

Hw #5. DSP 11/7/20.

5.) Nyquist Rate.

Consider $x(t) \leftrightarrow X(\omega) = \begin{cases} X(\omega) & \omega < \omega_B \\ 0 & \text{else.} \end{cases}$

a.) $y(t) = x(t) + x(t-2)$

$$x(t-2) \leftrightarrow e^{-j\omega t} X(\omega)$$

↑ phase shift

$$x(t) \leftrightarrow X(\omega)$$

only affect
Amplitude.

$$x(t) + x(t-2) \leftrightarrow X(\omega) (1 + e^{-j\omega t})$$

$$\text{Nyquist Rate } \omega_{nyq} = 2\omega_B \text{ Rad/s}$$

b.) $y(t) = x(2t)$

$$x(2t) \leftrightarrow \frac{1}{2} X\left(\frac{\omega}{2}\right)$$

affect Amp el Band.
down sampling.

$$\text{Nyquist Rate} = 2 \cdot 2 \cdot \omega_B = 4\omega_B \text{ Rad/s.}$$

$$\omega > \omega_B, |\omega| > \frac{\omega_B}{2} = \omega_B$$

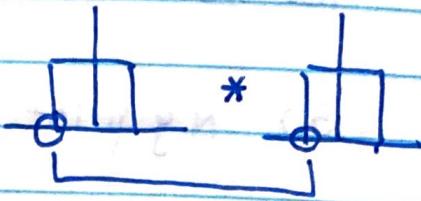
$$\omega = 2\omega_B \text{ (Band-limited freq.)}$$

$$\omega < 1.5\omega_B \text{ (aliasing)}$$

Sampling rate must be at least twice the signal frequency.

$$c.) y(t) = (x(t))^2 = x(t) \cdot x(t)$$

$$Y(w) = \frac{1}{2\pi} \int x(u) x(w-u)$$



$$\text{Nyquist Rate: } 2 \cdot 2w_B = 4w_B \text{ rad/s}$$

$$(s-3)x + (s+3)x = (2s)x$$

$$d.) y(t) = x(t) \cos(\omega_0 t)$$

$$y(t) \leftrightarrow \frac{1}{2\pi} \cdot x(w) * [\pi S(w-\omega_0) + \pi S(w+\omega_0)]$$

$$= \frac{1}{2} [x(w-\omega_0) + x(w+\omega_0)]$$

$$\text{Nyquist Rate} = 2 \cdot (w_B + \omega_0) \text{ rad/s}$$

$$e.) y(t) = x(t) * \frac{1}{t} \sin(\omega_c t)$$

$$x_1 = \frac{1}{t} \sin(\omega_c t) \leftrightarrow X_1(w) = \begin{cases} \pi & w < |w_c| \\ 0 & \text{else.} \end{cases}$$

Nyquist Rate = $2w_c^{\frac{1}{2}}$ rad/s since X_1 attenuate all information outside of $w < |w_c|, |w| < w_B$

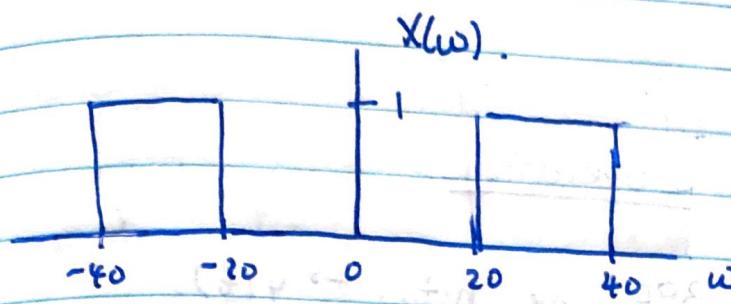
Case 2:

Nyquist Rate = $2w_B$ rad/s since $w < |w_B|, |w_c| > w_B$.

so as a result. Nyquist Rate = $\min\{2w_B, w_c \cdot 2\}$

6.) Aliasing: $X(t)$ has CTFT of

$$X(\omega) = \begin{cases} 1 & -20 \leq |\omega| \leq 40 \\ 0 & \text{else.} \end{cases}$$



a.) What is the Nyquist Rate?

$$2W_B = 2 \cdot 40 = 80 \text{ rad/s.}$$

b.) What if $x[n] = x(t \frac{2\pi}{20})$? $X(\omega/\tau)$

$$\tau = \frac{2\pi}{20}$$

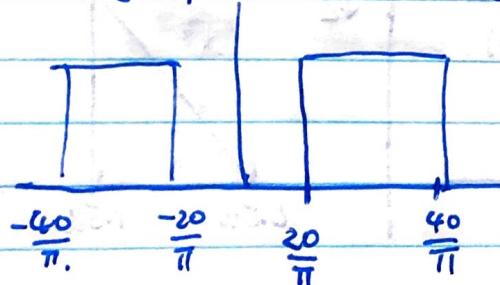
$$f = 20/2\pi$$

$$W_s < 2W_B$$

$\omega_s = 20$ rad/s there will be aliasing by the theorem, but to double check $X(\frac{\omega}{\tau})$

$$x(t \frac{2\pi}{20}) \Rightarrow$$

down sampling.



it's actually not aliasing, so the Conclusion here is no aliasing.

7.) Filtering and Sampling.

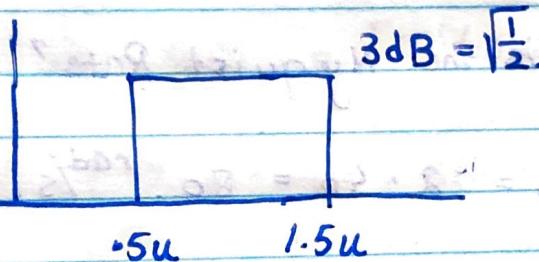
$x(t)$ is a real signal.

$X(\omega)$ is known

$$u > 0.$$

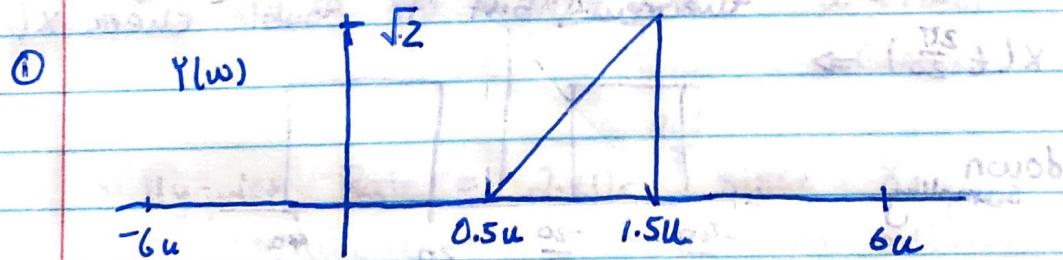
We have 2 operations:

- ①. $x(t)$ is BPF by $h(t)$. $\rightarrow y(t)$.



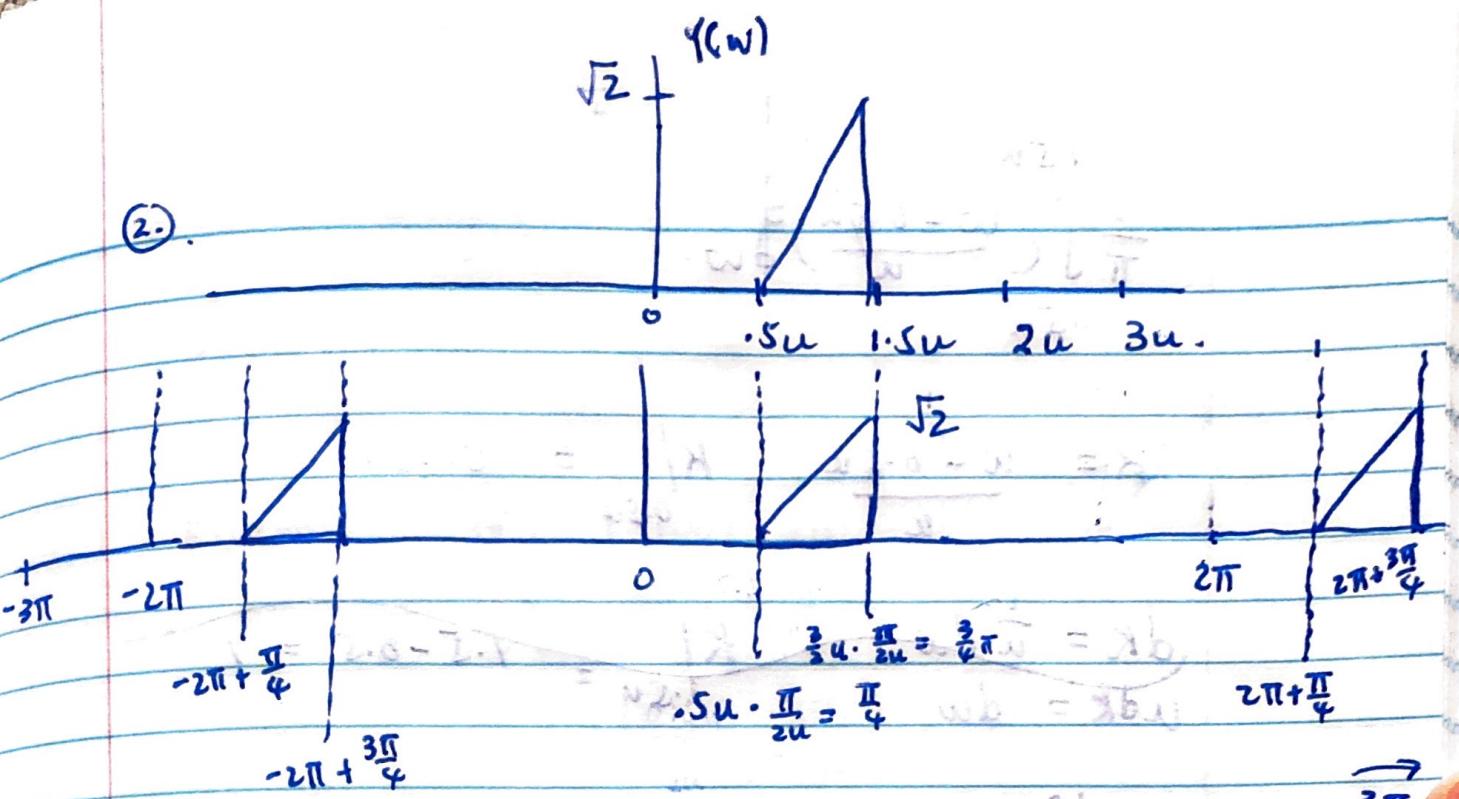
- ②. Sample the Result by $T = \frac{\pi}{2u}$, $y(n) =$

$$y(t) = x(t) * h(t) \leftrightarrow Y(\omega) = X(\omega) H(\omega).$$

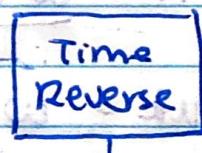
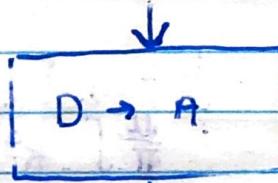


$$\textcircled{2} \quad y[n] = y\left(t \frac{\pi}{2u}\right) \leftrightarrow \text{DTFT}\{y(n)\} = \frac{1}{T} Y\left(\frac{\omega}{T}\right)$$

2.



c.) $y[n] \rightarrow H(w) = \begin{cases} \frac{1}{\sqrt{2}} & \frac{\pi}{4} < w < \frac{3\pi}{4} \\ 0 & \text{elsewhere} \end{cases}$



$x(t)$

d.) $\int |x(t)|^2 dt = \frac{1}{2\pi} \int |x(w)|^2 dw$

$$= \frac{1}{2\pi} \int_{-1.5u}^{0.5u} \left(\frac{1}{u}(w) - 0.5 \right)^2 dw$$

$$= \frac{1}{\pi} \int_{+1.5u}^{-0.5u} \left(\frac{w-0.5u}{u} \right)^2 dw$$

$$\frac{1}{\pi} \int_{-0.5u}^{1.5u} \left(\frac{w - 0.5u}{u} \right)^2 dw.$$

$$K = \frac{w - 0.5u}{u} \quad K \Big|_{0.5u} = 0.5.$$

$$dK = \frac{1}{u} dw \quad K \Big|_{1.5u} = 1.5 - 0.5 = 1$$

$$u dk = dw$$

$$\frac{1}{\pi} \int_{-0.5u}^{1.5u} \left(\frac{w}{u} \right)^2 - 2 \frac{w}{u} \cdot \frac{1}{2} + 0.25 dw.$$

$$-\frac{1}{\pi} \int_{-0.5u}^{1.5u} \left(\frac{w - 0.5u}{u} \right)^2 dw$$

$$K = \frac{w - 0.5u}{u} \quad \frac{1}{\pi} \int_0^u k^2 dk$$

$$uk = w - 0.5u = u \frac{1}{3} k^3 \Big|_0^1$$

$$u dk = dw$$

$$K(w=0.5u) = 0$$

$$K(w=1.5u) = 1$$

$$\boxed{\frac{u}{3\pi}}$$

Energy is $\frac{u}{3\pi}$.

energy of $y(t)$.

$$y(t) \leftrightarrow Y(\omega)$$

$$\text{Energy.} : \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega.$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega) \cdot X(\omega)|^2 d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (\sqrt{2})^2 |X(\omega)|^2 d\omega$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$\Rightarrow \boxed{\frac{u}{3\pi}}$$

8.) We know there's a signal $x[n]$

We want $x[n-k]$

$$k \neq \mathbb{Z}$$

What is the ideal freq. Response
of this filter?

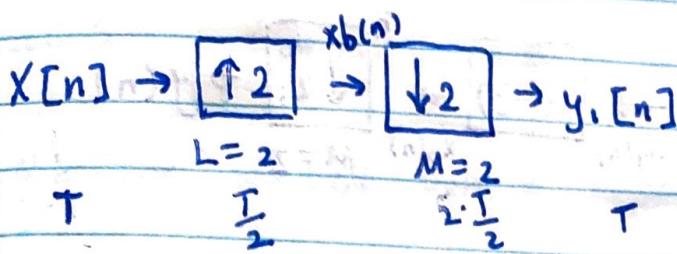
$$x[n] \leftrightarrow x(\omega)$$
$$x[n-k] \leftrightarrow e^{-jk\omega} x(\omega)$$

$$h_k[n] \leftrightarrow \boxed{e^{-jk\omega}} = H(\omega)$$

in a sense:

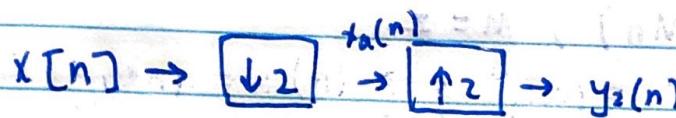
$$x[n] \rightarrow \boxed{h_k[n]} \rightarrow x[n-k]$$

9.1 Downsampling + Upsampling.



$$X_b(n) = X\left[\frac{n}{L}\right] \iff X_b(w) = X(wL)$$

$$\begin{aligned} Y_1(n) &= X_b(Mn) \iff Y_1(w) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(\frac{wL}{M} - \frac{2\pi i}{M}\right) \\ &= \frac{1}{M} \sum_{i=0}^{M-1} X(w - \pi i) \\ &= \frac{1}{M} (X(w) + X(w - 2\pi)) \end{aligned}$$

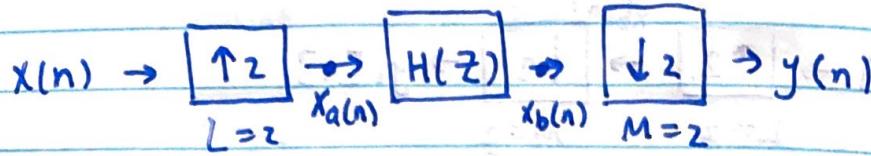


$$\begin{aligned} X_a[n] &= X[Mn] \iff X_a(w) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(\frac{w}{M} - \frac{2\pi i}{M}\right) \\ &= \frac{1}{2} \sum_{i=0}^1 X\left(\frac{w}{2} - \pi i\right) \\ &= \frac{1}{2} (X(\frac{w}{2}) + X(\frac{w}{2} - \pi)) \end{aligned}$$

$$\begin{aligned} Y_2(n) &= X_a[n/L] \iff Y_2(w) = X_a(Lw) \\ &= \frac{1}{2} (X(w) + X(w - \pi)) \end{aligned}$$

$$\therefore Y_2(w) \neq Y_1(w).$$

now:



$$X_a(n) = X\left(\frac{\omega}{L}\right) \text{ for } n = \pm L, \pm 2L, \pm 3L, \dots$$

$$X_a(\omega) = X(L\omega), \quad L = 2.$$

$$X_b(n) = X_a(n) * h(n) \leftrightarrow X_a(\omega) \cdot H(\omega)$$

$$\begin{aligned} X_b(\omega) &= X(L\omega)H(\omega) \\ &= X(2\omega)H(\omega) \end{aligned}$$

$$y(n) = X_b(Mn), \quad M = 2.$$

$$Y(\omega) = \frac{1}{M} \sum_{i=0}^{M-1} X_b\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right).$$

$$= \frac{1}{2} \sum_{i=0}^1 X_b\left(\frac{\omega}{2} - \frac{2\pi i}{2}\right)$$

$$= \frac{1}{2} (X_b\left(\frac{\omega}{2}\right) + X_b\left(\frac{\omega}{2} - \pi\right)) \quad \begin{array}{l} \text{in DTFT,} \\ \text{signal is always} \\ \text{periodic} \end{array}$$

$$= \frac{1}{2} (X(\omega)H\left(\frac{\omega}{2}\right) + X(\omega - 2\pi)H\left(\frac{\omega}{2} - \pi\right))$$

$$= \frac{1}{2} (X(\omega)H\left(\frac{\omega}{2}\right) + X(\omega)H\left(\frac{\omega}{2} - \pi\right))$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{2} \left(H\left(\frac{\omega}{2}\right) + H\left(\frac{\omega}{2} - \pi\right) \right) \underbrace{H\left(\frac{1}{2}(\omega - 2\pi)\right)}$$

$$z = e^{j\left(\frac{\omega}{2} - \pi\right)}$$

$$= e^{j\frac{\omega}{2}} \cdot e^{-j\pi}$$

$$-z = e^{j\frac{\omega}{2}} \quad \widehat{Y(z)} = \frac{1}{2} \left(H\left(\frac{z}{2}\right) + H\left(-\frac{z}{2}\right) \right)$$

10.7 Polyphase Signal Processing.

Consider transfer function $H(z)$.

a.) Find $H_e(z)$ where $h_e(n) = h(2n)$.

$$h(n) \rightarrow \boxed{\downarrow 2} \rightarrow h_e(n)$$

$M=2$.

$$\begin{aligned} h_e(n) = h(2n) &\leftrightarrow \frac{1}{2} \sum_{i=0}^{2-1} H\left(\frac{\omega}{2} - 2\pi i/\omega\right) \\ &= \frac{1}{2} \left(H\left(\frac{\omega}{2}\right) + H\left(\frac{\omega}{2} - \pi\right) \right) \\ \underline{H_e(\omega)} &= \frac{1}{2} \left(H\left(\frac{\omega}{2}\right) + H\left(-\frac{\omega}{2}\right) \right) \end{aligned}$$

$$h_e(n) = h(2n) \leftrightarrow H_e(z) = \sum_{n=0}^{\infty} h(2n) z^{-n}, |z| \neq 0$$

b.) Find $H_o(z)$ where $h_o(n) = h(2n+1)$

$$H_o(z) = \sum_{n=0}^{\infty} h(2n+1) z^{-n}, |z| \neq 0.$$

$$\begin{aligned} c.) H(z) &= \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} h(2n) z^{-n} + \sum_{n=0}^{\infty} h(2n+1) z^{-n} \\ &= \sum_{n=0}^{\infty} h(2n) z^{-2n} + \sum_{n=0}^{\infty} h(2n+1) z^{-(2n+1)} \\ &= \sum_{n=0}^{\infty} h(2n) (z^2)^{-n} + \sum_{n=0}^{\infty} h(2n+1) (z^2)^{-n-1} \\ &= \sum_{n=0}^{\infty} h(2n) (z^2)^{-n} + z^{-1} \sum_{n=0}^{\infty} h(2n+1) (z^2)^{-n} \\ &= H_e(z^2) + z^{-1} H_o(z^2) \end{aligned}$$

$|z| > 0$

$$\text{d.) } H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} h_1(M_n)(z^M)^{-n} + \sum_{n=0}^{\infty} h_2(M_n+1)(z^M)^{-n} \cdot z^{-1}$$

$$+ \sum_{n=0}^{\infty} h_3(M_n+2)(z^M)^{-n} z^{-2} + \dots$$

$$= H_1(z^M) + z^{-1} H_2(z^M) + z^{-2} H_3(z^M) + \dots$$

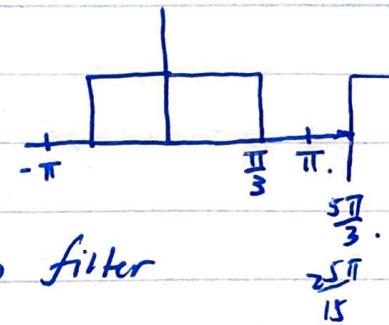
$$+ z^{-(M-1)} H_M(z^M)$$

$$H(z) = \sum_{n=0}^{\infty} h_M(M_n+M-1) z^{-n}.$$

11.) FIR filter Design.

Design an FIR filter w/ linear phase
that approximates the ideal freq. Resp.

$$H_d(\omega) = \begin{cases} 1 & , \quad |\omega| < \frac{\pi}{3} \\ 0 & , \quad \pi \geq |\omega| \geq \frac{\pi}{3}. \end{cases}$$



a.) Determine the coefficient of a 15 tap filter
Based on the IDFT method.

$$N = 15$$

I assume that
 $h[n]$ is Real and

Equally Spaced Value for $A_d(w)$

exhibit symmetry about
 $M = \frac{N-1}{2}$ sample

$$\omega = \frac{2\pi k}{N} = \frac{2\pi k}{15}, \{k=0, \dots, N-1\}$$

$$A_d = \begin{bmatrix} 1 & & & & 0 \\ & \frac{2\pi k}{15} < \frac{\pi}{3} = \frac{5\pi}{15} & & & \frac{2\pi k}{15} > \frac{\pi}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{3}{15} & \frac{4}{15} & \frac{6}{15} & \frac{8}{15} & \frac{10}{15} & \frac{12}{15} & \frac{14}{15} & \frac{16}{15} & \frac{18}{15} & \frac{20}{15} & \frac{22}{15} & \frac{24}{15} & \frac{26}{15} \end{bmatrix}$$

$$h[n] = \frac{1}{15} \sum_{k=0}^{14} A[0] + \sum_{k=1}^M 2A[k] \cos \frac{2\pi(n-k)k}{N}$$

$$h[n] = \frac{1}{15} \left[A[0] + \sum_{k=1}^M 2A[k] \cos\left(\frac{2\pi(n-k)}{N} k\right) \right]$$

$$M = N - 1 = 15 - 1 = 7.$$

$$N = 15$$

$$\begin{aligned} h[14] &= h[0] = \frac{1}{15} \left[1 + \sum_{k=1}^7 2A[k] \cos\left(\frac{2\pi(0-k)}{15} k\right) \right] \\ &= \frac{1}{15} \left[1 + \sum_{k=1}^7 2 \cos\left(\frac{-14\pi}{15} k\right) \right] \\ &= \frac{1}{15} \left[1 + 2 \cos\left(-\frac{14}{15}\pi\right) + 2 \cos\left(-\frac{28}{15}\pi\right) \right] \end{aligned}$$

$$h[13] = h[1] = \frac{1}{15} \left[1 + 2 \cos\left(-\frac{12}{15}\pi\right) + 2 \cos\left(-\frac{24}{15}\pi\right) \right]$$

$$h[12] = h[2] = \frac{1}{15} \left[1 + 2 \cos\left(-\frac{10}{15}\pi\right) + 2 \cos\left(-\frac{20}{15}\pi\right) \right]$$

$$h[11] = h[3] = \frac{1}{15} \left[1 + 2 \cos\left(-\frac{8}{15}\pi\right) + 2 \cos\left(-\frac{16}{15}\pi\right) \right]$$

$$h[10] = h[4] = \frac{1}{15} \left[1 + 2 \cos\left(-\frac{6}{15}\pi\right) + 2 \cos\left(-\frac{12}{15}\pi\right) \right]$$

$$h[9] = h[5] = \frac{1}{15} \left[1 + 2 \cos\left(-\frac{4}{15}\pi\right) + 2 \cos\left(-\frac{8}{15}\pi\right) \right]$$

$$h[8] = h[6] = \frac{1}{15} \left[1 + 2 \cos\left(-\frac{2}{15}\pi\right) + 2 \cos\left(-\frac{4}{15}\pi\right) \right]$$

$$h[7] = h[7] = \frac{1}{15} \left[1 + 2 \cos(0) + 2 \cos(0) \right] = \frac{1}{3}.$$