Today's lecture

- poles-zeros
- properties of ROC
- properties of z transform
- Examples

Announcements

- MT 1 next Thursday (10/8 on Webex at <u>10:10 am</u>)
- Posted last year's exam & solutions and transform tables.
- 1 sided 1 page crib sheet

True / False Questions

DTFT converges everywhere. F

z transform converges everywhere. F

ROC always contains the <u>unit circle</u>. F

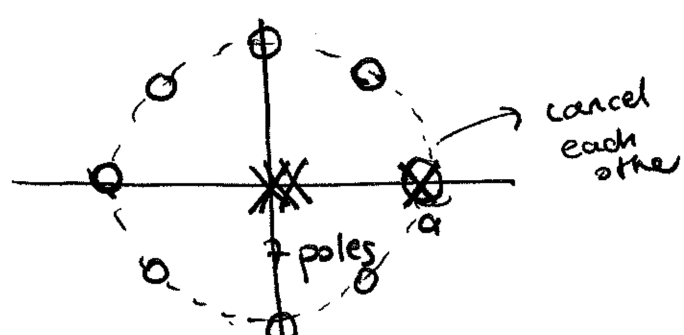
If ROC contains the unit circle, then the DTFT exists. T

ROC may contain zeros. T

ROC may contain <u>poles</u>. F

Last lecture's example:

$$X(z) = \frac{z^N - a^N}{z^{N-1}(z-a)}$$



Assume $N=8$, $a>0$

$z^N = a^N \to z^8 = a^8$, $z_1 = a$, $z_2 = -a$, $z_3, z_4 = \pm ja$

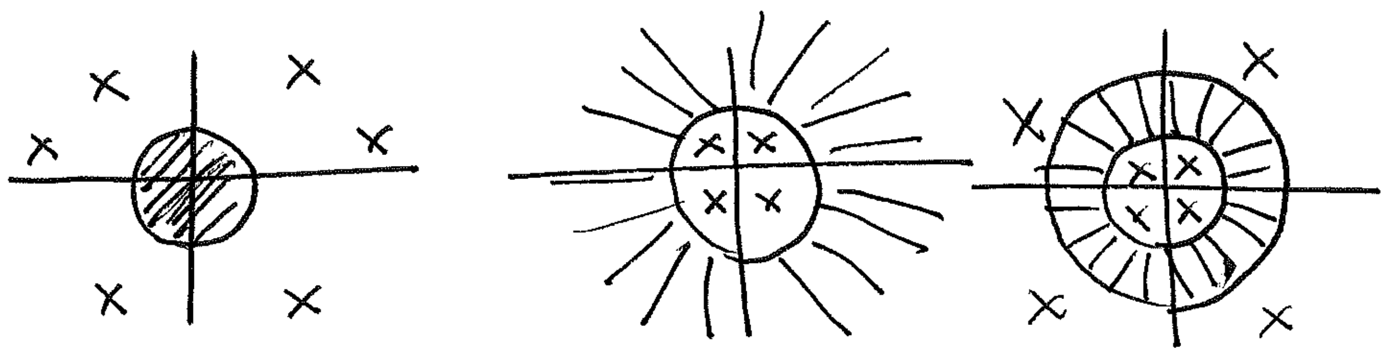
$z_5^8 = (a \cdot e^{j\pi/4})^8 = a^8$

<u>Exercise</u> $x[n] = 2^n \cos(3n)\, u[n]$

Determine the z-transform and the ROC along with the pole-zero locations.

Properties of ROC

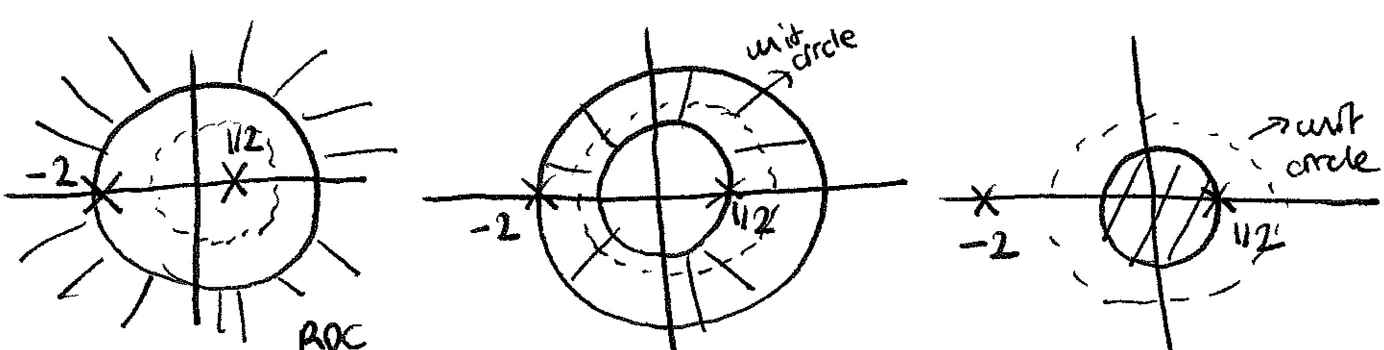
1. ROC is a ring or disc centered at 0.
2. ROC does not contain poles. (otherwise it does not converge).
3. If $x[n]$ is finite length, ROC will be the entire z-plane (except possibly $z = 0, +\infty$)



ROC cannot contain any poles.

4. ROC tells if the system is <u>stable</u>. If the ROC contains the <u>unit circle</u>, then the system is <u>stable</u>.

5. For causal systems, the impulse response is right-sided.



CAUSAL
NOT STABLE

STABLE
~~CAUSAL~~
NOT CAUSAL

NOT STABLE
NOT CAUSAL

Properties of z-transform

1. Linearity

$$a x_1[n] + b x_2[n] \xleftrightarrow{z\text{-transform}} a X_1(z) + b X_2(z)$$

ROC is the <u>intersection</u> of ROCs of $X_1(z)$ and $X_2(z)$.

2. Time shift

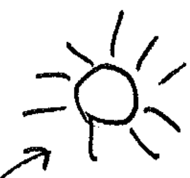
$$x[n-\underset{\text{integer}}{n_0}] \longleftrightarrow z^{-n_0} X(z) \qquad (\text{Recall in DTFT } z = e^{jw})$$

If $n_0 > 0$ this will introduce new poles in ROC.

If $n_0 < 0$ this will introduce new zeros in ROC.

<u>Example</u> Consider the z-transform given as

$$X(z) = \frac{1 + 2z^{-1}}{1 + z^{-1}}$$



Assume that $x[n]$ is a <u>right-sided signal</u>. Determine $x[n]$.

$$X(z) = \frac{1}{1+z^{-1}} + 2 z^{-1} \cdot \frac{1}{1+z^{-1}}$$

$$\frac{1}{1+z^{-1}} \xleftrightarrow{z} (-1)^n u[n], \quad |z| > 1$$

$$x[n] = (-1)^n u[n] + 2 \underbrace{(-1)^{n-1} u[n-1]}_{} \quad n_0 = 1$$ using time shift property

3. Scaling $\quad a^n x[n] \xrightarrow{z\text{-transform}} X\left(\frac{z}{a}\right)$

Let's show why this is true. Let $y[n] = a^n x[n]$

$$Y(z) = \sum_{n=-\infty}^{+\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{+\infty} a^n x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

4. Time reversal $\quad x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right)$

'Inversion of poles and zeros across unit circle!'

5. Convolution

$$x[n] * h[n] \xleftrightarrow{z} X(z)H(z)$$

6. Differentiation $\quad n x[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz}$

Why? $\quad X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} x[n]\,(-n)\, z^{-n-1}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} (n x[n])\, z^{-n}$$

7. Initial value theorem

$$x[0] = \lim_{z\to\infty} X(z)$$

Transfer Function (z transform of impulse function)

$$H(z) = \frac{N(z)}{D(z)} \qquad \begin{array}{l} N(z)=0 \rightarrow \text{zeros} \\ D(z)=0 \rightarrow \text{poles} \end{array}$$

* For stability we require $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$

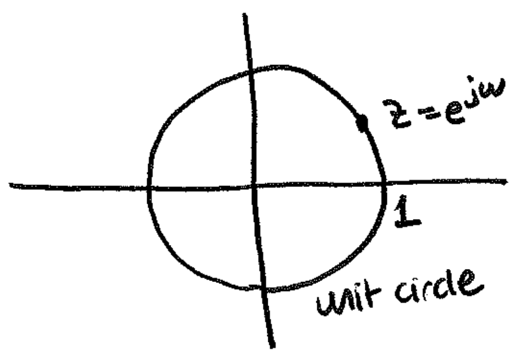
$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$$

$$z = e^{j\omega} \Rightarrow H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}$$

? triangle inequality

$$|H(e^{j\omega})| = \left| \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{+\infty} |h[n] e^{-j\omega n}|$$

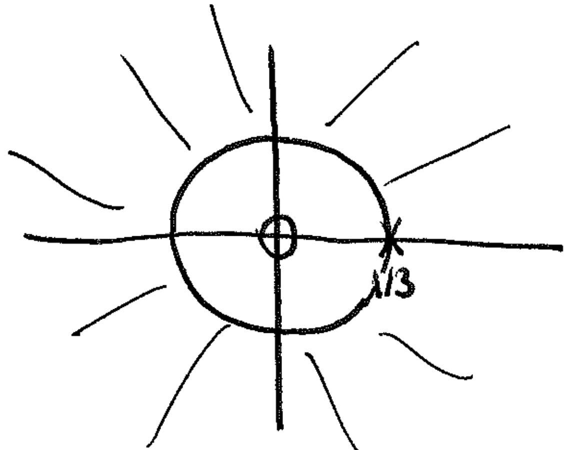
$$= \sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$



ROC contains unit circle if $h[n]$ is stable.

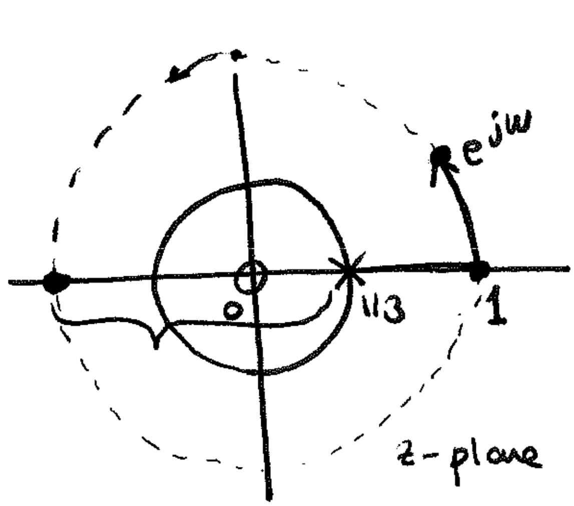
<u>Example</u> $h[n] = \left(\frac{1}{3}\right)^n u[n] \Rightarrow H(z) = \frac{z}{z - 1/3}$, $|z| > 1/3$

$$H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - 1/3} \quad \text{(DTFT)}$$



Can we plot the magnitude response? Yes

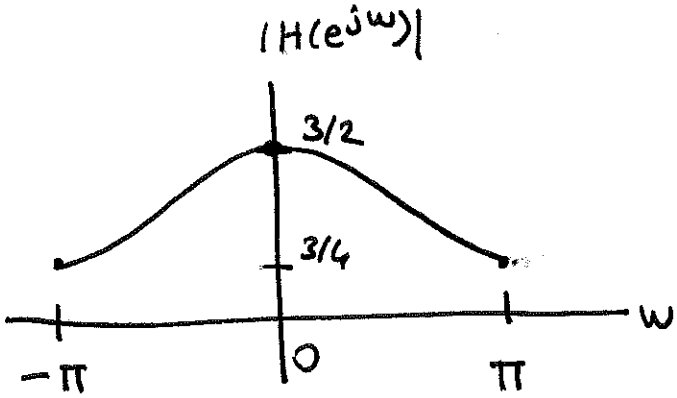
$|H(e^{j\omega})|$



$z = e^{jw} \quad |z| = 1$

$$|H(e^{jw})| = \frac{|N(e^{jw})|}{|D(e^{jw})|}$$ DTFT magnitude

$$= \frac{\prod_{\text{zeros}} \left(\text{length of vector from each } 0 \text{ to } e^{jw}\right)}{\prod_{\text{poles}} \left(\text{length of vector from each } \times \text{ to } e^{jw}\right)}$$



$$|H(e^{j0})| = \frac{1}{2/3} = \frac{3}{2}$$

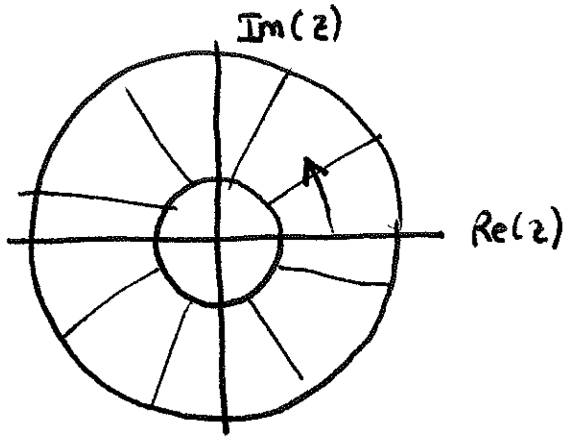
$$|H(e^{j\pi})| = \frac{1}{4/3} = \frac{3}{4}$$

Inverse z-transform

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

'Cauchy integral theorem'

complex contour integral
for some $|z|=r$ in the ROC



Example transform pairs

$$\alpha^n u[n] \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}}, \quad |z|>|\alpha|$$

$$\alpha^n \cos w_0 n \; u[n] \xleftrightarrow{z} \frac{1-(\cos w_0)\alpha z^{-1}}{1-(2\cos w_0)\alpha z^{-1}+\alpha^2 z^{-2}}, \quad |z|>|\alpha|$$

$$\alpha^n \sin w_0 n \; u[n] \xleftrightarrow{z} \frac{(\sin w_0)\alpha z^{-1}}{1-(2\alpha\cos w_0)z^{-1}+\alpha^2 z^{-2}}, \quad |z|>|\alpha|$$

'See Example 3.2.5 from textbook.'

Example $X(z) = \dfrac{7-13z^{-1}}{1-2z^{-1}-3z^{-2}}$, $|z|>1$

What is the inverse z-transform of $X(z)$?

$$X(z) = \frac{7-13z^{-1}}{1-2z^{-1}-3z^{-2}} = \frac{A}{1-3z^{-1}} + \frac{B}{1+z^{-1}}$$

$(1+z^{-1})$ $\quad$ $(1-3z^{-1})$

A, B real numbers

$$7-13z^{-1} = A(1+z^{-1}) + B(1-3z^{-1})$$

$$\begin{aligned} 7 &= A+B \\ -13 &= A-3B \end{aligned} \implies A=2,\ B=5$$

$$X(z) = \underbrace{\frac{2}{1-3z^{-1}}}_{|z|>3} + \underbrace{\frac{5}{1+z^{-1}}}_{|z|>1} \quad , \ \underline{|z|>3}$$

$$x[n] = 2.3^n.u[n] + 5.(-1)^n u[n]$$

Example $X(z) = \dfrac{3z}{z^2+2z+4}$ (right sided $x[n]$)

$x[n]$?

$$X(z) = \frac{3z^{-1}}{1+2z^{-1}+4z^{-2}} \longrightarrow$$ Use the relation from previous page:

$$= \frac{(\sin w_0)\,\alpha\, z^{-1}}{1-(2\alpha\cos w_0)z^{-1} + \alpha^2 z^{-2}} \ \Bigg| \ . \sqrt{3}/2$$

$\alpha = 2$, $\cos w_0 = -\frac{1}{2}$ $\rightarrow w_0 = \frac{2\pi}{3}$

$$= \frac{\sqrt{3}\,2\,z^{-1}}{1+2z^{-1}+4z^{-2}} \cdot \frac{\sqrt{3}}{2} \xleftrightarrow{\text{inverse } z} \frac{\sqrt{3}}{2} . 2^n . \sin\left(\frac{2\pi}{3}n\right) u[n]$$