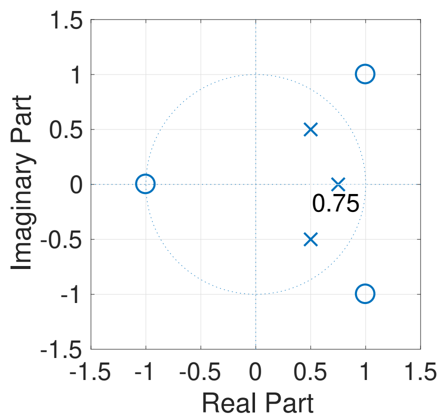
Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2020

Homework #3: due Monday, Oct. 26$^{\text{th}}$, at the beginning of class.

4. (20 points) **Discrete time system analysis and z-transform.**

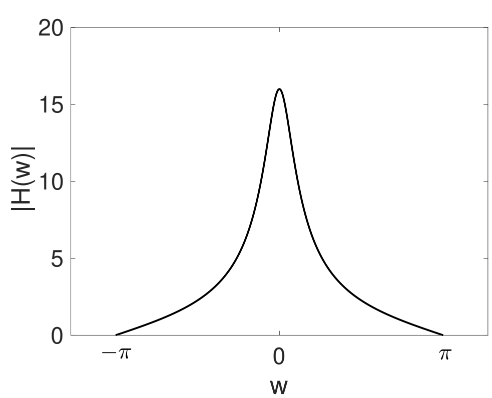
   (a) Estimate the transfer function $H(z)$ corresponding to the below pole-zero plot.



Note that the system has 3 poles at $z = 0.75, 0.5+0.5j, 0.5-0.5j$, and 3 zeros at $z = -1, 1+j, 1-j$. Hence, the transfer function can be written as

$$\begin{aligned}
H(z) &= \frac{(z+1)(z-1-j)(z-1+j)}{(z-0.75)(z-0.5-0.5j)(z-0.5+0.5j)} \\
&= \frac{(z+1)(z^2-2z+2)}{(z-0.75)(z^2-z+0.5)} \\
&= \frac{z^3-z^2+2}{z^3-1.75z^2+1.25z-0.375} \\
&= \frac{1-z^{-1}+2z^{-3}}{1-1.75z^{-1}+1.25z^{-2}-0.375z^{-3}}.
\end{aligned}$$

   (b) Sketch the magnitude of the frequency response for $\omega \in (-\pi, \pi)$.
   The pole at $z = 0.5+0.5j$ will cancel the zero at $z = 1+j$. Similarly, $z = 0.5-0.5j$ will cancel the zero at $z = 1-j$. Hence, the frequency response will be $H(\omega) = \frac{e^{j\omega}+1}{e^{j\omega}-0.75}$.



(c) What is the governing system difference equation for this filter?
$H(z) = \frac{Y(z)}{X(z)}$ and hence, $Y(z) = X(z)H(z)$. Using the result from part (a), we have

$$(1 - z^{-1} + 2z^{-3})X(z) = (1 - 1.75z^{-1} + 1.25z^{-2} - 0.375z^{-3})Y(z).$$

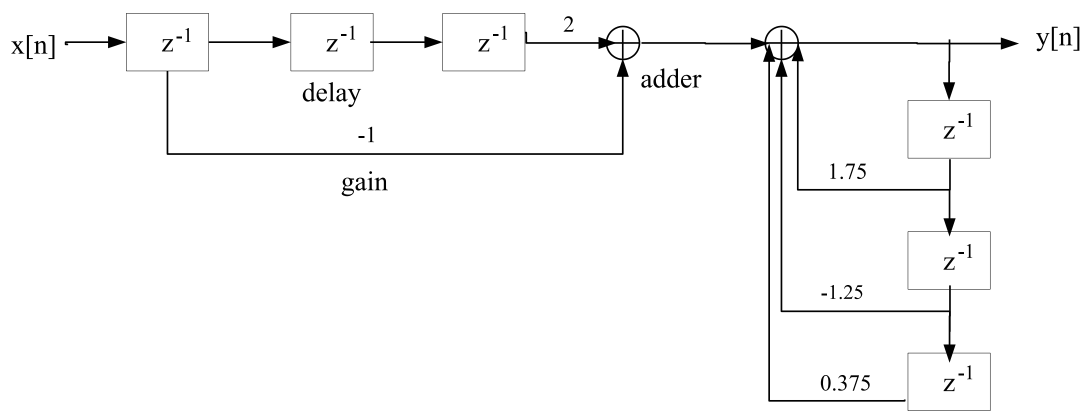
Taking the inverse z-transform of the above relation, we have

$$x[n] - x[n-1] + 2x[n-3] = y[n] - 1.75y[n-1] + 1.25y[n-2] - 0.375y[n-3].$$

Equivalently,

$$y[n] = 1.75y[n-1] - 1.25y[n-2] + 0.375y[n-3] + x[n] - x[n-1] + 2x[n-3].$$

(d) Draw a block diagram implementation of this discrete time filter using simple delay elements (one clock cycle flip flops), constant multipliers, and adders.



(e) Find the output $y[n]$ of this system for the following inputs:
Eigenfunctions; if $x[n]$ is an eigenfunction "in the form a^n", then $y[n] = H(a)a^n$. Note that $H(3) = \frac{160}{117}$ and $H\left(\frac{1}{3}\right) = -\frac{416}{25}$. Therefore, we have the following outputs.

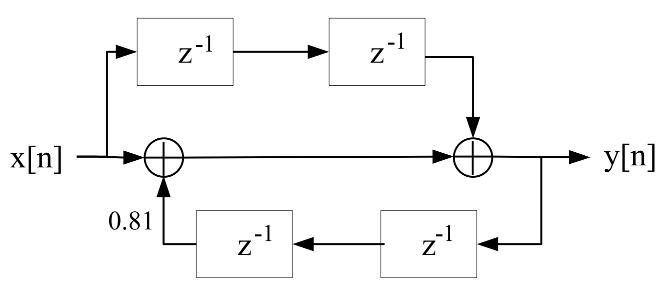
i. $x[n] = 3^n$. Hence, $y[n] = H(3)3^n = \frac{160}{117}3^n$.

ii. $x[n] = 3^n 3^{-10}$. This is indeed similar to previous part where input is scaled by 3^{-10}. Hence, $y[n] = \frac{160}{117}3^n 3^{-10}$.

iii. $x[n] = \left(\frac{1}{3}\right)^n 3^{10}$. Hence, $y[n] = H\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^n 3^{10} = \left(-\frac{416}{25}\right)3^{10-n}$.

5. (20 points) **Digital filter design.**

Consider an implementation of a simple digital filter, as shown in the below block diagram using simple delay elements, 2-input adders, and a single constant multiplier.



Provide the following items.

(a) A valid difference equation that describes the system.

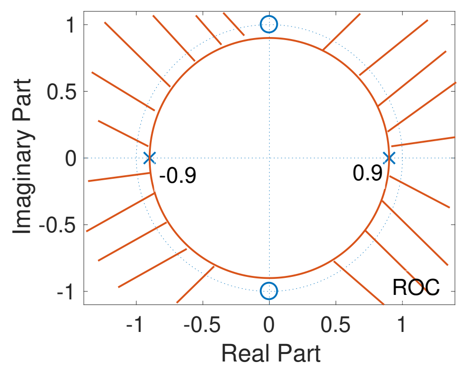
$$x[n] + x[n-2] = y[n] - 0.81y[n-2].$$

(b) The corresponding transfer function $H(z)$.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1-0.81z^{-2}} = \frac{z^2+1}{(z-0.9)(z+0.9)}.$$

Note that ROC: $|z| > 0.9$ because it is a causal system (ROC extends outside from the largest pole).

(c) A pole zero plot of $H(z)$, including the unit circle. Is this system stable?



Stable since it includes the unit circle.

(d) Plot its amplitude response $H(\omega)$ for $\omega \in (-\pi, \pi)$. What kind of filter is this?

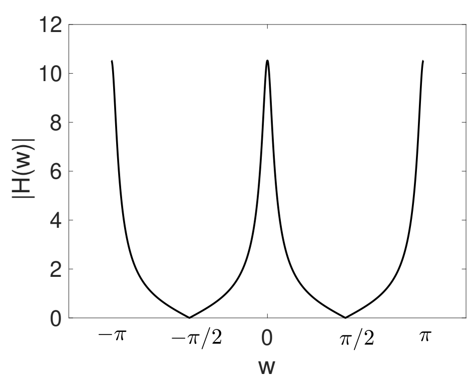
Note that $z = e^{j\omega}$

$\omega = 0$, then $z = 1$ and $H(1) = \frac{2}{0.19} \approx 10$

$\omega = \pi$, then $z = -1$ and $H(-1) = \frac{2}{0.19} \approx 10$

$\omega = \frac{\pi}{2}$, then $z = j$ and $H(j) = 0$

The frequency response will behave as shown below. Hence, it is a "band-stop filter".



6. (10 points) **Circular convolution.**

Circular convolution of $x[n]$ and $h[n]$ is given by

$$y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[(n-k)_N].$$

Note that this function is $N = 6$-periodic.

This process is similar to linear convolution, except the signal wraps around periodically.

Consider the first pair. In the $n = 0$ position we have,

$$y[0] = x[0] \circledast h[0] = \sum_{k=-\infty}^{\infty} x[k]h[6-k] = \sum_{k=0}^{5} x[k]h[(0-k)_6] = -1+2+0+2-6+4 = 1.$$

Similarly, in the next position we have

$$y[1] = \sum_{k=0}^{5} x[k]h[(1-k)_6] = 2-1+2+0+2-6 = -1.$$

Continuing the process we get the final result as shown below (left).

Now consider the second pair. In the $n = 0$ position we have,

$$y[0] = \sum_{k=-1}^{2} x[k]h[(0-k)_6] = -1\cdot 2+1\cdot 1-2\cdot(-1)-1\cdot 0+0\cdot 0+0\cdot 0 = 1.$$

At $n = 1$, we have

$$y[1] = \sum_{k=-1}^{2} x[k]h[(1-k)_6] = -1\cdot 0+1\cdot 2-2\cdot 1-1\cdot(-1)+0\cdot 0+0\cdot 0 = 1.$$

Continuing the process we get the final result as shown below (right). Note that since we have a length 4 and length 3 signal, the length of the convolution is $4+3-1 = 6$. Therefore, the circular and linear convolutions are the same. In other words, the zeros that pad the end of the signals prevent the unwanted intersections with “wrapped around" pieces of the signal.