Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2020

Exam #1.
October 8, 2020, 10:10-11:30 AM

Show all work for full credit.

- Closed book, closed notes.
- 1 one-sided crib sheet is allowed.
- Electronic devices (including calculators) are not permitted.
- Reduce expressions involving exponentials, sines, and cosines as much as possible.
- If the sinc function is needed, use the definition $\mathrm{sinc}(x) = \frac{\sin x}{x}$.
- Geometric series formula: $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$.
- Finite sum formula: $\sum_{n=M}^{N-1} a^n = \frac{a^M - a^N}{1-a}, \quad a \neq 1$.
- When in doubt, show your work.

Good luck!

1		**20**
2		**35**
3		**45**
Total		**100**

Append the below phrase into your handwritten solutions and upload a single pdf file that includes your handwritten crib sheets to Gradescope.

I am aware of the Academic Integrity policy. I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Name

Signature

1. (20 points.) **Discrete-time system properties.** Consider the system given by the input-output relationship

$$y[n] = x[n-1]\cos^2\left(\frac{\pi}{8}(n+1)+\frac{\pi}{4}\right)$$

You need to prove if each of the below statements (a)-(c) is true, and otherwise give a counter example.

Determine if the given system is

(a) (5 points.) linear.

Let $x_1[n] \Longleftrightarrow y_1[n]$ and $x_2[n] \Longleftrightarrow y_2[n]$ for the given system. A linear system would satisfy $z[n] = ax_1[n] + bx_2[n] \Longleftrightarrow ay_1[n] + by_2[n]$. Let's check if this is true:

The output for the given input $z[n] = ax_1[n] + bx_2[n]$ is $y[n] = z[n-1]\cos^2\left(\frac{\pi}{8}(n+1)+\frac{\pi}{4}\right) = (ax_1[n-1]+bx_2[n-1])\cos^2\left(\frac{\pi}{8}(n+1)+\frac{\pi}{4}\right) = ax_1[n-1]\cos^2\left(\frac{\pi}{8}(n+1)+\frac{\pi}{4}\right)+bx_2[n-1]\cos^2\left(\frac{\pi}{8}(n+1)+\frac{\pi}{4}\right) = ay_1[n]+by_2[n]$. **Therefore, it is linear.**

(b) (5 points.) time-invariant.

Let $x[n] \Longleftrightarrow y_1[n]$. We want to check if $z[n] = x[n-n_0] \Longleftrightarrow y_2[n] = y_1[n-n_0]$. The output for the given input $z[n] = x[n-n_0]$ is $y_2[n] = z[n-1]\cos^2\left(\frac{\pi}{8}(n+1)+\frac{\pi}{4}\right) = x[n-1-n_0]\cos^2\left(\frac{\pi}{8}(n+1)+\frac{\pi}{4}\right)$. However, $y_1[n-n_0] = x[n-1-n_0]\cos^2\left(\frac{\pi}{8}(n-n_0+1)+\frac{\pi}{4}\right) \neq y_2[n]$. **Therefore, it is time varying.**

(c) (5 points.) causal.

Yes. The output at time n is determined by the value of the input at time $n-1$.

(d) (5 points.) Determine the output $y[n]$ if the input is $x[n] = \cos\left(\frac{\pi}{8}n\right)$.

$$y[n] = \cos\left(\frac{\pi}{8}n - \frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}(n+1)+\frac{\pi}{4}\right)$$

2. (35 points.) **Z-transform.** Consider the system which has the following transfer function:

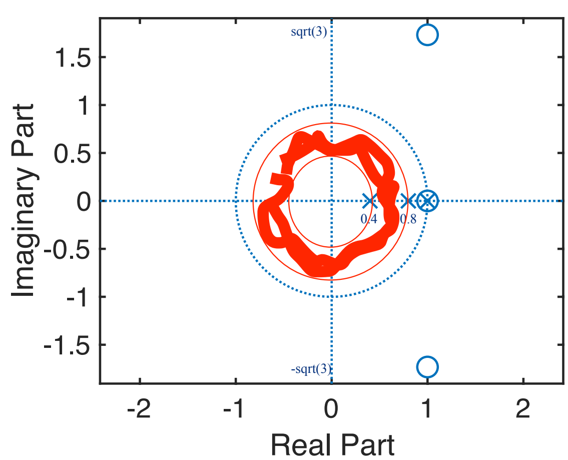
$$H(z) = \frac{1-3z^{-1}+6z^{-2}-4z^{-3}}{(1-z^{-1})(1-0.4z^{-1})(1-0.8z^{-1})}, \quad \text{ROC: } a<|z|<b$$

(a) (10 points.) Plot the pole-zero diagram for the given system. Indicate the ROC.

The transfer function can be rewritten as

$$H(z) = \frac{z^3-3z^2+6z-4}{(z-1)(z-0.4)(z-0.8)} = \frac{(z-1)(z^2-2z+4)}{(z-1)(z-0.4)(z-0.8)}, \quad \text{ROC: } a<|z|<b.$$

The system has 3 poles (0.4, 0.8, 1) and 3 zeros ($1, 1+\sqrt{3}, 1-\sqrt{3}$). Note that the zero and pole at 1 cancel each other.



Since the ROC is $a<|z|<b$ for finite a and b, it is disk shaped.

(b) (5 points.) Determine the finite constants a and b.
Since the ROC cannot contain any poles $a = 0.4$, $b = 0.8$.

(c) (3 points.) Is this system stable? Explain your reasoning.
It is not stable because the ROC does not contain the unit circle.

(d) (2 points.) Is this system causal? Explain.
It is not causal because the ROC is not outwards.

(e) (10 points.) Determine the impulse response $h[n]$ of the system.
Rewriting (a) we obtain

$$H(z) = \frac{z^2 - 2z + 4}{(z-0.4)(z-0.8)} = 1 - \frac{0.8z - 3.68}{(z-0.4)(z-0.8)} = 1 - \frac{A}{z-0.4} - \frac{B}{z-0.8}, \quad \text{ROC: } 0.4 < |z| < 0.8$$

where you can find out that $A = 8.4$ and $B = -7.6$.
Hence, $h[n] = \delta[n] - A(0.4)^{n-1}u[n-1] + B(0.8)^{n-1}u[-n]$. Note that we used the time-shift property.

(f) (5 points.) Discuss whether $h[n]$ is even or not.
It is not even because $h[n] \neq h[-n]$.

3. (45 points.) **Mixed bag.** The parts of this problem are independent of each other. The idea here is to use your knowledge of the Linear time-invariant (LTI) systems, discrete-time signals, Discrete Time Fourier Transform (DTFT) properties, such as oddness, evenness, Parseval's relation, $\sum_{n=-\infty}^{\infty} x[n] = X(0)$ and $x[0] = \frac{1}{2\pi}\int_{2\pi} X(\omega)\mathrm{d}\omega$, stability, causality, etc. You can refer to the tables to verify your solutions.

(a) (7 points.) True/false and fill in the blank questions. You do not need to explain your reasoning. Read each question carefully.

T LTI systems can be completely characterized by its impulse response.

F $x[n]\delta[n-1] = x[1]$

T $x[n] * \delta[n+1] = x[n+1]$ where $*$ denotes convolution.

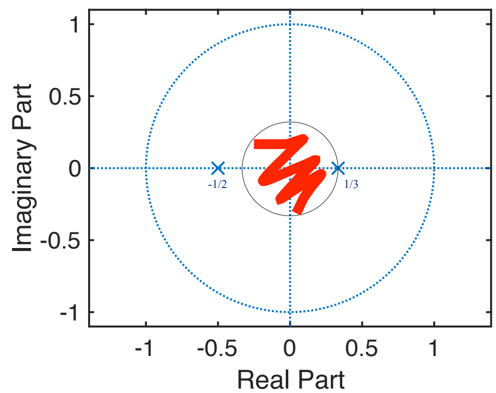
T If $x[n]$ is an odd signal, then $x[0] = 0$.

F For stable systems, the ROC is towards outwards.

T If $x[n]$ is real and even, then its DTFT $X(\omega)$ is also even.

F The DTFT of a rectangular pulse is a sinc waveform.

(b) (10 points.) The pole-zero diagram of the causal signal $x[n]$ has two poles at -2 and 3. Plot the ROC for the time reversed signal $x[-n]$. Indicate the ROC.
This is quite similar to the example in Lecture 11. However, note that the original signal $x[n]$ here is causal. Therefore, the ROC for $x[-n]$ is inwards.



(c) (13 points.) Derive the discrete-time signal $x[n]$ that has DTFT $X(\omega) = \frac{1}{(1-0.5e^{-j\omega})^2}$ using DTFT properties. You can refer to the tables to verify your solutions.

Let $Y(\omega) = \frac{1}{1-0.5e^{-j\omega}}$. Hence, $y[n] = 0.5^n u[n]$ You can note that multiplication in frequency domain is convolution in time domain (and you obtain a right sided signal by convolving a right sided signal with itself): $x[n] = y[n] * y[n] = \sum_{k=-\infty}^{\infty} 0.5^k u[k] 0.5^{(n-k)} u[n-k] = \sum_{k=0}^{n} 0.5^n = (n+1)0.5^n$, $n \geq 0$. Equivalently, $x[n] = (n+1)0.5^n u[n]$.

(d) (15 points.) Compute and plot the DTFTs $X_1(\omega)$, $X_2(\omega)$, $X_3(\omega)$ of

$$x_1[n] = \{1,1,\underline{1},1,1\}, \qquad x_2[n] = \{1,0,1,0,\underline{1},0,1,0,1\}$$

$$x_3[n] = \{1,0,0,1,0,0,\underline{1},0,0,1,0,0,1\}.$$

Determine the relation between $X_1(\omega)$, $X_2(\omega)$, $X_3(\omega)$.

$$X_1(\omega) = \sum_{n=-2}^{2} x_1[n]e^{-j\omega n} = \sum_{n=-2}^{2} e^{-j\omega n} = 1 + 2\cos(\omega) + 2\cos(2\omega)$$
$$X_2(\omega) = \sum_{n=-4}^{4} x_2[n]e^{-j\omega n} = \sum_{l=-2}^{2} e^{-j\omega 2l} = 1 + 2\cos(4\omega) + 2\cos(2\omega)$$
$$X_3(\omega) = \sum_{n=-6}^{6} x_3[n]e^{-j\omega n} = \sum_{m=-2}^{2} e^{-j\omega 3m} = 1 + 2\cos(6\omega) + 2\cos(3\omega)$$

You can observe that $X_2(\omega) = X_1(2\omega)$ and $X_3(\omega) = X_1(3\omega)$.

Plot is shown in the figure

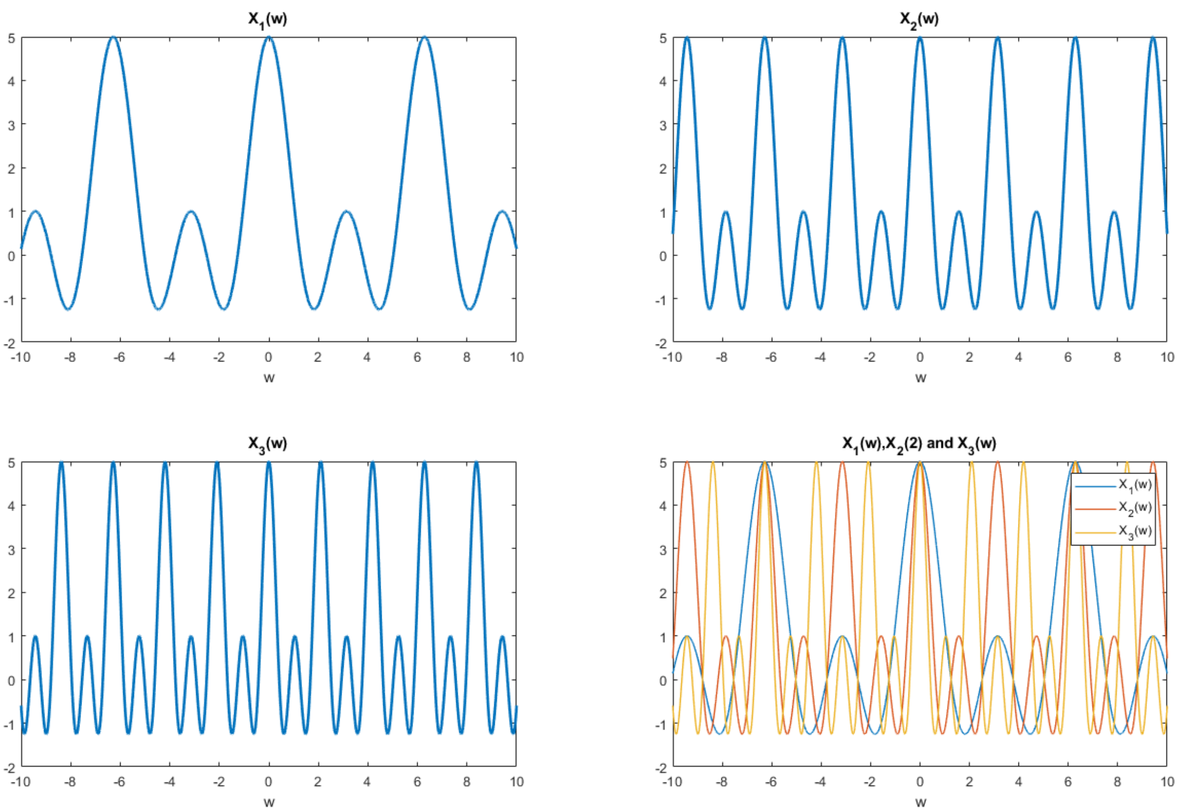


Figure 1: Plot for problem 3