Math. Program., Ser. B 94: 279–294 (2003)

Digital Object Identifier (DOI) 10.1007/s10107-002-0320-3

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On the separation of split cuts and related inequalities

Received: October 23, 2000 / Accepted: October 03, 2001
Published online: September 5, 2002 –

Abstract. The *split cuts* of Cook, Kannan and Schrijver are general-purpose valid inequalities for integer programming which include a variety of other well-known cuts as special cases. To detect violated split cuts, one has to solve the associated *separation problem*. The complexity of split cut separation was recently cited as an open problem by Cornuéjols & Li [10].

In this paper we settle this question by proving strong $\mathcal{NP}$-completeness of separation for split cuts. As a by-product we also show $\mathcal{NP}$-completeness of separation for several other classes of inequalities, including the *MIR-inequalities* of Nemhauser and Wolsey and some new inequalities which we call *balanced split cuts* and *binary split cuts*. We also strengthen $\mathcal{NP}$-completeness results of Caprara & Fischetti [5] (for $\{0, \frac{1}{2}\}$-*cuts*) and Eisenbrand [12] (for *Chvátal-Gomory cuts*).

To compensate for this bleak picture, we also give a positive result for the *Symmetric Travelling Salesman Problem*. We show how to separate in polynomial time over a class of split cuts which includes all *comb inequalities with a fixed handle*.

Key words. cutting planes – separation – complexity – travelling salesman problem – comb inequalities

1. Introduction

A wide variety of general-purpose cutting planes (valid linear inequalities) have been proposed for integer and mixed-integer programming over the years. These include, in historical order, Gomory's *fractional* and *mixed-integer* cuts [15, 16]; the *intersection* cuts of Balas [1]; the *Chvátal-Gomory* cuts (see Chvátal [8] and Nemhauser & Wolsey [24]); the *disjunctive* cuts (see Balas [2]); the *split cuts* of Cook, Kannan & Schrijver [9]; the *MIR-inequalities* of Nemhauser & Wolsey [25]; the *matrix cuts* of Lovász & Schrijver [23]; the *lift-and-project* cuts of Balas, Ceria & Cornuéjols [4] and the $\{0, \frac{1}{2}\}$-*cuts* of Caprara & Fischetti [5].

Although this array of inequalities is rather bewildering, it is known that many of them are essentially the same. For example, the Gomory fractional cuts are equivalent to Chvátal-Gomory cuts, and the Gomory mixed-integer cuts are equivalent to both split cuts and MIR inequalities. The relationships between all of these inequalities are made precise in the recent excellent paper by Cornuéjols & Li [10].

A question of interest for a given class of inequalities is how easy they are to generate in the context of a cutting plane algorithm. This leads us to consider the so-called *separation problem* associated with each class of inequalities, i.e., the problem of detecting

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when an inequality of a given type is violated by a fractional solution of an LP relaxation of the original problem (see Grötschel, Lovász & Schrijver [17]).

Some progress has been made in the last ten years on determining the complexity of these separation problems. On the positive side, it is known that the separation problems for both matrix cuts and lift-and-project cuts can be solved in polynomial time [4, 23]. On the negative side, the separation problems for both $\{0, \frac{1}{2}\}$-cuts and Chvátal-Gomory cuts are strongly $\mathcal{NP}$-complete (see [5] and [12]). The complexity of separation for the remaining cuts was cited as an open problem by Cornuéjols & Li [10].

In this paper we prove strong $\mathcal{NP}$-completeness of separation for split cuts and MIR inequalities. The key observation leading to our result is that split cut separation can be formulated in a nontrivial way as a (nonlinear) mixed-integer program with parity constraints. As a by-product, we also show $\mathcal{NP}$-completeness of separation for two new classes of inequalities, which we call *balanced split cuts* and *binary split cuts*. We also strengthen the $\mathcal{NP}$-completeness results of Caprara & Fischetti [5] and Eisenbrand [12] by showing that separation of both $\{0, \frac{1}{2}\}$-cuts and Chvátal-Gomory cuts is strongly $\mathcal{NP}$-complete even when variables are constrained to be non-negative (as is normally the case in practice).

To compensate for this bleak picture, we also give a positive separation result for the well-known *Symmetric Travelling Salesman Problem* (STSP). For this problem the split cuts include the well-known *comb* inequalities as a special case, along with various other more general facet-inducing inequalities. Although the complexity of both comb and split cut separation in the case of the STSP is unknown, we show how to separate in polynomial time over a class of split cuts which includes all *(generalized) comb inequalities with a fixed handle*. This result is achieved by showing that all $\{0, \frac{1}{2}\}$-cuts for the STSP are split cuts associated with so-called *natural* disjunctions.

The structure of the remainder of the paper is as follows. In Section 2 we review the definitions of various known cuts, define the new cuts (binary split cuts and balanced split cuts), and show how the various cuts are related. In Section 3 we establish the new $\mathcal{NP}$-completeness results. In Section 4, we apply the concepts to the STSP and derive the new separation result. Finally, conclusions are given in Section 5.

Throughout the paper, we assume that a mixed-integer linear program (MILP) has n integer-constrained variables, p continuous variables, and m linear constraints. The vector of integer-constrained variables will be denoted by x and the vector of continuous variables by y. The feasible region of the LP relaxation is assumed to be the polyhedron

$$P := \{(x, y) \in \mathbb{R}^{n+p} : Ax + Gy \leq b\},$$

where A and G are integral matrices of appropriate dimension ($m \times n$ and $m \times p$, respectively) and b is an m-vector of integral right hand sides. This implies that the convex hull of feasible MILP solutions is the polyhedron

$$P_I := \text{conv}\{x \in Z^n, y \in \mathbb{R}^p : Ax + Gy \leq b\}.$$

We have $P_I \subseteq P$ and we assume in this paper that containment is strict. A *cutting plane* is a linear inequality which is *valid* for P_I (i.e., satisfied by all $(x, y) \in P_I$), but violated by some $(x, y) \in P \setminus P_I$.

In the majority of interesting cases it can be assumed that all variables are constrained to be non-negative. When this is true, we will say that the *non-negativity assumption* holds and we will consider the non-negativity constraints as part of the system $Ax + Gy \leq b$.

Finally, given a graph $G = (V, E)$, we define for any $S \subseteq V$ the edge sets $E(S) := \{(i, j) \in E : i, j \in S\}$ and $\delta(S) := \{(i, j) \in E : i \in S, j \in V \setminus S\}$.

2. Relationships between various classes of cuts

In this section we review the definitions of some of the classes of cuts mentioned in Section 1, introduce two new classes, and show how the various classes all relate to each other.

We begin with the cuts introduced by Chvátal [8], which have since become known as *Chvátal-Gomory* (or CG) cuts (see, e.g., [24]). The CG cuts, which are only defined for pure integer linear programs (ILPs), are valid inequalities of the form $(\lambda A)x \leq \lfloor \lambda b \rfloor$, where $\lambda \in \mathbb{R}^m_+$ is such that $\lambda A \in Z^n$ and $\lfloor \cdot \rfloor$ represents integer rounding downward. Obviously, we can require that λb is not integral, because otherwise the CG cut will be dominated by the inequalities defining P.

The CG cuts are related to the classical *fractional* cuts of Gomory [15]. In fact it is often claimed that they are equivalent. However, this is not exactly true – see [10] for a more precise statement. Indeed, we can define CG cuts even when variables are permitted to be negative, which is not the case for fractional cuts.

The components of the vector λ used in the derivation of a CG cut are called *CG multipliers*. By imposing restrictions on the CG multipliers, various special kinds of CG cut can be defined, see Caprara & Fischetti [5] and Caprara, Fischetti & Letchford [6]. In particular, a $\{0, \frac{1}{2}\}$-*Chvátal-Gomory cut*, or $\{0, \frac{1}{2}\}$-*cut* for short, is a CG cut in which all CG multipliers are either zero or one-half [5]. Again it should be noted that $\{0, \frac{1}{2}\}$-cuts can be defined even when variables are permitted to be negative.

Now we come to the most important class of cuts in this paper, the *split cuts* of Cook, Kannan & Schrijver [9]. The split cuts, which are defined for general MILPs, are derived via a so-called *disjunctive* argument as follows (see also Balas [2, 3]). Given any $c \in Z^n$ and $d \in Z$, each MILP solution must satisfy either $cx \leq d$ or $cx \geq d+1$. Then, if we define the polyhedra $P^L := \{(x, y) \in P : cx \leq d\}$ and $P^R := \{(x, y) \in P : cx \geq d+1\}$, we have that any inequality which is valid for P^L and P^R is also valid for P_I. (Here, 'L' and 'R' are meant to denote 'left' and 'right'.) Any inequality of this type is a split cut.

Closely related to the split cuts are the *MIR-inequalities* of Nemhauser & Wolsey [24]. Unlike the split cuts, however, the MIR-inequalities are only defined for problems where the non-negativity assumption holds. Given two inequalities valid for P of the form

$$\sum_{j=1}^{n} \pi^i_j x_j + \sum_{j=1}^{p} \mu^i_j y_j \leq \pi^i_0$$

for $i = 1, 2$, the associated MIR-inequality takes the form

$$\sum_{j=1}^{n} \lfloor \pi_j^2 - \pi_j^1 \rfloor x_j + \frac{1}{1-f_0} \left(\sum_{j=1}^{n} \pi_j^1 x_j + \sum_{j=1}^{p} \min(\mu_j^1, \mu_j^2) y_j - \pi_0^1 \right) \leq \lfloor \pi_0^2 - \pi_0^1 \rfloor$$

where $f_0 = \pi_0^2 - \pi_0^1 - \lfloor \pi_0^2 - \pi_0^1 \rfloor$.

It is often said (see, e.g., [9, 25]) that the split cuts, MIR-inequalities, along with the classical *mixed-integer cuts* of Gomory [16], are all equivalent. However, again this is not exactly true — see [10] for a more precise statement. In particular, we can define split cuts even when variables are permitted to be non-negative. MIR-inequalities and mixed-integer cuts, on the other hand, are only defined when the non-negativity assumption holds.

The so-called *lift-and-project* cuts of Balas, Ceria & Cornuéjols [4] are a simple kind of split cut, obtained when the disjunction is of the form $(x_i \leq 0) \vee (x_i \geq 1)$, where x_i is a binary variable.

Next we will introduce two more classes of cuts to further complicate an already complicated picture. Recall the definition of P^L and P^R in the above definition of split cuts. The system of inequalities defining P^L, namely

$$cx \leq d, \; Ax + Gy \leq b \tag{1}$$

will be referred to as the *left system* and the inequality $cx \leq d$ will be referred to as the *left term*. Similarly, the system of inequalities defining P^R, namely

$$-cx \leq -d - 1, \; Ax + Gy \leq b \tag{2}$$

will be called the *right system* and the inequality $-cx \leq -d - 1$ will be called the *right term*.

Note that, to show that an inequality is implied by the left system (or right system) in a split derivation, one can sum together the inequalities in the left system, multiplied by non-negative coefficients. We will call these coefficients *disjunctive multipliers* by analogy with the CG multipliers.

Definition 1. *A* balanced split cut *is a split cut in which the disjunctive multipliers for the left and right term are equal to each other.*

Note that the multipliers for the left and right term may be assumed to be equal to 1 in the derivation of a balanced split cut without loss of generality.

Definition 2. *A* binary split cut *is a split cut in which all disjunctive multipliers are in* $\{0, 1\}$ *for both systems.*

Given a binary split cut, we also define the set S_L, containing the inequalities in the left system whose disjunctive multipliers are 1, and a corresponding set S_R for the right system. Note that one may assume without loss of generality that the left term is in S_L (and the right term is in S_R), since otherwise the split cut would be implied by the original inequality system $Ax \leq b$. Thus, we may assume that binary split cuts are balanced without loss of generality. Also, one may assume that no inequality is in both S_L and S_R, because removing such an inequality from S_L and S_R leads to an equivalent or stronger binary split cut.

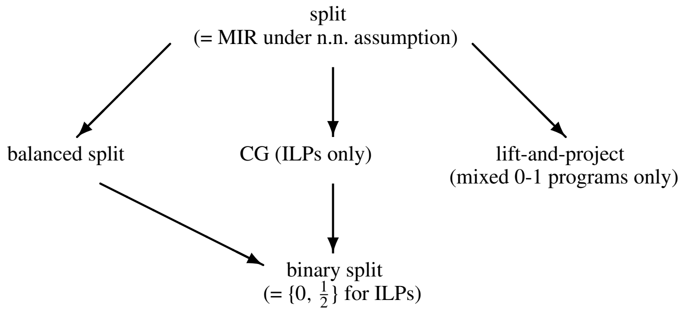

Fig. 1. Relationships between classes of cuts

The next proposition shows that, for ILPs, binary split cuts and $\{0, \frac{1}{2}\}$-cuts are essentially the same.

Proposition 1. *In the case of ILPs, the binary split cuts and* $\{0, \frac{1}{2}\}$*-cuts are equivalent. That is, given any cut in one family, there is a cut in the other family which is equivalent (or stronger).*

Proof. Consider first a binary split cut $\alpha x \leq \beta$ (note that β is integer). As stated before, we can assume that the left term is contained in S_L and the right term in S_R. Now consider the $\{0, \frac{1}{2}\}$-cut obtained by setting the CG multipliers to $\frac{1}{2}$ for the inequalities in $S_L \cup S_R$, but excluding the left and right terms. It is immediately verified that this cut has the form $\alpha x \leq \lfloor \beta + \frac{1}{2} \rfloor$, i.e., it is equivalent to the binary split cut.

Now consider a $\{0, \frac{1}{2}\}$-cut $\gamma x \leq \delta$. Let S be the set of inequalities in the original system $Ax \leq b$ with CG multiplier $\frac{1}{2}$ in the derivation of this cut. The same cut can be derived as a binary split cut from the disjunction $(\gamma x \leq \delta) \vee (\gamma x \geq \delta + 1)$. This is trivial for the left side: The left term is given a disjunctive multiplier of 1 and all other inequalities receive a disjunctive multiplier of zero. On the right side, the right term and the inequalities in S are given disjunctive multipliers of 1. The claim follows from the observation that the sum of the inequalities in S gives $2\gamma x \leq 2\delta + 1$. □

Thus, in the case of ILPs, binary split cuts yield nothing new. On the other hand, even in the case of 0-1 ILPs, there are balanced split cuts which are neither CG cuts, nor lift-and-project cuts. Consider, for example, the polyhedron $P := \{x \in [0, 1]^3 : 2x_1 + x_2 + x_3 \leq 2, -2x_1 + x_2 + x_3 \leq 0\}$. The inequality $x_2 + x_3 \leq 0$ is easily shown to be a balanced split cut, yet neither a CG cut nor a lift-and-project cut.

Figure 1 illustrates the relationships between all of the inequalities discussed. An arrow from one class to another indicates that the former class is a proper generalization of the latter, even in the case of pure 0-1 ILPs.

3. Separation

3.1. Known results

In order to use a class of valid linear inequalities as cutting planes, it is necessary to solve the so-called *separation problem* (Grötschel, Lovász & Schrijver [17]). For a given family of inequalities $\mathcal{F}$ and a given point $x^* \in \mathbb{R}^n$, the separation problem is to find an inequality in $\mathcal{F}$ which is violated by x^*, or to prove that none exists.

The only class of inequalities among those displayed in Figure 1 which are known to be separable in polynomial time are the lift-and-project cuts [4, 23]. On the other hand, the separation problems for $\{0, \frac{1}{2}\}$-cuts and CG cuts are strongly $\mathcal{NP}$-complete (see Caprara & Fischetti [5], Eisenbrand [12], respectively). However these two hardness results are only proved for ILPs without the non-negativity assumption. The complexity of separation for $\{0, \frac{1}{2}\}$-cuts and CG cuts under the non-negativity assumption, as well as for the remaining inequalities in Figure 1, was unknown until the present paper.

In the next subsection we show strong $\mathcal{NP}$-completeness of separation for split cuts, balanced split cuts, binary split cuts, CG cuts, MIR-inequalities and $\{0, \frac{1}{2}\}$-cuts, in all cases for pure ILPs under the non-negativity assumption.

3.2. *Hardness of separation*

In order to prove our hardness results it is necessary to express the separation problem for split cuts in a certain nontrivial form. A split cut is a valid inequality of the form $\alpha x + \beta y \le \gamma$, with

$$\alpha = \lambda^L c + \mu^L A = -\lambda^R c + \mu^R A, \tag{3}$$

$$\beta = \mu^L G = \mu^R G, \tag{4}$$

and

$$\gamma = \lambda^L d + \mu^L b = -\lambda^R (d + 1) + \mu^R b, \tag{5}$$

where $cx \le d \vee cx \ge d + 1$ ($c \in Z^n$, $d \in Z$) is the associated disjunction (which *has to be determined* in the separation problem), and (λ^L, μ^L), $(\lambda^R, \mu^R) \ge 0$ are the disjunctive multiplier vectors for the left and right system, respectively. Using this fact, the split cut can be expressed as:

$$\left((\mu^L + \mu^R)A + (\lambda^L - \lambda^R)c\right) x + (\mu^L + \mu^R)Gy \le (\mu^L + \mu^R)b + (\lambda^L - \lambda^R)d - \lambda^R. \tag{6}$$

This can be simplified in the following way. First, we assume without loss of generality that $\lambda^L + \lambda^R = 2$ and let $\lambda := \lambda^R$. (Note that $0 < \lambda < 2$ in any non-dominated cut, and that $\lambda = 1$ for a balanced split cut.) Second, we define the *slack vector* $s := b - Ax - Gy$. Then the split cut can be written as:

$$(\mu^L + \mu^R)s + (2 - 2\lambda)(d - cx) - \lambda \ge 0. \tag{7}$$

Therefore, if we are given a fractional point $(x^*, y^*) \in P$, with associated slack vector $s^* := b - Ax^* - Gy^* \ge 0$, then the problem of separating the *most violated* split cut can be stated as

$$\min (\mu^L + \mu^R)s^* + (2 - 2\lambda)(d - cx^*) - \lambda \tag{8}$$

subject to

$$(\mu^R - \mu^L)A = 2c, \tag{9}$$

$$(\mu^R - \mu^L)G = 0, \tag{10}$$

$$(\mu^R - \mu^L)b = 2d + \lambda, \tag{11}$$

$$c, d \text{ integer}, \tag{12}$$

$$\mu^L, \mu^R \geq 0, 0 < \lambda < 2. \tag{13}$$

The above discussion is summarized in the following proposition:

Proposition 2. *For a given $(x^*, y^*) \in P$, a split cut is violated if and only if the optimal solution value to (8)-(13) is strictly smaller than 0.*

It is also instructive to see that, in the case of balanced split cuts (with λ fixed to 1), the formulation (8)-(13) is a MILP. Moreover, in that case, the c and d variables can be eliminated and constraints (9) and (11) can be replaced with

$$(\mu^R - \mu^L)A \ \mathtt{mod}\ 2 = 0, \tag{14}$$

$$(\mu^R - \mu^L)b \ \mathtt{mod}\ 2 = 1, \tag{15}$$

where, given a (not necessarily integer) value x, $x \ \mathtt{mod}\ 2$ is interpreted to mean $x - 2\lfloor x/2 \rfloor$. The resulting problem is very similar to the formulation for $\{0, \frac{1}{2}\}$-separation given in [5]. However, whereas it is natural to observe that $\{0, \frac{1}{2}\}$-separation involves *parity* constraints, the fact that the same sort of constraints (i.e., (9) and (11)) appear for split cut separation is certainly surprising. As we show next, this is the key to our complexity result.

We will find the following lemma very useful.

Lemma 1. *Non-dominated split cuts arise for μ^L, μ^R such that, for $i = 1, \ldots, m$, $\mu_i^L \cdot \mu_i^R = 0$. Moreover, in the case of ILPs, $0 \leq \mu_i^L < 2$ and $0 \leq \mu_i^R < 2$ for $i = 1, \ldots, m$.*

Proof. Consider a solution of (8)-(13) in which $\mu_i^L, \mu_i^R > 0$ for some i, assuming without loss of generality $\mu_i^L > \mu_i^R$. Since the slacks are non-negative, a solution which is not worse can be obtained by replacing μ_i^L by $(\mu_i^L - \mu_i^R)$ and μ_i^R by 0.

Now suppose that we are dealing with an ILP. Recall that A and b are integral. Consider a solution of (8)-(13) in which $\mu_i^L \geq 2$ for some i. If we reduce μ_i^L by two and increase c and d accordingly to satisfy (9) and (11), the objective (8) decreases by $2\lambda s_i^* \geq 0$. Similarly, if $\mu_i^R \geq 2$ for some i and we reduce μ_i^R by two and decrease c and d accordingly, the objective (8) decreases by $(4 - 2\lambda)s_i^* \geq 0$. This reduction can be repeated until all multipliers are less than 2. □

Now we are ready for the main theorem.

Theorem 1. *The separation problem for split cuts is strongly $\mathcal{NP}$-complete even for pure ILPs under the non-negativity assumption.*

Proof. We reduce the well-known (and strongly $\mathcal{NP}$-complete) max-cut problem (in its cardinality version) to the problem mentioned. Let $G = (V, E)$ be a graph. Denote by M the node-edge incidence matrix of G, i.e., the matrix with $|V|$ rows and $|E|$ columns, with a one in row i and column j if and only if node i is incident on edge j. Let r denote the row vector consisting of $|E|$ ones. Define a matrix A, with $|V| + 1$ rows and $|E| + |V| + 1$ columns, as $A := \left(\begin{smallmatrix} M \\ r \end{smallmatrix} \middle| -2I\right)$. Also define a column vector b consisting of $|V|$ zeros followed by a -1.

Consider the polyhedron $P := \{x \in \mathbb{R}^{|E|+|V|+1} : Ax \leq b, x \geq 0\}$ and the associated split cuts. We let μ^L and μ^R denote the disjunctive multiplier vectors for the inequalities $Ax \leq b$, and π^L and π^R denote the disjunctive multipliers for the non-negativity inequalities. Note that $\mu^L, \mu^R \in \mathbb{R}_+^{|V|+1}$ and $\pi^L, \pi^R \in \mathbb{R}_+^{|E|+|V|+1}$.

Claim 1. If a vector $x^* \in P$ violates a split cut, then it violates a split cut in which $\pi_i^L = \pi_i^R = 0$ for $i = |E| + 1, \ldots, |E| + |V| + 1$.

To see this, note that each of the associated non-negativity inequalities is dominated by one of the inequalities in the system $Ax \leq b$, together with the non-negativity inequalities on the first $|E|$ variables.

After the elimination of the non-negativity inequalities mentioned in Claim 1, for $i = 1, \ldots, |V|$, variable $x_{|E|+i}$ appears only in one constraint – namely, the ith constraint of the system $Ax \leq b$ – and in this constraint it has a coefficient of -2.

Claim 2. If a vector $x^* \in P$ violates a split cut, then it violates a split cut in which μ^L and μ^R are binary.

To see this, notice that Claim 1 and Equation (9) imply $(\mu^R - \mu^L) \cdot (-2I) = 0 \bmod 2$. Equivalently, for $i = 1, \ldots, |V| + 1$ we have $(\mu_i^R - \mu_i^L) \in Z$. Together with Lemma 1 this implies that μ^L and μ^R are binary.

Claim 3. If a vector $x^* \in P$ violates a split cut, then it violates a split cut in which μ^L, μ^R, π^L and π^R are all binary.

To see this, notice that Claim 1 and Equation (9) imply $(\mu^R - \mu^L)A + (\pi^R - \pi^L)(-I) = 0 \bmod 2$. Since by Claim 2, μ^R and μ^L are integral, we have that $\pi^R - \pi^L$ must be integral too. Together with Lemma 1 this implies that π^L and π^R are binary.

Claim 4. If a vector $x^* \in P$ violates a split cut, then it violates a binary split cut.

By Claim 3, μ^L, μ^R, π^L and π^R can be assumed to be binary. Now notice that, since the vector b only contains one non-zero entry, Equation (11) reduces to $-\mu_{|V|+1}^R + \mu_{|V|+1}^L = 2d + \lambda$. This implies that $\lambda \in Z$, which, since $0 < \lambda < 2$, implies that $\lambda = 1$.

So from this point on, by Proposition 1, we can assume that the split cut is a $\{0, \frac{1}{2}\}$-cut derived from the inequalities $Ax \leq b$ and the non-negativity inequalities on $x_1, \ldots, x_{|E|}$. Let $\lambda \in \{0, \frac{1}{2}\}^{|V|+1}$ denote the vector of CG multipliers for the inequalities in the system $Ax \leq b$ and let $\varphi \in \{0, \frac{1}{2}\}^{|E|}$ denote the vector of CG multipliers for the first $|E|$ non-negativity inequalities. To obtain rounding on the right hand side, $\lambda_{|V|+1}$ must equal one. To obtain integer coefficients on the left hand side we must have $\varphi_i = 1/2$ if and only if $(\lambda M)_i \in \{0, 1\}$, since the coefficient of edge i in the $\{0, \frac{1}{2}\}$-cut has the form

$(\lambda M)_i + \lambda_{|V|+1} + \varphi_i$. Let $S \subseteq V$ be defined by $S := \{i \in V : \lambda_i = 1/2\}$. Then, the set of edges in G whose associated variables have $\varphi_i = 1/2$ is given by $E \setminus \delta(S)$. Furthermore, it is easy to show [5] that the $\{0, \frac{1}{2}\}$-cut is violated by a vector $x^* \in \mathbb{R}_+^{|E|+|V|+1}$ if and only if $\lambda s^* + \varphi x^* < 1$, where $s^* := b - Ax^*$ and $\varphi_{|E|+i} := 0$ for $i = 1, \ldots, |V|$.

We are now ready for the reduction. Let d_i denote the degree of vertex i in G. Define the vector $x^* \in \mathbb{R}_+^{|E|+|V|+1}$ as follows. For $i = 1, \ldots, |E|$, let $x_i^* = \varepsilon$, where ε is some small positive quantity that we will define later. For $i = 1, \ldots, |V|$, let $x^*_{|E|+i} = \varepsilon d_i/2$. Finally, let $x^*_{|E|+|V|+1} = (1 + \varepsilon|E|)/2$. Notice that, by construction, all of the inequalities in the system $Ax \leq b$ are tight (have zero slack) at x^*. Then, finding a $\{0, \frac{1}{2}\}$-cut violated by x^*, if any, is equivalent to finding, if any, $S \subseteq V$ such that $\sum_{i \in E \setminus \delta(S)} x_i^* = \varepsilon|E \setminus \delta(S)| < 1$. The max-cut problem, in its cardinality and recognition version, calls for a set $S \subseteq V$ such that $|\delta(S)| > K$, or, equivalently, $|E \setminus \delta(S)| < |E| - K$. Setting $\varepsilon := 1/(|E| - K)$ completes the reduction. □

Noting that the disjunctive multipliers have to be binary and that λ has to be one in the proof of Theorem 1 immediately yields several corollaries:

Corollary 1. *The separation problem for binary split cuts is strongly $\mathcal{NP}$-complete, even for pure ILPs under the non-negativity assumption.*

Corollary 2. *The separation problem for balanced split cuts is strongly $\mathcal{NP}$-complete, even for pure ILPs under the non-negativity assumption.*

Corollary 3. *The separation problem for MIR-inequalities is strongly $\mathcal{NP}$-complete, even for pure ILPs.*

Corollary 4. *The separation problem for CG cuts is strongly $\mathcal{NP}$-complete, even under the non-negativity assumption.*

Corollary 5. *The separation problem for $\{0, \frac{1}{2}\}$-cuts is strongly $\mathcal{NP}$-complete, even under the non-negativity assumption.*

Before moving on, we would like to make a couple of interesting observations. First, it is not difficult to show that, when P is defined in the proof of Theorem 1, then P is unbounded and has a unique vertex. Second, and paradoxically, it is not difficult to show how to minimize any linear function over the associated P_I (or prove unboundedness) in polynomial time. If the optimal value is indeed bounded, then the minimization problem amounts to finding a minimum weight *odd circuit* in G. This can be done using the algorithm of Gerards & Schrijver [14].

Therefore, all of the above-mentioned separation problems remain strongly $\mathcal{NP}$-complete even when P has only one vertex, and even when optimization over P_I is polynomial-time solvable.

3.3. Separation when the disjunction is fixed

The results of the previous subsection paint a rather bleak picture. On a more positive note, results in [3] and [4] imply that, for a *fixed* disjunction, the separation of split cuts can be carried out in polynomial time, even if the system $Ax + Gy \leq b$ contains exponentially many constraints, provided an efficient separation procedure is known for this system. We give an explicit proof of this here to make the presentation self contained:

Theorem 2. *Split cuts associated with system $Ax + Gy \leq b$ and a given disjunction $(cx \leq d) \vee (cx \geq d + 1)$ can be separated in polynomial time provided an efficient separation procedure is known for $Ax + Gy \leq b$ as well as for the homogenized system $Ax + Gy \leq 0$.*

Proof. Separation of split cuts means testing, for a given $(x^*, y^*) \in \mathbb{R}^{n+p}$, if (x^*, y^*) lies in the convex hull of $\{(x, y) \in \mathbb{R}^n : cx \leq d, Ax + Gy \leq b\} \cup \{x \in \mathbb{R}^n : -cx \leq -d - 1, Ax + Gy \leq b\}$. As shown in [3], this is equivalent to finding a solution to the linear system

$$\begin{aligned}
x^L + x^R &= x^*,\\
y^L + y^R &= y^*,\\
t^L + t^R &= 1,\\
cx^L &\leq dt^L,\\
Ax^L + Gy^L &\leq bt^L,\\
-cx^R &\leq (-d - 1)t^R,\\
Ax^R + Gy^R &\leq bt^R,\\
x^L, x^R, y^L, y^R, t^L, t^R &\geq 0,
\end{aligned}$$

where x^L and x^R are n-dimensional vectors, y^L and y^R are p-dimensional vectors and t^L, t^R are scalars. If a solution exists, no split cut is violated, otherwise the dual solution of the above system (with a suitably-defined objective function) yields the multipliers to derive a violated split cut; see [4] for details.

In the case where A and G have exponentially many rows, the above linear system can be solved efficiently if one can test efficiently, for a given pair of vectors $((x^L)^*, (y^L)^*)$, and a given scalar $(t^L)^*$, if $a^i(x^L)^* + g^i(y^L)^* > b_i(t^L)^*$ for some row (a^i, g^i) of (A, G). If $(t^L)^* \neq 0$, this is equivalent to testing if $a^i \frac{(x^L)^*}{(t^L)^*} + g^i \frac{(y^L)^*}{(t^L)^*} > b_i$ and an efficient separation procedure for $Ax + Gy \leq b$ is sufficient. Otherwise, if $(t^L)^* = 0$, we have to find, if any, a row (a^i, g^i) of (A, G) such that $a^i(x^L)^* + g^i(y^L)^* > 0$, i.e., we need a separation procedure for the homogenized system. □

Note that, for most cases of practical interest (such as the one that we will consider in the next section), an efficient separation procedure for $Ax + Gy \leq b$ can easily be adapted to the homogenized system $Ax + Gy \leq 0$. Indeed, when P is bounded (i.e., a polytope), the only point satisfying $Ax + Gy \leq 0$ is the origin and the associated separation problem is trivial.

Theorem 2 implies as a corollary the already-mentioned fact that *lift-and-project* cuts can be separated in polynomial time (because at most n fixed disjunctions are needed to derive them). In particular, the so-called *odd hole* inequalities for the stable set problem are lift-and-project cuts with respect to a standard formulation and therefore we obtain the well-known result that odd hole inequalities are separable in polynomial time [4, 17].

At this point we would like to mention what we call *natural* disjunctions, defined only for pure ILPs.

Definition 3. *A (two-term) disjunction is* natural *if it is of the form* $a^i x \leq f$ *or* $a^i x \geq f + 1$ *for some row* $a^i x \leq b_i$ *of the original system* $Ax \leq b$ *and for some integer* f.

The resulting *natural split cuts* are a generalization of lift-and-project cuts, yet Theorem 2 implies that they can still be separated in polynomial time. Some other implications of Theorem 2 are discussed in Letchford [22].

We note in passing that there may be $\{0, \frac{1}{2}\}$-cuts which are not natural split cuts. For example, the famous *blossom inequalities* for matching problems (Edmonds [11]) are easily shown to be of this type. Note that, if the blossom inequalities *were* natural split cuts, as an immediate corollary one would get a *compact* (polynomial-size) LP formulation of the matching problem – thus solving a long-standing open problem (see Yannakakis [31]).

Thus, the natural split cuts should be placed immediately above the lift-and-project cuts in Figure 1.

In the next section we examine the situation with the Symmetric Travelling Salesman Problem.

4. The symmetric travelling salesman problem

Given a complete undirected graph $G = (V, E)$ with non-negative costs c_{ij} for each $(i, j) \in E$, the well-known *Symmetric Travelling Salesman Problem* (STSP) is that of finding a minimum weight Hamiltonian circuit (tour) in G.

At present, the most successful approach to solving large-scale STSP instances to optimality is the so-called *branch-and-cut* method [30], which is based on adding cutting planes to the following ILP formulation of this problem. Define a binary variable x_{ij} for each edge $(i, j) \in E$, taking the value 1 if (i, j) is in the tour and 0 otherwise, and let $x(F)$ for any $F \subseteq E$ denote $\sum_{(i,j)\in F} x_{ij}$. The formulation is:

$$\min \sum_{(i,j)\in E} c_{ij} x_{ij} \tag{16}$$

subject to

$$x(\delta(\{i\})) = 2 \qquad (\forall i \in V) \tag{17}$$

$$x(E(S)) \leq |S| - 1 \qquad (\forall S \subset V : 2 \leq |S| \leq |V| - 2) \tag{18}$$

$$x_{ij} \in \{0, 1\} \qquad (\forall (i, j) \in E). \tag{19}$$

Constraints (17) are called *degree equations* and Constraints (18) are the well-known *subtour elimination constraints* (SECs). Note that there are an exponential number of SECs and that the SECs with $|S| = 2$ impose upper bounds of 1 on the variables.

The polytope $P := \{x \in \mathbb{R}_+^{|E|} :$ (17), (18) hold$\}$ is known as the *subtour elimination polytope*. The associated integer polytope, $P_I := \text{conv}\{x \in P \cap Z^{|E|}\}$, is called the *travelling salesman polytope*. The most successful cutting planes are those which induce *facets* (faces of maximal dimension) of this polytope.

Among the inequalities known to induce facets of P_I, apart from the SECs themselves, are the *comb* inequalities of Grötschel & Padberg [19, 20]. Let $p \geq 3$ be an odd integer. Let $H \subset V$ and $T_j \subset V$ for $j = 1, \ldots, p$ be such that $H \cap T_j \neq \emptyset$ and

$T_j \setminus H \neq \emptyset$ for $j = 1, \ldots, p$, and $T_i \cap T_j = \emptyset$ for $i, j = 1, \ldots, p, i \neq j$. The comb inequality is:

$$x(E(H)) + \sum_{j=1}^{p} x(E(T_j)) \leq |H| + \sum_{j=1}^{p} |T_j| - (3p+1)/2. \tag{20}$$

The set H is called the *handle* of the comb and the T_j are called *teeth*. When $|T_j| = 2$ for all teeth, the comb inequalities are known as *2-matching* inequalities, see [11, 19, 20].

In [6] it was shown that the comb inequalities, and even some more general facet-inducing inequalities known as *extended comb inequalities*, are a special case of $\{0, \frac{1}{2}\}$-cuts. The vertices in the handle are precisely those whose degree equations receive a CG multiplier of 1/2. By analogy, we extend the use of the term *handle* to mean the set of vertices whose degree equations have CG multiplier 1/2, in *any* $\{0, \frac{1}{2}\}$-cut for the STSP, and denote the handle by H. Letting $\mathcal{F}$ denote the family of SECs having CG multiplier 1/2, and using the (convenient although imprecise) notation $\lfloor \alpha x \rfloor$ for $\lfloor \alpha \rfloor x$, the $\{0, \frac{1}{2}\}$-cut has the form:

$$\left\lfloor \frac{2x(E(H)) + x(\delta(H)) + \sum_{S_i \in \mathcal{F}} x(E(S_i))}{2} \right\rfloor \leq \left\lfloor \frac{2|H| + \sum_{S_i \in \mathcal{F}} (|S_i| - 1)}{2} \right\rfloor, \tag{21}$$

noting that once the Constraints (17) and (18) having CG multiplier 1/2 are fixed, the CG multipliers for the non-negativity constraints are uniquely defined as the left hand side coefficients must be integer (see [5]).

Our main goal in this section is to show that any $\{0, \frac{1}{2}\}$-cut for the STSP is a natural (binary) split cut. To show this, we will need a few preliminary definitions and lemmas.

Definition 4. *A family of vertex sets $S_1, \ldots, S_k \subset V$ is said to be* nested *(or* laminar*) if, for all i, j, $S_i \cap S_j \neq \emptyset$ implies either $S_i \subseteq S_j$ or $S_j \subseteq S_i$.*

The following lemma was stated, but not proved, in [6]. Here we give an explicit proof as this result will be used extensively in the sequel.

Lemma 2. *Let $S_1, \ldots, S_k \subset V$ be the sets of vertices whose SECs have CG multiplier 1/2 in the derivation of a $\{0, \frac{1}{2}\}$-cut for the STSP. Then, we can assume without loss of generality that the sets $S_1, \ldots S_k \subset V$ form a nested family.*

Proof. We use a standard uncrossing argument. Suppose that two sets S_i and S_j both have CG multiplier 1/2, and $S_i \cap S_j$, $S_i \setminus S_j$ and $S_j \setminus S_i$ are all non-empty. The sum of the two SECs can be written as $x(E(S_i \setminus S_j)) + x(E(S_j \setminus S_i)) + 2x(E(S_i \cap S_j)) \leq |S_i \setminus S_j| + |S_j \setminus S_i| + 2|S_i \cap S_j| - 2$. Now consider what happens if we reduce the multipliers of both sets to zero and, instead, increase the multipliers of the two SECs on $S_i \setminus S_j$ and $S_j \setminus S_i$, together with the degree equations on the vertices in $S_i \cap S_j$, by one-half. The sum of these is $x(E(S_i \setminus S_j)) + x(E(S_j \setminus S_i)) + 2x(E(S_i \cap S_j)) + x(\delta(S_i \cap S_j)) \leq |S_i \setminus S_j| + |S_j \setminus S_i| + 2|S_i \cap S_j| - 2$. Thus, the new CG cut is at least as strong as the original $\{0, \frac{1}{2}\}$-cut. Moreover, if any CG multiplier is now equal to one, we can obtain a still stronger cut by setting it to zero instead. The resulting cut is a $\{0, \frac{1}{2}\}$-cut. □

Definition 5. *Let $S_1, \ldots, S_k \subset V$ be a nested family. A set S_i which is not contained in any other set S_j is said to be at* nesting level 1. *Recursively, for $r > 1$, a set S_i which is contained in a set S_j at nesting level $r-1$ but not contained in any other set at nesting level $\geq r$ is said to be at* nesting level r.

Given a nested family $\mathcal{F} := S_1, \ldots, S_k$, we partition $\mathcal{F}$ into two sub-families $\mathcal{O}$ and $\mathcal{E}$, where $\mathcal{O}$ contains all sets S_i which are at an *odd* nesting level, and $\mathcal{E}$ contains all sets S_i which are at an *even* nesting level. We next show that the SECs associated with sets at an odd nesting level contribute more to the right hand side of a $\{0, \frac{1}{2}\}$-cut than the SECs associated with sets at an even nesting level.

Lemma 3. *If $\mathcal{F}$ is a nested family and $\mathcal{O}$ and $\mathcal{E}$ are defined as above, then:*

$$\sum_{S_i \in \mathcal{O}} (|S_i| - 1) - \sum_{S_i \in \mathcal{E}} (|S_i| - 1) \geq 0. \tag{22}$$

Moreover, if $\mathcal{F}$ is to be used to derive a non-dominated $\{0, \frac{1}{2}\}$-cut for the STSP, then the left hand side of (22) must be odd.

Proof. Let S_i be a set at nesting level r, r odd, and let $\mathcal{E}_i$ denote the (possibly empty) family of sets at nesting level $r+1$ which are contained in S_i. By definition,

$$|S_i| \geq \sum_{S_j \in \mathcal{E}_i} |S_j|.$$

It follows immediately that, if $|\mathcal{E}_i| \geq 1$ then

$$|S_i| - 1 \geq \sum_{S_j \in \mathcal{E}_i} (|S_j| - 1). \tag{23}$$

Moreover, (23) holds trivially when $|\mathcal{E}_i| = 0$. We then obtain (22) by summing (23) over all sets in $\mathcal{O}$.

The fact that the left hand side of (22) must be odd follows from the fact that we want rounding down to occur on the right hand side of the $\{0, \frac{1}{2}\}$-cut, yet the right hand side of each degree equation is 2, i.e., even. $\square$

We are now ready to show the main result of this section:

Theorem 3. *Any $\{0, \frac{1}{2}\}$-cut for the STSP is a natural binary split cut.*

Proof. First, note that the $\{0, \frac{1}{2}\}$-cut (21), below denoted also by $\alpha x \leq \beta$, can be rewritten as:

$$x(E(H)) + \left\lfloor \frac{x(\delta(H)) + \sum_{S_i \in \mathcal{O}} x(E(S_i)) + \sum_{S_i \in \mathcal{E}} x(E(S_i))}{2} \right\rfloor$$
$$\leq |H| + \left\lfloor \frac{\sum_{S_i \in \mathcal{F}} (|S_i| - 1)}{2} \right\rfloor. \tag{24}$$

In particular, given a generic edge $e \in E$, let $h_e := 1$ if $e \in E(H)$ and $h_e := 0$ if $e \notin E(H)$. Let $\omega_e := |\{S_i \in \mathcal{O} : e \in E(S_i)\}|$ and $\varepsilon_e := |\{S_i \in \mathcal{E} : e \in E(S_i)\}|$ denote, respectively, the number of SECs at an odd and even nesting level containing edge e. Moreover, let S be the *smallest* set in $\mathcal{F}$ such that $e \in E(S)$. Note that $\omega_e = \varepsilon_e$ if S is at an *even* nesting level (and also if $e \notin E(S)$ for all $S \in \mathcal{F}$), and $\omega_e = \varepsilon_e + 1$ if S is at an *odd* nesting level. From (24), we get

$$\alpha_e = \begin{cases} h_e + \lfloor \frac{1}{2}(\omega_e + \varepsilon_e + 1) \rfloor = h_e + \omega_e & \text{if } e \in \delta(H), \\ h_e + \lfloor \frac{1}{2}(\omega_e + \varepsilon_e) \rfloor = h_e + \varepsilon_e & \text{if } e \notin \delta(H). \end{cases} \tag{25}$$

To derive $\alpha x \le \beta$ as a natural binary split cut we use the disjunction:

$$x(E(H)) \le |H| - \left\lceil \frac{\sum_{S_i \in \mathcal{O}}(|S_i| - 1) - \sum_{S_i \in \mathcal{E}}(|S_i| - 1)}{2} \right\rceil$$

$$\vee$$

$$-x(E(H)) \le -|H| + \left\lfloor \frac{\sum_{S_i \in \mathcal{O}}(|S_i| - 1) - \sum_{S_i \in \mathcal{E}}(|S_i| - 1)}{2} \right\rfloor.$$

Note that this disjunction is natural, because the quantity $x(E(H))$ is the left hand side of the SEC for H.

It remains to be shown that the $\{0, \frac{1}{2}\}$-cut is valid for the left system and the right system. By adding to the left term the SECs for the sets at odd nesting level, we obtain

$$x(E(H)) + \sum_{S_i \in \mathcal{O}} x(E(S_i)) \le |H| + \left\lfloor \frac{\sum_{S_i \in \mathcal{F}}(|S_i| - 1)}{2} \right\rfloor,$$

which dominates the $\{0, \frac{1}{2}\}$-cut as the coefficient of each edge e is $h_e + \omega_e \ge h_e + \varepsilon_e$. On the other hand, by adding to the right term the SECs for the sets at even nesting level, together with the degree equations for the vertices in H, we obtain

$$x(E(H)) + x(\delta(H)) + \sum_{S_i \in \mathcal{E}} x(E(S_i)) \le |H| + \left\lfloor \frac{\sum_{S_i \in \mathcal{F}}(|S_i| - 1)}{2} \right\rfloor$$

which also dominates the $\{0, \frac{1}{2}\}$-cut, as the coefficient of each edge e is $h_e + \varepsilon_e + 1 \ge h_e + \omega_e$ if $e \in \delta(H)$ and $h_e + \varepsilon_e$ if $e \notin \delta(H)$. □

Now let us consider the implications of these results for *separation*. It is well-known that, although there are an exponential number of SECs, the separation problem for them can be solved in polynomial time (see, e.g., Padberg & Grötschel [27]). Polynomial-time separation algorithms are also known for the 2-matching inequalities (Padberg & Rao [28]), for comb inequalities with a fixed number of teeth (Carr [7]) and for a certain generalization of comb inequalities when the edges whose variables are positive induce a planar graph (Letchford [21], see also Fleischer & Tardos [13]).

As mentioned in the previous section, the separation problem for $\{0, \frac{1}{2}\}$-cuts is strongly $\mathcal{NP}$-complete in general [5]. However, it is unknown whether this is true in the case of the STSP. Also, the complexity of separation is unknown even for comb inequalities. However, considering Theorems 2 and 3, we have:

Corollary 6. *Given a* fixed *handle H, one can separate in polynomial time over a class of inequalities which includes all* $\{0, \frac{1}{2}\}$*-cuts with that handle.*

Corollary 7. *Given a* fixed *handle H, one can separate in polynomial time over a class of inequalities which includes all extended comb inequalities with that handle.*

These separation results are new.

At first sight it might be thought that these corollaries rely on the use of the ellipsoid method (which, while polynomial, is inefficient in practice). However, by considering a *compact* LP formulation for the subtour polytope (see e.g. [31]), one can derive the same results without requiring the ellipsoid method. Moreover, the standard simplex method provides a practically useable, though theoretically non-polynomial, method for separation.

To use the separation result in a practical cutting plane algorithm for the STSP, a heuristic must be devised for finding suitable handles H with which to form a disjunction. We do not address this question in detail here, but we would like to note that:

- A set H is more likely to be a 'good' handle candidate if $x^*(\delta(H))$ is close to an odd integer.
- Good heuristics for finding a handle of a *comb* already exist, see for example Grötschel & Holland [18] or Padberg & Rinaldi [29].

To close this section, we mention an open question. Although we have proved that the natural, 'handle-type' disjunctions can be used to derive the $\{0, \frac{1}{2}\}$-cuts, we have not shown that the $\{0, \frac{1}{2}\}$-cuts are the *only* cuts which can be derived in this way. Are there any other (non-redundant and preferably facet-inducing) cuts which can be derived from 'handle-type' disjunctions?

5. Conclusion

We have explored the relationships between many different classes of cutting planes for ILPs and MILPs and given some new results about the complexity of the associated separation problems. These results settle the complexity status of all inequalities discussed. Moreover, we have given a new separation result for (a generalization of) the comb inequalities for the STSP, which is the latest in a series of incremental advances on comb separation [28, 30, 7, 13, 21].

Future research could include searching for special cases where separation of a given class of cuts can be performed in polynomial time, as done in [5, 6] for $\{0, \frac{1}{2}\}$-cuts. Moreover, the practical use of 'handle-type' disjunctions for the STSP should be investigated.

Acknowledgements. The work of the first author was partially supported by CNR and MURST, Italy. Thanks are due to the anonymous referee.

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