Least squares (LS) parameter estimation and model order selection

In this assignment we want to estimate the optimal order for the parameter estimate of an AR process. For an AR model with roots $\underline{r} = [r_1\ r_2 \ldots r_{k^*}]$, the coefficients for the all-pole transfer function are $\underline{a_{ar}} = [a_0\ a_1 \ldots a_{k^*}]$ which is also the output from Matlab's poly($\underline{r}$), and the AR process is defined as

$$a_0 y_t + a_1 y_{t-1} \ldots + a_{k^*} y_{t-k^*} = e_t,$$

where $a_0 = 1$. Denoting $w_1 = -a_1, \ldots, w_{k*} = -a_{k^*}$ we have

$$y_t = w_1 y_{t-1} + w_2 y_{t-2} \ldots + w_{k^*} y_{t-k^*} + e_t. \quad (1)$$

Using (1) for time instances N_2 to N, we have N_1 equations (since $N_2 = N - N_1 + 1$) and k^* unknowns.

$$\begin{bmatrix} y_{N_2} \\ y_{N_2+1} \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} y_{N_2-1} & y_{N_2-2} & \cdots & y_{N_2-k^*} \\ y_{N_2} & y_{N_2-1} & \cdots & y_{N_2+1-k^*} \\ \vdots & \vdots & \vdots & \vdots \\ y_{N-1} & y_{N-2} & \cdots & y_{N-k^*} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{k^*} \end{bmatrix} + \begin{bmatrix} e_{N_2} \\ e_{N_2+1} \\ \vdots \\ e_N \end{bmatrix}$$

or in more compact notation

$$\underline{d} = A\underline{w}^* + \underline{e}. \quad (2)$$

The least squares estimate of parameter vector $\underline{w}^*$ can be obtained from equation (2) using

$$\underline{w} = (A^T A)^{-1} A^T \underline{d}, \quad (3)$$

or with Matlab's backslash operator

$$\underline{w} = A \backslash \underline{d}. \quad (4)$$

We will run the parameter estimation (and model order estimation) for multiple realizations of the AR process. Additionally, you have to build the matrix A for each different order of parameter estimation. For example, when estimating the parameters of order k the matrix A will have dimensions $N_1 \times k$.

You will have to implement three different model order criteria: Final prediction error (FPE), Akaike Information Criterion (AIC), and Minimum Description Length (MDL). For details see `http://www.cs.tut.fi/~tabus/course/AdvSP/21006Lect8.pdf` page 28. Collect for each realization of the AR process the index (the order) of the minimum for each criteria. Study the resulting plots and use them to estimate the optimal model order.

1. Build matrix A from data $\underline{y}$ as in equation (2). Keep in mind that $\underline{d} = [y_{N_2}\ y_{N_2+1} \ldots y_N]^T$.

2. Compute the LS solution for the system of equations (2) in Matlab using both (3) and (4). Compare resulting $\underline{w}$ against $\underline{w}^*$. Is there any difference between the results of (3) and (4)?

2. Compute the residual $\underline{err} = \underline{d} - \hat{\underline{d}}$ where $\hat{\underline{d}} = A\underline{w}$.

3. Implement FPE, AIC and MDL criteria. Here you need the residual $\underline{err}$.

4. Study the resulting plots and give an estimate of the optimal model order. What do the plots represent? Can you see the actual model order from the definition of our AR process?