d^i)y^i \le c + s \tag{44}$$

are also valid for PI, where s is defined as in (43). So, applying Proposition 20, we see that

$$\sum_{i\in S}\min\{\lambda, t^i + d^i\}y^i + \sum_{i\in U}\phi_S(t^i + d^i)y^i \le \sum_{i\in S}\min\{\lambda, t^i + d^i\} - \lambda + s, \tag{45}$$

where $\phi_S(\cdot) = \phi_C(\cdot)$ is defined as in Proposition 20. Specifically, we have that

$$\phi_S(u) = \begin{cases} 0, & \text{if } u \le t^{[1]} + d^{[1]} - \lambda, \\ (j-1)\lambda + (u - [\sum_{i=1}^{j}(t^{[i]} + d^{[i]}) - \lambda]), & \text{if } \sum_{i=1}^{j}(t^{[i]} + d^{[i]}) - \lambda \le u \le \sum_{i=1}^{j}(t^{[i]} + d^{[i]}), \\ & \quad 1 \le j \le r_S - 1, \\ (j-1)\lambda, & \text{if } \sum_{i=1}^{j-1}(t^{[i]} + d^{[i]}) \le u \le \sum_{i=1}^{j}(t^{[i]} + d^{[i]}) - \lambda, \\ & \quad 1 \le j \le r_S - 1, \\ (r_S - 1)\lambda + (u - [\sum_{i=1}^{r_S}(t^{[i]} + d^{[i]}) - \lambda]), & \text{if } u \ge \sum_{i=1}^{r_S}(t^{[i]} + d^{[i]}) - \lambda, \end{cases} \tag{46}$$

where

$$r_S = |\{i \in S : t^i + d^i > \lambda\}|.$$

Using (43) to replace s in (45), we can rewrite this inequality as

$$\sum_{i\in S\cup U} s^i \ge \lambda + \sum_{i\in S}\max\{-t^i, d^i - \lambda\}(1 - y^i) + \sum_{i\in U} f_S(t,d)y^i + \sum_{i\in U} d^i, \tag{47}$$

where

$$f_S(t^0, d^0) = \phi_S(t^0 + d^0) - d^0 = \begin{cases} -d^0, & \text{if } t^0 + d^0 \le t^{[1]} + d^{[1]} - \lambda, \\ t^0 + (j-1)\lambda - \\ (\sum_{i=1}^{j}[t^{[i]} + d^{[i]}] - \lambda), & \text{if } \sum_{i=1}^{j}(t^{[i]} + d^{[i]}) - \lambda \le t^0 + d^0 \le \\ & \quad \sum_{i=1}^{j}(t^{[i]} + d^{[i]}), \\ & \quad j \le r_S - 1, \\ (j-1)\lambda - d^0, & \text{if } \sum_{i=1}^{j-1}(t^{[i]} + d^{[i]}) \le t^0 + d^0 \le \\ & \quad \sum_{i=1}^{j}(t^{[i]} + d^{[i]}) - \lambda, \\ & \quad j \le r_S - 1, \\ t^0 + (r_S - 1)\lambda - \\ (\sum_{i=1}^{r_S}[t^{[i]} + d^{[i]}] - \lambda), & \text{if } t^0 + d^0 \ge \sum_{i=1}^{r_S}(t^{[i]} + d^{[i]}) - \lambda. \end{cases} \tag{48}$$

We can extend (47) to the complete class of lifted cover inequalities.

Proposition 22. (Lifted Cover Inequalities) *Given a cover S of PI, order the $i \in S$ as in Proposition 12. Also, define μ^1 as in Proposition 12. Let (T, U) be any partition of $\mathcal{P} \setminus S$, and let (T', T'') be any partition of T. If $\mu^1 \geq 0$ and $|S| \geq 2$, then*

$$\sum_{i \in S \cup U} s^i \geq \lambda + \sum_{i \in S} \max\{-t^i, d^i - \lambda\}(1 - y^i) + \sum_{i \in U} f_S(t, d) y^i + \sum_{i \in U} d^i \\ + \frac{\lambda}{t^{[2]} + d^{[2]}} \sum_{i \in T'} (x^i - (\mu^1 - t^i) y^i) \tag{49}$$

is valid for X^{PI}, where $f_S(\cdot, \cdot)$ is defined as in (48).

When $T' = \emptyset$, validity follows from the argument discussed in the motivating paragraphs above. When $T' \neq \emptyset$, then the proof is similar to that of Proposition 12.

Example 1 (continued)

Again take $S = \{1, 2, 3\}$ as our cover, so that $\lambda = 3$, $t^{[1]} + d^{[1]} = t^1 + d^1 = 7$, and $\mu^1 = 4$. Then $r_S = 3$, and

$$\phi_S(u) = \begin{cases} 0 & \text{if } \quad u \leq 4, \\ u - 4 & \text{if } \ 4 \leq u \leq 7 \\ 3 & \text{if } \ 7 \leq u \leq 10 \\ 3 + (u - 10) & \text{if } 10 \leq u \leq 13 \\ 6 & \text{if } 13 \leq u \leq 16 \\ 6 + (u - 16) & \text{if } \quad u \geq 16, \end{cases} \tag{50}$$

Let $U = \{4\}$. Then $\phi_S(t^4 + d^4) = \phi_S(13) = 6$, and so $f_S(t^4, d^4) = \phi_S(t^4 + d^4) - d^4 = 6 - 10 = -4$. Then (22) yields

$$s^1 + s^2 + s^3 + s^4 \geq 3 + 2(1 - y^1) + (1 - y^2) + 2(1 - y^3) - 4y^4 + 10 \\ = 18 - 2y^1 - y^2 - 2y^3 - 4y^4, \tag{51}$$

a valid inequality for X^{PI}. □

Under certain conditions, lifted cover inequalities define facets of conv(X^{PI}). For example, we have

Proposition 23. *Given an inequality of the form (49), if*

1. $\lambda > 0$, $t^{[2]} + d^{[2]} > \lambda$, $t^i < \mu^1, i \in T$; *and*
2. *for each* $i \in U$, *there exists a* $k : 1 \leq k \leq r_S - 1$ *for which* $t^i + d^i = \sum_{j=1}^{k}(t^{[j]} + d^{[j]})$;

then (49) induces a facet of conv(X^{PI}).

Proof. Again it suffices to exhibit $3P - T$ points that are in the intersection of the hyperplane defined by (49) and of X^{PI}, and that are linearly independent of each other and of the $|T|$ rays $s^i = 1, i \in T$,

We again first assume that that $d^i + d^{[1]} \geq \lambda, i \in \{S \setminus [1]\} \cup U$. Note that if we add the qualification $x^i = y^i = 0, s^i = d^i, i \in U$, to each of the points presented in the proof of Proposition 16, then we have $3|S| + 2|T|$ points that are linearly independent of each other and of the extreme directions, and they all satisfy (49) at equality. Therefore, it suffices to show $3|U|$ points in the hyperplane defined by (49) that are linearly independent of each other, of the $3|S| + 2|T|$ points just mentioned, and of the $|T|$ extreme rays. That is, for each item $i' \in U$, it suffices to show three points in the hyperplane defined by (49) that are linearly independent of each other and the points and rays already mentioned.

Let $r_S = |\{i \in S : t^i + d^i > \lambda\}|$. Recall that, for each $i' \in U$,

$$t^{i'} + d^{i'} = \sum_{i=1}^{k}(t^{[i]} + d^{[i]}) = \mu^1 + \lambda + \sum_{i=2}^{k}(t^{[i]} + d^{[i]}),$$

for some k such that $1 \leq k \leq r_S - 1$. So for each $i' \in U$, consider the points

1.

$$\begin{aligned} x^{i'} = (d^{i'} - \lambda)^+, y^{i'} = 1, s^{i'} = \min\{t^{i'} + d^{i'} + \textstyle\sum_{i=k+1}^{|S|}(t^{[i]} + d^{[i]}) - c, d^{i'}\}& \\ = \min\{\lambda, d^{i'}\}& \end{aligned}$$
$$x^{[i]} = y^{[i]} = 0, s^{[i]} = d^{[i]}, i = 1, \ldots, k$$
$$x^{[k+1]} = d^{[k+1]} - (\lambda - d^{i'})^+, y^{[i]} = 1, s^{[i]} = (\lambda - d^{i'})^+,$$
$$x^{[i]} = d^{[i]}, y^{[i]} = 1, s^{[i]} = 0, i = k+2, \ldots, |S|$$
$$x^i = y^i = 0, s^i = d^i, i \notin S \cup i';$$

2.

$$x^{i'} = d^{i'}, y^{i'} = 1, s^{i'} = 0$$
$$x^{[i]} = y^{[i]} = 0, s^{[i]} = d^{[i]}, i = 1, \ldots, k+1$$
$$x^{[i]} = d^{[i]}, y^{[i]} = 1, s^{[i]} = 0, i = k+2, \ldots, |S|$$
$$x^i = y^i = 0, s^i = d^i, i \notin S \cup i';$$

3.

$$\begin{gathered} x^{i'} = d^{i'} + t^{[k+1]} + d^{[k+1]} - \lambda,\ y^{i'} = 1,\ s^{i'} = 0 \\ x^{[i]} = y^{[i]} = 0,\ s^{[i]} = d^{[i]},\ i = 1, \ldots, k+1 \\ x^{[i]} = d^{[i]},\ y^{[i]} = 1,\ s^{[i]} = 0,\ i = k+2, \ldots, |S| \\ x^i = y^i = 0,\ s^i = d^i,\ i \notin S \cup i'. \end{gathered}$$

It can be checked that the $3|U|$ points defined in this manner are linearly independent of each other and of the previously given points and rays.

As before, when the assumption $d^i + d^{[1]} \geq \lambda$, $i \in \{S \setminus [1]\} \cup U$, does not hold, the proof is similar. □

Although condition 2 in Proposition 23 is restrictive, it is not difficult to show that (49) induces faces of dimension at least $3P - |U|$ if this condition does not hold but condition 1 does. The proof of this proceeds along the same lines as that given for Proposition 23, and uses many of the same linearly independent points.

Example 1 (continued)

The inequality (51) satisfies the conditions of Proposition 23; in particular, note that $t^4 + d^4 = t^1 + d^1 + t^2 + d^2$. Therefore (51) defines a facet of $\mathrm{conv}(X^{PI})$. □

4.3. Lifting reverse cover inequalities

In this section we show that although that we cannot use results for CKP to strengthen inequalities of the form (33), we can use results for CKP to define a second class of reverse cover inequalities.

First, consider an inequality of the form (33). Then, given sets S and T' needed to define (33), we can use (3) and the inequalities (2) for $i \in S$ to obtain

$$\sum_{i \in S} (t^i y^i + (d^i - s^i)) + \sum_{i \in T'} (t^i y^i + x^i) \leq c. \tag{52}$$

Adding $d^i y^i$, $i \in S$, and $(c - t^i) y^i$, $i \in T'$, to both sides of (52) yields

$$\sum_{i \in S} ((t^i + d^i) y^i + (d^i - s^i)) + \sum_{i \in T'} (c y^i + x^i) \leq c + \sum_{i \in S} d^i y^i + \sum_{i \in T'} (c - t^i) y^i. \tag{53}$$

We can rewrite (53) as

$$\sum_{i \in S} (t^i + d^i) y^i + \sum_{i \in T'} c y^i \leq c + \left[\sum_{i \in S} (s^i + d^i y^i - d^i) + \sum_{i \in T} ((c - t^i) y^i - x^i) \right]. \tag{54}$$

Because of (12) (for $i \in S$) and (4) (for $i \in T'$), we know that

$$s = \sum_{i\in S}(s^i + d^i y^i - d^i) + \sum_{i\in T}((c - t^i)y^i - x^i) \geq 0. \tag{55}$$

Therefore, defining s as in (55), we will consider the instance of CKP defined by

$$\sum_{i\in S}(t^i + d^i)y^i + \sum_{i\in T'} cy^i \leq c + s. \tag{56}$$

By taking j' to be any item in T', we can let $C = S \cup j'$ and $D = T'$. Note that, for this choice of D, $\mu_D = c - \sum_{i\in S}(t^i + d^i) = \mu$. Also note that for this choice of D, $\psi_D(t^i + d^i) = t^i + d^i, i \in S = C \setminus j'$, since $t^i + d^i \leq c - \mu, i \in S$. Thus, applying (21) to the instance of CKP defined by (56) yields

$$\sum_{i\in S}(t^i + d^i)y^i + \sum_{i\in T}(c - \mu)y^i \leq \sum_{i\in S}(t^i + d^i) + s. \tag{57}$$

By using (55) to replace s in (57), we see that this inequality is exactly (33). Thus (33) cannot be strengthened by using CKP results.

We will discuss how to define another family of reverse cover inequalities that can be lifted to induce facets. Consider a reverse cover S and let U be some subset of $\mathcal{P} \setminus S$ such that $t^i + d^i \geq \mu, i \in U$. Thus, $S \cup i$ is a cover of PI, for all $i \in U$.

Now consider a given i' in U. If $y^{i'} = 0$, but $y^i = 1, i \in S$, then clearly $d^{i'}$ is a lower bound on $s^{i'}$ (and thus on $\sum_{i\in S\cup i'} s^i$ as well). However, if $y^{i'} = 1$ and $y^i = 1, i \in S$, then the only lower bound on $\sum_{i\in S\cup i'} s^i$ is determined by the capacity; specifically,

$$\sum_{i\in S\cup i'} s^i \geq \sum_{i\in S\cup i'} (t^i + d^i) - c = t^{i'} + d^{i'} - \mu$$

must be true.

This reasoning give rise to a family of valid inequalities incorporating this additional set U; specifically we obtain

$$\sum_{i\in S\cup U} s^i \geq -\sum_{i\in S} t^i(1 - y^i) + \sum_{i\in U} d^i + \sum_{i\in U}(t^i - \mu)y^i. \tag{58}$$

However, these inequalities are not facet inducing; for any inequality of the form (58), we obtain an inequality of the form (33) that dominates it by taking $T' = U$. (To see this, note that replacing s^i by $d^i - x^i$ for $i \in U$ yields inequalities that are stronger, because of (2), and recall that $t^{i'} - \mu = \sum_{i\in S}(t^i + d^i) - (c - t^{i'}), i' \in U$.)

Nevertheless, we can use lifting results for CKP to strengthen these inequalities and so define a second family of facet–inducing inequalities based on reverse covers of PI. To do this, again let S be a reverse cover of PI, and let $U \subseteq \mathcal{P} \setminus S$ be such that $t^i + d^i \geq \mu, i \in U$.

Again, because of (2) and (3), we know that

$$\sum_{i\in S\cup U} (t^i y^i + d^i - s^i) \leq c \tag{59}$$

is valid for PI; adding $\sum_{i \in S \cup U} d^i y^i$ to both sides of the inequality yields

$$\sum_{i \in S \cup U} (t^i + d^i) y^i \le c + \sum_{i \in S \cup U} (s^i + d^i y^i - d^i). \tag{60}$$

Because of (12), we know that

$$s = \sum_{i \in S \cup U} (s^i + d^i y^i - d^i) \ge 0, \tag{61}$$

and therefore inequalities valid for the instance of CKP defined by

$$\sum_{i \in S \cup} (t^i + d^i) y^i \le c + s \tag{62}$$

are also valid for PI, where s is defined as in (61).

By taking j' to be any item in U, we can let $C = S \cup j'$ and $D = U$. Observe that, as before, $\mu_D = c - \sum_{i \in S} (t^i + d^i) = \mu > 0$. Applying Proposition 21 to the instance of CKP defined by (62), we see immediately that

$$\sum_{i \in S} \psi_U (t^i + d^i) y^i + \sum_{i \in U} (t^i + d^i - \mu) y^i \le \sum_{i \in S} \psi_U (t^i + d^i) + s, \tag{63}$$

is valid for PI, where s is defined by (61), and $\psi_U(\cdot) = \psi_D(\cdot)$ is defined as in Proposition 21. Specifically, if we order the sums $t^i + d^i$, $i \in U$, so that $t^{[1]} + d^{[1]} \ge \ldots \ge t^{[|U|]} + d^{[|U|]}$, and let $r_U = |\{i \in U : t^i + d^i > \mu\}|$, then

$$\psi_U(u) = \begin{cases} u, & \text{if } \quad 0 \le u \le t^{[1]} + d^{[1]} - \mu \\ u - j\mu, & \text{if } \sum_{i=1}^{j} (t^{[i]} + d^{[i]}) \le u \le \sum_{i=1}^{j+1} (t^{[i]} + d^{[i]}) - \mu, \\ & \qquad j = 1, \ldots, r_U - 1, \\ \sum_{i=1}^{j} (t^{[i]} + d^{[i]}) - j\mu, & \text{if } \sum_{i=1}^{j} (t^{[i]} + d^{[i]}) - \mu \le u \le \sum_{i=1}^{j} (t^{[i]} + d^{[i]}), \\ & \qquad j = 1, \ldots, r_U - 1, \\ \sum_{i=1}^{r_U} (t^{[i]} + d^{[i]}) - r_U \mu, & \text{if } \quad \sum_{i=1}^{r_U} (t^{[i]} + d^{[i]}) - \mu \le u. \end{cases} \tag{64}$$

Now, using (61) to replace s in (63), we obtain the valid inequality

$$\sum_{i \in S \cup U} s^i \ge \sum_{i \in S} g_U(t^i, d^i)(1 - y^i) + \sum_{i \in U} d^i + \sum_{i \in U} (t^i - \mu) y^i, \tag{65}$$

where

$$g_U(t_0, d_0) = d_0 - \psi_U(t_0 + d_0) = \begin{cases} -t_0, & \text{if } \quad 0 \le t_0 + d_0 \le t^{[1]} + d^{[1]} - \mu \\ j\mu - t_0, & \text{if } \quad \sum_{i=1}^{j} (t^{[i]} + d^{[i]}) \le t_0 + d_0 \\ & \qquad \le \sum_{i=1}^{j+1} (t^{[i]} + d^{[i]}) - \mu, \\ & \qquad j = 1, \ldots, r_U - 1, \\ d_0 - [\sum_{i=1}^{j} (t^{[i]} + d^{[i]}) - j\mu], & \text{if } \sum_{i=1}^{j} (t^{[i]} + d^{[i]}) - \mu \le t_0 + d_0 \\ & \qquad \le \sum_{i=1}^{j} (t^{[i]} + d^{[i]}), \\ & \qquad j = 1, \ldots, r_U - 1, \\ d_0 - [\sum_{i=1}^{r_U} (t^{[i]} + d^{[i]}) - r_U \mu], & \text{if } \quad \sum_{i=1}^{r_U} (t^{[i]} + d^{[i]}) - \mu \le u. \end{cases} \tag{66}$$

We can expand (65) into the full, second class of reverse cover inequalities.

Proposition 24. (Lifted Reverse Cover Inequalities, Second Family) *Given a reverse cover S of PI and a set $U \subseteq \mathcal{P} \setminus S$ such that $t^i + d^i \geq \mu, i \in U$, order the $i \in U$ so that $t^{[1]} + d^{[1]} \geq \ldots \geq t^{[|U|]} + d^{[|U|]}$. Let $T = \mathcal{P} \setminus \{S \cap U\}$, and let (T', T'') be any partition of T. Then*

$$\sum_{i \in S \cup U} s^i \geq \sum_{i \in S} g_U(1 - y^i) + \sum_{i \in U}(t^i - \mu)y^i + \sum_{i \in U} d^i$$
$$+ \min_{i \in S}\{\tfrac{\psi_U(t^i + d^i)}{t^i + d^i}\} \sum_{i \in T'}(x^i - (\mu - t^i)y^i) \tag{67}$$

is valid for X^{PI}.

When $T' = \emptyset$, a proof of this proposition follows the motivating paragraphs above. When $T' \neq \emptyset$, this proposition can be proven by considering minimization and maximization problems similar to those in the proof of Proposition 12.

Example 1 (continued)

Choose $S = \{4\}$ as our reverse cover; thus $\mu = 16 - 13 = 3$. Let $U = \{2, 3\}$ and $T' = \{1\}$. Then

$$\psi_U(u) = \begin{cases} u & \text{if } \quad u \leq 3, \\ 3 & \text{if } 3 \leq u \leq 6 \\ u - 3 & \text{if } 6 \leq u \leq 9 \\ 6 & \text{if } \quad u \geq 9, \end{cases} \tag{68}$$

$\psi_U(t^4 + d^4) = \psi_U(13) = 6$, $g_U(t^4, d^4) = d^4 - \psi_U(t^4 + d^4) = 10 - 6 = 4$, and

$$\min_{i \in S}\{\frac{\psi_U(t^i + d^i)}{t^i + d^i}\} = \frac{\psi_U(t^4 + d^4)}{t^4 + d^4} = \frac{6}{13}.$$

Therefore (67) yields

$$s^2 + s^3 + s^4 \geq 4(1 - y^4) + (2 - 3)y^2 + (1 - 3)y^3 + 4 + 5 + \frac{6}{13}(x^1 - (3 - 2)y^1)$$
$$= 13 - 4y^4 - 1y^2 - 2y^3 + \frac{6}{13}(x^1 - y^1) \tag{69}$$

a valid inequality for X^{PI}. □

Note that if t^i and d^i are constant for all items, then lifted cover inequalities and the second family of lifted reverse cover inequalities coincide. Therefore, it is not surprising that propositions analogous to Propositions 10 and 15 hold for this second family of reverse cover inequalities.

Proposition 25. *For PI, every dominant fractional solution of X_{LP}^{PI} of the form (15) is cut off by an inequality of the form (67) with $T' = 0$.*

Proposition 26. *For PI, if t^i and d^i are constant for all items, then every dominant fractional solution of X_{LP}^{PI} of the form (16) is cut off by an inequality of the form (67).*

The proofs of the above two proposition are similar to those for Propositions 10 and 15.

Under certain conditions, this second family of reverse cover inequalities defines facets of conv(X^{PI}). For example, we have

Proposition 27. *Given an inequality of the form (67), if*

1. *$S \neq \emptyset$, $U \neq \emptyset$, $t^i < \mu, i \in T$, $t^i + d^i > \mu, i \in U$; and*
2. *there exists an $i \in S$ such that $t^i + d^i > \sum_{i \in U}(t^i + d^i) - \mu$;*

then (67) defines a facet of conv(X^{PI}).

As with previous propositions, this proposition can be proven by exhibiting $3P - |T|$ linearly independent points in X^{PI}. Moreover, as with Proposition 23, if Condition 1 (but not necessarily condition 2) of the proposition is satisfied, then this new family of reverse cover inequalities induces faces of dimension at least $3P - |U|$.

Example 1 (continued)

The inequality (69) satisfies the conditions of Proposition 27, and is therefore a facet of conv(X^{PI}). □

5. Conclusions and future directions

As mentioned in the introduction, one of the most successful techniques for solving integer programming problems in recent years has been the application of polyhedral results for structured relaxations of these problems. Restricting our focus to PI, a single-period production planning model, has allowed us to derive results for a multi-item, capacitated model. Because lot-sizing problems typically involve significant interaction between different periods, however, some care is needed in applying our results for PI to multi–period problems. That is, to effectively use our results for PI, it is actually necessary to define instances of PI that take into account demand and production decisions in more than just a single period. Methods to do this are nontrivial and are discussed in detail in Miller et al. [2000b]. Computational experience suggests that using these methods to apply our results for PI can be helpful in effectively solving multi–item, multi–period capacitated lot–sizing problems. We also expect that our results for PI may be useful in solving other problems that have substructures of the form PI, such as fixed charge network flow problems.

In special cases, it may be possible to find a complete description of conv(X^{PI}). For example, PI is polynomially solvable when both setup times and demands are constant for all items (Miller et al. [2002]). Even in this case, though, a complete description of conv(X^{PI}) remains open. However, if we impose the cost conditions $q^i > 0, h^i > 0, i = 1, \ldots, P$, we can state that inequalities (33) and (49) (or, equivalently, (33) and (67)) suffice to solve PI by linear programming. The extensive development needed to prove this result can be found in Miller et al. [2002]. We remark here that the expanded forms of the valid inequalities defined in Section 4 are essential for obtaining this result.

An interesting question is what other classes of inequalities define facets of conv(X^{PI}). Marchand and Wolsey [1999] and Miller, et al. [2000a] have defined additional families of valid inequalities for CKP, which could perhaps be used to derive additional facet–defining inequalities for PI. There may also be other ways to define facet–defining inequalities for PI.

In Miller [1999], a number of extensions to PI are considered. For example, when variable lower bounds are added to the formulation of MCL, it is possible to derive an extension of PI that is analogous to a single period of this new lot–sizing model. These extensions provide potentially useful relaxations for even more general models than those for which PI serves as a relaxation. Analysis of these and other generalizations of PI are areas of future research.

Appendix A: Proof of Lemma 13

Lemma 13

Let $r_S = |\{i \in S : t^i + d^i > \lambda\}|$, *and define*

$$\varphi_S(u) = \begin{cases} u, & \text{if } \quad 0 \le u \le \sum_{i=r_S+1}^{|S|}(t^{[i]} + d^{[i]}) \\ \sum_{i=r_S+1}^{|S|}(t^{[i]} + d^{[i]}) + (r_S - j)\lambda, & \text{if } \sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) \le u \le \sum_{i=j-1}^{|S|}(t^{[i]} + d^{[i]}) - \lambda, \\ & \quad j = 2, \dots, r_S, \\ \sum_{i=r_S+1}^{|S|}(t^{[i]} + d^{[i]}) + (r_S - j)\lambda + & \text{if } \sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) - \lambda \le u \le \sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}), \\ \quad (u - [\sum_{i=j}^{|S|}(t^{[i]} + d^{[i]}) - \lambda]), & \quad j = 2, \dots, r_S. \end{cases} \tag{26}$$

Then

$$\min_{(x,y,s)\in X^{PI}:x^{i'}=\bar{x}^{i'},y^{i'}=1} \left\{ \sum_{i\in S} s^i + \sum_{i\in S} \max\{-t^i, d^i - \lambda\} y^i \right\} = \sum_{i\in S} d^i - \varphi_S(c - (t^{i'} + \bar{x}^{i'})). \tag{27}$$

Proof. Assume that we are given an item i' and a value $\bar{x}^{i'}$. We will first construct a feasible solution to the minimization problem defined in (27) that has the value defined by the statement of Lemma 13, and then we will then show this solution is optimal as well as feasible. Define

$$j' = \min\{j : \sum_{i=j+1}^{|S|}(t^{[i]} + d^{[i]}) < c - (t^{i'} + \bar{x}^{i'})\}.$$

For convenience, we will first assume that $t^{[j']} + d^{[j']} \ge \lambda$ and that $t^{[j']} \le c - (t^{i'} + \bar{x}^{i'} + \sum_{i=j'+1}^{|S|}(t^{[i]} + d^{[i]}))$. Define $(\hat{x}, \hat{y}, \hat{s})$ as follows:

- $\hat{x}^{i'} = \bar{x}^{i'}, \hat{y}^{i'} = 1, \hat{s}^{i'} = (d^{i'} - \hat{x}^{i'})^+$;

- for $i = 1, \dots, j' - 1$, let
$$\hat{x}^{[i]} = \hat{y}^{[i]} = 0, \hat{s}^{[i]} = d^{[i]};$$
- for $i = j' + 1, \dots, |S|$, let
$$\hat{x}^{[i]} = d^{[i]}, \hat{y}^{[i]} = 1, \hat{s}^{[i]} = 0;$$
- if $\sum_{i=j'}^{|S|}(t^{[i]} + d^{[i]}) \leq c - (t^{i'} + \bar{x}^{i'}) + \lambda$, then let
$$\hat{x}^{[j']} = c - t^{[j']} - (t^{i'} + \bar{x}^{i'}) - \sum_{j'+1}^{|S|}(t^{[i]} + d^{[i]}), \hat{y}^{j'} = 1, \hat{s}^{[j']} = (d^{[j']} - \hat{x}^{[j']});$$
else let
$$\hat{x}^{[j']} = \hat{y}^{j'} = 0, \hat{s}^{[j']} = d^{[j']}.$$

Then it can be checked that $(\hat{x}, \hat{y}, \hat{s})$, is a feasible solution for the minimization problem defined by (27), and that

$$\sum_{i \in S} \hat{s}^i + \sum_{i \in S} \max\{-t^i, d^i - \lambda\}\hat{y}^i = \sum_{i \in S} d^i - \varphi_S(c - (t^{i'} + \bar{x}^{i'})).$$

Thus we have shown

$$\min_{(x,y,s) \in X^{PI}: x^{i'} = \bar{x}^{i'}, y^{i'} = 1} \left\{ \sum_{i \in S} s^i + \sum_{i \in S} \max\{-t^i, d^i - \lambda\} y^i \right\}$$
$$\leq \sum_{i \in S} d^i - \varphi_S(c - (t^{i'} + \bar{x}^{i'})).$$

Now note that the minimization problem defined by (27) can be solved greedily. To see this, note that the feasible solution $y^i = x^i = 0$, $s^i = d^i$, $i \in S$ has an objective function value $\sum_{i \in S} d^i$ for this problem. We can improve the solution by setting $y^i = 1$, $x^i = d^i$, $s^i = 0$, in *nondecreasing* order of $t^i + d^i$, as long as capacity allows. This is because the capacity used up by doing so is equal to $t^i + d^i$ for each i, whereas the decrease in the objective function is $\min\{t^i + d^i, \lambda\}$. That is, it is always efficient to setup and produce for items with the smallest values of $t^i + d^i$, and thus there always exists an optimal solution with $x^{[i]} = d^{[i]}$, $y^i = 1$, $s^i = 0$, $i = j' + 1, \dots, |S|$.

In an optimal solution, there might also exist production for item $[j']$. We can improve the solution by satisfying demand for $[j']$ from production rather than inventory if there is enough capacity left to produce more than $d^i - \lambda$ units of i, thus allowing us to decrease s^i by the amount of production. That is, an optimal solution with $y^{[j']} = 1$ exists if and only if

$$c - (\sum_{i=j'+1}^{|S|} (t^{[i]} + d^{[i]}) + t^{i'} + \bar{x}^{i'}) \geq t^{[j']} + d^{[j']} - \lambda,$$

or, equivalently,

$$\sum_{i=j'}^{|S|}(t^{[i]}+d^{[i]}) \leq c-(t^{i'}+\bar{x}^{i'})+\lambda. \tag{70}$$

"In this case we produce as much as possible of $|j'|$, i.e., we set $x^{[j']}=c-t^{[j']}-(\sum_{i=j'+1}^{|S|}(t^{[i]}+d^{[i]}+t^{i'}+\overline{x}^{i'}))$." Note that this is exactly the quantity specified for $\bar{x}^{j'}$ above. From this argument we see that $(\hat{x},\hat{y},\hat{s})$ is an *optimal* as well as feasible to the minimization problem in (27).

If $t^{[j']}+d^{i'} \geq \lambda$ but $t^{[j']} > c-(t^{[i']}+\bar{x}^{i'}+\sum_{i=j'+1}^{|S|}(t^{[i]}+d^{[i]}))$, the argument proceeds similarly; in particular, it is again the case that there exists an optimal solution with $y^{[j']}=1$ if and only if (70) holds. In this case, however, we need to reduce production for some items $[i]$ in $j'<i\leq|S|$, to free up enough capacity to set up for $[j']$. In this case, however, it is not difficult to see that $d^{[j']}-\lambda<0$, and so setting up for j' yields an optimal solution when (70) holds, even if no production occurs.

If $t^{[j']}+d^{[j']}<\lambda$ (note that this condition implies (70)), the proof is again similar. Since (70) always holds in this case, there always exists an optimal solution with $y^{[j']}=1$, although it may be necessary to reduce some production for some items $[i]$ with $j'<i\leq|S|$ in order to free enough capacity for this setup. □

Appendix B: Proof of Lemma 14

Lemma 14

$$\begin{aligned}&\max_{0\leq x^i\leq c-t^i}\{\varphi_S(c-(t^i+x^i))+\tfrac{\lambda}{(t^{[2]}+d^{[2]})}(x^i-(\mu^1-t^i))\}\\&=\textstyle\sum_{i\in S\setminus[1]}\min\{t^i+d^i,\lambda\}.\end{aligned} \tag{30}$$

Proof. First, note that when $x^i=\mu^1-t^i$, then

$$\begin{aligned}\varphi_S(c-(t^i+x^i))+\frac{\lambda}{(t^{[2]}+d^{[2]})}(x^i-(\mu^1-t^i)) &= \varphi_S(c-\mu^1)\\ &= \varphi_S(\sum_{i\in S\setminus[1]}(t^i+d^i))\\ &= \sum_{i\in S\setminus[1]}\min\{t^i+d^i,\lambda\},\end{aligned}$$

by definition of μ^1 and $\varphi_S(\cdot)$. Thus we have shown

$$\max_{0\leq x^i\leq c-t^i}\{\varphi_S(c-(t^i+x^i))+\frac{\lambda}{(t^{[2]}+d^{[2]})}(x^i-(\mu^1-t^i))\}\geq\sum_{i\in S\setminus[1]}\min\{t^i+d^i,\lambda\}.$$

To see that this value of x^i maximizes $\varphi_S(c-(t^i+x^i))+\frac{\lambda}{(t^{[2]}+d^{[2]})}(x^i-(\mu^1-t^i))$, first note that $\varphi_S(c-(t^i+x^i))=\sum_{i\in S\setminus[1]}\min\{t^i+d^i,\lambda\}$ for all x^i such $0\leq x^i<\mu^1-t^i$.

However, $\frac{\lambda}{(t^{[2]}+d^{[2]})}(x^i - (\mu^1 - t^i)) < 0$ for all x^i such $0 \leq x^i < \mu^1 - t^i$. Thus, no value of x^i in this range can maximize $\varphi_S(c - (t^i + x^i)) + \frac{\lambda}{(t^{[2]}+d^{[2]})}(x^i - (\mu^1 - t^i))\}$.

Now note that as x^i increases above $\mu^1 - t^i$, the value of $\frac{\lambda}{(t^{[2]}+d^{[2]})}(x^i - (\mu^1 - t^i))$ increases by $\frac{\lambda}{(t^{[2]}+d^{[2]})}$ for every unit of increase. However, if we increase x^i by Δ units to $\mu^1 - t^i + \Delta$, it can be checked that

$$\begin{aligned}\varphi_S(c - (t^i + x^i)) &= \varphi_S(c - \mu^1 - \Delta)\\ &= \varphi_S(\sum_{i \in S\setminus[1]} (t^i + d^i) - \Delta)\\ &\leq \varphi_S(\sum_{i \in S\setminus[1]} (t^i + d^i)) - \frac{\lambda}{(t^{[2]} + d^{[2]})}\Delta;\end{aligned}$$

that is, $\varphi_S(c - (t^i + x^i))$ decreases by *at least* $\frac{\lambda}{(t^{[2]}+d^{[2]})}$ for every unit of increase of x^i above $\mu^1 - t^i$. Thus, no value of x^i in the range $\mu^1 - t^i < x^i \leq c - t^i$ can define a larger value of $\varphi_S(c - (t^i + x^i)) + \frac{\lambda}{(t^{[2]}+d^{[2]})}(x^i - (\mu^1 - t^i))\}$ than that defined by $x^i = \mu^1 - t^i$. This completes the proof. □

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