9. Given an infinite collection An, n = 1, 2,… of intervals of the real line, their intersection is defined to be:

$$\bigcap_{n=1}^{\infty} A_n = \{x|(\forall n)(x \in A_n)\}$$

Give an example of a family of intervals An, n = 1, 2,…, such that

$A_{n+1} < A_n$ (Here, < means real subset of, I can't find the correct symbol) for all n and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

Solution: Let $A_n = (0, \frac{1}{n})$, n = 1, 2,…, then it satisfies the stated property.

Proof: We need to prove the following points:

(1) $A_{n+1} < A_n$

(2) $\bigcap_{n=1}^{\infty} A_n = \emptyset$

For any natural number n, $\forall x \in A_{n+1}$ by definition, 0<x<1/(n+1)<1/n, hence, $x \in A_n$, hence, according to set theory, we have $A_{n+1} < A_n$, hence, (1) is proved.

Because of $A_n = (0, \frac{1}{n})$, n = 1, 2,…, so $(\forall x \leq 0)[x! \in \bigcap_{n=1}^{\infty} A_n]$. Then, in order to prove (2), we only need to prove that,

$(\forall x > 0)[x! \in \bigcap_{n=1}^{\infty} A_n]$.

$(\forall x > 0)$, let $n_0 = \left\lfloor \frac{1}{x} \right\rfloor$ ➔ $\frac{1}{x} - 1 < n_0 \leq \frac{1}{x}$ ➔$\frac{1}{x} < n_0 + 1$

Hence, $(\forall n > n_0)$, we have, $n + 1 > n_0 + 1 > \frac{1}{x}$ ➔ $x > \frac{1}{n+1}$➔ $x! \in A_{n+1} = \bigcap_{k=1}^{n+1} A_k$

By the definition of limit,$(\forall x > 0)$, $x! \in lim_{n\to\infty} \bigcap_{k=1}^{n+1} A_k = \bigcap_{n=1}^{\infty} A_n$.

Combining $(\forall x \leq 0)[x! \in \bigcap_{n=1}^{\infty} A_n]$ with $(\forall x > 0)$, $x! \in lim_{n\to\infty} \bigcap_{k=1}^{n+1} A_k = \bigcap_{n=1}^{\infty} A_n$, we can conclude that $\bigcap_{n=1}^{\infty} A_n = \emptyset$, (2) is proved.

Hence, $A_n = (0, \frac{1}{n})$, n = 1, 2,…, satisfies the stated property.

Q.E.D