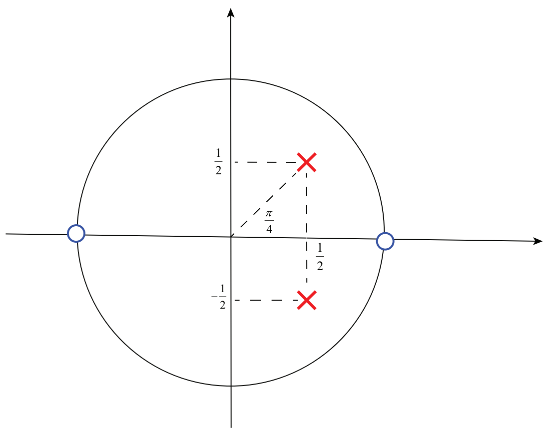
Exam 2 Solutions

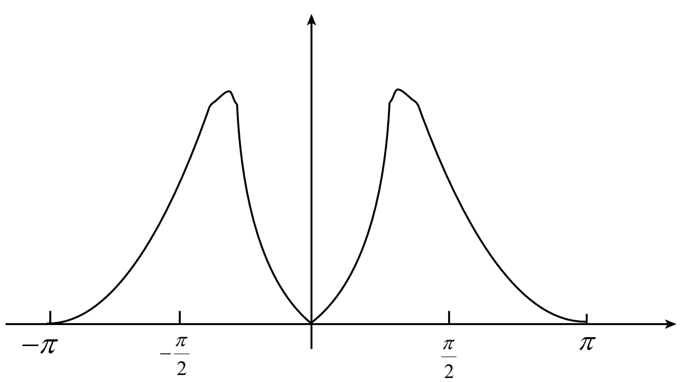
1. (25 points) **Filter design using the z-transform.** We are given a transfer function for a linear and time invariant discrete-time system:

$$H(z) = \frac{z^2 - 1}{z^2 - z + 0.5}$$

(a) (10 points.) Give the pole-zero diagram of $H(z)$.
Solution:



(b) (10 points.) Using the pole-zero diagram, plot the magnitude of the frequency response $|H(\omega)|$ for $\omega \in (-\pi, \pi)$. Indicate the values of $|H(0)|$, $|H\left(\frac{\pi}{2}\right)|$ and $|H\left(-\frac{\pi}{2}\right)|$.
Solution:



$$H(0) = 0$$

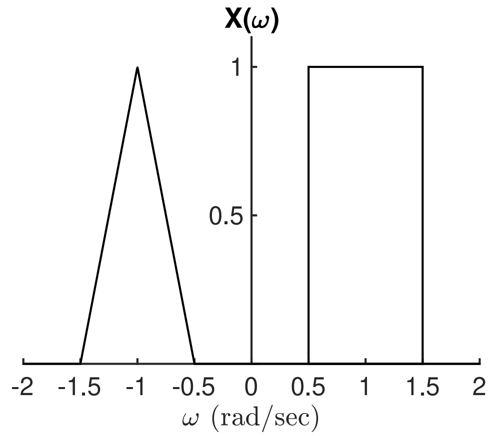
$$H(e^{j\frac{\pi}{2}}) = \frac{e^{j\frac{\pi}{2}\cdot 2} - 1}{e^{j\frac{\pi}{2}\cdot 2} - e^{j\frac{\pi}{2}} + 0.5} = \frac{-1-1}{-1-j+0.5} \to |H(e^{j\frac{\pi}{2}})| = \frac{2}{\sqrt{1.25}}$$

$$H(e^{-j\frac{\pi}{2}}) = \frac{e^{-j\frac{\pi}{2}\cdot 2} - 1}{e^{-j\frac{\pi}{2}\cdot 2} - e^{-j\frac{\pi}{2}} + 0.5} = \frac{-1-1}{-1+j+0.5} \to |H(e^{-j\frac{\pi}{2}})| = \frac{2}{\sqrt{1.25}}$$

$$H(e^{j\frac{\pi}{4}}) = \frac{e^{j\frac{\pi}{4}\cdot 2} - 1}{e^{j\frac{\pi}{4}\cdot 2} - e^{j\frac{\pi}{4}} + 0.5} = \frac{j-1}{j - \frac{1+j}{\sqrt{2}} + 0.5}$$

(c) (5 points.) What type of digital filter is $H(\omega)$? (Low pass, high pass, band pass) Explain your answer.
Solution: It is a band pass filter. Note that we have $H(0) = H(\pi) = 0$ and $H(\omega)$ peaks around $\omega = \frac{\pi}{4}$, i.e., the filter cancels both low and high frequencies and amplifies intermediate frequencies.

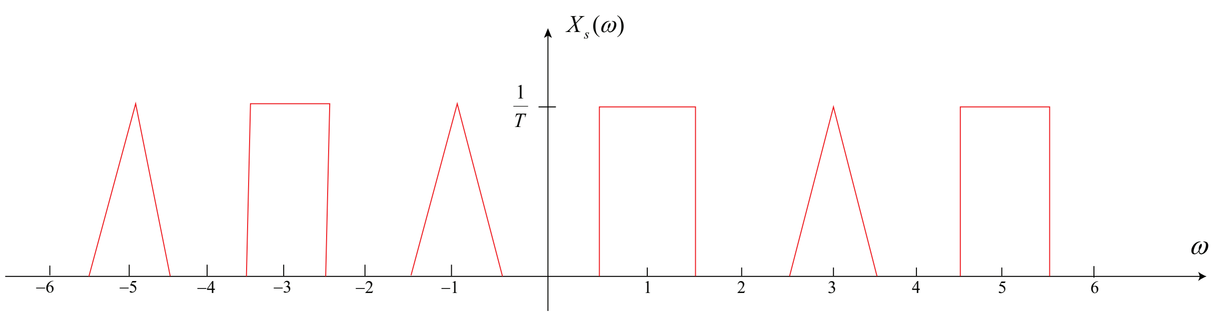
2. (30 points) **Sampling.** We consider a continuous time signal $x(t)$ with a Fourier transform $X(\omega)$ as shown below.



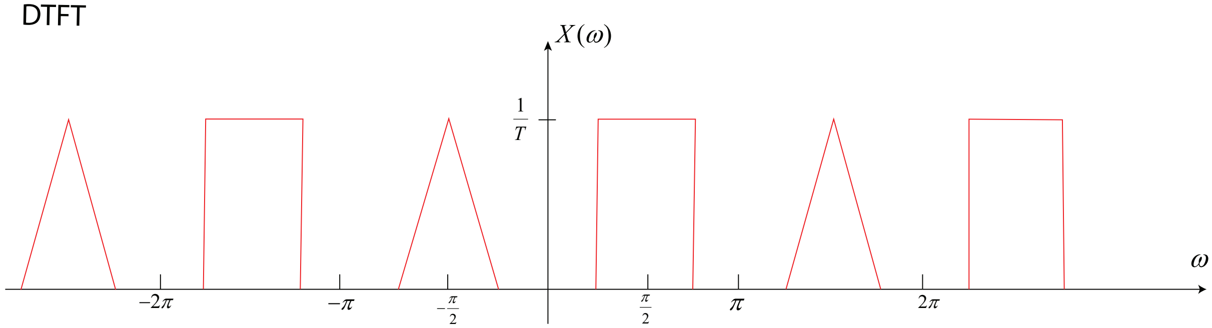
(a) (2 points.) What is the Nyquist rate of $x(t)$?
Solution: 3 rad/s (= 2× the maximum frequency)

(b) (2 points.) Is $x(t)$ real or complex valued?
Solution: Complex.
A formal explanation: Note that $X(\omega)$ is real but not even. Hence, $X^*(\omega) = X(\omega) \neq X(-\omega) = X^*(-\omega)$. Therefore, $X(-\omega) = \int x(t)e^{j\omega t}\mathrm{d}t$ and $X^*(-\omega) = \int x^*(t)e^{-j\omega t}\mathrm{d}t$ which would be equal to $X(\omega)$ only if $x(t)$ was real. However, this is not true.

(c) (6 points.) $x(t)$ is sampled with a sampling period of $T = \frac{2\pi}{4}$ seconds. Call the resulting signal $x[n]$. Plot the Fourier transform (DTFT) of $x[n]$ in the frequency range $\omega \in [-4\pi, 4\pi]$.
Solution: $T = \frac{2\pi}{4}$, $\omega_s = \frac{2\pi}{T} = 4$ rad/s



DTFT



The DTFT satisfies $X(\omega) = X_s\left(\frac{\omega}{T}\right)$.

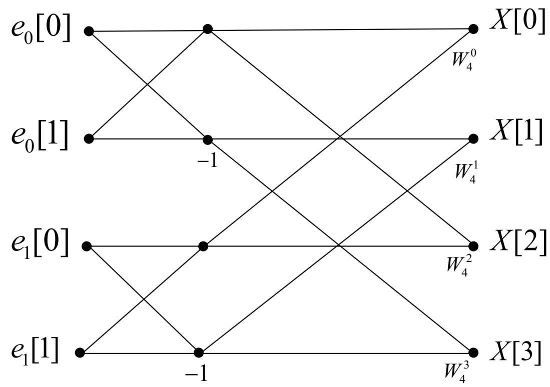
(d) (10 points.) Now assume that we want to compute a length $N = 4$ DFT of $x[n]$, i.e., $X[k]$, by applying radix-2 decimation in time (DIT) FFT (Fast Fourier Transform) algorithm on where the input to the block diagram is the polyphase components of $x[n]$, i.e., $e_0[n]$ and $e_1[n]$. Sketch the block diagram of the DIT FFT implementation and indicate the branch gains, inputs and outputs.

Solution: The polyphase components can be obtained by downsampling $x[n]$ by a factor of 2:

$$x[n] \rightarrow \boxed{\downarrow 2} \rightarrow e_0[n]$$

$$x[n+1] \rightarrow \boxed{\downarrow 2} \rightarrow e_1[n]$$

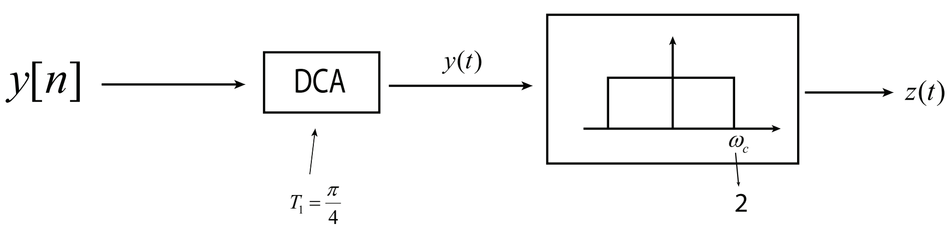
$X[k] = \sum_{l=0}^{1} e_0[l] W_2^{lk} + W_4^k \sum_{l=0}^{1} e_1[l] W_2^{lk}$, where both terms are 2 point DFTs.



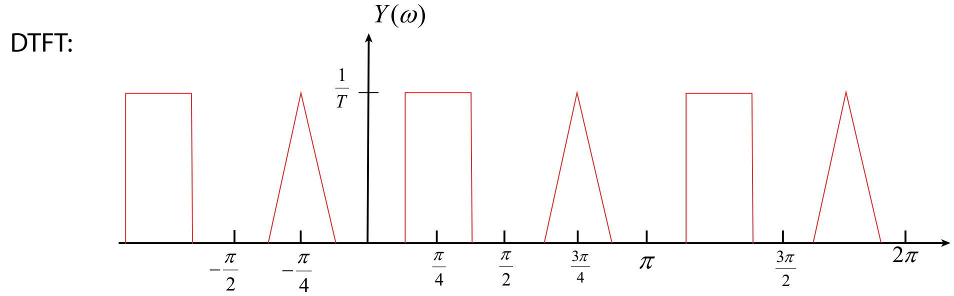
(e) (10 points.) Now assume that we upsample $x[n]$ by a factor of $L = 2$ and denote the resulting signal by $y[n]$. We convert the upsampled signal $y[n]$ from discrete time to continuous time $y(t)$ using a sampling period of $T_1 = \frac{\pi}{4}$. We then reconstruct a new continuous time signal $z(t)$ by interpolating $y(t)$ with an ideal low pass filter with cutoff $\omega_c = 2$ and gain $L = 2$. Sketch the block diagrams to indicate the relationship between $x[n]$ and $z(t)$. Plot the Fourier transforms of $y(t)$ and $z(t)$.

Solution:

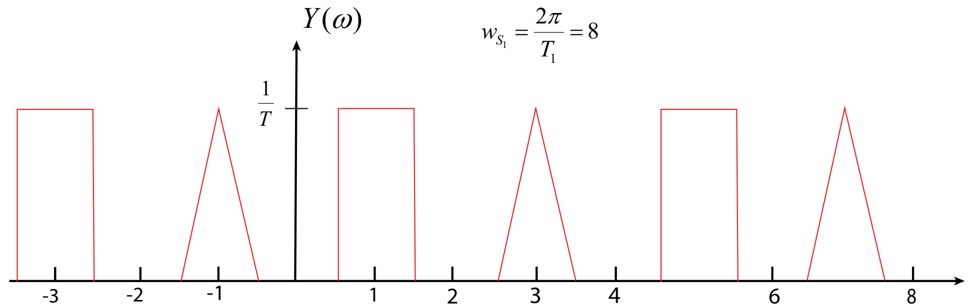
$x[n] \rightarrow \boxed{\uparrow 2} \rightarrow y[n]$

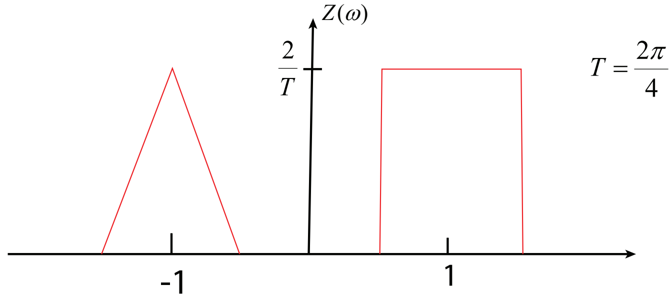


$$Y(\omega) = Y_S(\frac{\omega}{T_1})$$



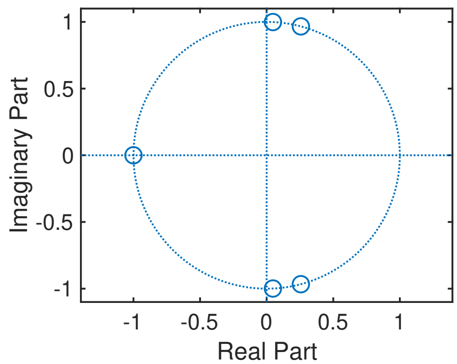
CTFT:





3. (15 points.) True/false and fill in the blank questions. You do not need to explain your reasoning. Read each question carefully.

   1. F Circular (or cyclic) convolution always produces the same result as linear convolution.
   2. T A length 8 Discrete Fourier Transform (DFT) computation requires 64 multiplications.
   3. T The twiddle factors (W_N^{nk}, complex roots of DFT) are uniformly distributed on the unit circle.
   4. F Prefiltering is always required in downsampling.
   5. T We can compute the inverse DFT of a length 4 signal using 2 stages of butterflies where each butterfly computes a length 2 DFT.
   6. T We can change the sampling rate by a non-integer factor by cascading interpolator and decimator operations.
   7. T For a length N DFT, $W_N^{k\frac{N}{2}} = -1$ if k is odd.
   8. F The complexity of a 8 point Decimation in Time (DIT) FFT algorithm is twice the complexity of a 4 point DIT FFT algorithm.
   9. T The zeros of a real Finite Impulse Response (FIR) linear phase filters can look like below.
   10. Equivalent systems of polyphase representations provide computational savings by filtering at a lower sampling rate.
   11. Polyphase decomposition of a signal can be obtained via downsampling .
   12. The purpose of sinc interpolation after upsampling is to cancel the zeros of the upsampled signal.
   13. Linear interpolation is as good as sinc interpolation if sampling rate is high (original signal is oversampled) .



14. If we have two length 8 signals $x[n]$ and $y[n]$ in the time domain, the length 15 Discrete Fourier Transform (DFT) of their linear convolution is the same as product of the DFTs of the zero padded signals $x[n]$ and $y[n]$ with 7 zeros.

15. Because of Covid-19 ____

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4. (30 points.) The parts of this problem are independent of each other. Read each question carefully. You can refer to the tables to verify your solutions.

   (a) (15 points.) **Downsampling and Upsampling.** Consider the two different ways of cascading a compressor $M = 2$ and an expander $L = 2$ as shown below. Show that $y_1[n]$ and $y_2[n]$ are different. Hint: You can give a counter example.

   Solution: We give a counter example. Assume $x[n] = \delta[n] + \delta[n-1]$. When we upsample $x[n]$ by a factor of 2, we get $z_1[n] = \delta[n] + \delta[n-2]$. Hence, when we downsample $z_1[n]$ by a factor of 2 we get $y_1[n] = \delta[n] + \delta[n-1]$.

   Similarly, when we downsample $x[n]$ by a factor of 2, we get $z_2[n] = \delta[n]$. Upsampling $z_2[n]$ by a factor of 2, we get $y_2[n] = \delta[n]$. Hence, $y_2[n] \neq y_1[n]$.

   (b) (15 points.) **Discrete Fourier Transform (DFT) of a DFT.** Let $X[k]$ be the N-point DFT of the sequence $x[n]$, $0 \le n \le N-1$. What is the N-point DFT of the sequence $y[n] = X[n]$, $0 \le n \le N-1$?

   Solution: The N-point DFT of $x[n]$ is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}.$$

Similarly, the N-point DFT of the sequence $y[n] = X[n]$ is

$$\begin{aligned}
Y[k] &= \sum_{n=0}^{N-1} y[n] W_N^{nk} \\
&\overset{(a)}{=} \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} x[m] W_N^{nm} \right] W_N^{nk} \\
&\overset{(b)}{=} \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} W_N^{n(m+k)} \\
&\overset{(c)}{=} \sum_{m=0}^{N-1} x[m] \delta_N[m+k], \quad k = 0, \dots, N-1 \\
&= \sum_{m=0}^{N-1} x[m] \delta[N-1-m-k], \quad k = 0, \dots, N-1,
\end{aligned}$$

where (a) follows from plugging in the definition of DFT of $X[n]$, (b) from rearranging the summations, (c) noticing that $\sum_{n=0}^{N-1} W_N^{n(m+k)} = \frac{1-e^{-j2\pi(m+k)}}{1-e^{-j\frac{2\pi}{N}(m+k)}} = 0$ if $m+k$ is not a multiple of N and is nonzero only when $m+k$ is a multiple of N. More specifically, $\sum_{n=0}^{N-1} W_N^{n(m+k)} = 0$ for $(m+k)_N \neq 0$ and $W_N^{n(m+k)} = 1$ for $(m+k)_N = 0$. Hence, $\sum_{n=0}^{N-1} W_N^{n(m+k)} = \delta_N[m+k]$ which is periodic with N.

Hence,

$$\{Y[0], Y[1], \dots, Y[N-2], Y[N-1]\} = \{x[N-1], x[N-2], \dots, x[1], x[0]\}.$$

Make up exam question.

(c) (15 points.) **Discrete Fourier Transform (DFT).** Let $X[k]$ be the N-point DFT of the sequence $x[n]$, $0 \leq n \leq N-1$. We define a $2N$-point sequence $y[n]$ as

$$y[n] = \begin{cases} x\left[\frac{n}{2}\right], & \text{n even} \\ 0, & \text{n odd.} \end{cases}$$

Determine the $2N$-point DFT of $y[n]$ in terms of $X[k]$.

Solution: The $2N$-point DFT of $y[n]$ is given as

$$\begin{aligned}
Y[k] &= \sum_{n=0}^{2N-1} y[n] W_N^{nk} \\
&= \sum_{l=0}^{N-1} y[2l] W_{2N}^{2lk} + \sum_{l=0}^{N-1} y[2l+1] W_{2N}^{(2l+1)k} \\
&= \sum_{l=0}^{N-1} x[l] W_{2N}^{l2k} \\
&= \sum_{l=0}^{N-1} x[l] W_N^{lk} \\
&= \begin{cases} X[k], & k = 0, \dots, N-1 \\ X[k-N], & k = N, \dots, 2N-1. \end{cases}
\end{aligned}$$

where we used the periodicity property $X[k+N] = X[k]$ for $k = N, \dots, 2N-1$.