Assignment 4

EEE554

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October 14, 2022

Question 1: *Problem 1*

1a. A is 1, see earlier pages for work

1b. $Z = V/U$, $S_Z = 0 < z < \frac{1}{2}$, see earlier pages for work

Intersection point of curve is $(\frac{1}{1-z}, \frac{1}{1-z} - 1)$

$F_Z(z) = Pr(V \leq uz) = 1 - Pr(V \geq uz)$

$Pr(V \geq uz) = \int_{u=\frac{1}{1-z}}^{2} \int_{v=uz}^{u-1} u + v \quad dvdu, \forall\ 0 < z < \frac{1}{2}$

$Pr(V \geq uz) = \int_{u=\frac{1}{1-z}}^{2} (u^2 - u - u^2 z + \frac{1}{2}u^2 - u + \frac{1}{2} - \frac{u^2 z^2}{2})du, \forall\ 0 < z < \frac{1}{2}$

$Pr(V \geq uz) = \int_{u=\frac{1}{1-z}}^{2} ((1 - z + \frac{1}{2} - \frac{1}{2}z^2)u^2 + -2u^2 + \frac{1}{2})du, \forall\ 0 < z < \frac{1}{2}$

$Pr(V \geq uz) = (\frac{3}{2} - z - \frac{1}{2}z^2)(\frac{1}{3})(2^3 - (\frac{1}{1-z})^3) - (4 - (\frac{1}{1-z})^2) + \frac{1}{2}(2 - \frac{1}{1-z}), \forall\ 0 < z < \frac{1}{2}$

$\Rightarrow F_Z(z) = 1 - (\frac{3}{2} - z - \frac{1}{2}z^2)(\frac{1}{3})(2^3 - (\frac{1}{1-z})^3) - (4 - (\frac{1}{1-z})^2) + \frac{1}{2}(2 - \frac{1}{1-z}), \forall\ 0 < z < \frac{1}{2}$

$$\therefore F_Z(z) = \begin{cases} 0 & z < 0 \\ 1 - (\frac{3}{2} - z - \frac{1}{2}z^2)(\frac{1}{3})(2^3 - (\frac{1}{1-z})^3) - (4 - (\frac{1}{1-z})^2) + \frac{1}{2}(2 - \frac{1}{1-z}) & 0 \leq z < \frac{1}{2} \\ 1 & otherwise \end{cases}$$

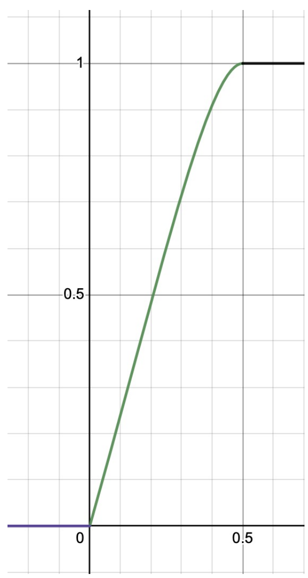

Figure 1: F_Z

1c. $W = U + V$, $S_W = 1 < w < 3$, see earlier pages for work
Intersection point of curve is $(\frac{w+1}{2}, \frac{w+1}{2} - 1)$
$F_W(w) = Pr(V \leq w - u)$

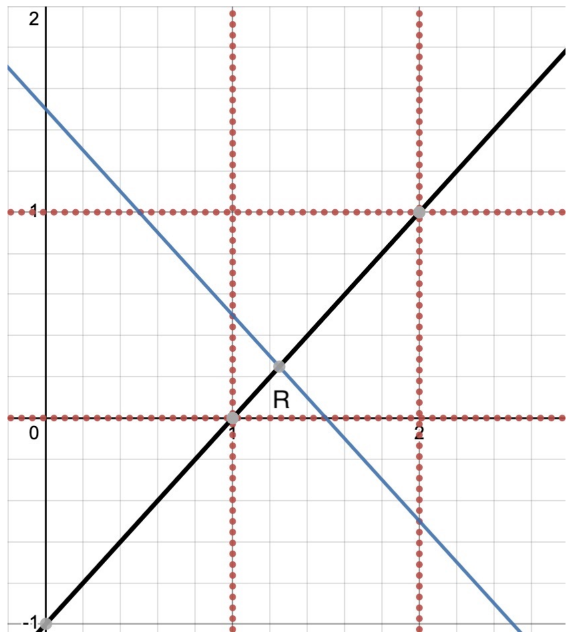

Figure 2: Case 1, blue is the $v = w - u$, black is $v = u - 1$, region R is the region of interest

Case 1: if $1 \leq w < 2$
$F_W(w) = \int_{u=1}^{\frac{w+1}{2}} \int_{v=0}^{u-1} u + v \; dvdu + \int_{u=\frac{w+1}{2}}^{w} \int_{v=0}^{w-u} u + v \; dvdu$, $\forall \; 1 < w < 2$
$F_W(w) = \int_{u=1}^{\frac{w+1}{2}} \frac{3}{2}u^2 - 2u + \frac{1}{2} \; du + \int_{u=\frac{w+1}{2}}^{w} -\frac{1}{2}u^2 + \frac{1}{2}w^2 \; du$, $\forall \; 1 < w < 2$
$F_W(w) = \frac{1}{2}((\frac{w+1}{2})^3 - 1) - ((\frac{w+1}{2})^2 - 1) + \frac{1}{2}(\frac{w+1}{2} - 1)) - \frac{1}{6}(w^3 - (\frac{w+1}{2})^3) + \frac{1}{2}w^2(w - \frac{w+1}{2})$, $\forall 1 < w < 2$

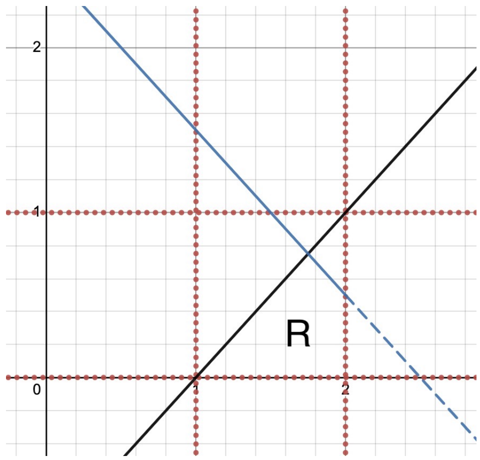

Figure 3: Case 2, blue is the $v = w - u$, black is $v = u - 1$, region R is the region of interest

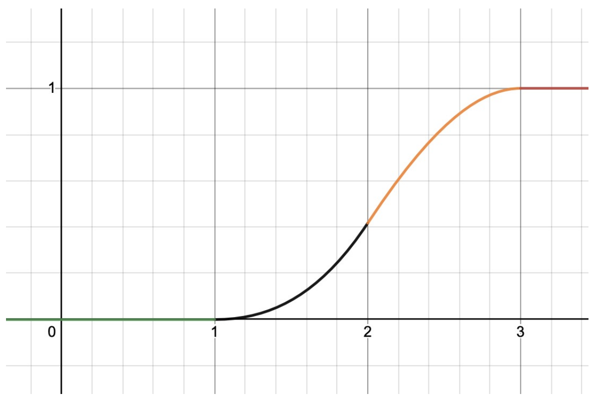

Figure 4: F_W for all w

Case 2: if $2 \leq w < 3$

$F_W(w) = \int_{u=1}^{\frac{w+1}{2}} \int_{v=0}^{u-1} u + v \; dvdu + \int_{\frac{w+1}{2}}^{2} \int_{v=0}^{w-u} u + v \; dvdu, \; \forall \; 2 < w < 3$

$F_W(w) = \int_{u=1}^{\frac{w+1}{2}} \frac{3}{2}u^2 - 2u + \frac{1}{2} \; du + \int_{\frac{w+1}{2}}^{2} -\frac{1}{2}u^2 + \frac{1}{2}w^2 \; du, \; \forall \; 2 < w < 3$

$F_W(w) = \frac{1}{2}((\frac{w+1}{2})^3 - 1) - ((\frac{w+1}{2})^2 - 1) + \frac{1}{2}(\frac{w+1}{2} - 1) - \frac{1}{6}(2^3 - (\frac{w+1}{2})^3) + \frac{1}{2}w^2(2 - \frac{w+1}{2}), \; \forall \; 2 < w < 3$

$\therefore$ Let $k = \frac{w+1}{2}$

$$F_W(w) = \begin{cases} 0 & w < 1 \\ \frac{1}{2}(k^3 - 1) - (k^2 - 1) + \frac{1}{2}(k - 1) + -\frac{1}{6}(w^3 - k^3) + \frac{1}{2}w^2(w - k) & 1 \leq w < 2 \\ \frac{1}{2}(k^3 - 1) - (k^2 - 1) + \frac{1}{2}(k - 1) + -\frac{1}{6}(2^3 - w^3) + \frac{1}{2}w^2(2 - k) & 2 \leq w < 3 \\ 1 & w \geq 3 \end{cases}$$

1d.

$$Z = \frac{V}{U}$$
$$W = V + U$$
$$\Rightarrow U(w,z) = \frac{W}{1+Z}, \text{ and } V(z,w) = \frac{ZW}{1+Z}$$

$$\because S_{uv} = \{\{u|1 \leq u < 2\}, \{v|0 \leq v < 1\}, \{(u,v)|v \leq u-1\}\}$$

$$\therefore S_{wz} = \{\{u(w,z)|1 \leq \frac{w}{1+z} < 2\}, \{v(w,z)|0 \leq \frac{zw}{1+z} < 1\}, \{(u,v)|\frac{zw}{1+z} \leq \frac{w}{1+z} - 1\}\}$$

Question 2: *Problem 2*

2a-2b please see attachment.

2c. Picture please see attachment below. When M is small, the transformation using histogram do not match the expected probability density function. As M grows by couple decades, the probability density function using histogram match the expected probability density function.