Real Analysis Written Examination

August 2015

1. Let f be an L^2 integrable function on $[0,1]$ and $F(x) = \int_0^x f(t)\,dt$.

(a). Prove that f is L^1 integrable on $[0,1]$ and therefore $F(x)$ is well-defined on $[0,1]$.

(b). Prove that

$$\sup_{0 \le x \le 1} |F(x)| \le \sqrt{\int_0^1 f^2(t)\,dt}$$

(c). Prove that F is uniformly continous on $[0,1]$.

(d). Find the total variation of the function F on $[0,1]$.

2. Consider the equation $x = \ln(x+N)$ for $x > -N$.

(a). Show that the equation has exactly two solutions for each $N > 1$.

(b). Let $x_1(N) < x_2(N)$ be the two solutions for each $N > 1$. Show that

$$\lim_{N\to\infty} x_1(N) = -\infty \quad \textit{and} \quad \lim_{N\to\infty} x_2(N) = \infty$$

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3. Let f be a continuous real valued function with two variables and at most polynomial growth, that is, there exists a constant N such that $|f(x,t)| \le N(1+x^2+t^2)^N$ for all $(x,t) \in R^2$. Consider the function g defined by

$$g(x) = \int_0^\infty f(x,t)e^{-t}\,dt$$

Prove that g is a continuous function on the real line R.

4. Let $E \subset R^n$ be a Lebesgue measurable subset of R^n.

(a). Prove that there is a set E_1 with zero measure $m(E_1) = 0$ and a G_δ set G such that $E = G - E_1$.

(b). Prove that there is a set E_2 with $m(E_2) = 0$ and an F_σ set F such that $E = F \cup E_2$.

5. Let $\mathbf{H}$ be an infinite dimensional Hilbert space and let $\mathbf{S} \subset \mathbf{H}$ be a subspace of $\mathbf{H}$.

(a). Show that $\mathbf{S}^\perp$, the orthogonal complement of $\mathbf{S}$ in $\mathbf{H}$, is a closed subspace of $\mathbf{H}$.

(b). Show that $\mathbf{H} = \mathbf{S} \bigoplus \mathbf{S}^\perp$ if and only if $\mathbf{S}$ is a closed subspace of $\mathbf{H}$.

(c). Either show that there is an infinite dimensional subspace $\mathbf{S}$ in $\mathbf{H}$ that is not a closed subspace or prove the contrary.

6. Let $\{ f_n \}_{n=1}^\infty$ be a sequence of L^1 integrable functions on $[0,1]$ and f_n converges to a bounded function f almost everywhere on $[0,1]$. Disprove by example or prove the following statements:

(a). $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$

(b). If, furthermore, $|f_n(x)| \le |f(x)|$ for almost all $x \in [0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$$

COMPLEX ANALYSIS part of Analysis Qualifier Exam, August, 2015. All your answers must be justified whether or not the problem asks for a proof.

1.Let $f(z)$ be a holomorphic function (meaning that its complex derivative $f'(z)$ exists) on a domain $D \subset \mathbb{C}$ and that its real and imaginary parts are C^∞. State and prove the Cauchy-Riemann equations for f.

2.Let γ be the counterclockwise circle with center 0 and radius 2. Find the value of the line integral

$$\int_\gamma \frac{e^{3z} - z}{z^3 - iz^2}\, dz.$$

3.State and prove (briefly) the maximum modulus theorem for holomorphic functions.

4.Find the Laurent series of

$$f(z) = \frac{1}{z^2 - 3z}$$

on a punctured disc $\Delta^*(3; R) = \{z \in \mathbb{C} \mid 0 < |z-3| < R\}$ and determine the largest R for which the series converges.

5.The functions $cos(z)$ and $sin(z)$ are defined and holomorphic for all $z \in \mathbb{C}$. Explain carefully why $\cos^2(z) + \sin^2(z) = 1$ for all $z \in \mathbb{C}$.

6.Let Δ be an open disc with center p and positive radius. Suppose f is holomorphic on the punctured disc $\Delta \setminus \{p\}$.
a)Define "f has an essential singularity at p".
b)Give an example of such an f.
c)Give a condition on or property of f (distinct from the definition) that is equivalent to f having an essential singularity at p.