ANALYSIS
QUALIFYING EXAM
FALL 2011

Part A. *Choose any 3 of the 6 problems and solve.* 𝔍𝔲𝔰𝔱𝔦𝔣𝔶 𝔢𝔞𝔠𝔥 𝔰𝔱𝔢𝔭*! May quote named-theorems. Throughout,* $\boldsymbol{m}$ *is the Lebesgue measure on* $\mathbb{R}$ *and* $f' = \frac{d}{dx} f$.

Q-A1. Suppose $f(x), xf(x) \in L_2(\mathbb{R})$. Prove that $f \in L_1(\mathbb{R})$.

Q-A2. Let $f(x)$ be non-decreasing on $[0,1]$ (you may assume - by a theorem - that f is differentiable almost everywhere).
(a) Show that f' is measurable.
(b) Show that f' is integrable and $\int_0^1 f'(x)dx \leq f(1) - f(0)$.

Q-A3. Let $(X, \mathfrak{M}, \mu)$ be a real measure space, where $\mathfrak{M}$ is a σ-algebra. Suppose $f, g : X \to \mathbb{R}$ are non-negative measurable functions. Denote the characteristic function of a set A by χ_A. Define, for each $E \in \mathfrak{M}$, the set map $\nu(E) = \int_X g\chi_E d\mu$.
(a) Show that ν is a measure on $\mathfrak{M}$;
(b) Show $\int_X f d\nu = \int_X fg d\mu$.

Q-A4. Suppose $f \in L_p[0,1]$ for some $1 \leq p \leq \infty$, and define

$$h(t) := \boldsymbol{m}\{x \in [0,1] : |f(x)| > t\}, \qquad \text{for each } 0 \leq t < \infty.$$

Show that $\int_0^\infty h(t)dt < \infty$.

Q-A5. A measure ν_1 is *absolutely continuous* with respect to ν_2, denoted $\nu_1 \ll \nu_2$, if every $\nu_2(A) = 0$ implies $\nu_1(A) = 0$.
Consider the *coin-flipping* (probability) measure ν on $\{0,1\}$ where both 0 and 1 are equally probable (prob. $\frac{1}{2}$ each). The product topological space $X = \{0,1\}^\infty$ is equipped with the product probability measure P, arising from ν. Let $\Phi : X \to \mathbb{R}$ be the map given by $\Phi(\boldsymbol{s}) = \sum_{n=0}^{\infty} s_k \left(\frac{1}{3}\right)^k$, where $\boldsymbol{s} = (s_0, s_1, s_2, \dots) \in X$ is treated as a binary sequence. Define the measure $\boldsymbol{\mu}$ on $\mathbb{R}$ by

$$\boldsymbol{\mu}(E) = P(\Phi^{-1}(E)), \qquad \forall E \subset \mathbb{R}.$$

Is $\boldsymbol{\mu} \ll \boldsymbol{m}$? Justify.

Q-A6.
(a) If $x_n \in \mathbb{R}, x_n \to 0, f \in L_2(\mathbb{R})$ and $f_n(x) := f(x + x_n)$, show $f_n \to f$ in $L_2(\mathbb{R})$.
(b) If $E \subset \mathbb{R}$ is measurable with $\boldsymbol{m}(E) > 0$, show the set $E - E := \{a - b : a, b \in E\}$ contains an open neighborhood of the origin.

Part B. *Choose any 4 of the 7 problems and solve.* Justify each step! May quote named-theorems. The identity map is denoted by I, i.e., $I(x) = x$.

Q-B1. Suppose V is the vector space of continuously differentiable functions on $[0,1]$. Treat V as a subspace of $C[0,1]$, under the *maximum norm* $\|\cdot\|_{max}$. Define the map $D: V \to C[0,1]$ by $D(f) = \frac{d}{dx}f$. Is D a bounded operator, i.e. $D \in \mathcal{B}(V, C[0,1])$? Justify.

Q-B2. Equip the linear space of continuous functions $C[0,1]$ with the L_1-norm, i.e., $\|f\| = \int |f|$. Is it a Banach space? Why, or why not? Explain.

Q-B3. Let H be a real Hilbert space. Assume $z_n, x_n, y_n \in \{x \in H : \|x\| \leq 1\}$ and $\langle x_n, y_n \rangle \to 1$ as $n \to \infty$. Given $T \in \mathcal{B}(H)$ is self-adjoint ($T^* = T$) with

$$1 = \|T\| = \lim_{n\to\infty} \|T(z_n)\|.$$

(a) Show that $x_n - y_n \to 0$;
(b) Show that $T(T(z_n)) - z_n \to 0$.

Q-B4. Suppose H is a Hilbert space and that $T: H \to H$ is a bounded linear operator (i.e., $T \in \mathcal{B}(H)$). If $\|T\| < 1$, prove $(I - T)$ is invertible (i.e. $(I-T)^{-1} \in \mathcal{B}(H)$ exists).

Q-B5. With ℓ_2 = square summable sequences, let $A := \{x \in \ell_2 : \sum_{n=1}^{\infty} n|x_n|^2 \leq 1\}$. Prove that A is compact in the ℓ_2-norm topology (recall this norm $\|x\| = \sqrt{\sum_n |x_n|^2}$).

Q-B6. Let $f_n : [0,1] \to \mathbb{R}$ be differentiable satisfying both

$$\int_0^1 f_n(x)dx = 0, \qquad \text{and} \qquad |f_n'(x)| \leq \frac{1}{\sqrt{x}}; \qquad 0 < x \leq 1.$$

Prove that $\{f_n\}_n$ has a uniformly convergent subsequence on $[0,1]$.

Q-B7. With $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$, show that there is a measurable function $f : \mathbb{S}^1 \to \mathbb{S}^1$ whose Fourier coefficients satisfy

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{2\pi i\theta}) e^{-2\pi in\theta} d\theta \neq 0, \qquad \forall n \in \mathbb{Z}.$$

Here $i = \sqrt{-1}$. *Hint: Baire Catagory Thm.*