9/28/20

Today's lecture

- Linear phase filters
- z-transform
  - relationship between DTFT and z-transform
  - region of convergence (ROC)
  - poles-zeros
  - properties of ROC
- properties of z-transform

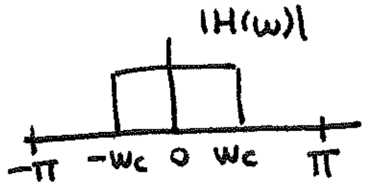
Announcements

- Homework 2 due 10/1 Thursday
- Hybrid class schedule w/ 4 teams

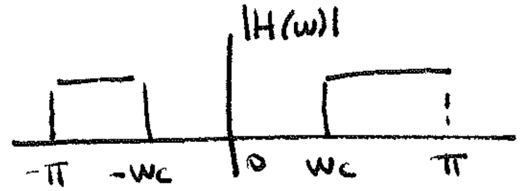
Team 1	Last Name	"A..." — "Da..."
Team 2	"	"Dav..." — "K..."
Team 3	"	"L..." — "N..."
Team 4	"	"O..." — "Z..."

- MT 1 10/8 Thursday

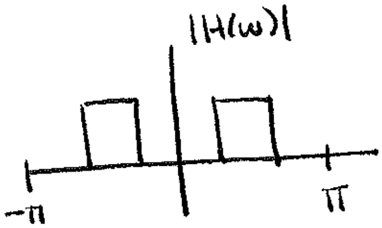
Ideal Filters



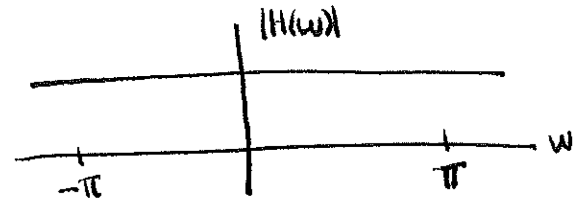
Low pass, with cutoff w_c



High pass



Bandpass



All pass

Phase response $\qquad H(w) = |H(w)|\, e^{j\angle H(w)}$

Question: Why is $\angle H(w) = -\underset{\substack{\downarrow \\ \text{constant}}}{c}w$ desirable?

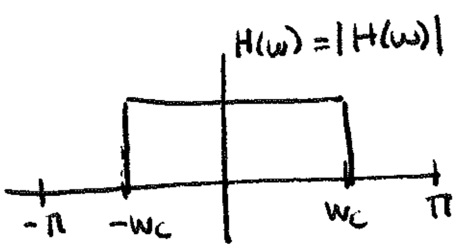
$$Y(w) = X(w)H(w) = X(w)\,|H(w)|\underbrace{e^{-jcw}}_{\text{phase shift in frequency}}$$

$$x[n] \overset{DTFT}{\longleftrightarrow} X(w)$$

$$x[n-c] \overset{DTFT}{\longleftrightarrow} X(w)e^{-jcw}$$

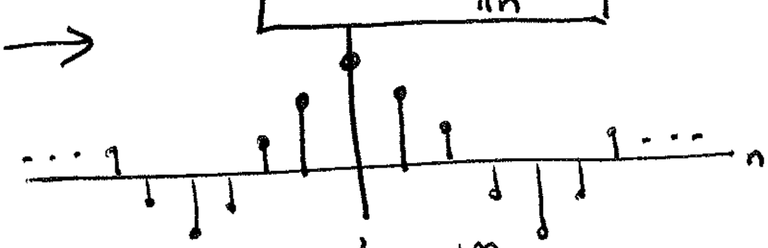
Question: Why cannot we get $c=0$? (0 delay)

$$Y(w) = X(w)\,|H(w)|$$



Ideal LPF $\rightarrow$

$$\boxed{h[n] = \frac{\sin w_c \pi n}{\pi n}}$$



noncausal

$\sum_{n=-\infty}^{+\infty} |h[n]| = \infty$ not stable

z - transform

Continuous	Discrete
$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$	$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$
FT (CTFT)	DTFT
$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$	$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$
Laplace Transform	z-transform

Let $x[n] = z^n$ where z is a complex number

$$y[n] = x[n] * h[n] \qquad x[n] \longrightarrow \boxed{\begin{matrix} h \\ \text{LTI} \end{matrix}} \longrightarrow y[n]$$

$$= \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k] z^{n-k} = z^n \underbrace{\sum_{k=-\infty}^{+\infty} h[k] z^{-k}}_{\text{z - transform}}$$

$$h[n] \xleftrightarrow{\text{z-transform}} H(z) = \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

Impulse function 'Transfer function'

$$z^n \longrightarrow \underset{\text{LTI}}{\boxed{h[n]}} \longrightarrow z^n H(z)$$

(Think about $z = e^{j\omega_0}$)

Recall

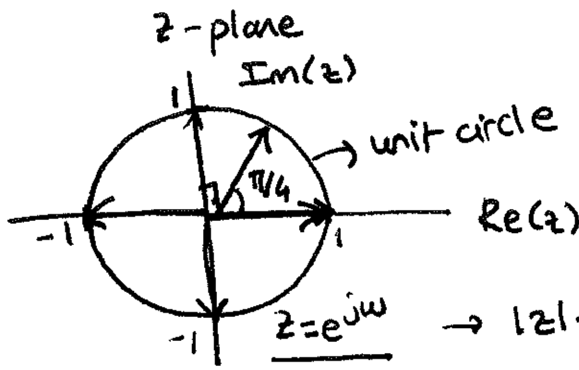
$$e^{j\omega_0 n} \longrightarrow \underset{\text{LTI}}{\boxed{h[n]}} \longrightarrow e^{j\omega_0 n} H(\omega_0)$$

Relationship between DTFT and z-transform

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \qquad \text{(DTFT)}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \Big|_{z=e^{j\omega}} = X(z)\Big|_{z=e^{j\omega}}$$

'DTFT is 2π periodic'



$\underline{z=e^{j\omega}} \rightarrow |z|=1 \qquad z = |z| . e^{j\angle z}$

Why do we need z transform?

- The DTFT does not always converge or exist.
  Not all signals satisfy the condition '$\sum_n |x[n]| < \infty$'

- The z-transform may converge in places where DTFT does not exist.

- The z-transform helps design filters. (pole-zero diagram)

The Region of Convergence (ROC)

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$ where z is a complex number.

$$z = r.e^{jw}$$

$$X(re^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] \, (re^{jw})^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \left(x[n]\, r^{-n}\right) e^{-jwn}$$

DTFT of $y[n]$

$$\rightarrow Y(\omega) = \sum_{n=-\infty}^{+\infty} y[n] e^{-jwn}$$

$$= \text{DTFT}\left(x[n] r^{-n}\right)$$

When does this converge?

As long as $\sum_{n=-\infty}^{+\infty} |x[n] r^{-n}| < \infty$, z-transform converges.

Special cases:

$r = 1$ DTFT exists

$r < 1$ DTFT may converge.

Example

$x[n] = u[n]$. Does the DTFT converge?

$$\sum_{n=-\infty}^{+\infty} |x[n]| = \sum_{n=0}^{\infty} 1 = \infty \Rightarrow \text{It does not converge.}$$

$x[n] = u[n] r^{-n}$ where r is some real constant.

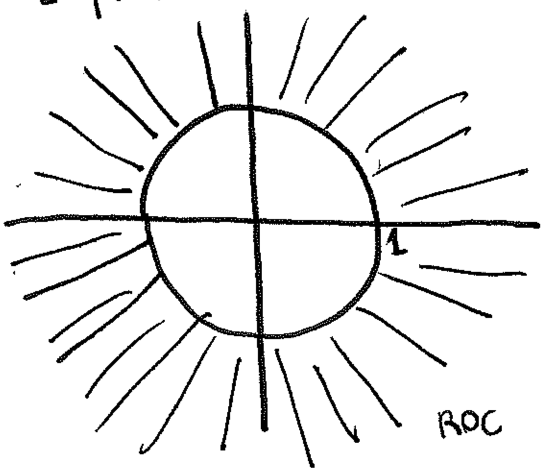
Determine the range of r for which the DTFT converges.

$$\sum_{n=-\infty}^{+\infty} |x[n]| = \sum_{n=0}^{\infty} |r^{-n}| = \frac{1}{1-|r^{-1}|}, \quad |r| > 1$$

$$\left(\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, |a| < 1 \right)$$

The ROC of the z-transform is '$|r|>1$',

z plane



z-plane

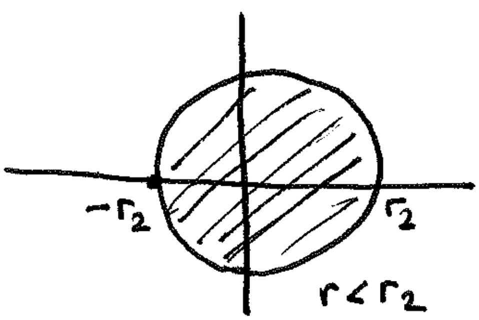
$$x[n] = u[n]\, r^{-n}$$

$$z = r e^{j\omega} \qquad z = \underbrace{|z|}_{r} \cdot e^{j\overbrace{\angle z}^{}_{\omega}}$$

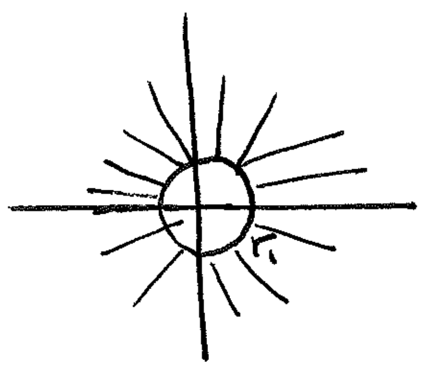
$$|z| > 1$$

<u>Note</u>: the convergece of the z-transform depends only on $|z| = |r|$

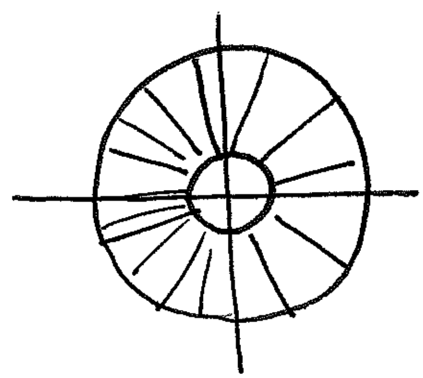
How does the ROC look like?



$|z| < r_2$

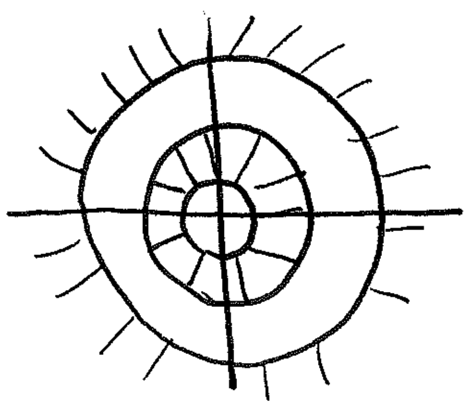


$|z| > r_1$



$r_1 < |z| < r_2$

Question: Can ROC look like this?



numerator

$$X(z) = \frac{N(z)}{D(z)} \rightarrow \text{polynomials in } z$$

denominator

			z Plane
$N(z) = 0 \Rightarrow$	$X(z) = 0$	"zeros"	$\circ$
$D(z) = 0 \Rightarrow$	$X(z) = \infty$	"poles"	$\times$

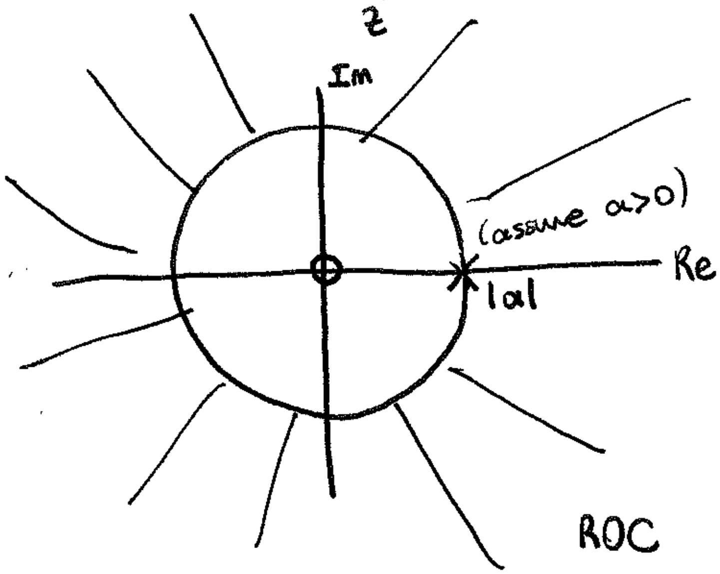
Example $x[n] = a^n u[n]$ where a is a real number.

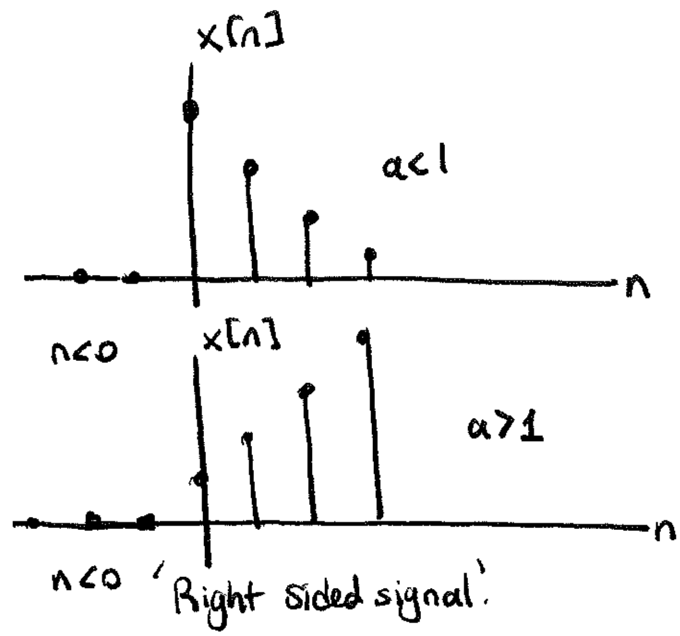
Compute the z-transform of $x[n]$.

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \Rightarrow \text{converges if } \left|\frac{a}{z}\right| < 1 \quad (\text{or } |z| > |a|)$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$



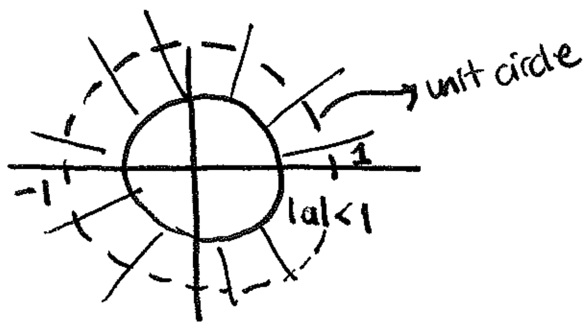


Q. When does the DTFT exist?

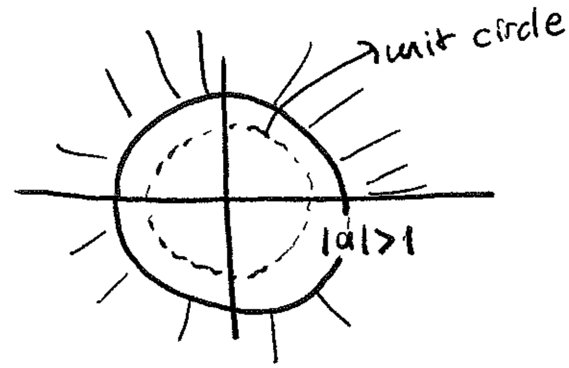
Recall that $z = re^{jw}$, $X(z) = \text{DTFT}(x[n]r^{-n})$

If $r=1$ DTFT exists. What does this mean?

$\Rightarrow$ ROC contains the "unit circle".



ROC contains the unit circle $\Rightarrow$ DTFT exists.



ROC does not contain the unit circle $\Rightarrow$ DTFT does NOT exist.

<u>Example</u>

$x[n] = -a^n u[-n-1]$ (a is real)

Compute the z-transform of $x[n]$.

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{n=-\infty}^{-1} -\left(\frac{a}{z}\right)^n$$

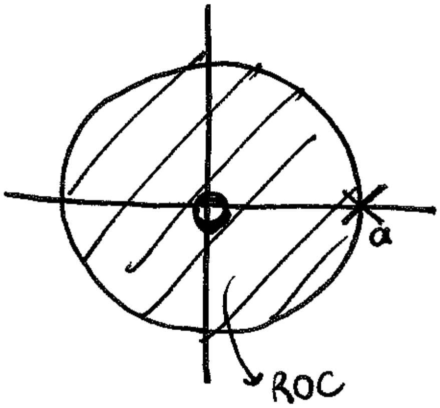
Change of variables $m=-n$

$$X(z) = \sum_{m=1}^{\infty} -\left(\frac{z}{a}\right)^m \quad \text{converges if } \left|\frac{z}{a}\right| < 1. \qquad \left(\begin{array}{c}\text{RECALL} \\ \displaystyle\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}, \ |r|<1\end{array}\right)$$

$$= \underbrace{-\frac{1}{1-\frac{z}{a}} + 1}_{\frac{-a}{a-z} + \frac{a-z}{a-z}} = \frac{z}{z-a}$$

Exactly the same as the right sided signal's z-transform

but $|z| < |a|$



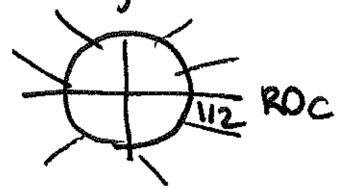
z-transform consists of both 'X(z)' and the 'ROC' that says where it is valid.

zeros : 0

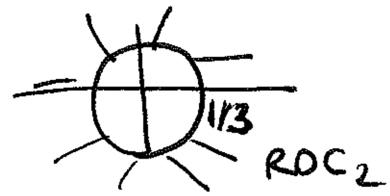
poles : a

Example $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$ 2 right-sided signals

$\left(\frac{1}{2}\right)^n u[n] \xrightarrow{\text{z-trans.}} \frac{z}{z - 1/2}$ ROC_1 $|z| > 1/2$



$\left(-\frac{1}{3}\right)^n u[n] \xrightarrow{\text{z-trans.}} \frac{z}{z + 1/3}$ ROC_2 $|z| > 1/3$

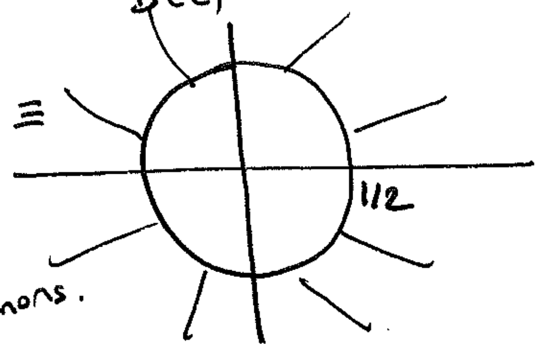


$$X(z) = \frac{z}{z - 1/2} + \frac{z}{z + 1/3} = \frac{A(z)}{B(z)}$$

$(z + 1/3)$ $(z - 1/2)$

determine poles & zeros using partial fractions.

ROC : $ROC_1 \cap ROC_2 \equiv$

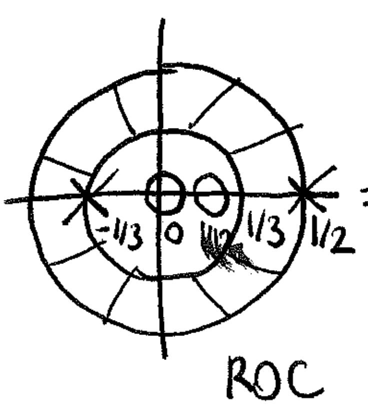


Example $x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$

$$X(z) = \frac{z}{z + 1/3} + \frac{z}{z - 1/2}$$

ROC : $|z| > 1/3 \cap |z| < 1/2$

$= 1/3 < |z| < 1/2$



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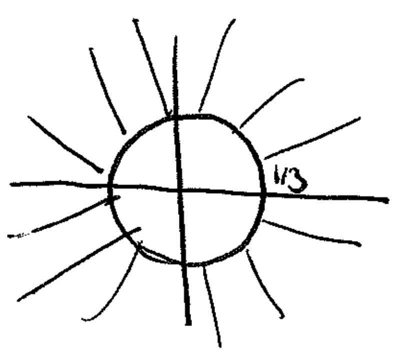

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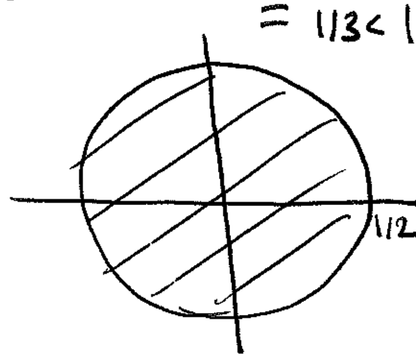


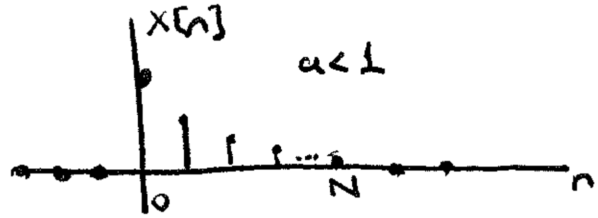
$$X(z) = \frac{z}{z+1/3} + \frac{z}{z-1/2} = \frac{z(z-1/2)+z(z+1/3)}{(z+1/3)(z-1/2)} \overset{\text{show}}{=} \frac{2z(z-1/12)}{(z+1/3)(z-1/2)}$$

$(z-1/2)$ $(z+1/3)$

zeros $0, \ 1/12$
poles $-1/3, \ 1/2$

<u>Example</u> Finite length exponential

$$x[n] = \begin{cases} a^n, & n \in [0, N-1] \\ 0, & \text{otherwise} \end{cases}$$



$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(\frac{a}{z}\right)^n = \frac{1-\left(\frac{a}{z}\right)^N}{1-\frac{a}{z}}$$

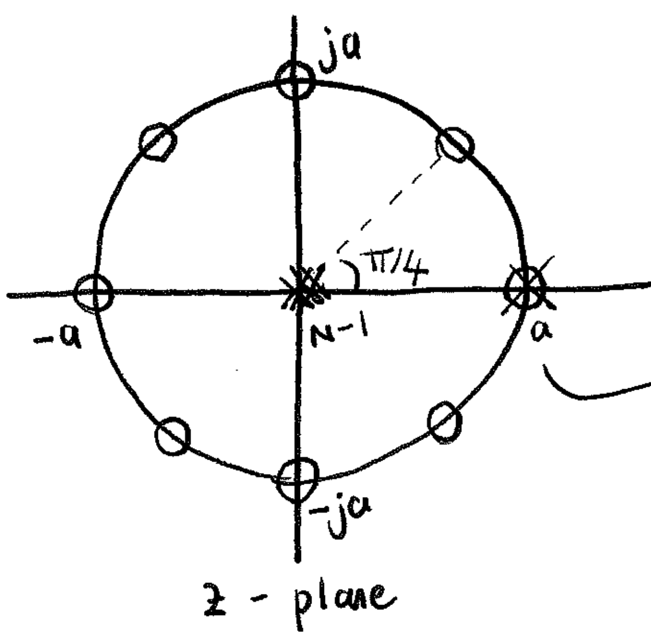
finite sum formula (valid for any a)

$$= \frac{z^N - a^N}{z^N\left(1-\frac{a}{z}\right)} = \frac{z^N - a^N}{z^{N-1}(z-a)}$$

$z^N - a^N$ ← numerator polynomial
$z^{N-1}(z-a)$ → denom. "

poles : 1 pole at $z=a$, $N-1$ poles at $z=0$

zeros : $z^N - a^N = 0 \Rightarrow z^N = a^N \Rightarrow |z| = |a|$ (N complex valued roots)



ROC ?

'N is some integer number'

these cancel each other

N-1 poles at $z=0$

N-1 complex zeros scattered around the complex plane. (evenly distributed) on circle of radius a