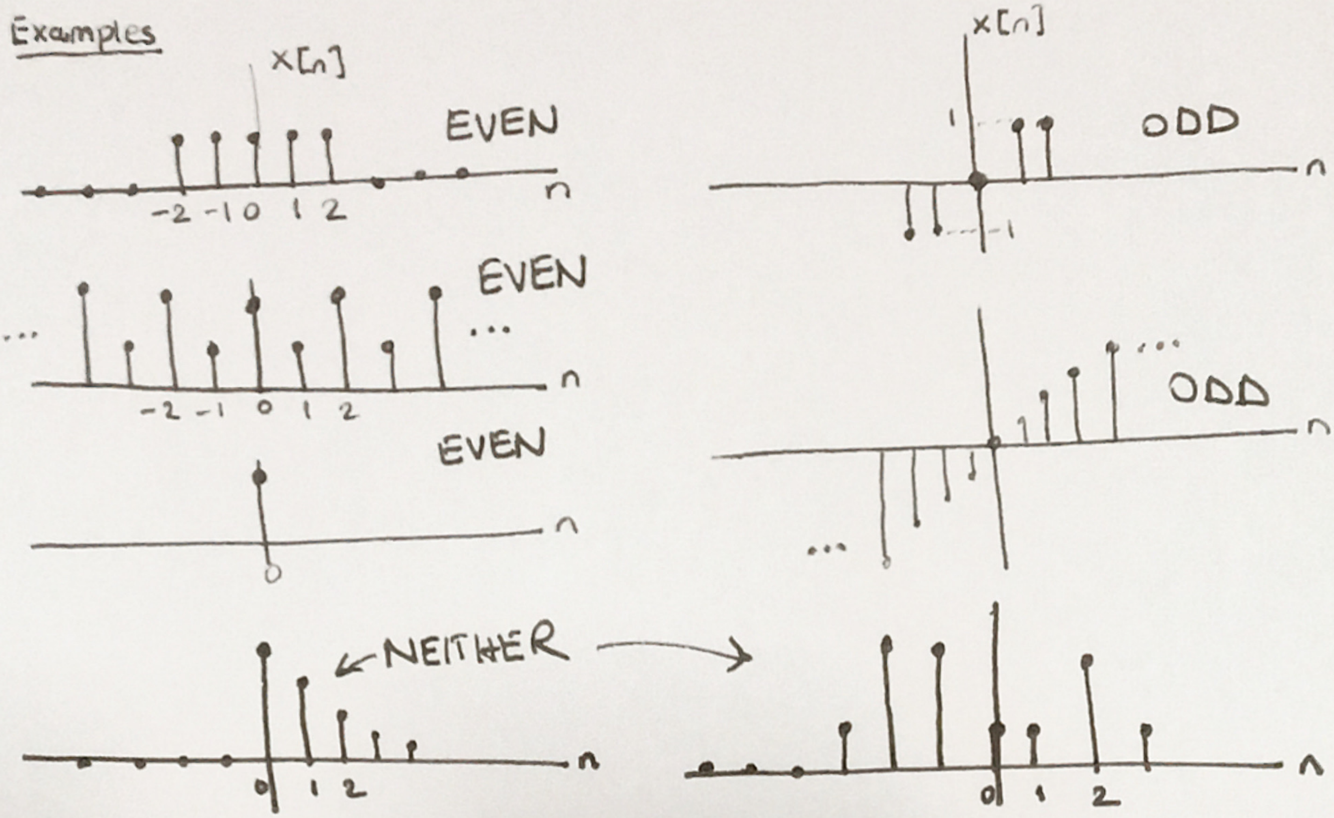
9/10/20

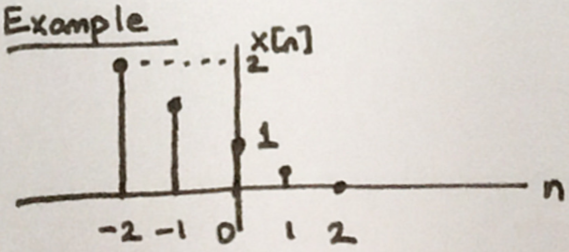
Today's Lecture

- Examples of discrete-time signal properties
- LTI system properties
- Review of Fourier Series

Examples



Example



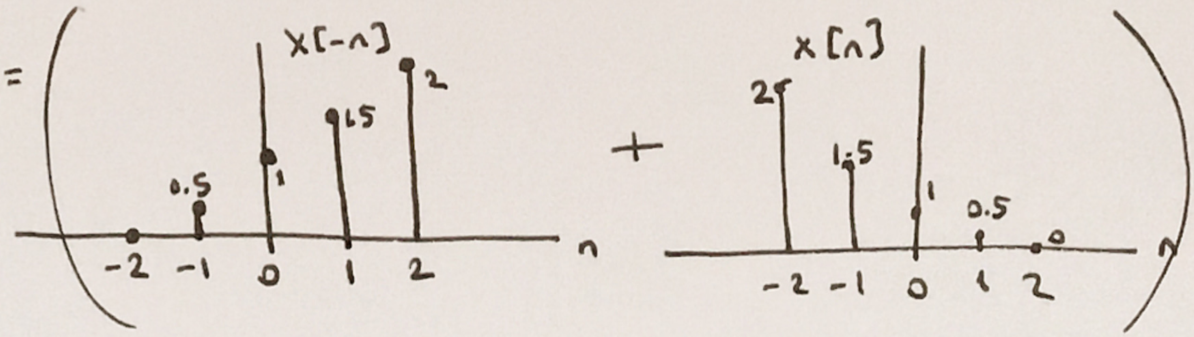
$x_e[n] = \text{Even}(x[n])$?

$x_o[n] = \text{Odd}(x[n])$?

$$x_e[n] = \frac{x[n] + x[-n]}{2}, \quad x_o[n] = \frac{x[n] - x[-n]}{2}$$

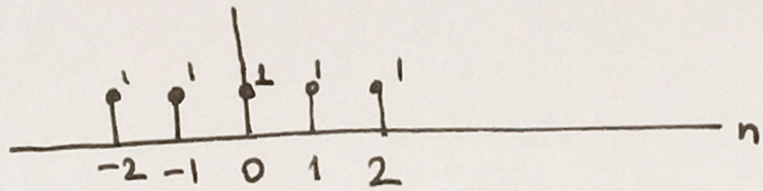
$$x[n] = x_e[n] + x_o[n]$$

$x_e[n] =$

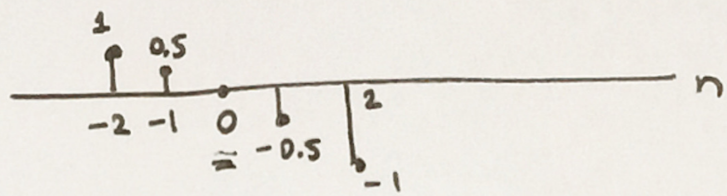


$\cdot \frac{1}{2}$

$=$



$x_o[n] =$



$x_o[n] = -x_o[-n]$ at $n=0$ $x_o[0] = -x_o[0]$

$= 0$

__Example__ (from previous lecture) $\boxed{y[n] = n\,x[n]}$

Is this a linear system?

$x_1[n] \longrightarrow y_1[n] = n\,x_1[n]$

$x_2[n] \longrightarrow y_2[n] = n\,x_2[n]$

We need to check whether __additivity__ & __homogeneity__ hold.

$$z[n] = a\,x_1[n] + b\,x_2[n] \longrightarrow y[n] = n.z[n]$$

$$= n(a x_1[n] + b x_2[n])$$

$$= a \underbrace{n x_1[n]}_{y_1[n]} + b \underbrace{n x_2[n]}_{y_2[n]}$$

$$= a\,y_1[n] + b\,y_2[n]$$

LINEAR

Properties of LTI systems

$$\delta[n] \longrightarrow \boxed{H} \longrightarrow h[n]$$

impulse function.

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = x[n] * h[n]$$

1. The impulse response $h[n]$ ($h(t)$ for CT)

For LTI systems, $h[n]$ completely characterizes the behavior. (not true for non LTI systems)

Example $y[n] = n\,x[n] \longrightarrow$ not time invariant.

What is the impulse response? $\Downarrow$

$y[n] \neq x[n] * h[n]$

$\delta[n] \longrightarrow 0$

(Check previous lecture)

2. Commutative property

$$x[n] * h[n] = h[n] * x[n] = y[n]$$

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

$$h[n] \longrightarrow \boxed{X} \longrightarrow y[n]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] = \sum_{m=-\infty}^{+\infty} x[n-m]\, h[m]$$

$\underline{m = n-k}$

$k = n-m$

$$= \sum_{m=-\infty}^{+\infty} h[m]\, x[n-m]$$

$$= h[n] * x[n]$$

3. Distributive property

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

4. Associative property

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

$$x[n] \longrightarrow \boxed{h_1[n]} \xrightarrow{(x[n] * h_1[n])} \boxed{h_2[n]} \xrightarrow{(x[n] * h_1[n]) * h_2[n]} y[n]$$

$$\equiv x[n] \longrightarrow \boxed{h_1[n] * h_2[n]} \longrightarrow y[n] \quad \text{(associative)}$$

$$\equiv x[n] \longrightarrow \boxed{h_2[n] * h_1[n]} \longrightarrow y[n] \quad \text{(commutative)}$$

$$\equiv x[n] \longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow y[n]$$

The order of LTI systems does not matter.

<u>Example</u>

$$x[n] \longrightarrow \boxed{\times 2} \xrightarrow{2x[n]} \boxed{(\cdot)^2} \longrightarrow 4x^2[n]$$

$$\neq$$

$$x[n] \longrightarrow \boxed{(\cdot)^2} \xrightarrow{x^2[n]} \boxed{\times 2} \longrightarrow 2x^2[n]$$

It is not linear. You cannot exchange the order for non LTI systems.

5. Causality

$y[n]$ does not depend on $x[n+k]$, $k>0$

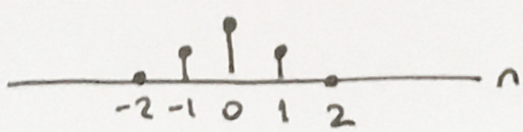
A system is causal <u>if and only if</u> $h[k]=0$, $k<0$.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=0}^{+\infty} h[k]x[n-k]$$

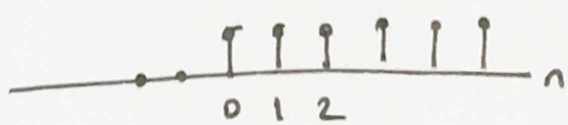
$$= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots$$

$h[n]$ (impulse response)



$\rightarrow$ <u>not causal.</u>

$h[n] = u[n]$



$\rightarrow$ <u>causal.</u>

6. Unit step response

Response to $u[n]$. Denoted by $s[n]$.

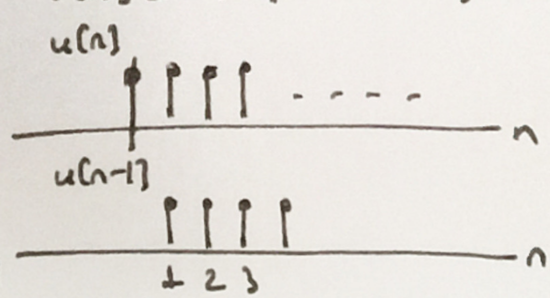
$\delta[n] \longrightarrow h[n]$

$u[n] \longrightarrow s[n]$

$u[n-1] \longrightarrow s[n-1]$

(time invariance)

$\delta[n] = u[n] - u[n-1]$



$\delta[n] = u[n] - u[n-1] \longrightarrow s[n] - s[n-1] = h[n]$

(linearity)

In CT, $h(t) = \frac{d}{dt} s(t)$, $s(t) = \int_{-\infty}^{t} h(\tau) d\tau$

In DT, $h[n] = s[n] - s[n-1]$, $s[n] = \sum_{k=-\infty}^{n} h[k]$

$$= \sum_{k=-\infty}^{n} \left(s[k] - s[k-1] \right)$$

$$= \ldots + \cancel{s[0]} - s[-1] + \cancel{s[1]} - \cancel{s[0]} + s[2] - \cancel{s[1]} + \ldots$$

Step response can characterize an LTI system.

7. Stability

An arbitrary relaxed system is Bounded Input Bounded Output (BIBO) stable if and only if $y[n]$ is bounded for every bounded input $x[n]$.

$x[n]$ is bounded if there exists a constant M such that $|x[n]| \leq M < \infty \quad \forall n$

$y[n]$ is bounded if there exists a constant N such that $|y[n]| \leq N < \infty \quad \forall n$.

<u>Example</u> (2.3.7 Textbook)

An LTI system with impulse response

$$h[n] = \begin{cases} a^n & , n \geq 0 \\ b^n & , n < 0 \end{cases}$$

Determine the range of values of a and b for which the LTI system is <u>stable</u>.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] \right| \leq \sum_{k=-\infty}^{+\infty} |h[k]x[n-k]|$$

$|a+b| \leq |a| + |b|$

$$= \sum_{k=-\infty}^{+\infty} |h[k]|.|x[n-k]|$$

we assume $|x[n]| \leq M \quad \forall n$

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]|.\underbrace{|x[n-k]|}_{\leq M} \leq M \sum_{k=-\infty}^{+\infty} |h[k]|$$

If $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$ then the system is stable.

$$\underbrace{\sum_{k=-\infty}^{-1} |b^k|}_{\text{converges when } |b|>1} + \underbrace{\sum_{k=0}^{+\infty} |a^k|}_{\text{converges when } |a|<1} < \infty$$

$$= \frac{1}{1-a}$$

The system is stable if both $|a| < 1$, $|b| > 1$.

8. Initial rest condition.

If $x[n]=0$ for $n<n_0$, $y[n]=0$ for $n<n_0$,
then the system satisfies the initial rest condition.

Linear Constant Coefficient Difference Equations
(LCCDE)

If the system is LTI and <u>causal</u>, then it can be fully characterized by LCCDEs.

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$$y[n] = \frac{1}{a_0}\left[\underbrace{\sum_{k=0}^{M} b_k\, x[n-k]}_{\text{current \& previous values of } \underline{\text{input}}} - \underbrace{\sum_{k=1}^{N} a_k\, y[n-k]}_{\text{previous values of output}}\right]$$

$N=0 \Rightarrow y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k\, x[n-k]$ (no recursion)

$h[n] = ?$ (Next lecture)

$$h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Finite impulse response!

<u>Hint</u>: Think about convolution sum.