Ann Oper Res (2007) 149:57–61
DOI 10.1007/s10479-006-0103-y

A brief history of lift-and-project

Sebastián Ceria

Published online: 9 December 2006

During the late 1970's and 1980's a novel concept was explored by a number of researchers interested in the solution of integer programming and combinatorial optimization problems. Rather than formulating the problem as an integer program on the natural space of decision variables, that may lead to a weak linear programming relaxation, these researchers proposed formulating the problem on a higher dimensional space, using auxiliary variables, but where the linear programming relaxation might lead to a stronger relaxation. Little did I know, when I decided to pursue my doctoral studies, that this same topic would become the cornerstone of my Ph.D. dissertation.

When I arrived to Carnegie Mellon to pursue my doctoral studies, as a fresh graduate in Applied Mathematics from the University of Buenos Aires in Argentina, I was immediately impressed by the quality of the faculty. I had heard several stories about the legendary Egon Balas from my undergraduate thesis advisor, Hugo Scolnik, who had shared an office with Egon in Brazil, while both were visiting Nelson Maculan. I had never heard from Gérard Cornuéjols, until I received an email from him prompting me to accept my fellowship offer at Carnegie Mellon (these were the early days of email, and thanks to some pioneering colleagues at the University of Buenos Aires, I was already hooked to email in 1988!).

Let me backtrack for a moment. I had read about Pittsburgh in an old (1969) copy of the Colliers Encyclopedia, that my father kept in our apartment (among the thousands of other books he kept). The city was described as an industrial town that was (in)famous for its steel mills and the pollution that they spread in the air, forcing the locals to change their shirts more than twice a day. Colliers spoke about Pittsburgh's renaissance period in the 50's and 60's but was not big on details on how the new era had changed the city. I had also seen the grim pictures of Pittsburgh and its steel mills in the movie "The Deer Hunter". The prospect of going to Carnegie Mellon sounded great, but I had my doubts about its location... Fortunately, to boost my confidence that I had made the right choice, just before I left, I received news that Pittsburgh had been chosen "The most livable city in America", by Rand McNally. I was off to a flying start, I thought.

S. Ceria (✉)
Chief Executive Officer, Axioma, Inc., New York, NY 10004
e-mail: sceria@axiomainc.com

I have to admit that I was pleasantly surprised when I arrived to Pittsburgh, and even more impressed when I reached the Carnegie Mellon campus. When I arrived to the Graduate School of Industrial Administration, I was received as an immediate member of the family. I had a great feeling about the whole experience from the start. After my first two semesters in the program, I had been exposed to "Graphs and Networks" that was taught by Egon Balas, and "Integer Programming", taught by Gérard Cornuéjols. The courses were excellent. After that experience, there was no doubt in my mind that the strength of the department was Combinatorial Optimization and that I wanted to work in that area.

I remember thinking at the time that it would be great to work with both Egon and Gérard on a paper related to Combinatorial Optimization and Integer Programming. I liked to solve combinatorial problems on graphs, but was particularly interested in how these problems could be solved by using Integer Programming.

I was surprised to discover that even though Egon and Gérard had their offices next to each other, they had never written a paper together! Fortunately for me, the program at CMU had a mandatory summer paper that first-year Ph.D. students had to complete, which gave me the golden opportunity I was looking for. The only small "detail" that remained was to find a good topic. Lady luck struck again, Lovász came to Carnegie Mellon to present his new work on "Cones of Matrices and Set Functions" (Lovász and Schrijver, 1991). Lovász was, as usual, brilliant, and presented his work with great clarity (their paper proved to be more challenging to fully understand). His seminar gave us plenty of great opportunities for a summer paper. Their paper's main concept was that it was possible to strengthen the linear programming relaxation of a mixed-integer program by "lifting" the problem to a higher dimensional space, that involved more variables, strengthening the formulation in the higher dimensional space by using the information that some variables had to be integer ($x_j^2 = x_j$, for $j = 1, \dots, n$), and then "projecting" the resulting polyhedron to the original space of variables. The "lifting" step was carried out by multiplying the original constraints of the problem by $(1 - x_j) \geq 0$ and $x_j \geq 0$, for all j corresponding to variables that had to be either 0 or 1.

We started exploring the first alternatives as soon as we could, and in the summer of 1989, I attended a seminar at Bellcore that had Lovász and Schrijver as the central speakers. Their main presentation was on "Cones of Matrices and Set Functions". Gérard came up with the idea of restricting Lovász and Schrijver's method to one variable per iteration, thus reducing the dimension of the higher dimensional space. The first important result was when we proved that even though we were multiplying one variable at a time, we were still obtaining the convex hull of integer points in a number of steps equal to the number of 0-1 variables, just like Lovász and Schrijver's method. Furthermore, we showed that the same procedure could be used for mixed 0-1 programs without any loss of generality. Egon was very excited about the results, and he managed to prove the equivalence with disjunctive programming and I was able to relate them to intersection cuts (Balas, 1971, 1979). Disjunctive Programming was one of Egon's seminal papers, that had been somewhat "ignored" by the community. This discovery allowed us to reuse all the machinery that Egon, Jeroslow and others had developed during that time. By using a similar result of (Jeroslow, 1980) for disjunctive programming, we were able to prove that our cutting plane algorithm based on this methodology converged in a finite number of steps. It was Egon, who, one afternoon, proposed to us the name "lift-and-project". We were immediately sold on it. In the meantime, I was working on the stable set polytope, searching for the "lift-and-project" rank of known inequalities. I was able to quickly establish the connections with perfect graphs (minimally imperfect graphs had lift-and-project rank 1), and those results caught the attention of Lovász and Schrijver. I was very lucky to have interactions with such great legends so early in my career.

Our results were not only interesting in "theory", but had potential practical implications. The fact that we were lifting the problem to a space that was double the size of the original one, in principle allowed us to generate cutting planes, through the solution of linear program. We came up with the notion of "the deepest cut", although we struggled for a while on how to properly scale these cuts. Egon's experience with disjunctive cuts and Gérard's insights with the geometrical implications on using different scalings were instrumental in allowing us to make significant headway. I had a strong interest in computations, and had a pretty reasonable background in computer programming. I knew how to program in Fortran, Pascal, and had learnt the first steps of C while in Argentina. Our first implementation of Lift-and-Project cuts was using LINDO. I built a rather rudimentary cutting plane algorithm that worked reasonably well for small problems. But the code was slow, and the behavior was somewhat erratic. Indeed, I was trying to solve some of the hardest integer programming problems around; set covering problems arising from the literature. Egon was discouraged with the computational results, to the point that he was ready to "give up" on the idea of using these cuts in computations. Apparently his past experiences with disjunctive cuts had not been successful, and he was afraid that since lift-and-project cuts were essentially equivalent, the results would be the same. I, on the other hand, decided that it was worth persevering, and I would not be disappointed.

During the following summer I continued with my research on lift-and-project. I analyzed the relationship of our procedure to the Reformulation and Linearization Technique of Sherali and Adams (1990), and proved some generalizations were possible. After the summer of 1990 I had completed my qualifiers (they were brutal) and was able to fully dedicate all my time to research, and I immersed myself completely into the world of lift-and-project. My most important discovery at the time was the fact that lift-and-project cuts could be easily "lifted", thus removing the need to solve a linear program to generate the cutting plane in the full space of variables, but rather restrict it to the space of "fractional" variables. This considerably reduced the dimension of the problems needed to be solved to generate cutting planes, and allowed us to incorporate lift-and-project cuts in a branch-and-cut framework. Egon also suggested a cut strengthening procedure he had developed for disjunctive cuts with Jeroslow (Jeroslow, 1980). The addition of the strengthening would prove to be a great contribution to the computational implementation of lift-and-project cuts.

In 1991, our first paper was complete. We called it "A lift-and-project cutting plane algorithm for mixed 0-1 programs" and submitted it to Mathematical Programming (Balas, Ceria, and Cornuéjols, 1993a). The response couldn't have been better. The paper was an instant success and went through what Egon and Gérard considered was "minor" refereeing (this was my first publication ever, so I did not know much about the process). The computational results were limited, but it was our intention not to cloud what we considered was an important "theoretical" discovery with an extensive computational analysis.

Once the paper was accepted for publication, I had the freedom that I needed to concentrate on computational work. I was about to embark on a two-year project to write a branch-and-cut code designed to solve generic mixed 0-1 programs that used lift-and-project cuts. The development of the code was fun, and Gérard and I quickly discovered that there were many choices when generating generic cuts. How many cuts should we generate for each fractional solution? Should we add them all at once or one at a time (we knew from our experiments that this was not a good idea)? How frequently should we generate cuts throughout the tree? Every node? Every five nodes? Finding out the right combinations would turn out to be an art, and to get things right required a lot of experimentation. I must have spent hundreds, or maybe even thousands, of hours looking at the output screens from CPLEX (from ILOG) to try to understand the kinds of problems where we could improve upon what CPLEX was doing.

And then we had to compare with some other methodologies that also relied on generic cutting planes, such as knapsack inequalities. I called our code MIPO for Mixed-Integer Programming Optimizer (A version of MIPO still exists and is maintained by Gábor Pataki).

I would like to say a few words on Computational Integer Programming research. I believe that the work we did during that time would be almost impossible to repeat today. Don't get me wrong, I don't mean to discourage young researchers to tackle challenging problems, but commercial providers of LP and MIP solvers, such as ILOG (who produces and distributes CPLEX) devote so much research and development to their products, that it would be almost a miracle for a researcher to single-handedly improve the results obtained by a general mixed-integer programming solver on generic mixed-integer programs, unless, of course, the research happens to be completely revolutionary (or manages to prove that P = NP). We were "lucky", that, at that time, even though this might sound surprising considering all the research that was done in integer programming in the seventies, commercial mixed-integer solvers were still in their infancy, and not anywhere as powerful as they are today. Even though commercial solvers still had the advantage of a fully integrated LP-solver (while we had to rely on a "loose" integration), we were still able to significantly improve the performance of basic solvers by using lift-and-project cuts, and later by adding Gomory's mixed-integer cuts (Balas et al., 1996b). Of course, since now commercial solvers include both Lift-and-Project and Gomory cuts, the state-of-the-art is not what it used to be!

The results we obtained with MIPO were impressive and became the basis we would use for our second paper (an earlier version of which was published in the SODA Proceedings (Balas, Ceria, and Cornuéjols, 1993b), which we called "Mixed 0-1 Programming with Lift-and-Project in a Branch-and-Cut framework" (Balas, Ceria, and Cornuéjols, 1996a). We decided to submit the paper to Management Science, since we wanted to reach a broader audience of OR researchers and practitioners. As far as I can remember, the refereeing process was once again quite painless, and the paper was accepted for publication. The paper was very well received in the OR community and became the basis for my "job talk" when I decided to pursue an academic career, and would ultimately help me land a job at Columbia Business School.

From the very first successful experiments, we tried to find the reasons behind the success of our computational experiments. We were always intrigued by the fact that the types of cutting planes we were using, even the more generic version of lift-and-project cuts, called disjunctive cuts, had been tried before without much luck. We found the following eight reasons to be quite convincing:

1. We used cutting planes that were based on generic geometrical properties of disjunctive relaxations.
2. We generated cutting planes in rounds, one for each fractional variable, and only reoptimized after a whole set of cuts had been added to the linear programming relaxation.
3. We generated a few rounds of cuts per node (most of the time only one), before branching (and thus changing the geometry of the fractional point).
4. We generated cuts by solving a "deepest cut" linear program, choosing the "best" cut that would cut-off the current fractional point
5. We constantly purged inactive cuts from the lp relaxation, thus improving the numerical stability and reducing degeneracy
6. We embedded the cutting planes in a branch-and-cut framework, and leveraged the fact that the cuts could be easily lifter to work in other nodes of the tree.
7. We kept a pool of cuts generated throughout the tree, and used this set to improve the relaxation at other nodes.

8. We benefited from modern computer technology that allowed us to keep a lot of information in memory without having to store information on disk.

We believe that our work in lift-and-project cuts, and the one that followed on Gomory cuts, helped change the overall mood of the community as it related to the reluctance of using generic integer cuts as part of the standard machinery to solve generic mixed-integer programs. It's now hard to find a state-of-the-art MIP solver that doesn't use one, or both, of these types of cuts as the default for solving such problems.

In spite of the reasons I listed above, I still believe that the best reason for the success of lift-and-project is that you can never go wrong if you work with two giants, like Egon Balas and Gérard Cornuéjols. I was lucky to be there and bring them together. Just once.

References

Balas, E. (1971). "Intersection Cuts—A New Type of Cutting Planes for Integer Programming." *Operations Research*, 19, 19–39.

Balas, E. (1979). "Disjunctive Programming." *Annals of Discrete Mathematics*, 5, 3–51.

Balas, E., S. Ceria, and G. Cornuéjols. (1993a). "A Lift-and-Project Cutting Plane Algorithm for Mixed 0-1 Programs." *Mathematical Programming*, 58, 295–324.

Balas, E., S. Ceria, and G. Cornuéjols. (1993b). "Solving Mixed 0-1 Programs by a Lift-and-Project Method." In *Proceedings of the Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pp. 232–242.

Balas, E., S. Ceria, and G. Cornuéjols. (1996a). "Mixed 0-1 Programming by Lift-and-Project in a Branch-and-Cut Framework." *Management Science*, 42, 1229–1246.

Balas, E., S. Ceria, G. Cornuéjols, and N.R. Natraj. (1996b). "Gomory Cuts Revisited." *Operation Research Letter*, 19, 1–9.

Balas, E. and R. Jeroslow. (1980). "Strengthening Cuts for Mixed Integer Programs." *European Journal of Operations Research*, 4, 224–234.

Jeroslow, R. (1980). "A cutting plane game for facial disjunctive programs." *SIAM J. Control and Optimization* 18, 264-–280.

Lovász, L. and A. Schrijver. (1991). "Cones of Matrices and Set-Functions and 0-1 Optimization." *SIAM J. Optimization*, 1, 166–190.

Sherali, H. and W. Adams. (1990). "A Hierarchy of Relaxations Between the Continuous and Convex Hull Representations for Zero-one Programming Problems." *SIAM J. Disc. Math.*, 3, 411–430.