9/21/20

Today's Lecture

- Examples on Frequency Response
- Discrete-time Fourier Transform (DTFT)

Last lecture

$$e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix} LTI \\ h[n] \end{matrix}} \longrightarrow H(\omega_0)\, e^{j\omega_0 n}$$

complex exponential

eigenvalue   eigenfunction.

* Complex exponentials are eigenfunctions of LTI systems.

$$y[n] = x[n] * h[n]$$

$$x[n] = e^{j\omega_0 n}$$

$$y[n] = H(\omega_0)\, e^{j\omega_0 n}$$

* The output $y[n]$ has the <u>same frequency</u> ω_0 as the input.

* Only the <u>magnitude and phase</u> are changed.

Frequency Response Interpretation

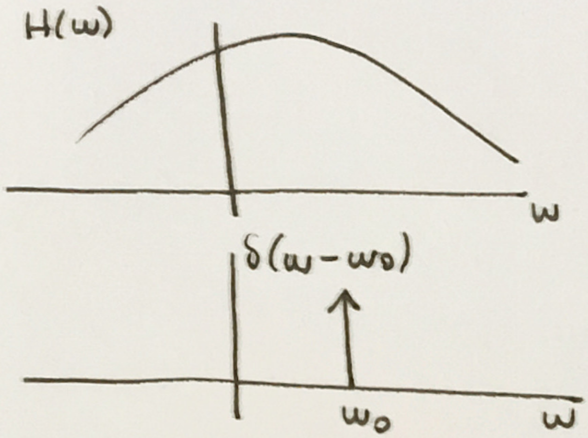
Let's consider CT systems.

$$x(t) = e^{j\omega_0 t} \xrightarrow{F} X(\omega) = 2\pi\delta(\omega-\omega_0)$$

$$x(t) \longrightarrow \boxed{\begin{matrix}LTI\\ h(t)\end{matrix}} \longrightarrow y(t)$$

$$Y(\omega) = X(\omega)H(\omega) = 2\pi\delta(\omega-\omega_0)H(\omega)$$

$$= 2\pi\delta(\omega-\omega_0)\underbrace{H(\omega_0)}_{\text{some constant}}$$

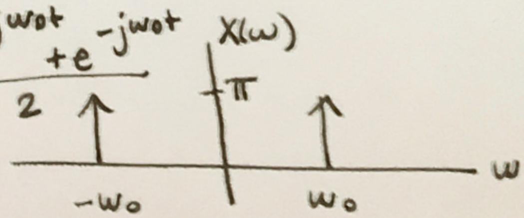


$y(t) = H(\omega_0)e^{j\omega_0 t}$

has the same fundamental frequency (we are not introducing new frequencies)

Example

$$x(t) = \cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$



Assume an LTI system

$h(t)$ is real

$\Rightarrow H(\omega) = H^*(-\omega)$

$$Y(\omega) = X(\omega)H(\omega)$$

$$= \pi(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))H(\omega)$$

$$= \pi(\delta(\omega+\omega_0)\underbrace{H(-\omega_0)}_{H^*(\omega_0)} + \delta(\omega-\omega_0)H(\omega_0))$$

$$Y(\omega) = \pi\left(\delta(\omega+\omega_0)\,|H(\omega_0)|\,e^{-j\angle H(\omega_0)} + \delta(\omega-\omega_0)\,|H(\omega_0)|\,e^{j\angle H(\omega_0)}\right)$$

$$e^{-j\omega_0 t} \xrightarrow{F} 2\pi\delta(\omega+\omega_0)$$

$$e^{j\omega_0 t} \xrightarrow{F} 2\pi\delta(\omega-\omega_0)$$

$$y(t) = \left(\frac{1}{2}e^{-(j\omega_0 t + j\angle H(\omega_0))} + \frac{1}{2}e^{(j\omega_0 t + j\angle H(\omega_0))}\right)|H(\omega_0)|$$

$$= \cos(\omega_0 t + \angle H(\omega_0)) \cdot |H(\omega_0)|$$

Example A Discrete Time system

$$\cos 2n \longrightarrow \boxed{\text{LTI}} \longrightarrow \cos 3n$$

$$\frac{e^{j2n} + e^{-j2n}}{2}$$

Is this possible?

NO!

For LTI systems we do not introduce new frequencies.

Filter interpretation: We often interpret '$H(\omega)$' as what the system does to frequencies of the input signal.

Discrete-time Fourier Transform (DTFT)

For the continuous time Fourier transform

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt$$

Frequency range $\omega \in (-\infty, \infty)$

Discrete-time Fourier transform

$$X(\omega) = \underbrace{\sum_{n=-\infty}^{+\infty}}_{\text{DT}} x[n] e^{-j \underbrace{\omega}_{\text{continuous}} \overbrace{n}^{\text{integer}}}$$

Frequency range $\omega \in [-\pi, \pi]$

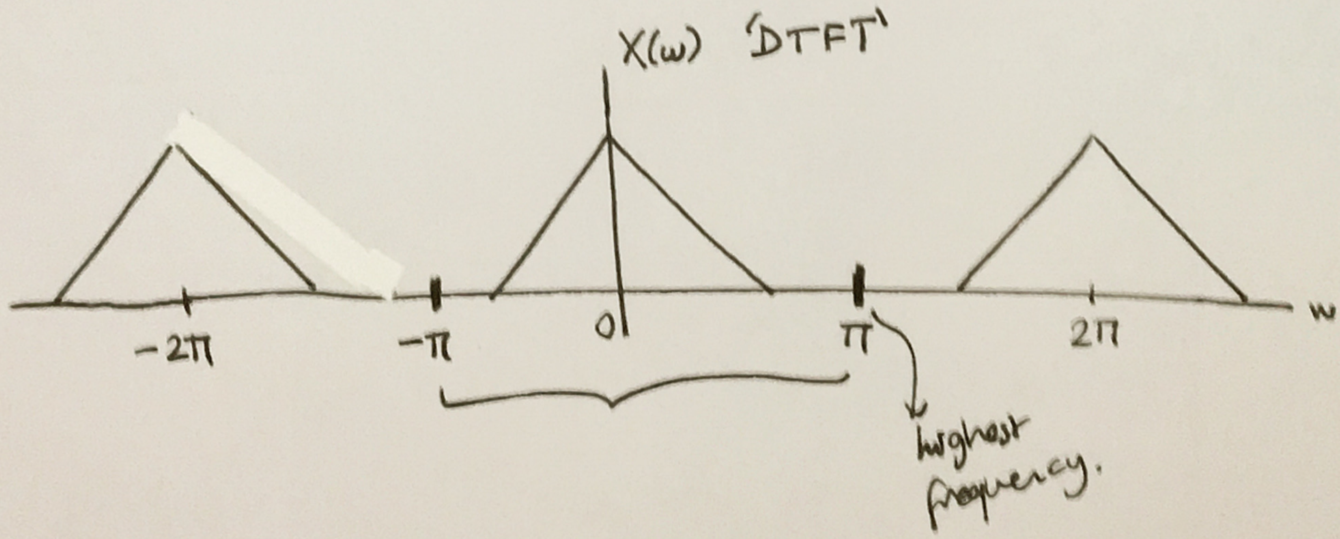
Q. Why do we have a fixed set of frequencies in discrete time?

$$X(\omega + 2\pi) = \sum_{n=-\infty}^{+\infty} x[n] \, e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \, e^{-j\omega n} \cdot \underbrace{e^{-j2\pi n}}_{1 \cdot e^{j \angle \text{Phase}}} \qquad (n \text{ integer})$$

therefore, $= X(\omega)$ for all ω.

$\therefore X(\omega)$ is periodic. Period is $\underline{2\pi}$.



If $x[n]$ is real

$$X(w) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} \qquad X^*(w) = \sum_{n=-\infty}^{+\infty} x^*[n] e^{jwn}$$

DTFT

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{jwn}$$

$$X^*(-w) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} = X(w)$$

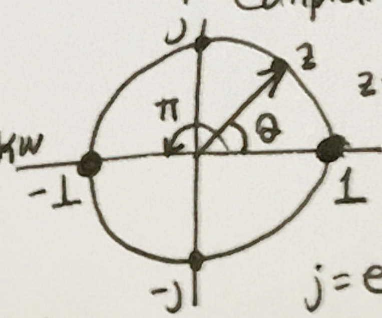
[Other notations $X(jw)$, $X(e^{jw})$]

* What is the highest frequency (w) in a discrete time signal?

$$w = \pi \longrightarrow f = \frac{w}{2\pi} = \frac{1}{2}$$

$$e^{j\pi n} = \cos \pi n + j \underbrace{\sin \pi n}_{0} = (-1)^n$$

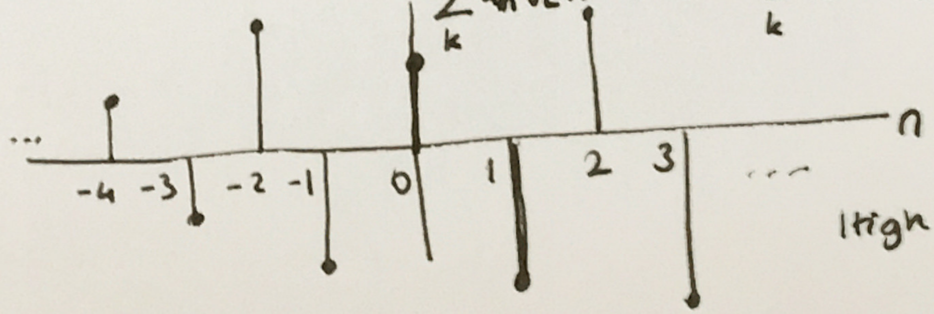
Complex plane



$z = e^{j\theta}$

$j = e^{j\frac{\pi}{2}}$

$$\sum_k a_k \delta[n-k] \xleftrightarrow{FT} \sum_k a_k e^{-jkw}$$



High frequency signal.

<u>Inverse DTFT</u> (periodic with 2π)

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{X(w)}_{\text{DTFT}} e^{jwn} \, dw$$

Try to show this (exercise)!

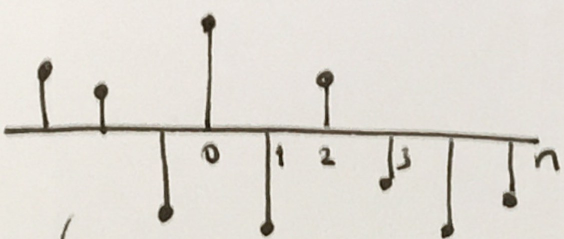
Recall the Fourier Series expansion for a periodic signal (with period T)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j\frac{2\pi}{T}kt} \quad , \quad a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi}{T}kt}\, dt$$

(sign flip)

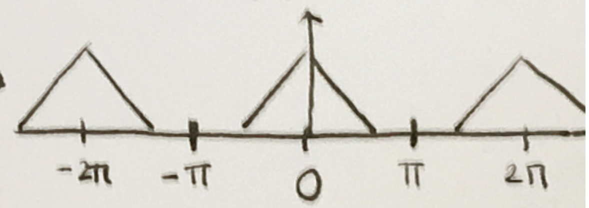
DTFT is very similar to a FS expansion for a signal with period $T = 2\pi$.

x[n] DT



$\xleftrightarrow{\text{DTFT}}$

'X(w)'

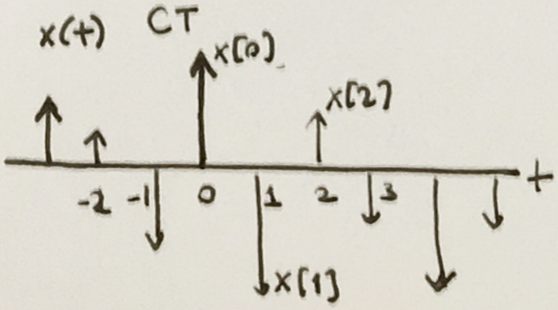


Periodic with 2π

a_k?

FS

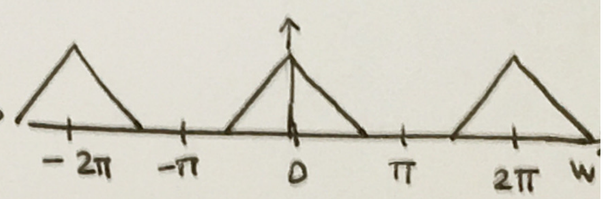
x(t) CT



Weighted impulse train

$\xleftrightarrow{\text{CTFT}}$

X(w)



Periodic in w since we have an impulse train

$\delta(t-n) \xleftrightarrow{\text{CTFT}} e^{-jwn}$ (continuous time)

$\delta(t-t_0) \longleftrightarrow e^{-jwt_0}$

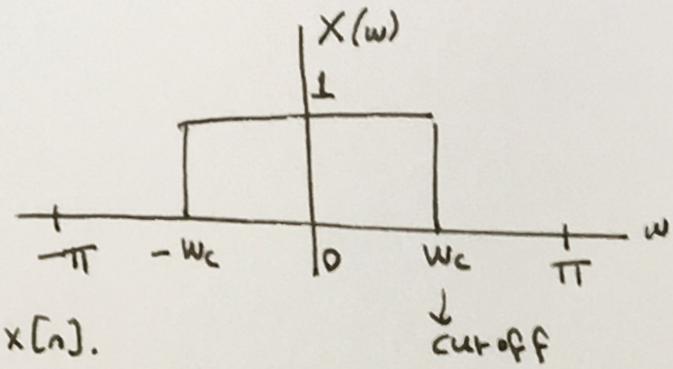
Continuous time

$$\sum_{n=-\infty}^{+\infty} x[n]\delta(t-n) \xleftrightarrow{\text{CTFT}} \underbrace{\sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}}_{\text{DTFT}}$$

constants

	Continuous Time	Discrete Time
Periodic	Fourier Series	? → Discrete Fourier Transform (DT Fourier Series)
Non-periodic (in time domain)	Fourier Transform (CTFT)	Discrete Time Fourier Transform (always periodic in frequency domain)

Example Consider the DTFT $X(\omega)$ (of a DT signal $x[n]$) given as below:

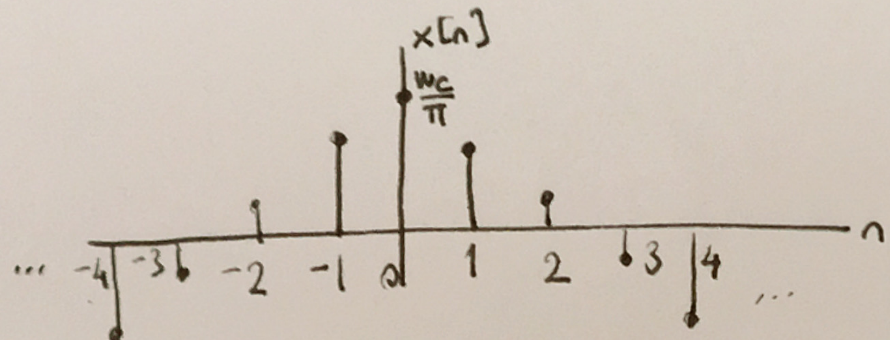


Determine $x[n]$.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n}\,d\omega$$

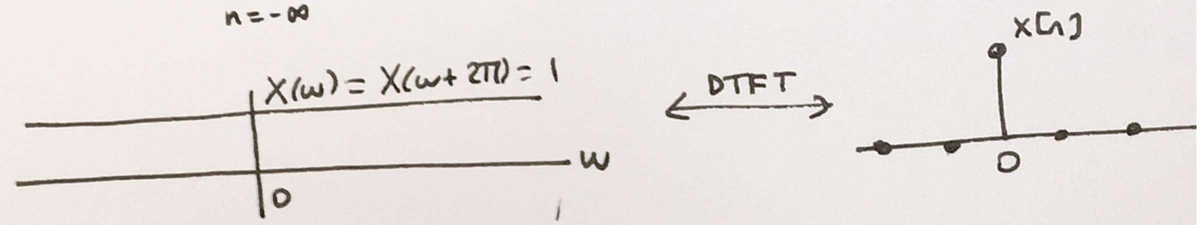
$$= \frac{1}{2\pi}\cdot\frac{1}{jn}\cdot\left(e^{j\omega_c n} - e^{-j\omega_c n}\right) = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi}\,\text{sinc}(\omega_c n)$$

$\text{sinc}(\omega) = \frac{\sin \omega}{\omega}$

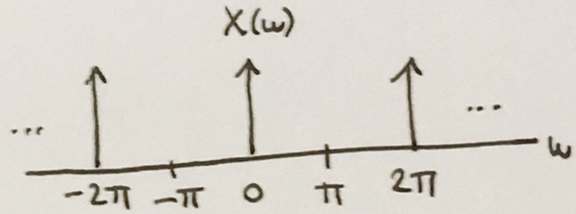


Example $x[n] = \delta[n]$ DTFT $X(\omega)$?

$$X(\omega) = \sum_{n=-\infty}^{+\infty} \delta[n] e^{-j\omega n} = 1$$

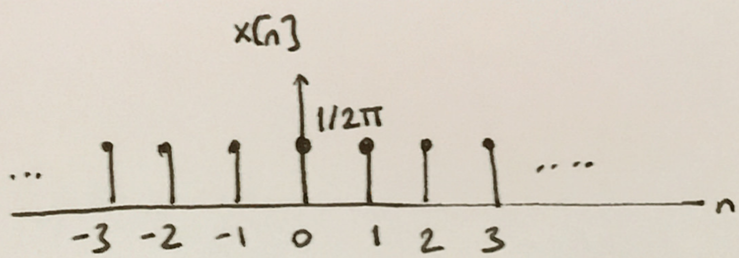


Example $X(\omega) = \delta(\omega)$ for DT $x[n]$ $x[n]$?



Always periodic in frequency domain

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} \, d\omega = \frac{1}{2\pi}$$



Note that the signal is constant in discrete time