9/24/20

Today's lecture

- DTFT
  - examples, convergence
- Filters / frequency response
  - linear phase
- z-transform

Reminder for 'Attendance'

Refer to Syllabus!

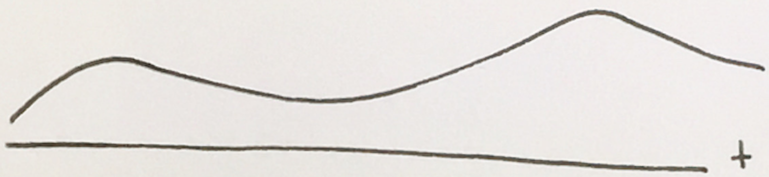
Fourier Transforms of Sampled Signals

$p(t)$ CT impulse train

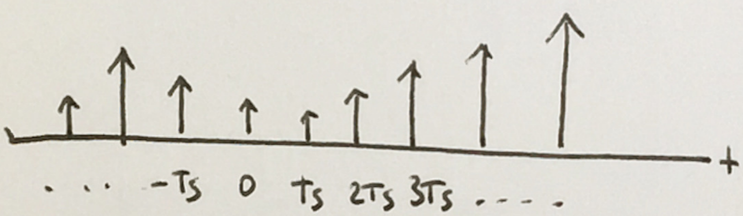
periodic

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$x(t)$ CT signal



$x_s(t) = x(t)\,p(t)$ (Sampled signal)



$$\boxed{x[n] = x_s\left(\frac{t}{T_s}\right)} \; ?$$

CT

$$x_s(t) = x(t)\,p(t) \xleftrightarrow{FT} X_s(\omega) = \frac{1}{2\pi} \underbrace{X(\omega)}_{\text{CTFT}} * P(\omega)$$

$$\underbrace{P(\omega)}_{FT} = F(p(t)) = F\left(\sum_k \underbrace{c_k}_{\text{FS coefficient}} e^{jk\omega_0 t}\right) = \sum_k 2\pi c_k \,\delta(\omega - k\omega_0)$$

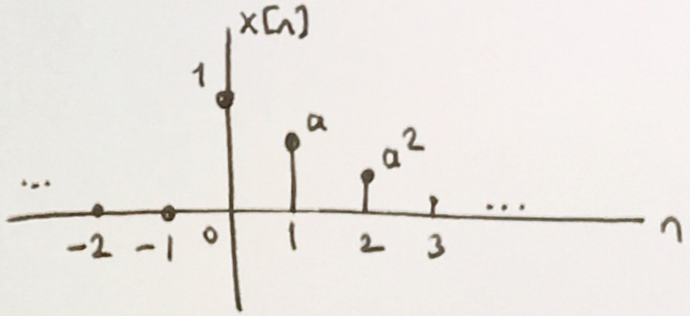
$\omega_0 = \frac{2\pi}{T_s}$

where $\omega_0 = \frac{2\pi}{T_s}$

$$\boxed{c_k = \frac{1}{T_s}\int_{T_s} \underbrace{p(t)}_{\delta(t)} e^{-jk\omega_0 t}\,dt} = \frac{1}{T_s} \quad \text{for all } k.$$

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * \sum_k \frac{2\pi}{T_s}\delta(\omega - k\omega_0) = \underbrace{\frac{1}{T_s}\sum_k X(\omega - k\omega_0)}_{\text{replicated, scaled versions of } X(\omega)\text{, spaced every } \omega_0 \text{ apart}}$$

Example $x[n] = a^n u[n]$, $a < 1$



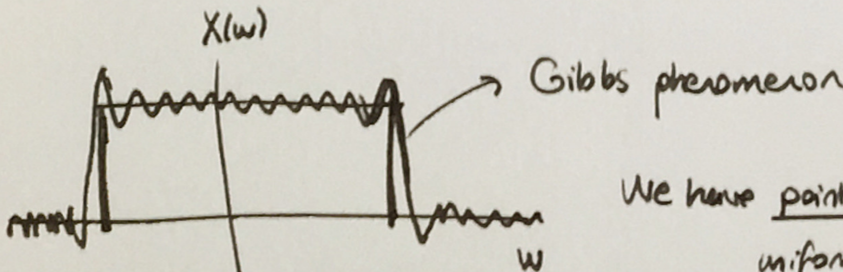
What is the DTFT $X(\omega)$?

Convergence of DTFT

1. If $\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$, then DTFT converges '<u>uniformly</u>'

2. If $\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$, then DTFT has '<u>mean square convergence</u>'

Recall Last lecture's example (FS coefficients <u>for a square wave</u>)



$x[n] = \text{sinc}(w_c n) \xrightarrow{\text{DTFT}} \text{rect}$

We have <u>pointwise</u> convergence but not uniform convergence.

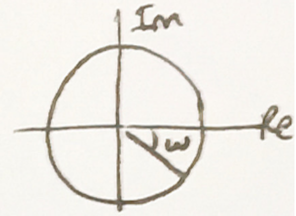
For our square wave example $\sum_{n=-\infty}^{+\infty} |x[n]| = \sum_{n=-\infty}^{+\infty} \frac{w_c}{\pi} \text{sinc}(w_c n)$

$= \infty$

$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$

For $x[n] = a^n u[n]$, $\underline{a<1}$

$$\sum_{n=-\infty}^{+\infty} |x[n]| = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} < \infty \qquad \text{(uniform convegence)}$$



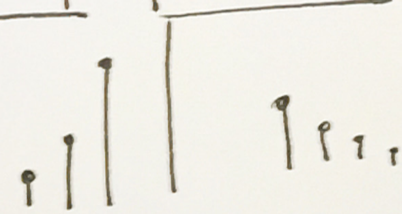
$$X(w) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-jwn} = \sum_{n=0}^{\infty} a^n e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \underbrace{(a e^{-jw})}_{b}{}^n \qquad \text{Let } \boxed{b = a e^{-jw}} \qquad \boxed{|e^{-jw}| = 1}$$

$|b| < 1$

$$= \sum_{n=0}^{\infty} b^n \, , \quad |b| < 1$$

$b > 1$

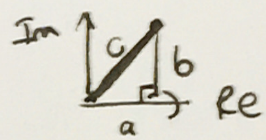


$$= \frac{1}{1-b} = \frac{1}{1 - a e^{-jw}} \qquad \text{complex valued.}$$

$X(w) = |X(w)| e^{j \underline{\angle X(w)}}$ ~~frequency response~~ = DTFT

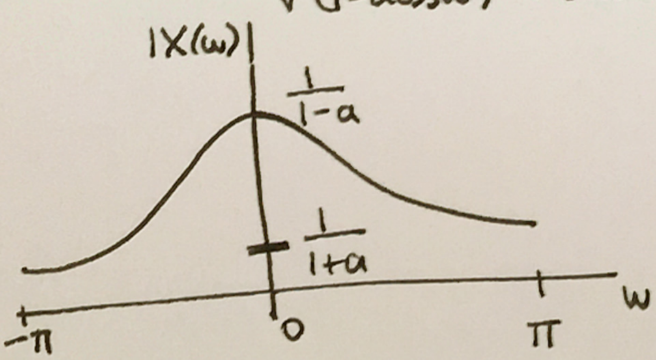
$|X(w)|$: magnitude response

$\underline{\angle X(w)}$: phase response



$$|X(w)| = \left| \frac{1}{1 - a e^{-jw}} \right| = \frac{1}{|1 - a\cos w + j a \sin w|} \qquad c^2 = a^2 + b^2$$

$$= \frac{1}{\sqrt{(1 - a\cos w)^2 + (a \sin w)^2}} = \frac{1}{\sqrt{1 - 2a\cos w + a^2}}$$

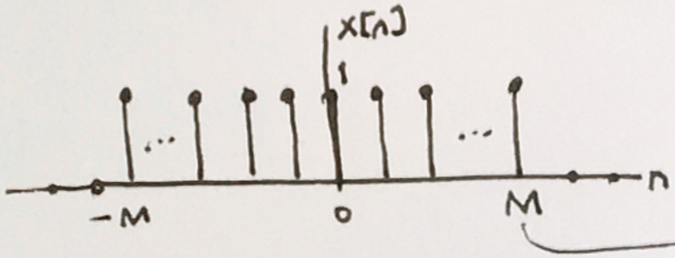


$$|X(0)| = \frac{1}{\sqrt{1 - 2a + a^2}} = \frac{1}{1-a}$$

$$|X(\pi)| = \frac{1}{\sqrt{1 + 2a + a^2}} = \frac{1}{1+a}$$

'A Low Pass Filter'

Example Pulse in the time domain. Compute DTFT $X(w)$?



2M+1 taps.

$$X(w) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-jwn}$$

$$= \sum_{n=-M}^{M} 1 \cdot e^{-jwn}$$

$m = n+M$

$$= \sum_{m=0}^{2M} 1 \cdot e^{-jwm} \cdot e^{jwM}$$

small

Capital

$$= e^{jwM} \cdot \frac{1 - e^{-jw(2M+1)}}{1 - e^{-jw}}$$

$$\sum_{n=0}^{M} a^n = \frac{1 - a^{M+1}}{1 - a}$$

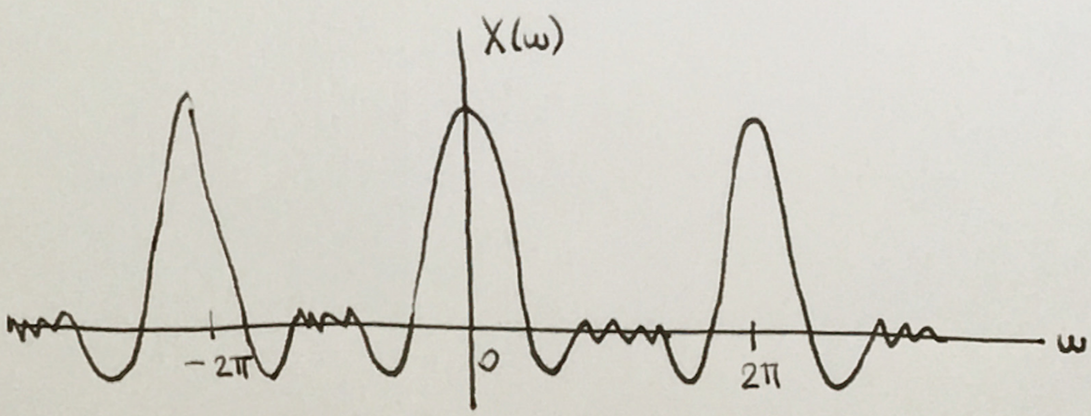
finite sum formula

$$X(w) = e^{jwM} \cdot \frac{e^{-jw\left(\frac{2M+1}{2}\right)}}{e^{-j\frac{w}{2}}} \cdot \frac{e^{jw\left(\frac{2M+1}{2}\right)} - e^{-jw\left(\frac{2M+1}{2}\right)}}{e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \cancel{e^{jwM}} \cdot \frac{\cancel{e^{-jwM}} \cdot \cancel{e^{-j\frac{w}{2}}}}{\cancel{e^{-j\frac{w}{2}}}} \cdot \frac{2j \sin\left(w\left(\frac{2M+1}{2}\right)\right)}{2j \sin\left(\frac{w}{2}\right)} = \frac{\sin\left(w\left(\frac{2M+1}{2}\right)\right)}{\sin\left(\frac{w}{2}\right)}$$

※ Note that $X(\omega)$ is not sinc. In DTFT, $X(\omega)$ has to be a periodic waveform (sinc is not periodic.)



Periodic.

Convolution

$$y[n] = x[n] * h[n] \quad \xleftrightarrow{\text{DTFT}} \quad Y(w) = X(w)H(w)$$

$$Y(w) = \sum_{n=-\infty}^{+\infty} y[n]\, e^{-jwn} \qquad \text{DTFT}$$

conv.

$$= \sum_{n=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k] \right) e^{-jwn}$$

$$= \sum_{k=-\infty}^{+\infty} x[k] \underbrace{\sum_{n=-\infty}^{+\infty} h[n-k]\, e^{-jwn}}_{H(w)e^{-jwk}} = H(w) \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\, e^{-jwk}}_{X(w)}$$

time shift vs phase shift

$$x[n-k] \longleftrightarrow X(w) \cdot e^{-jwk}$$

Frequency Response

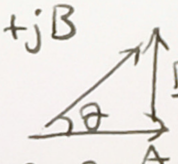
$$Ae^{jw_0 n} \longrightarrow \boxed{\begin{matrix} H \\ (LTI) \end{matrix}} \longrightarrow A \underbrace{|H(w_0)|}_{\text{amplitude scaling}} e^{j(w_0 n + \underbrace{\angle H(w_0)}_{\text{phase shift}})}$$

Example

$$X(w) = \frac{1}{1 - ae^{-jw}} = \frac{1}{1 - a\cos w + ja\sin w}$$

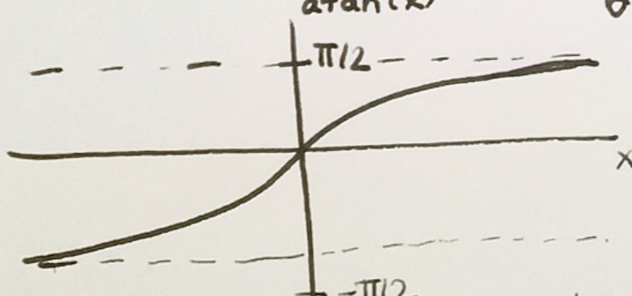
$$\angle X(w) = -\operatorname{atan}\Big(\underbrace{\frac{a \sin w}{1 - a\cos w}}_{\theta(w)}\Big)$$

$A + jB$



$\tan\theta = \frac{B}{A}$.

atan(x)



which is an increasing function of x.

$$\theta(w) = \frac{a\sin w}{1 - a\cos w}$$

$\theta'(w) > 0$ if $\underline{\cos w > a}$ Show!

$\theta'(w) < 0$ if $\cos w < a$