10/22/20

Today's lecture

- Circular convolution examples / properties
- FFT
- Decimation in time FFT
- Decimation in frequency FFT
- Radix-2 FFT and generalizations

Last lecture's example: 2 length 4 signals: $\{x[0], x[1], x[2], x[3]\}$

$\{h[0], h[1], h[2], h[3]\}$

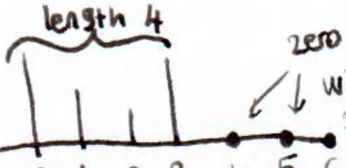
Linear convolution: $\sum_{k=0}^{3} x[k]h[n-k]$ $\rightarrow$ this has length $4+4-1=7$

Circular convolution: (length 7) $y[n] = \sum_{k=0}^{6} x[k]h_7[n-k]$

zero pad with 3 zeros, i.e. $h[6]=h[5]=h[4]=0$

$$\underset{\substack{7\times 1\\ \text{vector}}}{\begin{bmatrix} y[0]\\ y[1]\\ y[2]\\ y[3]\\ y[4]\\ y[5]\\ y[6] \end{bmatrix}} = \underset{7\times 7 \text{ circulant matrix}}{\left[\begin{array}{cccc|ccc} h[0] & 0 & 0 & 0 & h[3] & h[2] & h[1]\\ h[1] & h[0] & 0 & 0 & 0 & h[3] & h[2]\\ h[2] & h[1] & h[0] & 0 & 0 & 0 & h[3]\\ h[3] & h[2] & h[1] & h[0] & 0 & 0 & 0\\ 0 & h[3] & h[2] & h[1] & h[0] & 0 & 0\\ 0 & 0 & h[3] & h[2] & h[1] & h[0] & 0\\ 0 & 0 & 0 & h[3] & h[2] & h[1] & h[0] \end{array}\right]} \underset{\substack{7\times 1\\ \text{vector}}}{\begin{bmatrix} x[0]\\ x[1]\\ x[2]\\ x[3]\\ x[4]\\ x[5]\\ x[6] \end{bmatrix}} \quad \left.\begin{matrix} x[4]\\ x[5]\\ x[6]\end{matrix}\right\} = 0$$

= Linear convolution (for this specific example)

$x[n], h[n]$ 

zero pad with 3 zeros $\longrightarrow$ Multiply their 7 point DFTs $\longrightarrow$ 7 point IDFT $\longrightarrow$ $x[n] * h[n]$ linear conv.

Time shift

$$x_N[n-m] \overset{\text{(length } N)}{\xleftrightarrow{\text{DFT}}} W_N^{km} X[k]$$

$$\sum_{n=0}^{N-1} x_N[n-m] W_N^{kn} = \sum_{\ell=-m}^{N-1-m} x_N[\ell] \overbrace{W_N^{k(\ell+m)}}^{W_N^{k\ell} W_N^{km}}$$

$\ell = n - m$

$$= W_N^{km} \sum_{\ell=-m}^{N-1-m} x_N[\ell] W_N^{k\ell}$$

(redundant) $$= W_N^{km} \sum_{\ell=-m}^{N-1-m} x_N[\ell] W_N^{(k+N)\ell}$$

$$= W_N^{km} \underbrace{\sum_{\ell=0}^{N-1} x_N[\ell] W_N^{k\ell}}_{X[k]}$$

Piazza example on circular convolution

$x[n]$: $x[0]=1$, $x[1]=2$, $x[2]=3$, $h[n]$: $h[0]=h[1]=h[2]=1/3$

A. Linear convolution (conv)

$$\sum_{k=0}^{2} x[k]\,h[n-k] = h[n]+2h[n-1]+3h[n-2]$$

$y[0]=1/3$ $y[1]=1$ $y[2]=2$ $y[3]=5/3$ $y[4]=1$ (length 5)

B. $N=3$ circular (cconv)

$$\sum_{k=0}^{2} x[k]\,h_3[n-k]$$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \end{bmatrix} = \begin{bmatrix} h[0] & h[2] & h[1] \\ h[1] & h[0] & h[2] \\ h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

C. $N=5$ circular

$$\sum_{k=0}^{4} x[k]\,h_5[n-k]$$

(zero pad with 2 zeros)

i.e $h[3]=h[4]=0$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} h[0] & h[4] & h[3] \\ h[1] & h[0] & h[4] \\ h[2] & h[1] & h[0] \\ h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 2 \\ 5/3 \\ 1 \end{bmatrix}$$

(same as A)

D. $N=7$ circular

$$\sum_{k=0}^{6} x[k]\,h_7[n-k]$$

(zero pad with 4 zeros)

i.e. $h[3]=h[4]=h[5]=h[6]=0$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} h[0] & h[6] & h[5] \\ h[1] & h[0] & h[6] \\ h[2] & h[1] & h[0] \\ h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \\ h[6] & h[5] & h[4] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 2 \\ 5/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Complexity of DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] \underbrace{e^{-j\frac{2\pi}{N}kn}}_{W_N^{kn}} \quad , \; k = 0, \ldots, N-1$$

N^2 complex multiplications

$N(N-1)$ complex additions

Fast Fourier Transform (FFT)

* Reduce the number of operations to $N \log N$.

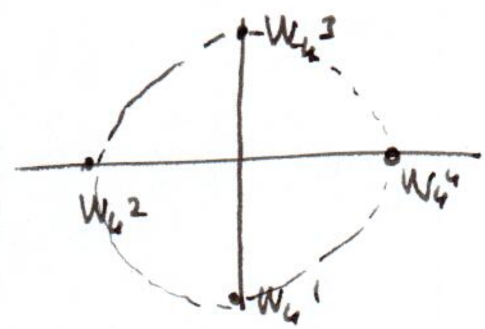
$N = 10^3$ DFT $\sim 10^6$

FFT $\sim 10^4$ (100 times faster than DFT)

* How?

- Decomposition into smaller DFTs
- Simplifications: $W_N^{kN} = 1$

$$W_N^{(k_{odd})\frac{N}{2}} = -1$$



$N=4$

* Periodicity

$$W_N^{n(k+N)} = W_N^{k(n+N)} = W_N^{kn}$$

* Conjugation

$$W_N^{k(N-n)} = W_N^{-kn} = \left(W_N^{kn}\right)^{*} \text{ (conjugation)}$$

Recall that there are only N unique entries in DFT matrix.

"dftmtx"
(4) (16) (32)

Decimation in time (N even)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$$= \sum_{n \text{ even}} x[n] W_N^{nk} + \sum_{n \text{ odd}} x[n] W_N^{nk}$$

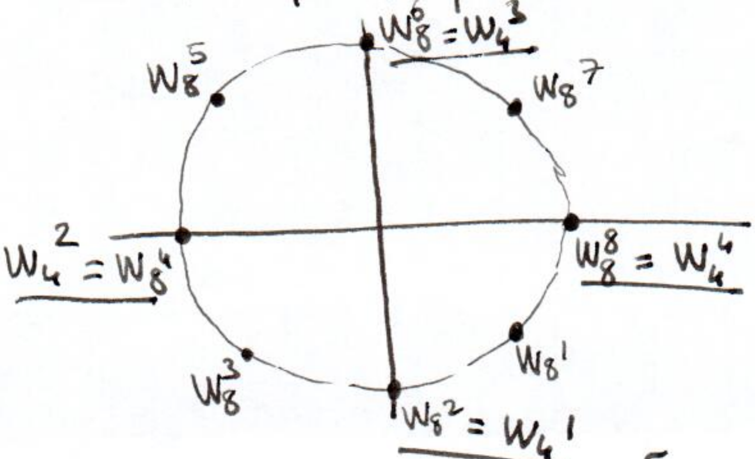
$$= \sum_{l=0}^{N/2-1} x[2l] W_N^{2lk} + \sum_{l=0}^{N/2-1} x[2l+1] W_N^{(2l+1)k}$$

$$= \sum_{l=0}^{N/2-1} x[2l] (W_N^2)^{lk} + \sum_{l=0}^{N/2-1} x[2l+1] (W_N^2)^{lk} \cdot W_N^k$$

$$= \underbrace{\sum_{l=0}^{N/2-1} x[2l] W_{N/2}^{lk}}_{\substack{\text{length } N/2 \text{ DFT} \\ \text{of even terms} \\ \parallel \\ G[k]}} + W_N^k \underbrace{\sum_{l=0}^{N/2-1} x[2l+1] W_{N/2}^{lk}}_{\substack{\text{length } N/2 \text{ DFT} \\ \text{of odd terms} \\ \parallel \\ H[k]}}$$

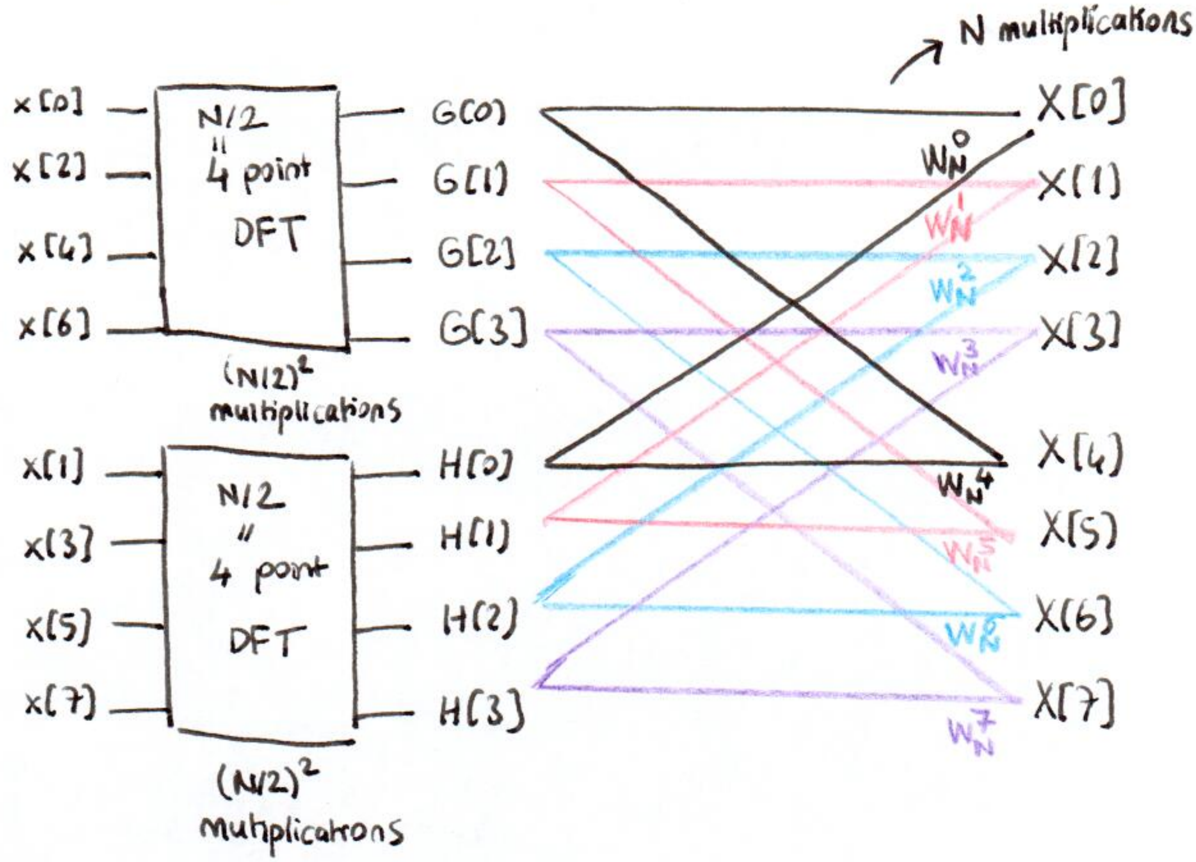
$$X[k] = G[k] + W_N^k H[k], \quad k = 0, \dots, N-1$$

Consider for example $N = 8$



$$\underbrace{\begin{array}{ll} G[0] = G[4], & H[0] = H[4] \\ G[1] = G[5] & H[1] = H[5] \\ G[2] = G[6] & H[2] = H[6] \\ G[3] = G[7] & H[3] = H[7] \end{array}}_{\text{periodic with } N/2 = 4}$$

Assume $N=8$ (given) $\quad X[k] = G[k] + W_N^k H[k]\ ,\ k=0,\dots,7$

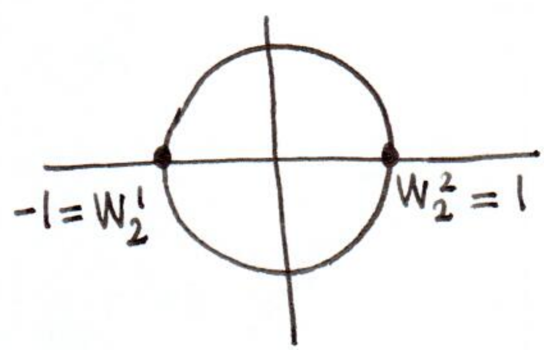


Total number of multiplications: $\quad 2\cdot\left(\frac{N}{2}\right)^2 + N \approx \frac{N^2}{2}$

Length 2 DFT

$$\underbrace{x[0]\ ,\ x[1]}_{\text{time domain}} \longrightarrow \underbrace{X[0]\ ,\ X[1]}_{k\ \text{domain}}$$

$N=2 \qquad W_2^1 = -1\ ,\ W_2^2 = 1$

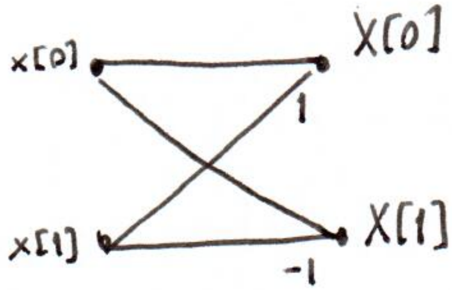


$$X[k] = \sum_{n=0}^{1} x[n] W_2^{nk}$$

$X[0] = x[0] + x[1]$ (no multiplication needed)

$X[1] = x[0] - x[1]$

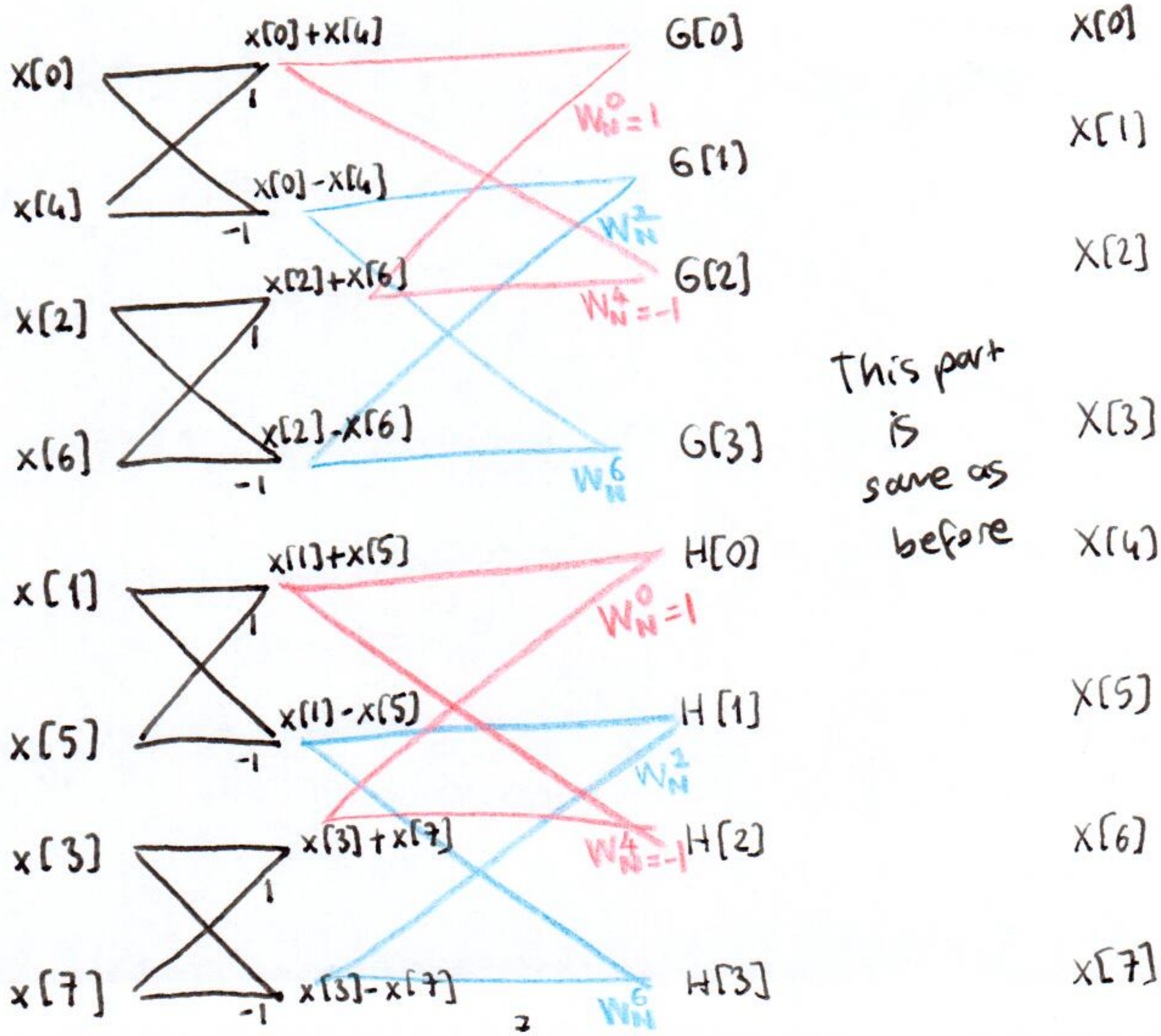
2 point DFT



Recall that $G[k] = \sum_{\ell=0}^{N/2-1} x[2\ell] W_{N/2}^{\ell k}$ (length N/2 DFT of even terms)

$$H[k] = \sum_{\ell=0}^{N/2-1} x[2\ell+1] W_{N/2}^{\ell k}$$ (length N/2 DFT of odd terms)

$$X[k] = G[k] + W_N^k H[k] \quad , k=0, \ldots, N-1$$



Let's verify some branches:

$$G[0] = x[0] + x[2] + x[4] + x[6]$$

$$G[1] = x[0] + x[2]W_N^2 + x[4]W_N^4 + x[6]W_N^6 \qquad (\text{Recall } N=8)$$

$$= (x[0] - x[4]) + W_N^2\,(x[2] - x[6])$$

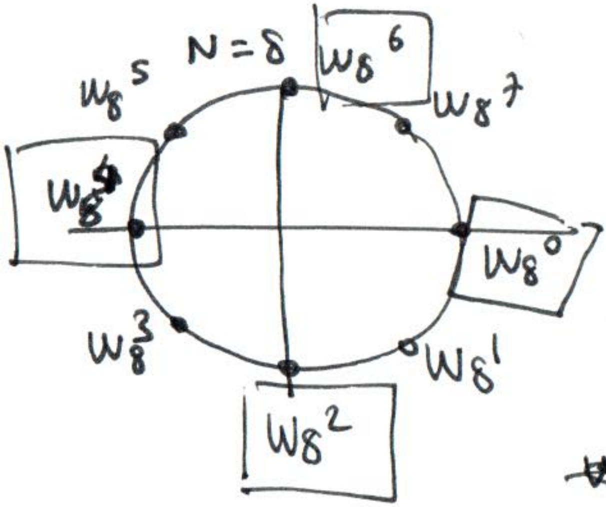
The rest of $G[k]$, $H[k]$, please verify.

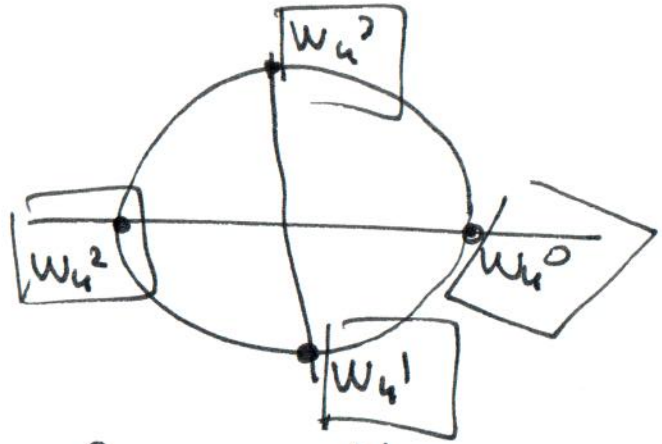
Radix-2 FFT

- The number of data points is a power of 2. In the above example $N = 2^3$. 3 is the number of stages.
- Consider a pattern from the figure

$$W_N^{\ell+N/2} = -W_N^{\ell}$$

- We divide the number of multiplications we need by 2.
- This reduced complexity approach can provide savings in terms of storage space.

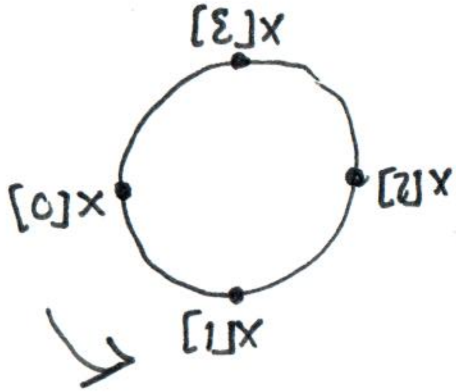




$\cancel{W_8}$ $W_8^2 = W_4^1$

$W_8^4 = W_4^2$

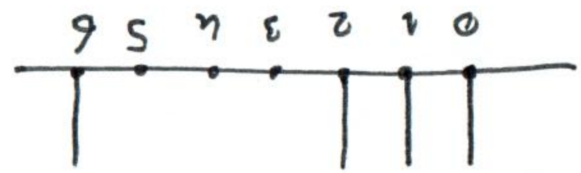
$W_8^6 = W_4^3$



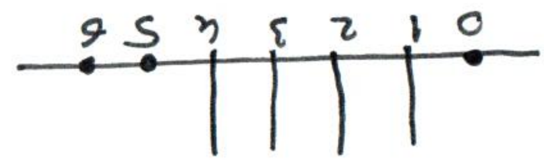
$\tilde{y}[n] \Leftarrow \tilde{y}[t] = \tilde{y}[0]$

$n = t$

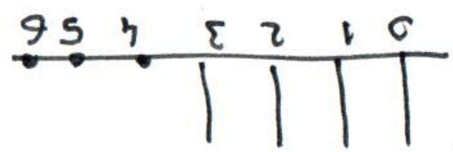
$h_t[n-1]$



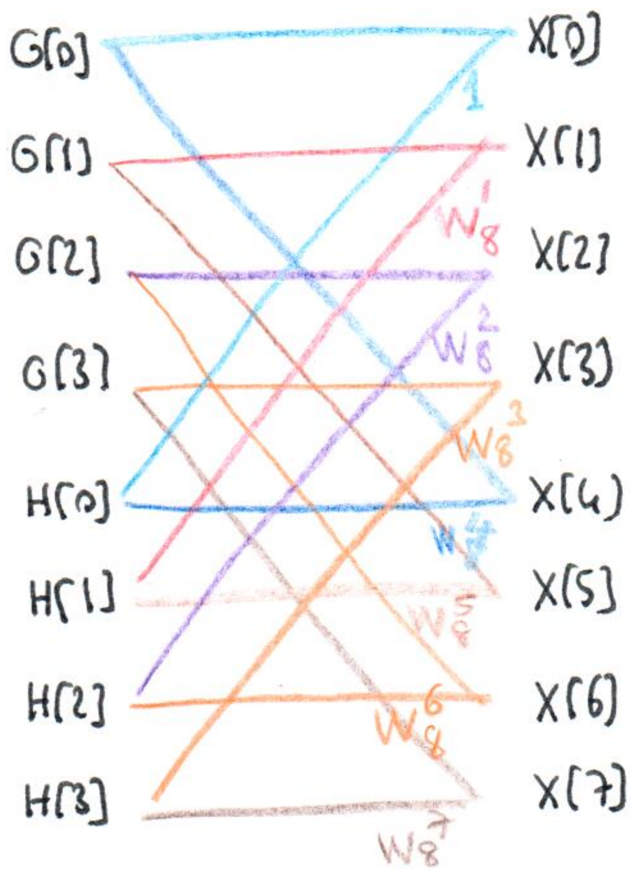
$h_t[n+1]$



$h_t[n-1] = \tilde{y}[n]$

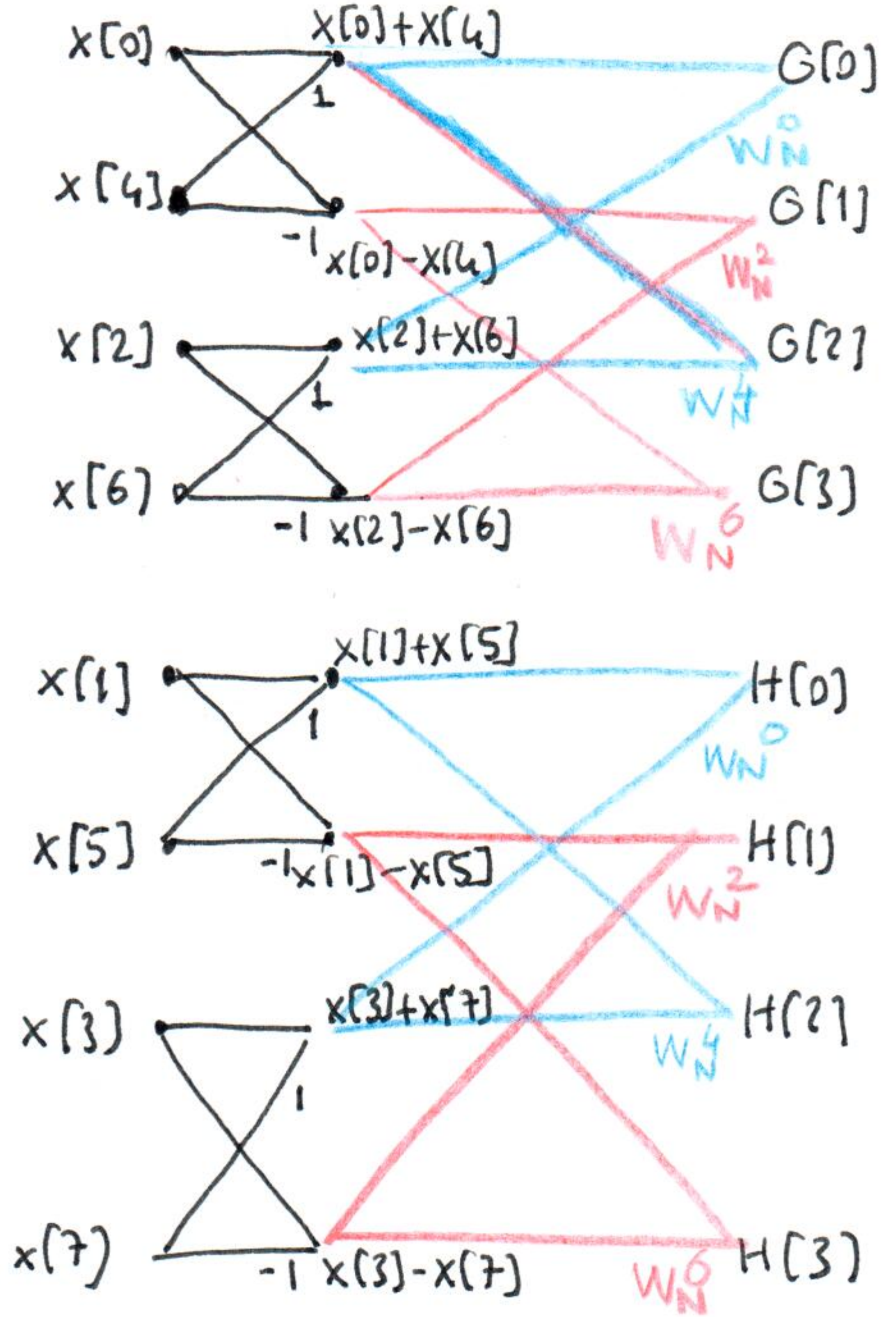


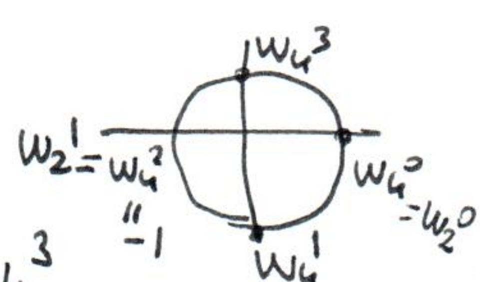
$h_t[n]$



G[0]
G[1]
G[2]
G[3]
H[0]
H[1]
H[2]
H[3]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
1
W_8^1
W_8^2
W_8^3
W_8^4
W_8^5
W_8^6
W_8^7

$$X[k] = G[k] + W_N^k H[k], \quad k = 0, \dots, N-1$$





$$G[0] = x[0] + x[2] + x[4] + x[6]$$

$$G[1] = x[0] + \underline{x[2]W_4^1} + x[4]W_4^2 + \underline{x[6]W_4^3}$$

$$= x[0] - x[4] + W_4^1 \left(x[2] + x[6]\underbrace{W_4^2}_{-1}\right)$$

$$= x[0] - x[4] + \underline{W_4^1}(x[2] - x[6])$$