10/26/20

Today's lecture

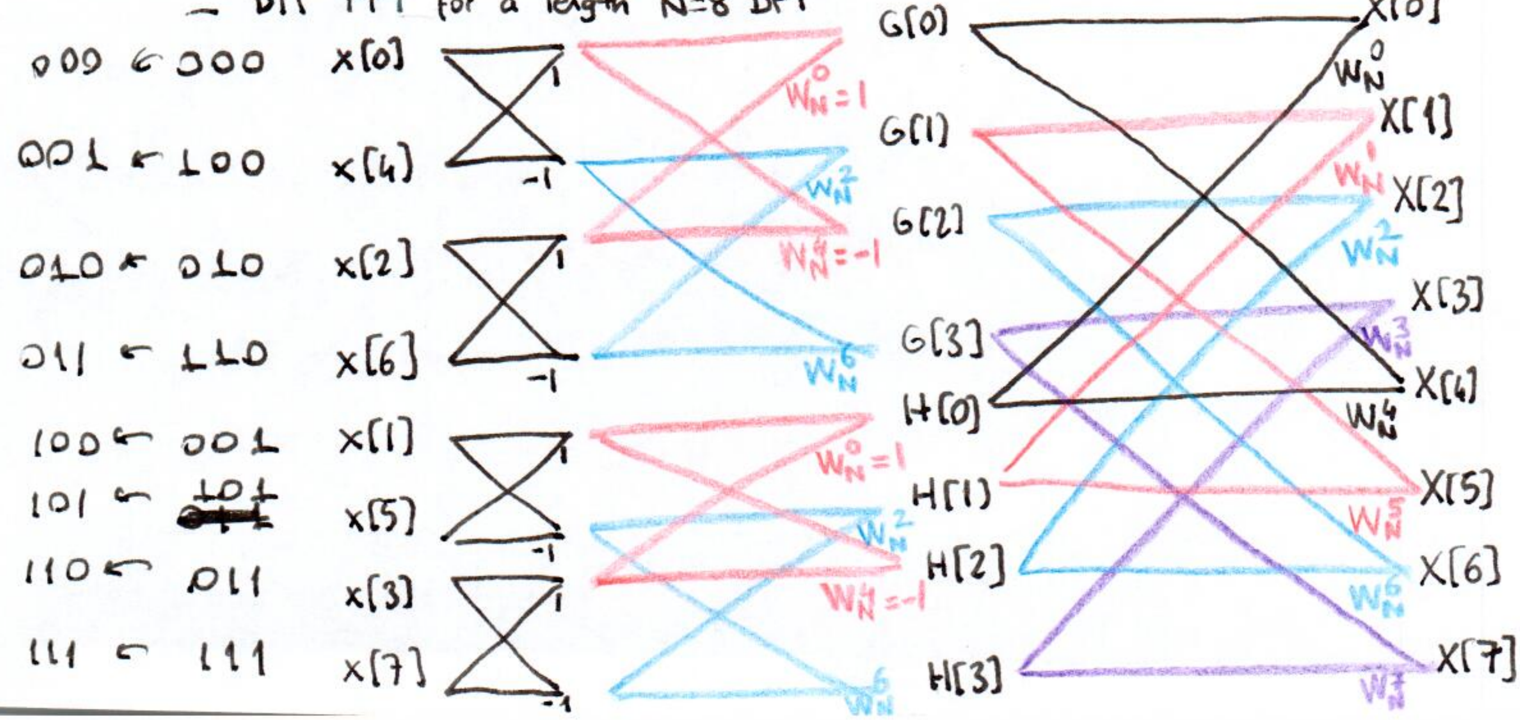
- Decimation in frequency (DIF) FFT
- The sampling theorem

Readings : 8-1, 8-2 FFT
6-1, 6-2 Sampling

Question on Piazza: Post the solutions today.

Last lecture : DIT FFT

- The radix-2 DIT : It divides a DFT of size N into 2 interleaved DFTs of size $\frac{N}{2}$ with each recursive stage.
- The radix-2 DIT first computes the DFTs of the even indexed input $x[2\ell]$ and odd indexed input $x[2\ell+1]$, $\ell = 0, 1, \dots, \frac{N}{2}-1$, and then combines those two results to produce the DFT.
- Applying radix-2 DIT recursively the overall runtime is reduced to $O(N\log N)$
- DIT FFT for a length N=8 DFT



$$
\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccccccc} 0^{th} & & 2^{nd} & & 4^{th} & & 6^{th} & \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & W_8 & W_8^2 & W_8^3 & -1 & -W_8 & -W_8^2 & -W_8^3 \\
1 & W_8^2 & -1 & -W_8^2 & 1 & W_8^2 & -1 & -W_8^2 \\
1 & W_8^3 & -W_8^2 & W_8 & -1 & -W_8^3 & W_8^2 & -W_8 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -W_8 & W_8^2 & -W_8^3 & -1 & W_8 & -W_8^2 & W_8^3 \\
1 & -W_8^2 & -1 & W_8^2 & 1 & -W_8^2 & -1 & W_8^2 \\
1 & -W_8^3 & -W_8^2 & -W_8 & -1 & W_8^3 & W_8^2 & W_8
\end{bmatrix}
\end{array}
\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}
$$

8x8 matrix "Dftmtx" $N=8$

Even Columns

$$
\begin{bmatrix} I_{4\times 4} \\ I_{4\times 4} \end{bmatrix}
\underbrace{\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & W_8^2 & -1 & -W_8^2 \\
1 & -1 & 1 & -1 \\
1 & -W_8^2 & -1 & W_8^2
\end{bmatrix}}_{F_4}
\quad \text{where} \quad
I_{4\times 4} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

DFT for $N=4$

Odd Columns

$$
\begin{bmatrix} I_{4\times 4} \\ -I_{4\times 4} \end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
W_8 & W_8^3 & -W_8 & -W_8^3 \\
W_8^2 & -W_8^2 & W_8^2 & -W_8^2 \\
W_8^3 & W_8 & -W_8^3 & -W_8
\end{bmatrix}
=
\begin{bmatrix} I_{4\times 4} \\ -I_{4\times 4} \end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & W_8 & 0 & 0 \\
0 & 0 & W_8^2 & 0 \\
0 & 0 & 0 & W_8^3
\end{bmatrix}
F_4
$$

$$F_8 = \underbrace{\begin{bmatrix} I_{4\times4} & I_{4\times4} \\ I_{4\times4} & -I_{4\times4} \end{bmatrix}}_{8\times8 \text{ matrix}} \underbrace{\begin{bmatrix} I_{4\times4} & O \\ O & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & W_8 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \end{bmatrix}}_{8\times8 \quad \text{'Twiddle factors'}} \underbrace{\begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix}}_{8\times8}$$

What about when N is not even?

1 Cooley-Tukey FFT

FFT for general N

2 Good-Thomas FFT

FFT to get rid of the twiddle factors $W_8^{\ell r}$.

Decimation in Frequency (N even)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

Let's consider the even samples of $X[k]$:

$$X[2\ell] = \sum_{n=0}^{N-1} x[n] W_N^{n.2\ell}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{n.2\ell} + \sum_{n=N/2}^{N-1} x[n] W_N^{n.2\ell}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{n\ell} + \sum_{n=0}^{N/2-1} x\left[n+\tfrac{N}{2}\right] \underbrace{W_N^{\left(n+\frac{N}{2}\right)2\ell}}_{W_N^{n2\ell} \cdot \underbrace{W_N^{N\ell}}_{1}}$$

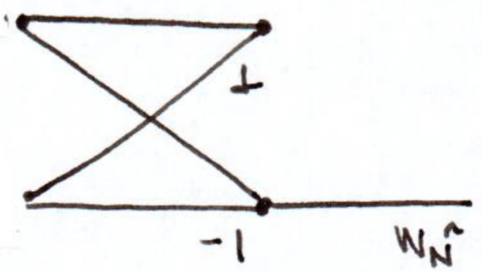
$$W_N^{n2\ell} \to W_{N/2}^{n\ell}$$

$$X[2\ell] = \sum_{n=0}^{N/2-1} \left(x[n] + x\left[n+\frac{N}{2}\right]\right) W_{N/2}^{n\ell}$$

which is like a N/2 DFT of the summed input (top half & bottom half)

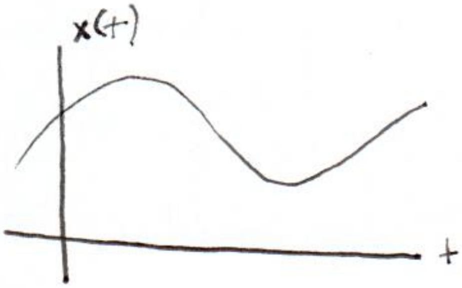
For the odd samples of $X[k]$ we have

$$X[2\ell+1] = \sum_{n=0}^{N/2-1} \left(x[n] - x\left[n+\frac{N}{2}\right]\right) \underbrace{W_N^n}_{\text{Twiddle factor}} W_{N/2}^{n\ell}$$

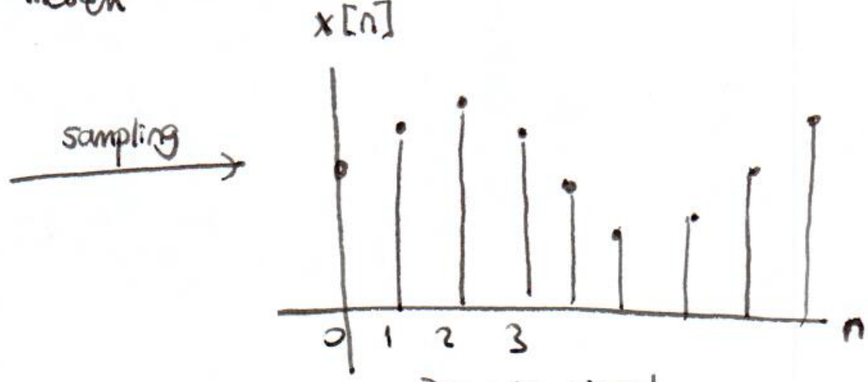


butterfly for DIF FFT

The Sampling Theorem



Continuous signal



Discrete signal

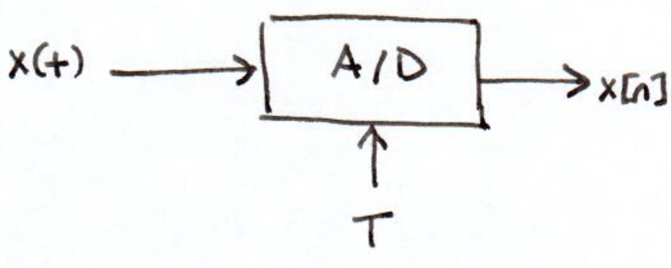
When the inputs are quantized
$\Rightarrow$ digital signal.

Periodic Sampling

$x[n] = x(nT)$ where n is an integer, T is sampling period

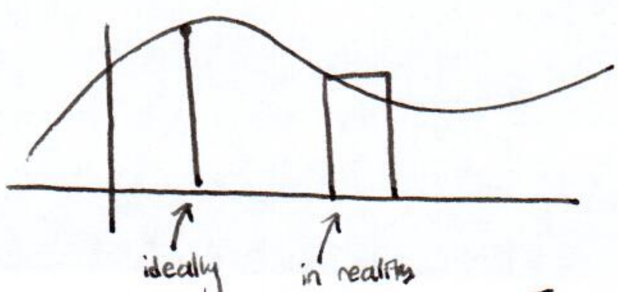
$$W_s = \frac{2\pi}{T}$$ (radians) sampling frequency

$$f_s = \frac{1}{T}$$ (Hertz, Hz) sampling frequency



Non-ideal effects

1. Ideally we multiply $x(t)$ with a shifted impulse train. Instead of sampling $x(t)$ we end up sampling $x(t) * h(t)$
(impulse response of the filter)

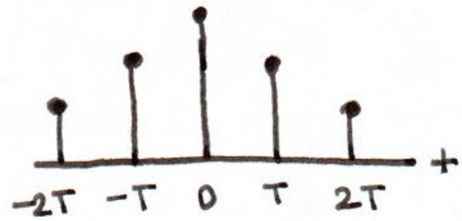


2. Noise or distortion

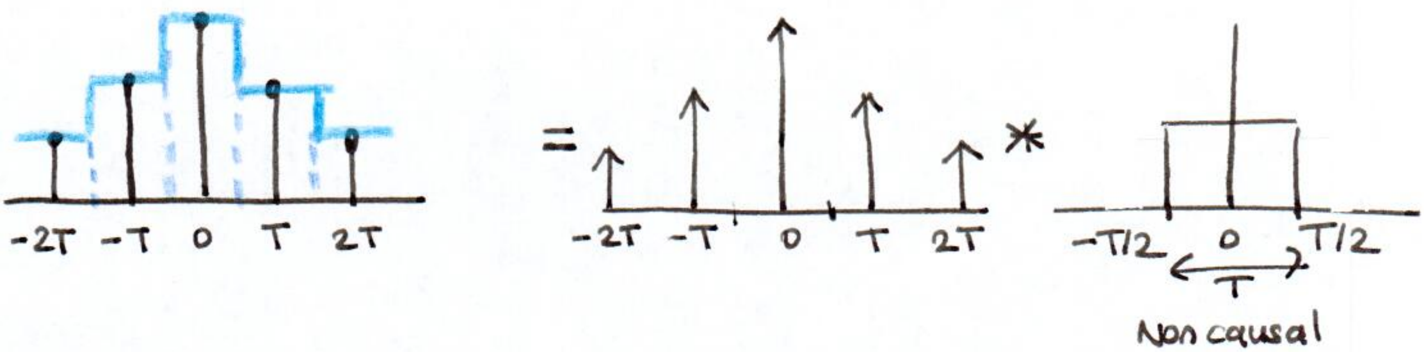
$$x[n] = y(nT) + z[n]$$
(noise)

Reconstructing CT signal $x_c(t)$ given $x[n]$.

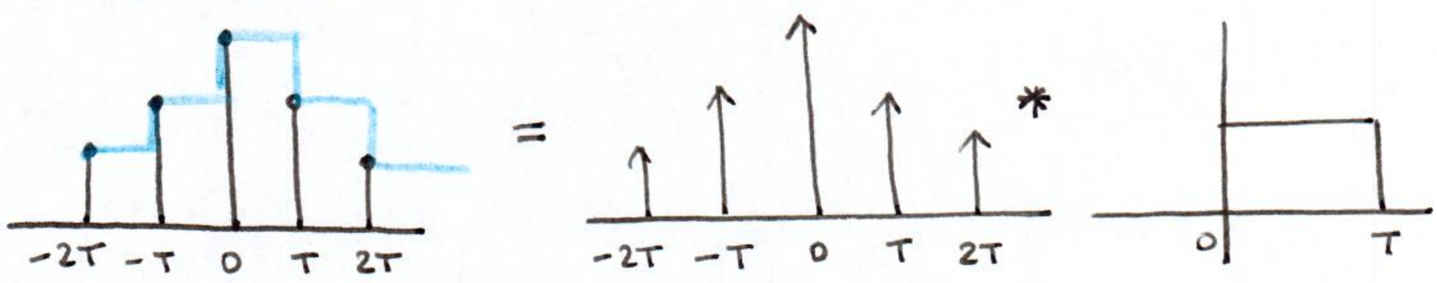
Assume $x[n] = x(nT)$



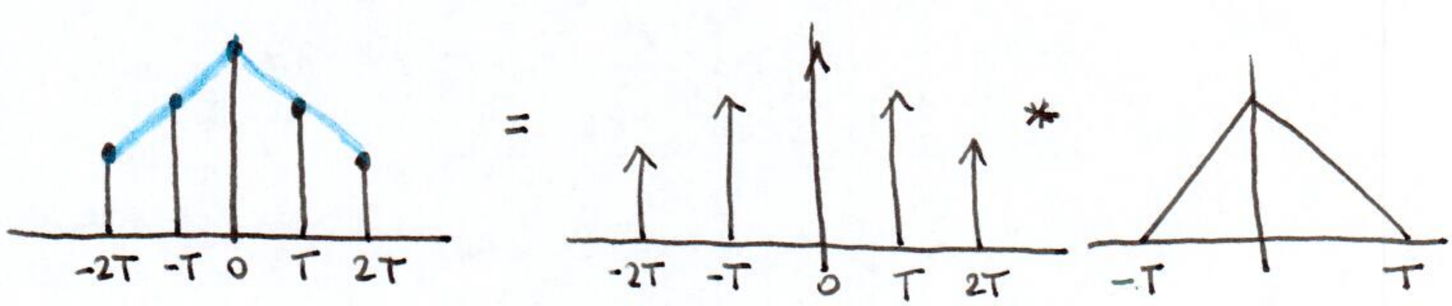
1. Nearest neighbor



2. Zero-order hold

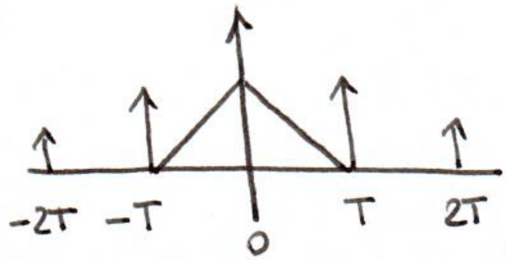


3. First-order hold



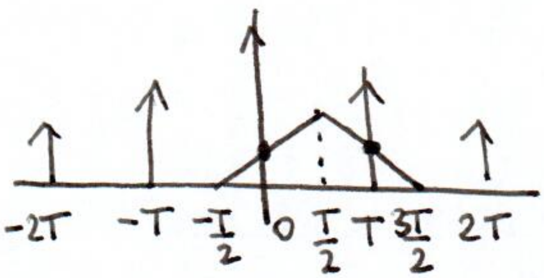
- Zero-order hold and nearest neighbor only consider individual samples.

- First-order hold linearly weights 2 consecutive samples:



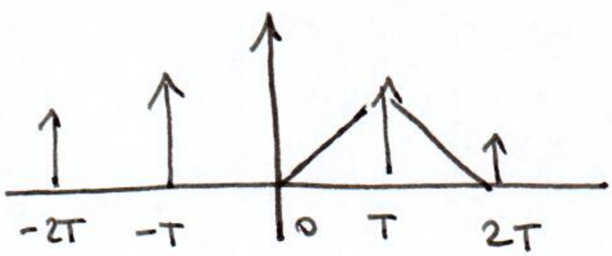
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

$$y(0) = \int_{-T}^{T} x(\tau) h(\tau) d\tau = x(0)$$



$$y\left(\tfrac{T}{2}\right) = \int_{-\frac{T}{2}}^{\frac{3T}{2}} x(\tau) h\left(\tau - \tfrac{T}{2}\right) d\tau$$

$$= \underline{\frac{x(0) + x(T)}{2}}$$



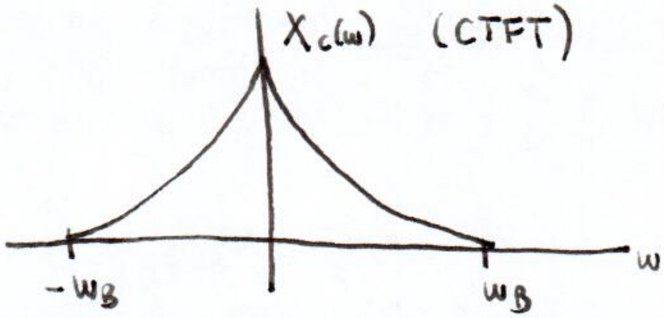
$$y(T) = \int_{0}^{2T} x(\tau) h(\tau - T) d\tau$$

$$= x(T)$$

Question: What is the correct interpolator?

Answer: Sinc interpolation

- We require the input signal to be bandlimited.

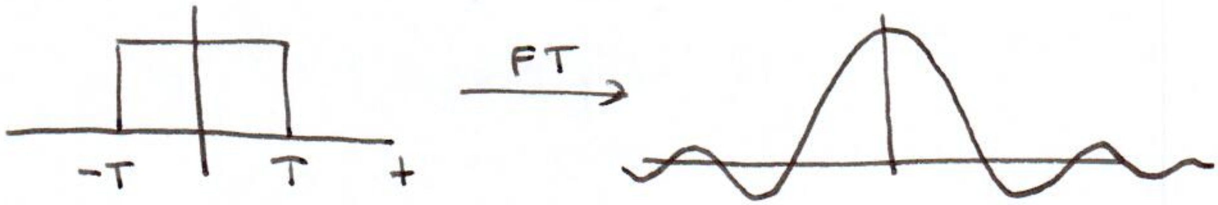


$$X_c(\omega) = 0, \quad |\omega| > \omega_B$$

* If we have finite duration in time (time-limited) → not band-limited.

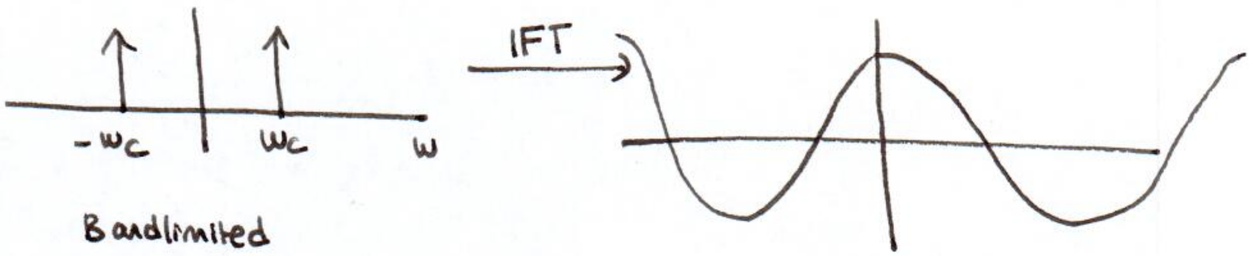
bandlimited signal → not time limited.

For example



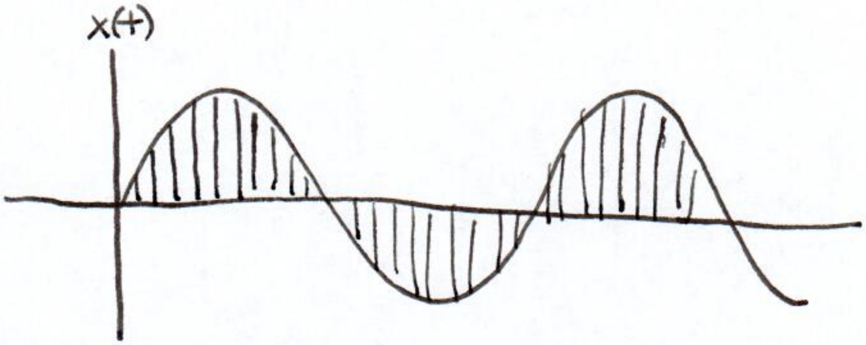
Time limited

Not band-limited

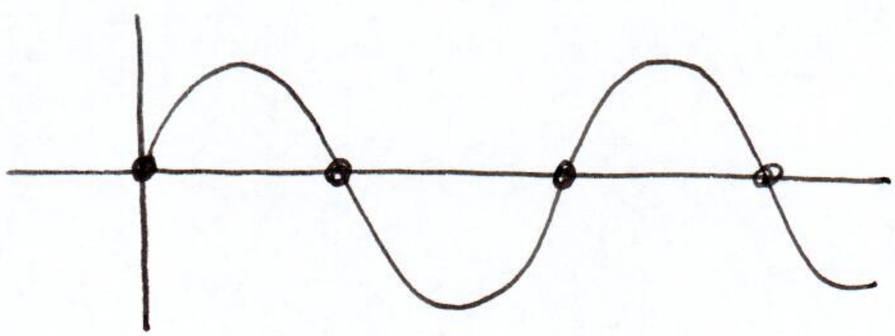


Bandlimited

Not time-limited



← Good example
samples are close enough



← Bad example
samples are far away

Nyquist - Shannon Sampling Theorem

A bandlimited signal with maximum frequency w_B can be perfectly reconstructed from its evenly spaced samples if the sampling frequency w_s satisfies

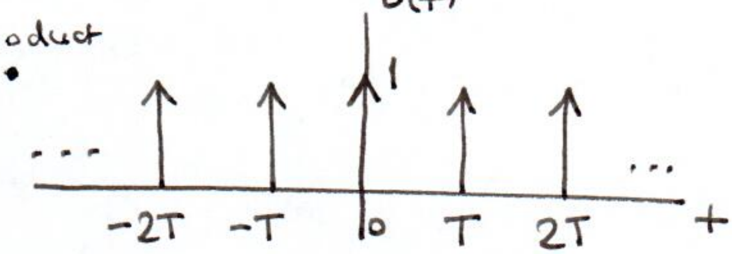
$$w_s > \underbrace{2w_B}_{\text{The Nyquist rate}}$$

↓ Strict inequality

continuous

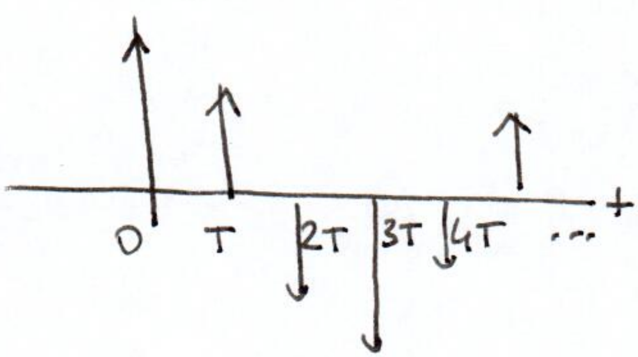
$x_c(t)$

product ·



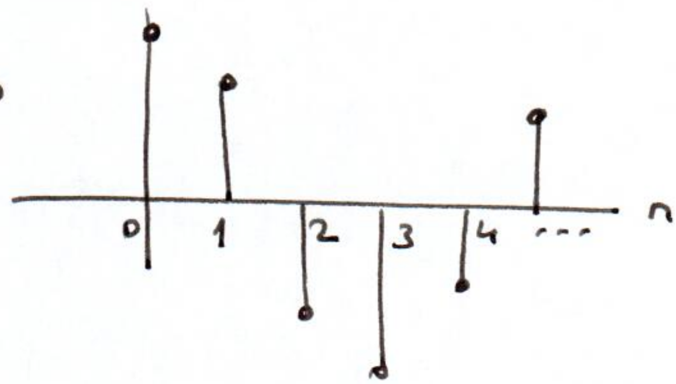
sampled

$x_s(t) = x_c(t) \cdot d(t)$



$\Longrightarrow$

$x[n]$



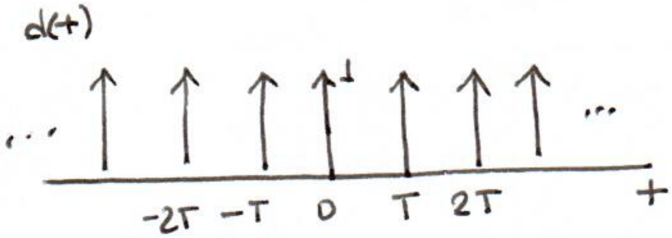
$d(t)$ is periodic $\Longrightarrow$ FS expansion $\quad d(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j\frac{2\pi}{T}kt}$

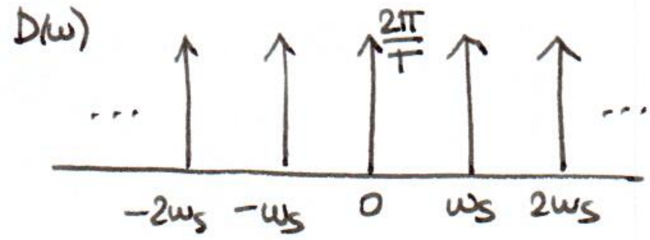
where $\quad a_k = \frac{1}{T}\int_{-T/2}^{T/2} d(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$

$$= \frac{1}{T}$$

$$x_s(t) = x_c(t) \cdot d(t) \xleftrightarrow{FT} \frac{1}{2\pi} X_c(\omega) * D(\omega) \qquad \text{CTFT}$$

$$d(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j\frac{2\pi}{T}kt} \xleftrightarrow{FT} D(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$





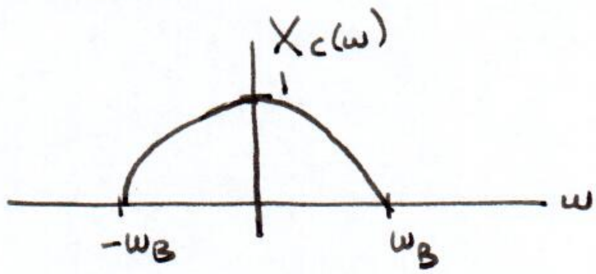
$$\omega_s = \frac{2\pi}{T}$$

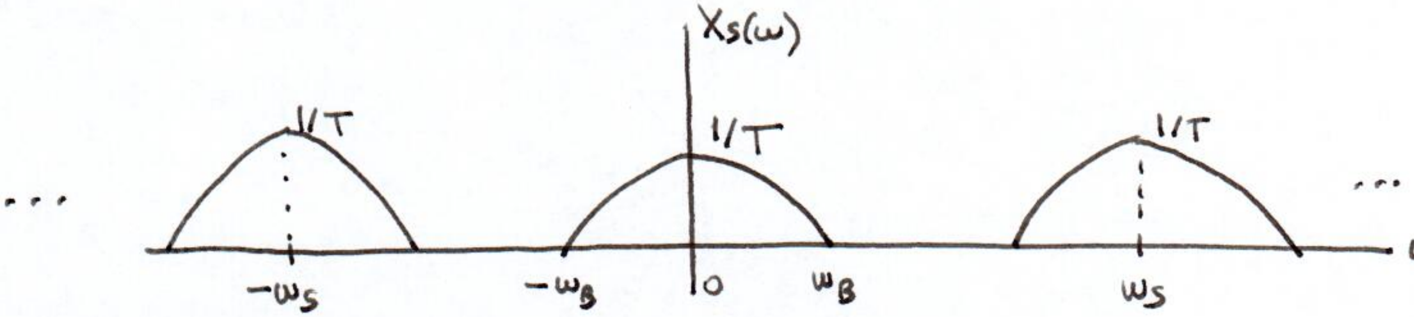
Therefore,

$$X_s(\omega) = \frac{1}{2\pi} X_c(\omega) * D(\omega)$$

$$= \frac{1}{2\pi} X_c(\omega) * \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - \omega_s k)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(\omega - \omega_s k)$$





* Need to low pass filter $X_s(\omega)$ to get $X_c(\omega)$ (original signal)

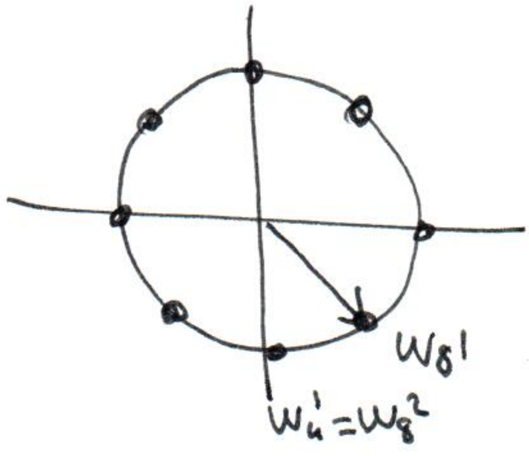
Question: When can we perfectly recover $X_c(\omega)$?

$$y[n] = \underline{x[0]} \; \overset{\frown}{x[2]} \; \underline{x[4]} \; \overset{\frown}{x[6]} \rightarrow \text{odd}$$

even

$$z[n] = \underline{x[1]} \; \overset{\frown}{x[3]} \; \underline{x[5]} \; \overset{\frown}{x[7]} \rightarrow \text{odd}$$

even



FS

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi}{T}kt}\, d$$

$$= \frac{1}{T}$$