Analysis Qualifying Exam, Jan. 2010

Part I : Complex Analysis questions

1. Find the values of the following contour integrals:

   a) $\oint_c ze^{1/z}\,dz$; C is the square with vertices at $z = 1, i, -1, -i$, respectively, and oriented counterclock-wise.

   b) $\int_c \mathrm{Log}(z)\,dz$; $\mathrm{Log}(z)$ is the principle logarithmic function ($\theta = \mathrm{Arg}(z) \in (-\pi, \pi)$) and C is the contour given by the horizontal line connecting $z = i$ to $z = 1 + i$, and then the vertical line connecting $z = 1 + i$ to $z = 1$.

   c) $\int_c \frac{z}{(z-i)^2}\,dz$; C is the circle of radius 2 centered at $z = i$, oriented counterclock-wise.

2. Suppose $f(z)$ is entire and its image lies in the left-half plane $\{x < 0\}$, describe all such functions. Prove your assertion.

3. Let $f(z)$ be analytic on $D(1/2, 1/2)$ - the open disk with radius $1/2$ centered at $z = 1/2$. Suppose $f(z)$ has the property that $f(1/n) = 1/n^3$ for all integers $n \geq 2$. Is there only one such analytic function? If so, prove it. If not so, display at least two such functions.

4. Compute the integral
$$\int_0^{2\pi} \frac{d\theta}{1 + \sin^2\theta}.$$

5. Let $f : D(0,1) \longrightarrow D(0,1)$ be an analytic function such that $f(0) = 0$, where $D(0,1)$ is the open unit disk centered at the origin.

a) Prove that $|f(z)| \leq |z|$ for all $z \in D(0,1)$.

b) Prove that if there is a point $z_0 \in D(0,1)$ such that $|f(z_0)| = |z_0|$, then f must be a rotation.

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Part I Real & Functional Analysis

Your solutions to the following problems should contain enough details to convince the grader, for example, if you use Lebesgue Dominated Convergence Theorem, you should mention it and verify the conditions required by the theorem. If you use a result that has no name or you forget the name, write enough about it to show you know it.

(10 points) 1. Let E_k, $k = 1, 2, \cdots$, be measurable sets in $[0,1]$ with $m(E_k) = 1$ ("m" stands for Lebesgue measure). Prove that $m(\bigcap_{k=1}^{\infty} E_k) = 1$.

2. Let $\{f_k(x)\}_{k=1}^{\infty}$ be a sequence of nonnegative measurable functions defined on $[0,1]$. $\forall$ integers $k, n \geq 1$, define $E_k^n = \{x \in [0,1] \mid f_k(x) \geq \frac{1}{n}\}$.

(6 pts) (i) Let $B = \{x \in [0,1] \mid \lim_{k\to\infty} f_k(x) \neq 0$ or does not exist$\}$.

Express B in terms of E_k^n (involving "$\bigcup$" (union) and "$\bigcap$" (intersection)).

(6 pts) (ii) Show that if $f_k(x) \xrightarrow{\text{as } k\to\infty} 0$ a.e on $[0,1]$, then $\forall$ fixed $n \geq 1$, $\lim_{l\to\infty} m(\bigcup_{k=l}^{\infty} E_k^n) = 0$

(10pts) 3. Let $f(x)$ be a <u>nonnegative</u> measurable function defined on $[0,1]$. Suppose $\exists$ constant M such that

$$\int_0^1 f^k(x)\,dx \le M, \quad \forall\ k \ge 1.$$

Prove that $m(\{x \in [0,1] \mid f(x) > 1\}) = 0$

4. Suppose $f \in L^1(0,1)$ (i.e. Lebesgue integrable on $(0,1)$).

Define $F(x) = \int_x^1 \frac{f(t)}{t}\,dt, \quad \forall\ x \in (0,1]$.

Show that

(8pts) (i) $\lim_{x \to 0^+} x F(x) = 0$;

(8pts) (ii) $\int_0^1 F(x)\,dx = \int_0^1 f(t)\,dt$;

(8pts) (iii) $F(x)$ is differentiable a.e on $(0,1)$, find $F'(x)$.

(10pts) 5. Suppose $f, f_k \in L^2(0,1)$, $k = 1, 2, \cdots$, and assume

$$\|f_k - f\|_{L^2(0,1)} \to 0 \quad \text{as} \quad k \to \infty$$

Show that $\forall\ g \in L^2(0,1)$, we have

$$\int_0^1 f_k g\,dx \to \int_0^1 f g\,dx \quad \text{as } k \to \infty.$$

"one"

(10pts) 6. Suppose $f \in AC[a,b]$, $\forall\ [a,b] \subset \mathbb{R}$ ("$AC[a,b]$" stands for "absolute continuous on $[a,b]$") and $f' \in L^1(\mathbb{R})$.

Show that $\lim_{x \to \infty} f(x)$ exists as a finite number.

(10 pts) 7. Let $\{f_k(x)\}_{k=1}^{\infty}$ be a sequence of differentiable functions defined on $[0,1]$. Suppose $f_k(0)=0$ and

$$|f_k'(x)| \le \text{const } M, \quad \forall x \in [0,1], \ \forall k \geq 1.$$

Prove that after passing to a subsequence, f_k converges uniformly on $[0,1]$ as $k \to \infty$.

(10 pts) 8. Let $H = L^2(-1,1)$ be equipped with the standard inner product. Let $M = \{ f \in H \mid f \text{ is an even function on } (-1,1)\}$

Find the orthogonal complement $M^{\perp}$ of M.

$$(M^{\perp} = \{ g \in H \mid (f,g)_H = 0, \quad \forall f \in M\})$$

(10 pts) 9. Suppose $K(x,y) \in L^2((0,1)\times(0,1))$. Consider the integral equation

$$(\star) \cdots \quad u(x) = 2010 + \lambda \int_0^1 K(x,y)u(y)\,dy, \quad x \in (0,1),$$

where λ is a real constant. Prove that $\exists$ small $\varepsilon > 0$ such that whenever $|\lambda| < \varepsilon$, $(\star)$ has one and only one solution u in the space $L^2(0,1)$.

Hint: Define a mapping $T: L^2(0,1) \to L^2(0,1)$ by

Tu = right hand of $(\star)$; think of a solution of $(\star)$ as a fixed point

(12 pts) 10. Let $p \in (1,\infty)$ and $\{f_k\} \subset L^p(0,1)$ with

$$\|f_k\|_{L^p(0,1)} \le \text{Const } M, \quad \forall k \geq 1.$$

Suppose $f_k \to f$ in measure on $(0,1)$ as $k \to \infty$. Prove that

$$f_k \to f \text{ in } L^1(0,1) \text{ as } k \to \infty.$$