9/14/20

Today's Lecture

- Review of Fourier Series
  - Continuous Time Fourier Transform
  - Properties

Example (from last lecture)

Linear constant coefficient difference equations (LCCDE)

$$y[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k \, x[n-k] \qquad \text{(no recursion)}$$

What is $h[n] = ?$

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] = \sum_{k=0}^{M} h[k] x[n-k] \qquad \text{(LTI and causal)}$$

$$h[n] = \begin{cases} \dfrac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Fourier analysis

$$\begin{tikzcd} \begin{matrix}\text{Function} \\ x(t)\end{matrix} \arrow[r, bend left, "\text{analysis (decomposition)}"] & \begin{matrix}\text{Oscillatory components} \\ \{a_k\}\end{matrix} \arrow[l, bend left, "\text{synthesis (rebuilt)}"] \end{tikzcd}$$

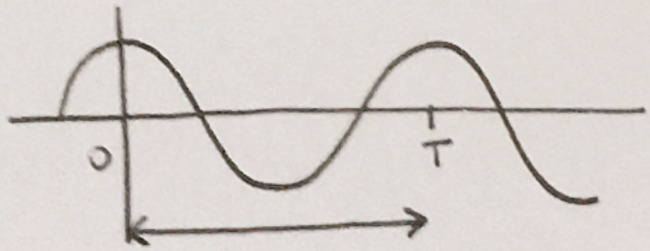
Applications in SP: equalization, sound spectrograms, spectroscopy

Other scientific applications in physics, statistics, protein structure analysis, etc...

Fourier Series

$x(t)$ should be a periodic signal. Let its period be T.

$$x(t) = x(t+T) \quad \text{for all } t.$$



$$\cos(\omega_0 t) = \cos\left(\frac{2\pi}{T} t\right)$$

$$T = \frac{2\pi}{\omega_0}, \quad \omega_0 = \frac{2\pi}{T}$$

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{j\omega_0 t \cdot k} \qquad \text{(Synthesis)}}$$

$$= a_0 \cdot 1 + a_1 \cdot e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{j\omega_0 t 2} + a_{-2} e^{-j\omega_0 t 2} + \cdots$$

The FS coefficients of $x(t)$:

$$\boxed{a_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad \text{(Analysis)}}$$

Let's show why the above is true. We will use the synthesis equation.

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{j(k-n)\omega_0 t} dt$$

$$= \int_0^T \sum_{k=-\infty}^{+\infty} a_k \left(\cos((k-n)\omega_0 t) + j \sin((k-n)\omega_0 t)\right) dt$$

If $k=n$ then $\int_0^T \left(\cos(\underbrace{(k-n)}_{0}\omega_0 t) + j\sin(\underbrace{(k-n)}_{0}\omega_0 t)\right) dt = \int_0^T 1\, dt = T$

If $k \neq n$ then $\int_0^T \cos((k-n)\omega_0 t)\, dt = \int_0^T \sin((k-n)\omega_0 t)\, dt = 0$

$$\Rightarrow \int_0^T x(t) e^{-jn\omega_0 t} dt = a_n T + \sum_{k \neq n} a_k \cdot 0$$

Properties of FS : $x(t)$ periodic with period T , $x(t) \longleftrightarrow \{a_k\}$

$$w_0 = \frac{2\pi}{T}$$

1. Linearity

$$x(t) \longleftrightarrow \{a_k\}$$
$$y(t) \longleftrightarrow \{b_k\}$$

then $\alpha x(t) + \beta y(t) \longleftrightarrow \{\alpha a_k + \beta b_k\}$

2. Time shifting

$$y(t) = x(t - t_0) \longleftrightarrow a_k e^{-jk w_0 t_0} = b_k$$

Note that $|b_k| = |a_k|$.

$$x(t) = \sum a_k e^{jw_0 tk}$$
$$x(t - t_0) = \sum a_k e^{jw_0(t-t_0)k}$$
$$= \sum \underbrace{a_k e^{-jw_0 k t_0}}_{b_k} \cdot e^{jw_0 tk}$$

3. Differentiation

$$x'(t) = \frac{d\,x(t)}{dt} \longleftrightarrow \{ jk w_0 a_k \}$$

4. Parseval's theorem

$$\underbrace{\frac{1}{T}\int_0^T |x(t)|^2 dt}_{\text{Average power of signal}} = \underbrace{\sum_{k=-\infty}^{+\infty} |a_k|^2}_{\text{Power of FS coefficients}}$$

5. Convolution

$$x(t) \overset{FS.}{\longleftrightarrow} \{a_k\}$$
$$y(t) \longleftrightarrow \{b_k\}$$

$$\int_0^T x(\tau) y(t-\tau) d\tau \longleftrightarrow T a_k b_k$$ (We will show this!)

$$x(t)y(t) \longleftrightarrow \sum_{l=-\infty}^{+\infty} a_l b_{k-l} = a * b$$

$$\int_0^T \underbrace{\left(\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0\tau}\right)}_{x(\tau)} \underbrace{\left(\sum_{l=-\infty}^{+\infty} b_l e^{jl\omega_0(t-\tau)}\right)}_{y(t-\tau)} d\tau$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_k b_l \int_0^T e^{jk\omega_0\tau} \cdot e^{jl\omega_0(t-\tau)} d\tau$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_k b_l e^{jl\omega_0 t} \underbrace{\int_0^T e^{j(k-l)\omega_0\tau} d\tau}_{= \begin{cases} T & \text{when } k=l \\ 0 & \text{when } k \neq l \end{cases}}$$

exercise! (similar to page 2)

$$= \sum_{k=-\infty}^{+\infty} \underbrace{a_k b_k T}_{\text{FS coefficients for the convolution of } x(t) \text{ and } y(t)} e^{jk\omega_0 t}$$

Examples

Question: Assume $x(t)$ is a real and periodic signal. What can we say about $\{a_k\}$?

$$x(t) = \underbrace{x^*(t)}_{\text{complex conj.}} \quad \text{because } x(t) \text{ is real.}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}$$

Let $l = -k$ so that

$$= \sum_{l=-\infty}^{+\infty} a_{-l}^* e^{jl\omega_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$$

$\Rightarrow \boxed{a_k = a_{-k}^*}$ for real and periodic $x(t)$.

e.g. $a_1 = 1+j$, $a_{-1} = 1-j$ $\quad a_{-1}^* = 1+j = a_1$

When $x(t)$ is a real signal, we can write

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2 \underbrace{A_k}_{\text{amplitude}} \cos(k w_0 t + \underbrace{\theta_k}_{\text{phase shift}})$$

$$= a_0 + 2 \sum_{k=1}^{+\infty} \left[B_k \cos(k w_0 t) - C_k \sin(k w_0 t) \right]$$

<u>Example</u>. $x(t) = 1 - 2\sin(w_0 t)$, $\{a_k\}$?

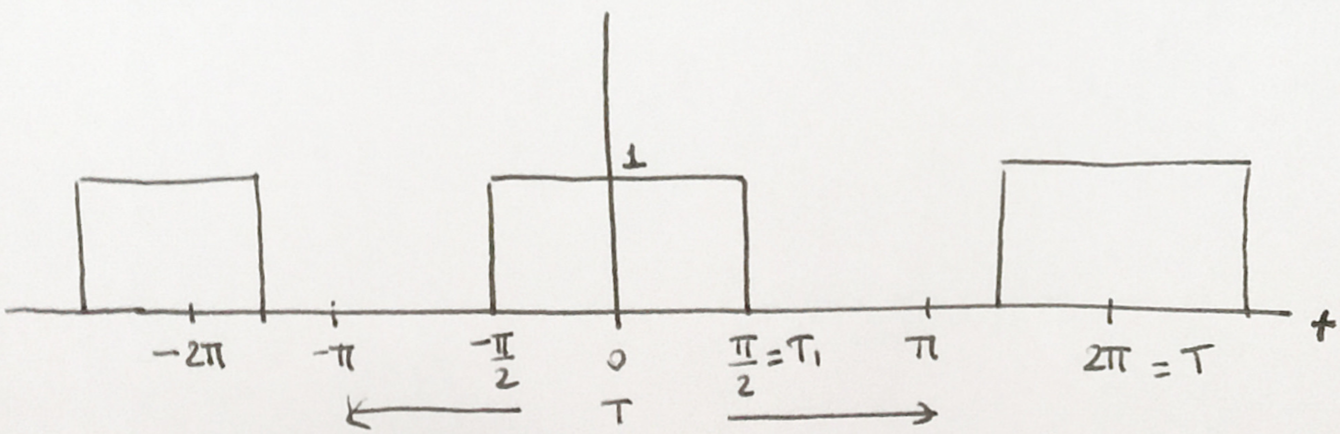
$$= 1 - 2\left(\frac{1}{2j}(e^{jw_0t} - e^{-jw_0t})\right)$$

$$= 1 + je^{jw_0t} - je^{-jw_0t} \qquad (\text{recall } j^2 = -1)$$

$a_0 = 1$, $a_1 = j$, $a_{-1} = -j$, $a_k = 0$, $k \neq 0, 1, -1$

<u>Example</u>:

$x(t)$ (periodic)



$T = 2\pi$, $w_0 = \frac{2\pi}{T} = 1$, $\{a_k\}$?

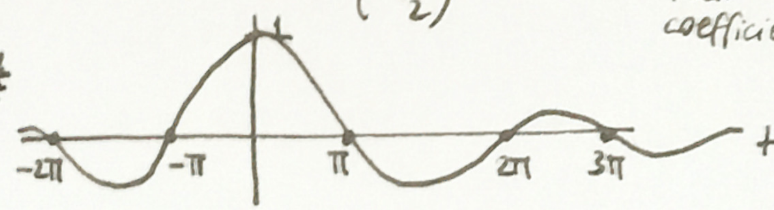
$$a_k = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, e^{-jkt}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-jkt}\, dt$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot e^{-jkt}\, dt = -\frac{1}{2\pi(jk)}\, e^{-jkt}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

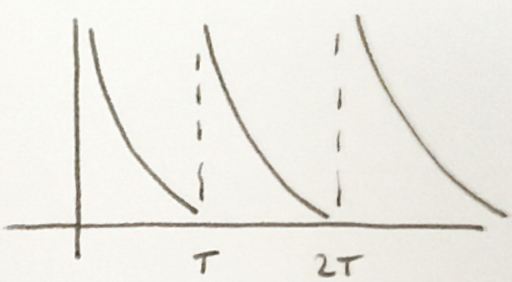
$$= -\frac{1}{2\pi(jk)}\left(e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right)$$

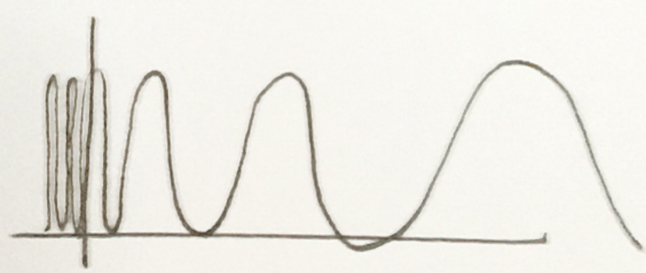
$$= \frac{1}{\pi k}\underbrace{\left(\frac{e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}}}{2j}\right)}_{\sin\left(\frac{k\pi}{2}\right)} = \underbrace{\frac{1}{2}\,\text{sinc}\left(\frac{k\pi}{2}\right)}_{\text{real coefficients}}$$

Note: $\text{sinc}\, t = \dfrac{\sin t}{t}$

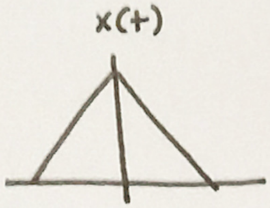


A sufficient condition for FS of a function to converge at a given point t is if the function's left and right derivatives at t exist. In this case, FS converges to the average of the left & right limits.



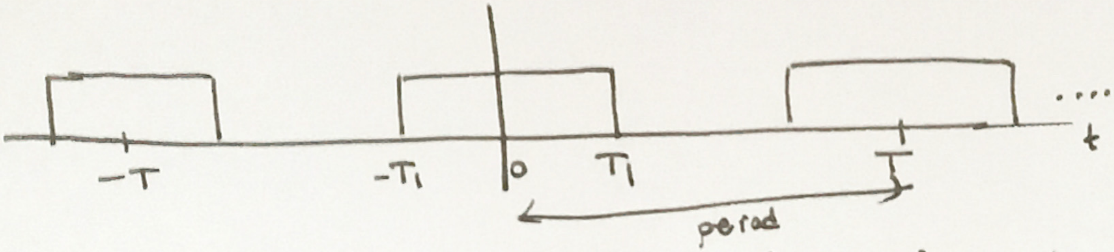


Fourier Transform



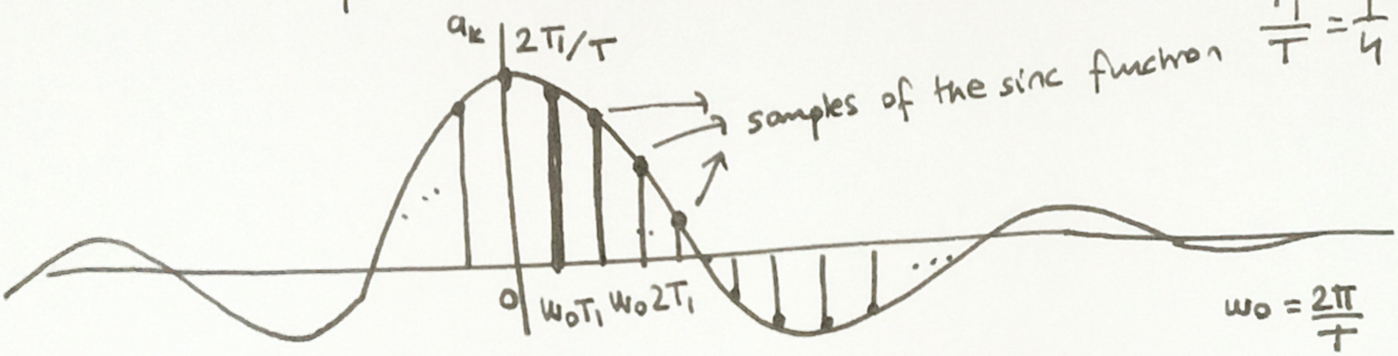
aperiodic signal $\Rightarrow$ cannot compute FS.

<u>Square wave</u>



$w_0 = \frac{2\pi}{T}$

$a_k = \frac{2T_1}{T} \text{ sinc } (k w_0 T_1)$ (<u>Exercise</u>) (in the previous example $\frac{T_1}{T} = \frac{1}{4}$

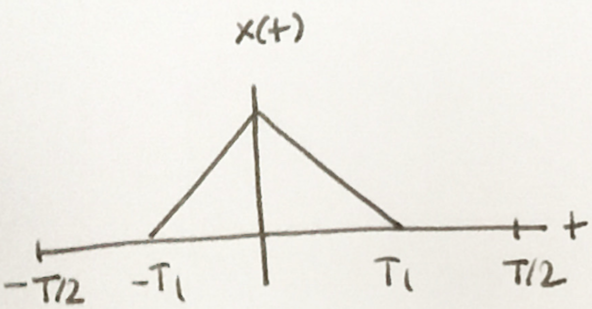


$w_0 = \frac{2\pi}{T}$

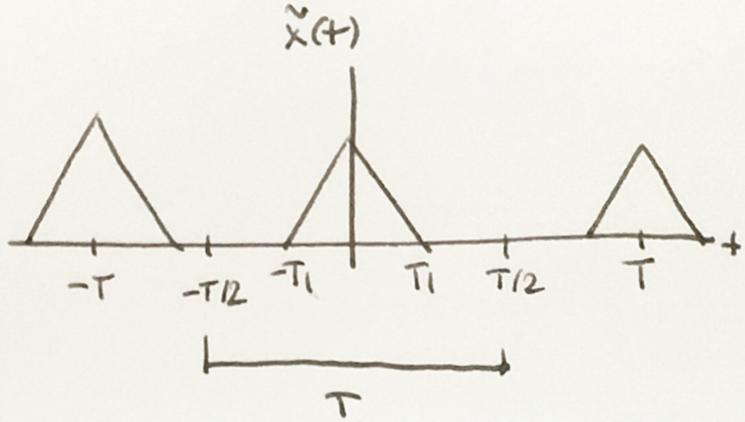
FS coefficients are samples of sinc function.

As T increases w_0 decreases, samples get infinitely close.

Aperiodic signal



Periodic signal



$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad , \quad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$\tilde{x}(t) \longrightarrow$ periodic signal

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$x(t) \longrightarrow$ aperiodic signal

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} X(k\omega_0)$$

$X(k\omega_0) \downarrow$ FT

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$ Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$ Inverse Fourier Transform

Continuous Time Signals

$x(t) \longrightarrow$ periodic $\longrightarrow$ FS $\quad a_k$ (discrete) in k

$x(t) \longrightarrow$ aperiodic $\longrightarrow$ FT $\quad X(\omega)$ (continuous) in ω

FT converges when $x(t)$ has finite energy (i.e. finite number of extrema, finite number of discontinuities)

Question: Can we compute the FT of periodic signals?

Answer: YES. (eg square wave)

Properties of FT

1. Linearity

$$x(t) \xrightarrow{FT} X(\omega)$$

$$y(t) \xrightarrow{FT} Y(\omega)$$

$$ax(t) + by(t) \xrightarrow{FT} aX(\omega) + bY(\omega)$$

2. Time shift

$$x(t - t_0) \xrightarrow{FT} X(\omega)e^{-j\omega t_0}$$

Note that

$$|X(\omega)e^{-j\omega t_0}| = |X(\omega)|$$

3. Conjugation

$$x^*(t) \xrightarrow{FT} X^*(-\omega)$$ (FT integral)

4. Differentiation and integration

$$x'(t) \xrightarrow{FT} j\omega X(\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \longrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$ (will not prove this).

5. Duality (can be seen via a change in variables)

6. Parseval

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$