Part 1
Complex Analysis

(1) a)Find the radius of the largest disc, D centered at the origin such that

$$f(z) = \frac{1}{2 - \sin z}$$

is holomorphic on D.
b) Find the first 4 non-zero terms of the power series expansion about the origin of $f(z)$ on D.

(2) Let $Z = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{Z}, n > 1\}$. Let f be a bounded holomorphic function on $(|z| < 1) \setminus Z$. Show that f extends holomorphically to $|z| < 1$.

(3) Use the residue calculus to evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx.$$

(4) Prove that the equation

$$ze^{2-z} = 1$$

has exactly one solution in $|z| < 1$ and that solution is real.

Part 2
Real and Functional Analysis

(1) Give an example of a function which is continuous on $[0,1]$ but not of bounded variation. Prove that the function satisfies these properties. Your answer should include a definition of bounded variation

(2) a)Let f,g be measurable functions on $\mathbb{R}$. Show that $h(x,y)=f(x)g(y)$ is measurable on $\mathbb{R}^2$.
b) If E_1,E_2 are measurable subsets of $\mathbb{R}$, show that $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$.

(3) Each of the integrals

$$\int_1^\infty \left(\int_0^1 (e^{-xy} - 2e^{-2xy})\, dy \right)\, dx)$$

and

$$\int_0^1 \left(\int_1^\infty (e^{-xy} - 2e^{-2xy})\, dx \right)\, dy$$

exists but are not equal to each other. What can you conclude from this?

(4) Suppose that $p \geq 1$ and that f_n is a sequence of functions which converges to a function f in $L^p[0,1]$. show that a subsequence f_{n_k} converges to f a.e.
Hint: Show that $f_n \to f$ in measure.

(5) If $f \in L^2[0,2\pi]$ show that

$$\lim_{k\to\infty} \int_0^{2\pi} f(t) \sin kt\, dt = 0.$$

(6) Suppose that $K(x,y)$ is continuous on $[0,1]\times[0,1]$ and is bounded in absolute value by $M>0$. Define a mapping

$$A : C[0,1] \to C[0,1]$$

as follows

$$Af(x) = \int_0^1 K(x,y) f(y)\, dy.$$

Show that the equation

$$f = \lambda A f$$

has a unique solution if $|\lambda| < M^{-1}$.