3a.) $x(n) = \delta[n]$

$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}$ $\quad k = 0, \cdots, N-1$

$= (W_N^{0})^N$

$= 1 \quad \forall k.$

3b.) $x[n] = \delta[n-n_0]$

$k = 0, \cdots, N-1$

$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$

$= W_N^{k \cdot 1} \sum_{n=0}^{N-1} W_N^{kn}$ $\quad k = 0, \cdots, N-1.$

$= W_N^{k} \cdot 1$ $\quad k = 0, \cdots, N-1$

$= e^{j\frac{2\pi}{N}k}$ $\quad k = 0, \cdots, N-1.$

3c.) $x[n] = a^n, \quad 0 \leq n \leq N-1.$

$X[k] = \sum_{n=0}^{N-1} a^n W_N^{nk} = \sum_{n=0}^{N-1} (aW_N^{k})^n$

$= \dfrac{1-(aW_N^{k})^N}{1-aW_N^{k}} = \dfrac{1-a^N}{1-aW_N^{k}}.$ $\quad |aW_N^{k}| < 1$

$|a| < 1$

$k = 0, \cdots, N-1$

3d.) $x(n) = \begin{cases} 1 & 0 \le n \le \frac{N}{2}-1 \\ 0 & \frac{N}{2} \le n \le N-1. \end{cases}$

$$X[K] = \sum_{n=0}^{N-1} x(n) W_N^{nk} \qquad K = 0, \cdots, N-1.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + \underbrace{\sum_{n=N/2}^{N-1} x[n] W_N^{nk}}_{=0}.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} W_N^{nk} = \sum_{n=0}^{\frac{N}{2}-1} (W_N^{k})^n$$

$$= \frac{1-(W_N^{k})^{\frac{N}{2}}}{1-W_N^{k}}$$

For $K=0$ $\quad X[0] = \sum_{n=0}^{\frac{N}{2}-1} 1 = \frac{N}{2}.$

For $K=$ even $\quad X[4] = \dfrac{1-(W_N)^{\frac{4N}{2}}}{1-W_N^{4}} = 0$

For $K=$ odd. $\quad \dfrac{1-(-1)}{1-e^{-j\frac{2\pi}{N}k}} = \dfrac{2}{e^{-j\frac{\pi}{N}k}\left(e^{j\frac{\pi}{N}k}-e^{-j\frac{\pi}{N}k}\right)} = \dfrac{2e^{j\frac{\pi}{N}k}}{2\sin(\frac{\pi}{N}k)} = \dfrac{e^{j\frac{\pi}{N}k}}{\sin(\frac{\pi}{N}k)}$

$$X[K] = \begin{cases} N/2 & K=0. \\ 0 & K \text{ even} \\ \dfrac{e^{j\frac{\pi}{N}k}}{\sin(\frac{\pi}{N}k)} & K \text{ odd} \end{cases} \qquad K = \{0, \cdots, N-1\}$$

3 e.) $x[n] = e^{j\frac{2\pi}{N}k_0 n}$.

$X[k] = \sum_{n=0}^{N-1} W_N^{-k_0 n} W_N^{kn} =$ $\qquad k = 0, \dots, N-1.$

$k = k_0 \Rightarrow \sum_{n=0}^{N-1} 1 = N$

$k \neq k_0 \Rightarrow \quad X[k] = \dfrac{1 - W_N^{(k-k_0)N}}{1 - W_N^{k-k_0}} = \dfrac{1 - e^{-j2\pi(k-k_0)}}{1 - e^{j\frac{2\pi}{N}(k-k_0)}}$

$$= \frac{e^{-j\pi(k-k_0)}\left[e^{j\pi(k-k_0)} - e^{-j\pi(k-k_0)}\right]}{e^{j\frac{\pi}{N}(k-k_0)}\left[e^{j\frac{\pi}{N}(k-k_0)} - e^{-j\frac{\pi}{N}(k-k_0)}\right]}$$

$$= e^{-j\pi(k-k_0)(1-\frac{1}{N})} \frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))}$$

$$X[k] = \begin{cases} N & k = k_0 \\ e^{-j\pi(k-k_0)(1-\frac{1}{N})} \dfrac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} & k = \{0, \dots, N-1\} - \{k_0\} \end{cases}$$

3f.) $x[n] = \cos(\frac{2\pi}{N} k_0 n)$, $0 \le n \le N-1$.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \qquad k = \{0, \dots, N-1\}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{2} e^{j\frac{2\pi}{N}k_0 n} + \frac{1}{2} e^{-j\frac{2\pi}{N}k_0 n}\right] W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2} W_N^{-k_0 n} W_N^{nk} + \frac{1}{2} W_N^{k_0 n} W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2} (W_N^{k-k_0})^n + \frac{1}{2} (W_N^{k+k_0})^n$$

$$= \frac{1}{2}\left[\frac{1-(W_N^{k-k_0})^N}{1-W_N^{k-k_0}} + \frac{1-(W_N^{k+k_0})^N}{1-W_N^{k+k_0}}\right]$$

$X[k]$:

$\downarrow$

$k = k_0$: $\sum_{n=0}^{N-1} \frac{1}{2} + \frac{1}{2} W_N^{2k_0 n} = \frac{N}{2} + \cancel{\frac{1}{2}\frac{1-(W_N^{2k_0})^N}{1-W_N^{2k_0}}} \rightarrow 0 = \frac{N}{2}$

$k = -k_0$ $\sum_{n=0}^{N-1} \frac{1}{2} W_N^{+2k_0 n} + \frac{1}{2} = 0 + N/2 = N/2.$

$k =$
$\{0, \dots, N-1\} - \{\pm k_0\}$ $\frac{1}{2}\left[\frac{1-(e^{-j\frac{2\pi}{N}(k-k_0)N})}{1-e^{-j\frac{2\pi}{N}(k-k_0)}} + \frac{1-(e^{-j\frac{2\pi}{N}(k+k_0)N})}{1-e^{-j\frac{2\pi}{N}(k+k_0)}}\right]$

$$= \frac{1}{2}\left[\frac{e^{-j\pi(k-k_0)}}{e^{-j\frac{\pi}{N}(k-k_0)}}\frac{(\sin(\pi(k-k_0))}{(\sin(\frac{\pi}{N}(k-k_0))} + \frac{e^{-j\pi(k+k_0)}}{e^{-j\frac{\pi}{N}(k+k_0)}}\frac{\sin(\pi(k+k_0))}{\sin(\frac{\pi}{N}(k+k_0))}\right]$$

$$= \frac{1}{2} e^{-j\pi(k-k_0)(1-\frac{1}{N})} \frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} + \frac{1}{2} e^{-j\pi(k+k_0)(1-\frac{1}{N})} \frac{\sin(\pi(k+k_0))}{\sin(\pi/N(k+k_0))}$$

39.) $x(n) = \sin(\frac{2\pi}{N} k_0 n)$, $0 \le n \le N-1$.

$$X[k] = \sum_{n=0}^{N-1} x(n) W_N^{nk} \qquad \forall k \in \{0, \dots, N-1\}$$

$$= \sum_{n=0}^{N-1} \sin(\tfrac{2\pi}{N} k_0 n) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2j}\left(e^{j\frac{2\pi}{N}k_0 n} - e^{-j\frac{2\pi}{N}k_0 n}\right) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2j}\left(W_N^{-k_0 n} - W_N^{+k_0 n}\right) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2j}\left(W_N^{(k-k_0)n} - W_N^{(k+k_0)n}\right).$$

For $k = k_0$

$$= \sum_{n=0}^{N-1} \frac{1}{2j}\left(W_N^{0} - W_N^{2k_0 n}\right)$$

$$= \frac{N}{2j} - 0$$

$$\cancel{= \frac{1}{2j}\underbrace{\left(\frac{1 - W_N^{2k_0 N}}{1 - W_N^{2k_0}}\right)}_{=0} - \frac{N}{2j}}$$

$$\cancel{= \frac{-N}{2j}}$$

For $k = -k_0$.

$$= \sum_{n=0}^{N-1} \frac{1}{2j}\left(W_N^{-2k_0 n} - 1\right)$$

$$= 0 - \frac{N}{2j}$$

$$= -N/2j$$

$$
X[K] = \begin{cases} +\dfrac{N}{2j} & K = k_0 \\ -\dfrac{N}{2j} & K = -k_0 \\ \dfrac{1}{2j}\,\dfrac{1-(W_N^{(k_0 N)})}{1-W_N^{k_0+k}} - \dfrac{1}{2j}\,\dfrac{1-(W_N^{+k_0 N})}{1-W_N^{k-k_0}} \end{cases}
$$

Simplified more in next page!

$K = \{0, \cdots, N-1\} - \{\pm k_0\}$

~~3g.)~~

3g.)

For $k = \{0, \cdots, N-1\} - \{\pm k_0\}$

$$X[k] = \sum_{n=0}^{N-1} \frac{1}{2j}\left(W_N^{(k-k_0)n} - W_N^{(k+k_0)n}\right)$$

$$= \frac{1}{2j}\left[\frac{1-W_N^{(k-k_0)N}}{1-W_N^{k-k_0}} - \frac{1-W_N^{(k+k_0)N}}{1-W_N^{(k+k_0)}}\right]$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j2\pi(k-k_0)}}{1-e^{-j\frac{2\pi}{N}(k-k_0)}} - \frac{1-e^{-j2\pi(k+k_0)}}{1-e^{-j\frac{2\pi}{N}(k+k_0)}}\right]$$

$$= \frac{1}{2j}\frac{e^{-j\pi(k-k_0)}}{e^{-j\frac{\pi}{N}(k-k_0)}}\frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} - \frac{1}{2j}\frac{e^{-j\pi(k+k_0)}}{e^{-j\frac{\pi}{N}(k+k_0)}}\frac{\sin(\pi(k+k_0))}{\sin(\frac{\pi}{N}(k+k_0))}$$

$$= \frac{1}{2j}\frac{e^{-j\pi(k-k_0)(1-\frac{1}{N})}\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} - \frac{1}{2j}\frac{e^{-j\pi(k+k_0)(1-\frac{1}{N})}\sin(\pi(k+k_0))}{\sin(\frac{\pi}{N}(k+k_0))}$$

3h.) $x(n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad 0 \le n \le N-1$

$$X[k] = \sum_{n=0}^{N-1} x[n] \, W_N^{nk}$$

$n = 2r$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] \, W_N^{2rk} + \sum_{r=\frac{N}{2}}^{N-1} \cancel{x[2r+1]}^{\,0} \, W_N^{2rk+k}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} W_N^{2rk} + 0$$

$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} (W_N^{2k})^r$$

for $k=0$: $X[0] = \sum_{r=0}^{\frac{N}{2}-1} 1 = N/2$

$k \neq 0$: $X[k] = \dfrac{1 - (W_N^{2k})^{\frac{N}{2}}}{1 - W_N^{k}} = 0$

For odd k: $X[k] = 0$

For even k: $X[k] = 0$

$$X[k] = \begin{cases} N/2 & k = 0 \\ 0 & k = \{0, \cdots, N-1\} - \{0\} \end{cases}$$

HW #4 DSP.

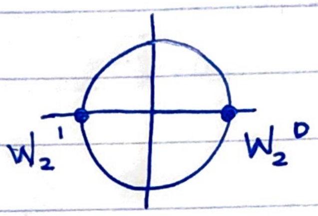
4.) DFT matrix.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

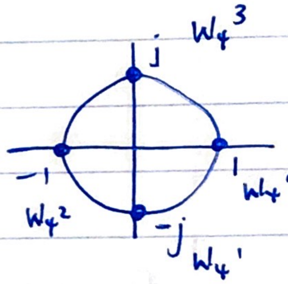
a.) Plot $N=2$ in complex plane.

$$W_2^{k} = e^{-j\frac{2\pi}{2}k} = e^{-j\pi k}$$

$k = \{0, 1\}$



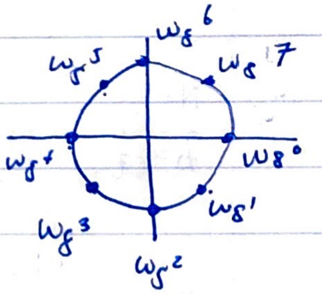
$N = 4$



$$W_4^{k} = \{W_4^0, W_4^1, W_4^2, W_4^3\}$$

$k = \{0, 1, 2, 3\}$

$N=8$



$k = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$W_8^{k} = \{W_8^0, W_8^1, W_8^2, \dots, W_8^7\}$

$W_8^{k} = \{1, W_8^1, -j, W_8^3, -1, W_8^5, j, W_8^7\}$

4b.) Write $N \times N$ DFT matrix for $N=2$.
in terms of W_N's

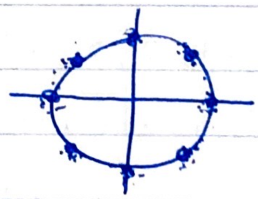
$N=2$
$K=\{0,1\}$

$$\begin{bmatrix} W_2^{0} & W_2^{0} \\ W_2^{0\cdot 1} & W_2^{1\cdot 1} \end{bmatrix}$$

$$\begin{bmatrix} W_2^{0} & W_2^{0} \\ W_2^{0} & W_2^{1} \end{bmatrix}$$

Repeat for W_N's where $N=8$.

$N=8$
$K=\{0,\dots,7\}$

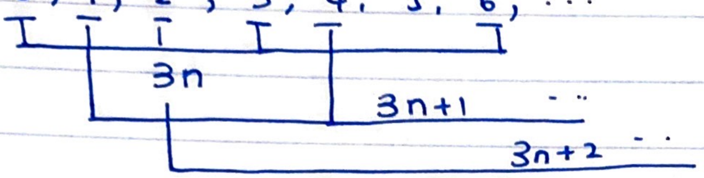


$$\begin{array}{c} n=0 \\ n=1 \\ n=2 \\ n=3 \\ n=4 \\ n=5 \\ n=6 \\ n=7 \end{array}
\begin{bmatrix}
W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} \\
W_8^{0} & W_8^{1} & W_8^{2} & W_8^{3} & W_8^{4} & W_8^{5} & W_8^{6} & W_8^{7} \\
W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} & W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} \\
W_8^{0} & W_8^{3} & W_8^{6} & W_8^{1} & W_8^{4} & W_8^{7} & W_8^{2} & W_8^{5} \\
W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} \\
W_8^{0} & W_8^{5} & W_8^{2} & W_8^{7} & W_8^{4} & W_8^{1} & W_8^{6} & W_8^{3} \\
W_8^{0} & W_8^{6} & W_8^{4} & W_8^{2} & W_8^{0} & W_8^{6} & W_8^{4} & W_8^{2} \\
W_8^{0} & W_8^{7} & W_8^{6} & W_8^{5} & W_8^{4} & W_8^{3} & W_8^{2} & W_8^{1}
\end{bmatrix}$$

5.) Given: 24 point ($N=24$) $x[n]$

3 FFT Chip ($K=8$ per chip)

Plan: $\{0, 1, 2, 3, 4, 5, 6, \dots$



For chip 1, we can have input of $x(n)$ where index n: $3n$ $n \in \{0, \dots 7\}$

For chip 2, we can have input of $x(n)$ where index n: $3n+1$ $n \in \{0, \dots, 7\}$

For chip 3, we can have input of $x(n)$ where index n: $3n+2$ $n \in \{0, \dots, 7\}$.

$$X[K] = \sum_{r=0}^{\frac{N}{3}-1} X[3n] W_N^{3Kn} + \sum_{r=\frac{N}{3}}^{\frac{2N}{3}-1} X[3n+1] W_N^{(3n+1)K} + \sum_{r=\frac{2N}{3}}^{N-1} x[3n+2] W_N^{(3n+2)K}$$

$r = 3n$

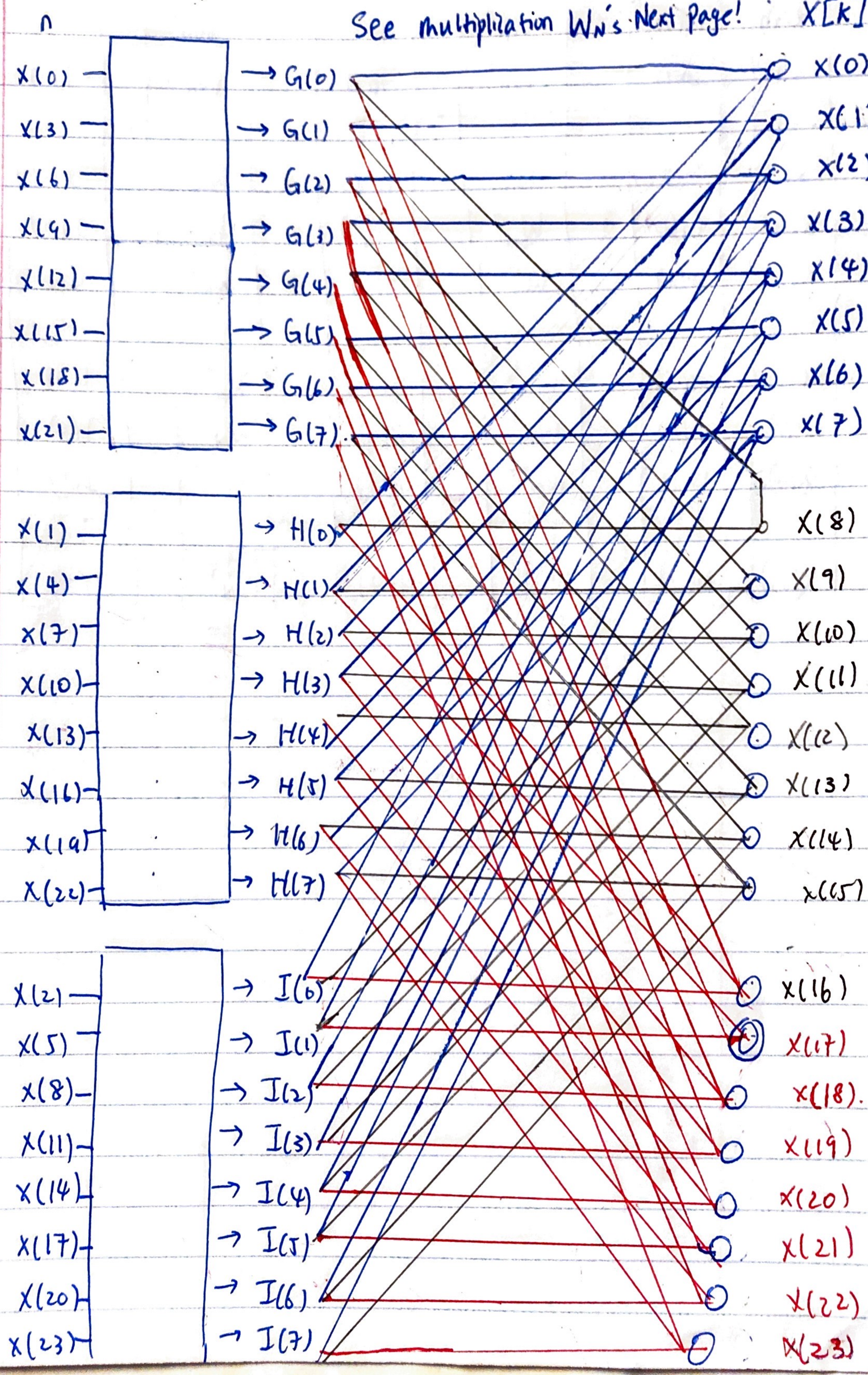
$$= \overbrace{\sum_{r=0}^{\frac{N}{3}-1} X[3n]}^{G[K]} + \overbrace{\sum_{r=0}^{\frac{N}{3}-1} X[3n+1] W_N^K}^{W_N^K H[K]} + \overbrace{\sum_{r=0}^{\frac{N}{3}-1} X[3n+2] W_N^{2K}}^{W_N^{2K} I[K]}$$

For $K \; \forall \{0, \dots, 7\}$

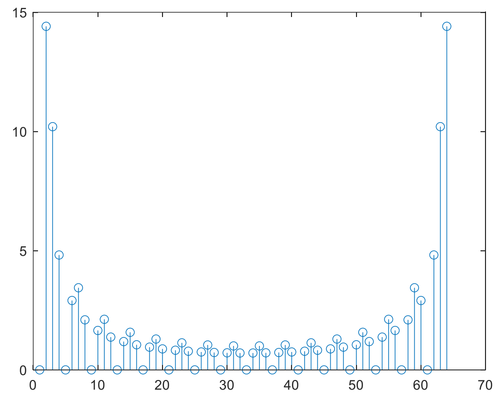
$$X[K+8] = \sum_{n=0}^{\frac{N}{3}-1} x[3n] + \sum_{n=0}^{\frac{N}{3}-1} x[3n+1] W_N^{(K+8)} + \sum_{n=0}^{N/3-1} X[3n+2] W_N^{2(K+8)}$$

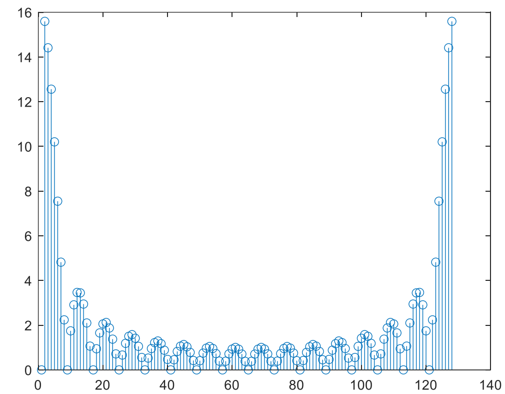
$$X[K+16] = \sum_{n=0}^{\frac{N}{3}-1} X[3] + \sum_{n=0}^{\frac{N}{3}-1} x[3n+1] W_N^{K+16} + \sum_{n=0}^{\frac{N}{3}-1} X[3n+2] W_N^{2(K+16)}$$

See multiplication W_N's Next page!

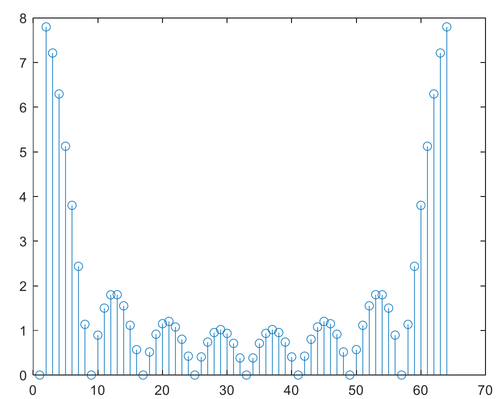


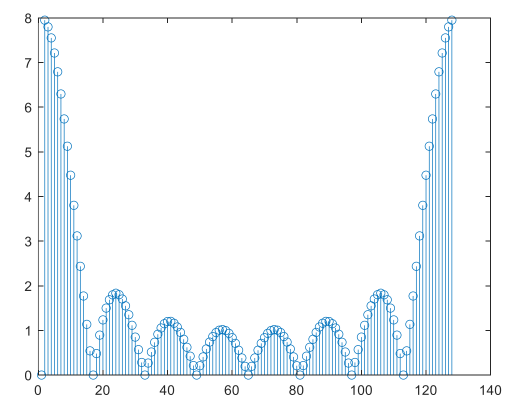
$$
\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(7) \\ X(8) \\ X(9) \\ \vdots \\ X(15) \\ X(16) \\ X(17) \\ \vdots \\ X(23) \end{bmatrix}
=
\begin{bmatrix} G(0) \\ G(1) \\ \vdots \\ G(7) \\ G(0) \\ G(1) \\ \vdots \\ G(7) \\ G(0) \\ G(1) \\ \vdots \\ G(7) \end{bmatrix}
+
\begin{bmatrix} W_N^{0} \\ W_N^{1} \\ \vdots \\ W_N^{7} \\ W_N^{8} \\ W_N^{9} \\ \vdots \\ W_N^{15} \\ W_N^{16} \\ W_N^{17} \\ \vdots \\ W_N^{23} \end{bmatrix}
\cdot
\begin{bmatrix} H(0) \\ H(1) \\ \vdots \\ H(7) \\ H(0) \\ H(1) \\ \vdots \\ H(7) \\ H(0) \\ H(1) \\ \vdots \\ H(7) \end{bmatrix}
+
\begin{bmatrix} W_N^{0} \\ W_N^{2} \\ \vdots \\ W_N^{14} \\ W_N^{16} \\ W_N^{18} \\ \\ W_N^{30} \\ W_N^{32} \\ W_N^{34} \\ \vdots \\ W_N^{46} \end{bmatrix}
\cdot
\begin{bmatrix} I(0) \\ I(1) \\ \vdots \\ I(7) \\ I(0) \\ I(1) \\ \vdots \\ I(7) \\ I(0) \\ I(1) \\ \vdots \\ I(7) \end{bmatrix}
$$

<table>
<tr><td>N = 64, Length = 16</td><td>N=128, Length = 16</td></tr>
<tr><td>


Frequency interval between successive samples for the plot in part a is $\frac{2\pi}{64}$
At w = 0, the value is 16
Interval between null samples is $\frac{2\pi}{64} * 4 = \frac{2\pi}{16}$
The null sample interval is always $\frac{2\pi}{Length}$
</td><td>


Frequency interval between successive samples for the plot in part a is $\frac{2\pi}{128}$
At w = 0, the value is 16
Interval between null samples is $\frac{2\pi}{128} * 8 = \frac{2\pi}{16}$
The null sample interval is always $\frac{2\pi}{Length}$
</td></tr>
<tr><td colspan="2">The one with greater N has a faster sampling rate than the smaller N, the one with higher N look closer to the continuous time signal.</td></tr>
</table>

<table>
<tr><td>N = 64, Length = 8</td><td>N = 128, Length = 8</td></tr>
<tr><td>


Frequency interval between successive samples for the plot in part a is $\frac{2\pi}{64}$
At w = 0, the value is 8
Interval between null samples is $\frac{2\pi}{64} * 8 = \frac{2\pi}{8}$
The null sample interval is always $\frac{2\pi}{Length}$
</td><td>


Frequency interval between successive samples for the plot in part a is $\frac{2\pi}{128}$
At w = 0, the value is 16
Interval between null samples is $\frac{2\pi}{128} * 16 = \frac{2\pi}{8}$
The null sample interval is always $\frac{2\pi}{Length}$
</td></tr>
<tr><td colspan="2">The one with greater N has a faster sampling rate than the smaller N, the one with higher N look closer to the continuous time signal.</td></tr>
</table>