Name: Key

Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2019

Exam #1. Closed book, closed notes.
October 10, 2019, 10:00-11:20 AM

Show all work for full credit.

- Electronic devices are not permitted.
- Reduce expressions involving exponentials, sines, and cosines as much as possible.
- If the sinc function is needed, use the definition $\text{sinc}(x) = \frac{\sin x}{x}$.
- Geometric series formula: $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ if $|a| < 1$.
- When in doubt, show your work.

Good luck!

1		**15**
2		**20**
3		**20**
4		**25**
5		**20**
Total		**100**

ıts.) Consider a linear time invariant (LTI) system with input $x[n]$, impulse response $h[n]$, and .. $y[n]$ as shown below.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

(a) (5 points.) Compute the output for the system given as below. Your answer should be given as a function of $x[n]$, $h[n]$ and $y[n]$ in the most compact form.

$$x[n] \longrightarrow \boxed{h[n-2]} \longrightarrow \ ?\ y[n-2]$$

$x[n] * h[n-2] = x[n-2] * h[n] = y[n-2]$ since we have an LTI system.

(b) (5 points.) Compute the output for the cascade system given as below where $*$ is the convolution operator. Your answer should be given as a function of $x[n]$, $h[n]$ and $y[n]$ in the most compact form.

$$x[n] * h[n+1] \longrightarrow \boxed{\delta[n-2]} \longrightarrow \boxed{\delta[n+1]} \longrightarrow \ ?\ y[n]$$

Remember that $h[n] * \delta[n] = h[n]$.

Consider the cascaded impulse response:

$$h[n+1] * \delta[n-2] * \delta[n+1] = (h[n+1] * \delta[n-2]) * \delta[n+1]$$

$$= h[n-1] * \delta[n+1]$$

$$= h[n]$$

Hence $x[n] * h[n] = y[n]$

(c) (5 points.) The response of the system to the input $x_1[n] = \sin(\pi/2n)$ is $y_1[n] = 2\sin(\pi/2n)$. Determine the response $y_2[n]$ of the system to $x_2[n] = e^{j\frac{\pi}{2}(n-1)}$. **Hint:** Consider the eigenfunctions of discrete time LTI systems.

$$x_2[n] = e^{\frac{j\pi(n-1)}{2}} \longrightarrow \boxed{h[n]} \longrightarrow ?$$

Remember that DT complex exponentials are eigenfunctions of LTI systems.

- From the given information, $H\left(\frac{\pi}{2}\right) = H\left(-\frac{\pi}{2}\right) = 2$ where H is the frequency response. This implies that

$$z[n] = e^{j\frac{\pi}{2}n} \longrightarrow \boxed{h[n]} \longrightarrow H\left(\frac{\pi}{2}\right)e^{j\frac{\pi}{2}n} = 2e^{j\frac{\pi}{2}n} = 2z[n]$$

Note that $x_2[n] = z[n-1] \longrightarrow \boxed{h[n]} \longrightarrow 2z[n-1] = 2e^{j\frac{\pi}{2}(n-1)}$

2. (20 points.) You can use any method you prefer in each case, but simplify your answer as much as possible.

   (a) (5 points.) Find $y[n] = h[n] * x[n]$ for $x[n] = a^n u[-n]$ and $h[n] = u[n+3] - u[n-2]$. Assume $a > 1$ is a real number.

$$y[n] = \sum_{k=-\infty}^{+\infty} \left(u[k+3] - u[k-2]\right) a^{n-k} u[k-n]$$

$$= \sum_{k=-3}^{1} a^{n-k} u[k-n] = a^n \sum_{k=-3}^{1} a^{-k} u[k-n]$$

If $n < -3 \Rightarrow y[n] = a^n \sum_{k=-3}^{1} a^{-k} = a^n \cdot \frac{a^3 - a^{-2}}{1 - a^{-1}}$

If $-3 \le n \le 1 \Rightarrow y[n] = a^n \sum_{k=n}^{1} a^{-k} = a^n \cdot \frac{a^{-n} - a^{-2}}{1 - a^{-1}}$

If $n > 1 \Rightarrow y[n] = 0$

   (b) (5 points.) Compute the output $y[n] = h[n] * x[n]$ for $x[n] = u[n+1]$ and $h[n] = (-2)^n u[n]$.

$$y[n] = \sum_{k=-\infty}^{+\infty} u[k+1] (-2)^{n-k} u[n-k] = \sum_{k=-1}^{n} (-2)^{n-k} = (-2)^n \sum_{k=-1}^{n} \left(-\tfrac{1}{2}\right)^k \quad \text{if } n \ge -1$$

$$= (-2)^n \frac{(-1/2)^{-1} - (-1/2)^{n+1}}{1 - (-1/2)}$$

$$= \frac{2}{3}(-2)^n \left(-2 - (-1/2)^{n+1}\right)$$

$$= \frac{2}{3}(-2)^{n+1} + \frac{1}{3} \quad \text{if } n \ge -1$$

$$y[n] = \frac{1}{3} u[n+1] + \frac{2}{3}(-2)^{n+1} u[n+1]$$

   (c) (10 points.) A discrete time LTI system is described by the following difference equation:

$$y[n] - 0.25y[n-1] + 0.5y[n-2] + y[n-3] = 3x[n] - 3x[n-2].$$

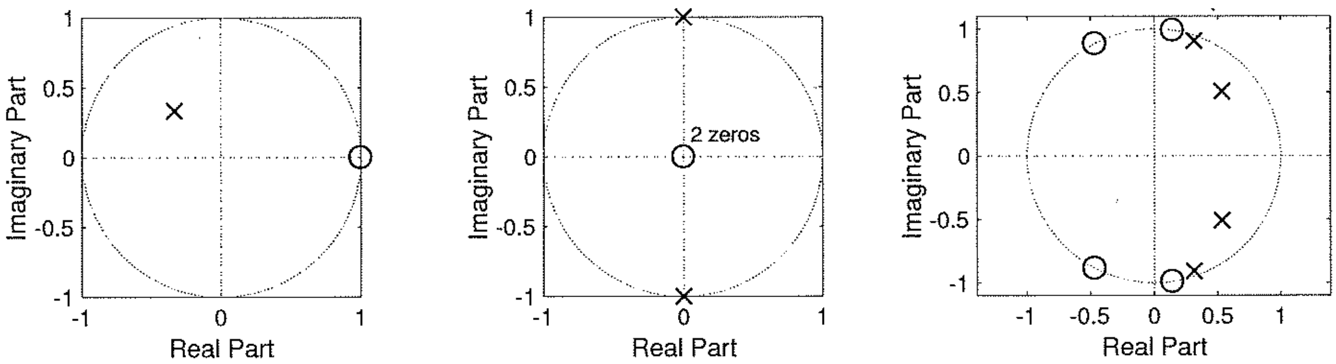
   What is the output of this system $y[n]$ for $x[n] = j^n$? **Hint:** Eigenfunctions.

$$Y(z) - 0.25 z^{-1} Y(z) + 0.5 z^{-2} Y(z) + z^{-3} Y(z) = 3X(z) - 3z^{-2} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 - 3z^{-2}}{1 - 0.25z^{-1} + 0.5z^{-2} + z^{-3}}$$

$$y[n] = j^n H(j) = j^n \frac{3+3}{1 + 0.25j - 0.5 + j} = \frac{6}{1.25j + 0.5} \cdot j^n$$

3. (20 points.) Consider the following pole-zero diagrams (z planes) in Figure 1 corresponding to the z-transforms of the right-sided (i.e. $h[n] = 0$ for $n < 0$) impulse responses $h_1[n]$, $h_2[n]$ and $h_3[n]$.



(a) $H_1(z)$. (b) $H_2(z)$. (c) $H_3(z)$.

Figure 1: Three pole-zero plots.

(a) (10 points.) For each of $h_1[n]$, $h_2[n]$ and $h_3[n]$, discuss whether $h[n]$ can be a real valued impulse response or not. Explain your reasoning. Note that you do not need to compute any explicit formula for the z-transforms.

Only $h_2[n]$ and $h_3[n]$ can be ~~a~~ real valued impulse responses.

- $h_1[n]$ cannot be real valued because it has one complex pole. (i.e. $H_1(z) = \frac{z-1}{z-\alpha} = \frac{1-z^{-1}}{1-\alpha z^{-1}}$ $\Rightarrow$ from Table 3, you can observe that $h_1[n]$ is $\alpha^n u[n] - \alpha^{n-1} u[n-1]$ where α is complex valued)
- $h_2[n]$, $h_3[n]$ can be real valued because their poles are in the form of $a \mp jb$ (i.e. their denominators will look like $z^2 - 2az + a^2 + b^2$. Again, from Table 3, you can observe that $h_2[n], h_3[n]$ are real valued sinusoids.

(b) (5 points.) Determine a z-transform $H_2(z)$ that is consistent with the pole-zero plot in Figure 1-(b) (middle plot). It's OK to leave the result as the product of factors in the numerator and denominator instead of multiplying it all out.

$$H_2(z) = \frac{z^2}{(z-j)(z+j)} = \frac{z^2}{z^2+1} = \frac{1}{1+z^{-2}}$$

From Table 3, we can observe that the z-transform ~~[illegible]~~ resembles the z-transform of a cosine (line 9)

(c) (3 points.) Determine an impulse response $h_2[n]$ that is consistent with the pole-zero plot. Note that your $H_2(z)$ should resemble something in the z-transform table.

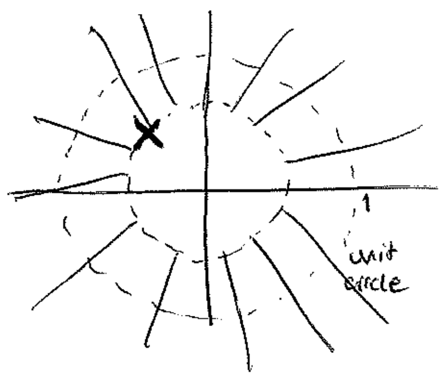
From Table 3, we can infer that

$h_2[n] = (\cos \omega_0 n) u[n]$ where $\cos \omega_0 = 0$

$\Rightarrow \omega_0 = \frac{\pi}{2}$

(d) (2 points.) Is a system which has $h_1[n]$ as impulse response (with the pole-zero plot in Figure 1-(a) (left plot)) stable? Is this system causal?

Since we have a right-sided impulse response, the ROC will look like below.



The ROC contains the unit circle

$\Rightarrow$ system is stable.

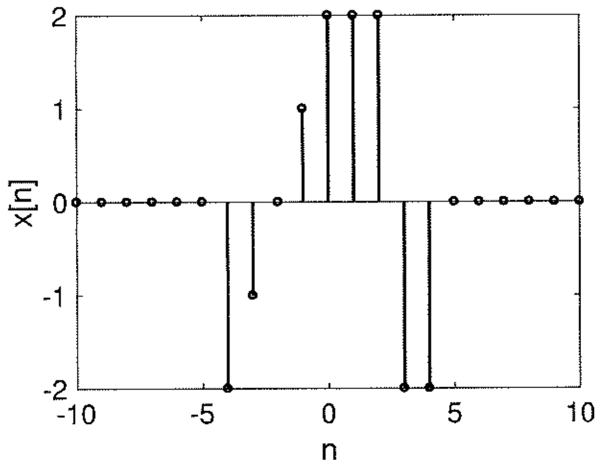
The system is causal because the ROC is towards outwards.

4. (25 points.) Discrete time signal and DTFT pairs

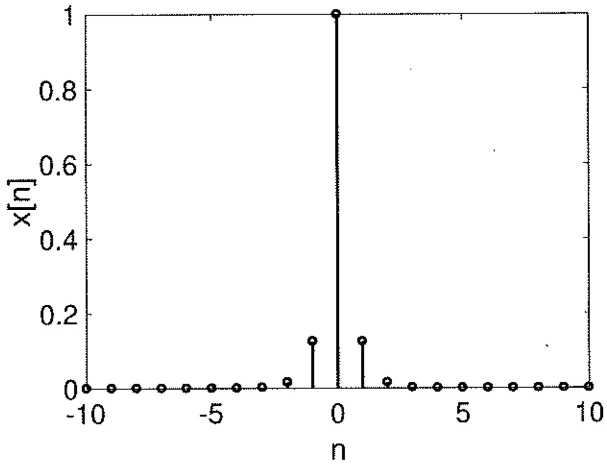
Six different real discrete time signals $x_1[n]$, $x_2[n]$, $x_3[n]$, $x_4[n]$, $x_5[n]$, $x_6[n]$ are shown in Figure 2 in time domain. The magnitudes of the Discrete Time Fourier transforms (DTFTs) of six signals $X_a(\omega)$, $X_b(\omega)$, $X_c(\omega)$, $X_d(\omega)$, $X_e(\omega)$, $X_f(\omega)$ are given in Figure 3 (over one period). Match each signal in Figure 2 with its corresponding DTFT magnitude in Figure 3, and summarize your findings in the following table and then explain your reasoning for each of the six signals below it. The idea here is to use your knowledge of DTFT properties, such as oddness, evenness, Parseval's relation, $\sum_{n=-\infty}^{\infty} x[n] = X(0)$ and $x[0] = \frac{1}{2\pi}\int_{2\pi} X(\omega)d\omega$.

$x[n]$	$X(\omega)$
$x_1[n]$	$X_d(\omega)$
$x_2[n]$	$X_b(\omega)$
$x_3[n]$	$X_c(\omega)$
$x_4[n]$	$X_a(\omega)$
$x_5[n]$	$X_e(\omega)$
$x_6[n]$	$X_f(\omega)$

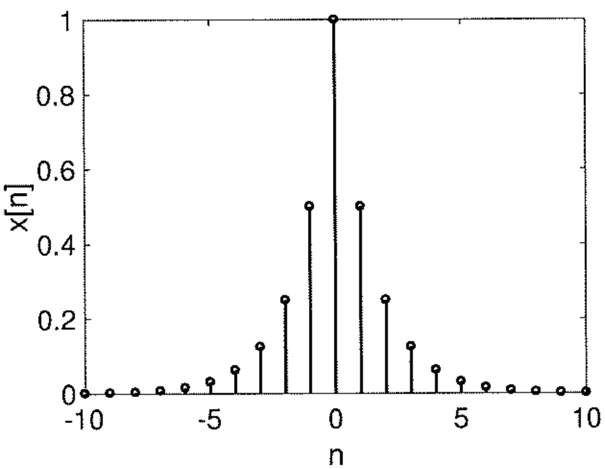
- All signals are real, so their DTFT should be even.
- $x_5[n]$ is a time shifted triangle, so its DTFT should be somehow close to sinc^2 multiplied by a phase shift, but we only focus on its magnitude, so $X_e(\omega)$.
- $x_1[n]$ has a mean equals to zero, so its DTFT should be zero at $\omega = 0$, so it is $X_d(\omega)$.
- $x_3[n]$ is wide in time-domain, so its DTFT should be narrow in frequency domain, so it is $X_c(\omega)$.
- $x_2[n]$ is narrow in time-domain, so its DTFT should be wide in frequency domain, so it is $X_b(\omega)$.
- $x_4[n]$ is close to $x_1[n]$, but is not zero-mean. So its DTFT should be $X_a(\omega)$.
- $x_6[n]$ is a rectangular pulse, so its DTFT should be close to sinc, which is $X_f(\omega)$.



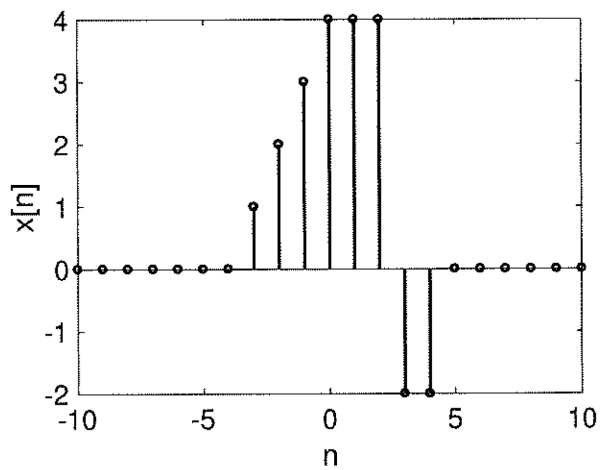
(a) $x_1[n]$.



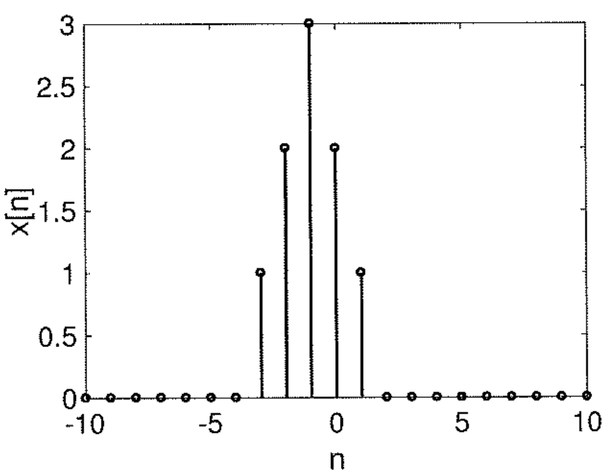
(b) $x_2[n]$.



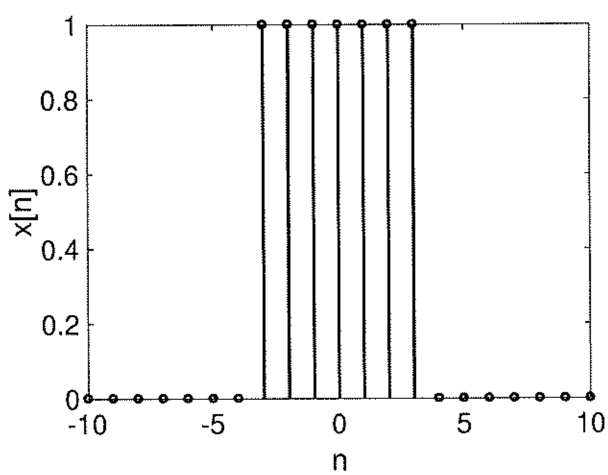
(c) $x_3[n]$.



(d) $x_4[n]$.

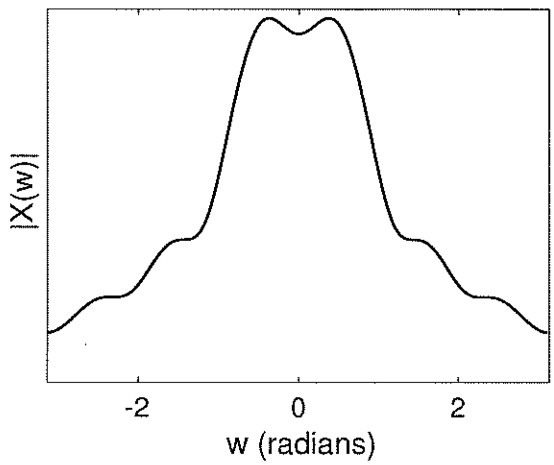


(e) $x_5[n]$.

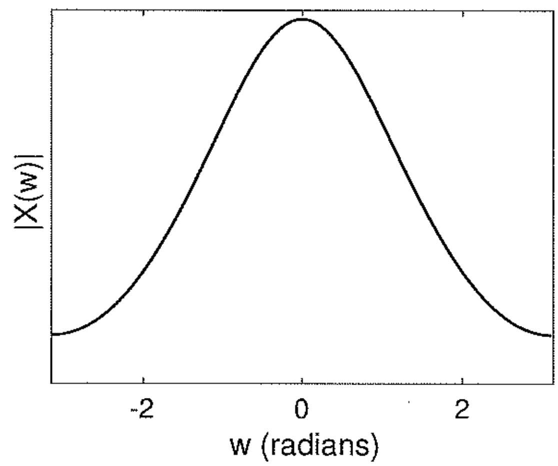


(f) $x_6[n]$.

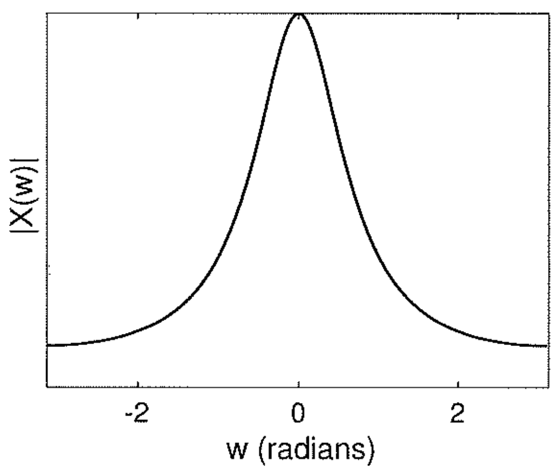
Figure 2: Time domain.



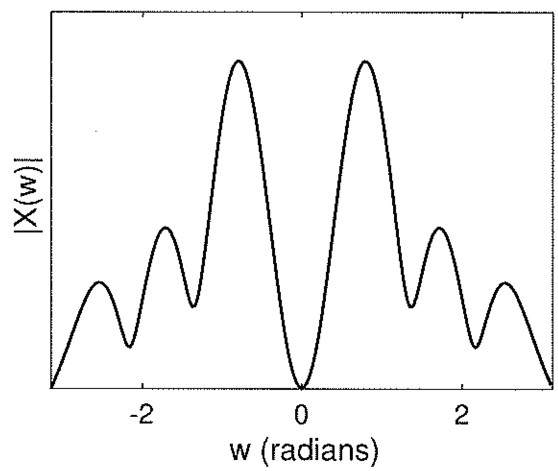
(a) $X_a(\omega)$.



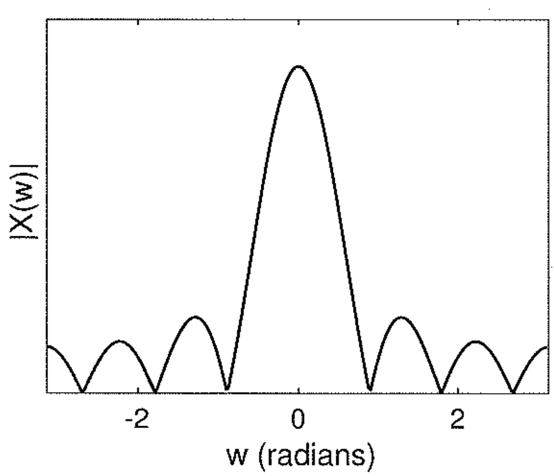
(b) $X_b(\omega)$.



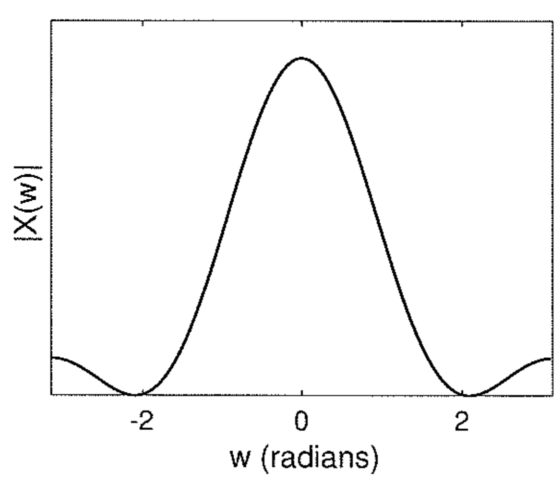
(c) $X_c(\omega)$.



(d) $X_d(\omega)$.



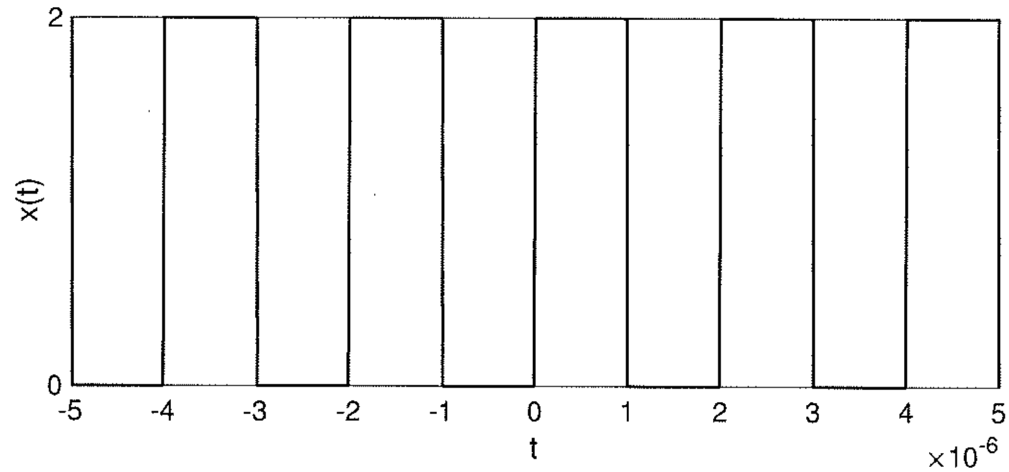
(e) $X_e(\omega)$.



(f) $X_f(\omega)$.

Figure 3: Frequency domain.

5. (20 points.) We have a clock signal $x(t)$ that is 2 Volt for T seconds and then 0 Volts for T seconds, then repeats, as shown below. Note that the clock rate is $1/T = 1$ MHz.



(a) (5 points.) Find a Fourier series representation of $x(t)$. Call these coefficients a_k.

x(t) has period 2T where $T = 10^{-6}$ sec.

$$a_k = \frac{1}{2T}\int_0^T 2e^{-j\frac{\pi}{T}kt}\,dt = \frac{1}{T}\cdot\frac{-T}{j\pi k}e^{-j\frac{\pi}{T}kt}\Big|_0^T$$

$$= \frac{1}{j\pi k}(1 - e^{-j\pi k}) \quad \text{when } k \neq 0$$

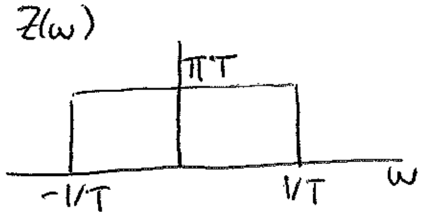
$$= \frac{1}{j\pi k}(1-(-1)^k)$$

$$a_0 = \frac{1}{2T}\int_0^T 2\,dt = 1$$

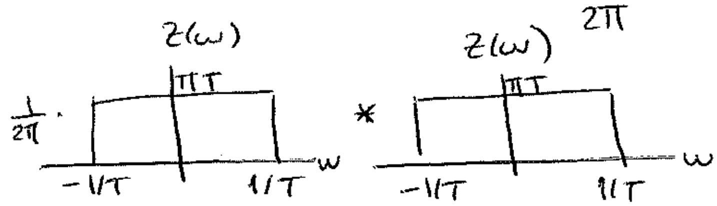
(b) (8 points.) The clock signal must propagate through a wire to get to its intended destination. The wire has a impulse response of $h(t) = \text{sinc}^2(t/T)$. Plot the frequency response of the channel, $H(\omega)$.
Hint:

$$\frac{\sin(At)}{\pi t} \overset{FT}{\longleftrightarrow} \begin{cases} 1, & |\omega| \le A \\ 0, & |\omega| > A. \end{cases}$$

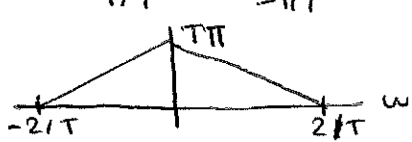
Note that $\underbrace{\text{sinc}(t/T)}_{z(t)} = \frac{\sin(t/T)}{t/T} = \pi T\frac{\sin(t/T)}{\pi t} \overset{FT}{\longleftrightarrow} \begin{cases} \pi T, & |\omega| \le 1/T \\ 0, & |\omega| > 1/T \end{cases}$



Since $h(t) = z(t)^2 \overset{FT}{\longleftrightarrow} \frac{1}{2\pi} Z(\omega) * Z(\omega)$



H(W) =



(c) (7 points.) Find an expression for the Fourier series coefficients b_k of the output clock signal $y(t) = x(t) * h(t)$.

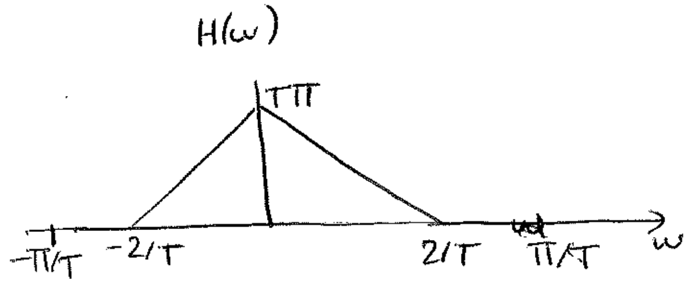
Note that $Y(\omega) = X(\omega) H(\omega)$

$$a_k = \frac{1}{2T} \int_0^T x(t) e^{-j\frac{2\pi}{2T}kt} dt = \frac{1}{2T} X(\omega) \Big|_{\omega = \frac{\pi k}{T}}$$

This connection between a_k and $X(\omega)$ helps us see that

$$b_k = a_k H\left(\frac{\pi}{T} k\right)$$

Note that $H\left(\frac{\pi}{T} k\right)$ has nonzero values only at $k=0$ ⇐



$$H\left(\frac{\pi}{T} \cdot 0\right) = T\pi$$

$$\Rightarrow \quad b_0 = a_0 T\pi = T\pi$$

$$b_k = 0 \quad \text{for } k \neq 0$$