11/12/20

Today's lecture

- Linear phase filters
- Different filter design techniques

1. Frequency sampling
2. Least-square approximation
3. Chebyshev approximation

Linear Phase Filters

$h[n]$: length N FIR

has linear phase $\theta(\omega) = k_1 + k_2\omega$

$$H(\omega) = e^{-j\omega M} \sum_{n=0}^{N-1} h[n] e^{j\omega(M-n)} \qquad \text{where } M = \frac{N-1}{2}$$

Rewriting this we can show that

$$H(\omega) = e^{-j\omega M}\Big((h[0]+h[N-1])\cos\omega M + j(h[0]-h[N-1])\sin\omega M$$
$$+$$
$$(h[1]+h[N-2])\cos\omega(M-1) + j(h[1]-h[N-2])\sin\omega(M-1)$$
$$+$$
$$\vdots \quad \Big)$$

Now assume that we have even symmetry, i.e., $h[N-n-1] = h[2M-n]$
$= h[n]$

Then all the sine terms will drop away:

$$H(\omega) = e^{-j\omega M}\Big((h[0]+h[N-1])\cos\omega M + (h[1]+h[N-2])\cos\omega(M-1) + \cdots \Big)$$

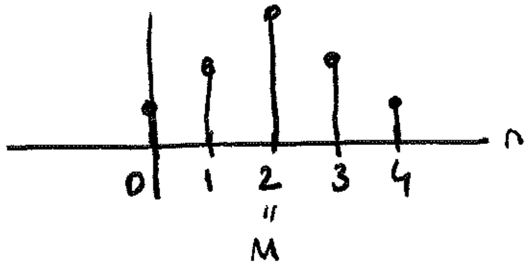
Amplitude response:

$$A(\omega) = \sum_{n=0}^{M-1} 2h[n]\cos(\omega(M-n)) + h[M] \qquad \text{if } N \text{ is ODD}$$

$$(n \to M-n) = \sum_{n=1}^{M} 2h[M-n]\cos(\omega n) + h[M] \qquad \text{'Type I } N \text{ ODD'}$$

Example

$N=5, \quad h[n]=h[4-n], \quad M=\frac{N-1}{2}=2$



Other Types:

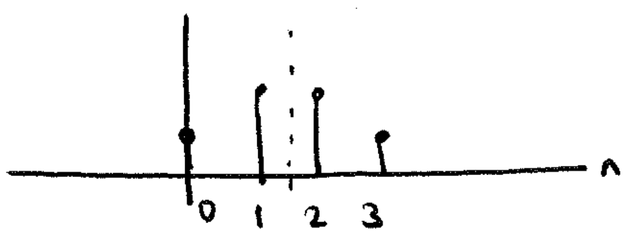
$$A(\omega) = \sum_{n=0}^{N/2-1} 2\,h[n]\cos(\omega(M-n))$$

$$(n \to \tfrac{N}{2}-n) \quad = \sum_{n=1}^{N/2} 2\,h\left[\tfrac{N}{2}-n\right]\cos\left(\omega\left(n-\tfrac{1}{2}\right)\right) \qquad \text{'Type II} \quad N \text{ EVEN'}$$

change of variables $n \to N/2-n$

Example

$N=4, \quad h[n]=h[3-n]$



$M=\frac{N-1}{2}=\frac{3}{2}$

Recall that $\theta(\omega)=k_1+k_2\omega$ and let $k_1=\frac{\pi}{2}$

$$H(\omega) = A(\omega)e^{j\theta(\omega)} = A(\omega)e^{j\frac{\pi}{2}}\cdot e^{jk_2\omega} = jA(\omega)e^{-jM\omega}$$

$h[n] = -h[N-n-1]$ (odd symmetry)

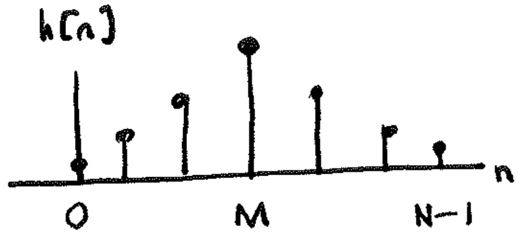
$$A(\omega) = \sum_{n=0}^{M-1} 2\,h[n]\sin(\omega(M-n)) \qquad \text{'Type III} \quad N \text{ ODD'}$$

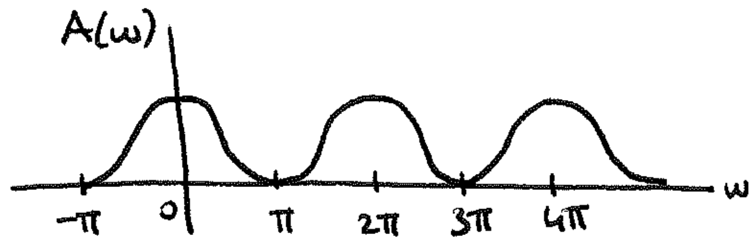
$$A(\omega) = \sum_{n=0}^{N/2-1} 2\,h[n]\sin(\omega(M-n)) \qquad \text{'Type IV} \quad N \text{ EVEN'}$$

$$= \sum_{n=1}^{N/2} 2\,h\left[\tfrac{N}{2}-n\right]\sin\left(\omega\left(n-\tfrac{1}{2}\right)\right)$$

4 possible types that give linear phase.

Type I N ODD





Has odd length

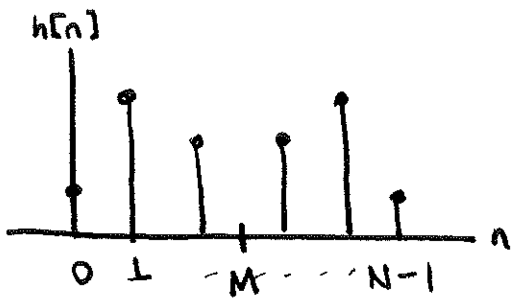
Even symmetric about M

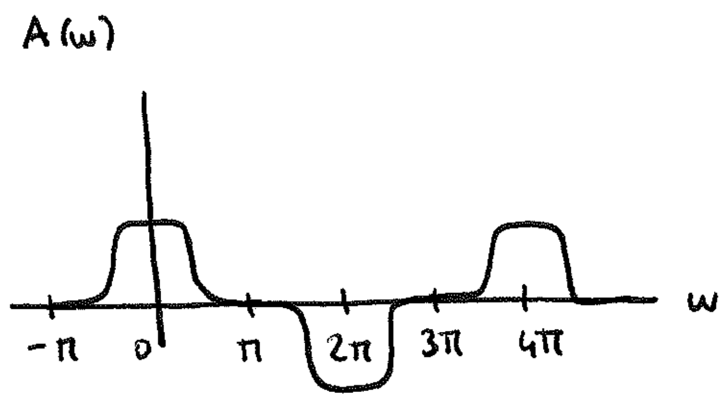
Even about $w=0, \pi$

Periodic with 2π

Low pass characteristics

Type II N EVEN





Has even length

Even symmetric about $M = \frac{N-1}{2}$

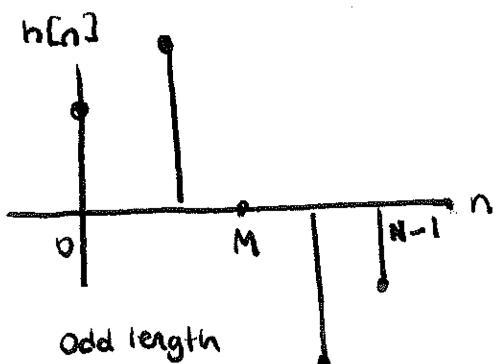
Even about $w=0$

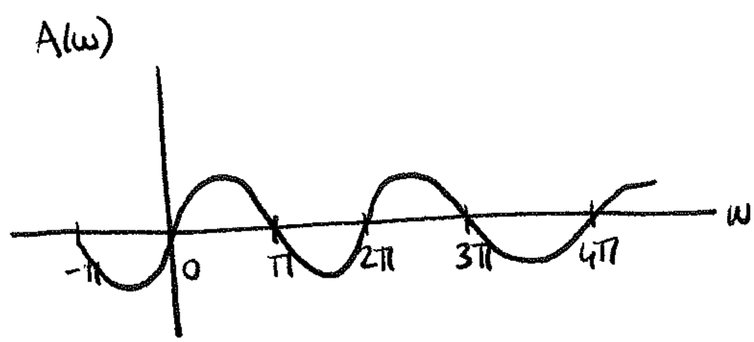
Odd about $w=\pi$

Periodic with 4π

Note: DTFT $H(\omega)$ is always 2π periodic. $A(\omega)$ is a continuous, real valued, smooth function based on both $H(\omega)$ and $\angle H(\omega)$: $A(\omega) = \mp|H(\omega)|$

Type III N ODD





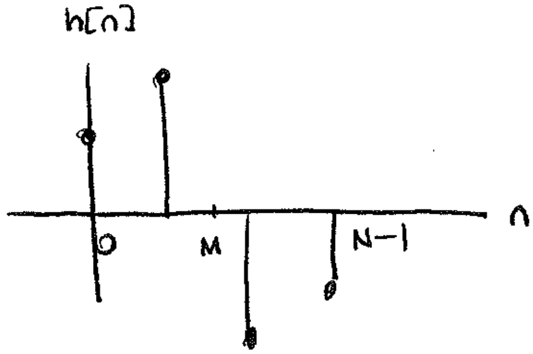
Odd length

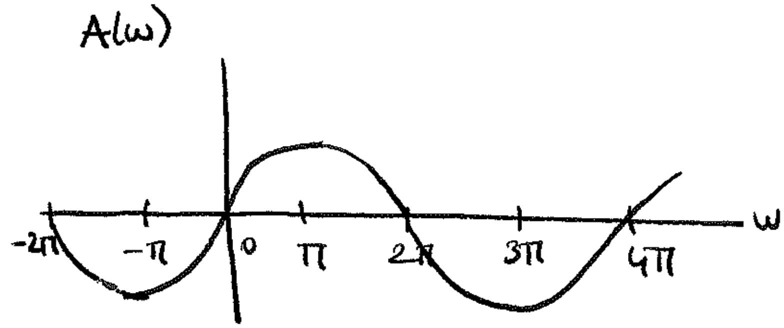
Odd symmetric about M

Odd about $w=0, \pi$

Periodic with 2π

Type IV N EVEN





Even length

Odd symmetric about $M = \frac{N-1}{2}$

Odd about $\omega = 0$

Even about $\omega = \pi$

Periodic with 4π

Notes

- Types III, IV have $A(0) = 0 \Rightarrow$ bad for LPF
- Types II, III have $A(\pi) = 0 \Rightarrow$ bad for HPF
- Type III is good for BPF.

Zero Locations for Linear Phase FIR Filters

$$H(z) = \sum_{n=0}^{N-1} h[n] z^{-n}$$

$$= z^{-(N-1)} \left(h[0] z^{N-1} + h[1] z^{N-2} + \cdots + h[N-1] \right)$$

1. If $h[n]$ is real, then if $H(z) = 0 \Rightarrow H(z^*) = 0$

Why? $$H(z^*) = (z^*)^{-(N-1)} \left(h[0](z^*)^{N-1} + h[1](z^*)^{N-2} + \cdots + h[N-1] \right)$$

($h[n]$ real) $$= (z^*)^{-(N-1)} \left(h^*[0](z^*)^{N-1} + h^*[1](z^*)^{N-2} + \cdots + h^*[N-1] \right)$$

$$= (H(z))^* = 0$$

2. If $h[n] = \pm h[N-1-n]$ where $+$ for Types I–II,
$-$ for Types III–IV

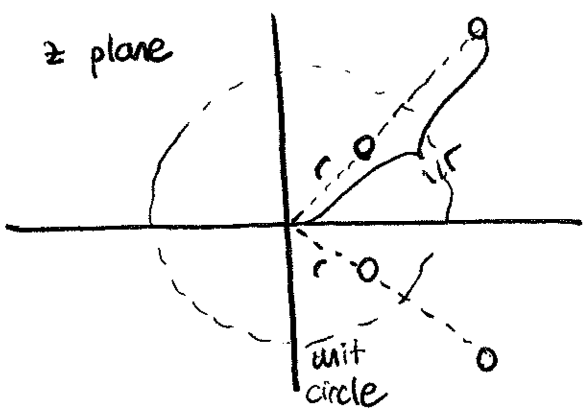
then if $H(z)=0 \Rightarrow H\left(\frac{1}{z}\right)=0$

Why? $H\left(\frac{1}{z}\right) = z^{N-1}\left(h[0]\, z^{-(N-1)} + h[1]\, z^{-(N-2)} + \cdots + h[N-1]\right)$

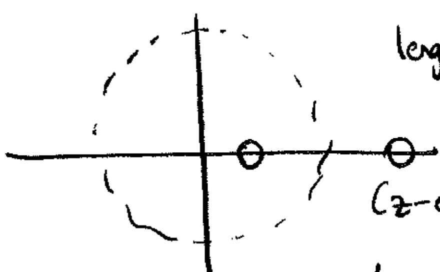
$= \pm z^{N-1}\left(h[N-1]\, z^{-(N-1)} + h[N-2]\, z^{-(N-2)} + \cdots + h[0]\right)$

$= \pm z^{N-1} H(z) = 0$

The zeros of linear phase, real FIR filters look like:



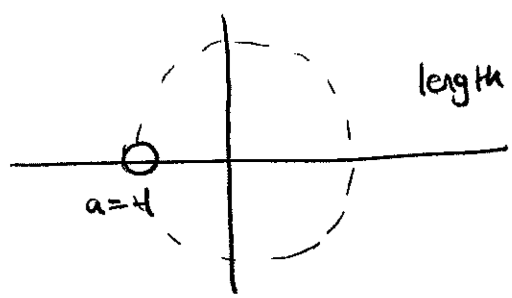
length 5 filter,



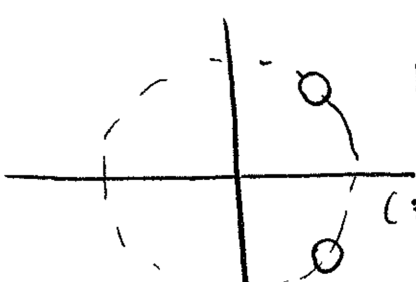
length 3

$(z-a)\left(z-\frac{1}{a}\right)$

a real

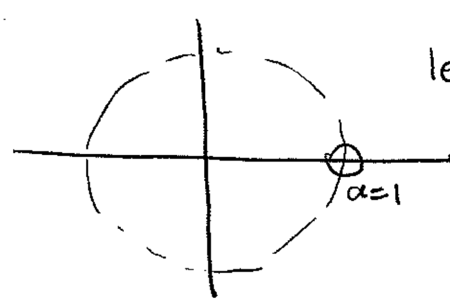


length 2



length 3

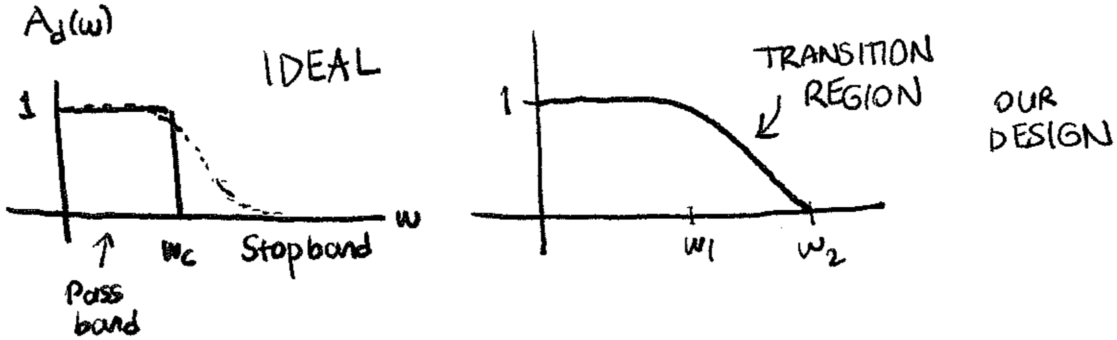
$(z-e^{-j\omega})(z-e^{j\omega})$



length 2

* We could decompose or cascade these decompositions.

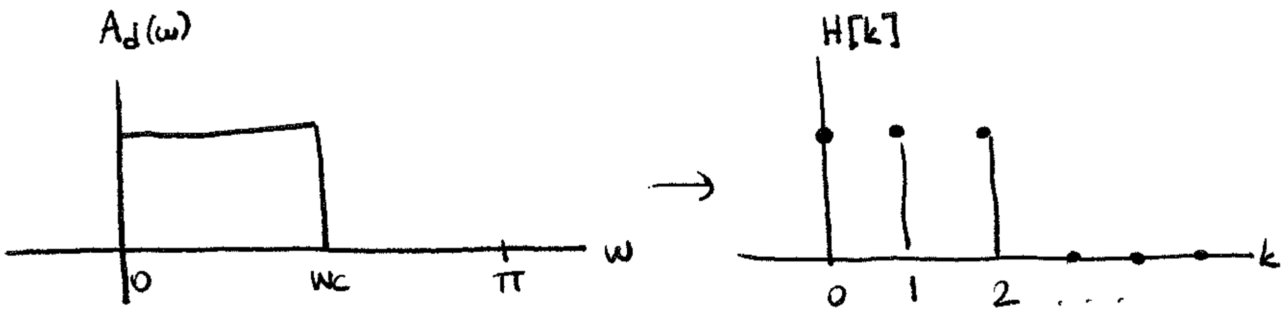
Different FIR Design Techniques



Different approximations

- Least-squares (LS): to minimize the average or squared error in frequency domain.
- Chebyshev approximation: to minimize the maximum error over certain regions of frequency response.
- Butterworth: Taylor series approximation to desired response.

1. Frequency Sampling Design



Assume that we take N equally spaced samples in the frequency domain at

$$\frac{2\pi k}{N}, \quad k = 0, \ldots, N-1$$

The inverse DFT

$$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k]\, e^{j\frac{2\pi}{N}kn} \quad , \quad n=0, \ldots, N-1$$

We have N equations and N unknowns $\Rightarrow$ We can solve this.

N log N operations to implement DFT. (using FFT algorithm)

Now assume that $h[n]$ is real, linear phase.

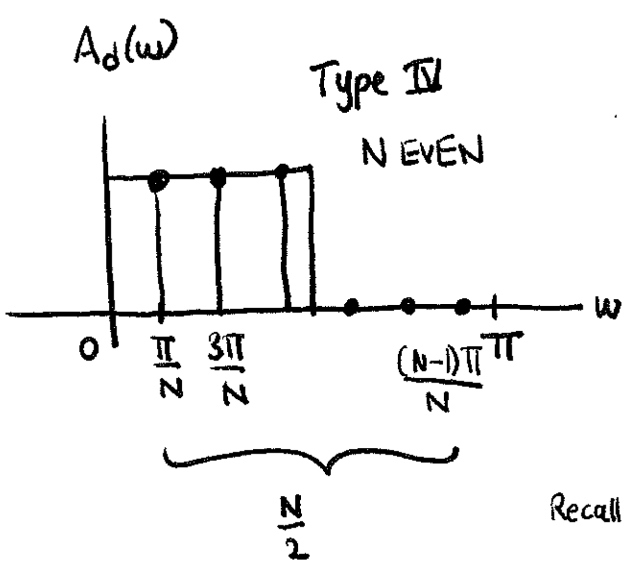
$\Rightarrow$ We can get a gain of 4. Why?*

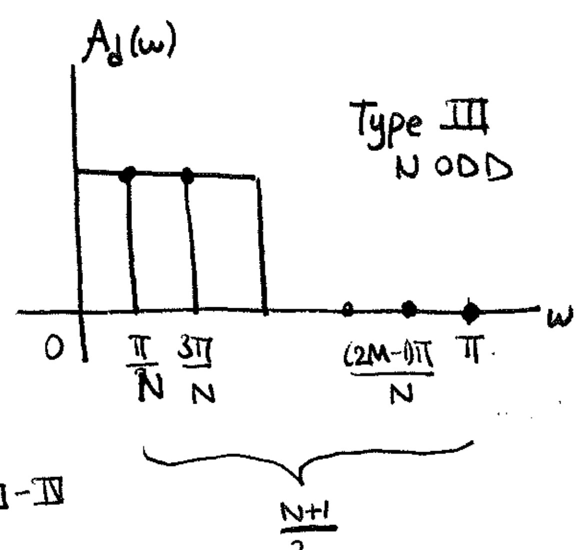
$$h[n] = \frac{1}{N}\left[A[0] + \sum_{k=1}^{M} 2A[k] \cos \frac{2\pi(n-M)k}{N}\right] \qquad \text{Type I}$$

where $M = \frac{N-1}{2}$ and $A[0], \ldots, A[M]$ are samples of amplitude response

* because $M \approx \frac{N}{2}$ and we only have the "cos" terms.

We have equivalent forms for other types (Types II, III, IV)





Recall for Types III–IV

$$\sum_n h[n] = 0 \Rightarrow \underbrace{H(0) = \sum h[n]}_{=0}$$

Note: We could also use non-uniform sampling. However, in that case we can no longer use IDFT.

2. Least-Squares (LS) error in frequency domain.

a) $L \geq N$ discrete frequency samples where L is the length of the filter

$$E = \sum_{k=0}^{L-1} |A(w_k) - A_d(w_k)|^2 \quad \text{mean-squared error}$$

$$w_k = \frac{2\pi k}{L}, \quad k = 0, \ldots, L-1, \text{ with amplitude samples } A_d(w_k).$$

From Parseval's theorem,

$$E = \sum_{n=-\frac{(L-1)}{2}}^{\frac{L-1}{2}} |h[n] - h_d[n]|^2$$

where $h_d[n]$ is a length-L FIR filter.

Using $M = \frac{N-1}{2}$ we rewrite E:

$$E = \sum_{n=-M}^{M} |\underset{\text{length } N}{h[n]} - \underset{\text{length } L}{h_d[n]}|^2 + \underbrace{\sum_{n=M+1}^{(L-1)/2} 2\,|h_d[n]|^2}_{\text{residual error}}$$

To minimize E, take $h[n]$ to agree with $h_d[n]$ on first N filter taps, i.e., truncate $h_d[n]$.

To adjust the 'residual error', choose N appropriately.

Solving a LS problem

Recall that the amplitude response of a linear phase filter is

$$A(w) = \sum_{n=0}^{M-1} 2\,h[n]\cos(w(M-n)) + h[M] \qquad (\text{Type I } N \text{ ODD})$$

$$\underset{\substack{L\times 1\\ \text{vector}}}{a} = \underset{\substack{L\times(M+1)\\ \text{matrix}}}{F}\ \underset{\substack{(M+1)\times 1\\ \text{vector}}}{h}$$

where $L > M+1$, i.e., F is a tall matrix.

$$F = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$F^T a = \overbrace{F^T F}^{\text{invertible}} h$$ where F^T is the transpose.

$$\hat{h} = \underbrace{(F^T F)^{-1}}_{\substack{(M+1)\times(M+1)\\ \text{matrix}}} F^T a$$ is the solution that minimizes

$$(Fh - a_d)^T (Fh - a_d)$$