10. Give an example of a family of intervals A_n, n = 1, 2,…, such that $A_{n+1} < A_n$ (Here, < means real subset of, I can't find the correct symbol) for all n, and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Solution: Let $A_n = (-\frac{1}{n}, \frac{1}{n})$, n = 1, 2,…, then $A_{n+1} < A_n$ for all n, and $\bigcap_{n=1}^{\infty} A_n = \{0\}$ and thus has the stated property.

Proof: We need to prove the following two points:

(1) $A_{n+1} < A_n$

(2) $\bigcap_{n=1}^{\infty} A_n = \{0\}$

For any natural number n, $\forall x \in A_{n+1}$ by definition, -1/n<-1/(n+1)<x<1/(n+1)<1/n, hence, $x \in A_n$, hence, according to set theory, we have $A_{n+1} < A_n$, (1) is proved.

Because of $A_n = (-\frac{1}{n}, \frac{1}{n})$, n = 1, 2,…, clearly, for any n, $0 \in A_n$, hence, $0 \in \bigcap_{n=1}^{\infty} A_n$.

Next we only need to prove $(\forall x \neq 0)(x! \in \bigcap_{n=1}^{\infty} A_n)$.

so $(\forall x \leq 0)[x! \in \bigcap_{n=1}^{\infty} A_n]$. Then, in order to prove (2), we only need to prove that, $(\forall x > 0)[x! \in \bigcap_{n=1}^{\infty} A_n]$.

$(\forall x \neq 0)$, let $n_0 = \left\lfloor \frac{1}{|x|} \right\rfloor$ ➔ $\frac{1}{|x|} - 1 < n_0 \leq \frac{1}{|x|}$ ➔ $\frac{1}{|x|} < n_0 + 1$

Hence, $(\forall n > n_0)$, we have, $n + 1 > n_0 + 1 > \frac{1}{|x|}$ ➔ $|x| > \frac{1}{n+1}$ ➔ $|x|! \in A_{n+1} = \bigcap_{k=1}^{n+1} A_k$ ➔ $x! \in \bigcap_{k=1}^{n+1} A_k$

By the definition of limit, $(\forall x \neq 0)[x! \in lim_{n\to\infty} \bigcap_{k=1}^{n+1} A_k = \bigcap_{n=1}^{\infty} A_n]$.

Combining $0 \in \bigcap_{n=1}^{\infty} A_n$ with $(\forall x \neq 0)[x! \in lim_{n\to\infty} \bigcap_{k=1}^{n+1} A_k = \bigcap_{n=1}^{\infty} A_n]$, we can conclude that $\bigcap_{n=1}^{\infty} A_n = \{0\}$, (2) is proved.

Hence, $A_n = (-\frac{1}{n}, \frac{1}{n})$, n = 1, 2,…, has the stated property.

Q.E.D