SGN 21006 Advanced Signal Processing: Lecture 2 Random Signals

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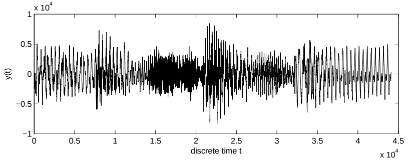
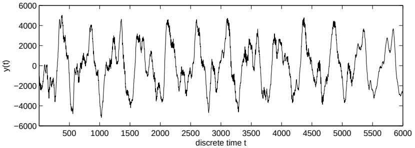
Studying signal waveforms





(Top) One second of audio signal sampled at 44100 Hz, with 16 bits per sample;

(Bottom) Zoom onto the first 6000 samples

- Finding periodicities
- Finding compact representations:
  - Deterministic function + random component
  - Regression on other signals + random (exploiting correlations to other given signals)
  - A parametric model including random components

Signal representations

- Deterministic + random

$$
\begin{aligned}
X(t) &= f(t) + e(t) \\
&= \sum_{k=1}^{K_1} a_k \sin(\omega_k t + \phi_k) + e_1(t) \qquad \text{Sum of sinusoids} \\
&= \sum_{k=1}^{K_2} b_k \phi_k(t) + e_2(t) \qquad \text{Decomposition on bases } \{\phi_k\}_{k=1}^{K_2}
\end{aligned}
$$

  where $X(t)$ is the given signal, $f(t)$ is a deterministic signal and $e(t), e_1(t), e_2(t)$ are random components ("errors" or "residuals")
- Regression + random

$$
X(t) = \sum_{k=1}^{K_3} X_k(t) + e(t)
$$

  where the random signal $X(t)$ is regressed on other (random) signals $X_1(t), \dots, X_{K_3}(t)$.
- Parametric models
  $X(t) = a_1 X(t-1) + \dots a_{n_a} X(t-n_a) + b_0 e(t) + \dots + b_{n_b} e(t-n_b)$

Random variables

- A random variable X takes values in a set (continuous or discrete). The cumulative distribution function $F_X(x) = Prob(X \leq x)$ can be used for describing probabilities of X falling in an interval: $Prob(a < X \leq b) = F_X(b) - F_X(a)$.
- for a discrete random variable defined on $m, m+1, m+2, \ldots, M$ the probability mass function $p(j) = Prob(X \leq j) - Prob(X < j) = Prob(X = j)$ is easy to use. We have the normalization condition $\sum_{j=m}^{M} Prob(X = j) = 1$.
- continuous random variables $x \in (-\infty, \infty)$ are fully described by the probability density function (pdf) $p(x)$, which obeys the normalization condition $\int_{-\infty}^{\infty} p(x)dx = 1$.
    1. The cumulative distribution function is $F_X(x) = Prob(X \leq x) = \int_{-\infty}^{x} p(y)dy$.
    2. Probability of the variable X falling in the interval $(a, b]$ is $Prob(a < X \leq b) = F_X(b) - F_X(a)$.

Examples of continuous distributions

- Normal distribution, or Gaussian distribution, denoted $\mathcal{N}(\mu, \sigma^2)$, with mean μ and standard deviation σ, with the pdf

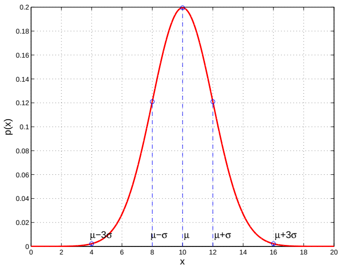
$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

  1. the cumulative distribution

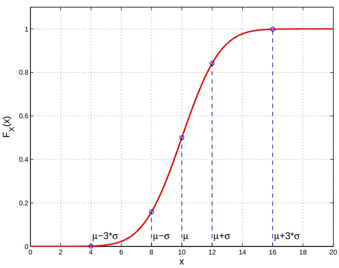
$$F_X(x) = Prob(X \leq x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy$$

     does not have a closed form expression. It is often computed using the cdf of the distribution having parameters $\mu = 0, \sigma = 1$, denoted $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$

Gaussian $\mathcal{N}(\mu = 10, \sigma = 2)$

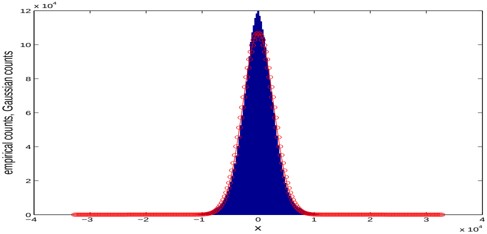


Probability distribution function



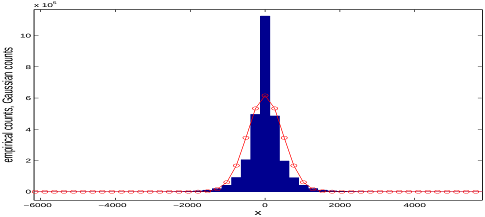
Cumulative distribution function

Fitting a Gaussian distribution $\mathcal{N}(\mu, \sigma)$ to the audio data of Page 1



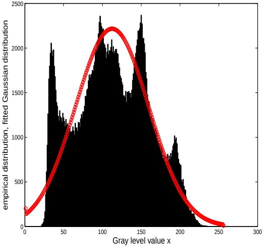
- Estimate the sample mean $\hat{\mu}_1 = \frac{1}{N}\sum_{k=1}^{N} Y(k) = -0.063824 \approx 0$
- Estimate the sample standard deviation $\hat{\sigma}_1 = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(Y(k) - \hat{\mu}_1)^2} = 2742.3$
- Since the range of Y is too large, $Y_k \in \{-2^{15}, 2^{15}\}$, let us take intervals in the range $\{-2^{15}, 2^{15}\}$, of length 256; there are 256 such intervals. Count how many times Y_k falls in a given interval, call it empirical count (blue stems).
- Compute the probability of a Gaussian $\mathcal{N}(\hat{\mu}_1, \hat{\sigma}_1)$ random variable to fall in an interval $(a, b]$, as $Prob(Y \in (a, b]) = F_X(b) - F_X(a)$. Obtain the Gaussian counts as $N \cdot Prob(Y \in (a, b])$, represented in red.
- The empirical counts and the Gaussian counts are reasonably close, so Gaussian distribution is a good approximation. In this representation we have $Y_t = e_t \sim \mathcal{N}(\hat{\mu}_1, \hat{\sigma}_1)$. Is this the best representation?

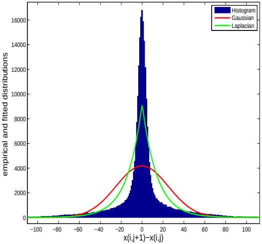
Fitting a Gaussian distribution $\mathcal{N}(\mu, \sigma)$ to the difference $Y(t) - Y(t-1)$



- Estimate the sample mean $\hat{\mu}_2 = \frac{1}{N-1}\sum_{k=2}^{N}(Y(k) - Y(k-1)) = 0.0006464 \approx 0$
- Estimate the sample standard deviation of $e(k) = Y(k) - Y(k-1)$ as $\hat{\sigma}_2 = \sqrt{\frac{1}{N-1}\sum_{k=2}^{N}((Y(k) - Y(k-1)) - \hat{\mu}_2)^2} = 470$
- Repeat the interval construction and the counting as in previous page
- In this representation we have $Y(k) = Y(k-1) + e(k)$ where $e(k) \sim \mathcal{N}(\hat{\mu}_2, \hat{\sigma}_2)$. Here $\hat{\sigma}_2 = 470 << \hat{\sigma}_1 = 2742.3$, so the random component needed for explaining data is much smaller. Prediction by this model will be more accurate!
- The empirical counts and the Gaussian counts are not as close as earlier, so Gaussian distribution may be changed for example to a Laplace distribution.
- Even tighter representations, of the form $Y(t) = a_1 Y(t-1) + \ldots + a_{n_a} Y(t-n_a) + e(t)$ will be discussed in linear prediction lectures.

Fitting a Gaussian distribution $\mathcal{N}(\mu, \sigma)$ to the gray levels on "Barbara" image

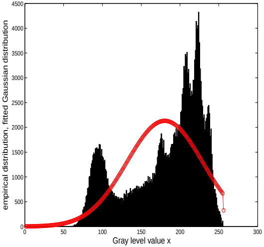


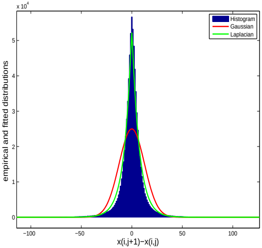


- (Middle) Fitting a Gaussian to the gray levels $y(i,j)$. Estimate the sample mean $\hat{\mu}_2 = \frac{1}{N-1}\sum_{k=2}^{N} Y(k) = 112.45$. Estimate the sample standard deviation $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{k=2}^{N}(Y(k)-\hat{\mu})^2} = 47.2$. Model: $y(i,j) = \varepsilon(i,j) \sim \mathcal{N}(112.4, 47.2)$
- (Right) Fitting Gaussian and Laplace distributions to the differences $y(i,j+1) - y(i,j)$. Interval constructions and counts as in previous page for the Gaussian distribution (red curve). Model: $y(i,j+1) = y(i,j) + \varepsilon(i,j+1)$ with $\varepsilon(i,j+1) \sim \mathcal{N}(-0.01, 25)$.
- Similar estimation of parameters and counting for the Laplace distribution (green curve).

$$p(x;\mu,b) \quad = \quad \frac{1}{2b} e^{-|x-\mu|/b}$$

Fitting a Gaussian distribution $\mathcal{N}(\mu, \sigma)$ to the gray levels on "Lena" image

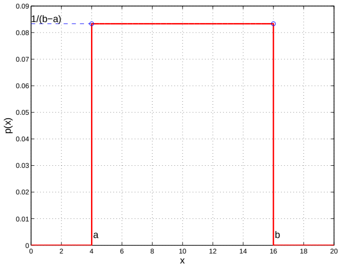


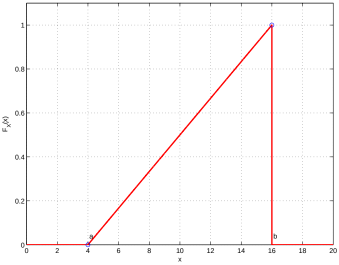


- (Middle) Fitting a Gaussian to the gray levels $y(i,j)$. Estimate the sample mean $\hat{\mu}_2 = \frac{1}{N-1}\sum_{k=2}^{N} Y(k) = 180.2$. Estimate the sample standard deviation $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{k=2}^{N}(Y(k) - \hat{\mu})^2} = 49.05$. Model: $y(i,j) = \varepsilon(i,j) \sim \mathcal{N}(180, 49)$
- (Right) Fitting Gaussian and Laplace distributions to the differences $y(i,j+1) - y(i,j)$. Interval constructions and counts as in previous page for the Gaussian distribution (red curve). Model: $y(i,j+1) = y(i,j) + \varepsilon(i,j+1)$ with $\varepsilon(i,j+1) \sim \mathcal{N}(-0.03, 12.5)$
- Similar estimation of parameters and counting for the Laplace distribution (green curve).

$$p(x;\mu,b) = \frac{1}{2b} e^{-|x-\mu|/b}$$

Uniform distribution $\mathcal{U}(a, b)$





Probability distribution function   Cumulative distribution function

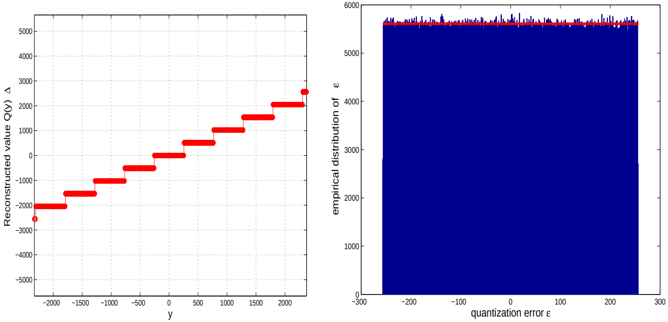
- Uniform distribution, denoted $\mathcal{U}(a, b)$, with mean $\mu = \frac{a+b}{2}$ and standard deviation $\sigma = \sqrt{(b-a)^2/12}$, with the pdf

$$p(x) = \begin{cases} \frac{1}{b-a} & if \quad x \in \{a, b\} \\ 0 & if \quad x \notin \{a, b\} \end{cases}$$

- the cumulative distribution

$$F_X(x) = \begin{cases} \frac{x-a}{b-a} & if \quad x \in \{a, b\} \\ 0 & if \quad x \notin \{a, b\} \end{cases}$$

Uniform quantization



- The quantized value is computed as $Q(y) = \text{round}\ (y/\Delta)$ and has a dynamic range Δ times smaller than the initial dynamic range.
- The reconstructed value is $\hat{y} = Q(y)\Delta$. The quantization error is $\varepsilon = y - \hat{y}$.
- For the audio signal we have $y \in \{-2^{15}, 2^{15}\}$. Let's take $\Delta = 512$.
- The quantization error ε belongs to $\{-256, \ldots, 256\}$, hence the uniform distribution $\mathcal{U}(a, b)$ has parameters $a = -256, b = 256$. The histogram of ε is shown in blue, while the ideal uniform counts, corresponding to the uniform distribution $\mathcal{U}(a, b)$, are shown in red.

Joint probabilities; Vector random variables

- the joint probability of X and Y is $p(x,y) = Prob(X = x; Y = y)$
- Two situations:
  1. X and Y are independent iff for all x, y
     $$Prob(X = x; Y = y) \quad = \quad Prob(X = x)Prob(Y = y)$$
  2. when X and Y are NOT independent, the factorization involves conditional probabilities, by the rule of Bayes
     $$\begin{aligned} Prob(X = x; Y = y) &= Prob(X = x|Y = y)Prob(Y = y) \\ &= Prob(Y = y|X = x)Prob(X = x) \end{aligned}$$
- for a vector of random variables, $\underline{X} = [X_1, X_2, \ldots, X_n]^T$, having pdf $p(\underline{x})$, the mean is
  $$\underline{\mu} = E[\underline{X}] = \int \underline{x}p(\underline{x})dx_1 \ldots dx_n,$$
  the correlation matrix is
  $$R \quad = \quad E[\underline{x}\underline{x}^T]$$
  and the covariance matrix is
  $$\Sigma \quad = \quad E[(\underline{x} - E[\underline{x}])(\underline{x} - E[\underline{x}])^T = R - \underline{\mu}\underline{\mu}^T$$

Gaussian random vectors

- consider the vector of random variables, $\underline{X} = [X_1, X_2, \ldots, X_n]^T$, having the Gaussian pdf

$$p(\underline{x}) = \frac{1}{(2\pi)^{N/2}(\det\Sigma)^{1/2}} e^{-(\underline{x}-\underline{\mu})^T\Sigma^{-1}(\underline{x}-\underline{\mu})/2}$$

  with mean $\underline{\mu}$ and covariance matrix Σ.

- A linear transformation B applied to a Gaussian vector $\underline{x}$ of mean $\underline{\mu}_x$ and covariance matrix Σ_x, results in a Gaussian vector $\underline{y} = B\underline{x}$, having the mean

$$\underline{\mu}_y = B\underline{\mu}_x$$

  and covariance matrix

$$\Sigma_y = B\Sigma_x B^T$$

Uncorrelated Gaussian random vectors

- If the n_x-vector $\underline{x}$ and n_y-vector $\underline{y}$ are zero mean and uncorrelated and jointly Gaussian, then they are also independent.
  Proof: Denote the joint vector $\underline{z} = [\underline{x}^T \quad \underline{y}^T]^T$. The Gaussian assumption states:

$$\begin{aligned} p(\underline{x}) &= \frac{1}{(2\pi)^{n_x/2}(\det \Sigma_x)^{1/2}} e^{-\underline{x}^T \Sigma_x^{-1} \underline{x}/2} \\ p(\underline{y}) &= \frac{1}{(2\pi)^{n_y/2}(\det \Sigma_y)^{1/2}} e^{-\underline{y}^T \Sigma_y^{-1} \underline{y}/2} \\ p(\underline{z}) &= \frac{1}{(2\pi)^{(n_x+n_y)/2}(\det \Sigma_z)^{1/2}} e^{-\underline{z}^T \Sigma_z^{-1} \underline{z}/2} \end{aligned}$$

We need to show that $p(\underline{z}) = p(\underline{x})p(\underline{y})$.
If $\underline{x}$ and $\underline{y}$ are uncorrelated $E[\underline{x}\underline{y}^T] = 0$. The covariance matrix of $\underline{z}$ is

$$\Sigma_z = E[\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \begin{bmatrix} \underline{x}^T & \underline{y}^T \end{bmatrix}] = \begin{bmatrix} E[\underline{x}\underline{x}^T] & E[\underline{x}\underline{y}^T] \\ E[\underline{y}\underline{x}^T] & E[\underline{y}\underline{y}^T] \end{bmatrix} = \begin{bmatrix} E[\underline{x}\underline{x}^T] & 0 \\ 0 & E[\underline{y}\underline{y}^T] \end{bmatrix} = \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_y \end{bmatrix}$$

The quadratic form in the exponential of $p(\underline{z})$ is

$$\underline{z}^T \Sigma_z^{-1} \underline{z} = \begin{bmatrix} \underline{x}^T & \underline{y}^T \end{bmatrix} \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_y \end{bmatrix}^{-1} \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = x^T \Sigma_x^{-1} \underline{x} + y^T \Sigma_y^{-1} \underline{y}$$

and hence

$$e^{-z^T \Sigma_z^{-1} \underline{z}/2} = e^{-(x^T \Sigma_x^{-1} \underline{x} + y^T \Sigma_y^{-1} \underline{y})/2} = e^{-x^T \Sigma_x^{-1} \underline{x}/2} e^{-y^T \Sigma_y^{-1} \underline{y}/2}$$

The determinant of Σ_z also factorizes as $\det \Sigma_z = \det \Sigma_x \det \Sigma_y$ and finally $p(\underline{z}) = p(\underline{x})p(\underline{y})$ which means $\underline{x}$ and $\underline{y}$ are independent.

Expectation for continuous variables

- for continuous random variables $x \in (-\infty, \infty)$, fully described by the probability density function (pdf) $p(x)$ the expectation of a function $g(X)$ is $E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)dx$
- Important expectations
  1. mean, or first moment, or expected value of X
  $$\mu = E[X] = \int_{-\infty}^{\infty} xp(x)dx$$
  2. variance, or expected value of $(X-\mu)^2$
  $$\sigma^2 = var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx$$
  3. second moment, or expected value of X^2
  $$E[X^2] = \int_{-\infty}^{\infty} x^2 p(x)dx = \sigma^2 + \mu^2$$

Expectation for discrete variables

- for discrete random variables X taking values in the set $m, m+1, m+2, \ldots, M$ with the probability mass function $p(j) = Prob(X \leq j) - Prob(X < j) = Prob(X = j)$ the expectation of a function $g(X)$ is $E[g(X)] = \sum_{j=m}^{M} g(j)p(j)$
- Important expectations
  1. mean, or first moment, or expected value of X
  $$\mu = E[X] = \sum_{j=m}^{M} jp(j)$$
  2. variance, or expected value of $(X - \mu)^2$
  $$\sigma^2 = var(X) = E[(X - m)^2] = \sum_{j=m}^{M} (j - \mu)^2 p(j)$$
  3. second moment, or expected value of X^2
  $$E[X^2] = \sum_{j=m}^{M} j^2 p(j) = \sigma^2 + \mu^2$$

Properties of expectation operator

For simplicity we take the case of discrete random variables, but the results are holding for continuous random variables as well.

- The linearity property: given two random variables, X and Y, with joint pmf $g(x,y) = Prob(X = x; Y = y)$ and two constants a and b, then

  $$\boxed{\mathrm{E[aX+bY] = aE[X] + bE[Y]}}$$

  Proof:

  $$\begin{aligned} E[aX+bY] &= \sum_x \sum_y (ax+by) Prob(X=x; Y=y) \\ &= a\sum_x \sum_y x Prob(X=x; Y=y) + b\sum_x \sum_y y Prob(X=x; Y=y) \\ &= a\sum_x x Prob(X=x) + b\sum_y y Prob(Y=y) = aE[X] + bE[Y] \end{aligned}$$

- The expectation of a product of two independent random variables is equal to the product of expectations $E[XY] = E[X]E[Y]$
  Proof: If X is independent of Y, then
  $Prob(X = x, Y = y) = Prob(X = x)Prob(Y = y)$ and

  $$\begin{aligned} E[XY] &= \sum_x \sum_y xy Prob(X=x; Y=y) \\ &= \sum_x x Prob(X=x) \sum_y y Prob(Y=y) = E[X]E[Y] \end{aligned}$$

Properties of expectation operator

- In general the expectation of a product of two random variables is NOT equal to the product of expectations, $E[XY] \neq E[X]E[Y]$ unless the random variables are independent.
- The difference $E[XY] - E[X]E[Y]$ is equal to the crosscorrelation function

$$\begin{aligned} R(X,Y) &= E[(X - E[X])(Y - E[Y])] = E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\ &= E[XY] - 2E[X]E[Y] + E[X]E[Y] = E[XY] - E[X]E[Y] \end{aligned}$$

- For independent variables the cross-correlation is 0.
  Proof: $R(X,Y) = E[XY] - E[X]E[Y] = 0$.

Gaussian vectors with diagonal covariance matrix have independent components

- Take a random vector $\underline{X} = [X_1 \quad \dots \quad X_n]^T$ with mean $\underline{\mu} = [\mu_1 \quad \dots \quad \mu_n]^T$ and diagonal covariance matrix

$$R = E[(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})^T] = \begin{bmatrix} E[(X_1 - \mu_1)^2] & 0 & \dots & 0 \\ 0 & E[(X_2 - \mu_2)^2] & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & E[(X_n - \mu_n)^2] \end{bmatrix}$$

Proof that Gaussian vectors with diagonal covariance matrix have independent components:

$$R^{-1} = \left(E[(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})^T]\right)^{-1} = \begin{bmatrix} 1/\sigma_1^2 & 0 & \dots & 0 \\ 0 & 1/\sigma_2^2 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1/\sigma_n^2 \end{bmatrix}$$

Hence $(\underline{x} - \underline{\mu})^T R^{-1} (\underline{x} - \underline{\mu}) = \sum_{i=1}^n \frac{(x_i - \mu_i)^2}{\sigma_i^2}$ and $\det R = \prod_{i=1}^n \sigma_i^2$

$$\begin{aligned} p(\underline{x}) &= \frac{1}{(2\pi)^{n/2}(\det R)^{1/2}} e^{-(\underline{x}-\underline{\mu})^T R^{-1}(\underline{x}-\underline{\mu})/2} = \frac{1}{(2\pi)^{n/2}(\det R)^{1/2}} e^{-\sum_{i=1}^n \frac{(x_i-\mu_i)^2}{2\sigma_i^2}} \\ &= \prod_{i=1}^n \frac{1}{(2\pi)^{1/2}\sigma_i} e^{-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}} = \prod_{i=1}^n p(x_i) = p(x_1, x_2, \dots, x_n) \rightarrow \text{INDEPENDENCE} \end{aligned}$$

Uncorrelated jointly Gaussian variables are independent

- Take $X_1 \quad \ldots \quad X_n$ uncorrelated jointly Gaussian.
- "uncorrelatedness" means that $E[(X_i - \mu_i)(X_j - \mu_j)] = 0$ for all i, j, thus the covariance matrix of the vector $[X_1 \; X_2 \; \ldots X_n]$ is diagonal.
- "jointly Gaussian" means that $p(\underline{x}) = \frac{1}{(2\pi)^{n/2}(\det R)^{1/2}} e^{-(\underline{X}-\underline{\mu})^T R^{-1}(\underline{X}-\underline{\mu})/2}$.
- From the previous page it results that $X_1 \quad \ldots \quad X_n$ are independent.
- Hence uncorrelated Gaussian variables are independent
- The opposite is true in general: independent variables are uncorrelated
- To the extent that the distributions can be approximated well by Gaussiaan distribution, one can identify in general "uncorrelated" and "independent", at least at the level of heuristic descriptions.

De-correlating random vectors: the Karhunen-Loeve transform

- Consider a random vector $\underline{X} = [X_1 \quad \dots \quad X_n]^T$ with mean $\underline{\mu} = [\mu_1 \quad \dots \quad \mu_n]^T$ with an arbitrary covariance matrix R_X, which is not a diagonal matrix.
- The non-diagonal element (i,j) of the matrix R_X is

$$R_X(i,j) = E[(X_i - \mu_i)(X_j - \mu_j)^T]$$

- The random variables are not independent! We want to find, by using a simple linear transformation, a vector of n random variables which are uncorrelated (and if Gaussian distributed, are also independent).
- We want to find the transformation matrix B so that $\underline{Y} = B(\underline{X} - \mu)$ has the elements uncorrelated with each other. That will happen if the covariance matrix $E[\underline{Y}\underline{Y}^T]$ is diagonal.

$$\begin{aligned} R_Y &= E[\underline{Y}\underline{Y}^T] = E[B(\underline{X} - \mu)(B(\underline{X} - \mu))^T] \\ &= E[B(\underline{X} - \mu)(\underline{X} - \mu)^T B^T] = E[BR_X B^T] \end{aligned}$$

In general the matrix B will be complex valued, so all transposition operations $(\cdot)^T$ must be replaced by "complex conjugation and transpose" operation $(\cdot)^H$.

$$R_Y = E[\underline{Y}\underline{Y}^H] = E[BR_X B^H]$$

The Karhunen-Loeve transform

- Consider the eigenvalue decomposition of R_x, where each eigenvalue-eigenvector pair $\lambda_i, \underline{q}_i$ obeys $R_x \underline{q}_i = \lambda_i \underline{q}_i$. Denote $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. One can choose such eigenvectors that the matrix $Q = [\underline{q}_1 \; \underline{q}_2 \; \ldots \underline{q}_n]$ is unitary, so that

$$\begin{aligned} Q^H Q &= I \\ R_x Q &= Q\Lambda \\ R_x &= Q\Lambda Q^H \\ \Lambda &= Q^H R_x Q \end{aligned}$$

- Taking $B = Q^H$ we have the desired transformation

$$\begin{aligned} \underline{Y} &= Q^H(\underline{X} - \underline{\mu}) \\ R_Y &= E[\underline{Y}\,\underline{Y}^H] = Q^H R_X Q = \Lambda \end{aligned}$$

  and, since the matrix Λ is diagonal, the components of the vector $\underline{Y}$ are not correlated, $E[Y_i Y_j] = 0$.

- if the distribution of the initial vector $\underline{X}$ was Gaussian, also the transformed vector $\underline{Y}$ is Gaussian distributed, and hence the components of the vector $\underline{Y}$ are independent!
- the dependent components of $\underline{X}$ are transformed easily by the KL transform into independent components, over which the study is much simpler, since each component can be studied separately!
- the results of the analysis over the components of $\underline{Y}$ can then be phrased in terms of the initial random variables in $\underline{X}$ by the inverse transform

$$\underline{X} = Q\underline{Y} + \mu$$

Simulation of random variables in Matlab

- Uniform distribution $\mathcal{U}(0,1)$ on the open interval (0,1):
  rand(n) returns a n-by-n matrix containing pseudorandom values drawn from the standard uniform distribution on the open interval (0,1). rand(m,n) or rand([m,n]) returns an m-by-n matrix.
- normal distribution $\mathcal{N}(0,1)$ of zero mean and standard deviation $\sigma = 1$:
  r = randn(n) returns an n-by-n matrix containing pseudorandom values drawn from the standard normal distribution. randn(m,n) or randn([m,n]) returns an m-by-n matrix.
- Inverse transform method
  We want to generate n samples from any given cumulative distribution function, say $F_X(x)$. The inverse method requires to generate n samples $u_1, \dots, u_n$ from the uniform distribution $\mathcal{U}(0,1)$ (e.g. by using *rand(n)* function) and then obtain the desired samples as $x_1 = F_X^{-1}(u_1), \dots, x_n = F_X^{-1}(u_n)$, where the function F_X^{-1} is defined as

$$F_X^{-1}(u) = \arg\min_x \{F(x) \geq u, u \in [0,1]\}$$

Proof: We need to show that $Prob(X \leq x) = F_X(x)$, where $X = F_X^{-1}(U)$ and where U satisfies $Prob(U \leq u) = u$. We evaluate: $Prob(X \leq x) = Prob(F_X^{-1}(U) \leq x)$. If the function $F_X(x)$ is continuous, the function $F_X^{-1}(\cdot)$ is the proper inverse function of $F_X(x)$.
$Prob(X \leq x) = Prob(F_X^{-1}(U) \leq x) = Prob(U \leq F_X(x)) = F_X(x)$. The case of discontinuous $F_X(x)$ can be shown to lead to same result, that $F_X^{-1}(U)$ is distributed as $F_X(\cdot)$.