Name: ______________________________ RIN: ______________

Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2019

Final. Closed book, closed notes.
December 17, 2019, 3:00-6:00 PM

Show all work for full credit.

- Calculators are allowed. Other electronic devices are not permitted.
- Reduce expressions involving exponentials, sines, and cosines as much as possible.
- If the sinc function is needed, use the definition $\mathrm{sinc}(x) = \frac{\sin x}{x}$.
- Geometric series formula: $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$.
- Finite sum formula: $\sum_{n=M}^{N-1} a^n = \frac{a^M - a^N}{1-a}, \quad a \neq 1$.

When in doubt, show your work. Good luck!

1		**30**
2		**25**
3		**30**
4		**25**
5		**25**
6		**35**
7		**30**
Total		**200**

1. (30 points.) True/false and fill in the blank questions. You do not need to explain your reasoning. Read each question carefully.

    1. ______ The computational complexities of computing FFT and inverse FFT are not the same.

    2. ______ DFT matrix is real valued.

    3. ______ DFT is also known as DTFS (discrete time Fourier series).

    4. ______ The least squares (LS) and the recursive least squares (RLS) algorithms are deterministic approximations to the Wiener filter (if the processes are wide sense stationary).

    5. ______ Using linear phase filters, we can design low pass, band pass or high pass filters.

    6. ______ In practice, sinc interpolation is very easy to implement.

    7. ______ Region of convergence (ROC) for which the z-transform converges cannot contain any poles.

    8. ______ We can apply the bilinear transformation or the impulse invariance method to convert an analog filter to a digital filter.

    9. ______ Gradient descent algorithm always converges to the global minimum.

    10. ____________________ are optimal linear discrete-time filters that are used to produce the minimum mean-square error (MMSE) of stationary signal and noise processes.

    11. Rounding and analog-to-digital conversion are examples of ____________________.

    12. The least mean square (LMS) algorithm is a stochastic ____________________ to find filter taps that minimizes an error function.

    13. As long as $x[n] = x_c(nT)$ was obtained by sampling without aliasing, we can ____________________ by any integer factor.

    14. ____________________ model can be regarded as smoothing the data.

    15. When we design a digital IIR filter using the impulse invariance approach, ____________________ can be a big problem if the sampled signal contains high frequencies.

2. (25 points.) Sampling. We wish to modulate a signal $x(t) = \text{sinc}\left(\frac{t}{10}\right)$ where t is in seconds. We use a new type of modulator which produces a transmit signal $s(t) = x(t) \cdot [\cos(\omega_0 t) + \cos(2\omega_0 t)]$.

   (a) (3 pts) Plot the Fourier Transform $X(\omega)$.

   Hint:

$$\frac{\sin(At)}{\pi t} \overset{FT}{\longleftrightarrow} \begin{cases} 1, & |\omega| \leq A \\ 0, & |\omega| > A. \end{cases}$$

   (b) (5 pts) Plot $S(\omega)$ for the output of the modulator. What is the requirement on the value ω_0 for $x(t)$ to be recoverable from $s(t)$?

(c) (6 pts) The signal $s(t)$ is sent through a channel that attenuates it by 60 dB and delays it by 0.1 seconds.

   i. What is the impulse response $h(t)$ of this channel?

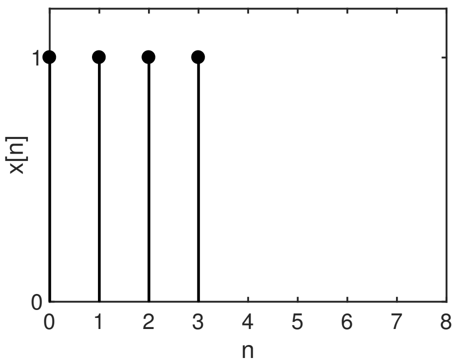
   ii. What is the output $y(t)$ in terms of $x(t)$?

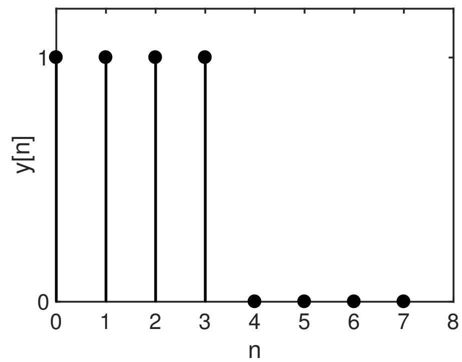
(d) (6 pts) Design a filter to recover $x(t)$ perfectly (some delay is acceptable) from $y(t)$, assuming the condition you found in part (b) is met.

(e) (5 pts) If we were to instead sample the output $y(t)$ to get a digital signal $y[n]$ for processing (rather than using the above filter), what would be the minimum required sampling frequency according to the sampling theorem?

3. (30 points.) Circular convolution.

   Consider the two discrete-time signals, $x[n]$ and $y[n]$. The first signal $x[n]$ is a rectangular pulse of length 4. The second signal $y[n]$ is the sequence $x[n]$ zero-padded to length 8. They are shown below.





   In the following questions, you can either compute or plot. Make sure your plots are labeled correctly.

   (a) (5 pts) What is the magnitude of the length 4 DFT of $x[n]$, i.e. $|X[k]|$?

   (b) (5 pts) What is the magnitude of the length 8 DFT of $y[n]$, i.e. $|Y[k]|$?

   (c) (5 pts) What is the length 8 circular convolution of $x[n]$ with itself?

(d) (5 pts) What is the length 8 circular convolution of $y[n]$ with itself?

(e) (5 pts) What is the linear convolution of $x[n]$ with itself?

(f) (5 pts) Which convolutions for the discrete-time sequences you obtained in parts (c)-(e) are related and why?

4. (25 points.) Inverse z-transform.

   Each part of this problem can be considered separately.

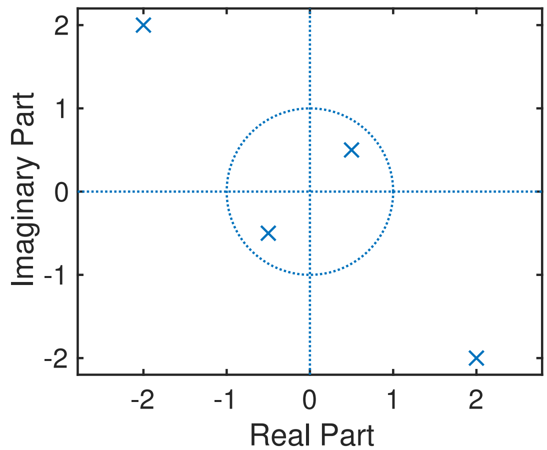
   (a) (10 pts) The z-transform of a signal with ROC $|z| > 2$ is given as

   $$X(z) = \frac{2z+1}{z^2 - 3z + 2}.$$

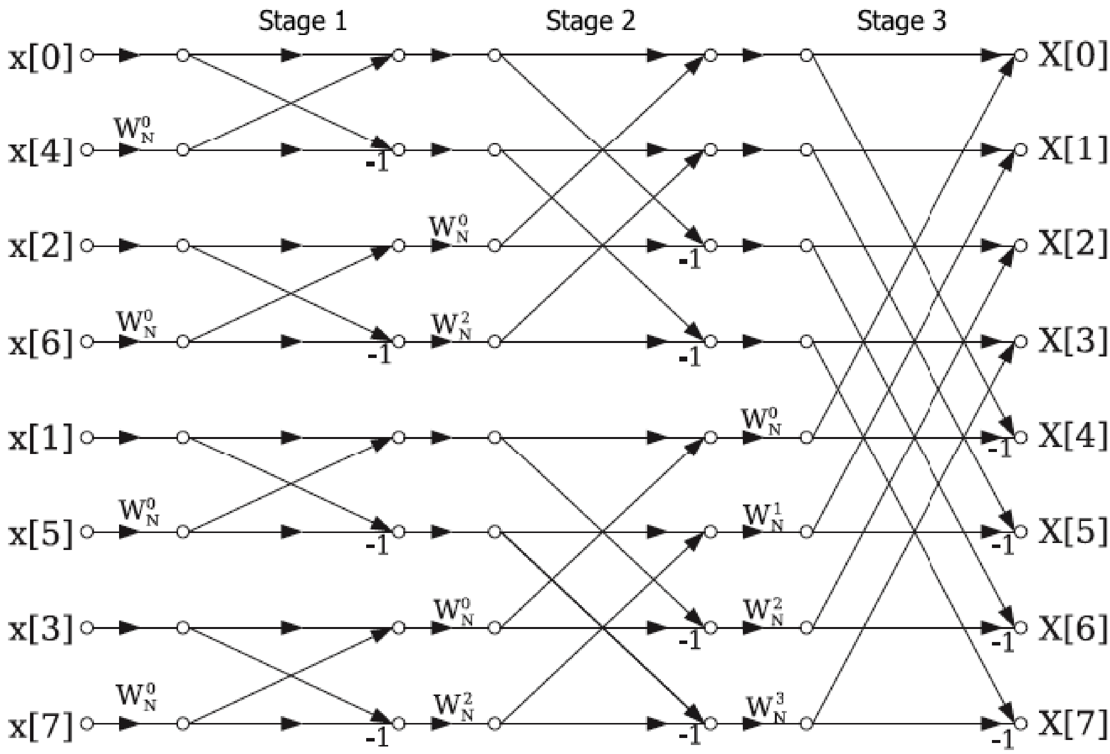
   Determine $x[n]$.

   (b) (3 pts) Why do we need to define a region-of-convergence (ROC) for the z-transform?

(c) (12 pts) Assume that the right sided signal $y[n]$ has a z-transform has 4 poles at $2-2j$, $-2+2j$, $0.5+0.5j$ and $-0.5-0.5j$. The corresponding z-plane is shown below. Determine the locations of the poles for the z-transform of $y[-n]$. Plot the corresponding z-plane. Your plot should also indicate the ROC.



5. (25 points.) FFT Algorithm. Suppose we compute an $N = 8$-point DFT of $x[n]$, $n = 0, \ldots, N-1$ using the eight-point decimation-in-time FFT algorithm with the twiddle factors as shown below.



(a) (5 pts) For the above algorithm, how many multiplications you need at each stage? Explain.

(b) (5 pts) Assume that $x[1] = x[3] = x[5] = x[7] = 0$. What does it imply in terms of the $N = 8$-point DFT of $x[n]$, i.e. $X[k]$? In this case, how many multiplications do you need in total to implement the FFT?

(c) (7 pts) Now assume that the inputs are $X[k]$, $k = 0, \ldots, 7$ and you want to compute $x[n]$, $n = 0, \ldots, 7$. How would you modify the above algorithm if you are only allowed to change the branch gains? You can indicate the new gains on the plot above.

(d) (8 pts) Now assume that $x[n]$ is a complex-valued sequence given as

$$x[n] = x_1[n] + jx_2[n], \quad n = 0, \ldots, 7$$

where $x_1[n]$ and $x_2[n]$ are two real-valued sequences of length 8. Denote by $X_1[k]$ and $X_2[k]$ the DFTs of these sequences. Use DFT properties to determine $X_1[k]$ and $X_2[k]$ from $X[k]$. Show your work.

6. (35 points.) Adaptive filtering. Consider an autoregressive process that satisfies the difference equation

$$x[n] - x[n-1] - 0.1x[n-2] = v[n]$$

where $\{x[n]\}$ is a wide sense stationary (WSS) process, and $v[n]$ is a white noise process with variance $\sigma_v^2 = 0.64$. Let $r_x[l] = \mathbb{E}[x[n]x[n-l]]$ be the autocorrelation function of the process $\{x[n]\}$ where "$\mathbb{E}$" is the statistical expectation.

(a) (5 pts) Generate a system of equations for the above process:

$$x[n]x[n-l] + a_1x[n-1]x[n-l] + \dots a_Mx[n-M]x[n-l] = v[n]x[n-l], \quad l = 1, \dots, M \quad (*)$$

What are the values of M, $a_1, \dots, a_M$?

(b) (5 pts) Take the statistical expectation of both sides in equation $(*)$ and write down the Yule-Walker equation with $l = 0$.

(c) (10 pts) Take the statistical expectation of both sides in equation $(*)$, to determine the Yule-Walker equations. Write down these equations in matrix form such that $R_x \cdot \mathbf{a} = \mathbf{r_x}$ where R_x is an $M \times M$ autocorrelation matrix, **a** is an $M \times 1$ vector that contain a_1 to a_M and **r** is an $M \times 1$ vector that contain the “-" of autocorrelations from $r_x[1]$ to $r_x[M]$. Determine $r[0]$ to $r[M]$.

(d) (10 pts) Assume that we observe the signal $y[n]$ which is a noisy version of $x[n]$ and is given by

$$y[n] = x[n] + w[n]$$

where $w[n]$ is a white noise process with variance $\sigma_w^2 = 1$. We want to design a filter to recover an estimate of $x[n]$ from $y[n]$, i.e. the desired output $d[n] = x[n]$. Design a Wiener filter with length 3 to estimate $x[n]$. Note: Recall that the Wiener filter coefficients $\hat{h}$ satisfy $R_y\hat{h} = \mathbf{p}$ where R_y is the autocorrelation matrix of $y[n]$ and $\mathbf{p} = [\mathbb{E}[y[n]d[n]], \mathbb{E}[y[n-1]d[n]], \mathbb{E}[y[n-2]d[n]]]^\top$ is the cross correlation vector between $y[n]$'s and $d[n]$.

i. What is R_y in terms of R_x?

ii. Determine $\mathbb{E}[d[n]d[n]]$.

iii. Determine $\mathbf{p}$.

(e) (5 pts) Determine the expression for the minimum mean square error (MMSE) for the length 3 filter in part (d).

7. (30 points.) IIR filter design.

   Consider an IIR system with input $x[n]$ and output $y[n]$ described by the difference equation

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-k].$$

   (a) (5 pts) Determine the transfer function $H(z)$ of the system.

   (b) (15 pts) Now consider the digital filter with $H_d(z) = \frac{B(z)}{A(z)}$ where $B(z) = b_0 + b_1 z^{-1} + \ldots b_M z^{-M}$ and $A(z) = 1 + a_1 z^{-1} + \ldots + a_N z^{-N}$. Propose a procedure to determine the impulse response $h_d[n]$.

(c) (10 pts) Describe a procedure that computes the discrete time frequency response $H\left(\frac{2\pi}{N}k\right)$ for $k = 0, 1, \ldots, N-1$, in terms of $A(z)$ and $B(z)$ using the FFT algorithm.

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