10/1/2020

HW #2.) Aiden Chen ①

$\sqrt{5}$
? 6c
$\sqrt{7}$
$\sqrt{}$ 8

5). Fourier Series and Fourier Transform.

a.) Consider $x[n] = \{-1, 0, \underset{\uparrow}{1}, 2, 4\}$ with Fourier Transform $X(W) = X_R(\omega) + j X_I(W)$
Determine $Y(W) = X_I(W) + X_R(W)e^{j2w}$.

Find $X(W)$.

$$X(W) = \sum_{K=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{K=-2}^{2} x[n] e^{-j\omega n}$$

$$= x[-2] e^{j\omega 2} + 0 + x[0] e^{j\omega(0)} + x[1] e^{-j\omega 1} + x[2] e^{-j2w}$$

$$= -1 e^{j\omega 2} + e^{j0} + 2e^{-j\omega} + 4e^{-2j\omega}.$$

$$= 1 + -1e^{j2\omega} + 4e^{j(-2)\omega} + 2e^{-j\omega}.$$

$$= 1 + -1e^{j2\omega} + 2e^{-j\omega} + 4e^{j(-2)\omega}.$$

$$= 1 + -1(\cos(2\omega) + j\sin(2\omega))$$
$$+ 2(\cos(\omega) - j\sin(\omega))$$
$$+ 4(\cos(2w) - j\sin(2W))$$

$$= 1 + \cos(\omega)(2) + \cos(2\omega)(-1+4)$$
$$+ j\sin(\omega)(-2) + j\sin(2\omega)(-1+-4)$$

$$X(W) = 1 + 2\cos(\omega) + 3\cos(2\omega) + j(-2)\sin(\omega) + j(-5)\sin(2w)$$

$$X_R(W) = Re\{X(W)\} = 1 + 2\cos(\omega) + 3\cos(2\omega)$$
$$X_I(w) = Im\{X(\omega)\} = -2\sin(w) - 5\sin(2\omega)$$

Find $Y(\omega)$

$$Y(W) = X_I(W) + X_R(W)e^{j2\omega}$$

$$Y(w) = -2\sin(w) - 5\sin(2w) + (1 + 2\cos(w) + 3\cos(2w))e^{j2w}$$

$$= -2\left(\frac{1}{2j}e^{jw} - \frac{1}{2j}e^{-jw}\right) + 5\left(\frac{-1}{2j}e^{j2w} + \frac{1}{2j}e^{-j2w}\right)$$

$$+ e^{j2w} + (e^{jw} + e^{-jw})e^{j2w} + \frac{3}{2}(e^{j2w} + e^{-j2w})e^{j2w}$$

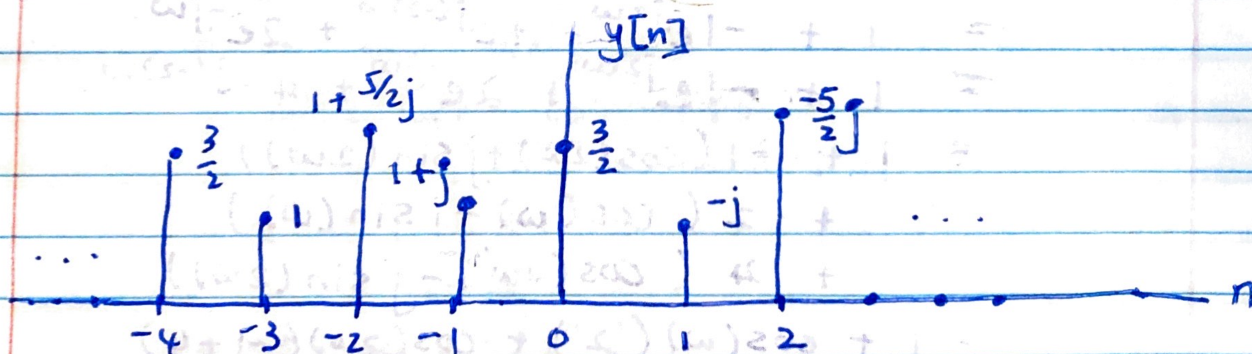
$$= je^{jw} - je^{-jw} + \frac{5}{2}je^{j2w} + -\frac{5}{2}je^{-j2w}$$

$$+1e^{jw} \quad +1e^{j2w} \quad + e^{j3w} + \frac{3}{2}e^{j4w} +$$

$$= e^{-j2w}\left(-\frac{5}{2}j\right) + e^{-jw}(-j) + \frac{3}{2} + e^{jw}(1+j) + \left(1+\frac{5}{2}j\right)e^{j2w} + e^{j3w} + e^{j4w}\frac{3}{2}$$

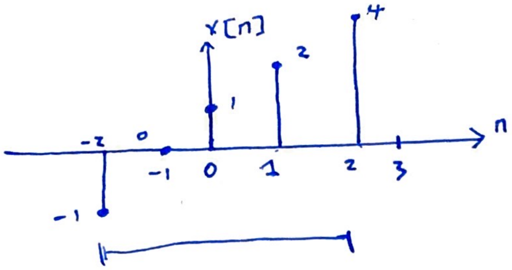
Find y[n]

$$y[n] = \frac{-5}{2}j\,\delta[n-2] + (-j)\,\delta[n-1] + (1+j)\,\delta[n+1] + \left(1+\frac{5}{2}j\right)\delta[n+2] + \delta[n+3] + \frac{3}{2}\delta[n+4] + \frac{3}{2}\delta[n]$$



5a.)

$x[n] = \{-1, 0, \underline{1}, 2, 4\}$



$T = 5$

$N = 5$

$y[n] = \sum_{k=-\infty}^{\infty} x[n-k5]$

i.) What is the period?

$N = 5.$

ii.) Find Fourier Coefficient of $y[n]$ using $X(\omega)$.

$a_K = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}dt \Rightarrow Ta_K = \int_T x(t)e^{-\frac{2\pi}{T}kt}dt$

Since y now is continuous, we can extend our knowledge from C.T → D.T. periodic.

$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_T x(t)e^{-j\omega t}dt$

$\therefore\ a_K = \frac{1}{N}Y(k\frac{2\pi}{N}) = \frac{1}{N}\left[X_I(k\frac{2\pi}{N}) + X_R(k\frac{2\pi}{N})e^{j2(k\frac{2\pi}{N})}\right]$

$= \frac{1}{5}\left[X_I(k\frac{2\pi}{5}) + X_R(k\frac{2\pi}{5})e^{j2(k\frac{2\pi}{5})}\right]$

$a_K = \{1.2\angle 0,\ 0.84\angle -3.02\ \text{RAD},\ 0.737\angle -7.97E{-2}\ \text{RAD},\ 0.699\angle 3.05\ \text{RAD},\ 1.1\angle 8.6E{-2}\ \text{RAD}\}$

6 a.) Compute DTFT of $x(\omega)$.

$x[n] = \text{Sinc}[n] \cdot \text{Sinc}[n]$ Period $= 2\pi$.

$\uparrow x_1[n]$

Given

$$\frac{W}{\pi}\text{Sinc}\left(\frac{Wn}{\pi}\right) = \frac{\sin(Wn)}{\pi n} \overset{F}{\longleftrightarrow} X(\omega) = \begin{cases} 1, & 0 \le \omega \le W \\ 0 & W < |\omega| < \pi \end{cases}$$

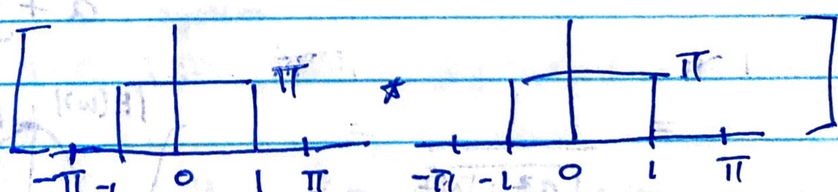
Then: $\rightarrow$ $$\frac{\sin(n)}{n} \overset{F}{\longleftrightarrow} X(\omega) = \begin{cases} \pi, & 0 \le \omega \le W \\ 0, & W \le |\omega| < \pi \end{cases}$$

$$X_1(\omega) = \begin{cases} \pi, & 0 \le \omega \le W \\ 0, & W \le |\omega| < \pi \end{cases}$$

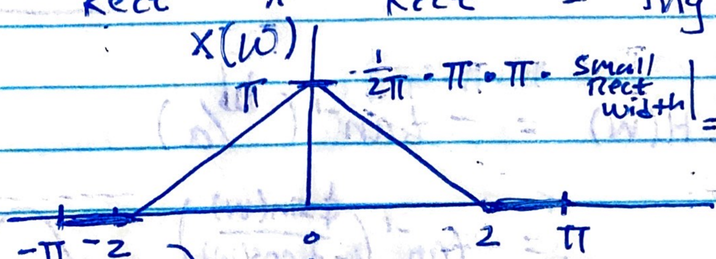
where $\boxed{W = 1}$ when matching $\frac{\sin(n)}{n} = \frac{\sin(Wn)}{n}$

$x[n] = \text{sinc}[n] \cdot \text{sinc}[n]$

$= \frac{1}{2\pi} X(\omega) * X_1(\omega)$

$= \frac{1}{2\pi}$ []

Rect $*$ Rect $=$ Trig.

$=$ 

$\frac{1}{2\pi} \cdot \pi \cdot \pi \cdot$ small rect width $\Big|_{=2} = \pi$

$$X[n] = \begin{cases} \pi + \pi/2(\omega) & -2 \le \omega \le 0 \\ \pi + \frac{\pi}{2}(-\omega) & 0 \le \omega \le 2 \\ 0 & -\pi < \omega < 2, \; 2 < \omega < \pi \end{cases}$$

diff of 2/width

sum of lower and upper limit

6b.) $\sum_{n=-\infty}^{\infty} x^2[n]$

Given $\sum_{n=-\infty}^{\infty} (x[n])^2 = \frac{1}{T}\int_T |X(\omega)|^2 \, d\omega$

$T = 2\pi$.

Then $\sum_{n=-\infty}^{\infty} (x[n])^2 = \frac{1}{2\pi}\int_{2\pi} (X(\omega))^2 \, d\omega$.

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (X(\omega))^2 d\omega$

$= \left(\frac{1}{2\pi}\int_0^{\pi} (X(\omega))^2 \, d\omega\right) 2$

$= \frac{1}{\pi}\int_0^{\pi} (X(\omega))^2 d\omega$

$= \frac{1}{\pi}\int_0^{2} (X(\omega))^2 \, d\omega$

$= \frac{1}{\pi}\int_0^{2} \left(\pi + \frac{\pi}{2}(-\omega)\right)^2 d\omega$.

$= \frac{1}{\pi}\int_0^{2} \pi^2 + \pi^2(-\omega) + \frac{\pi^2}{4}\omega^2 \, d\omega$

$= \frac{1}{\pi}\left[\pi^2\omega + \frac{-\pi^2}{2}\omega^2 + \frac{\pi^2}{4}\cdot\frac{1}{3}\omega^3\right]\Big|_0^2$

$= \frac{1}{\pi}\left[\pi^2 \cdot 2 + {}^{-\pi^2}\!/_2\, 4 + \frac{\pi^2}{12}\, 8\right.$

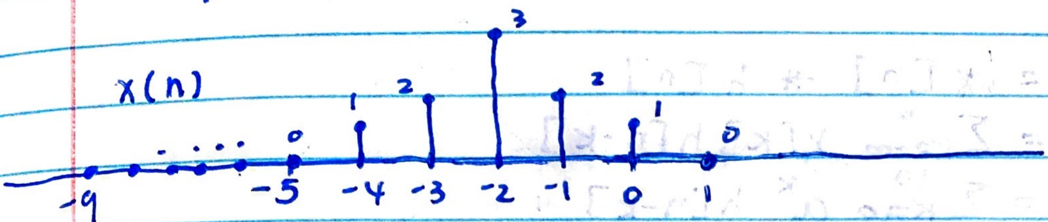
$= 2\pi + {}^{-\pi}\!/_2 \cdot 4 + {}^{\pi}\!/_{12} \cdot 8$

$= 2\pi + -2\pi + {}^{\pi}\!/_3 \cdot 2$

$= {}^{2\pi}\!/_3$

6(c) Given $x[n] = \begin{cases} 3-|n+2|, & n=-5,\dots,1 \\ 0, & n=-9,\dots,-6. \end{cases}$

compute and plot $X(w)$



$$X(w) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$

$$= \sum_{n=-4}^{0} x[n]e^{-jwn}$$

$$= 1e^{+jw4} + 2e^{+jw3} + 3e^{+jw2} + 2e^{+jw1} + 1e^{jw0}$$

$a = e^{jw2}$
$b = e^{jw1}$
$c = 1$

$$X(W) = e^{jw4} + 2e^{jw3} + 3e^{jw2} + 2e^{jw1} + 1$$

$$= \underbrace{(e^{jw2})^2}_{a} + \underbrace{2e^{jw2}\cdot 1}_{2ac} + \underbrace{(e^{jw1})^2}_{b} + \underbrace{2e^{jw1}\cdot 1}_{2bc} + \underbrace{1^2}_{c} + \underbrace{2e^{jw3}}_{2ab}$$

$$= (a+b+c)^2$$

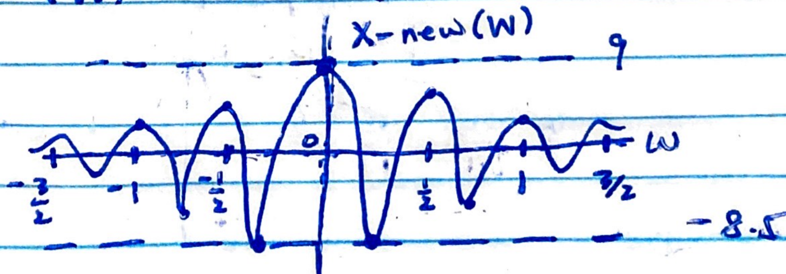
$$= (e^{jw2} + e^{jw1} + 1)^2$$

Then, $x[n+10] = X_{new}[n]$

$X_{new} = x[n+10] \rightarrow e^{-jw(-10)} X(W)$ (by time shift.)

$$= e^{jw10} X(W)$$

$$X_{new}(W) = e^{jw10}(e^{jw2} + e^{jw1} + 1)^2$$



6d.) $x[n] = a^n u[n]$, $h[n] = \beta^n (u(n) - u(n-3))$ find α, β

such that $y[n] = x[n] * h[n]$ is stable.

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[n-k]\, h[k]\right| \le \sum_{k=-\infty}^{\infty} \underbrace{|x[n-k]|}_{\le M} |h[k]|$$

check α

$$\sum_{k=-\infty}^{\infty} |x[n-k]| |h[k]| \le \sum_{k=-\infty}^{\infty} |a^{n-k} u[n-k]| |h[k]|$$

$\boxed{|a| < 1 \text{ to converge}}$

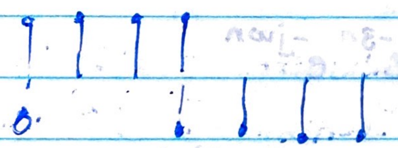
$\le M$

$$M \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |\beta^n (u[n] - u[n-3])|$$

check β

$= 1$ for $n \ge 0$ $\quad$ $= 1$ for $n \ge 3$

$$\sum_{k=0}^{2} |\beta^n| = 1 + |\beta^1| + |\beta|^2 < \infty$$



as long as $\beta < \infty$ the system is stable.

6e.) LTI sys. $h[n] = \frac{1}{2} e^{-n} u[n] + \frac{1}{2} e^{-3n} u[n]$

$x[n] = e^{-n} u[n]$

$y[n] = ?$

$x[n] \rightarrow \boxed{H(w)} \rightarrow y[n] = x[n] * h[n]$

6ei.) Find $H(w)$

$$H(w) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2} e^{-n} u[n] + \frac{1}{2} e^{-3n} u[n] \right) e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} e^{-n} e^{-jwn} + \frac{1}{2} e^{-3n} e^{-jwn}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{-n(1+jw)} + e^{-n(jw+3)} \right)$$

$$= \frac{1}{2} \sum \left(\frac{1}{e^{(1+jw)}} \right)^n + \left(\frac{1}{e^{3+jw}} \right)^n$$

$$= \frac{\frac{1}{2} \cdot 1}{1 - \frac{1}{e^{1+jw}}} + \frac{\frac{1}{2} \cdot 1}{1 - \frac{1}{e^{3+jw}}}$$

$$H(w) = \frac{1}{2} \left(\frac{1}{1 - e^{-(1+jw)}} + \frac{1}{1 - e^{-(3+jw)}} \right) \quad \forall w.$$

6eii) Find $Y(\omega)$

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-n} u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{e^{1+j\omega}}\right)^n$$

$$X(\omega) = \frac{1}{1-e^{-(1+j\omega)}}$$

$$H(\omega) = \frac{\frac{1}{2}}{1-e^{-(1+j\omega)}} + \frac{\frac{1}{2}}{1-e^{-(3+j\omega)}}$$

$$Y(\omega) = H(\omega)X(\omega) = \frac{\frac{1}{2}}{(1-e^{-(1+j\omega)})^2} + \frac{\frac{1}{2}}{(1-e^{-(3+j\omega)})(1-e^{-(1+j\omega)})}$$

$$= \frac{\frac{1}{2}}{(1-e^{-(1+j\omega)})^2} + \frac{a}{(1-e^{-(3+j\omega)})} + \frac{b}{(1-e^{-(1+j\omega)})}$$

$$\frac{1}{2} = a(1-e^{-1(1+j\omega)}) + b(1-e^{-(3+j\omega)})$$

$$\frac{1}{2} = a - ae^{-1(1+j\omega)} + b - be^{-(3+j\omega)}$$

$$a+b = \frac{1}{2}$$

$$ae^{-(1+j\omega)} + be^{-(3+j\omega)} = 0$$

$$a + be^{-2} = 0$$

$$a = -be^{-2}$$

$$-be^{-2} + b = \frac{1}{2}$$

$$b = \frac{1}{2}\frac{1}{1-e^{-2}}$$

$$a = -\frac{1}{2}\frac{1}{1-e^{-2}}(e^{-2})$$

$$= -\frac{1}{2}\frac{e^{-2}}{1-e^{-2}}$$

$$Y(\omega) = \frac{\frac{1}{2}}{(1-e^{-(1+j\omega)})^2} + \frac{-\frac{1}{2}\frac{e^{-2}}{1-e^{-2}}}{1-e^{-(3+j\omega)}} + \frac{\frac{1}{2}\frac{1}{1-e^{-2}}}{1-e^{-(1+j\omega)}}$$

6eiii) Find $y[n]$

Given $Y(\omega) = \dfrac{\frac{1}{2}}{(1-e^{-(1+j\omega)})^2} + \dfrac{-\frac{1}{2}\frac{e^{-2}}{1-e^{-2}}}{1-e^{-(3+j\omega)}} + \dfrac{\frac{1}{2}\cdot\frac{1}{1-e^{-2}}}{1-e^{-(1+j\omega)}}$

And we know:

$$e^{-n}u[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-e^{-(1+j\omega)}}$$

$$e^{-3n}u[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-e^{-(3+j\omega)}}$$

Find $y[n]$.

$$y[n] = DTFT^{-1}\{Y(\omega)\} = F^{-1}\left\{\frac{\frac{1}{2}}{(1-e^{-(1+j\omega)})^2}\right\} + F^{-1}\{\dots\} + F^{-1}\{\dots\}$$

$$= F^{-1}\left\{\frac{\frac{1}{2}}{(1-e^{-(1+j\omega)})^2}\right\} + \underbrace{-\frac{1}{2}\frac{e^{-2}}{1-e^{-2}}e^{-3n}u[n]} + \underbrace{\frac{1}{2}\frac{1}{1-e^{-2}}e^{-n}u[n]}$$

$$= F^{-1}\left\{\tfrac{1}{2}X(\omega)\cdot X(\omega)\right\} + (\ ) + (\ )$$

$$= \tfrac{1}{2}\left[x[n] * x[n]\right] + (\ ) + (\ )$$

$$= \tfrac{1}{2}\left[\sum_{k=-\infty}^{\infty} e^{-k}u[k]\, e^{-(n-k)}u[n-k]\right] + (\ ) + (\ )$$

$\hookrightarrow n>k$

$$= \tfrac{1}{2}\left[\sum_{k=0}^{n} \cancel{e^{-k}}\cdot e^{-n}\cdot \cancel{e^{k}}\cdot 1\right] + (\ ) + (\ )$$

$$= \tfrac{1}{2}e^{-n}(n+1) + -\frac{1}{2}\frac{e^{-2}}{1-e^{-2}}e^{-3n}u[n] + \frac{1}{2}\frac{1}{1-e^{-2}}e^{-n}u[n]$$

$\hookrightarrow n\geq 0$ $\qquad$ $\hookrightarrow n\geq 0$ $\qquad$ $\hookrightarrow n\geq 0$.

7.)

7ai Consider an LTI w/ impulse Response of $h[n] = (\frac{1}{4})^n u[n]$.

1.) Determine and sketch Magnitude of $H(W)$, $|H(W)|$.

$$H(W) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn}$$

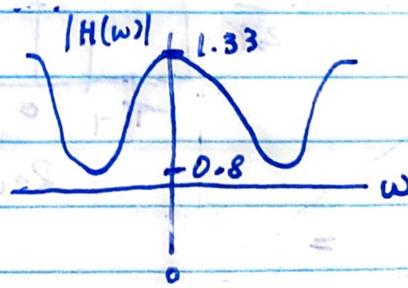
$$= \sum_{n=0}^{\infty} (\tfrac{1}{4})^n e^{-jwn}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{4e^{jw}}\right)^n = \frac{1}{1-\frac{1}{4}e^{-jw}}$$

$$|H(W)| = \frac{1}{1-\frac{1}{4}(\cos w - j\sin w)} = \frac{1}{\underbrace{1-\frac{1}{4}\cos w}_{a} + j\underbrace{\tfrac{1}{4}\sin(w)}_{W}}$$

$$= \frac{1}{a+jW}$$

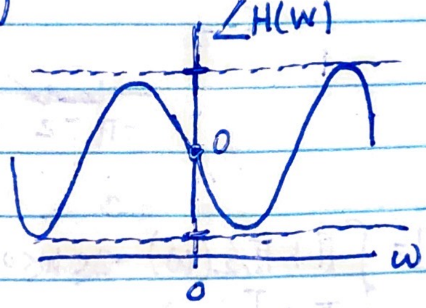
$$= \frac{1}{\sqrt{a^2+W^2}}$$



7aii.)

$$\angle H(W) = -\tan^{-1}(W/a)$$

$$= -\tan^{-1}\left(\frac{\frac{1}{4}\sin(W)}{1-\frac{1}{4}\cos(W)}\right)$$



7b.) An FIR filter is described by Relation.

$$2y[n] = x[n] + x[n-1]$$

7b i.) is this system LTI?

let $x_1[n] \rightarrow \frac{1}{2}x_1[n] + \frac{1}{2}x_1[n-1] = y_1[n]$

$x_2[n] \rightarrow \frac{1}{2}x_2[n] + \frac{1}{2}x_2[n-1] = y_2[n]$

$z[n] = a x_1[n] + b x_2[n]$

$$z[n] \rightarrow y[n] = \frac{1}{2}z[n] + \frac{1}{2}z[n-1]$$

$$= \frac{1}{2}(a x_1[n] + b x_2[n]) + \frac{1}{2}(a x_1[n-1] + b x_2[n-1])$$

$$= a\left(\frac{1}{2}x_1[n] + \frac{1}{2}x_1[n-1]\right) + b\left(\frac{1}{2}x_2[n] + \frac{1}{2}x_2[n-1]\right)$$

∴ Linear $\qquad a y_1[n] + b y_2[n]$

$$x_1[n] \xrightarrow{\text{shift}} x_1[n-n_0] \xrightarrow{\text{system}} y_1[n] = \frac{1}{2}(x_1[n-n_0] + x_1[n-n_0-1])$$

$$x_1[n] \xrightarrow{\text{sys.}} y_2[n] = \frac{1}{2}(x[n] + x[n-1]) \xrightarrow{\text{shift}} y_2[n-n_0] = \frac{1}{2}(x[n-n_0] + x[n-n_0-1])$$

$$y_1[n] = y_2[n-n_0]$$

∴ Yes, Time-invariant!

∴ Yes, the system is LTI.

$y[n] \stackrel{?}{=} \lambda x[n]$

7bii.) Find eigenfunction if possible

$x_1[n] = \delta[n]$ $\qquad 2y[n] = x[n] + x[n-1]$

$2\lambda\delta[n] = \delta[n] + \delta[n-1] \quad \forall n.$

let $n=0 \Rightarrow \lambda = 1+0 = 1$

$n=1 \rightarrow 0 = 1 \quad \times$

$\therefore$ λ does not exist, so $x_1[n]$ not an eigenfunction $\square$

$x_2[n] = u[n]$ $\qquad y[n] \stackrel{?}{=} \lambda x[n]$

$2\lambda u[n] = u[n] + u[n-1] \quad \forall n$

let $n=0 \Rightarrow 2\lambda = 1+0 \Rightarrow \lambda = \frac{1}{2}$

$n=1 \Rightarrow 2\lambda = 1+1 \Rightarrow \lambda = 2$

$\therefore$ λ does not exist, so $x_2[n]$ not an eigenfunction $\square$

$x_3[n] = e^{j\omega n}$ $\qquad y[n] \stackrel{?}{=} \lambda x[n]$

$$2\lambda e^{j\omega n} = e^{j\omega n} + e^{j\omega(n-1)}$$

$$2\lambda = 1 + e^{-j\omega}$$

$$\lambda = \frac{1+e^{-j\omega}}{2} \quad \forall n.$$

$\therefore$ λ exist and same for all n, $x_3[n]$ is an eigenfunction.

8.) Z-transform. Find Z-transform and the ROC.

8a.) $x[n] = \{3, 0, 0, 0, 0, \underset{\uparrow}{\underline{6}}, 1, 4\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-5}^{2} x[n] z^{-n}$$

$$= x[-5] z^{+5} + x[0] z^{-0} + x[1] z^{-1} + x[2] z^{-2}$$

$$= 3z^5 + 6 + z^{-1} + 4z^{-2} \qquad \boxed{\text{ROC: } z \in \{\mathbb{C}\} - \{0\}}$$

$$\boxed{X(z) = 3z^5 + 6 + \frac{1}{z} + 4\frac{1}{z^2}}$$

8b.) $x[n] = ((\frac{1}{10})^n + 10^n) u[n]$

$$x[n] = (\frac{1}{10})^n u[n] + 10^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (\frac{1}{10})^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} 10^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (\frac{1}{10} \cdot \frac{1}{z})^n + \sum_{n=0}^{\infty} (10 \cdot \frac{1}{z})^n$$

$$= \frac{1}{1 - \frac{1}{10z}} + \frac{1}{1 - \frac{10}{z}}$$

$$X(z) = \boxed{\frac{1}{1 - \frac{1}{10} z^{-1}} + \frac{1}{1 - 10 z^{-1}}} \qquad \boxed{\begin{aligned} &\text{ROC: ROC①} \cap \text{ROC②} \\ &= |z| > 10. \end{aligned}}$$

↳ ROC① $|z| > \frac{1}{10}$

↳ ROC②. $|z| > 10$