SGN 21006 Advanced Signal Processing: Lecture 9 Spectrum estimation

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Overview

- Energy spectral density - for finite energy deterministic signals
- Power spectral density - definitions for random signal
- Power spectral density - properties

The slides follow Chapters 1 and 2 from P. Stoica, R. Moses, "Spectral analysis of signals ", available on-line at `http://user.it.uu.se/~ps/SAS-new.pdf`

Power spectral estimation

- Goal: Given a finite number of values $y_1, y_2, \ldots, y_N$ of a stationary signal, estimate the power over narrow frequency bands.
- Applications:
  - the power over frequency represents a signature of the signal, by which it can be identified, classified, compared.
  - music decomposition over the semitones of the scale for automatic music transcription
  - speech decomposition over bark or mel scales, for speech recognition
  - monitoring the functioning of industrial machines or building and bridge structure
  - characterization of physiological waves (EEG,EKG)
  - estimate the direction of targets in underwater surveillance
- First we introduce the definitions of spectral density for deterministic signals, then for random signals, and then we state some properties of the spectral density.
- The non-parametric and parametric estimation methods are discussed in next lectures.

Deterministic signals

- Given a discrete-time signal $\{y_t\}_{t=-\infty}^{\infty}$ having finite energy

$$\sum_{t=-\infty}^{\infty} y_t^2 < \infty$$

then the Discrete-time Fourier transform (DTFT) can be defined as

$$\boxed{Y(\omega) \quad = \quad \sum_{t=-\infty}^{\infty} y(t)e^{-j\omega t}}$$

Here $Y(\omega)$ has a complex value and is the transform coefficient at frequency ω. The absolute value $|Y(\omega)|$ is the amplitude and $\arg(Y(\omega))$ is the phase corresponding to transform coefficient $Y(\omega)$ at frequency ω.

- The Parseval inequality holds for the DTFT:

$$\sum_{t=-\infty}^{\infty} y_t^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(\omega)|^2 d\omega$$

The lefthand side (l.h.s) is the total energy of the signal, the r.h.s is the integral of the squared amplitude over frequency. Hence the term $|Y(\omega)|^2$ has the significance of a density of energy over the frequency axis.

- Energy spectral density definition

$$\boxed{S(\omega) \quad = \quad |Y(\omega)|^2}$$

and from Parseval identity the distribution of energy sums up to the total energy of the signal $\sum_{t=-\infty}^{\infty} y_t^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)d\omega$.

Deterministic signals

- Property: Define for all k the "autocorrelation" of the deterministic signal (note the missing division by the length of the signal, which is ∞)

$$\rho(k) = \sum_{t=-\infty}^{\infty} y_t y_{t-k} = \rho(-k)$$

Then

$$\boxed{S(\omega) \quad = \quad \sum_{k=-\infty}^{\infty} \rho(k) e^{-j\omega k}}$$

- Proof:

$$\begin{aligned}
S(\omega) &= |Y(\omega)|^2 = Y(\omega)Y^*(\omega) = \sum_{t=-\infty}^{\infty} y(t)e^{-j\omega t} \sum_{\tau-\infty}^{\infty} y(\tau)e^{+j\omega\tau} \\
&= \sum_{t=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} y(t)y(\tau)e^{j\omega(\tau-t)} \overset{(k=t-\tau)}{=} \sum_{t=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} y(t)y(t-k)e^{-j\omega k} \\
&= \sum_{k=-\infty}^{\infty} e^{-j\omega k} \sum_{t=-\infty}^{\infty} y(t)y(t-k) = \sum_{k=-\infty}^{\infty} \rho(k)e^{-j\omega k}
\end{aligned}$$

- The two equivalent definitions of $S(\omega)$, the first as $S(\omega) = |Y(\omega)|^2$ and second $S(\omega) = \sum_{k=-\infty}^{\infty} \rho(k)e^{-j\omega k}$ will be used for defining power spectral density for random signals.

Power spectral density for random signals

- Motivating the definition for the case of random signals:
  - For different realizations of a random signal (e.g. generated by the same ARMA model using different realizations of the noise) we would like to have similar evaluations of the distribution of the power over frequencies. We need a deterministic function to characterize the content of power over different frequencies.
  - For a signal that is formed of a sum of k sinusoids (having frequencies $\omega_1, \ldots, \omega_k$, phases $\phi_1, \ldots, \phi_k$, and amplitudes $\alpha_1, \ldots, \alpha_k$) plus white noise, the autocorrelation function is obtained as the sum of the sinusoids having the same frequencies $\omega_1, \ldots, \omega_k$ and amplitudes $\alpha_1^2, \ldots, \alpha_k^2$ (but phases information $\phi_1, \ldots, \phi_k$ is lost). (See the lecture about spectrum estimation for line models)
  - Hence one can recover the powers of the signal over various frequencies from its autocorrelation function. Taking the Fourier transform of the autocorrelation function, $\mathcal{F}[r(\cdot)](\omega)$, we obtain the coefficients of the sinusoidal components over various frequencies.

Power spectral density: Definition 1

- We are given a discrete-time signal $\{y_t\}_{t=-\infty}^{\infty}$ which is a sequence of random variables with zero mean.
- First power spectral density definition:

$$\boxed{P(\omega) = \sum_{k=-\infty}^{\infty} r(k)e^{-j\omega k}}$$

where $r(k)$ is the auto covariance function $r(k) = E[y_t y_{t+k}]$.
- The inverse Fourier transform will recover the correlation from the spectrum:

$$r(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\omega)e^{j\omega k}d\omega$$

- The autocorrelation at lag 0 is

$$r(0) = E[y_t^2] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\omega)d\omega$$

This relation reinforces the power density interpretation: autocorrelation at lag 0 is the power of the signal, which is equal to the integral of the power density over all frequencies.
- One can interpret $P(\omega)d\omega$ as the infinitesimal power in the band $(\omega - \frac{d\omega}{2}, \omega + \frac{d\omega}{2})$

Power spectral density: Definition 2

- The second definition generalizes to random variables the definition $S(\omega) = |Y(\omega)|^2$ used for deterministic signals of finite energy.
- Define the finite Discrete Fourier transform of a sequence $y_1, y_1, \ldots, y_N$ as

$$Y_N(\omega) = \sum_{t=1}^{N} y_t e^{-j\omega t}$$

- Second power spectral density definition is then

$$P(\omega) = \lim_{N\to\infty} E\left\{\frac{1}{N}|Y_N(\omega)|^2\right\}$$

or written directly as a function of the data

$$\boxed{P(\omega) = \lim_{N\to\infty} E\left\{\frac{1}{N}\left|\sum_{t=1}^{N} y_t e^{-j\omega t}\right|^2\right\}}$$

Power spectral density: Equivalence of the Definitions

- Theorem: The Definitions 1 and 2 of the power spectral density are equivalent if the covariance function decays sufficiently rapidely, i.e if $\lim_{N\to\infty} \frac{1}{N}\sum_{k=-N}^{N} |k||r(k)| = 0$.
- Preliminary result (double summation formula) (Proof is left as an exercise) For any arbitrary function f

$$\sum_{t=1}^{N}\sum_{s=1}^{N} f(t-s) = \sum_{\tau=-N+1}^{N-1} (N-|\tau|)f(\tau)$$

- Proof of the theorem We have

$$\begin{aligned} E\left|\sum_{t=1}^{N} y_t e^{-j\omega t}\right|^2 &= E\sum_{t=1}^{N} y_t e^{-j\omega t}\sum_{s=1}^{N} y_s e^{j\omega s} = \sum_{t=1}^{N}\sum_{s=1}^{N} E[y_t y_s]e^{-j\omega(t-s)} \\ &= \sum_{t=1}^{N}\sum_{s=1}^{N} r(t-s)e^{-j\omega(t-s)} = \sum_{\tau=-N+1}^{N-1} (N-|\tau|)r(\tau)e^{-j\omega\tau} \end{aligned}$$

where the last equality comes from the preliminary result on double sums. Now the second definition gives

$$\lim_{N\to\infty} E\left\{\frac{1}{N}\left|\sum_{t=1}^{N} y_t e^{-j\omega t}\right|^2\right\} = \lim_{N\to\infty}\frac{1}{N}\sum_{\tau=-N+1}^{N-1}(N-|\tau|)r(\tau)e^{-j\omega\tau}$$

$$= \sum_{\tau=-\infty}^{\infty} r(\tau)e^{-j\omega\tau} - \lim_{N\to\infty}\frac{1}{N}\sum_{\tau=-N+1}^{N-1}|\tau|r(\tau)e^{-j\omega\tau} = \sum_{\tau=-\infty}^{\infty} r(\tau)e^{-j\omega\tau} = P(\omega)$$

since the term $\lim_{N\to\infty}\frac{1}{N}\sum_{\tau=-N+1}^{N-1}|\tau|r(\tau)e^{-j\omega\tau}$ is zero if $r(k)$ decays rapidly.

Properties of power spectral density

- The power spectral density is real valued and nonnegative for all frequencies ω

$$\boxed{P(\omega) \geq 0 \quad \forall \omega}$$

which can be seen from the second definition $P(\omega) = \lim_{N\to\infty} E\left\{\frac{1}{N}\left|Y_N(\omega)\right|^2\right\}$ where the squared value is always nonnegative.

- For real valued signals y_t the function $P(\omega)$ is even, i.e.,

$$\boxed{P(\omega) = P(-\omega) \quad \forall \omega \in (-\pi, \pi)}$$

Proof:

$$\begin{aligned} P(\omega) &= \sum_{k=-\infty}^{\infty} r(k)e^{-j\omega k} = r(0) + \sum_{k=1}^{\infty} r(k)(e^{-j\omega k} + e^{j\omega k}) = r(0) + 2\sum_{k=1}^{\infty} r(k)cos(\omega k) \\ &= r(0) + 2\sum_{k=1}^{\infty} r(k)cos(-\omega k) = P(-\omega) \end{aligned}$$

Properties of power spectral density

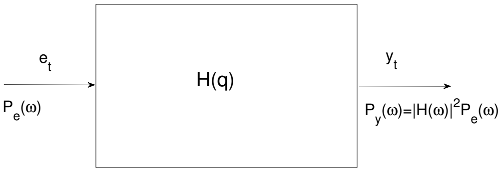
- *Theorem* The transfer of power spectral density $P_e(\omega)$ of the input through an asymptotically stable linear system having transfer function

$$H(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k} \qquad (1)$$

to the power spectral density $P_y(\omega)$ of the output is given by

$$\boxed{P_y(\omega) = |H(\omega)|^2 P_e(\omega) \quad \forall \omega}$$

where $H(\omega) = H(z)\Big|_{z=e^{j\omega}}$.



Properties of power spectral density

Proof of the power transfer theorem

- The linear system output is the convolution of the input e_t with the impulse response h_k:

$$y_t = H(q)e_t = \sum_{k=-\infty}^{\infty} h_k e_{t-k}$$

- The autocorrelation of the output is

$$\begin{aligned} r_y(k) &= E y_t y_{t-k} = E \sum_{m=-\infty}^{\infty} h_m e_{t-m} \sum_{s=-\infty}^{\infty} h_s e_{t-k-s} = \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} h_m h_s E[e_{t-m} e_{t-k-s}] \\ &= \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} h_m h_s r_e(k+s-m) \end{aligned}$$

- the power spectrum of the output is

$$\begin{aligned} P_y(\omega) &= \sum_{k=-\infty}^{\infty} r_y(k) e^{-j\omega k} = \sum_{k=-\infty}^{\infty} e^{-j\omega k} \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} h_m h_s r_e(k+s-m) \\ &= \left[\sum_{m=-\infty}^{\infty} h_m e^{-j\omega m}\right] \left[\sum_{s=-\infty}^{\infty} h_m e^{j\omega s}\right] \left[\sum_{k+s-m=-\infty}^{\infty} r(k+s-m) e^{-j\omega(k+s-m)}\right] \\ &= H(\omega) H^*(\omega) P_e(\omega) = |H(\omega)|^2 P_e(\omega) \end{aligned}$$

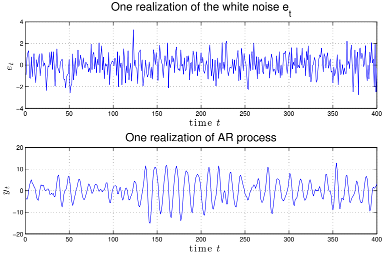
Example of power spectral density

Example

- We consider an AR(6) process, $y_t = \frac{1}{A(q)} e_t$, where the polynomial $A(z)$ has the following roots: $z_{1,2} = 0.9e^{\pm j\pi/3}$, $z_{3,4} = 0.7e^{\pm j3\pi/4}$, $z_{5,6} = 0.99e^{\pm j\pi/6}$, and the white noise e_t has zero mean and standard deviation $\sigma_e = 1$.
- The power spectral density of the white noise is $P_e(\omega) = \sigma_e^2 = 1$.
- The power spectral density of the random process y_t obtained by passing white noise e_t through the linear system with transfer function $\frac{1}{A(z)}$ is

$$P_y(\omega) = \frac{1}{|A(e^{j\omega})|^2} \sigma_e^2$$

- short realizations of the white noise and of the random process y_t are shown below



Example of power spectral density

- the power spectral density $P_y(\omega)$, and an estimate of the power spectral density obtained from a short realization of the random process y_t are shown below, first separated, then overlapped. The estimation $\hat{P}_y(\omega)$ is our next topic.