

Assignment 4

EEE554

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October 12, 2022

Question 1: Problem 1

1a. A is 1, see earlier pages for work

1b. $Z = V/U$, $S_Z = 0 < z < \frac{1}{2}$, see earlier pages for work

Intersection point of curve is $(\frac{1}{1-z}, \frac{1}{1-z} - 1)$

$$F_Z(z) = Pr(V \leq uz) = 1 - Pr(V \geq uz)$$

$$Pr(V \geq uz) = \int_{u=\frac{1}{1-z}}^2 \int_{v=uz}^{u-1} u + v \quad dvdu, \forall 0 < z < \frac{1}{2}$$

$$Pr(V \geq uz) = \int_{u=\frac{1}{1-z}}^2 (u^2 - u - u^2z + \frac{1}{2}u^2 - u + \frac{1}{2} - \frac{u^2z^2}{2})du, \forall 0 < z < \frac{1}{2}$$

$$Pr(V \geq uz) = \int_{u=\frac{1}{1-z}}^2 ((1 - z + \frac{1}{2} - \frac{1}{2}z^2)u^2 - 2u^2 + \frac{1}{2})du, \forall 0 < z < \frac{1}{2}$$

$$Pr(V \geq uz) = (\frac{3}{2} - z - \frac{1}{2}z^2)(\frac{1}{3})(2^3 - (\frac{1}{1-z})^3) - (4 - (\frac{1}{1-z})^2) + \frac{1}{2}(2 - \frac{1}{1-z}), \forall 0 < z < \frac{1}{2}$$

$$\Rightarrow F_Z(z) = 1 - (\frac{3}{2} - z - \frac{1}{2}z^2)(\frac{1}{3})(2^3 - (\frac{1}{1-z})^3) - (4 - (\frac{1}{1-z})^2) + \frac{1}{2}(2 - \frac{1}{1-z}), \forall 0 < z < \frac{1}{2}$$

$$\therefore F_Z(z) = \begin{cases} 0 & z < 0 \\ 1 - (\frac{3}{2} - z - \frac{1}{2}z^2)(\frac{1}{3})(2^3 - (\frac{1}{1-z})^3) - (4 - (\frac{1}{1-z})^2) + \frac{1}{2}(2 - \frac{1}{1-z}) & 0 \leq z < \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}$$

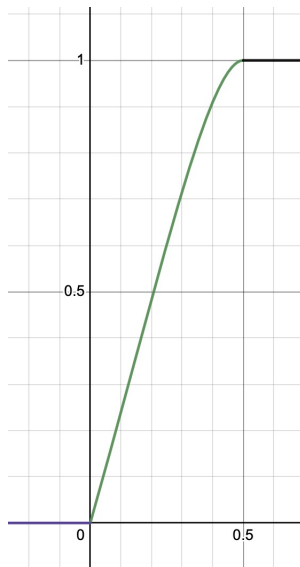


Figure 1: F_Z

1c. $W = U + V$, $S_W = 1 < w < 3$, see earlier pages for work
Intersection point of curve is $(\frac{w+1}{2}, \frac{w+1}{2} - 1)$
 $F_W(w) = Pr(V \leq w - u)$

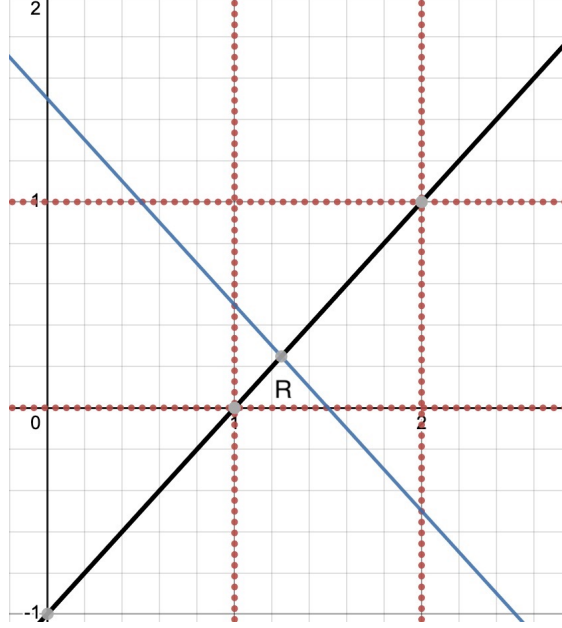


Figure 2: Case 1, blue is the $v = w - u$, black is $v = u - 1$, region R is the region of interest

Case 1: if $1 \leq \frac{w+1}{2} < 2$

$$F_W(w) = \int_{u=1}^{\frac{w+1}{2}} \int_{v=0}^{u-1} u + v \, dv \, du + \int_{u=\frac{w+1}{2}}^w \int_{v=0}^{w-u} u + v \, dv \, du, \forall 1 < w < 2$$

$$F_W(w) = \int_{u=1}^{\frac{w+1}{2}} \left(\frac{3}{2}u^2 - 2u + \frac{1}{2} \right) du + \int_{u=\frac{w+1}{2}}^w \left(-\frac{1}{2}u^2 + \frac{1}{2}w^2 \right) du, \forall 1 < w < 2$$

$$F_W(w) = \frac{1}{2} \left(\left(\frac{w+1}{2} \right)^3 - 1 \right) - \left(\left(\frac{w+1}{2} \right)^2 - 1 \right) + \frac{1}{2} \left(\frac{w+1}{2} - 1 \right) - \frac{1}{6} \left(w^3 - \left(\frac{w+1}{2} \right)^3 \right) + \frac{1}{2} w^2 \left(w - \frac{w+1}{2} \right),$$

$\forall 1 < w < 2$

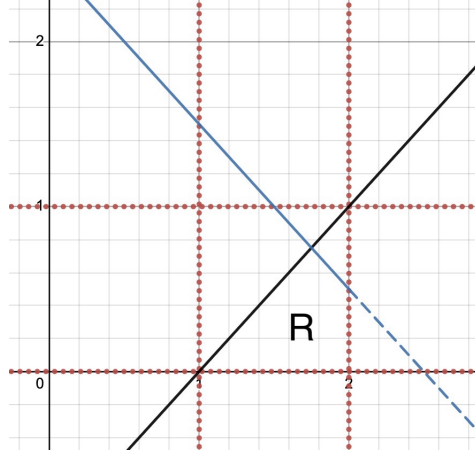


Figure 3: Case 2, blue is the $v = w - u$, black is $v = u - 1$, region R is the region of interest

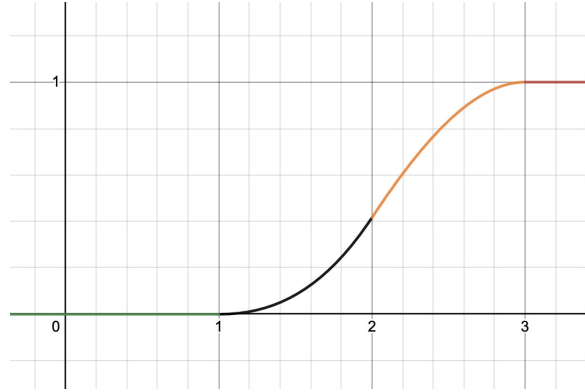


Figure 4: F_W for all w

Case 2: if $2 \leq \frac{w+1}{2} < 3$

$$F_W(w) = \int_{u=1}^{\frac{w+1}{2}} \int_{v=0}^{u-1} u + v \, dv \, du + \int_{\frac{w+1}{2}}^2 \int_{v=0}^{w-u} u + v \, dv \, du, \forall 2 < w < 3$$

$$F_W(w) = \int_{u=1}^{\frac{w+1}{2}} \left(\frac{3}{2}u^2 - 2u + \frac{1}{2} \right) du + \int_{\frac{w+1}{2}}^2 \left(-\frac{1}{2}u^2 + \frac{1}{2}w^2 \right) du, \forall 2 < w < 3$$

$$F_W(w) = \frac{1}{2} \left(\left(\frac{w+1}{2} \right)^3 - 1 \right) - \left(\left(\frac{w+1}{2} \right)^2 - 1 \right) + \frac{1}{2} \left(\frac{w+1}{2} - 1 \right) - \frac{1}{6} \left(2^3 - \left(\frac{w+1}{2} \right)^3 \right) + \frac{1}{2} w^2 \left(2 - \frac{w+1}{2} \right), \forall 2 < w < 3$$

\therefore Let $k = \frac{w+1}{2}$

$$F_Z(z) = \begin{cases} 0 & w < 1 \\ \frac{1}{2}(k^3 - 1) - (k^2 - 1) + \frac{1}{2}(k - 1) + -\frac{1}{6}(w^3 - k^3) + \frac{1}{2}w^2(w - k) & 1 \leq w < 2 \\ \frac{1}{2}(k^3 - 1) - (k^2 - 1) + \frac{1}{2}(k - 1) + -\frac{1}{6}(2^3 - w^3) + \frac{1}{2}w^2(2 - k) & 2 \leq w < 3 \\ 1 & w \geq 3 \end{cases}$$

1d.

$$\begin{aligned}
Z &= \frac{V}{U} \\
W &= V + U \\
\Rightarrow U(w, z) &= \frac{W}{1 + Z}, \text{ and } V(z, w) = \frac{ZW}{1 + Z} \\
\therefore S_{uv} &= \{\{u | 1 \leq u < 2\}, \{v | 0 \leq v < 1\}, \{(u, v) | v \leq u - 1\}\}
\end{aligned}$$

$$\therefore S_{wz} = \{\{u(w, z) | 1 \leq \frac{w}{1+z} < 2\}, \{v(w, z) | 0 \leq \frac{zw}{1+z} < 1\}, \{(u, v) | \frac{zw}{1+z} \leq \frac{w}{1+z} - 1\}\}$$

Question 2: *Problem 2*

2a-2b please see attachment.

2c. Picture please see attachment below. When M is small, the transformation using histogram do not match the expected probability density function. As M grows by couple decades, the probability density function using histogram match the expected probability density function.