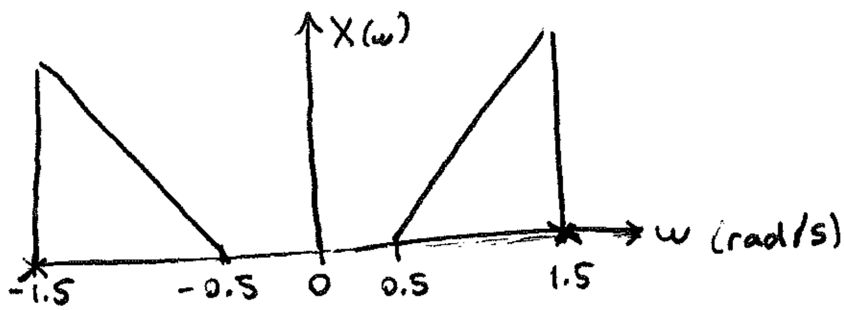
11/2/20

Today's Lecture

- Changing the sampling rate
  downsampling and upsampling
- Polyphase and multirate signal processing

Readings : 6 Sampling , 11 Upsampling , downsampling

Question on Piazza:

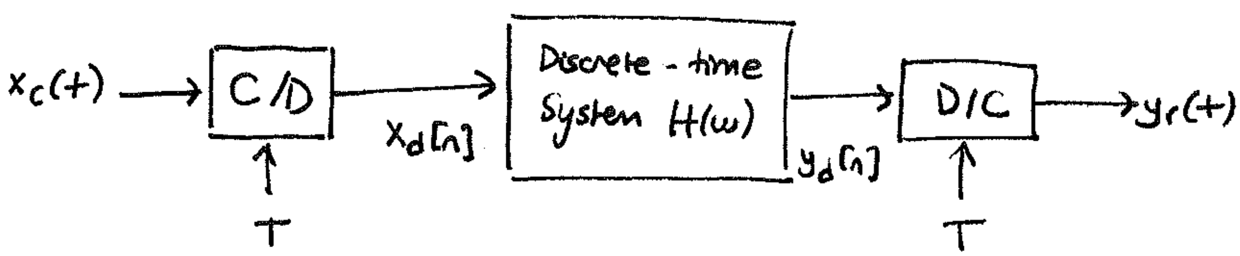


What is the Nyquist rate?

twice the maximum frequency $\Rightarrow$ 3 rad/s

$\frac{3}{2\pi}$ Hz

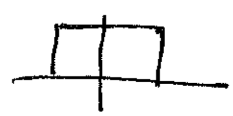
Last lecture (Discrete-time processing of continuous time signals)



$X_c(\omega)$ CT

$$X_s(\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(\omega - k\omega_s) \qquad \omega_s = \frac{2\pi}{T} \qquad \text{CT}$$

$$X_d(\omega) = X_s\left(\frac{\omega}{T}\right) \qquad \text{DT}$$



$$Y_d(\omega) = X_d(\omega)\, H(\omega) \qquad \text{DT}$$

$$Y_r(\omega) = Y_d(\omega T)\underbrace{H_r(\omega)}_{\text{CT filter}} \longleftrightarrow y_r(t) = \sum_{n=-\infty}^{+\infty} y_d[n]\,\mathrm{sinc}\left(\frac{t-nT}{T}\cdot\pi\right)$$

Changing the Sampling Rate

$$x[n] = x_c(nT) \longrightarrow x'[n] = x_c(nT') \qquad T' \neq T$$

$$x_c(nT) \rightarrow x[n] \rightarrow x_c(nT')$$

Sample at T'

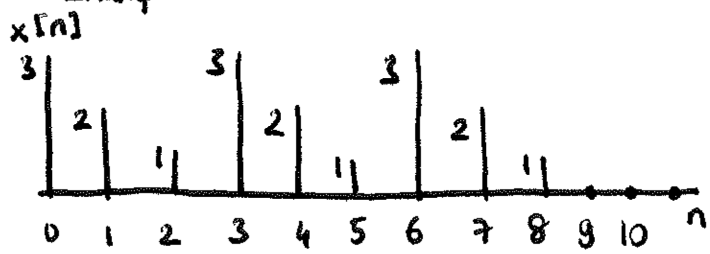
1. Downsampling by an Integer Factor M

downsampled

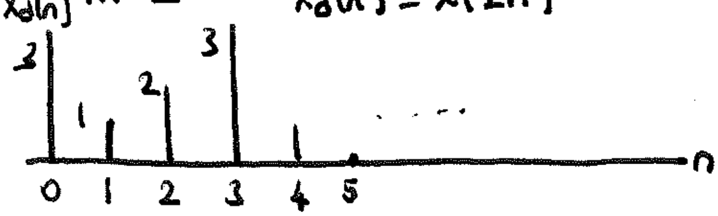
$$x_d[n] = x[nM] = x_c(nMT)$$

$$\underset{T}{x[n]} \longrightarrow \boxed{\downarrow M} \longrightarrow \underset{MT}{x_d[n] = x[nM]} \text{ (Compressor)}$$

Example

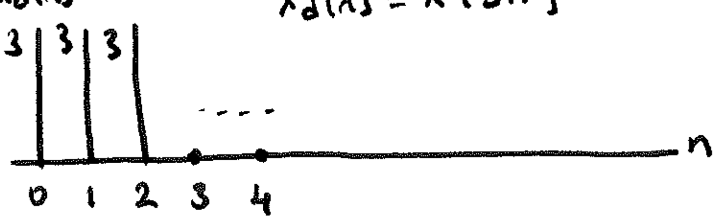


$x_d[n]$ $M=2$ $\qquad x_d[n] = x[2n]$



$M=3$

$x_d[n]$ $\qquad x_d[n] = x[3n]$



Question: When can we perfectly recover $x_c(t)$ from $x_d[n]$?

Answer: if the original signal is sampled at M times the Nyquist rate.

Frequency domain

$$\omega_s = \frac{2\pi}{T}$$

$$* \quad X(\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right) = X_s\left(\frac{\omega}{T}\right) \qquad \text{DTFT for } x[n]$$

Downsampled signal $x_d[n]$

$$X_d(\omega) = \frac{1}{T'}\sum_{l=-\infty}^{+\infty} X_c\left(\frac{\omega}{T'} - \frac{2\pi l}{T'}\right) \quad \text{where } T' = MT$$

$$= \frac{1}{MT}\sum_{l=-\infty}^{+\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi l}{MT}\right)$$

Note that from $*$

$$X\left(\frac{\omega - 2\pi m}{M}\right) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi(m+kM)}{MT}\right), \quad m = 0, \dots, M-1$$

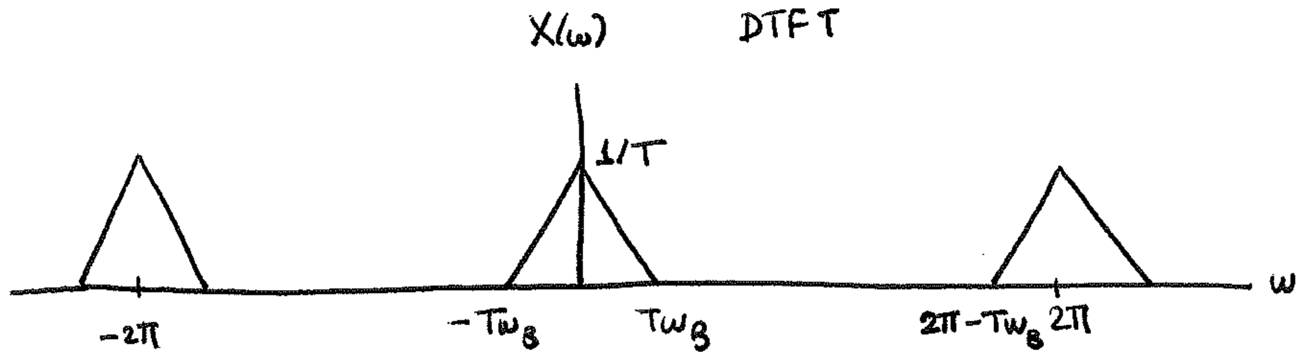
$$\frac{1}{M}\sum_{m=0}^{M-1} X\left(\frac{\omega - 2\pi m}{M}\right) = \frac{1}{M}\sum_{m=0}^{M-1}\frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi(m+kM)}{MT}\right)$$

let $l = m + kM$

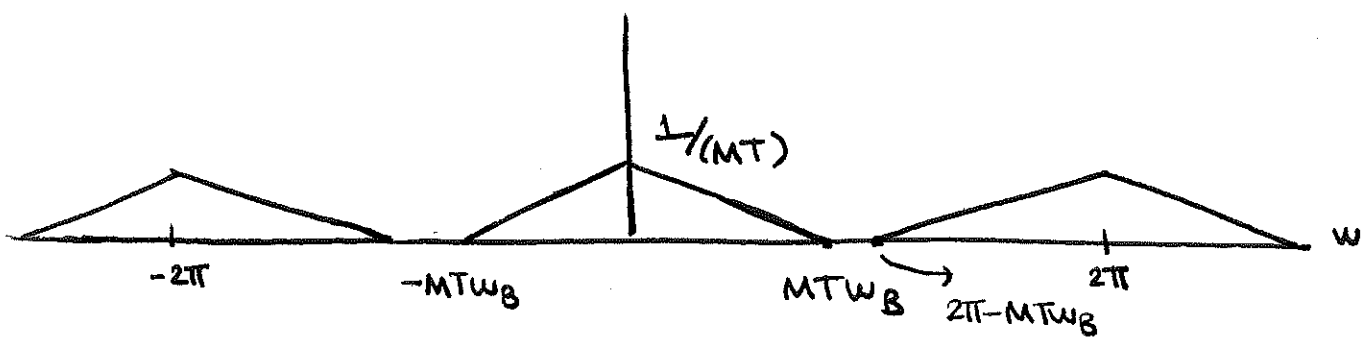
$$= \frac{1}{MT}\sum_{l=-\infty}^{+\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi l}{MT}\right) = X_d(\omega)$$

$$\boxed{X_d(\omega) = \frac{1}{M}\sum_{m=0}^{M-1} X\left(\frac{\omega - 2\pi m}{M}\right)}$$

* Independent copies of $X(\omega)$ scaled by M on frequency axis, shifted by $\pm 2\pi$, $\pm 4\pi$, ...



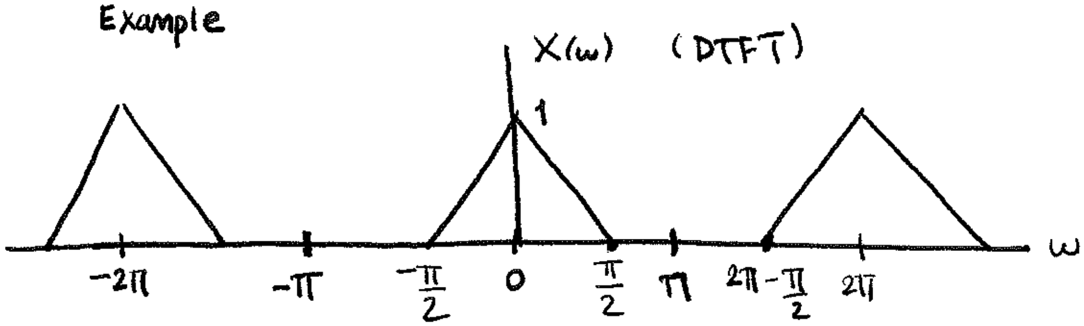
$$X_d(\omega) = \frac{1}{M}\sum_{m=0}^{M-1} X\left(\frac{\omega - 2\pi m}{M}\right)$$



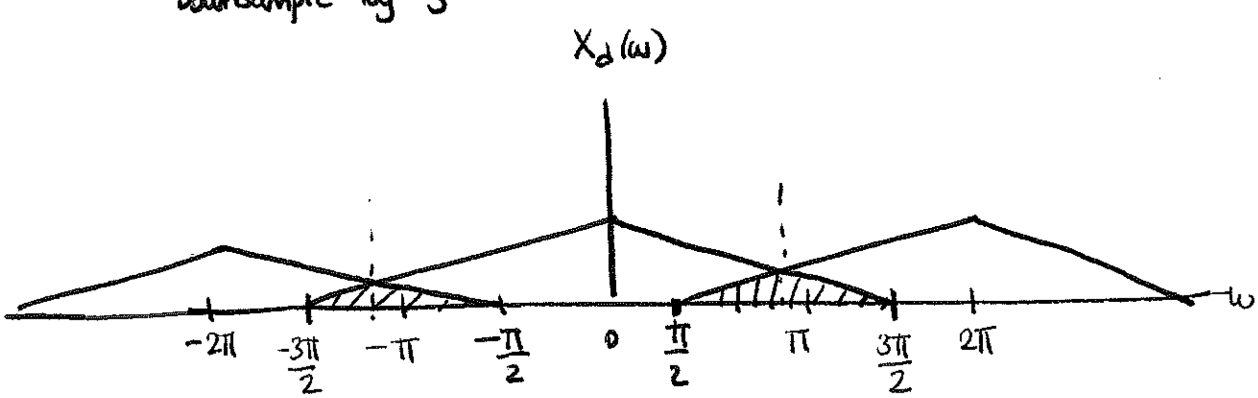
Shrink & spread

Aliasing can occur.

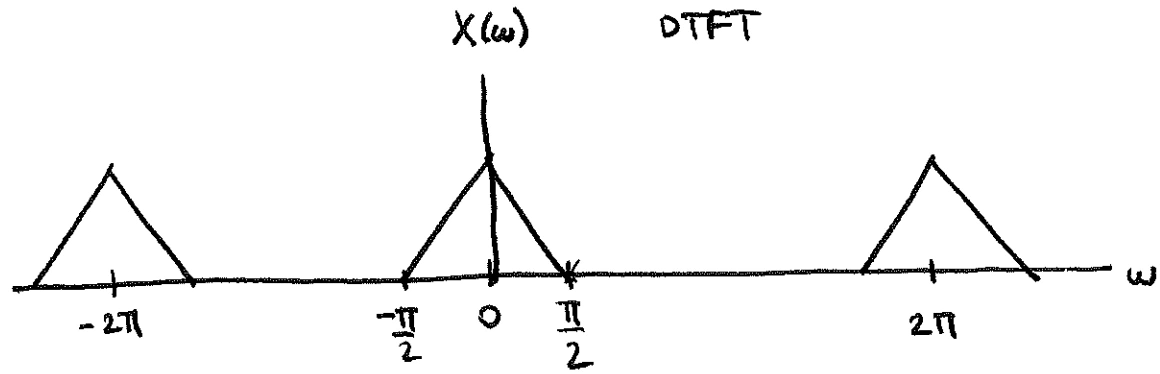
Example



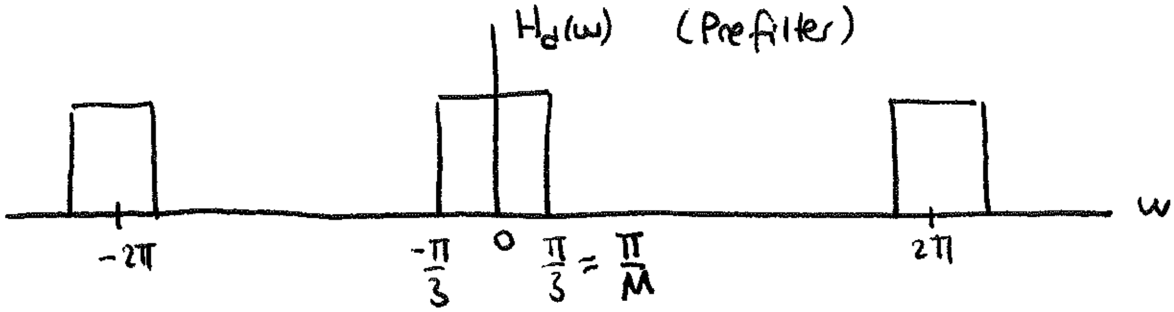
Downsample by 3

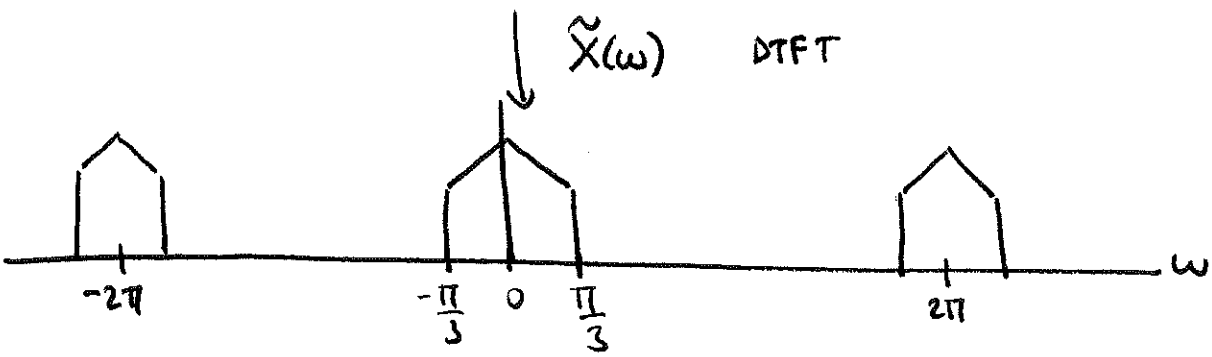


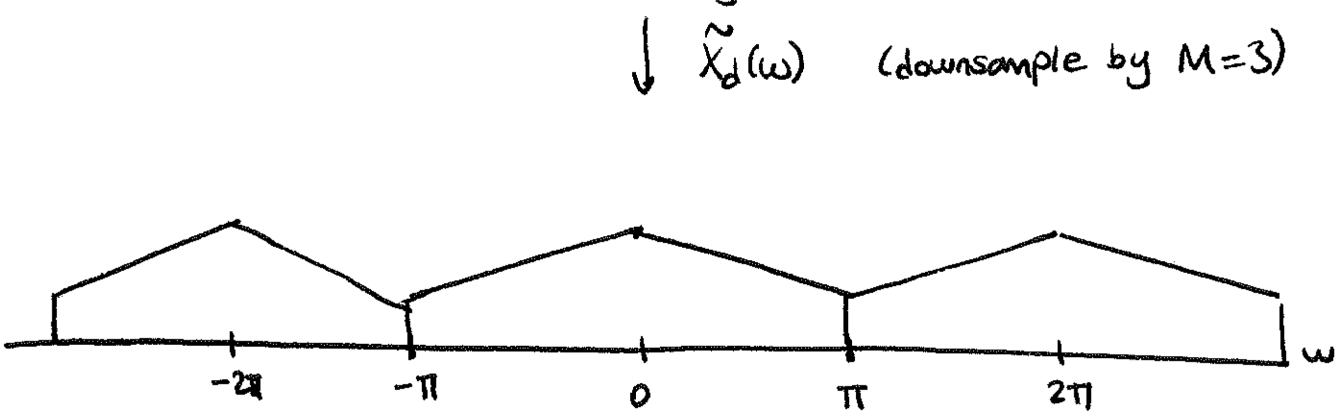
To prevent aliasing, we prefilter by a low pass filter with cutoff $\omega_c = \frac{\pi}{M}$.



We want to downsample $X(\omega)$ by a factor of $M=3$.







The downsampled signal no longer represents the original signal $X(\omega)$.

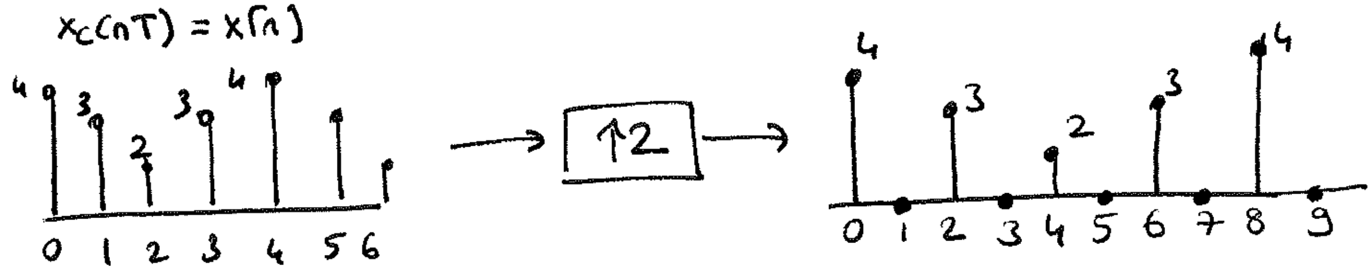
2. Upsampling by an integer factor

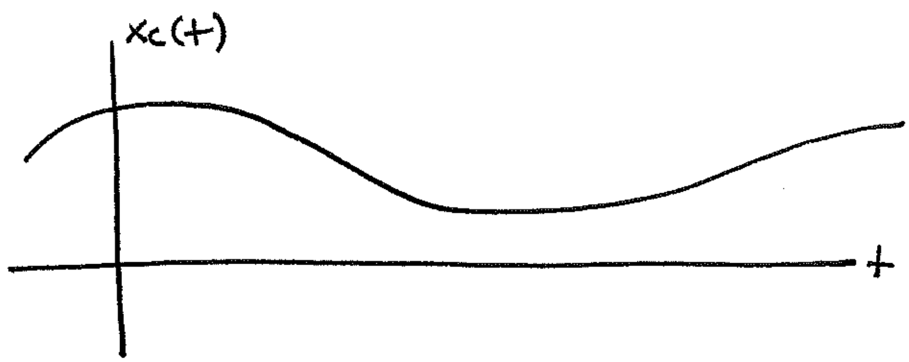
$$x[n] = x_c(nT) \longrightarrow x_i[n] = x_c\left(\frac{nT}{L}\right)$$

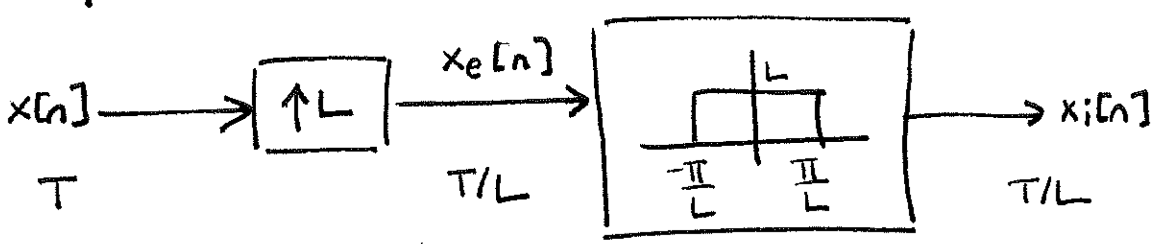
interpolation

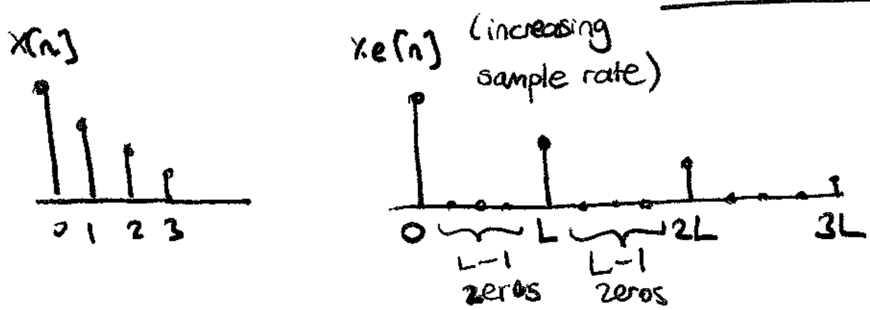
$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_e[n] \quad \text{(expander)}$$

$x_c(nT) = x[n]$









$$x_e[n] = \begin{cases} x\left[\frac{n}{L}\right] & , \quad n = 0, \pm L, \pm 2L, \dots \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-kL]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

$$= x[n] * \delta[n]$$

Frequency domain

$$X_e(\omega) = \sum_{n=-\infty}^{+\infty} x_e[n]\,e^{-j\omega n} \qquad \text{DTFT}$$

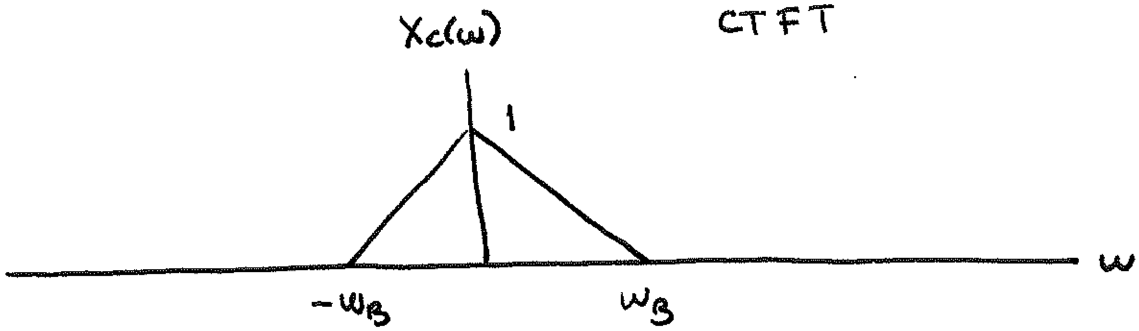
$$= \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-kL]\,e^{-j\omega n}$$

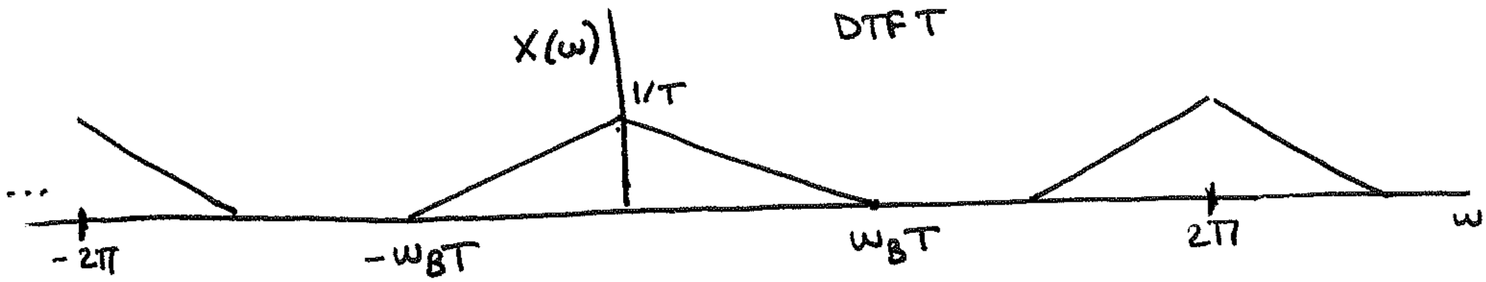
$$= \sum_{k=-\infty}^{+\infty} \underbrace{\sum_{n=-\infty}^{+\infty} x[k]\,\delta[n-kL]\,e^{-j\omega n}}_{\substack{x[k] \quad \text{if } n = kL \\ 0 \quad \text{otherwise}}}$$

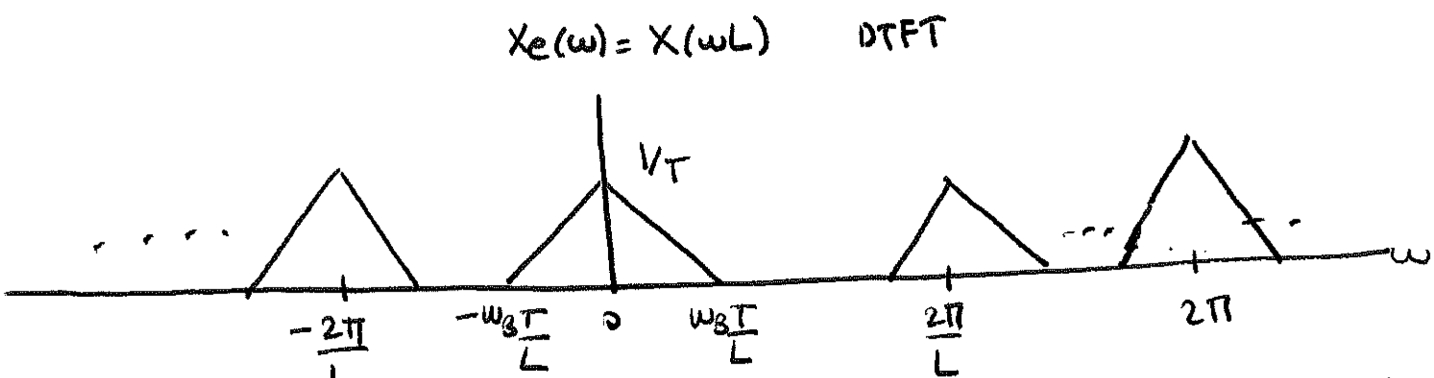
$$= \sum_{k=-\infty}^{+\infty} x[k]\,e^{-j\omega kL} = X(\omega L) \qquad \text{DTFT}$$

$$\boxed{X_e(\omega) = X(\omega L)}$$

* The DTFT of the expanded version is frequency axis scaled by L.

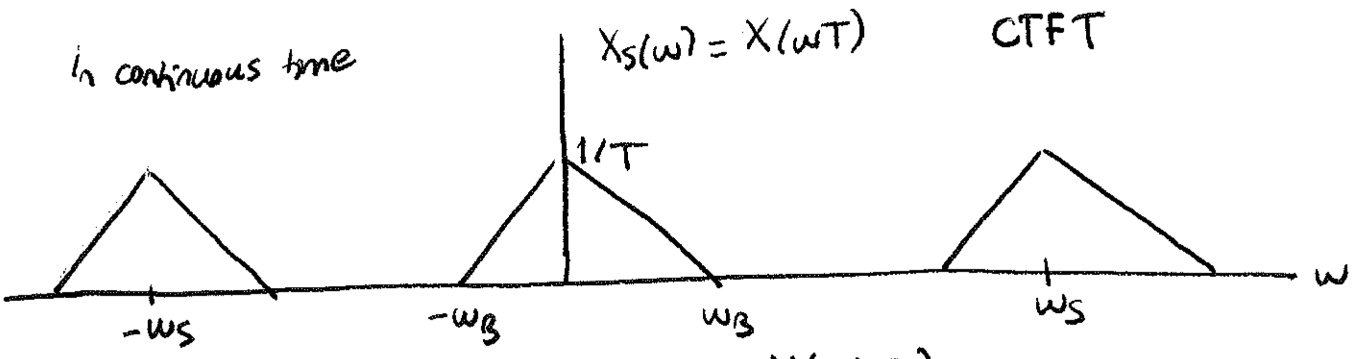


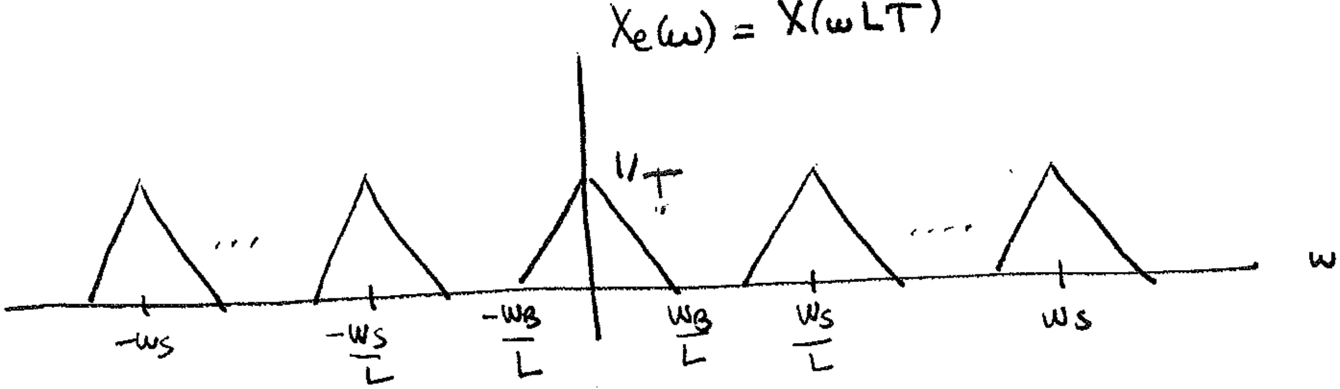




Note that $X_e(\omega)$ is still periodic with 2π. However, there are L copies of X within 2π.

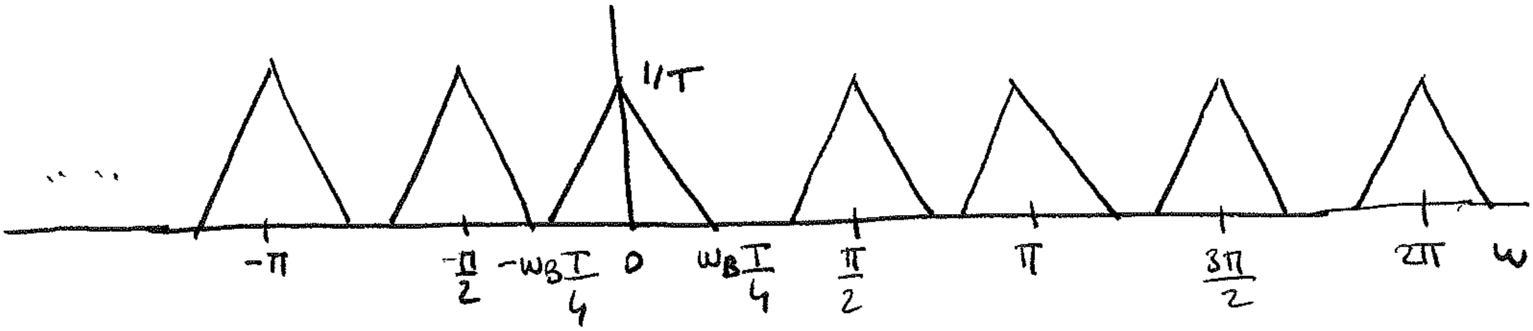
In continuous time



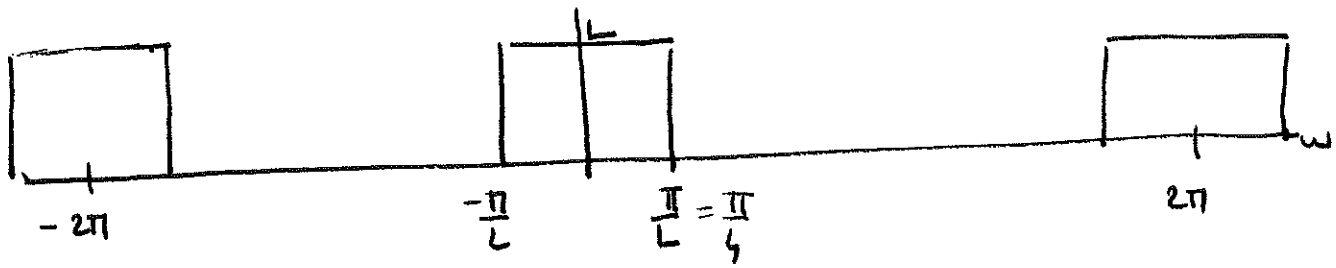


Example $L = 4$ (upsampling) $\omega_S = \frac{2\pi}{T}$

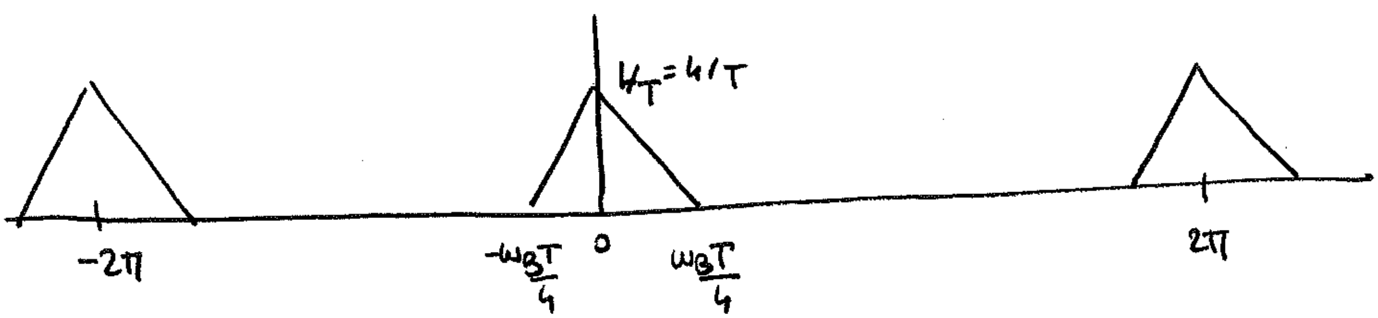
$X_e(\omega)$ (DTFT)



• $H_i(\omega)$



$= X_i(\omega)$



* If we had sampled the original signal at $\frac{T}{L}$, it would have looked like this.

* Provided that $x[n] = x_c(nT)$ was sampled without aliasing, we can upsample by any factor L.