10/29/20

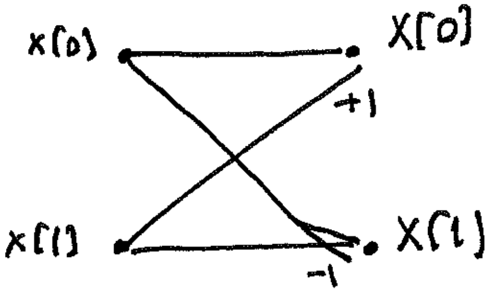
Today's Lecture

- The sampling theorem
- Discrete-time processing of continuous time signals
- Changing the sampling rate
  - downsampling
  - upsampling

Readings: 6-1, 6-2 Sampling

6-4, 6-5, 11.1-11.4 Upsampling / downsampling

Question on Piazza:



2 multiplications

length N input $\rightarrow \frac{N}{2}$ butterflies per stage

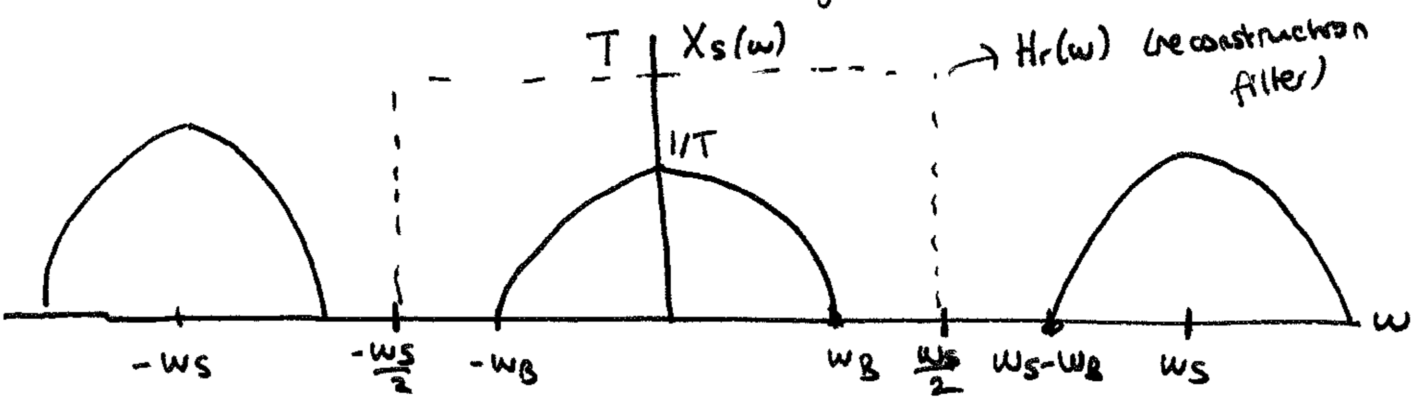
Complexity $\frac{N}{2}\log_2 N$

$\log_2 N$ stages

Recall from last time:

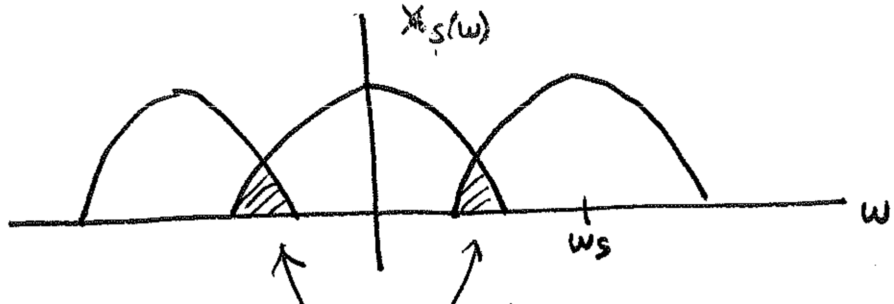
$$X_s(\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(\omega - \omega_s k)$$

$\frac{1}{T}$: sampling period; X_c: continuous time signal; ω_s: sampling frequency $= \frac{2\pi}{T}$



We can recover $X_c(\omega)$ if $\omega_s > 2\omega_B$.

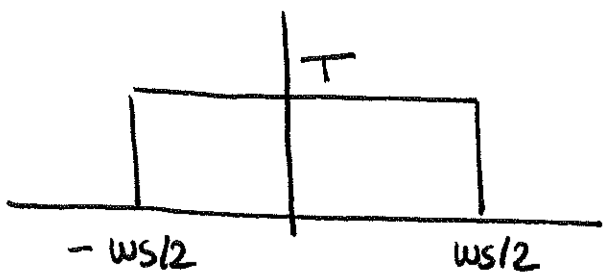
What if not? $(\omega_s < 2\omega_B)$ When we sample slowly, we have aliasing.



Aliasing (High frequency components combine with low frequency components from a different copy.

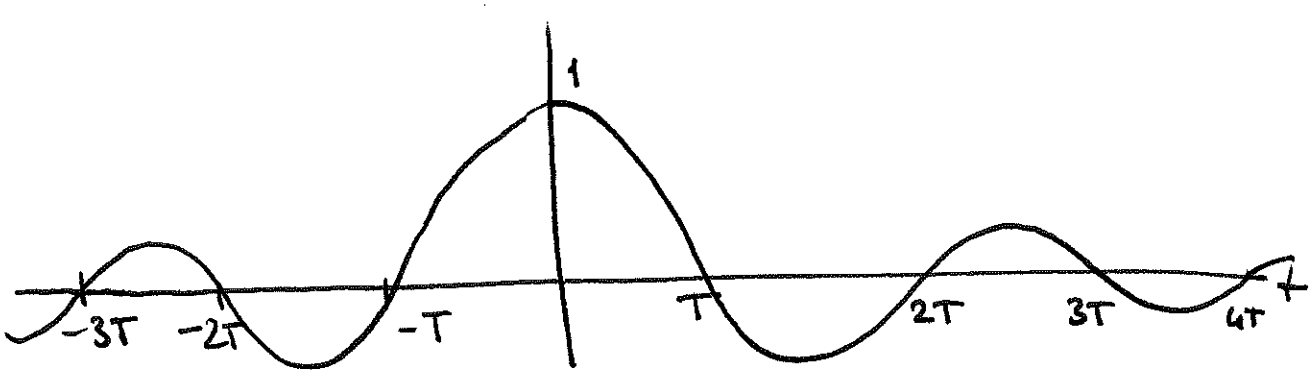
Reconstruction (assuming $\omega_s > 2\omega_B$)

$H_r(\omega)$ reconstruction filter



$X_s(\omega) . H_r(\omega) = X_c(\omega)$ (CTFT)

Time domain : $h_r(t) = \text{sinc}\left(\frac{\pi t}{T}\right)$

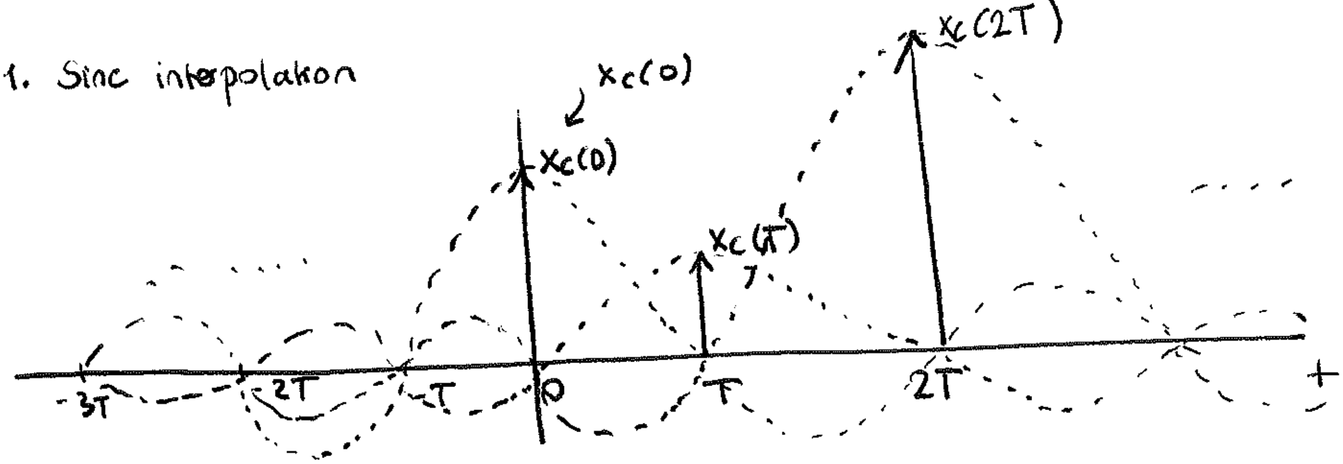


Reconstruction process is a convolution in time domain:

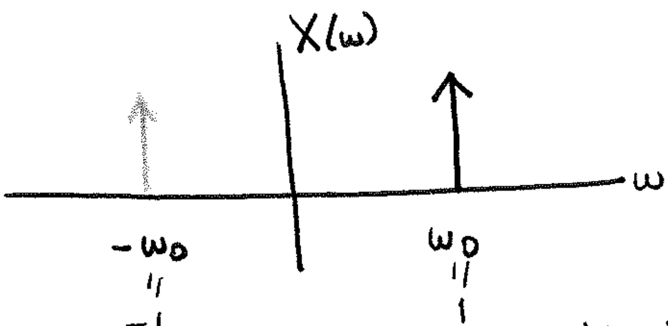
$$x_c(t) = \underbrace{x_s(t)}_{\sum_k \delta(t-kT) * x_c(t)} * h_r(t) = \sum_{n=-\infty}^{+\infty} x_c(nT) . \text{sinc}\left(\frac{\pi}{T}(t-nT)\right)$$

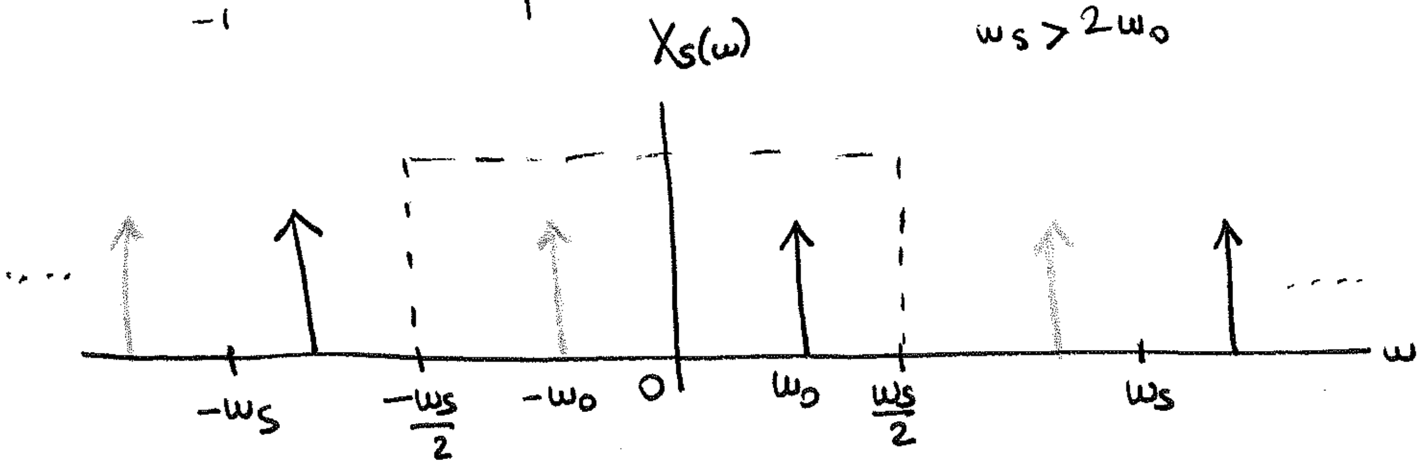
Examples

1. Sinc interpolation



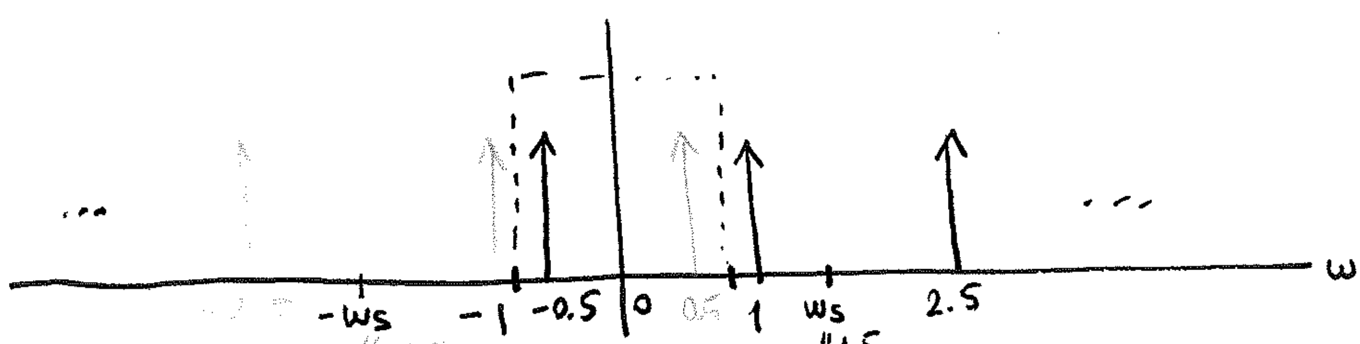
2. $x(t) = \cos(\omega_0 t)$



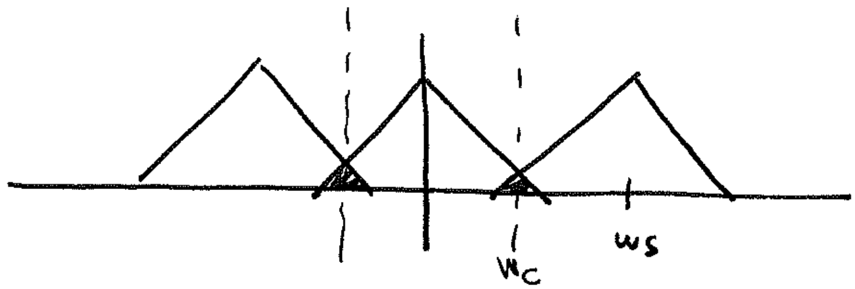


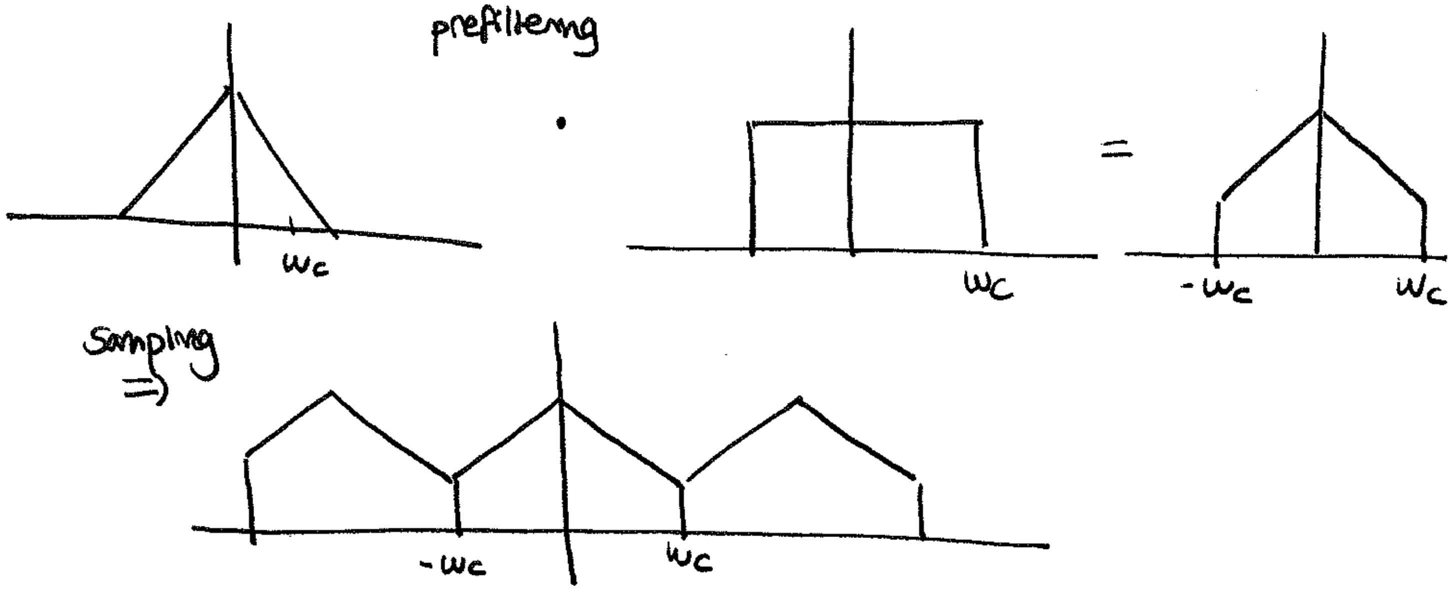
$\omega_s > 2\omega_0$

Let's assume now $\omega_0 = 1$ and $\omega_s = 1.5$ (below Nyquist rate)



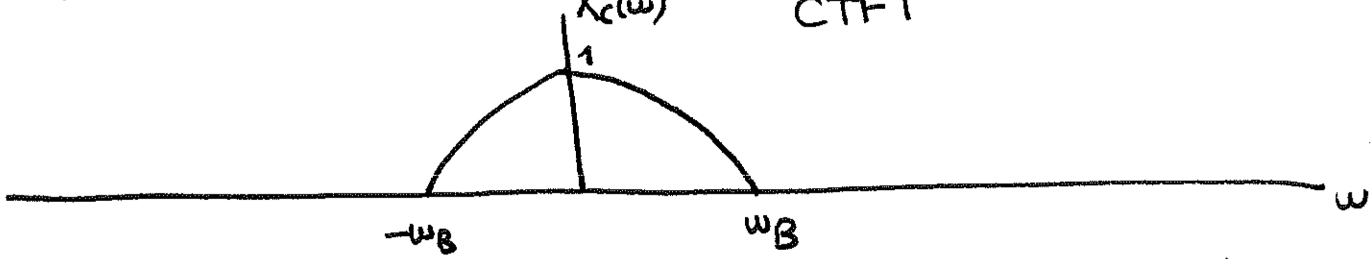
Idea: Prefiltering to prevent aliasing



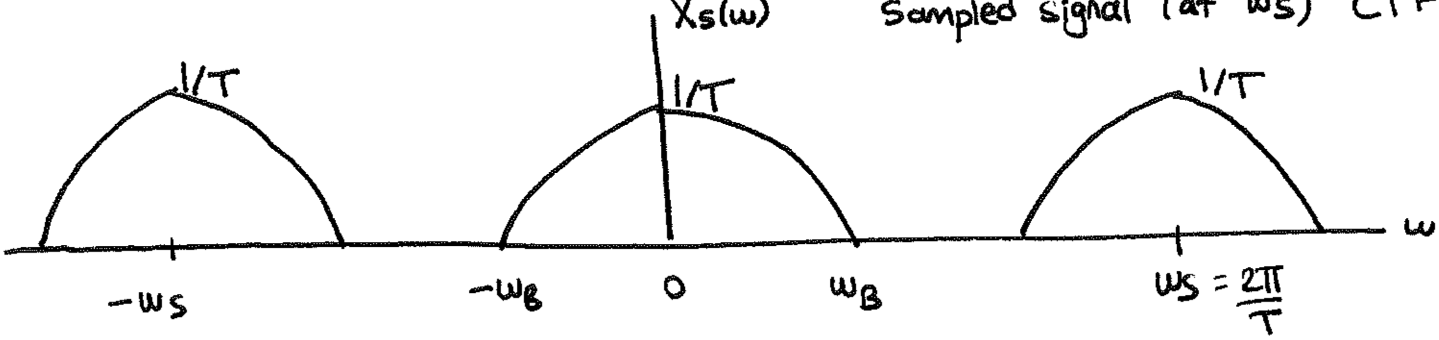


Original CT signal

CTFT



Sampled signal (at ω_s) CTFT

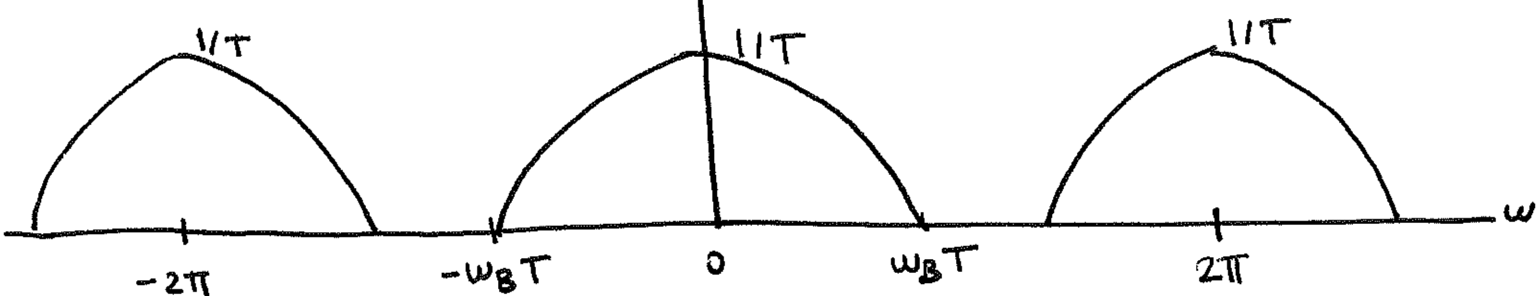


$X_d(\omega) = X_s\left(\frac{\omega}{T}\right)$ DTFT

$x[n] = x(nT)$ $\omega_s = \frac{2\pi}{T}$

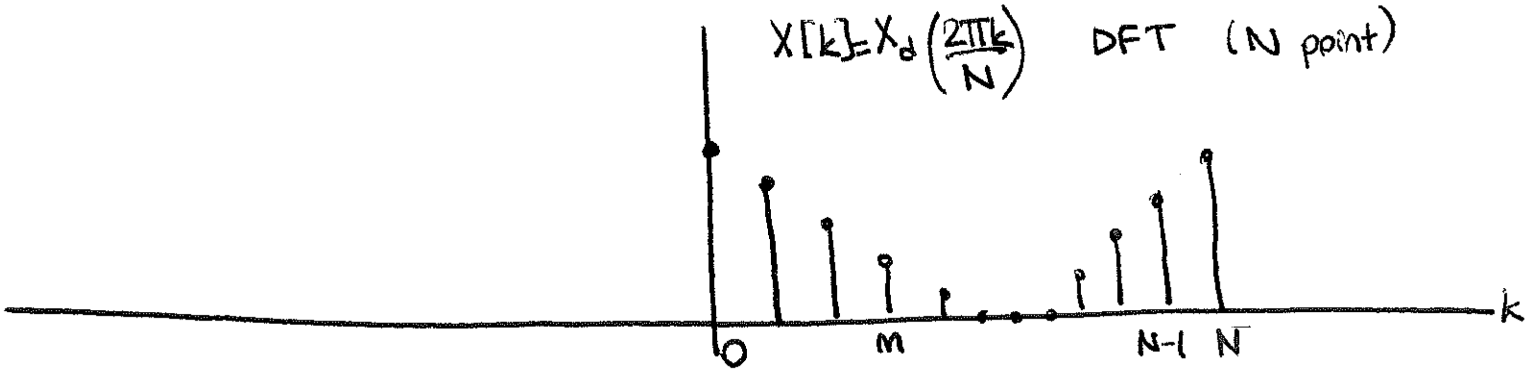
$X_d(\omega)$ DTFT (2π periodic)

$= X_d(\omega + 2\pi)$



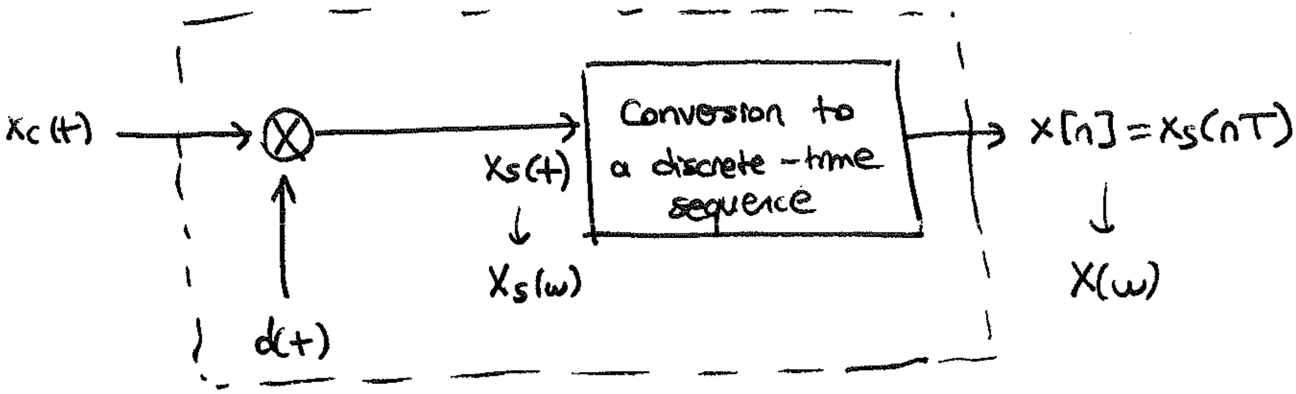
$= \omega_B \cdot \frac{2\pi}{\omega_s}$ (2π periodic)

$X[k] = X_d\left(\frac{2\pi k}{N}\right)$ DFT (N point)



Sample m: $\frac{2\pi m}{N} \longrightarrow$ will correspond to

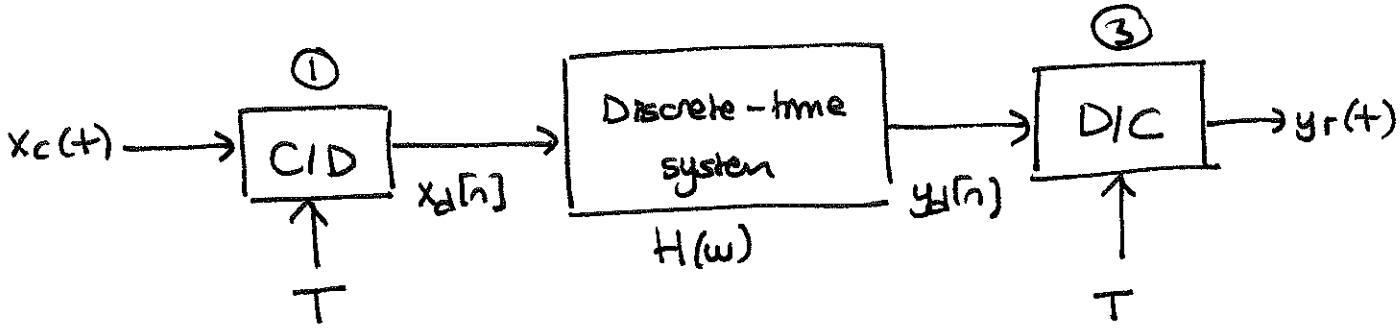
$$\frac{2\pi m}{N} \cdot \underbrace{\frac{\omega_s}{2\pi}}_{1/T} = \frac{m\,\omega_s}{N}$$



C/D converter

$$X_s(\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(\omega - k\omega_s) \quad , \quad \omega_s = \frac{2\pi}{T} \qquad \text{CTFT.}$$

Discrete-time processing of continuous time signals

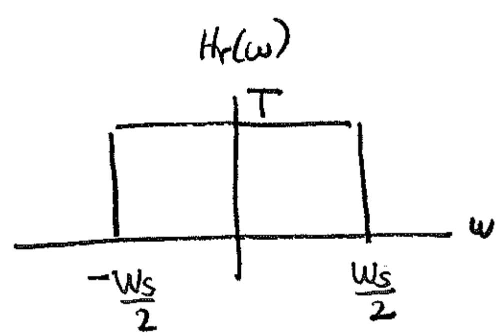


① $$X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right) \qquad \text{DTFT} \qquad \boxed{X_d(\omega) = X_s\left(\frac{\omega}{T}\right)}$$

③ $$y_r(t) = \sum_{n=-\infty}^{+\infty} y_d[n] \, \text{sinc}\left(\frac{t-nT}{T} \cdot \pi\right) \qquad \text{sinc interpolation}$$

$$Y_r(\omega) = H_r(\omega) \underbrace{Y_d(\omega T)}_{= Y_s(\omega)} \qquad \text{CTFT}$$

↳ sampled in CT



$$Y_r(\omega) = \begin{cases} T\, Y_d(\omega T) & , \quad |\omega| < \frac{\omega_s}{2} = \frac{\pi}{T} \\ 0 & , \quad \text{otherwise} \end{cases}$$

② $Y_d(\omega) = X_d(\omega) \underbrace{H(\omega)}_{}$ DTFT

Frequency response of the DT system

$Y_r(\omega) = H_r(\omega)\, Y_d(\omega T)$ reconstructed signal CTFT

$$= H_r(\omega)\, X_d(\omega T)\, H(\omega T)$$

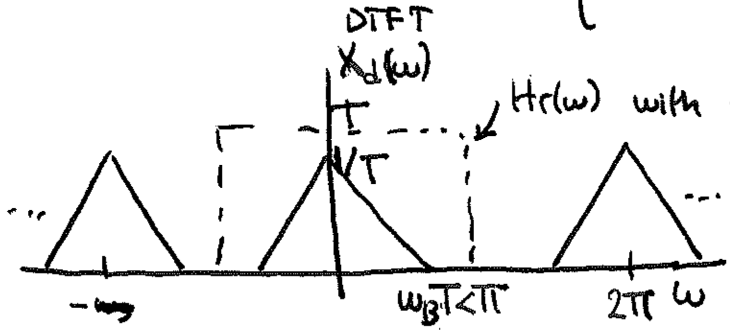
$$= H_r(\omega)\, H(\omega T) \cdot \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{T} - \underbrace{\frac{2\pi}{T}}_{\omega_s} k\right) \quad \text{from ①}$$

If $X_c(\omega) = 0$ for $|\omega| > \frac{\omega_s}{2}$, then

$$Y_r(\omega) = \begin{cases} H_r(\omega)\, H(\omega T) \frac{X_c(\omega/T)}{T} & , \quad |\omega| < \frac{\omega_s}{2} = \frac{\pi}{T} \\ 0 & , \quad \text{otherwise} \end{cases}$$

$= H_{eff}(\omega) X_c(\omega)$ (The overall CT system is equivalent to an LTI system with $H_{eff}(\omega)$)

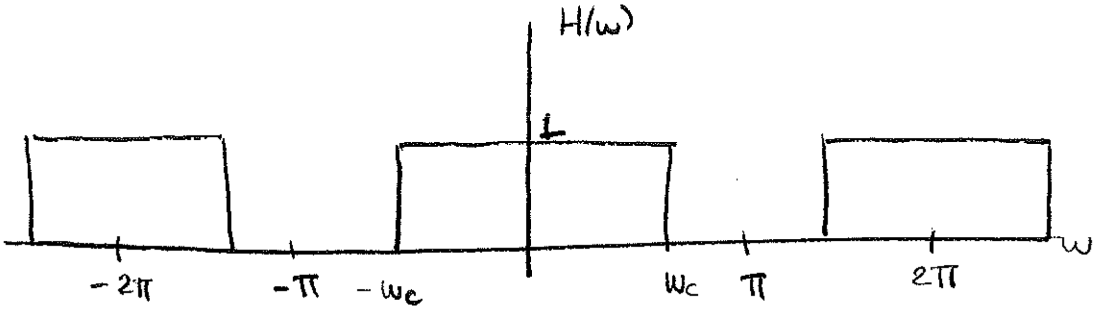
where $$H_{eff}(\omega) = \begin{cases} H(\omega T) & , \quad |\omega| < \frac{\pi}{T} \\ 0 & , \quad \text{else} \end{cases}$$



$H_r(\omega)$ with cutoff at $\frac{\omega_s}{2}$ in continuous time

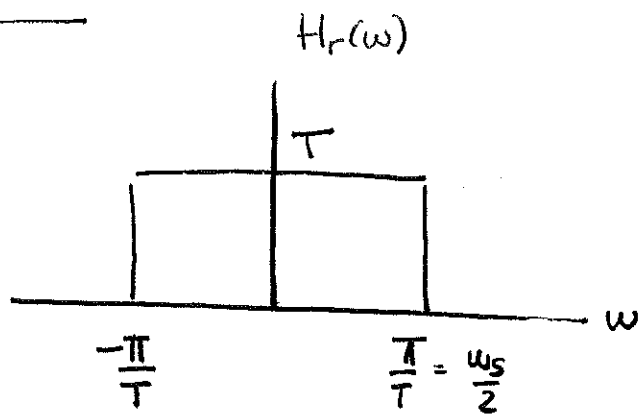
When we multiply $X_d(\omega T)$ with $H_r(\omega)$ we only retain $X_c(\omega)$ since we are above Nyquist rate and $H_r(\omega)$ has cutoff at $\omega_s/2$.

Assume that the frequency response of the discrete-time system is



$$H(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \omega_c \end{cases} \quad \text{(DTFT)}$$

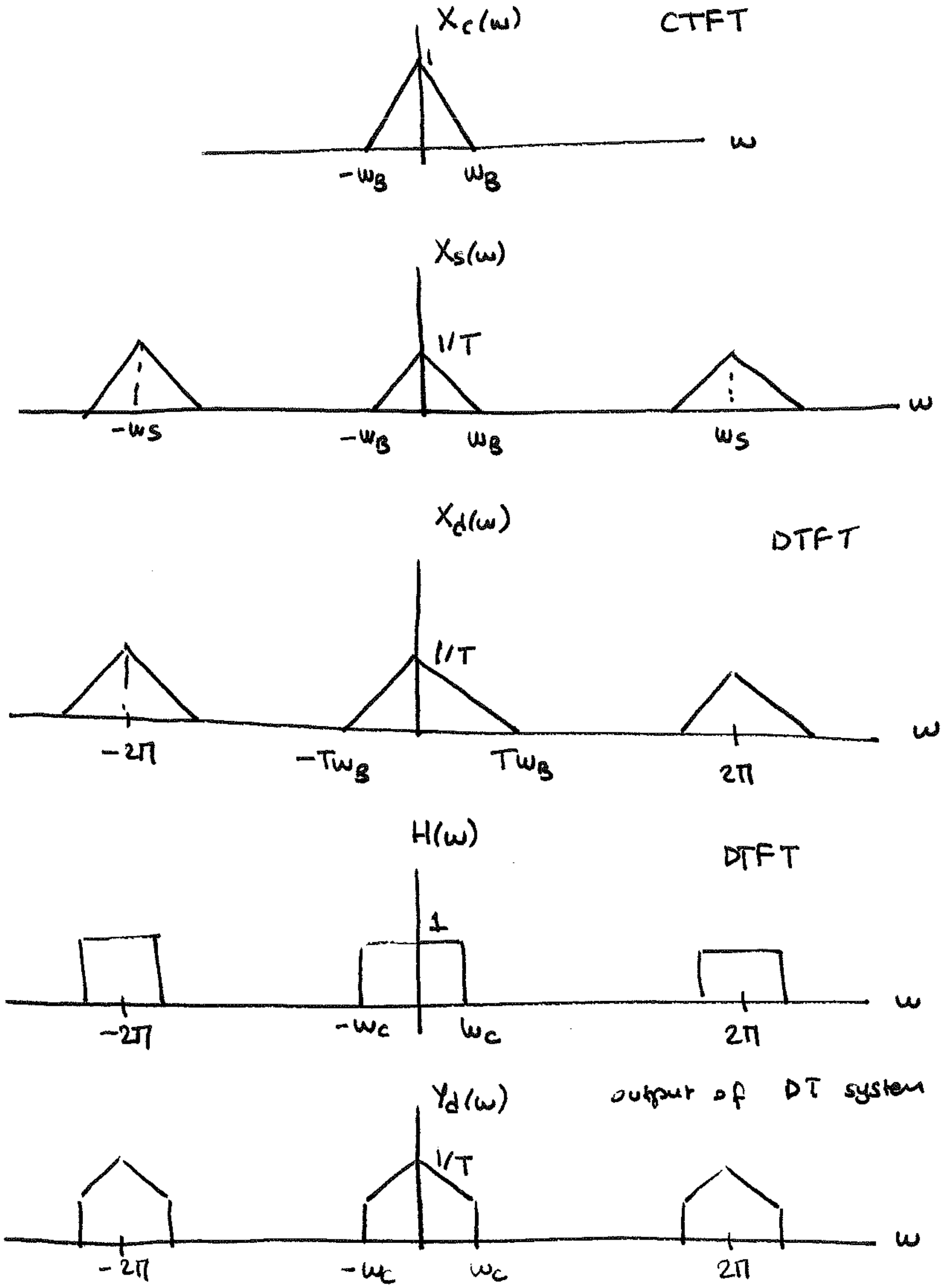
At Nyquist rate



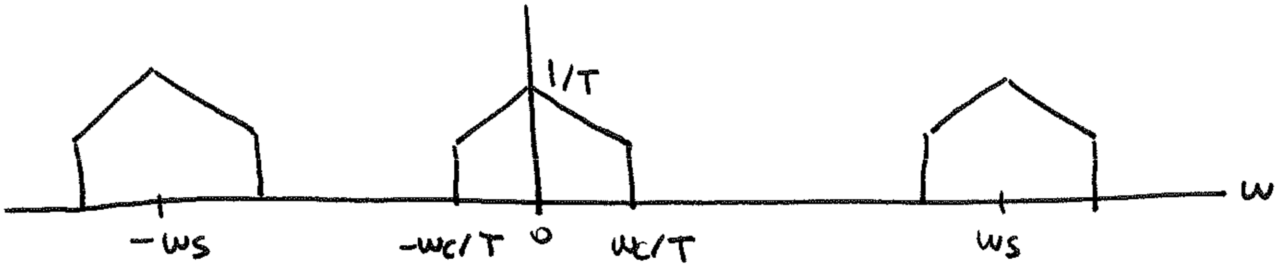
$$H_{eff}(\omega) = \begin{cases} H_r(\omega)H(\omega T), & |\omega| < \frac{\omega_s}{2} = \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} H(\omega T), & |\omega| < \frac{\pi}{T} = \frac{\omega_s}{2} \\ 0, & \text{otherwise} \end{cases}$$

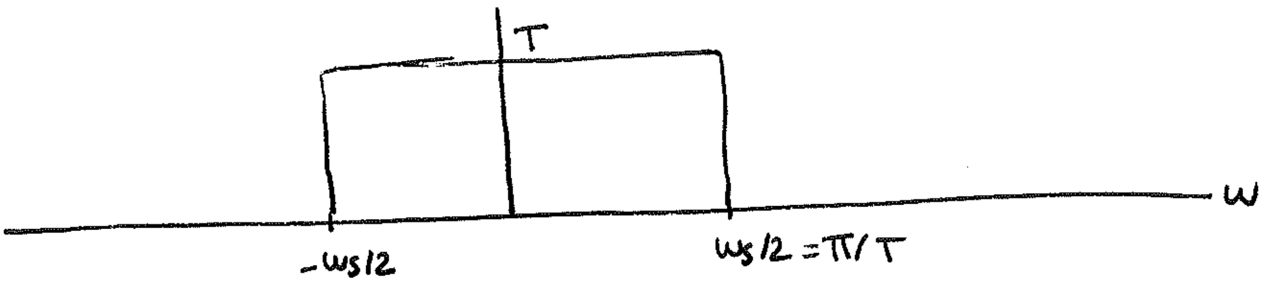
What if $\omega_c < T\omega_B$?



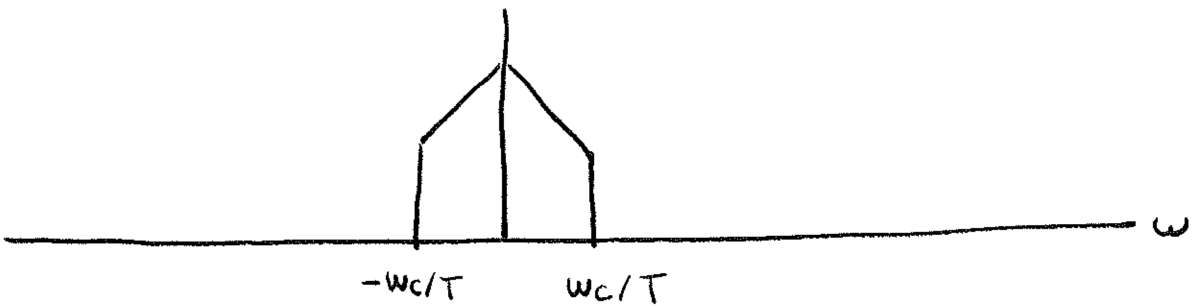
$Y_s(\omega) = Y_d(\omega T)$



$H_r(\omega)$



$Y_r(\omega)$

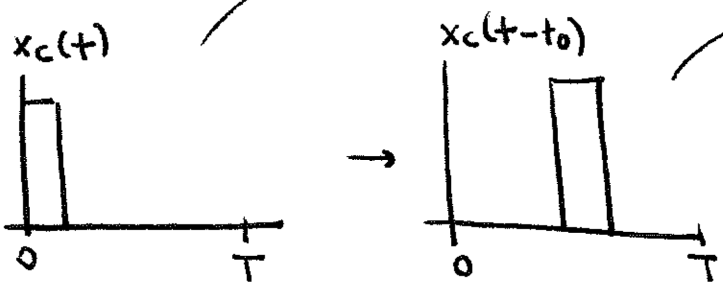


We require the discrete system to be <u>LTI</u>.

Sampler must be above the Nyquist rate (for the input signal)

For example, $x[n] = \delta[n]$ is not bandlimited.

$x[n] = 0$



not time invariant.