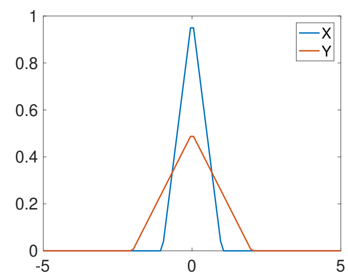
Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2020

Homework #5: due Thursday, Nov. 19^{th}, at the beginning of class.

5. (10 points) **Nyquist rate.** Consider a bandlimited continuous-time signal $x(t)$ such that $X(\omega) = 0$ for $\omega > \omega_B$. Determine the Nyquist rate for the following continuous time signals:

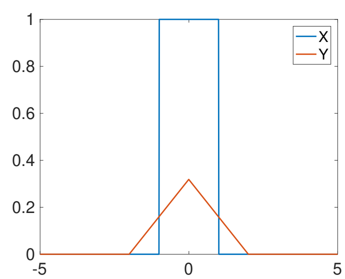
   (a) $Y(\omega) = X(\omega) + e^{-j2\omega}X(\omega)$. Note that $X(\omega)$ is bandlimited by ω_B, and $e^{-j2w}X(\omega)$ is just a phase shifted version of $X(\omega)$, i.e. no change in frequency content. Therefore, $e^{-j2\omega}X(\omega)$ is also bandlimited by ω_B. The Nyquist rate is $2\omega_B$.

   (b) $Y(\omega) = \frac{1}{2}X\left(\frac{\omega}{2}\right)$ from scaling property of the Fourier transform. Since $X(\omega)$ is bandlimited by ω_B, $X\left(\frac{\omega}{2}\right)$ is bandlimited by 2ω and the Nyquist rate is $4\omega_B$.



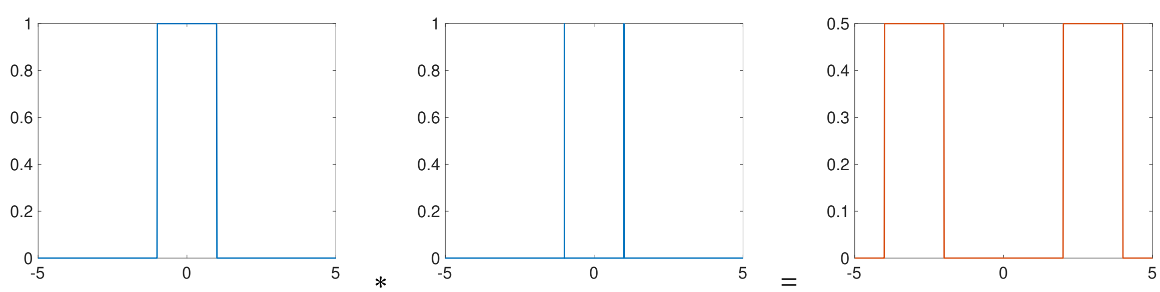
Example:5.b

   (c) $Y(\omega) = \frac{1}{2\pi}X(\omega) * X(\omega)$ because the Fourier transform of product of 2 time domain signals is proportional to the convolution of the Fourier transforms of the individual signals. Because of the convolution, the length of $Y(\omega)$ is twice the length of $X(\omega)$, hence the maximum frequency is $2\omega_B$ and the Nyquist rate is $4\omega_B$.



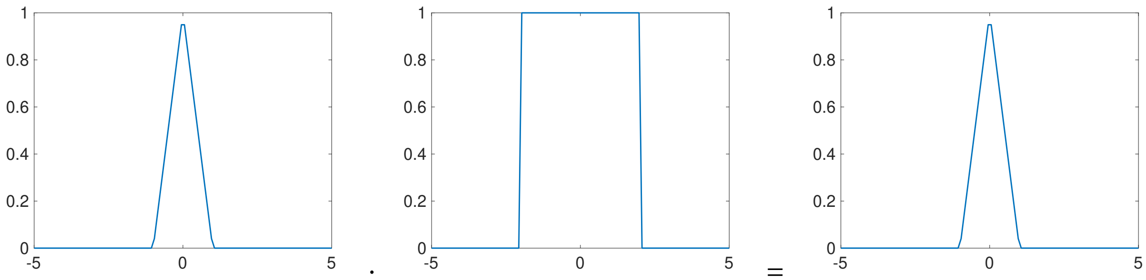
Example:5.c

   (d) Note that Fourier transform of $\cos(\omega_0 t)$ is $\frac{1}{2}\delta(\omega-\omega_0) + \frac{1}{2}\delta(\omega+\omega_0)$. Therefore, $Y(\omega) = X(\omega) * \left[\frac{1}{2}\delta(\omega-\omega_0) + \frac{1}{2}\delta(\omega+\omega_0)\right] = \frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$. This means we have two copies of $X(w)$ centered at $\pm\omega_0$. Therefore, the Nyquist rate is $2(\omega_0+\omega_B)$.

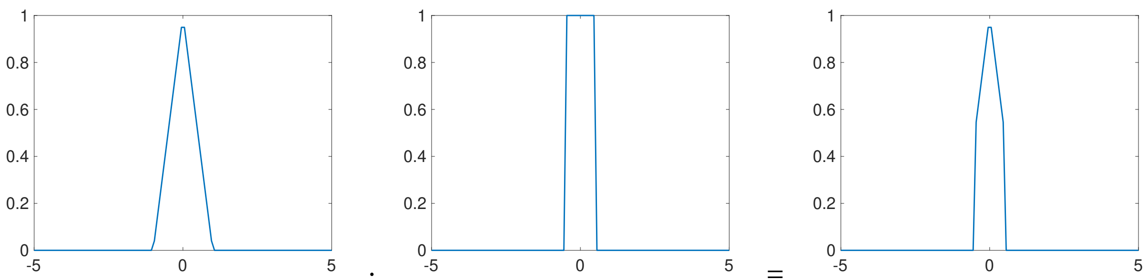


Example:5.d

(e) $Y(\omega) = X(\omega) RectangularPulse(\omega_c)$ because Fourier transform of sinc is a pulse with cutoff ω_c. If $\omega_c \geq \omega_B$, the signal bandwidth is still ω_B. However, if $\omega_c < \omega_B$, we low pass filter the signal and the bandwidth is w_c. Thus the Nyquist rate is $\min\{2\omega_B, 2\omega_c\}$.



Example 5.e.1

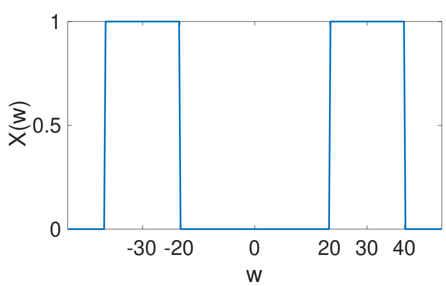


Example 5.e.2

6. (10 points) **Aliasing.** Consider a signal $x(t)$ whose continuous-time Fourier transform is

$$X(\omega) = \begin{cases} 1 & 20 < |\omega| < 40 \\ 0 & \text{otherwise} \end{cases}$$

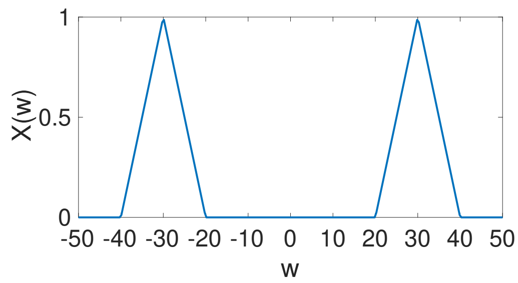
(a) The Fourier transform looks like



The Nyquist rate is twice the highest frequency, $\omega_s = 80$.

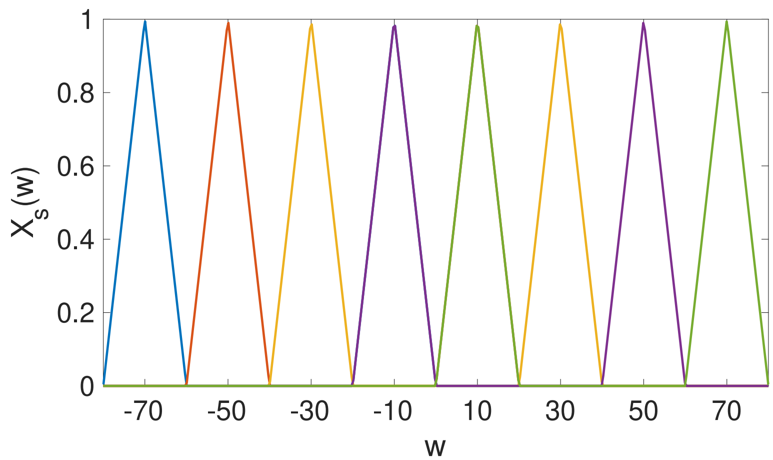
(b) The sampling rate is $\frac{2\pi}{T} = 20$ which is much less than 80. It seems like we should expect aliasing. However, in this case the frequency content does not come close to sampling the entire $[-40, 40]$ range.

To see why this is true, let's instead consider the triangular spectrum below (to avoid the edge effects caused by using the original rectangular spectrum in part a).

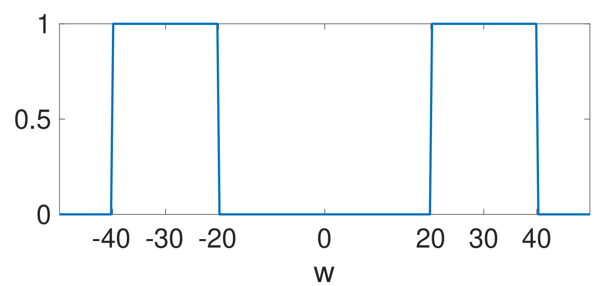


Example 6.b

For this example, when we sketch the spectrum of $x_s(t)$ obtained by impulse-train sampling, we will get copies at multiples of 20. We can see that the copies in fact do not overlap, sitting in the empty space between each other. Thus, there is actually no aliasing, and we could reconstruct $x(t)$ by bandpass (not lowpass) altering $x[n]$. (See below figure where each shifted copy of the Example $X(\omega)$ signal is in different color.)



i.e., we would use the filter



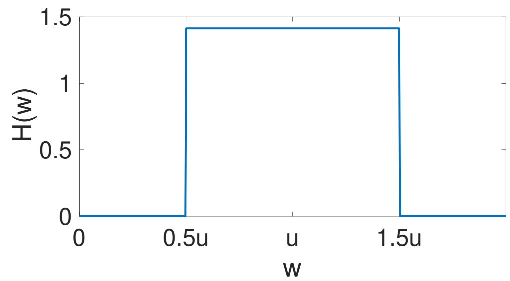
Note that we will have the same conclusion for the original $X(\omega)$, which has a rectangular waveform. I chose the triangular waveform for visual purposes only.

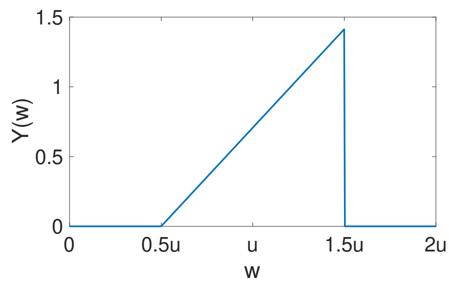
7. (10 points) **Filtering and Sampling.**

   (a) Passband gain: 3 dB

$$20\log_{10}|H(\omega)| = 3 \text{ dB}.$$

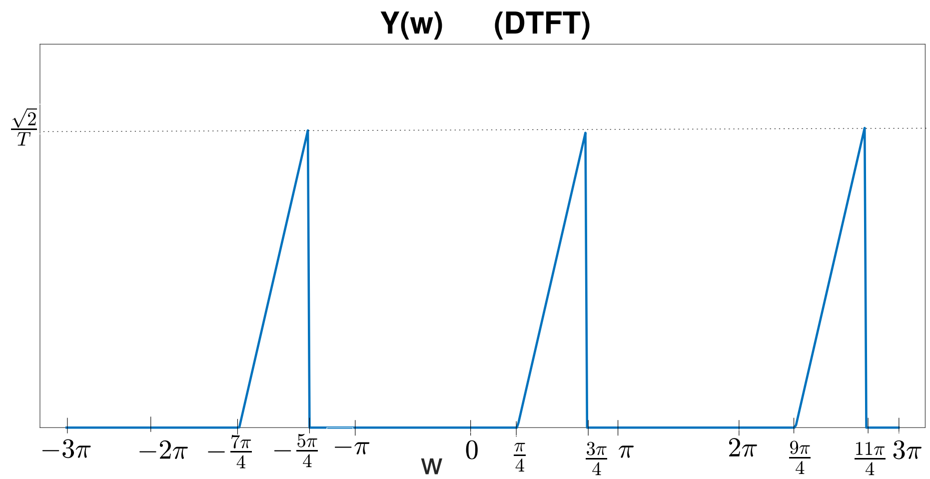
This implies $|H(\omega)|^2 = 2$, $0.5u < \omega < 1.5u$. Hence, $H(\omega)$ and the Fourier transform (continuous time) of $y(t)$ in the frequency range $[-6u, 6u]$ look like:



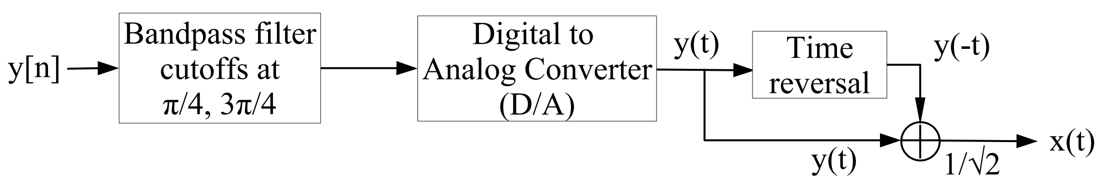


(b) Note that $y_d[n] = y(nT)$ where $T = \frac{\pi}{2u}$ seconds. The DTFT is $Y_d(\omega) = \frac{1}{T}Y\left(\frac{\omega}{T}\right)$ over a period of 2π. This means that the point $0.5u$ is mapped to $0.5u \cdot \frac{2\pi}{\omega_s} = 0.5u \cdot T = 0.5u \cdot \frac{\pi}{2u} = \frac{\pi}{4}$. Similarly for the other points on the x axis, i.e. $1.5u$ is mapped to $\frac{3\pi}{4}$.

The DTFT of $y[n]$ in the corresponding frequency range, namely $\omega \in [-3\pi, 3\pi]$ looks as below.



(c) A block diagram to get back the input $x(t)$ from $y[n]$:



(d) The energy of the input $x(t)$ can be computed using Parseval's theorem.

$$\int |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi}\int |X(\omega)|^2 \mathrm{d}w$$

$$= \frac{2}{2\pi}\int_{0.5u}^{1.5u}\left(\frac{\omega - 0.5u}{u}\right)^2 \mathrm{d}\omega = \frac{1}{\pi u^2}\int_0^u \omega^2 \mathrm{d}\omega = \frac{u}{3\pi}$$

The energy of the filtered signal $y(t)$ can be computed similarly. Since $Y(\omega) = X(\omega)H(\omega)$, $|Y(\omega)|^2 = |X(\omega)|^2|H(\omega)|^2 = |X(\omega)|^2 \cdot 2$ when $0.5u \leq \omega \leq 1.5u$ (where 2 is the passband gain).

$$\frac{1}{2\pi}\int_{0.5u}^{1.5u}|Y(\omega)|^2 \mathrm{d}\omega = \frac{2}{2\pi}\int_{0.5u}^{1.5u}|X(\omega)|^2 \mathrm{d}\omega = \frac{u}{3\pi}$$

Note: The decibel (dB) is a unit of measurement used to express the power or amplitude ratio on a logarithmic scale. The amplitude ratio in decibels (dB) is 20 times base 10 logarithm of the ratio. The power ratio in decibels (dB) is 10 times base 10 logarithm of the ratio.

8. (10 points) We know that for a discrete time signal $x[n]$, it's easy to create $x[n-k]$ when k is an integer. We can also think about the result when k is not an integer; for example, if $k = 0.5$, the effect would be the same as if we determined the continuous-time signal corresponding to the original samples, and took new samples exactly in between the original ones (a.k.a. a "half-sample delay"). However, we can accomplish this process directly using a digital filter $h_k[n]$. Determine the ideal frequency response of this filter, $H_k(\omega)$.

   $y[n] = x[n-k]$. In the frequency domain, this is just $Y(\omega) = e^{-j\omega k}X(\omega)$, so the ideal fractional delay filter is just $H_k(\omega) = e^{-j\omega k}$. Note that $|H_k(\omega)| = 1$, so this is an all-pass filter, even though there is definitely a phase change. However in practice, this filter may be hard to implement exactly.

9. (10 points) **Downsampling and Upsampling.**

Figure 1: Cascaded systems.

In the left figure, the output of the upsampler is $W_1(\omega) = X(2\omega)$. Thus, we have

$$\begin{aligned} Y_1(\omega) &= \frac{1}{2}\left(W_1\left(\frac{\omega}{2}\right) + W_1\left(\frac{\omega}{2} - \pi\right)\right) \quad \text{(see lecture notes)} \\ &= \frac{1}{2}\left(X(\omega) + X(\omega - 2\pi)\right) \end{aligned}$$

In the right figure, the output of the downsampler is $W_2(\omega) = \frac{1}{2}\left(X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2} - \pi\right)\right)$. Thus, we have

$$\begin{aligned} Y_2(\omega) &= W_2(2\omega) \\ &= \frac{1}{2}\left(X(\omega) + X(\omega - \pi)\right) \end{aligned}$$

Therefore, $Y_1(\omega) \neq Y_2(\omega)$. This implies $y_1[n] \neq y_2[n]$.

Now consider the system in Figure 2 with $H(z)$ being the transfer function of an LTI system. From

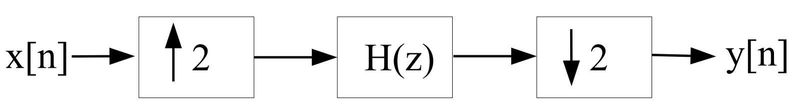


Figure 2: Cascaded system with a digital filter in the middle.

previous part, we have $W_1(\omega) = X(2\omega)$. Then, the output of the filter H is given by $W_3(\omega) = W_1(\omega)H(\omega) = X(2\omega)H(\omega)$ which is then input to the downsampler. Using the ideas from previous part,

$$\begin{aligned} Y(\omega) &= \frac{1}{2}\left(W_3\left(\frac{\omega}{2}\right) + W_3\left(\frac{\omega}{2} - \pi\right)\right) \\ &= \frac{1}{2}\left(X(\omega)H\left(\frac{\omega}{2}\right) + X(\omega - 2\pi)H\left(\frac{\omega}{2} - \pi\right)\right), \end{aligned}$$

where note that for DTFT we have $X(\omega - 2\pi) = X(\omega)$. Therefore, $Y(\omega) = \frac{X(\omega)}{2}\left(H\left(\frac{\omega}{2}\right) + H\left(\frac{\omega}{2} - \pi\right)\right)$.

The transfer function (this is z domain, remember the connection $z = e^{j\omega}$) of the whole system with input $x[n]$ and output $y[n]$ is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2}\left(H\left(\frac{z}{2}\right) + H\left(-\frac{z}{2}\right)\right)$$

10. (10 points) **Polyphase signal processing.** Consider a digital filter with transfer function $H(z)$.

(a) The transfer function $H_e(z)$ of even numbered samples $h_e[n] = h[2n]$:
$H_e(z) = \sum_{n=0}^{\infty} h[2n]z^{-n}$

(b) The transfer function $H_o(z)$ odd numbered samples $h_o[n] = h[2n+1]$:
$H_o(z) = \sum_{n=0}^{\infty} h[2n+1]z^{-n}$

(c) $H(z)$ in terms of $H_e(z)$ and $H_o(z)$ can be expressed as

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} h[2n]z^{-2n} + \sum_{n=0}^{\infty} h[2n+1]z^{-2n-1} \\ &= \sum_{n=0}^{\infty} h[2n](z^2)^{-n} + z^{-1}\sum_{n=0}^{\infty} h[2n+1](z^2)^{-n} \\ &= H_e(z^2) + z^{-1}H_o(z^2) \end{aligned}$$

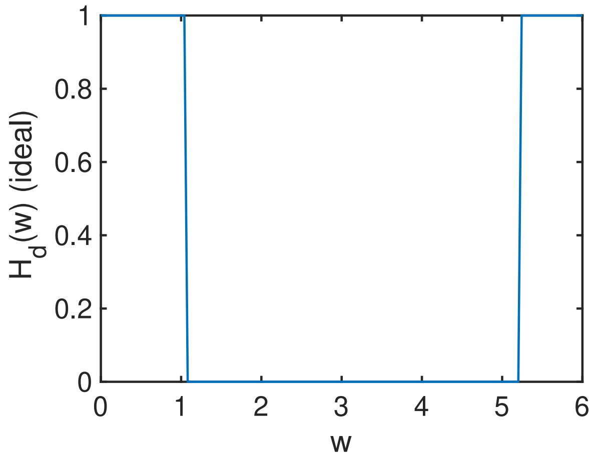
(d) We want to decompose $h[n]$ into M components: $h_1[n] = h[Mn]$, $h_2[n] = h[Mn+1], \ldots, h_M[n] = h[Mn+M-1]$.
$H(z)$ can be decomposed into a $M-$component polyphase filter structure with transfer function $H(z)$ that can be expressed in terms of the transfer functions of $h_k[n]$'s:

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} h[Mn]z^{-Mn} + \sum_{n=0}^{\infty} h[Mn+1]z^{-Mn-1} + \ldots + \sum_{n=0}^{\infty} h[Mn+M-1]z^{-Mn-M+1} \\ &= \sum_{n=0}^{\infty} h[Mn](z^M)^{-n} + z^{-1}\sum_{n=0}^{\infty} h[Mn+1](z^M)^{-n} + \ldots + z^{-M+1}\sum_{n=0}^{\infty} h[Mn+M-1](z^M)^{-n} \\ &= H_1(z^M) + z^{-1}H_2(z^M) + \ldots + z^{-M+1}H_M(z^M) \end{aligned}$$

From above relation, we can infer that $H_m(z) = \sum_{n=0}^{\infty} h[Mn+m-1]z^{-n}$.

11. (10 points) **FIR filter design.** Design an FIR digital filter with linear phase that approximates the ideal frequency response

$$H_d(\omega) = \begin{cases} 1, & \text{for} \quad |\omega| \le \frac{\pi}{3} \\ 0, & \text{for} \quad \frac{\pi}{3} < |\omega| \le \pi. \end{cases}$$



(a) The coefficients of a 15-tap filter based on the IDFT method can be found using the following relation:

$$H[k] = H_d(\omega)|_{\omega=\frac{2\pi k}{15}}, \quad k = 0, \ldots, 14.$$

Hence, incorporating the linear phase shift ($M = (N-1)/2 = 7$) we can infer that

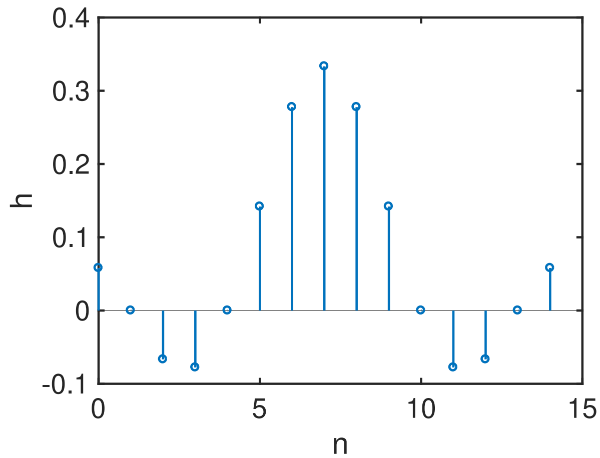
$$H[k] = \begin{cases} e^{-j7\frac{2\pi k}{15}}, & k = 0, 1, 2, 13, 14 \\ 0, & 3 \le k \le 12. \end{cases}$$

Therefore, taking the IDFT, we have

$$\begin{aligned} h[n] &= \frac{1}{N}\sum_{k=0}^{N-1} H[k] e^{j\frac{2\pi}{N}kn} \\ &= \frac{1}{15}\sum_{k=0}^{2} e^{-j\frac{14\pi k}{15}} e^{j\frac{2\pi}{15}kn} + \frac{1}{15}\sum_{k=13}^{14} e^{-j\frac{14\pi k}{15}} e^{j\frac{2\pi}{15}kn} \\ &= \frac{1}{15}\sum_{k=0}^{2} e^{-j\frac{14\pi k}{15}} e^{j\frac{2\pi}{15}kn} + \frac{1}{15}\sum_{l=1}^{2} e^{-j\frac{14\pi(15-l)}{15}} e^{j\frac{2\pi}{15}(15-l)n} \\ &= \frac{1}{15}\sum_{k=0}^{2} e^{-j\frac{14\pi k}{15}} e^{j\frac{2\pi}{15}kn} + \frac{1}{15}\sum_{l=1}^{2} e^{j\frac{14\pi l}{15}} e^{-j\frac{2\pi}{15}ln} \\ &= \frac{1}{15} + \frac{1}{15}\sum_{k=1}^{2} e^{-j\frac{14\pi k}{15}} e^{j\frac{2\pi}{15}kn} + \frac{1}{15}\sum_{k=1}^{2} e^{j\frac{14\pi k}{15}} e^{-j\frac{2\pi}{15}kn} \\ &= \frac{1}{15} + \frac{2}{15}\cos\left(\frac{(7-n)2\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{(7-n)4\pi}{15}\right). \end{aligned}$$

Note that $H[0] = 1$, $H[1] = H[14]$ and $H[2] = H[13]$ in frequency domain, and the other coefficients $H[k] = H_d(\omega)|_{\omega=\frac{2\pi k}{15}} = 0$ for $3 \le k \le 12$.

The coefficients $h[n]$ look like



(b) The frequency response of the filter as function of the magnitude response and the phase response is

$$H(\omega) = |H(\omega)|e^{j\Theta(H(\omega))} = \sum_{n=0}^{14} h[n]e^{-j\omega n}$$

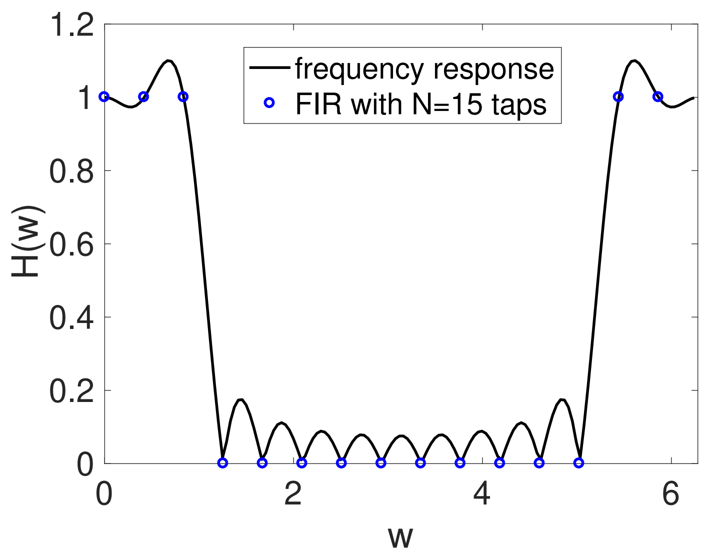
You can determine $H(\omega)$ by using the coefficients you get from part (a).
Note also that for the designed filter,

$$H(\omega)|_{\omega=\frac{2\pi k}{15}} = H[k].$$

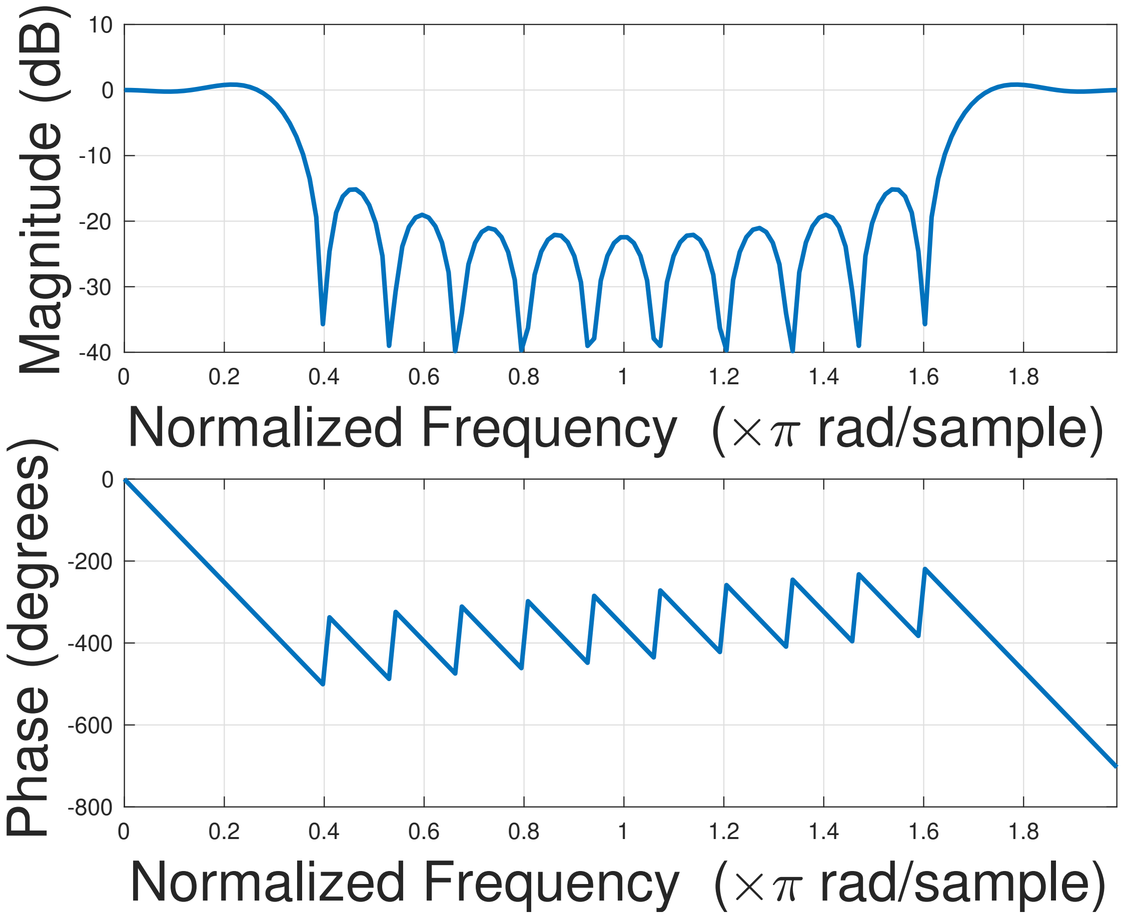
You can design this filter in Matlab using

$$FIR\_filter\_design.m$$

code on Piazza (Method I: Frequency sampling).



Or you can use the freqz function to directly create the below graphs:


Magnitude (dB)
10
0
-10
-20
-30
-40
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Normalized Frequency (×π rad/sample)
Phase (degrees)
0
-200
-400
-600
-800
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Normalized Frequency (×π rad/sample)