9/17/20

Today's Lecture

- Examples on Fourier Series and Fourier Transform (MATLAB)
- Frequency Response
- Discrete-time Fourier Transform (DTFT)

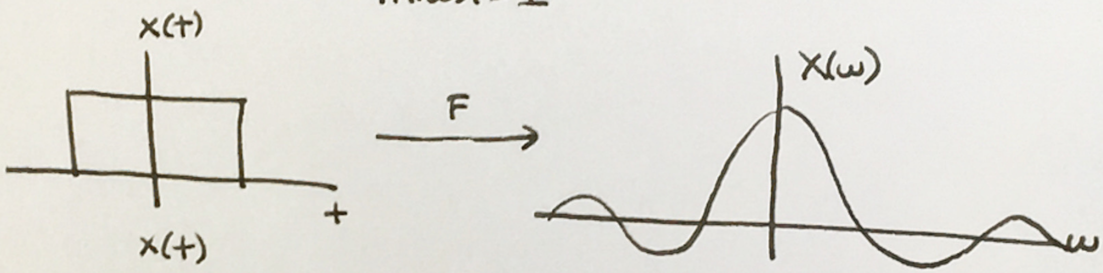
Note: Lecture 5, typo in page 4 (FS coefficients for $x(t) * y(t)$)

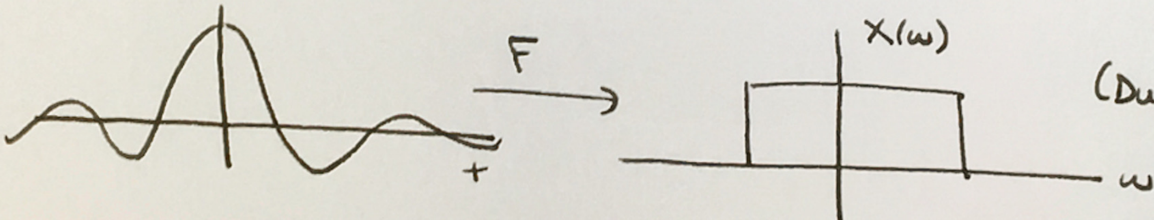
Examples $x(t) \xrightarrow{F} X(\omega)$

$\delta(t) \xrightarrow{F} 1$

$\delta(t-t_0) \xrightarrow{F} e^{-j\omega t_0}$ (Phase shift in frequency domain)

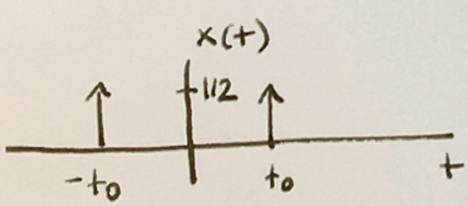
$|X(\omega)| = 1$





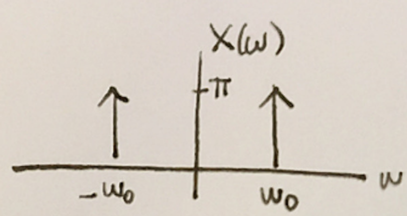
(Duality)

$1 \xrightarrow{F} 2\pi\,\delta(\omega)$ (from duality)



$\xrightarrow{F} \frac{1}{2}e^{j\omega t_0} + \frac{1}{2}e^{-j\omega t_0} = \cos(\omega t_0)$

$\cos(\omega_0 t) \xrightarrow{F}$



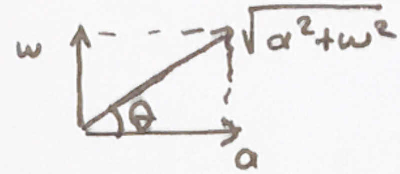
$e^{j\omega_0 t} \xrightarrow{F} 2\pi\,\delta(\omega-\omega_0)$

Example Magnitude and phase

$x(t) = e^{-at}u(t)$, $a>0$, $X(\omega)$?

↳ real number

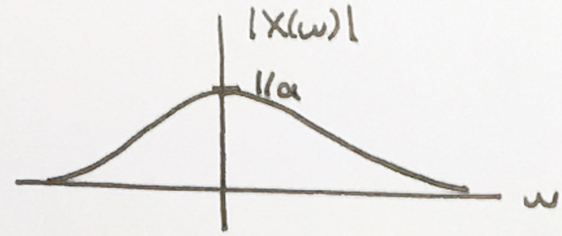
$$X(\omega) = \frac{1}{a+j\omega} \quad \text{(exercise)}$$



$$\tan\theta = \frac{\omega}{a}$$

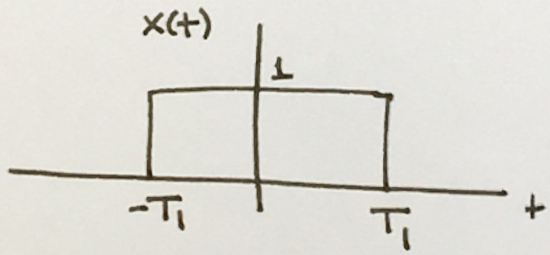
$$X(\omega) = \underbrace{|X(\omega)|}_{\text{magnitude of } X(\omega)} e^{j\overbrace{\angle X(\omega)}^{\text{phase of } X(\omega)}}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$



$$\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right) \quad \left(\text{If } X(\omega) = \frac{A(\omega)}{B(\omega)},\ \angle X(\omega) = \angle A(\omega) - \angle B(\omega)\right)$$

Example

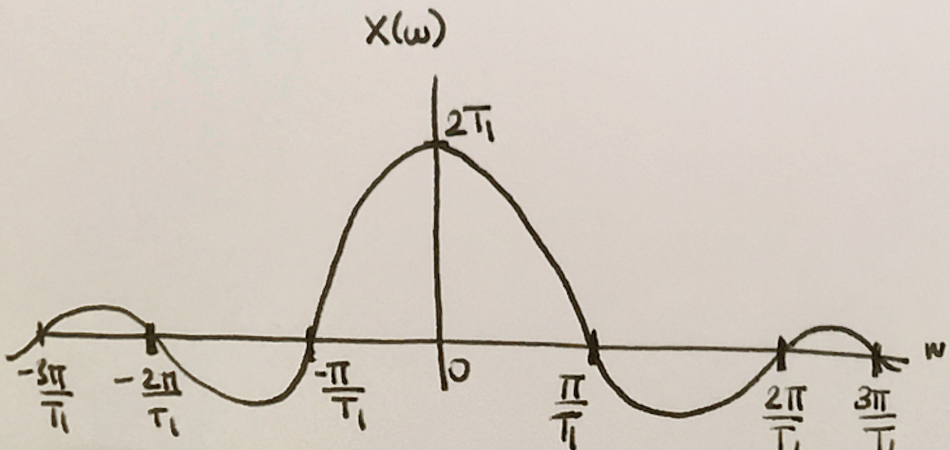


$X(\omega)$?

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{-T_1}^{T_1} 1 \cdot e^{-j\omega t}dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right) = \frac{2\sin(\omega T_1)}{\omega} = 2T_1 \,\text{sinc}(\omega T_1)$$

$$\text{sinc}(\omega) = \frac{\sin\omega}{\omega}$$



Frequency Response

1 - Solve differential equation (or difference equation)

2 - Compute impulse response $\underset{h[n]}{}$ → Take its Fourier Transform $H(\omega)$

$$h[n] \qquad\qquad H(\omega)$$

For an LTI system

$$x(t) \longrightarrow \underbrace{\boxed{h(t)}}_{\text{impulse response}} \longrightarrow y(t) = x(t) * h(t)$$

$$Y(\omega) = \int_{-\infty}^{+\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau \right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} x(\tau) \underbrace{\left[\int_{-\infty}^{+\infty} h(t-\tau) e^{-j\omega t} dt \right]}_{H(\omega)e^{-j\omega\tau} \text{ (phase shift in freq. domain)}} d\tau$$

$$= H(\omega) \underbrace{\int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau}_{X(\omega)} = X(\omega) H(\omega)$$

$$x(t) * h(t) \xrightarrow{\quad F \quad} X(\omega) H(\omega)$$

where $H(\omega) = F(h(t))$ is the Frequency Response.

Eigenfunctions of LTI systems

Recall: In linear algebra, we say the vector x is an eigenvector of the square matrix A if

$$Ax = \lambda x$$

where λ is a scalar called an eigenvalue.

x : is a special vector because, when operated on by A, it is unchanged except for a scale factor.

Can we extend this concept of eigenvectors to eigenfunctions of systems?

$$\underset{x[n]}{\text{Input}} \longrightarrow \boxed{H} \longrightarrow \underset{y[n]}{\text{Output}}$$

In other words, are the input sequences $\{x[n]\}$ such that

$$\boxed{y[n] = H(x[n]) = \lambda x[n]}$$

where λ is a scalar when H is a DT LTI system?

YES! if $y[n] = \lambda x[n]$ then $x[n]$ is an eigenfunction and the multiplier λ is an eigenvalue.

Example We have a DT system described by

$$3y[n] - y[n-1] = x[n]$$

Determine if each of the below $x[n]$ is an eigenfunction. If it is, then find the eigenvalue.

1. $x[n] = 2^{-n}$ $\quad y[n] \overset{?}{=} \lambda x[n] \quad \forall n$

$$3\lambda 2^{-n} - \lambda 2^{-(n-1)} = 2^{-n} \quad \Rightarrow \quad \boxed{\lambda = 1}$$

Yes.

2. $x[n] = e^{jn}$ $\quad y[n] \overset{?}{=} \lambda e^{jn} \quad \forall n$

$$3\lambda e^{jn} - \lambda e^{j(n-1)} = e^{jn}$$

$$(3\lambda - \lambda e^{-j})e^{jn} = e^{jn} \quad \Rightarrow \quad 3\lambda - \lambda e^{-j} = 1$$

Yes.

$$\boxed{\lambda = \frac{1}{3 - e^{-j}}}$$

3. $x[n] = u[n]$ $\quad y[n] \overset{?}{=} \lambda u[n]$

$$3\lambda u[n] - \lambda u[n-1] \overset{?}{=} u[n] \quad \forall n$$

not an eigenfunction

$n = 0 \Rightarrow u[0] = 1$, $u[-1] = 0 \Rightarrow \lambda = \frac{1}{3}$

$n = 1 \Rightarrow 3\lambda - \lambda = 1 \Rightarrow \lambda = 1/2$

Eigenfunctions of DT LTI Systems

* Complex exponentials are eigenfunctions of LTI systems.

Let $x[n] = e^{j\omega_0 n}$ for a DT LTI system with impulse response $h[n]$.

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\,x[n-k] \quad \text{(because LTI)}$$

$$= \sum_{k=-\infty}^{+\infty} h[k]\,e^{j\omega_0(n-k)}$$

$$= \sum_{k=-\infty}^{+\infty} h[k]\,e^{-j\omega_0 k}\,e^{j\omega_0 n}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{+\infty} h[k]\,e^{-j\omega_0 k}$$

$$= \underbrace{e^{j\omega_0 n}}_{x[n]}\ \underbrace{H(\omega_0)}_{\text{Frequency response of the LTI system at frequency } \omega_0}$$

$$\underbrace{e^{j\omega_0 n}}_{\text{complex exponential}} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow \underbrace{H(\omega_0)}_{\text{Eigenvalue associated with } e^{j\omega_0 n}} e^{j\omega_0 n}$$

Examples Sinusoids

$$\cos(\omega_0 n) = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)$$

$$e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow H(\omega_0)e^{j\omega_0 n}$$

$$e^{-j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow H(-\omega_0)e^{-j\omega_0 n}$$

$$x[n] = \cos(\omega_0 n) \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \frac{1}{2}\left(H(\omega_0)e^{j\omega_0 n} + H(-\omega_0)e^{-j\omega_0 n}\right)$$

Assume that $h[n]$ is real $\Rightarrow h[n] = h^*[n]$

$$H(\omega) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} \Rightarrow H^*(\omega) = \sum_{n=-\infty}^{+\infty} h^*[n]e^{j\omega n}$$

$$\left(\text{In CT } H(\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}\,dt\right) \qquad = \sum_{n=-\infty}^{+\infty} h[n]\,e^{j\omega n}$$

$$H^*(-\omega) = H(\omega) \rightarrow \boxed{H(-\omega) = H^*(\omega)}$$

Then

$$y[n] = \frac{1}{2}\left(H(\omega_0)e^{j\omega_0 n} + H^*(\omega_0)e^{-j\omega_0 n}\right)$$

$$= \text{Real}\left(H(\omega_0)e^{j\omega_0 n}\right) = \text{Real}\left(|H(\omega_0)|e^{j\angle H(\omega_0)} \cdot e^{j\omega_0 n}\right)$$

$$= |H(\omega_0)| \cdot \text{Real}\left(e^{j\overbrace{(\omega_0 n + \angle H(\omega_0))}}\right)$$

$$= |H(\omega_0)| \cdot \cos\left(\omega_0 n + \angle H(\omega_0)\right)$$

Note: The output has the same frequency as the input, only the magnitude and phase are changed.