09/08/20

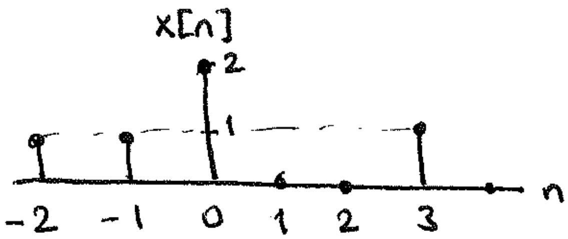
Announcements

- Homework 1 posted (Piazza & MATLAB Grader) (due Sep 14)
- This week is online

Today's Lecture

- Discrete-time systems, properties
- Linear time-invariant (LTI) system properties

Discrete-time signal



$x[n] = \{ \ldots, 0, 1, 1, \underline{2}, 0, 0, 1, 0, \ldots \}$
(n=0)

For example $\delta[n] = \{ \ldots, 0, 0, \underline{1}, 0, 0, \ldots \}$

$u[n] = \{ \ldots, 0, 0, \underline{1}, 1, 1, 1, \ldots \}$

Systems

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] \qquad x \longrightarrow y$$

Input output

H : frequency response

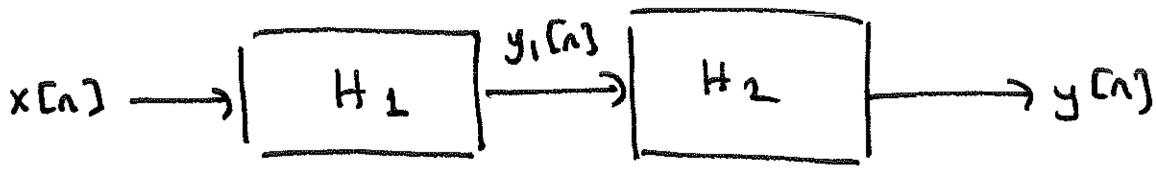
h : impulse response

$$y[n] = H(x[n])$$

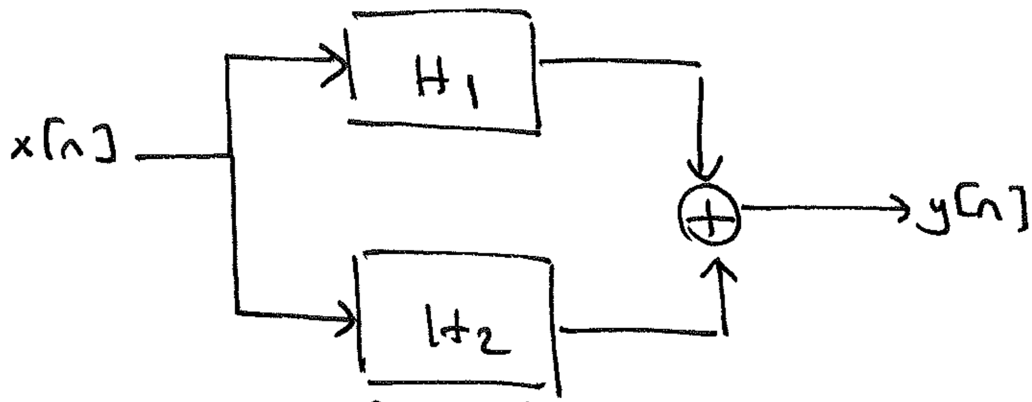
Example : moving average $y[n] = \frac{1}{5} \sum_{k=-2}^{2} x[n+k]$

$\rightarrow$ smooths out input $x[n]$.

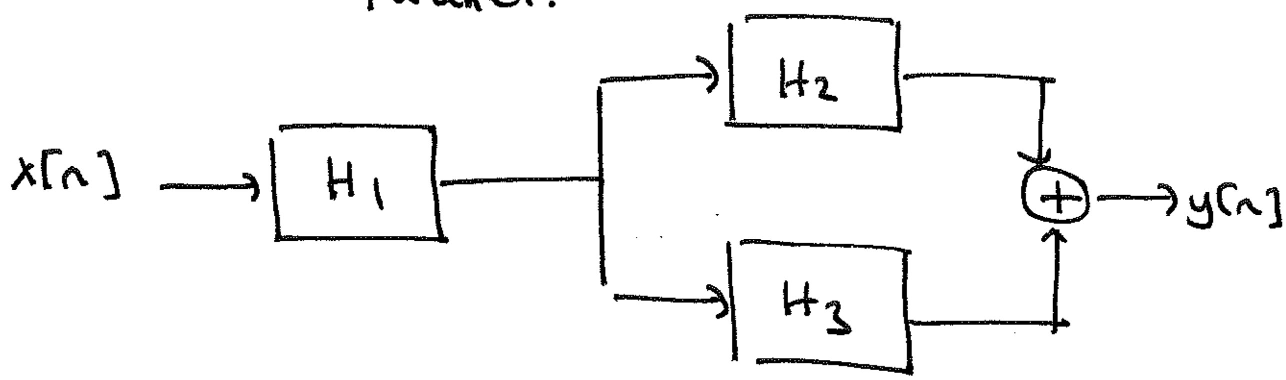
Connecting Systems



Cascade, serial



Parallel.



Properties of Systems

<u>1. Linearity</u> $\quad x_1 \longrightarrow y_1 \quad , \quad x_2 \longrightarrow y_2$

(Superposition) $a x_1[n] + b x_2[n] \longrightarrow a y_1[n] + b y_2[n]$

Additivity: $\quad x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n]$

Homogenity: $\quad \underset{\substack{\downarrow \\ \text{real constant}}}{a} x[n] \longrightarrow a\, y[n]$

Example $y[n] = x[n] - 3x[n-2]$

Is this system linear?

$$x_1[n] \longrightarrow y_1[n] = x_1[n] - 3x_1[n-2]$$

$$x_2[n] \longrightarrow y_2[n] = x_2[n] - 3x_2[n-2]$$

What is the response when $z[n] = ax_1[n] + bx_2[n]$

$$z[n] \longrightarrow \underbrace{z[n]}_{ax_1[n]+bx_2[n]} - 3\underbrace{z[n-2]}_{ax_1[n-2]+bx_2[n-2]}$$

$$= a\underbrace{(x_1[n] - 3x_1[n-2])}_{y_1[n]} + b\underbrace{(x_2[n] - 3x_2[n-2])}_{y_2[n]}$$

$$= ay_1[n] + by_2[n]$$

$\longrightarrow$ It is linear.

2. Causality : A system is causal if the output at time n depends on the inputs up to time n.

Example

$y[n] = x[n-2]$ Yes, causal

$y[n] = x[n+1]$ No.

$y[n] = x[n] - 3x[n-2]$, Yes

} These have memory

Memoryless : A system is memoryless if the output at time n only depends on the input at time n.

$y[n] = 2x[n]$ (memoryless)

$y[n] = \sin(x[n])$ (memoryless)

Example : If a system has <u>memory</u> is it not linear?

$$y[n] = x[n-1] \longrightarrow \text{linear}$$

$$y[n] = (x[n+1])^2 \longrightarrow \text{linear or not ?}$$

$$x_1[n] \longrightarrow y_1[n] = (x_1[n+1])^2$$
$$x_2[n] \longrightarrow y_2[n] = (x_2[n+1])^2$$

$$z[n] = a x_1[n] + b x_2[n] \longrightarrow a y_1[n] + b y_2[n] \ ?$$

$$\begin{aligned} \cancel{z[n]} = \quad y[n] &= (z[n+1])^2 \\ &= (a x_1[n+1] + b x_2[n+1])^2 \\ &= a^2 (x_1[n+1])^2 + 2ab\, x_1[n+1]\, x_2[n+1] \\ &\quad + b^2 (x_2[n+1])^2 \\ &\neq a\,(x_1[n+1])^2 + b\,(x_2[n+1])^2 \end{aligned}$$

<u>has memory & is Nonlinear</u>

Example $\quad y[n] = n\,x[n]$. <u>Is this system linear?</u>

Example inputs

$$x_1[n] = \delta[n] \longrightarrow y_1[n] = n\,\delta[n] = 0 \quad \text{for all } n.$$

$$x_2[n] = \delta[n-1] \longrightarrow y_2[n] = n\,\delta[n-1] = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$x[n] = 5 \longrightarrow y[n] = n.5$$

<u>Exercise</u> : Can you show if this system is <u>linear</u>?
(solve this next lecture)

Example: $y[n] = x[n] + 1$ Is this linear?

$x[n] = 1 \longrightarrow y[n] = 2$

$2x[n] \longrightarrow 2y[n] = 2.2 = 4$ (homogeneity)

but $y[n] = x[n] + 1 = 2 + 1 = 3 \neq 4$ not linear

3. **Time-invariance**: System behaves in the same way regardless of when the input is applied.

$x[n] \longrightarrow y[n]$

$x[n-n_0] \longrightarrow y[n-n_0]$, n_0 an integer.

For example, circuits might have time-varying capacitors, inductors, or internet traffic

Example: $y[n] = x[n] - 2x[n-1]$ Is this system time invariant?

$z[n] = x[n-n_0] \longrightarrow z[n] - 2z[n-1]$

$= x[n-n_0] - 2x[n-n_0-1]$

$y[n-n_0] = x[n-n_0] - 2x[n-n_0-1]$

$x[n-n_0] \longrightarrow y[n-n_0] \Rightarrow$ time invariant.

also linear.

Example: $y[n] = x[n^2]$ Is this time invariant?

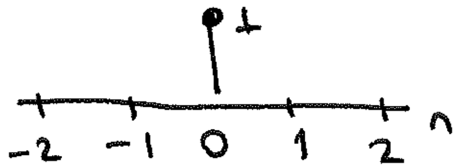
$z[n] = x[n-n_0] \longrightarrow z[n^2] = x[n^2-n_0]$

$y[n-n_0] = x[(n-n_0)^2] \neq x[n^2-n_0]$

time varying!

or we can give a counter example.

$$x[n] = \delta[n] \longrightarrow y[n] = \delta[n] = \delta[n^2]$$



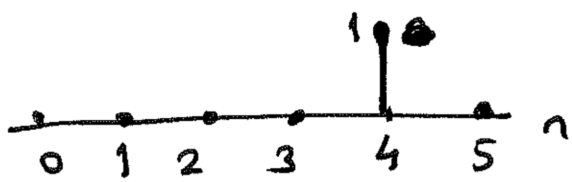
$$z[n] = x[n-4] \longrightarrow y[n] = z[n^2] = \delta[n^2 - 4]$$

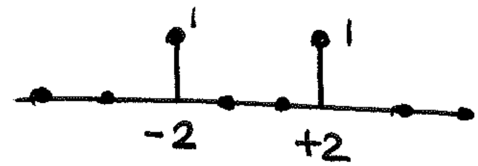
Actual output

$$= \delta[n-4]$$

$$n^2 - 4 = 0$$

$$n = \pm 2$$





$$\neq \delta[n-4] = y[n-4]$$

Linear Time Invariant Systems (LTI)

counter examples / non LTI systems such as nonlinear control, nonlinear optimization, nonlinear quantizers, amplifiers, ADC

Superposition:
$$x_i[n] \longrightarrow y_i[n]$$

$$\sum_i a_i x_i[n] \longrightarrow \sum_i a_i y_i[n]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

$$y[n] = x[n] * \underbrace{h[n]}_{\text{impulse response.}}$$

(only for LTI systems)

LTI system

$$\underset{\text{unit impulse}}{\delta[n]} \longrightarrow \boxed{H} \longrightarrow \underset{\text{impulse function}}{h[n]}$$

What is the response to $x[n]$? (for an LTI system)

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

$$H(\delta[n]) = h[n]$$

$$H(x[n]) = H\left(\sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]\right)$$

$$= \sum_{k=-\infty}^{+\infty} H\big(\underbrace{x[k]}_{a_k}\,\delta[n-k]\big) \quad \text{from linearity}$$

$$= \sum_{k=-\infty}^{+\infty} \overbrace{a_k}^{x[k]}\, \underbrace{H(\delta[n-k])}_{\substack{\shortparallel \\ h[n-k] \text{ from time-invariance}}} \quad \text{from linearity}$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

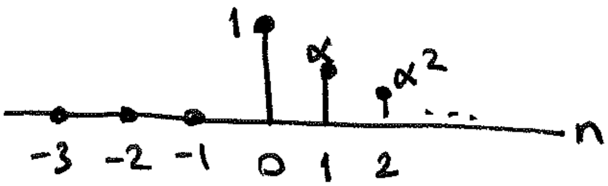
$$= x[n] * h[n]$$

Example: $x[n] = \alpha^n u[n], \quad \alpha \in (0,1)$

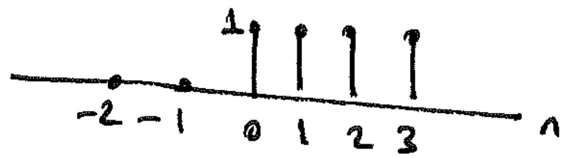
$h[n] = u[n]$ for an LTI system.

$y[n] = ?$ $\quad y[n] = x[n] * h[n]$

$x[n]$



$h[n]$



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

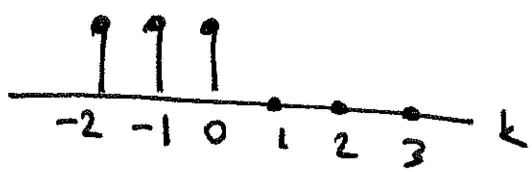
$$= \sum_{k=0}^{+\infty} x[k]\,h[n-k]$$

$$= \sum_{k=0}^{n} x[k] \cdot 1$$

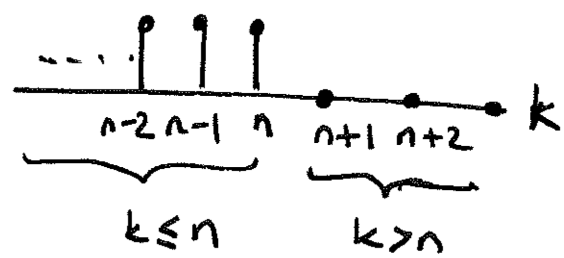
$$= \sum_{k=0}^{n} \alpha^k \quad , \; n \geq 0$$

$$= 1 + \alpha^1 + \dots + \alpha^n$$

$h[-k]$



$h[n-k]$



$$\sum_{k=0}^{n} \alpha^k \quad \text{where} \quad \alpha \in (0,1)$$

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} \cdot u[n]$$