Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2020

Final.

December 16, 2020, 11:30-2:30 PM

Show all work for full credit.

- Closed book, closed notes.
- 1 two-sided and 1 one-sided (or 3 one-sided) crib sheet is allowed.
- Electronic devices (including calculators) are not permitted.
- Reduce expressions involving exponentials, sines, and cosines as much as possible.
- If the sinc function is needed, use the definition $\text{sinc}(x) = \frac{\sin x}{x}$.
- Geometric series formula: $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1.$
- Finite sum formula: $\sum_{n=M}^{N-1} a^n = \frac{a^M - a^N}{1-a}, \quad a \neq 1.$
- When in doubt, show your work.

Good luck!

1		**40**
2		**40**
3		**40**
4		**40**
5		**40**
6		**50**
Total		**250**

Append the below phrase into your handwritten solutions and upload a single pdf file that includes your handwritten crib sheets to Gradescope.

I am aware of the Academic Integrity policy. I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

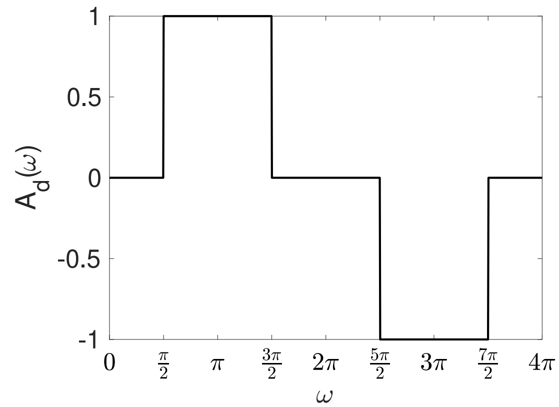
Name

Signature

1. (40 points.) **Linear phase filters.** We want to design a linear phase filter that approximates the ideal amplitude response

$$|A_d(\omega)| = \begin{cases} 1, & \omega \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \text{ and } \omega \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right], \\ 0, & \text{otherwise,} \end{cases}$$

which is as shown below.

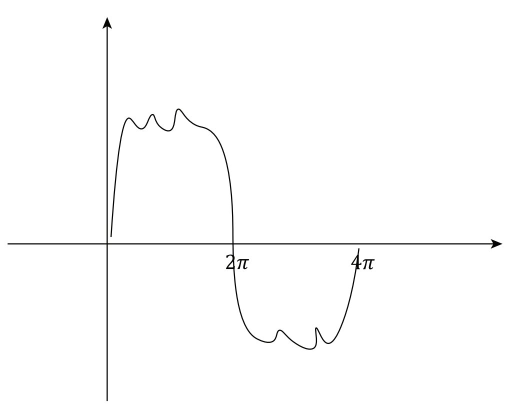


(a) (4 points.) What is the type of the filter shown above (low-pass, high-pass, band-pass)?
Solution: band-pass

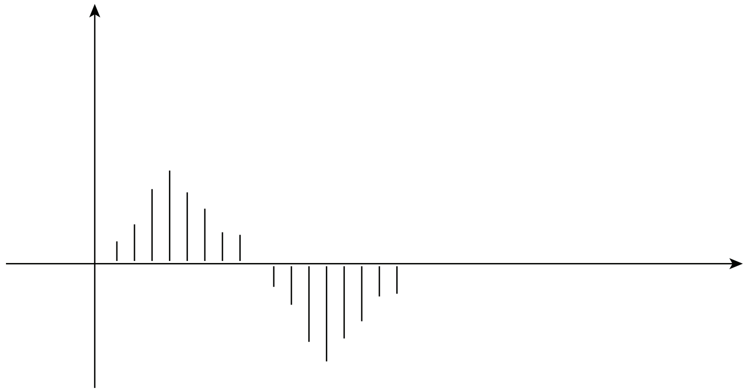
(b) (4 points.) What is the type of the FIR digital filter $H(\omega)$ with linear phase we should design?
Solution: type IV

(c) (8 points.) We design the filter using the frequency-sampling method by taking $N = 16$ samples equally-spaced by $\frac{2\pi}{N}$ starting at $\omega = 0$. Determine the length 16 vector A_d of samples of desired amplitude response.
Solution: {0,0,0,0,1,1,1,1,1,1,1,1,1,0,0,0}

(d) (8 points.) Roughly sketch the amplitude response $A(\omega)$. You do not need to be very precise here. Is the response symmetric about $\omega = 0$? Is it symmetric about $\omega = \pi$? Explain your reasoning.
Solution:



(e) (8 points.) Determine and sketch the filter coefficients $h[n]$.
Solution:



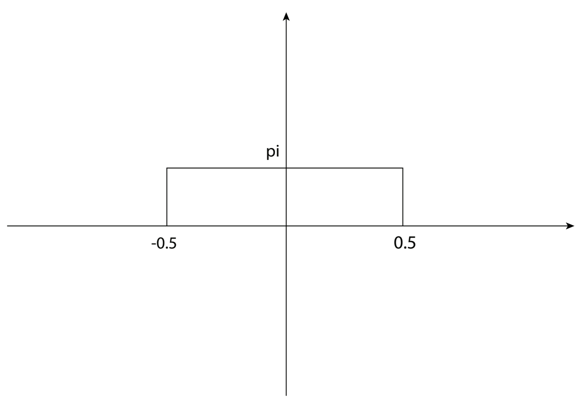
(f) (8 points.) What is the value of the frequency response of the designed filter at $\omega = \frac{4\pi}{5}$, i.e., $H\left(\frac{4\pi}{5}\right)$? Determine both the magnitude and the phase response.
Solution: magnitude is 1, phase is -5.5π

2. (40 points.) **Sampling.** Consider the continuous time signal

$$y(t) = \text{sinc}\left(\frac{t}{2}\right).$$

(a) (2 points.) Plot $Y(\omega)$, the Fourier transform of $y(t)$.
Solution:

$$y(t) = \text{sinc}(t/2) = \pi \cdot \frac{\text{sinc}(t/2)}{\pi} \rightarrow \begin{cases} \pi & -\frac{1}{2} \leq \omega \leq \frac{1}{2} \\ 0 & \text{else} \end{cases}$$



(b) (2 points.) Is $y(t)$ bandlimited? Explain.
Solution: Yes, because $\omega = 0$ for $1/2 < \omega < -1/2$.

(c) (4 points.) Determine the Nyquist rate (the minimum sampling rate to prevent aliasing) for the following continuous time signal

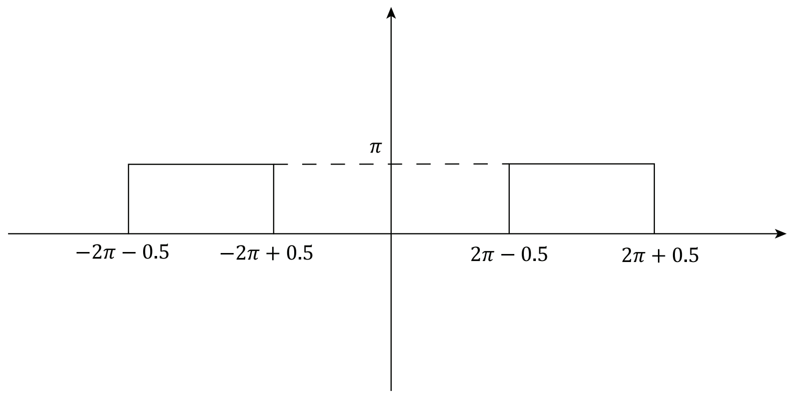
$$x(t) = y(t)\cos(\omega_c t), \quad \text{where} \quad \omega_c = 2\pi.$$

Solution: $4\pi + 1$
The Fourier transform of $\cos(\omega_c t)$ is equivalent to $\pi\left[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\right]$. Multiplication in time domain is convolution in the frequency domain. This duplicates the rectangle wave centered at -2π and 2π, hence the maximum frequency is $2\pi + 1/2$. By the Nyquist rate, the minimum sampling rate would be $2(2\pi + 1/2) = 4\pi + 1$

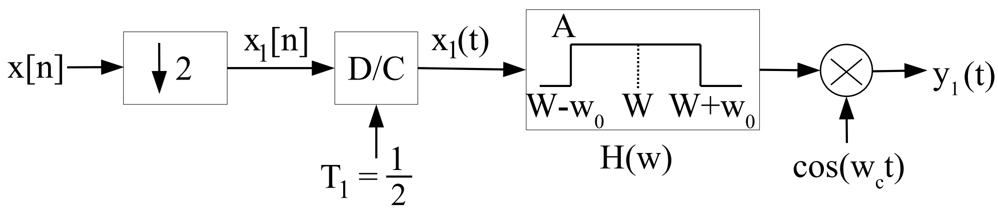
(d) (2 points.) Plot $X(\omega)$, the Fourier transform of $x(t)$.
Solution:

Lorem ipsum

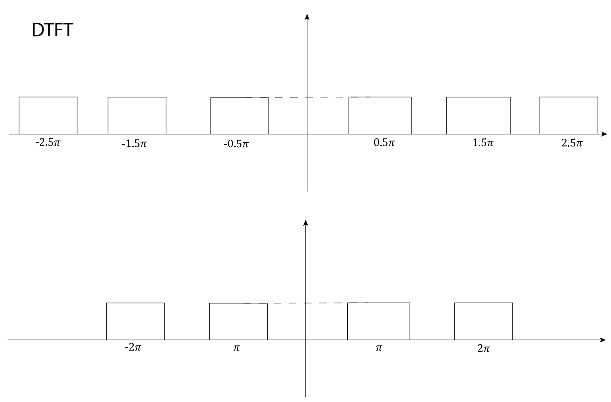
2



Not symmetric about $\omega = 0$, symmetric about $\omega = \pi$



(e) (5 points.) Assume that $x[n]$ is obtained from $x(t)$ by sampling it at a period $T = \frac{1}{4}$. Next $x[n]$ is downsampled by a factor of $M = 2$ to obtain $x_1[n]$ as shown above. Plot the Fourier transform of $x_1[n]$ in the frequency range $\omega \in [-4\pi, 4\pi]$.
Solution:

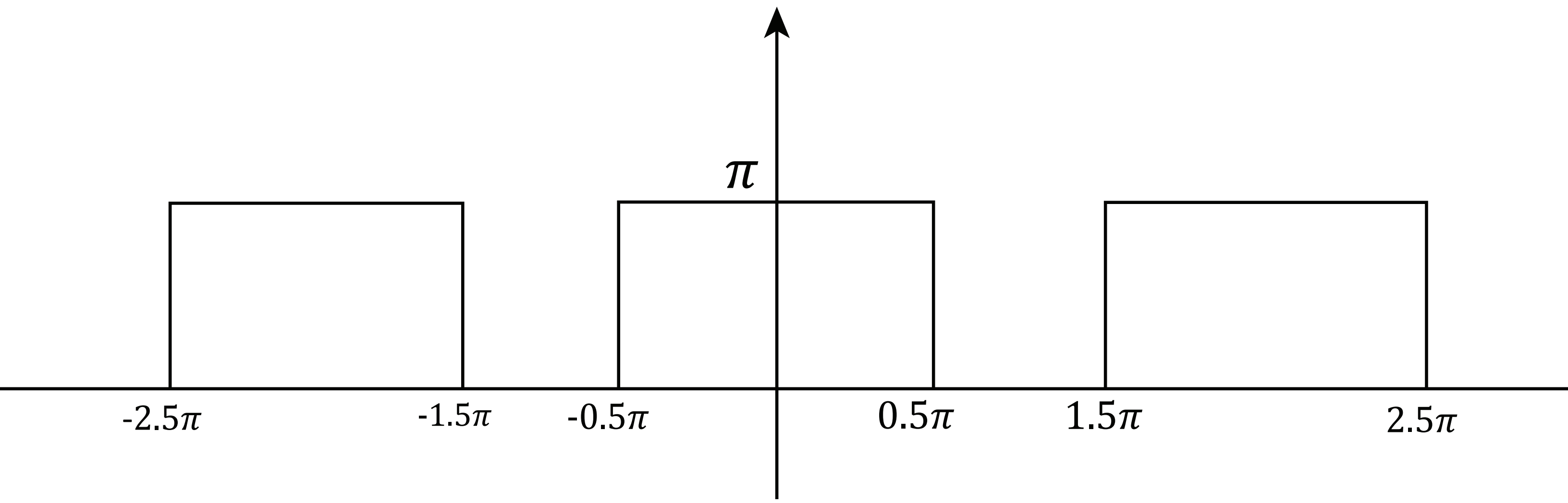


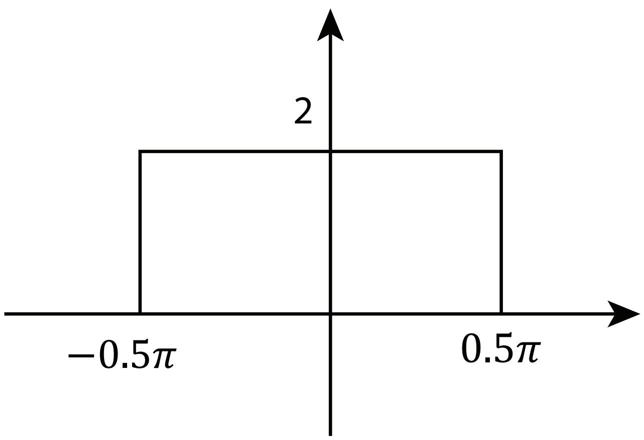
(f) (5 points.) Next we take the $N = 64$ point DFT of $x_1[n]$. What is the frequency sample at $k = 2$, $X_1[2]$?
Solution: 0

(g) (8 points.) As shown in the block diagram, $x_1[n]$ is converted from discrete sequence to continuous with $T_1 = \frac{1}{2}$, and a new signal is reconstructed with an ideal band-pass filter with cutoff frequencies $W - \omega_0$ and $W + \omega_0$ and gain $A = 2$ as shown above.
Determine the frequencies W, ω_0 such that the reconstructed signal $y_1(t)$ resembles $x(t)$. Is it possible to perfectly recover the original signal $x(t)$? Explain why.
Solution: $W = 2\pi$, $\omega_0 = 0.5$

(h) (4 points.) In the block diagram, why is the output of the band-pass filter multiplied with $\cos(\omega_c t)$?
Solution: To make sure the sample is within the same frequency domain

(i) (8 points.) Plot the Fourier transforms $X_1(\omega)$ and $Y_1(\omega)$ of the continuous time signals.
Solution:





3. (40 points.) **Digital filter design.** The transfer function of a digital filter with the region of convergence (ROC) $|z| > 2\sqrt{2}$ is given as

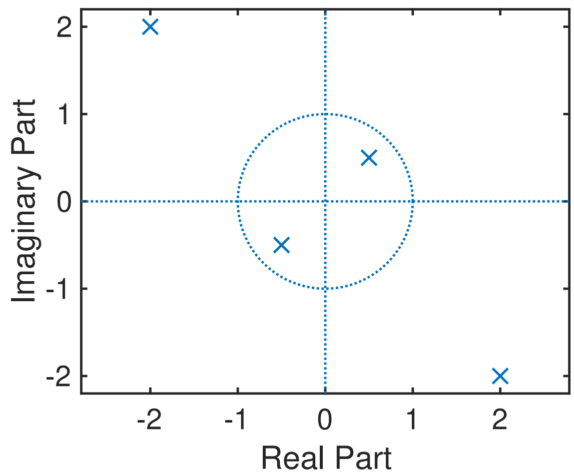
$$H(z) = \frac{1}{(z^2 + 8j)(z^2 - 0.5j)}, \quad \text{where} \quad |z| > 2\sqrt{2}.$$

(a) (4 points.) Determine the locations of the poles and zeros of the filter.
Solution: Poles: -0.5-0.5j, 0.5+0.5j, -2+2j, 2-2j

(b) (4 points.) Plot the pole zero diagram for $H(z)$. Indicate the ROC.
Solution: $ROC : |z| > 2\sqrt{2}$

(c) (4 points.) Is $h[n]$ stable? Is it causal?
Solution: Causal (because the ROC is outwards), not stable (because the unit circle is not included in the ROC.)

(d) (8 points.) Determine $h[n]$ by computing the inverse z-transform of $H(z)$. You can refer to the tables.
Solution:

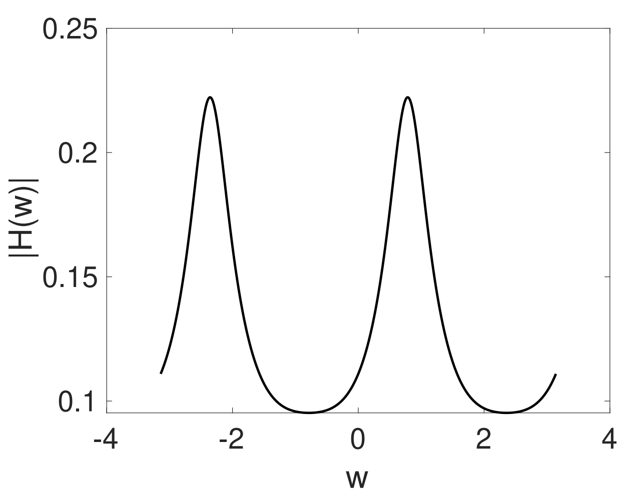


The final result follows from applying a partial fraction along with the time shift property:

$$h[n] = \frac{2}{17} j((8j)^{\frac{n-2}{2}} - (-0.5j)^{\frac{n-2}{2}}) \cos\left(\frac{\pi}{2}(n-2)\right) u[n-2].$$

(e) (5 points.) Estimate the magnitude response $|H(\omega)|$. Determine the type of the filter (e.g., low-pass, band-pass, high-pass, etc.).

Solution: The magnitude response characteristics is shown below. Note the asymmetry about the y axis. We accepted low-pass, high-pass and band-pass answers as long as you provided a justification for the figure you inferred. Note also that y axis never hits 0.



(f) (5 points.) Determine the locations of the poles of the time reversed filter $h[-n]$.

Solution: For the time reversed filter the poles are inverted about the unit circle. Computing the inverses we obtain the following set of poles: $-0.25 - 0.25j$, $0.25 + 0.25j$, $1 - j$, $-1 + j$

(g) (5 points.) Is $h[-n]$ stable? Is it causal?

Solution: not stable (because it does not contain the unit circle), not causal (because the ROC is inwards)

(h) (5 points.) Determine whether the given filter is a linear phase FIR filter. Explain your reasoning.

Solution: It is not linear phase (it is not FIR either). A linear phase is always stable and all the poles are located at the origin.

4. (40 points.) **Adaptive filtering.** Consider the autoregressive (AR) process generated by the difference equation

$$x[n] = 0.6x[n-1] - 0.4x[n-2] + v[n]$$

where $\{x[n]\}$ is a wide sense stationary (WSS) process, and $v[n]$ is a white noise process with variance $\sigma_v^2 = 1$. Let

$$r_x[l] = E[x[n]x[n-l]]$$

be the autocorrelation function of the process $\{x[n]\}$ where "E" denotes the statistical expectation. For your reference, the inverse of the autocorrelation matrix is

$$\mathbf{R}^{-1} = \begin{bmatrix} 1 & -0.6 & 0.4 \\ -0.6 & 1.2 & -0.6 \\ 0.4 & -0.6 & 0.1 \end{bmatrix}.$$

(a) (10 points.) Use the Yule-Walker equations to solve for the values of $r_x[0]$, $r_x[1]$, and $r_x[2]$.

(b) (10 points.) Determine the coefficients of the optimum length $N = 3$ linear predictor that minimizes the mean square error.

(c) (8 points.) Determine the minimum mean square error (MMSE).

(d) (12 points.) Now assume that there is $w[n]$ which is a white noise process with variance $\sigma_w^2 = 1$. Assume that the sequences $\{v[n]\}$ and $\{w[n]\}$ are uncorrelated. Then the observed process

$$x[n] + w[n]$$

is an autoregressive–moving-average (ARMA) model.

Determine the coefficients of the numerator polynomial (MA) component in the transfer function.

Solution:

(a) Multiply both sides by $x[n-l]$:

$$x[n]x[n-l] - 0.6x[n-1]x[n-l] + 0.4x[n-2]x[n-l] = v[n]x[n-l].$$

Take the statistical expectation of above equation to generate a system of equations for $l = 1, 2$:

$$r_x[1] - 0.6r_x[0] + 0.4r_x[1] = E[v[n]x[n-1]] = 0$$
$$r_x[2] - 0.6r_x[1] + 0.4r_x[0] = E[v[n]x[n-2]] = 0.$$

Hence, we can rewrite the above equations in matrix form:

$$\begin{bmatrix} r_x[0] & r_x[1] \\ r_X[1] & r_x[0] \end{bmatrix} \begin{bmatrix} -0.6 \\ 0.4 \end{bmatrix} = - \begin{bmatrix} r_x[1] \\ r_x[2] \end{bmatrix}$$

Yule-Walker equation with $l = 0$:

$$r_x[0] - 0.6r_x[1] + 0.4r_x[2] = E[v[n]x[n]]$$

where

$$E[v[n]x[n]] = E[v[n](0.6x[n-1] - 0.4x[n-2] + v[n])] = E[v^2[n]] = \sigma_v^2 = 1$$

Therefore, $r_x[0] - 0.6r_x[1] + 0.4r_x[2] = 1$. Combining this with $r_x[2] = 0.6r_x[1] - 0.4r_x[0]$ (from the 2nd line of matrix equation) $r_x[0] - 0.6r_x[1] + 0.4(0.6r_x[1] - 0.4r_x[0]) = 1$. Hence, $0.84r_x[0] - 0.36r_x[1] = 1$. We have the following relation $0.84r_x[0] - 0.36 \cdot 0.6\frac{r_x[0]}{1.4} = 1$ using the first line of matrix equation we have $1.4r_x[1] = 0.6r_x[0]$. Hence, solving for the above set of equations we obtain $r_x[0] = 1/(0.84 -$

$0.36*0.6/1.4) = 1.458$, $r_x[1] = 0.625$, and $r_x[2] = -0.2075$. The inverse of the auto-correlation matrix is as follows,

$$\mathbf{R}^{-1} = \begin{bmatrix} 1.458 & 0.625 & -0.2075 \\ 0.625 & 1.458 & 0.625 \\ -0.2075 & 0.625 & 1.458 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -0.6 & 0.4 \\ -0.6 & 1.2 & -0.6 \\ 0.4 & -0.6 & 0.1 \end{bmatrix}$$

(b) Note that the desired output signal is $d[n] = x[n]$. Hence, the cross correlation vector entries are given as $p[-k] = E[d[n]x[n-k] = r_x[k]$ for $k = 0,1,2$. The optimal MMSE estimator is given by the vector $\hat{\mathbf{h}} = \mathbf{R}^{-1}\mathbf{p}$. Note that $\mathbf{p} = [1.458, 0.625, -0.2075]^\intercal$ and $\mathbf{R}^{-1}$ is given above. The final result is $\hat{\mathbf{h}} = [1,0,0]^T$.

(c) The MMSE is given as

$$\sigma_d^2 - \mathbf{p}^\intercal \mathbf{R}^{-1}\mathbf{p} = e = 0$$

where we used $\sigma_d^2 = E[x[n]^2] = r_x[0] = 1.458$.

(d) Note that for the ARMA model

$$x[n] - 0.6x[n-1] + 0.4x[n-2] = v[n] + w[n] - 0.6w[n-1] + 0.4w[n-2].$$

Using this relation we can compute $r_x[0] - 0.6r_x[1] + 0.4r_x[2] = E[(v[n] + w[n] - 0.6w[n-1] + 0.4w[n-2])(v[n] + w[n] - 0.6w[n-1] + 0.4w[n-2] + 0.6x[n-1] - 0.4x[n-2])] = \sigma_v^2 + \sigma_w^2(1 + 0.6^2 + 0.4^2) = 2.52$.

5. (40 points.) **Linear minimum mean-square error estimation.** Consider a system such that we observe the following random process

$$y[n] = Ax[n] + v[n], \quad n = 0,\ldots,N-1$$

where $x[n]$ is a known input sequence, A is a random variable with zero mean $E[A] = 0$ and variance $E[A^2] = \sigma_A^2$. The process $v[n]$ is an additive white noise process with variance σ_v^2. You can assume that A and $v[n]$ are uncorrelated.

We devise a filter to estimate A, and the estimate of A is given by

$$\hat{A} = \sum_{n=0}^{N-1} h[n]y[n]$$

which minimizes the mean square error

$$e = E[(A - \hat{A})^2].$$

Determine the coefficients of h that achieves our objective. Your final result needs to be given as a function of the parameters σ_A^2, σ_v^2 and $x[n]$.

Solution: The partial derivative of the error needs to satisfy

$$\frac{\partial e}{\partial h[i]} = 0$$

which implies that

$$E\left[2\left(A - \sum_{n=0}^{N-1} h[n]y[n]\right)y[i]\right] = 0, \quad i = 1,\ldots,N-1.$$

Therefore, we infer that

$$E\left[Ay[i]\right] = E\left[\sum_{n=0}^{N-1} h[n]y[n]y[i]\right], \quad i = 1,\ldots,N-1$$

where

$$E\left[Ay[i]\right] = E\left[A(Ax[n] + v[n])\right] = x[n]E\left[A^2\right] + E\left[Av[n])\right] = \sigma_A^2 x[n] \tag{1}$$

$$
\begin{aligned}
E\left[\sum_{n=0}^{N-1} h[n]y[n]y[i]\right] &= E\left[\sum_{n=0}^{N-1} h[n](Ax[n]+v[n])(Ax[i]+v[i])\right] \\
&= E\left[\sum_{n=0}^{N-1} h[n](A^2x[n]x[i]+Ax[n]v[i]+v[n]Ax[i]+v[n]v[i])\right] \\
&= \sigma_A^2\sum_{n=0}^{N-1} h[n]x[n]x[i]+\sigma_v^2 h[i]. \qquad (2)
\end{aligned}
$$

Combining (1) and (2) we obtain the following relationship

$$\sigma_A^2 x[n] = \sigma_A^2\sum_{n=0}^{N-1} h[n]x[n]x[i]+\sigma_v^2 h[i].$$

Rewriting the above equation in matrix form

$$\sigma_A^2\mathbf{x} = (\sigma_A^2\mathbf{x}\mathbf{x}^\intercal+\sigma_v^2 I)\mathbf{h},$$

where $\mathbf{x}=[x[0],\ldots,x[N-1]]^\intercal$ and $\mathbf{h}=[h[0],\ldots,h[N-1]]^\intercal$ and I is an $N\times N$ identity matrix. The solution of the above equation is given by

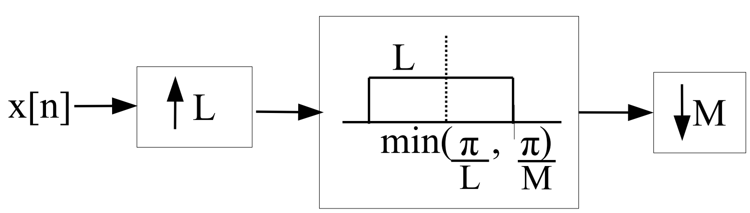
$$\mathbf{h} = (\sigma_A^2\mathbf{x}\mathbf{x}^\intercal+\sigma_v^2 I)^{-1}\sigma_A^2\mathbf{x}.$$

6. (50 points.) The parts of this problem are independent of each other. Read each question carefully. You can refer to the tables to verify your solutions.

   (a) (10 points.) **Speech sampling.** For most phonemes, almost all of the energy of human speech signal $x[n]$ is contained in the 100 Hz–4 kHz range, allowing a sampling rate of 8 kHz. Assume that $x[n]$ is generated with a sampling rate of $f_s = 8{,}000$ Hz. Now we resample $x[n]$ to $3{,}000$ Hz by a non-integer factor of 0.375. We call the resulting signal $y[n]$. Design a scheme (block diagram) with ideal filters that converts $x[n]$ to $y[n]$. Explain the steps are required to accomplish the process.

   Solution:

   This is fractional sampling. Let $M=8$ and $L=3$ and implement the following scheme that causes a net reduction in the sampling rate by a factor of 0.375.



   (b) (6 points.) **Sampling without aliasing.** Determine the Nyquist rate for the continuous time signal

$$z(t) = \begin{cases} 1, & |t|\le T \\ 0, & |t|>T. \end{cases}$$

   Solution:

   Note that $z(t)$ is not bandlimited. Therefore, the Nyquist rate is ∞.

(c) (6 points.) **IIR filter design.** Use the bilinear transformation with $T = 0.1$ to convert the analog filter with transfer function (in the Laplace domain)

$$H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital IIR filter. Compare the locations of the zeros in $H(z)$ with the locations of the zeros obtained by applying the impulse invariance method in the conversion of $H(s)$.

Solution:

Homework 6 question 1. The bilinear transform involves plugging in $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} = 20\frac{1-z^{-1}}{1+z^{-1}}$ into the transfer function

$$H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

That is,

$$\begin{aligned}
H_{\text{bilinear}}(z) &= \frac{20\frac{1-z^{-1}}{1+z^{-1}} + 0.1}{\left(20\frac{1-z^{-1}}{1+z^{-1}} + 0.1\right)^2 + 9} \\
&= \frac{20(1-z^{-1})(1+z^{-1}) + 0.1(1+z^{-1})^2}{(20(1-z^{-1}) + 0.1(1+z^{-1}))^2 + 9(1+z^{-1})^2} \\
&= \frac{20(1-z^{-2}) + 0.1(1+2z^{-1}+z^{-2})}{400(1-z^{-1})^2 + 4(1-z^{-2}) + 9.01(1+z^{-1})^2} \\
&= \frac{20.1 + 0.2z^{-1} - 19.9z^{-2}}{413.01 - 781.98z^{-1} + 405.01z^{-2}} \\
&= \frac{(1+z^{-1})(1-0.995z^{-1})}{1 - 2az^{-1} + (a^2+b^2)z^{-2}}
\end{aligned}$$

where $a = 0.9467$, $b = 0.2905$, i.e. zeros at -1 and 0.995 and poles at $0.9467 \pm j0.2905$. Let $r = \sqrt{a^2+b^2} = 0.99$ and $a = r\cos(\omega_0) = 0.9467$. Solving $\cos(\omega_0) = 0.9467/0.99$ gives $\omega_0 = 0.3$. Therefore, the transfer function for the digital filter is given as

$$H_{\text{bilinear}}(z) = \frac{(1+z^{-1})(1-rz^{-1})}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}, \quad \omega_0 = 0.3, \quad r = 0.99.$$

Note that

$$H(s) = \frac{1}{2}\left[\frac{1}{s + 0.1 - j3} + \frac{1}{s + 0.1 + j3}\right].$$

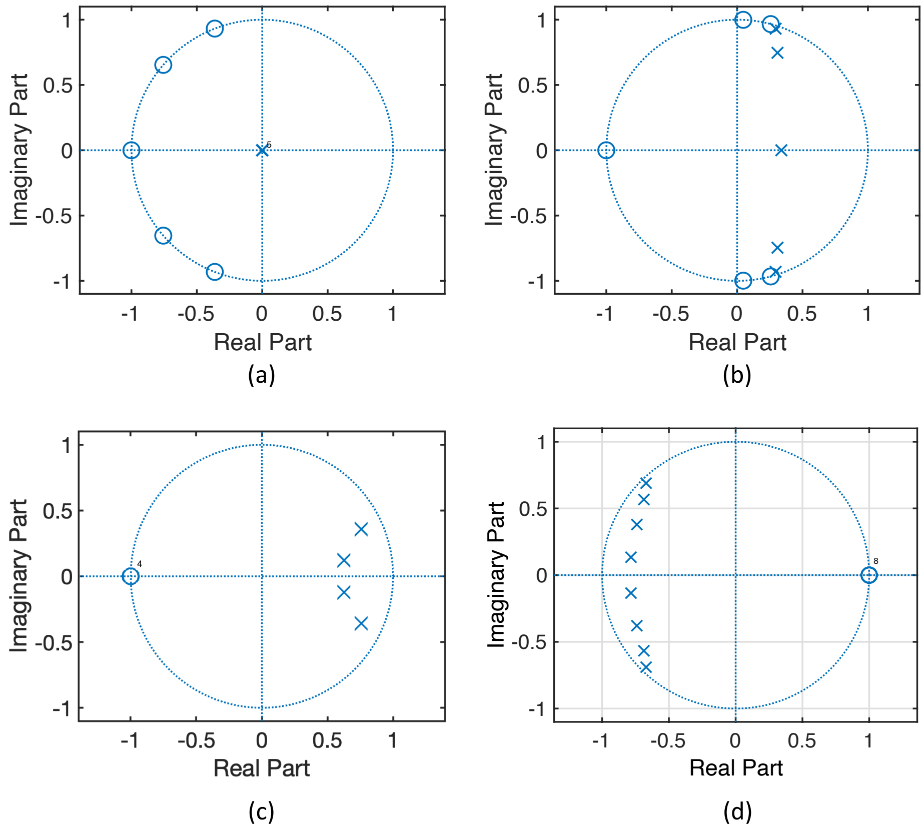
$H(s)$ has two zeros at -0.1 and ∞ and two poles $-0.1 \pm j3$. The matched z-transform for the impulse invariance (which involves plugging in $z = e^{sT}$) maps these into:

$$\begin{aligned}
\tilde{z}_1 &= e^{-0.1T} = e^{-0.01} = 0.99 \\
\tilde{z}_2 &= e^{-\infty T} = 0 \\
\tilde{p}_1 &= e^{(-0.1+j3)T} = 0.99e^{j0.3} \\
\tilde{p}_2 &= 0.99e^{-j0.3}.
\end{aligned}$$

From the impulse invariance method we obtain

$$\begin{aligned}
H_{\text{impulse invariance}}(z) &= \frac{1}{2}\left[\frac{1}{1 - e^{-0.1T}e^{j3T}z^{-1}} + \frac{1}{1 - e^{-0.1T}e^{-j3T}z^{-1}}\right] \\
&= \frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}, \quad \omega_0 = 0.3, \quad r = 0.99.
\end{aligned}$$

The poles are the same, but the zero is different.



(d) (12 points.) **Digital filter design.** We provide the z planes for several digital filters below which are either finite impulse response (FIR) or infinite impulse response (IIR). Identify the type of each filter. Based on the pole zero diagrams, justify whether each given filter is

(i) FIR (and/or linear phase) or IIR.

(ii) Low pass/band pass/high pass.

(iii) Determine the filter length if it is FIR.

Solution:

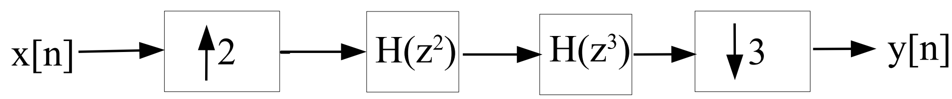
(a) (i) FIR and linear phase), (ii) Low pass (all the zeros are at high frequencies), (iii) $N = 6$.

(b) (i) IIR, (ii) Low pass (Elliptic) (Here, answers like bandpass and bandstop are also accepted.)

(c) (i) IIR, (ii) Low pass (Butterworth) (all poles are at low frequencies and the high frequencies are cancelled by the zeros at π.)

(d) (i) IIR, (ii) High pass (all the poles are at high frequencies and low frequencies are cancelled by the zeros at 0.).

(e) (16 points.) **Equivalent systems.** Consider the system relation shown below.



The DTFTs of $x[n]$ and $h[n]$ are given as

$$X(\omega) = \begin{cases} 1, & \omega \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ 0, & \text{otherwise.} \end{cases} \qquad H(\omega) = \begin{cases} 1, & \omega \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right], \\ 0, & \text{otherwise,} \end{cases}$$

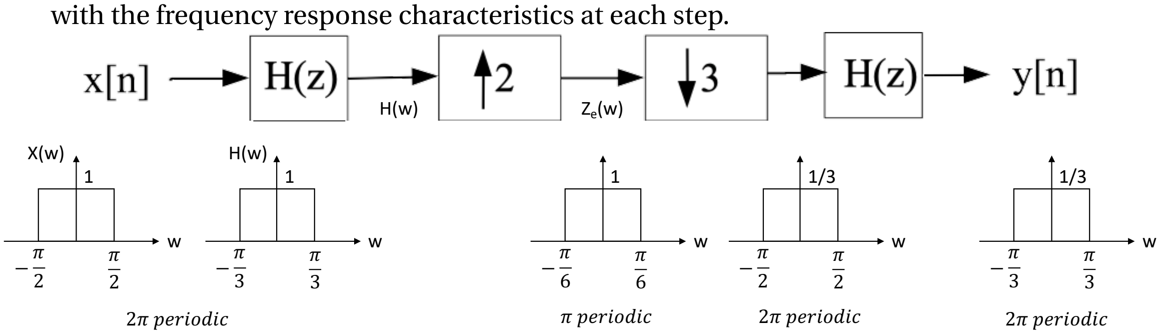
which are 2π periodic.

i. (8 points.) Determine $Y(\omega)$.

ii. (8 points.) As the designer you were told to interpolate only one of the signals $x[n]$ and $y[n]$ using sinc interpolation and the other one using linear interpolation. Which interpolation would you pick for $x[n]$ versus $y[n]$? Explain.

Solution:

To solve this problem we can use the equivalent system representation (Lecture 18) as shown below with the frequency response characteristics at each step.



i. The downsampled signal (by a factor of M) satisfies

$$Z_d(\omega) = \frac{1}{M} \sum_{m=0}^{M-1} Z\left(\frac{\omega - 2\pi m}{M}\right).$$

Similarly, the upsampled signal (by a factor of L) satisfies

$$Z_e(\omega) = Z(\omega L).$$

Therefore, $Y(\omega) = \frac{1}{3} \sum_{m=0}^{2} Z_e\left(\frac{\omega - 2\pi m}{3}\right) H(\omega)$ where $Z_e(\omega) = X(2\omega)H(2\omega)$. Hence,

$$Y(\omega) = \frac{1}{3} \sum_{m=0}^{2} X\left(2\frac{\omega - 2\pi m}{3}\right) H\left(2\frac{\omega - 2\pi m}{3}\right) H(\omega) = \frac{1}{3} \sum_{m=0}^{2} H\left(2\frac{\omega - 2\pi m}{3}\right) H(\omega) = \frac{1}{3} H(\omega).$$

If the analytical steps are hard to follow, you can directly plot the frequency responses as shown in the figure above.

ii. You would pick linear interpolation for the oversampled signal $x[n]$ and sinc interpolation for $y[n]$.