11/13/20

Today's lecture

FIR filter design techniques

IIR filter design techniques

Last time

1. Frequency Sampling Design.

2. Least Square approximation

$$A(\omega) = \sum_{n=0}^{M-1} 2\, h[n] \cos(\omega (M-n)) + h[M] \qquad (\text{Type I}, N \text{ ODD})$$

a) $L \geq N$ discrete frequency samples where L is the length of the filter

$$E = \sum_{n=-\frac{(L-1)}{2}}^{\frac{L-1}{2}} |\underbrace{h[n]}_{\text{length } N} - \underbrace{h_d[n]}_{\text{length } L}|^2$$

$$\underset{L\times 1}{a} = \underset{L\times(M+1)}{F}\ \underset{(M+1)\times 1}{h} \qquad \Rightarrow \qquad \hat{h} = \overbrace{(F^T F)^{-1} F^T}^{\text{pseudo inverse}} a \quad \text{minimizes}$$

$$(F\hat{h} - a_d)^T (F\hat{h} - a_d) = E_{min}$$

b) Integral squared error approximation

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |A(\omega) - A_d(\omega)|^2 \, d\omega \qquad \text{as} \quad L \to \infty$$

$$A(\omega) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n} \qquad (\text{DTFT})$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega) e^{j\omega n} \, d\omega$$

Using Parseval's theorem

$$E = \sum_{n=-\infty}^{+\infty} |h[n] - h_d[n]|^2$$

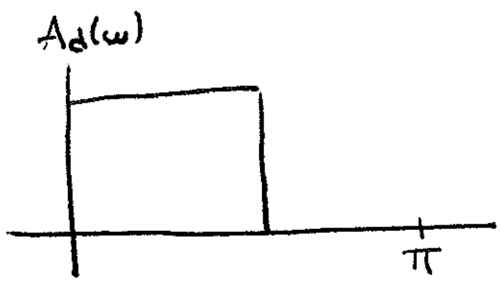
$$= \sum_{n=-M}^{M} |h[n] - h_d[n]|^2 + \sum_{n=M+1}^{\infty} 2\, h_d[n]^2 \qquad \frac{N-1}{2} = M$$

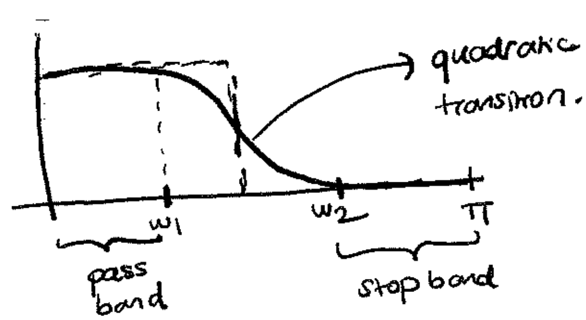
Using an approach similar to LS, assume that $h[n]$ is a truncated version of $h_d[n]$.

<u>Example</u> Consider Type I linear phase filters.

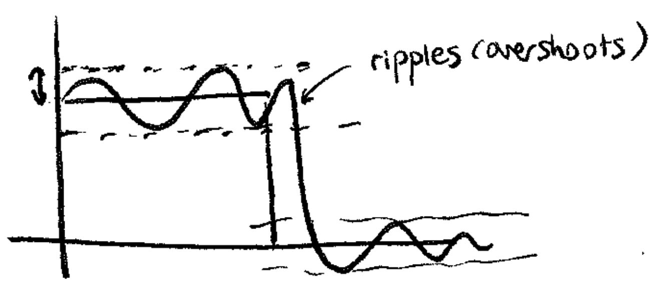
$$A(\omega) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n} = h[0] + \sum_{n=1}^{\infty} 2\, h[n] \cos(\omega n) \qquad \text{(DTFT)}$$

$$h[n] = \frac{1}{\pi} \int_0^{\pi} A(\omega) \cos(\omega n)\, d\omega \qquad \text{(inverse DTFT)}$$





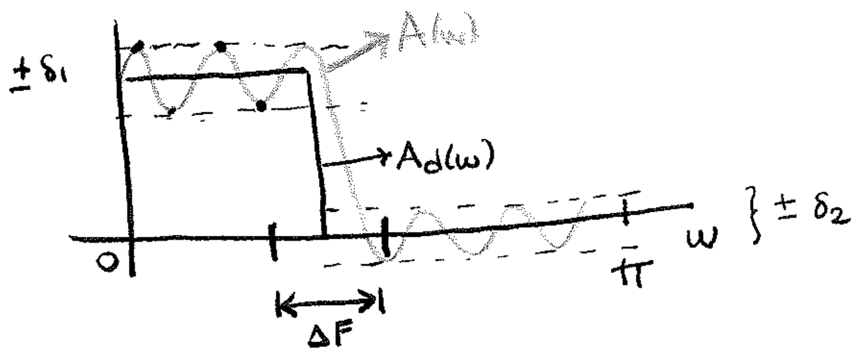
Note: LS gives optimal mean-square error. However, it does not keep the maximum error low.



3. Chebyshev Approximation

Goal: to reduce the maximum error. We want to minimize

$$E = \max_{\omega \in [0,\pi]} |A(\omega) - A_d(\omega)|$$



The optimal filters are equiripple.

An acceptable frequency response will have

- Linear phase
- ΔF transition band
- A deviation of $\pm\delta_1$ in the passband
- A deviation of $\pm\delta_2$ in the stopband

The Approximation Problem

* Given a desired, real-valued $A(\omega)$ defined and continuous on $[0,\pi]$.

$$A(\omega) = Q(\omega) \sum_{k=0}^{r-1} c_k \cos(\omega k) \qquad (\text{for all 4 Types})$$

$$Q(\omega) = \begin{cases} 1 & \text{I} \\ \cos(\omega/2) & \text{II} \\ \sin(\omega) & \text{III} \\ \sin(\omega/2) & \text{IV} \end{cases}$$

* A positive weight function $W(\omega)$, defined and continuous on $[0,\pi]$.

* If we can find c_k, then we can recover $h[n]$ from $A(\omega)$.

* Alternation Theorem

The polynomial of degree L that minimizes the maximum error will have <u>at least</u> $L+2$ extrema.

The error function for the best weighted Chebyshev approximation to a given continuous function $A_d(\omega)$ on $[0, \pi]$ is

$$E(\omega) = W(\omega)\,|A(\omega) - A_d(\omega)|$$

Recall that $A(\omega) = Q(\omega) \sum_{k=0}^{r-1} c_k \cos(\omega k) \longrightarrow$ degree $r-1$

$\Rightarrow$ This has at least $r+1$ extremal frequencies on $[0, \pi] : \{w_1, \ldots, w_{r+1}\}$

Extremal frequencies:

$$E(w_m) = -E(w_{m+1}) = \delta, \quad m = 1, \ldots, r$$

$$|E(w_m)| = \max_{w \in [0,\pi]} E(\omega)$$

<u>Chebyshev Polynomials</u> : 2 sets of polynomials:

$$\cos(n\theta) = T_n(\cos\theta)$$

$$\sin(n\theta) = U_{n-1}(\cos\theta)\sin\theta$$

Change of variables $x = \cos w$

$$A(\omega) = A(\cos^{-1}(x)) = \sum_{k=0}^{r-1} c_k \underbrace{\cos(k\cos^{-1}x)}_{\text{Chebyshev polynomials}}$$

Example We want to approximate $A_d(\omega) = \omega^2$ by $A(\omega) = d_0 + d_1\omega$ over $\omega \in [0,1]$ using Chebyshev approximation.

$A(\omega)$ has degree 1 $\longrightarrow$ 3 extremal points

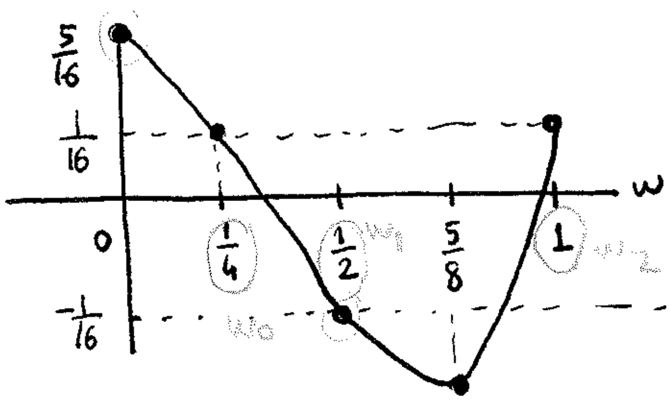
$$\min_{d_0, d_1} \max_{W \in [0,1]} |\omega^2 - (d_0 + d_1 \omega)|$$

Guess : $T_0 = \{ \underbrace{\tfrac{1}{4}}_{w_0}, \underbrace{\tfrac{1}{2}}_{w_1}, \underbrace{1}_{w_2} \}$ 0 is the iteration index.

$$w_i^2 = d_0 + d_1 w_i + (-1)^{i-1}\delta_0, \quad i = 1,2,3$$

$$\begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{1}{4} & 1 \\ 1 & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{\text{invertible } 3\times3 \text{ matrix}} \begin{bmatrix} d_0 \\ d_1 \\ \delta_0 \end{bmatrix} \Rightarrow \begin{bmatrix} d_0 \\ d_1 \\ \delta_0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{16} \\ \frac{5}{4} \\ \frac{1}{16} \end{bmatrix}$$

$$E_0(\omega) = A_d(\omega) - (d_0 + d_1\omega) = \omega^2 - \left(-\tfrac{5}{16} + \tfrac{5}{4}W\right)$$

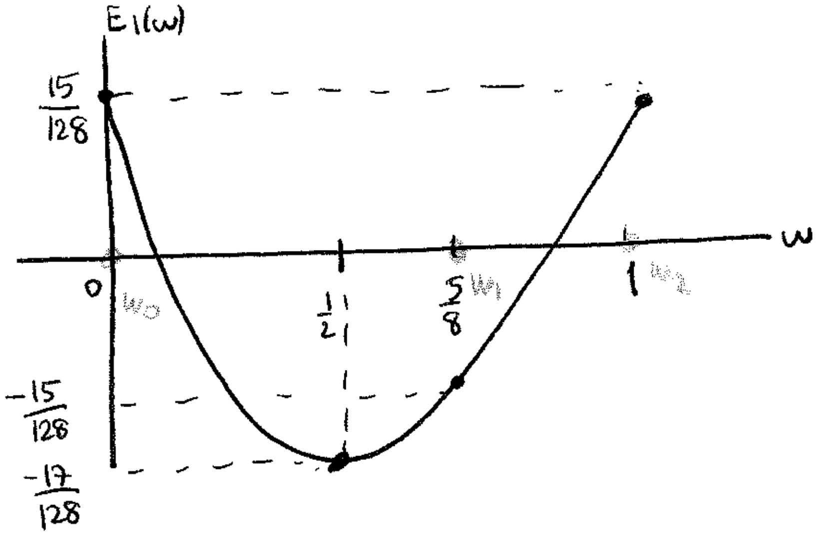


$$\max E_0(\omega) = \frac{5}{16} \neq \frac{1}{16} = \delta_0$$

$$T_1 = \{0, \tfrac{5}{8}, 1\}$$

$$\begin{bmatrix} 0 \\ \frac{25}{64} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & \frac{5}{8} & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \delta_1 \end{bmatrix} \Rightarrow \begin{bmatrix} d_0 \\ d_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} -\frac{15}{128} \\ 1 \\ \frac{15}{128} \end{bmatrix}$$

$$E_1(\omega) = \omega^2 - (d_0 + d_1\omega) = \omega^2 - \left(-\frac{15}{128} + \omega\right)$$

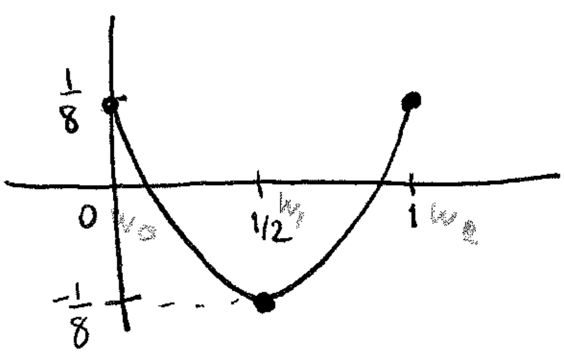


$$\max E_1(\omega) = \frac{17}{128} > \frac{15}{128} = \delta_1$$

$$T_2 = \{0, \tfrac{1}{2}, 1\}$$

$$\begin{bmatrix} 0 \\ \frac{1}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \delta_2 \end{bmatrix} \Rightarrow \begin{bmatrix} d_0 \\ d_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ 1 \\ \frac{1}{8} \end{bmatrix}$$

$$E_2(\omega) = \omega^2 - \left(-\frac{1}{8} + \omega\right)$$



$$\max E_2(\omega) = \frac{1}{8} = \delta_2$$

done!

Extremal frequencies are T_2.

Remez Exchange Algorithm

Lemma: $E(\omega) = A_d(\omega) - \underbrace{\sum_{k=0}^{r-1} c_k \cos \omega k}_{r \text{ cosines}}$ can be made to take

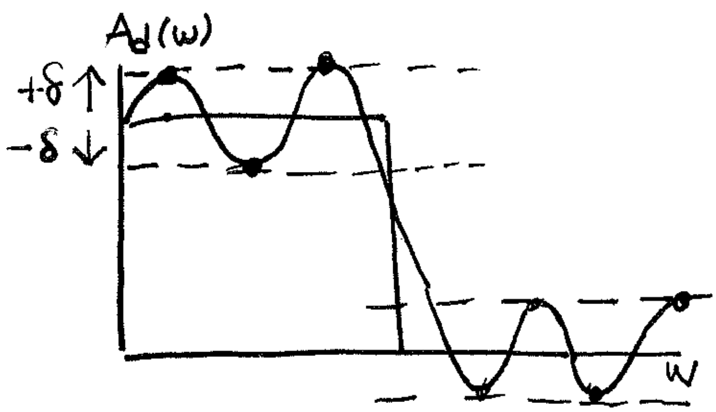
on values $\pm\delta$ for any given set $\{w_1, \dots, w_{r+1}\}$

Hence,

$$A_d(w_i) = \sum_{k=0}^{r-1} c_k \cos w_i k + (-1)^i \delta \quad , \quad i = 0, \dots, r+1$$

This has a unique solution for $c_k, k=0,\dots,r-1$ and δ.

These $r+1$ frequencies are the extremal ones.



• Extremal frequencies

Example:
$N=15$ taps filter, linear phase
$r=8$ cosines
$\Rightarrow r+1 = 9$ extremal frequencies

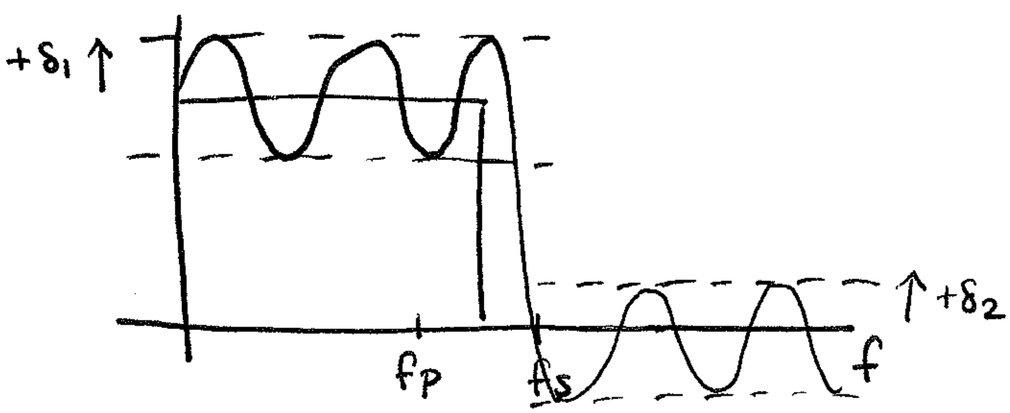
The Algorithm

Given $T_0 = \{W_1, W_2, \dots, W_{r+1}\}$ initial guesses for extremal frequencies (at iteration $k=0$)

The Algorithm

1. Solve the linear equations.
2. Interpolate to find frequency response on all of $[0,\pi]$
3. Search $[0,\pi]$ to see if/where the magnitude of error $> \delta_k$.
4. If max error $= \delta_k$, done. Otherwise take $r+1$ maximal error points as T_{k+1}.
5. Go back to step 1.

Question: How do we force the equiripple filter to satisfy the constraints in the passband and stopband given the number of filters taps N.



Parameters: N: # of taps
fp : passband edge
fs : stopband edge
δ_1 : deviation in passband
δ_2 : deviation in stopband

$$N \approx \frac{-20\log_{10}\sqrt{\delta_1\delta_2}-13}{14.6(f_s-f_p)}+1$$

Note: If the transition is steep (i.e., f_s-f_p small) then N is large
If the deviation is small then N should be large.

FIR filter design

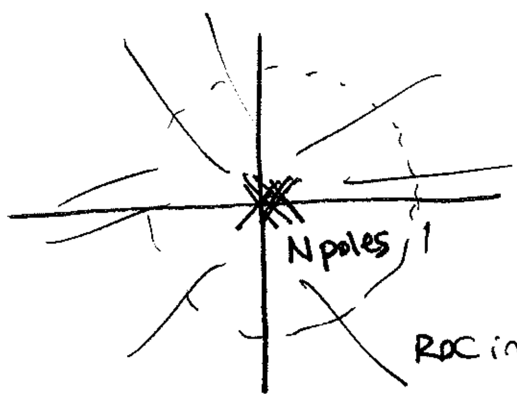
1. Can achieve linear phase (symmetric response) which is not possible in IIR.

2. Easy to implement

3. Remez exchange algorithm to design linear phase FIR filters

4. Always stable.

Why?



$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

$$= \frac{z^N a_0 + z^{N-1} a_1 + \cdots + a_N}{z^N}$$

ROC includes the unit circle.

FIR disadvantages

1. May need to have very large number of taps (N) to achieve good approximations to desired frequency response.

2. Delay can be very large.

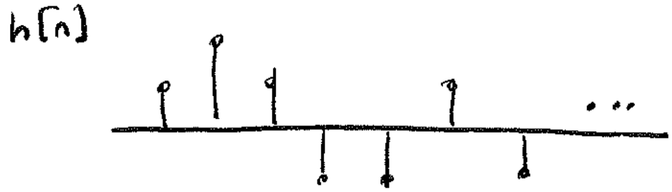
<u>Infinite Impulse Response (IIR) Filter Design</u>

$$y[n] = \sum_{m=0}^{M} b[m]\,x[n-m] - \sum_{k=1}^{N} a[k]\,y[n-k]$$

$$Y(z) = b[0]X(z) + b[1]X(z)z^{-1} + \cdots + b[M]X(z)z^{-M}$$
$$- a[1]Y(z)z^{-1} - a[2]Y(z)z^{-2} - \cdots - a[N]Y(z)z^{-N}$$

Hence,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b[0] + b[1]z^{-1} + \dots + b[M]z^{-M}}{1 + a[1]z^{-1} + \dots + a[N]z^{-N}} = \frac{B(z)}{A(z)}$$



Differences from FIR

1. We cannot do linear phase because it corresponds to symmetry around some point.

2. Low orders of IIR filters are sufficient to implement designs with tight specifications.

Design Process of $H(\omega)$

1. Start with a desired response $H_d(\omega)$
2. Choose class of filters (IIR, orders N, M)
3. Choose a distance measure between $H_d(\omega)$ and $H(\omega)$
4. Find the optimal filter that minimizes the error.

* For IIR filters it might be preferred to start with an analog filter and convert it to digital.

Digital IIR Filter Design

Prony's Method (1790s)

Given a desired IIR $h_d[n]$, $0 \le n < \infty$

$$y[n] = -\sum_{k=1}^{N} a[k]\,y[n-k] + \sum_{m=0}^{M} b[m]\,x[n-m]$$

$$h_d[n] \longrightarrow H_d(z) = \sum_{n=0}^{\infty} h_d[n]\,z^{-n}$$

$$= h_d[0] + h_d[1]z^{-1} + h_d[2]z^{-2} + \cdots$$

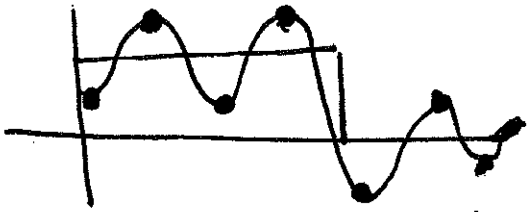
$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{B(z)}{A(z)} = H(z)$$

We want to achieve $H_d(z)\,A(z) = B(z)$

$$(1 + a_1 z^{-1} + \cdots + a_N z^{-N})(h_d[0] + h_d[1]z^{-1} + \cdots\cdots)$$

$$= b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_M \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} h_d[0] & 0 & - & - & - & \cdots & 0 \\ h_d[1] & h_d[0] & 0 & - & \cdots & - & 0 \\ h_d[2] & h_d[1] & h_d[0] & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ h_d[M] & h_d[M-1] & h_d[M-2] & \cdots & & \cdots & h_d[M-N] \\ \vdots & & & & & & \\ h_d[K] & h_d[K-1] & \cdots & & & & h_d[K-N] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$



'7' extrema
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L+2

degree L polynomial