Analysis Qualifying Exam, August, 2014
the complex part

(40pts) 1. Short questions (show your work)

(i) Show that the real part of an analytic function must be harmonic

(ii) Is it possible to find an analytic function that maps the entire complex plane into and onto the open unit disk $D_1(0)$? ($D_1(0)$ is centered at 0.)

(iii) Fix α in the open disk $D_1(0)$. Find a conformal mapping $\varphi_\alpha(z)$ that carries $D_1(0)$ onto $D_1(0)$ such that $\varphi_\alpha(\alpha)=0$.

(iv) Let $f(z)$ be analytic on $D_1(0)$ such that

- $f(0)=0$
- $\frac{1}{n} \notin f(D_1(0))$, for $n=1,2,3,\cdots$.

What can you say about $f(z)$?

(25pts) 2. Compute

(i) $$\frac{1}{2\pi i}\int_\Gamma \frac{e^z-e^{-z}}{z^4}\,dz,$$ where $\Gamma=\{z\in\mathbb{C} \mid |z|=1\}$, oriented counter-clockwise.

(ii) $$\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx,$$ where t is a real constant.

3. (15 pts) Let $f(z)$ be analytic on the entire $\mathbb{C}$. Suppose

- $|f(z)|=1$ if $|z|=1$;
- f has only one zero (counting multiplicity) in the unit disk $D_1(0)$.

Show that $f(z) = cz$, where c is a constant

(20 pts) 4. Denote by $D_r(0)$ the open disk centered at the origin with radius r. Let $f(z), f_1(z), f_2(z), \cdots, f_n(z), \cdots$ be analytic on $D_2(0)$. Suppose

$f_n(z) \to f(z)$ as $n \to \infty$, uniformly on $\partial D_1(0)$ (boundary of $D_1(0)$). Prove

(i) $f_n \to f$, $f_n' \to f'$ as $n \to \infty$, uniformly on $\overline{D_{1/2}(0)}$ (closure)

(ii) if $f(z) \neq 0$, $\forall z \in \partial D_1(0)$, then for large n, $f(z)$ and $f_n(z)$ have the same number of zeros in $D_1(0)$ (counting multiplicity).

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Real Analysis

Q1. Let f be a real-valued differentiable monotone function on $[0, 1]$. Define

$$g(x) = \frac{1}{2}\left[f(x) + f(1-x)\right] \quad \text{for } 0 \leq x \leq 1.$$

Show that

$$\sigma^2(g) \leq \frac{1}{2}\sigma^2(f)$$

where

$$\sigma^2(u) \equiv \int_0^1 (u(x) - E(u))^2\, dx \;\text{ and }\; E(u) \equiv \int_0^1 u(x)dx.$$

Q2. (a) Find a counterexample showing that the Dominated Convergence Theorem fails if the integrability of the dominating function is removed from the theorem.

(b) Find an example showing that the inequality in Fatou's Lemma is *strict*.

Q3. Let $f(x)$ be a real-valued function defined on a finite interval $[a, b]$ such that $f \in L^1([a, b])$ and

$$\int_a^b x^n f(x)dx = 0, \quad n = 0, 1, 2, \cdots.$$

Show that $f(x) = 0$ a.e..

Q4. Let $0 < p < 1$ and $-\infty < q < 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be positive real-valued functions defined on $[0, 1]$ such that $f \in L^p([0, 1])$ and $g \in L^q([0, 1])$. Assume also that fg is integrable on $[0, 1]$. Show that

$$\left(\int_0^1 f^p dx\right)^{\frac{1}{p}} \left(\int_0^1 g^q dx\right)^{\frac{1}{q}} \leq \int_0^1 fg dx.$$

Q5. Let $f : [0, 1] \to \mathbb{R}$ be continuous, $g : [0, 1] \to \mathbb{R}$ be Lebesgue measurable, and $0 \leq g(x) \leq 1$ for a.e. $x \in [0, 1]$. Find the limit

$$\lim_{n\to\infty} \int_0^1 f(g^n(x))dx.$$

Q6. Let $f, f_n \in L^1([0, 1])$ for $1 \leq n < \infty$. Assume that

(1) $f_n(x) \to f(x)$ a.e.,

(2) $\|f_n\|_{L^1} \to \|f\|_{L^1}$.

Show that $\|f_n - f\|_{L^1} \to 0$.