Part 1
Complex Analysis

(1) Suppose that $f(z)$ is a holomorphic (complex analytic) function of

$$z \in D^* = \{z \in \mathbb{C} | \, |z| < 1 \setminus \{0\}\}$$

and that there exists a positive number M such that

$$|f(z)| \leq M$$

for all $z \in D^*$. Show that f extends holomorphically to $D = D^* \cup \{0\}$.

(2) Describe the set in $\mathbb{C}$ where the infinite series

$$\sum_{n=1}^{\infty} \frac{e^{nz}}{n}$$

converges.

(3) Use the residue calculus to evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$$

(4) Show that if the real part of $z > 0$ then

$$\int_0^{\infty} e^{-zt^2} \, dt = \frac{\sqrt{\pi}}{2\sqrt{z}}$$

(5) Suppose that $f(z)$ is holomorphic on $|z| < 1$ and satisfies

$$|f(z)| < 1$$

and $f(1/2) = 0$. Use the Schwarz Lemma to show that

$$|f(4/5)| \leq 1/2.$$

Part 2
Real and Functional Analysis

(1) Suppose that f is a function defined on $\mathbb{R}$ and define

$$\omega(\delta) = \sup\{|f(x) - f(y)| : |x - y| < \delta\}.$$

a. Show that $\omega(\delta)$ is decreasing in δ.
b. Show that f is uniformly continuous on $\mathbb{R}$ if and only if

$$\lim_{\delta \to 0} \omega(\delta) = 0.$$

(2) Let $f(x, y)$ be defined for $0 \leq x, y \leq 1$ and satisfy the following conditions: for each $x \in [0, 1]$, $f(x, y)$ is an integrable function of y and $\dfrac{\partial f(x,y)}{\partial x}$ is a bounded function of (x, y). Show that $\dfrac{\partial f(x,y)}{\partial x}$ is a measurable function of y foor each x and

$$\frac{d}{dx}\int_0^1 f(x, y)\, dy = \int_0^1 \frac{\partial f(x,y)}{\partial x}\, dy.$$

(3) a. Define what it means for a sequence of functions f_n defined on $D \subset \mathbb{R}$ to converge to a function f in L^P for $p \geq 1$.
b. Define what it means for a sequence of functions f_n defined on $D \subset \mathbb{R}$ to converge to a function f in measure.
c. If D has finite Lebesgue measure, show that a sequence which converges in L^p for some $p \geq 1$, converges in measure.
d. Show that the condition of finite measure in part c. is necessary.

(4) a. Define what it means for a function f defined on an interval $[a, b] \subset \mathbb{R}$ to be of bounded variation.
b. Define what it means for a function f defined on an interval $[a, b] \subset \mathbb{R}$ to be absolutely continuous.
c. Show that if f is absolutely continuous then it is of bounded variation.
d. If V denotes the set of absolutely continuous functions on $[a, b]$ with $f(a) = 0$ show that $||f|| = V_a^b$, the variation of f on the interval defines a norm on V.
e. Show that the set of absolutely continuous elements of V form a closed subspace.

(5) Show that the set $C[0, 1]$ with norm

$$||f|| = \max|f(x)|.$$

is a Banach space.

(6) For $0 < \alpha \leq 1$, consider the set, $H_\alpha[0,1]$, of all real valued functions on the interval $[0,1]$ with the property that there exists a constant $C > 0$ with

$$|f(x) - f(y)| \leq C|x-y|^\alpha.$$

On $H_\alpha[0,1]$ define a norm

$$||f||_\alpha = \max|f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

Show that

$$\{f \in C[0,1] \,|\, ||f||_a \leq 1\}$$

is a compact subset of the space $C[0,1]$.

(7) a. If X is a Banach space and X^* denotes its dual space show that if $x_1 \neq x_2 \in X$ there is an element $y \in X^*$ such that

$$y(x_1) \neq y(x_2).$$

b. Show that the map

$$i : X \to X^{**}$$

given by $i(x)(y) = y(x)$ for $x \in X$ and $y \in X^*$ is one to one.