REAL & FUNCTIONAL ANALYSIS QUALIFYING EXAM – JANUARY 2016

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Solve $\boxed{\textbf{At Least}}$ 5 Problems

Problem 1. Let $f(x), f_1(x), f_2(x), \ldots$ and $g(x), g_1(x), g_2(x), \ldots$ be almost everywhere finite valued measurable functions defined on a measurable set $E \subset \mathbb{R}^d$ with $m(E) < \infty$. Suppose that $f_k \to f$ and $g_k \to g$ in measure on E. Show that $f_k \cdot g_k \to f \cdot g$ in measure on E.

Problem 2. Let $f(x)$ is a non-negative integrable function defined on a measurable set $E \subset \mathbb{R}^n$. Let

$$E_k = \{x \in E : f(x) \geq 2^k\},\ k = 1, 2, \ldots.$$

Show that

$$\sum_{k=1}^{\infty} 2^k m(E_k) < \infty.$$

Problem 3. Let $f \in L([0,1])$. Show that

$$\lim_{n\to\infty} \int_0^1 n \ln\left(1 + \frac{|f(x)|^2}{n^2}\right) dx = 0.$$

Problem 4. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of monotone increasing functions defined on $[a, b]$. Suppose that the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent on $[a, b]$. Show that

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x), \quad \text{a.e. } x \in [a, b].$$

Problem 5. Is $L^{\infty}((0,1))$ separable? Justify your answer.

Problem 6. Let $\{x_n\}$ be a sequence of pairwise orthogonal vectors in a Hilbert space H. Show that the following are equivalent:

(a) $\sum_{n=1}^{\infty} x_n$ converges in the norm topology of H.
(b) $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$.
(c) $\sum_{n=1}^{\infty} < x_n, y >$ converges for every $y \in H$.

Problem 7. Let H be a Hilbert space. Show that if $T : H \to H$ is symmetric, i.e. $\langle x, Ty\rangle = \langle Tx, y\rangle$ for all $x, y \in H$, then T is linear and continuous.

1. Consider the function $f(z) = z\mathrm{Re}(z) + \overline{z}\mathrm{Im}(z) + \overline{z}$.
   (a) Determine all points where f is differentiable.
   (b) Determine all points where f is holomorphic.

2. Does there exist a function f holomorphic on some open disk $\{z \in \mathbb{C} : |z| < \epsilon\}$ with $\epsilon > 0$ such that $f(1/n) = f(-1/n) = 1/n^3$ for all integers $n > 1/\epsilon$? If yes, exhibit such a function. If no, prove that no such function exists.

3. Evaluate the following integrals.

(a) $\displaystyle\int_C z^n e^{1/z}\,dz$, where C is the unit circle oriented counterclockwise

(b) $\displaystyle\int_0^\infty \frac{x^2}{x^4+6x^2+25}\,dx$

4. A meromorphic function f is said to be *doubly periodic* if there exist two non-zero complex numbers ω_1 and ω_2 such that $\omega_2/\omega_1 \notin \mathbb{R}$ and $f(z + n_1\omega_1 + n_2\omega_2) = f(z)$ for all $z \in \mathbb{C}$ and all $n_1, n_2 \in \mathbb{Z}$. (If f has a pole at z, then the equation should be interpreted as saying that f also has a pole at each $z + n_1\omega_1 + n_2\omega_2$ for $n_1, n_2 \in \mathbb{Z}$.) Let $\mathcal{P}$ denote the parallelogram consisting of the four line segments from 0 to ω_1, ω_1 to $\omega_1 + \omega_2$, $\omega_1 + \omega_2$ to ω_2 and ω_2 to 0. (Note that $\mathcal{P}$ is indeed a parallelogram because of the condition that $\omega_2/\omega_1 \notin \mathbb{R}$.)

   (a) Suppose that f is an entire doubly periodic function. Prove that f is constant.

   (b) Let f be a doubly periodic meromorphic function and assume that f has no zeroes or poles on $\mathcal{P}$. Prove that the number of zeroes inside $\mathcal{P}$ (counted with multiplicity) is equal to the number of poles inside $\mathcal{P}$ (counted with multiplicity).