Analysis Qualifying Exam – August 2013

Real Analysis

Q1. Let $E \subset \mathbb{R}^d$ be a measurable set.

(a) Suppose that $|E| < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions on E, and suppose that f_n is finite almost everywhere for each n. Show that if $f_n \to f$ almost everywhere on E, then $f_n \to f$ in measure.

(b) Show by example that part (a) can fail if $|E| = \infty$.

Q2. Let $f \in L^2(0,\infty)$ be given.

(a) Prove that

$$\left(\int_0^x f(t)dt\right)^2 \leq 2\sqrt{x}\int_0^x \sqrt{t}f(t)^2dt, \qquad \forall\ x \in (0,\infty).$$

(b) Use part (a) to prove that $\|F\|_{L^2(0,\infty)} \leq 2\|f\|_{L^2(0,\infty)}$, where

$$F(x) = \frac{1}{x}\int_0^x f(t)dt.$$

(Note: You can use part (a) to prove part (b) even you cannot prove part (a))

Q3. Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be σ-finite measure spaces, and fix $1 < p < \infty$. Show that if f is a measurable function on $X \times Y$, then

$$\left(\int_Y\left(\int_X |f(x,y)|d\mu(x)\right)^p d\nu(y)\right)^{1/p} \leq \int_X\left(\int_Y |f(x,y)|^p d\nu(y)\right)^{1/p} d\mu(x)$$

assuming that the right hand side is finite.

Q4. Let X be a real normed space. Prove that the norm is induced by an inner product if and only if the norm satisfies the parallelogram law, i.e.

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \qquad \forall\ x,y \in X.$$

Q5. Let $\{x_n\}$ be a sequence of pairwise orthogonal vectors in a Hilbert space H. Show that the following are equivalent:

(a) $\sum_{n=1}^{\infty} x_n$ converges in the norm topology of H.

(b) $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$.

(c) $\sum_{n=1}^{\infty} < x_n, y >$ converges for every $y \in H$.

COMPLEX ANALYSIS part of Analysis Qualifier Exam, August, 2013. All your answers must be justified whether or not the problem asks for a proof.

1.Let γ be the circle in the complex plane with center 0 and radius 5 traversed in a counterclockwise manner. Find the value of

$$\int_\gamma \frac{e^{2z}}{z^3 - 6iz^2 - 9z}\, dz.$$

2.Let

$$f(z) = \frac{z}{4 - e^{2z}}.$$

a)Find the radius of convergence of the power series expression for $f(z)$ centered at the point 0.
b)Find the first four terms of this power series, i.e. the zeroth through the third order terms.

3.State and prove the maximum modulus theorem for holomorphic functions.

4.State and prove Liouville's theorem for holomorphic functions.

5.Let $f(z)$ be holomorphic on an open set $U \subset \mathbb{C}$ and let $z_0 \in \mathbb{C}$.
a)State a condition that implies that there is an open set $V \subset U$ containing z_0 such that $f(z)$ restricted to V has an inverse function.
b)If $f(z)$ has such a local inverse $f^{-1}(w)$, write down an integral formula that gives $f^{-1}(w)$ in terms of $f(z)$.

6.Let $f(z)$ be holomorphic on an open connected set $U \subset \mathbb{C}$.
a)Give an example of U and f such that f does not have a holomorphic anti-derivative on U.
b)Under what condition on U do all f have holomorphic antiderivatives? Give a general formula for an anti-derivative in this case and explain why your formula works.