Name: Instructor's Solutions

Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2019

Exam #2. Closed book, closed notes.
November 21, 2019, 10:00-11:20 AM

Show all work for full credit.

- Electronic devices are not permitted.
- Reduce expressions involving exponentials, sines, and cosines as much as possible.
- If the sinc function is needed, use the definition $\text{sinc}(x) = \frac{\sin x}{x}$.
- Useful ratio to dB conversion formula: $10\log_{10}(2) = 3$ dB.

Good luck!

1		25
2		25
3		25
4		25
Total		100

1. (25 points.) True/false and fill in the blank questions. You do not need to explain your reasoning. Read each question carefully.

   1. __T__ A fast Fourier transform (FFT) is an algorithm that computes the DFT of a sequence, or its inverse (IDFT).

   2. __T__ Cyclic convolution of two discrete time signals in time domain corresponds to multiplication of their discrete Fourier transforms (DFTs) in frequency domain.

   3. __T__ Parseval's relation states that the sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform. (F is also acceptable because the statement is loose as it does not involve scaling)

   4. __F__ We cannot compute the DFT for aperiodic signals. $\left(X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}\right.$ It is always true that $X[k] = X[k+N]$, in frequency domain$\left.\right)$

   5. __T__ Frequency response of a discrete time signal is 2π periodic.

   6. __T__ A linear-phase FIR filter is always even symmetric or odd symmetric about the middle tap.

   7. __T__ A length $N = 8$ FFT diagram can be implemented in 3 stages. See Lecture 14. We can divide a length N DFT into $\log_2 N$ stages, each involving N multiplications.

   8. __F__ A length 1000 DFT operation involves 10,000 multiplications. $N=1000$ DFT involves N^2 multiplications

   9. __F__ Linear phase FIR filters are not stable.

   10. __F__ The downsampler $\boxed{\downarrow 3}$ is not linear. $x[n] \to \boxed{\downarrow 3} \to x[3n] = y[n]$

   11. __T__ The upsampler $\boxed{\uparrow 2}$ is not time invariant. 

   12. __T__ If the original signal is sampled at M times just above the Nyquist rate, we do not need to prefilter it before we downsample it by a factor of M.

   13. __F__ The output of the cascaded system of a downsampler and an upsampler as shown below satisfies $y[n] = x[n]$.

   $$x[n] \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{\uparrow 2} \longrightarrow y[n]$$

   Homework 5, P9

   14. __F__ The Remez exchange algorithm tries to maximize the number of extremal frequencies in designing Chebyshev FIR filters. "minimize"

   15. __F__ We may have aliasing if the sampling rate is above the Nyquist rate.

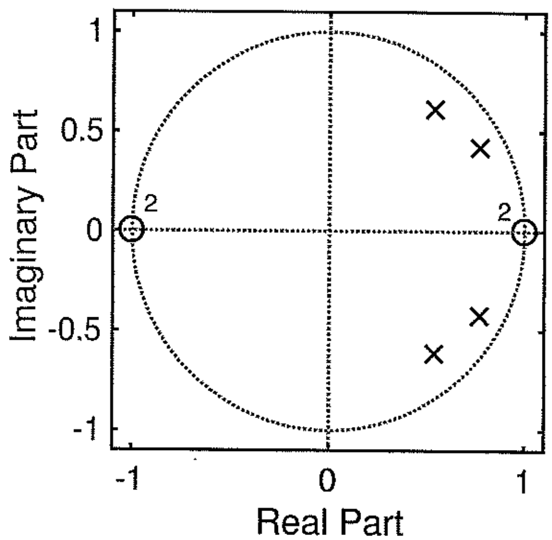
   16. __T__ Linear interpolation is good when the adjacent signal samples are very close to each other.

   17. __F__ We cannot do discrete time processing of continuous time signals.

18. The Nyquist rate is twice the bandwidth of a bandlimited signal.

19. The fundamental element of a radix-2 FFT is colloquially known as a butterfly

20. Given a transfer function $H(z)$ with pole-zero diagram as below, $H(\omega)$ is a bandpass filter.



21. Based on the pole-zero diagram as above, given that the above system is causal (or right-sided), then the filter is stable.

22. A twiddle factor, in FFT algorithms, is any of the trigonometric constant coefficients that are multiplied by the data in the course of the algorithm.

23. A bandlimited signal cannot be also time limited.

24. Ideal low pass filtering of a signal after upsampling is equivalent to interpolation.

25. DSP is ________.

2. (25 points.) Transfer function.

We are given a transfer function for a linear and time invariant discrete-time system:

$$H(z) = \frac{(z-j)(z+j)}{z(z-0.5)(z+0.5)} = \frac{Y(z)}{X(z)}$$

(a) (6 points.) Determine the difference equation corresponding to this system.

$X(z)(z-j)(z+j) = Y(z)\, z(z-0.5)(z+0.5)$

$X(z)(z^2+1) = Y(z)(z^3-0.25z)$

$z^2X(z) + X(z) = z^3Y(z) - 0.25zY(z)$

time shifting

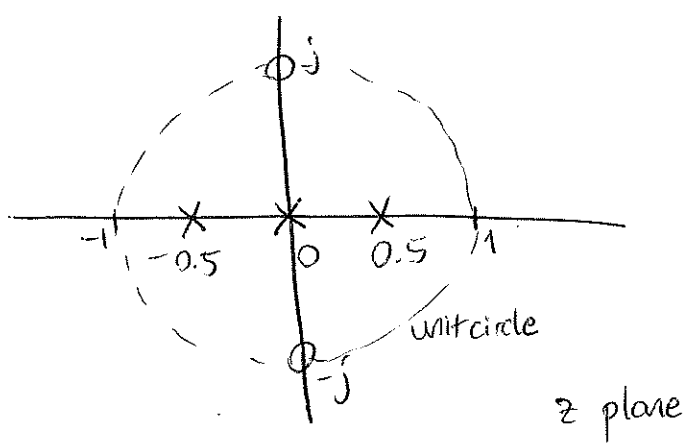
$x[n+2] + x[n] = y[n+3] - 0.25y[n+1]$

$y[n] = 0.25y[n-2] + x[n-1] + x[n-3]$

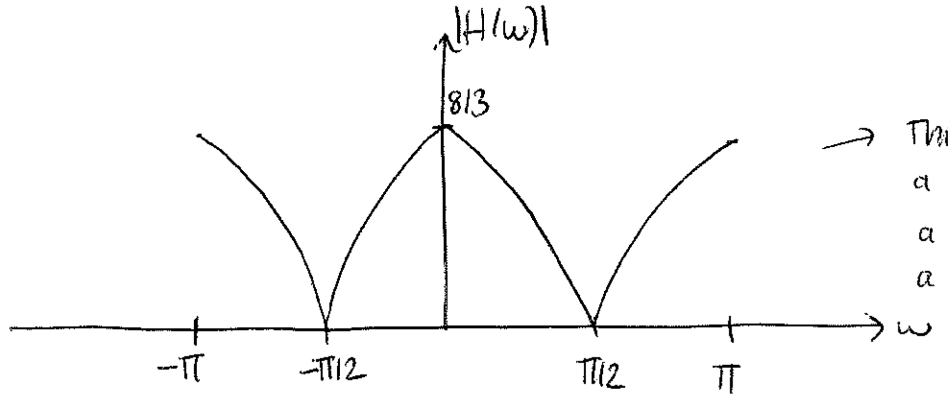
(b) (6 points.) Give the pole-zero diagram of $H(z)$.

Num $= (z-j)(z+j)$, zeros at $+j$ and $-j$

Den. $= z(z-0.5)(z+0.5)$, poles at $0, 0.5$ and -0.5



(c) (8 points.) Using the pole-zero diagram, plot the magnitude of frequency response of $H(\omega)$ for $\omega \in (-\pi, \pi)$. Your plot does not need to be very precise, only indicate the values of $H(0)$, $H\left(\frac{\pi}{2}\right)$ and $H\left(-\frac{\pi}{2}\right)$.



→ This is known as a notch filter, i.e. a band-stop filter with a narrow stopband.

$H(z)$ has zeros at $z = \pm j \Rightarrow H\left(\frac{\pi}{2}\right) = H\left(-\frac{\pi}{2}\right) = 0$

$\omega = 0 \Rightarrow z = e^{j\omega} = 1 \qquad H(z)\Big|_{z=1} = \dfrac{(1-j)(1+j)}{1\cdot(1-0.5)(1+0.5)} = \dfrac{2}{3/4} = \dfrac{8}{3}$

$\omega = \pi \Rightarrow z = e^{j\pi} = -1 \qquad H(z)\Big|_{z=-1} = \dfrac{(-1-j)(-1+j)}{(-1)(-1-0.5)(-1+0.5)} = \dfrac{-2}{3/4} = \dfrac{-8}{3}$

(d) (5 points.) Determine the value of $|H(\omega)|$ at $\omega = \pi/4$.

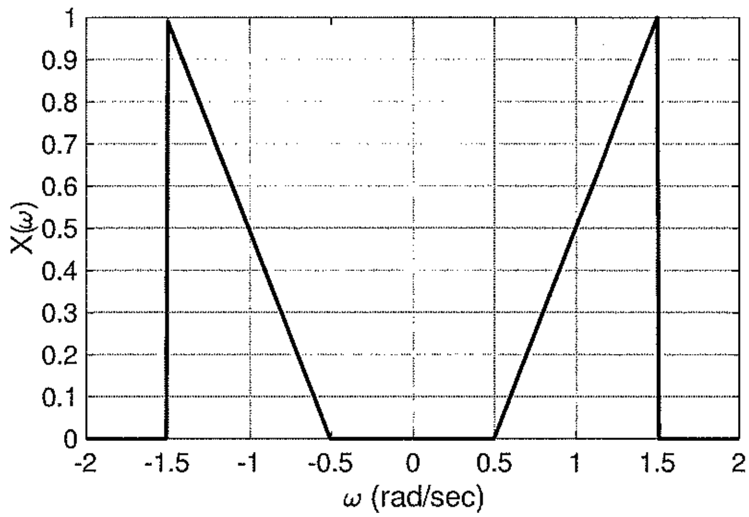
$$H(\omega) = \frac{(e^{j\omega} - j)(e^{j\omega} + j)}{e^{j\omega}(e^{j\omega} - 0.5)(e^{j\omega} + 0.5)}$$

$$= \frac{e^{2j\omega} + 1}{e^{j\omega}(e^{2j\omega} - 0.25)}$$

$$H\left(\frac{\pi}{4}\right) = \frac{e^{j\frac{\pi}{2}} + 1}{e^{j\frac{\pi}{4}}(e^{j\frac{\pi}{2}} - 0.25)} \Rightarrow \left|H\left(\frac{\pi}{4}\right)\right| = \frac{\sqrt{1^2 + 1^2}}{\sqrt{1^2 + 0.25^2}} = \frac{\sqrt{2}}{\sqrt{17/16}} = \frac{4\sqrt{2}}{\sqrt{17}}$$

$(e^{j\frac{\pi}{2}} = j)$

3. (25 points.) Sampling Theorem. We consider a real signal $x(t)$ with a Fourier transform $X(\omega)$ as shown below.

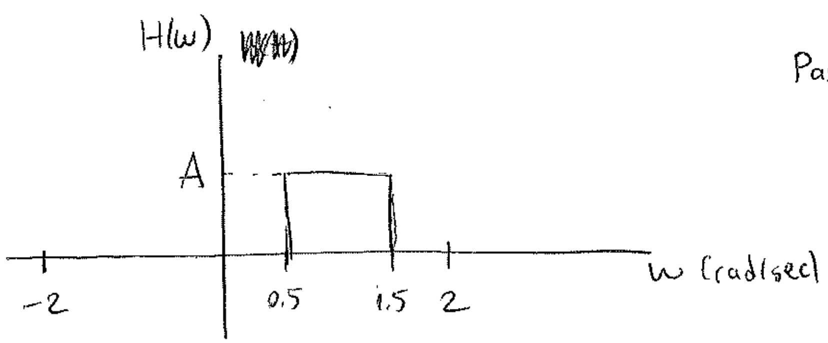


(a) (2 points.) What is the Nyquist rate of $x(t)$?

The Nyquist rate is twice the bandwidth

$\Rightarrow$ 3 rad/sec

(b) (3 points.) We have a complex filter $h(t)$, which has a flat passband extending only from 0.5 to 1.5 rad/sec. The passband has a 3 dB gain in the range $0.5 < \omega < 1.5$, but otherwise has a value of 0 ($-\infty$ in dB), including for all negative values of ω. Plot the Fourier transform (continuous time) of $h(t)$ in the frequency range $[-2, 2]$.

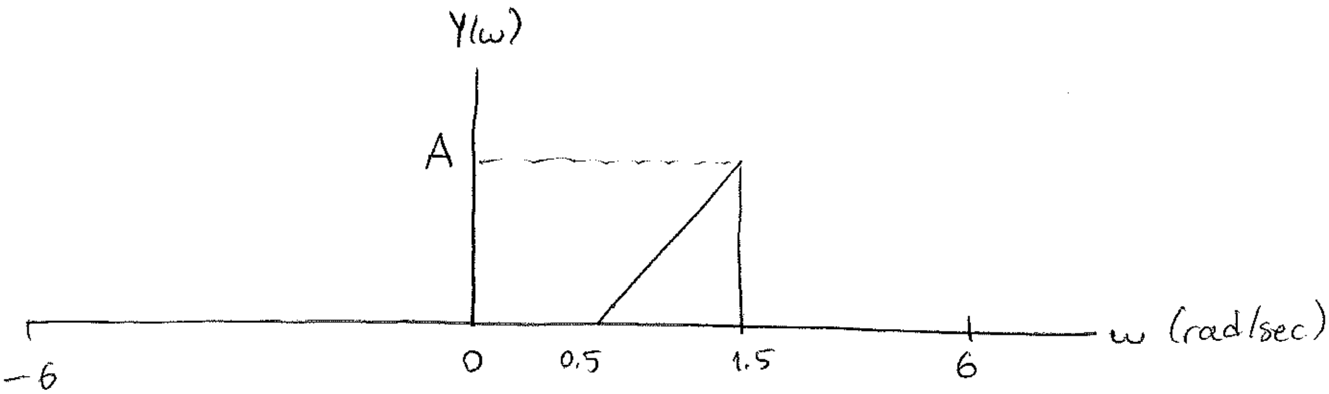


Passband gain $= 3\text{dB} = 10\log 2$

$= 20\log\sqrt{2}$

$\Rightarrow A = \sqrt{2}$

(c) (5 points.) $x(t)$ is bandpass filtered with the complex filter $h(t)$. Call the resulting signal $y(t)$. Plot the Fourier transform (continuous time) of $y(t)$ in the frequency range $[-6, 6]$.

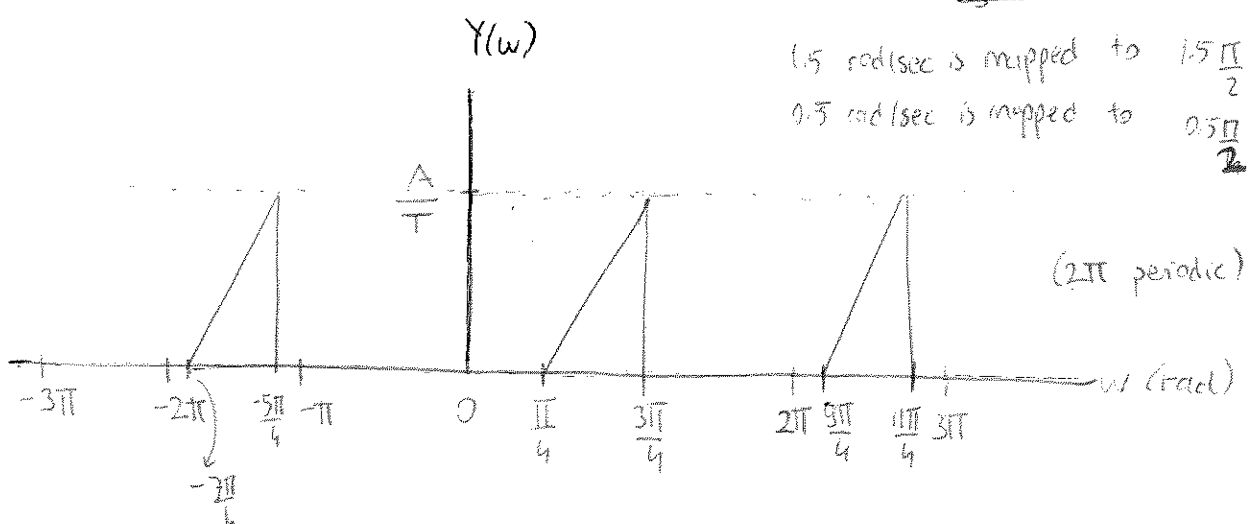


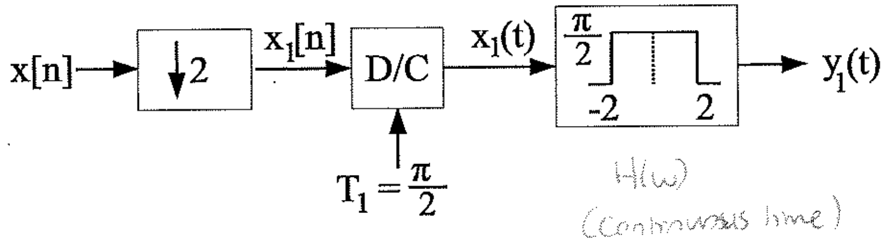
(d) (5 points.) The continuous time signal $y(t)$ is sampled with a sampling period of $T = \frac{\pi}{2}$ seconds. Call the resulting signal $y[n]$. Plot the Fourier transform (DTFT) of $y[n]$ in the corresponding frequency range, namely $\omega \in [-3\pi, 3\pi]$.

1.5 rad/sec is mapped to $1.5\frac{\pi}{2} = \frac{3\pi}{4}$

0.5 rad/sec is mapped to $0.5\frac{\pi}{2} = \frac{\pi}{4}$

(2π periodic)

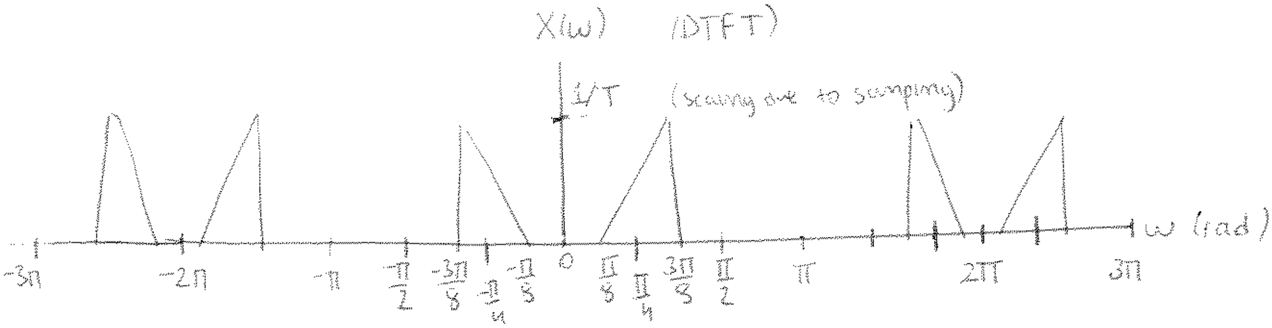


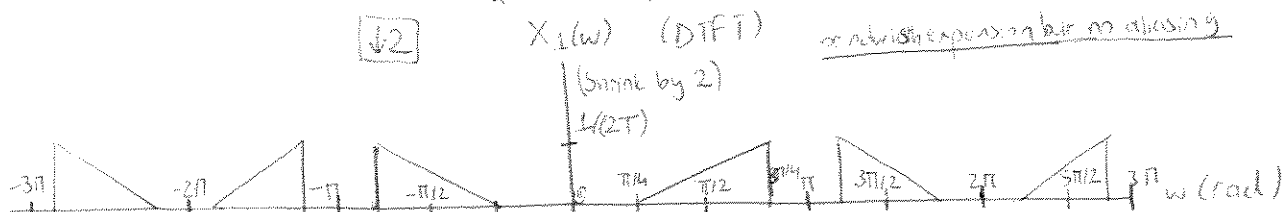


H(ω)
(continuous time)

(e) (5 points.) Now assume that $x[n]$ is obtained from the original signal $x(t)$ by sampling it with a sampling period of $T = \frac{\pi}{4}$. Next, $x[n]$ is downsampled by a factor of 2 as shown above. Call the resulting signal $x_1[n]$. Plot the Fourier transform (DTFT) of $x_1[n]$ in the frequency range $\omega \in [-3\pi, 3\pi]$.

$x[n] = x(nT)$ where $T = \frac{\pi}{4}$

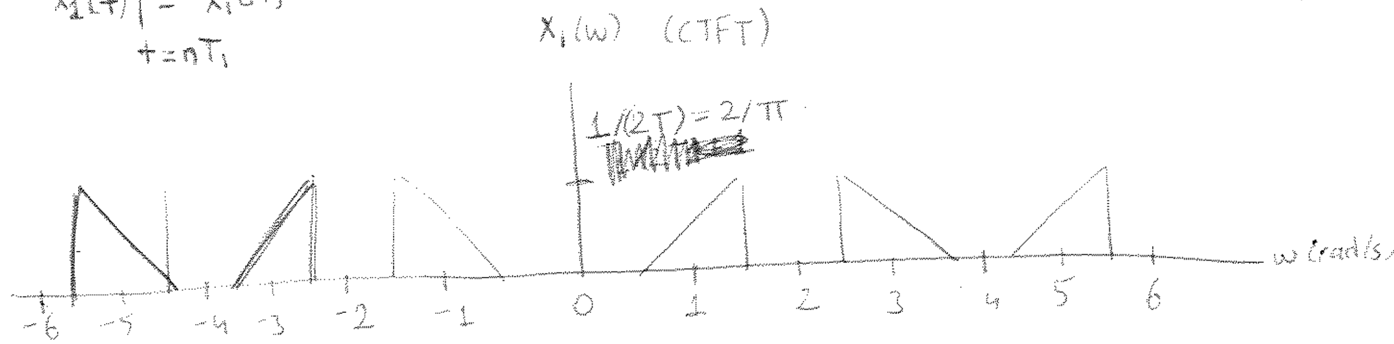




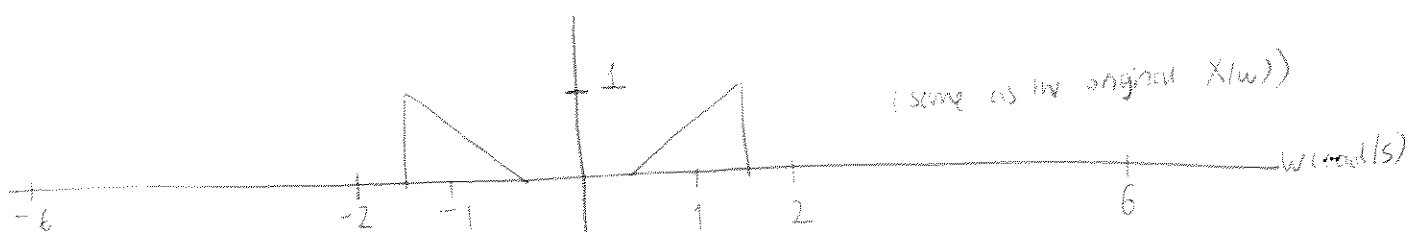
(f) (5 points.) The downsampled signal $x_1[n]$ is then converted from discrete sequence to continuous with $T_1 = \frac{\pi}{2}$, and a new signal is reconstructed with an ideal low-pass filter with cutoff frequency 2 and gain $\frac{\pi}{2}$, as illustrated above. What is the reconstructed signal $y_1(t)$? Plot the continuous time frequency responses of $X_1(\omega)$ and $Y_1(\omega)$.

$x_1(t)\big|_{t=nT_1} = x_1[n]$

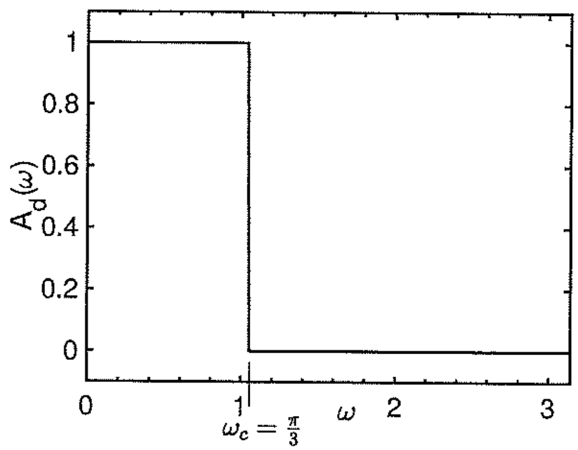
ω (rad) is mapped to $\frac{\omega}{T_1}$



$Y_1(\omega) = X_1(\omega)H(\omega)$ (CTFT)



4. (25 points.) Linear-phase FIR filter design. We want to design a Type-I, length $N = 15$ FIR digital filter with linear phase that approximates the ideal amplitude response as illustrated below:



$$A_d(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \frac{\pi}{3} \\ 0, & \text{for } \frac{\pi}{3} < |\omega| \le \pi. \end{cases}$$

$\frac{\pi}{3} = \frac{5\pi}{15}$

(a) (5 points.) Design the filter using the frequency-sampling method by taking $N = 15$ samples equally-spaced by $\frac{2\pi}{15}$ starting at $\omega = 0$. What is the vector A_d of desired amplitude response samples?

$A_d(\omega)\Big|_{w \le \frac{5\pi}{15}} = 1$, $A_d(\omega)\Big|_{\frac{5\pi}{15} < w \le \frac{15\pi}{15}} = 0$

$A_d = [\,1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\,]$

$w = 0$, $\frac{2\pi}{15}$, $\frac{4\pi}{15}$, $\frac{14\pi}{15}$, $\frac{16\pi}{15}$

1×15 vector

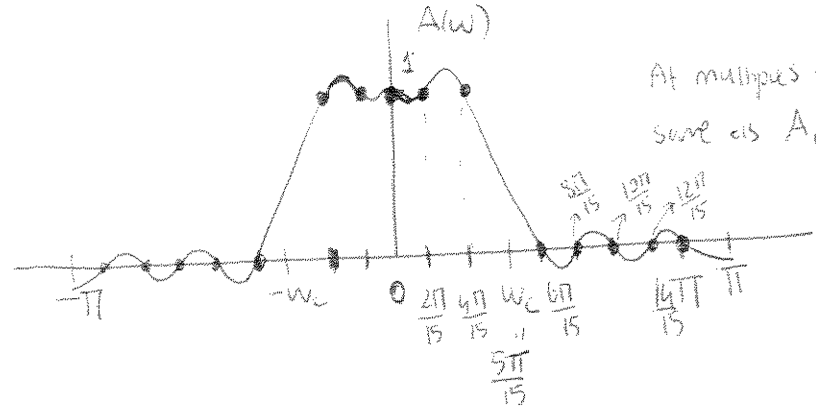
(b) (5 points.) What will the value of the frequency response of the designed filter be at $\omega = \frac{4}{5}\pi$? Determine both the magnitude and the phase response.

$w = \frac{4}{5}\pi = \frac{12}{15}\pi$ Note that $A_d(\omega) = 0$ for $\frac{\pi}{3} < w \le \pi$

$A_d(\omega)\Big|_{w = \frac{2\pi}{15} \times 6} = A[7] = 0 \Rightarrow H[7] = 0$

Linear phase $\Rightarrow e^{-j\frac{4\pi}{5} \cdot 7} \cdot A[7] = H[7]$

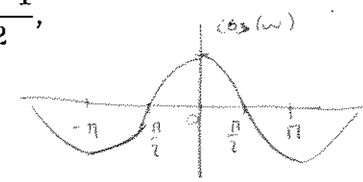
(c) (5 points.) Roughly sketch the amplitude response $A(\omega)$. You do not need to be very precise here. Is the response symmetric about $\omega = 0$? Is it symmetric about $\omega = \pi$? Explain your reasoning.



At multiples of $w = \frac{2\pi}{15}k$, $A(w)$ is the same as $A_d(w)$

Note for type-I linear phase $H(w) = e^{-jwM} A(w)$

such that $A(w) = h[M] + \underbrace{\sum_{n=0}^{M-1} 2h[n]\cos(w(M-n))}_{\text{even symmetric in } w=0 \text{ \& } \pi}$



(d) (10 points.) Recall that a Type-I linear-phase filter satisfies the equation

$$h[n] = \frac{1}{N}\left[A[0] + \sum_{k=1}^{M} 2A[k]\cos\left(\frac{2\pi(n-M)k}{N}\right)\right], \quad \text{where } M = \frac{N-1}{2},$$

where $A[k]$'s are samples of amplitude response of the filter $h[n]$.

1. (3 points.) Determine the filter coefficients $h[5]$, $h[7]$, and $h[9]$.

$M = \frac{N-1}{2} = 7$

$h[5] = h[9] = \frac{1}{15}\left[1 + \sum_{k=1}^{2} 2\cos\left(\frac{2\pi(5-7)k}{15}\right)\right]$ because cosine is even and $A[3], \dots, A[7] = 0$

$= \frac{1}{15}\left[1 + 2\cos\left(\frac{4}{15}\pi\right) + 2\cos\left(\frac{8}{15}\pi\right)\right]$

$h[7] = \frac{1}{15}\left[1 + 2\cos(0) + 2\cos(0)\right] = \frac{1}{3}$

2. (4 points.) How many equations you need to simultaneously solve in order to determine $h[n]$? Why?

Since it is Type I, length $N=15$, it is symmetric about $h[7]$. If we solve for $\{h[0], h[1], \dots, h[7]\}$, we can fully characterize $h[n]$.

$\Rightarrow M + 1 = 8$ equations

3. (3 points.) Is $h[n]$ real or complex valued? Does it have any symmetry properties? Explain.

You can see from above definition, since $A[k]$'s and cosines are real, $h[n]$ is real. We already ~~stated~~ explained in the previous parts that $h[n]$ has even symmetry.

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