Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2020

Homework #4: due Thursday, Nov. 5$^{\text{th}}$, at the beginning of class.

Analytical Problems: Clearly show your work and label your answers.

3. (20 points) **Discrete Fourier Transform (DFT).** Compute the N-point DFTs of the following signals:

   Recall the definition of $X[k]$: $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, $0 \le k \le N-1$.

   (a) $x[n] = \delta[n]$
   $X[k] = \sum_{n=0}^{N-1} \delta[n]e^{-j\frac{2\pi}{N}kn} = 1$, for $0 \le k \le N-1$.

   (b) $x[n] = \delta[n-n_0]$, where $0 < n_0 < N$
   $X[k] = \sum_{n=0}^{N-1} \delta[n-n_0]e^{-j\frac{2\pi}{N}kn} = e^{-j\frac{2\pi}{N}kn_0}$, $0 \le k \le N-1$.

   (c) $x[n] = a^n$, where $0 \le n \le N-1$
   $X[k] = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1}\left(ae^{-j\frac{2\pi}{N}k}\right)^n = \frac{1-\left(ae^{-j\frac{2\pi}{N}k}\right)^N}{1-ae^{-j\frac{2\pi}{N}k}} = \frac{1-a^N}{1-ae^{-j\frac{2\pi}{N}k}}$, $0 \le k \le N-1$.

   (d) $x[n] = \begin{cases} 1, & 0 \le n \le N/2-1, \\ 0, & N/2 \le n \le N-1 \end{cases}$, where N is even

   $X[k] = \sum_{n=0}^{N/2-1} e^{-j\frac{2\pi}{N}kn} = \frac{1-\left(e^{-j\frac{2\pi}{N}k}\right)^{N/2}}{1-e^{-j\frac{2\pi}{N}k}} = \frac{1-(-1)^k}{1-e^{-j\frac{2\pi}{N}k}}$, $0 \le k \le N-1$.

   (e) $x[n] = e^{j\frac{2\pi}{N}k_0 n}$, where $0 \le n \le N-1$
   $X[k] = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}k_0 n}e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} = N\delta[k-k_0]$, $0 \le k \le N-1$.

   (f) $x[n] = \cos\left(\frac{2\pi}{N}k_0 n\right)$, where $0 \le n \le N-1$

   $$X[k] = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{N}k_0 n\right)e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \frac{e^{\frac{2\pi}{N}k_0 n} + e^{-\frac{2\pi}{N}k_0 n}}{2} e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1$$
   $$= \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} + \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+k_0)n} = \frac{N}{2}\delta[k-k_0] + \frac{N}{2}\delta[k+k_0-N], \quad 0 \le k \le N-1,$$

   where the last step follows from part (e).

   (g) $x[n] = \sin\left(\frac{2\pi}{N}k_0 n\right)$, where $0 \le n \le N-1$

   $$X[k] = \sum_{n=0}^{N-1} \sin\left(\frac{2\pi}{N}k_0 n\right)e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \frac{e^{\frac{2\pi}{N}k_0 n} - e^{-\frac{2\pi}{N}k_0 n}}{2j} e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1$$
   $$= \frac{1}{2j}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} - \frac{1}{2j}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+k_0)n} = \frac{N}{2j}\delta[k-k_0] - \frac{N}{2j}\delta[k+k_0-N], \quad 0 \le k \le N-1.$$

   (h) $x[n] = \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd} \end{cases}$, where $0 \le n \le N-1$

Since the odd components are 0, we can rewrite $X[k]$ using the even components. When N is even,

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} = \sum_{l=0}^{N/2-1} x[2l]e^{-j\frac{2\pi}{N}k2l} = \sum_{l=0}^{N/2-1} e^{-j\frac{2\pi}{N}k2l}$$
$$= \frac{1-e^{-j\frac{2\pi}{N}k2\frac{N}{2}}}{1-e^{-j\frac{2\pi}{N}k2}} = \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{4\pi}{N}k}}$$

when the limit exists. Note that this equals 0 when k is not a multiple of $N/2$. When k is a multiple of $N/2$ the finite sum formula gives $0/0$.
Similarly, when N is odd we have

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} = \sum_{l=0}^{(N-1)/2} x[2l]e^{-j\frac{2\pi}{N}k2l} = \sum_{l=0}^{(N-1)/2} e^{-j\frac{2\pi}{N}k2l}$$
$$= \frac{1-e^{-j\frac{2\pi}{N}k(N+1)}}{1-e^{-j\frac{2\pi}{N}k2}}$$

When k is a multiple of $N/2$, we can instead evaluate the sum directly:

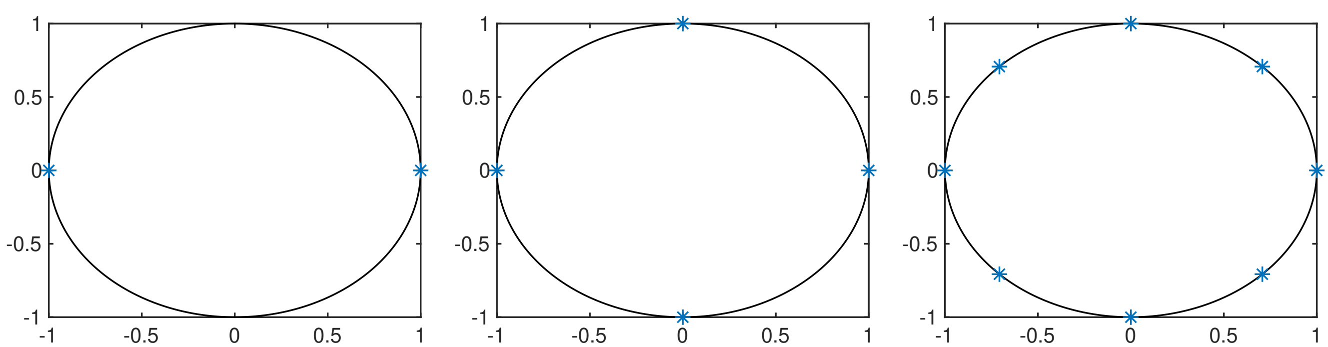
$$X[0] = \sum_{n=0}^{N-1} x[n] = \begin{cases} \frac{N}{2}, & \text{N even} \\ \frac{N+1}{2}, & \text{N odd} \end{cases}$$
$$X\left[\frac{N}{2}\right] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}\frac{N}{2}n} = \sum_{n=0}^{N-1} x[n](-1)^n = X[0]$$

4. (10 points) **DFT Matrix.** Recall the definition of DFT which is given as

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn},$$

where W_N^k for $k = 0, \ldots, N-1$ are called the N^{th} roots of unity.

(a) Plot the roots of unity for $N = 2$, $N = 4$, and $N = 8$ in the complex plane.
$N = 2 \rightarrow W_2 = -1$, $W_2^2 = 1$ (2 roots on unit circle (Left plot)
$N = 4 \rightarrow W_4 = -j$, $W_4^2 = -1$, $W_4^3 = j$, $W_4^4 = 1$ (4 roots on unit circle (Middle plot))
$N = 8 \rightarrow W_8 = e^{-j\pi/8}$, $W_8^2 = e^{-j\pi/4}$, $W_8^3 = e^{-j3\pi/8}$, $W_8^4 = -1$, $W_8^5 = e^{-j5\pi/8}$, $W_8^6 = e^{-j3\pi/4}$, $W_8^7 = e^{-j7\pi/8}$, $W_8^8 = 1$ (8 roots on unit circle (Right plot))



Note: These are complex planes. x-axis is the real axis and y-axis is the imaginary axis.

(b) Write down the $N \times N$ complex DFT matrix F for $N = 2$ and $N = 8$ in terms of W_N's.
For $N = 2$, we have the following 2×2 matrix:

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Note that this matrix contains the powers of W_2. ($W_2 = -1$, $W_2^2 = 1$)
For $N = 8$, we have the following 8×8 matrix:

$$F_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8 & W_8^2 & W_8^3 & -1 & -W_8 & -W_8^2 & -W_8^3 \\ 1 & W_8^2 & -1 & -W_8^2 & 1 & W_8^2 & -1 & -W_8^2 \\ 1 & W_8^3 & -W_8^2 & W_8 & -1 & -W_8^3 & W_8^2 & -W_8 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -W_8 & W_8^2 & -W_8^3 & -1 & W_8 & -W_8^2 & W_8^3 \\ 1 & -W_8^2 & -1 & W_8^2 & 1 & -W_8^2 & -1 & W_8^2 \\ 1 & -W_8^3 & -W_8^2 & -W_8 & -1 & W_8^3 & W_8^2 & W_8 \end{bmatrix}$$

Note that this matrix contains the powers of W_8.

5. (20 points) **Fast Fourier Transform (FFT).** Recall (from Lecture 14) that the DFT summation can be split into sums over the odd and even indexes of the input signal. The splitting into sums over even and odd time indexes is called decimation in time.

   In this problem we consider a slightly different approach. Assume that you are given three 8-point FFT chips and asked to build a system that computes a 24-point DFT. Show explicitly how you should interconnect those chips to be able to compute a 24-point DFT.

   Consider the definition of $X[k]$: $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, $0 \le k \le N-1$. We want to combine this idea with the DIT FFT. Hence, we split the DFT sum into 3 components each length 8:

$$\begin{aligned} X[k] &= \sum_{l=0}^{N/3-1} x[3l]e^{-j\frac{2\pi}{N}k3l} + \sum_{l=0}^{N/3-1} x[3l+1]e^{-j\frac{2\pi}{N}k(3l+1)} + \sum_{l=0}^{N/3-1} x[3l+2]e^{-j\frac{2\pi}{N}k(3l+2)}, 0 \le k \le N-1 \\ &= \sum_{l=0}^{N/3-1} x[3l]W_{N/3}^{lk} + \sum_{l=0}^{N/3-1} x[3l+1]W_{N/3}^{lk}W_N^k + \sum_{l=0}^{N/3-1} x[3l+2]W_{N/3}^{lk}W_N^{2k} \\ &= X_1[k] + W_N^k X_2[k] + W_N^{2k} X_3[k] \end{aligned}$$

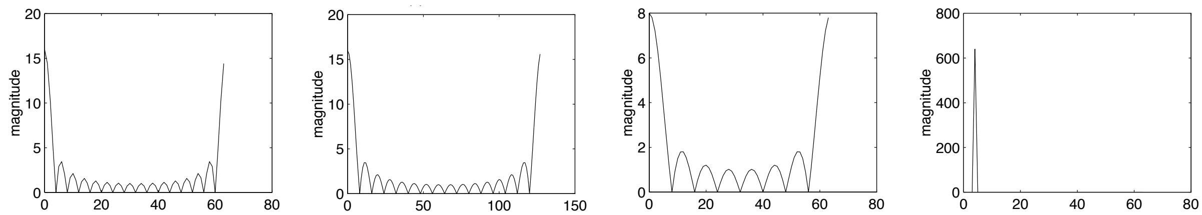
   where given $N = 24$, each of $X_1[k]$, $X_2[k]$ $X_3[k]$ in the above equation is a length $N/3 = 8$ FFT of the sequences $x[0], x[3], \ldots, x[21]$, $x[1], x[4], \ldots, x[22]$ and $x[2], x[5], \ldots, x[23]$.

6. (20 points) **Computation of the DFT using FFT. Write a FFT subroutine in Matlab to compute the following DFTs $X[k]$ and plot their magnitudes $|X[k]|$.**

   (a) The $N = 64$ point DFT of length 16 sequence $x[n] = \begin{cases} 1, & n = 0, 1, \ldots, 15, \\ 0, & \text{otherwise.} \end{cases}$

   (b) The $N = 128$ point DFT of length 16 sequence in the previous part,

   (c) The $N = 64$ point DFT of length 8 sequence $x[n] = \begin{cases} 1, & n = 0, 1, \ldots, 7, \\ 0, & \text{otherwise.} \end{cases}$



N=64 dc=16   N=128 dc=16   N=64 dc=8   N=64 dc=7.664e−14

(d) The $N = 64$ point DFT of length 64 sequence $x[n] = \begin{cases} 10e^{j\frac{\pi}{8}n}, & n = 0, 1, \ldots, 63, \\ 0, & \text{otherwise.} \end{cases}$

Answer the following questions for each of the above parts.

i What is the frequency interval between successive samples for the plots in parts (a)-(d)?

The frequency interval between successive samples for the plots in parts (a), (b), (c) and (d) are $1/64$, $1/128$, $1/64$ and $1/64$ respectively.

ii What is the value of the spectrum at 0 frequency obtained from the plots in parts (a)-(d)?

16, 16, 8 and 0

iii What is the frequency interval between successive nulls in the spectrum in parts (a)-(d)? What is the relationship between this interval (between the successive nulls) and the length of each signal?

Given the frequency interval is $2\pi/N$,

(a) 4 samples between nulls $\rightarrow 4 \cdot \frac{2\pi}{64} = \frac{\pi}{8}$

(b) 8 samples between nulls $\rightarrow 8 \cdot \frac{2\pi}{128} = \frac{\pi}{8}$

(c) 8 samples between nulls $\rightarrow 8 \cdot \frac{2\pi}{64} = \frac{\pi}{4}$

(d) 1 sample between nulls $\rightarrow 1 \cdot \frac{2\pi}{64} = \frac{\pi}{32}$

iv What is the difference between the plots obtained for (a) and (b)? Explain.

Resolution is better with $N = 128$.