11/09/20

Today's Lecture

- Wrapping up polyphase and multi-rate signal processing
- FIR filter design
- Linear phase filters

Announcements

Midterm 2 11/16 on Webex

Why?

Polyphase decomposition:
- Downsampled signal has fewer coefficients
- Processing cost is reduced.
- Think about Radix-2 DFT (even and odd components)
- original system N-tap, each of $e_i[n]$ is N/M tap

Block diagram / chaining:
- Via chaining we can get rid of the delay elements $\boxed{z^M}$ and $\boxed{z^{-M}}$

Equivalent systems:

$$\to \boxed{\downarrow M} \to \boxed{H(z)} \to \quad = \quad \to \boxed{H(z^M)} \to \boxed{\downarrow M} \to$$

$$\to \boxed{H(z)} \to \boxed{\uparrow L} \to \quad = \quad \to \boxed{\uparrow L} \to \boxed{H(z^L)} \to$$

Recall: $h[n] = h_0[n] + h_1[n-1] + \dots + h_{M-1}[n-M+1]$

$e_k[n] = h_k[nM] = h[nM+k]$ 'polyphase components'

Frequency domain

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]\, z^{-k}$$

$$= \sum_{l=-\infty}^{+\infty} h[lM]\, z^{-lM} + \sum_{l=-\infty}^{+\infty} h[lM+1]\, z^{-(lM+1)} + \cdots$$

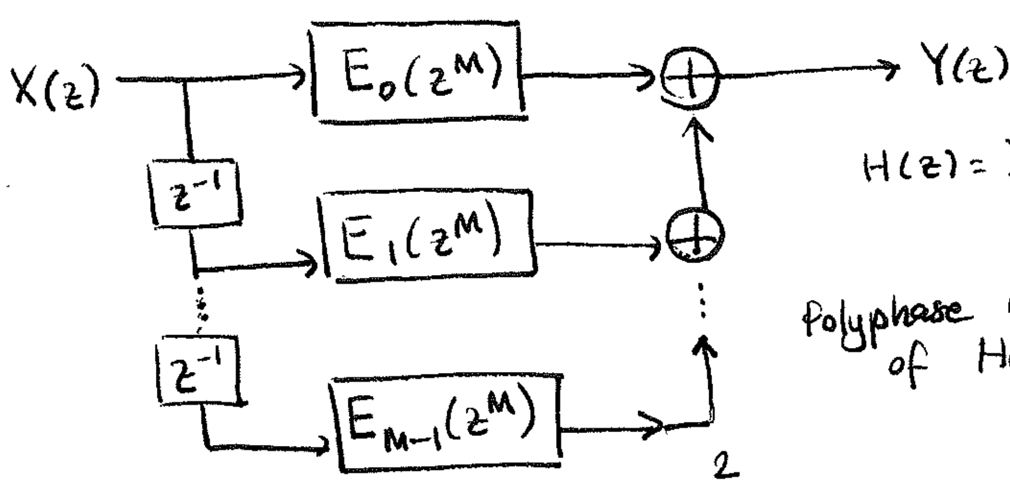
$$+ \sum_{l=-\infty}^{+\infty} h[lM+M-1]\, z^{-(lM+M-1)}$$

$$= \sum_{k=0}^{M-1} \sum_{l=-\infty}^{+\infty} h[lM+k]\, z^{-(lM+k)}$$

$$= \sum_{k=0}^{M-1} z^{-k} \sum_{l=-\infty}^{+\infty} \underbrace{h[lM+k]}_{e_k[l]}\, z^{-lM}$$

$$= \sum_{k=0}^{M-1} z^{-k} E_k(z^M)$$ where $E_k(z)$ is the z-transform of $e_k[n]$.

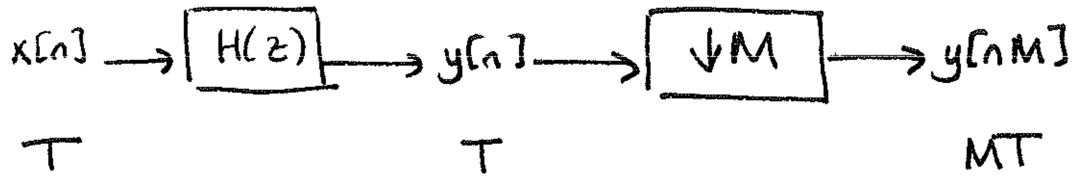
Using this we can establish a connection between the z-transform of $h[n]$ and the z-transform of its polyphase components.



$$H(z) = \frac{Y(z)}{X(z)}$$

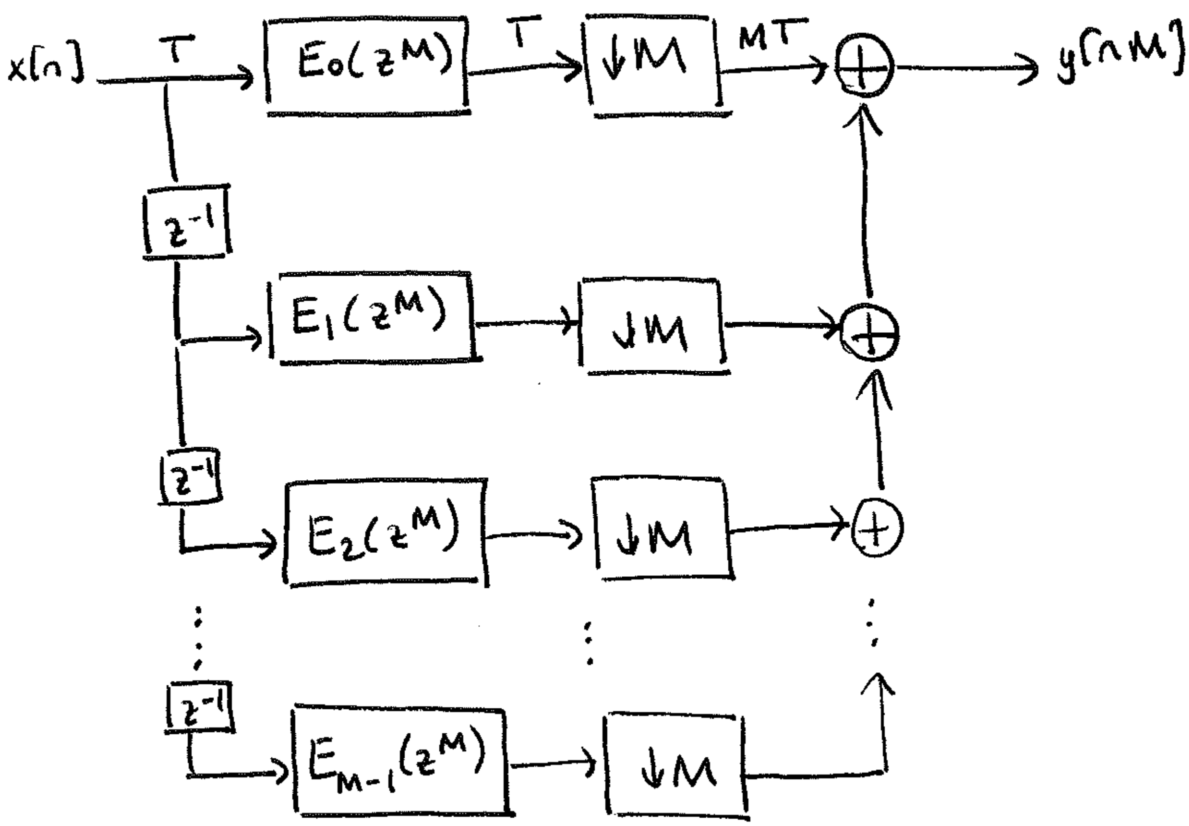
Polyphase realization of $H(z)$.

Polyphase Implementation of Decimation / Interpolation Filters

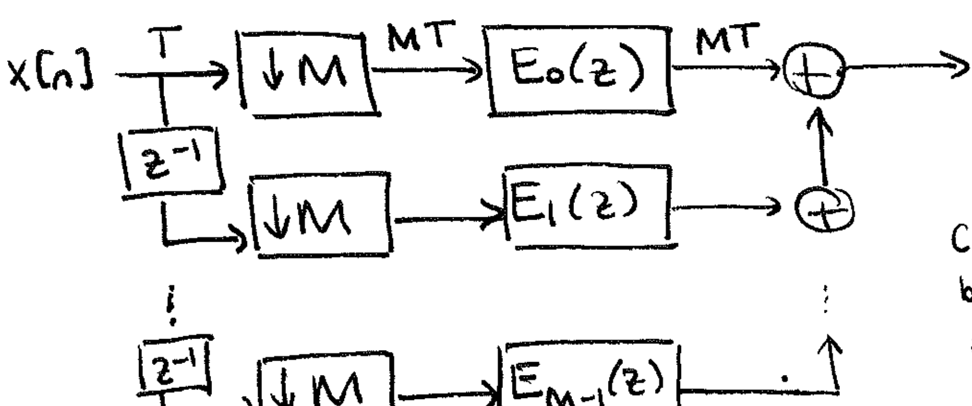


We generate and discard $\frac{M-1}{M}$ fraction of data.

Polyphase representation



A more efficient equivalent system: (sampling frequency reduced by a factor of M)

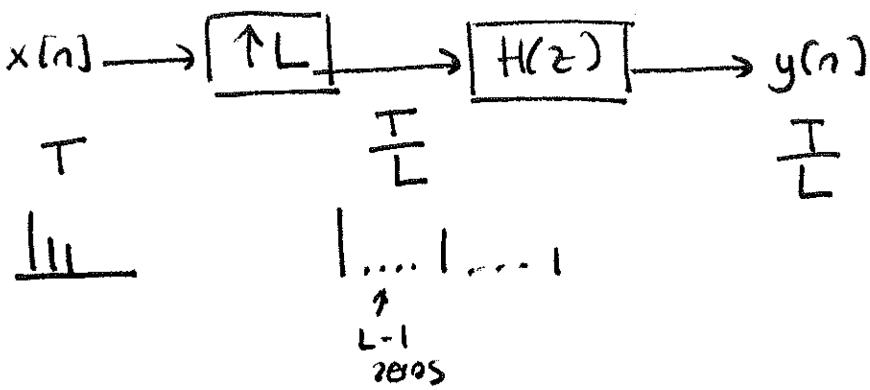


Computational savings by filtering at a low sampling rate.

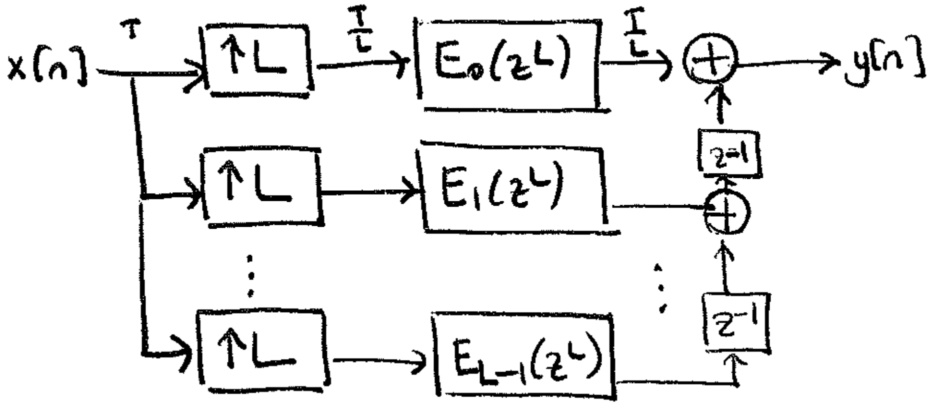
If the original system was an N-tap filter, each of $e_i[n]$ is an N/M tap filter.

If the original realization was N multiplications per unit time

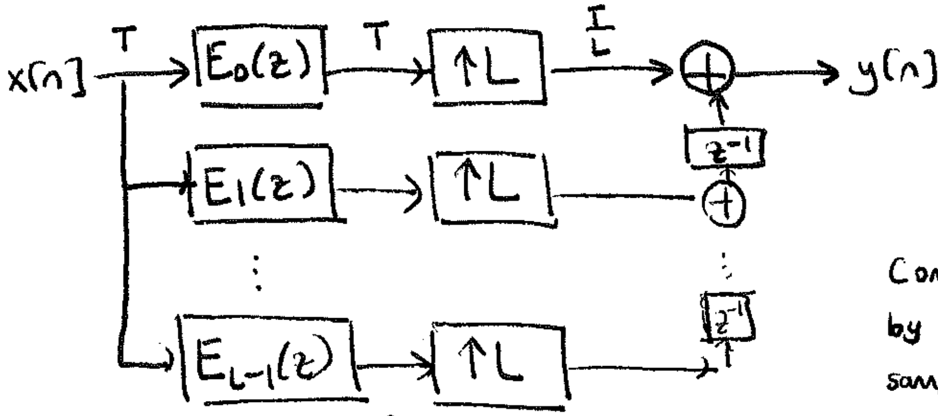
polyphase " N/M " "



We generate data and $\frac{L-1}{L}$ fraction of values are 0.



Using the second identity, flip the upsampler and the filters.



Computational savings by filtering at a low sampling rate.

No zeros are explicitly filtered.

Filter Design

LTI system

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega)\,\underline{H}(\omega)$$

$$Y(\omega) = |Y(\omega)| \cdot e^{j\angle Y(\omega)}$$

Magnitude $\quad |Y(\omega)| = |X(\omega)|\,|H(\omega)|$

Phase $\quad \angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$

Low Pass Filters
LPF

- sampling rate changes
- aliasing
- reconstruction (interpolation)

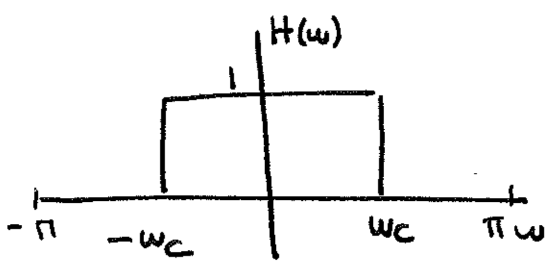
Band Pass Filters
BPF

- wireless transmitters / receivers (to prevent interference)

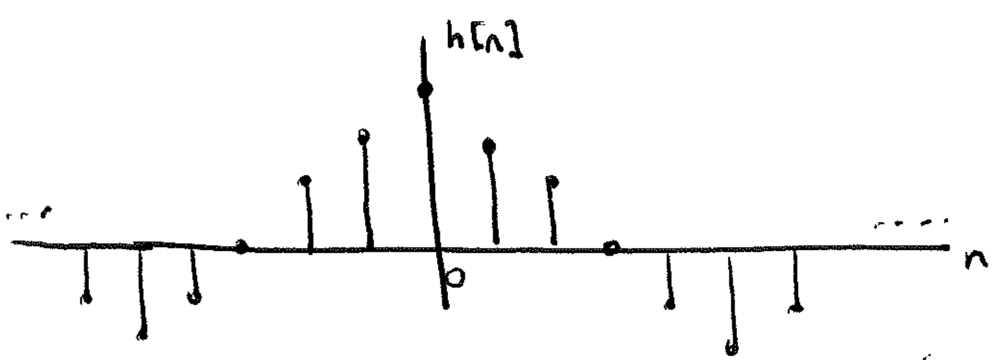
High Pass Filters
HPF

- cancelling low frequency noise
- audio amplifiers to boost volume.

Ideal low pass filter



$$H(\omega) = \begin{cases} 1, & \omega \le |w_c| \\ 0 & w_c < w \le \pi \end{cases}$$



$$h[n] = \frac{\sin w_c n}{\pi n}, \quad -\infty \le n \le \infty$$

CONS : infinite length impulse response (not practical)
non-causal ($h[n] \ne 0$ for $n<0$)

Goal: Approximate $h[n]$:

- Finite impulse response (FIR)
- Causal

Ideal LPF has no phase shift (phases of input are not altered)

Ideal delay: $h_d[n] = \delta[n - n_d]$, $n_d > 0$

$$H_d(\omega) = e^{-j\omega n_d} \quad , \quad -\pi \leq \omega \leq \pi$$

$$\Rightarrow \quad |H_d(\omega)| = 1 \quad \text{and} \quad \angle H_d(\omega) = \underbrace{-\omega n_d}_{\text{Phase is linear in } \omega} \quad , \quad -\pi \leq \omega \leq \pi$$

* We can tolerate linear distortion because each component of the input is moved in the same number of units.

As a result, we desire filters with linear phase:

$$H(\omega) = \begin{cases} e^{-j\omega n_d} & , \quad |\omega| < \omega_c \\ 0 & , \quad \omega_c \leq |\omega| \leq \pi \end{cases}$$

$$|H(\omega)| = \begin{cases} 1 & , \quad |\omega| < \omega_c \\ 0 & , \quad \omega_c \leq |\omega| \leq \pi \end{cases} \quad , \quad \angle H(\omega) = -\omega n_d \ , \ \omega \in [-\pi, \pi]$$

$$h[n] = \frac{\sin(\omega_c(n - n_d))}{\pi(n - n_d)} \quad , \quad -\infty < n < \infty$$

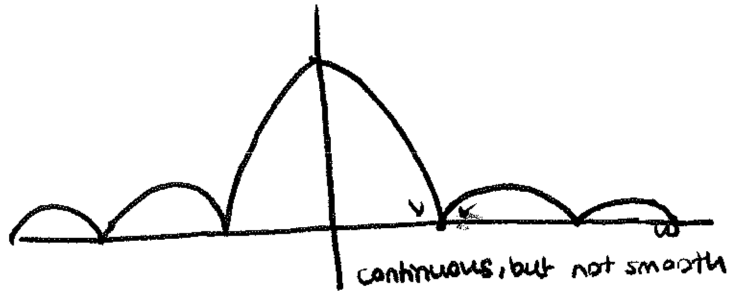
This is a LPF

output is delayed by n_d units.

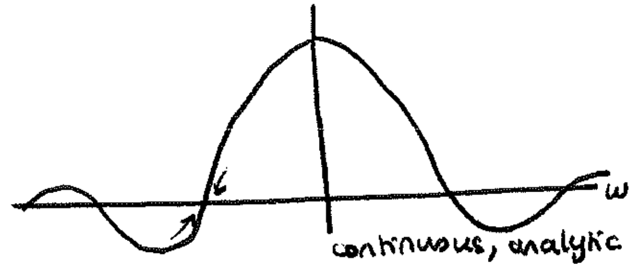
It is still not causal.

Arbitrary

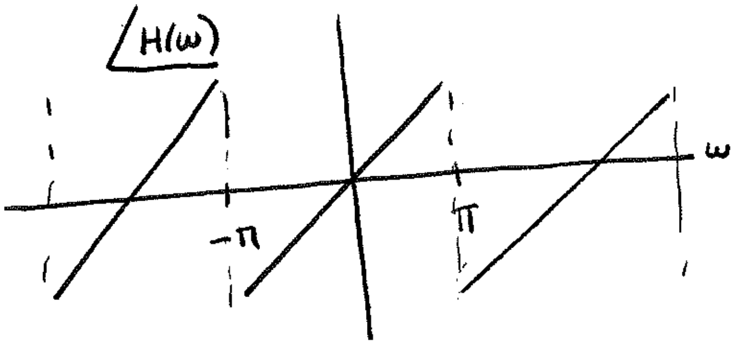
$|H(\omega)|$, magnitude response



$A(\omega) = \pm|H(\omega)|$, Amplitude response



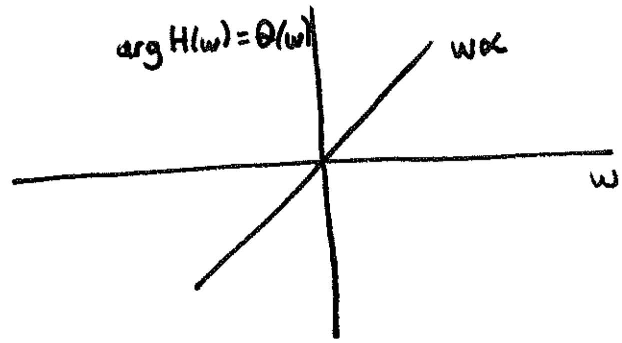
$\angle H(\omega)$



nd is −

'freqz'

$\arg H(\omega) = \theta(\omega)$

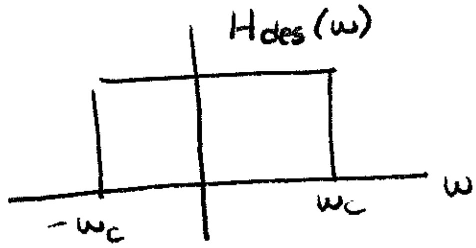


Linear Phase

Filter Design Process

1. Start with desired frequency response

$H_{des}(\omega)$



2. Choose a class of filters, e.g., length − N finite-impulse response (FIR) filter

3. Choose a measure of quality to approximate how close the design is to the desired performance

e.g., least-squares approximation

Chebyshev approximation

Taylor series approximation

4. Apply an algorithm to find the best realization of the filter.

We want real, causal and digital filters of the form

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}} = \frac{Y(z)}{X(z)}$$

✓ If $a_1 = a_2 = \dots = a_N = 0 \Rightarrow$ FIR filter (output only depends on input)

otherwise $\Rightarrow$ IIR Filter (infinite impulse response) where output also depends on the previous outputs

We have the following approximation problem:

$$\min_{a,b} \| \underbrace{E(z)}_{\text{error}} \| = \| H_{des}(z) - \underbrace{H(z)}_{\text{we design!}} \|$$

where $\| . \|$ denotes the norm, e.g. absolute value $\|x\| = |x|$ or the Euclidean norm $\|x\|$ of vector $x = (x_1, \dots, x_n)$: $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.

Linear Phase Filters

$h[n]$ is a length N filter (FIR) and assume that it has a linear phase: $\theta(\omega) = K_1 + K_2 \omega$.

$$H(\omega) = \sum_{n=0}^{N-1} h[n] e^{-j\omega n}$$

$$= e^{-j\omega M} \sum_{n=0}^{N-1} h[n] e^{-j\omega n} \cdot e^{j\omega M} \quad \text{where } M = \frac{N-1}{2}.$$

$$= e^{-j\omega M} \sum_{n=0}^{N-1} h[n] e^{j\omega (M-n)}$$

Next we observe that

$$\begin{aligned}
&\text{If } n=0 && h[0]e^{j\omega M} \\
&n=N-1 && h[N-1]e^{j\omega(M-N+1)} = h[N-1]e^{-j\omega M} \quad (\text{recall } M=\tfrac{N-1}{2}) \\
&n=1 && h[1]e^{j\omega(M-1)} \\
&n=N-2 && h[N-2]e^{j\omega(M-N+2)} = h[N-2]e^{-j\omega(M-1)} \\
& \vdots &&
\end{aligned}$$

We can rewrite $H(\omega)$ as

$$H(\omega) = e^{-j\omega M}\Big(\big(h[0]+h[N-1]\big)\cos \omega M + j\big(h[0]-h[N-1]\big)\sin \omega M$$
$$+$$
$$\big(h[1]+h[N-2]\big)\cos \omega(M-1) + j\big(h[1]-h[N-2]\big)\sin \omega(M-1)$$
$$+$$
$$\vdots \quad \Big)$$

If we have even symmetry, $h[n]=h[N-n-1]=h[2M-n]$, then all the sin terms will drop away:

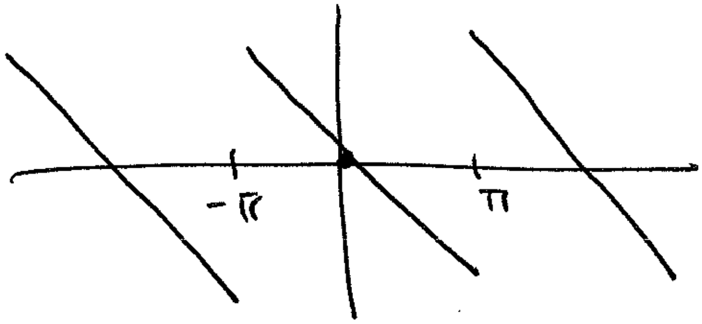
$$H(\omega) = e^{-j\omega M}\Big(\big(h[0]+h[N-1]\big)\cos \omega M + \big(h[1]+h[N-2]\big)\cos \omega(M-1) + \ldots \Big)$$

Amplitude response:

$$A(\omega) = \sum_{n=0}^{M-1} 2\,h[n]\cos(\omega(M-n)) + h[M] \qquad \text{if } N \text{ is odd}$$

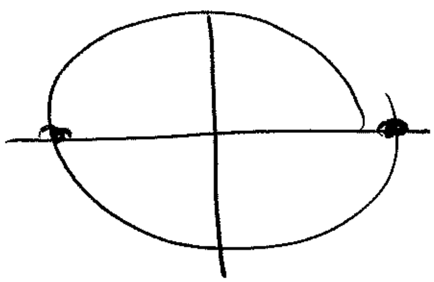
$$= \sum_{n=1}^{M} 2\,h[M-n]\cos(\omega n) + h[M] \qquad \text{'Type I } N \text{ Odd'}$$

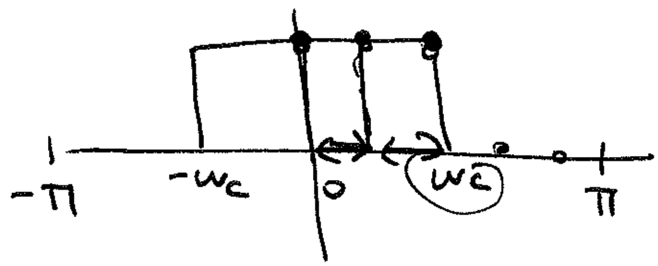
↑ change of variables $n \to M-n$



$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$

~~$H(\omega)$~~





$H(\omega) = +1\, e^{j\cdot 0} \rightarrow H(\omega) = -1$

$= 1\cdot e^{j\pi}$

$\cdot\, e^{j\pi} = -1$

N coefficient