Real Analysis Qualifying Exam – January 09, 2015

Problem 1. Let (X, d) be a metric space. Suppose that K is a compact subset of X and F is a closed subset of X. Assume $K \cap F = \emptyset$. Show that there exists a positive number $\delta > 0$, such that

$$d(x, y) \geq \delta > 0, \ \ \forall\, x \in K, y \in F.$$

Problem 2. Given $2 \leq p < \infty$. Show that for any real-valued functions $f, g \in L^p(\mathbb{R})$, it holds that

$$2\left(\left\|\frac{f}{2}\right\|_{L^p}^p + \left\|\frac{g}{2}\right\|_{L^p}^p\right) \leq \left\|\frac{f-g}{2}\right\|_{L^p}^p + \left\|\frac{f+g}{2}\right\|_{L^p}^p \leq \frac{1}{2}\left(\|f\|_{L^p}^p + \|g\|_{L^p}^p\right).$$

Problem 3. Suppose that $0 < \theta < 1$, $E \subset \mathbb{R}^n$ and $0 < |E| < \infty$, where $|\cdot|$ denotes the Lebesgue measure of E. Show that there is a cube $Q \subset \mathbb{R}^n$ such that

$$\theta|Q| < |E \cap Q|.$$

Problem 4. Suppose that $E \subset \mathbb{R}^n$ with $|E| < \infty$. Let f be a non-negative measurable function on E. Show that the following are equivalent:

(A) $f \in L^p(E)$;

(B) $\sum_{k=-\infty}^{\infty} 2^{kp}\big|\{x \in E : f(x) > 2^k\}\big| < \infty$.

Problem 5. Let H be a separable infinite dimensional Hilbert space and let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for H. Show that if $\{v_n\}_{n\in\mathbb{N}}$ is an orthonormal set in H such that $\sum_n \|u_n - v_n\|^2 < \infty$ then it is also an orthonormal basis for H.

Problem 6. Let $(X, \mathbb{M}, \mu)$ be a measure space and let $f, f_n \in L^p$, where $1 \leq p < \infty$. Prove that if $f_n \to f$ a.e., then $\|f_n - f\|_{L^p} \to 0$ if and only if $\|f_n\|_{L^p} \to \|f\|_{L^p}$.

COMPLEX ANALYSIS
QUALIFYING EXAM
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INSTRUCTIONS. *Choose any but* **EXACTLY** 5 *of the* 6 *problems and solve.* Justify each step completely*! Unreadable answers receive no credits. You may quote named-theorems. The letter z stands for a complex number throughout.*

Q-CA1.
(a) There is a 1-1 conformal map from $|z| < 1$ **onto** $\mathbb{C}$. Prove or disprove this statement.
(b) If a polynomial $P(z)$ satisfies $|P(z)| \leq 1$ for all $|z| = 1$, show that all its coefficients are bounded by 1 in magnitude.

Q-CA2.
(a) Find the number of roots of $z^6 - 5z^4 + 3z^2 - 1 = 0$ in the unit disk $|z| \leq 1$.
(b) Let f be satisfying all conditions:
(i) f analytic in $1 \leq |z| \leq 2$;
(ii) $|f(z)| \leq 1$ for $|z| = 1$;
(iii) $|f(z)| \leq 4$ for $|z| = 2$.
Prove that $|f(z)| \leq |z|^2$ throughout the annulus $1 \leq |z| \leq 2$.

Q-CA3.
(a) Provide an example of a complex function $h(z)$ with a non-isolated singularity.
(b) Suppose f and g are analytic and nonzero in $|z| < 1$ and $\frac{f'(1/n)}{f(1/n)} = \frac{g'(1/n)}{g(1/n)}$ for all $n \in \mathbb{N}$. Prove that $\frac{f}{g}$ is identically constant throughout the open disk $|z| < 1$.

Q-CA4. Use methods from complex analysis to evaluate the integral

$$\int_0^\infty \frac{\sin x}{x(1+x^2)}\, dx.$$

Q-CA5. Find a conformal map f sending the half-disk $S = \{z \in \mathbb{C} : |z| < 1;\ \Im(z) > 0\}$ onto the upper half-plane $\mathbb{H}$. *Here,* $\Im(z)$ = *imaginary part of* z.

Q-CA6. Suppose $f(z) = u + iv$ is a 1-1 conformal map from a domain R onto S. Prove that f preserves the Laplace equation from one domain to the other, i.e.
if some $T = T(x, y)$ satisfies $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ in R then $\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0$ in S.