ANALYSIS
QUALIFYING EXAM
SPRING 2012

Part A. *Choose any 3 of the 4 problems and solve.* *Justify each step! May quote named-theorems.*

Q-A1. Give an example of a function that is
(i) Lebesgue integrable but not Riemann integrable. Explain why.
(ii) Riemann integrable but not Lebesgue integrable. Explain why.

Q-A2. For this problem, consider just Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ together with the Lebesgue measure.
(i) State Fatou's lemma (no proof required).
(ii) State the Dominated Convergence Theorem (no proof required).
(iii) Give an example where $f_n(x) \to 0$ almost everywhere, but $\int f_n(x)dx = 1$.

Q-A3. Find the exact value of the limit

$$\lim_{n\to\infty} \int_0^\infty \left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \right) e^{-2x} dx.$$

Q-A4. Suppose $f(x), xf(x) \in L_2(\mathbb{R})$. Prove that $f \in L_1(\mathbb{R})$.

Part B. *Choose any 3 of the 5 problems and solve.* *Justify each step!* May quote named-theorems.

Q-B1. Is the metric space $C[0,1]$ of continuous functions under $d(f,g) = \int_0^1 |f-g|dx$ complete? Justify.

Q-B2. Given the integral equation

$$(*) \qquad f(t) = \frac{1}{2}\sin(f(t)) + \lambda \int_0^1 (t-s)^2 f(s)ds.$$

Show that there exists $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, the equation (*) has a unique solution $f \in C[0,1]$.

Q-B3. Consider the subset K of the Hilbert space $L_2[0,1]$ consisting of the functions $u_n(x) = (1 + 2^{-n})e^{2\pi i n x}$, where $n = 1, 2, 3, \ldots$.
(i) Show K does not contain an element of smallest norm.
(ii) Show that K is closed.

Q-B4. If $f \in L_1[-\pi, \pi]$, prove the fourier coefficients decay to 0, i.e. $\lim_{|n|\to\infty} \widehat{f}(n) = 0$.

Q-B5. Apply Fourier series to prove that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. *Hint. Parseval, $f(x) = x$ of period 1.*

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

COMPLEX ANALYSIS part of Analysis Qualifier Exam, January 13, 2012. All your answers must be justified whether or not the problem asks for a proof.

1.Let γ be the counterclockwise circle with center 0 and radius 3. Find the value of

$$\int_\gamma \frac{e^z - 1}{z^3(z-2)^2}\, dz.$$

2.Let $f(x+iy) = (sin(x)e^y + xy) + i(cos(x)e^y + \frac{x^2}{2} + \frac{y^2}{2})$. Is f a holomorphic function? Justify your answer.

3.Let D be a bounded domain in $\mathbb{C}$ with a C^1 boundary. Let $f(z)$ be a holomorphic function defined on an open set containing the closure of D. For any $z_0 \in D$, prove that $f(z)$ can be expressed as a convergent power series with center z_0 and positive radius of convergence.(Prove this result, do not simply quote a theorem that says that this is true.) Is this power series unique (explain)?

4.Suppose $f(z)$ is a holomorphic function on the punctured disk $D^* = \{z \mid 0 < |z-p| < R\}$ centered at the point $p \in \mathbb{C}$ of radius $R > 0$. Assume that $|f(z)|$ is bounded on D^*. Prove that $f(z)$ can be extended to also be holomorphic at p.

5.Suppose $f(z)$ is holomorphic on a disk $\Delta(p, R)$ of center p and radius $R > 0$ and that $f(p) = o$, $f(z) \neq 0$ for all $z \neq p$, $f'(p) = 0$, and $f''(p) = 0$. For small $\delta > 0$ and $w \in \mathbb{C}$ with $|w| < \delta$, how many solutions $z \in \Delta(p, R)$ are there for $f(z) = w$? Justify your answer precisely.