

Today's lecture

- Circular convolution examples / properties
- FFT
- Decimation in time FFT
- Decimation in frequency FFT
- Radix-2 FFT and generalizations

Last lecture's example: 2 length 4 signals: $\{x[0], x[1], x[2], x[3]\}$
 $\{h[0], h[1], h[2], h[3]\}$

Linear convolution: $\sum_{k=0}^3 x[k]h[n-k] \rightarrow$ this has length $4+4-1=7$

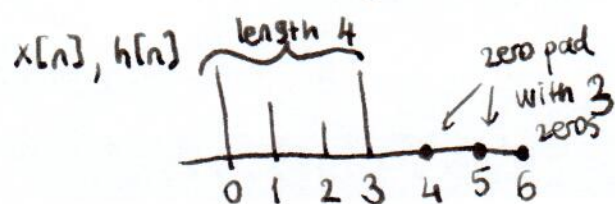
Circular convolution: (length 7)
 $y[n] = \sum_{k=0}^6 x[k]h_7[n-k]$
 zero pad with 3 zeros, i.e. $h[6]=h[5]=h[4]=0$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 & 1 & h[3] & h[2] & h[1] \\ h[1] & h[0] & 0 & 0 & 1 & 0 & h[3] & h[2] \\ h[2] & h[1] & h[0] & 0 & 1 & 0 & 0 & h[3] \\ h[3] & h[2] & h[1] & h[0] & 1 & 0 & 0 & 0 \\ 0 & h[3] & h[2] & h[1] & 1 & h[0] & 0 & 0 \\ 0 & 0 & h[3] & h[2] & 1 & h[1] & h[0] & 0 \\ 0 & 0 & 0 & h[3] & 1 & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \end{bmatrix} = 0$$

7x1 vector 7x7 circulant matrix 7x1 vector

= Linear convolution

(for this specific example)



Multiply their 7 point DFTs

7 point IDFT $\rightarrow x[n] * h[n]$ linear conv.

Time shift (length N)

$$x_N[n-m] \xleftrightarrow{\text{DFT}} W_N^{km} X[k]$$

$$\sum_{n=0}^{N-1} x_N[n-m] W_N^{kn} = \sum_{l=-m}^{N-1-m} x_N[l] W_N^{k(l+m)}$$

$$l = n - m$$

$$= W_N^{km} \sum_{l=-m}^{N-1-m} x_N[l] W_N^{kl}$$

$$\text{(redundant)} = W_N^{km} \sum_{l=-m}^{N-1-m} x_N[l] W_N^{(k+N)l}$$

$$= W_N^{km} \underbrace{\sum_{l=0}^{N-1} x_N[l] W_N^{kl}}_{X[k]}$$

Piazza example on circular convolution

$$x[n]: x[0]=1, x[1]=2, x[2]=3, h[n]: h[0]=h[1]=h[2]=1/3$$

A. Linear convolution (conv)

$$\sum_{k=0}^2 x[k] h[n-k] = h[n] + 2h[n-1] + 3h[n-2]$$

$$y[0]=1/3 \quad y[1]=1 \quad y[2]=2 \quad y[3]=5/3 \quad y[4]=1 \quad (\text{length } 5)$$

B. $N=3$ circular (ccconv)

$$\sum_{k=0}^2 x[k] h_3[n-k]$$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \end{bmatrix} = \begin{bmatrix} h[0] & h[2] & h[1] \\ h[1] & h[0] & h[2] \\ h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

C. $N=5$ circular

$$\sum_{k=0}^4 x[k] h_5[n-k]$$

(zero pad with 2 zeros)
i.e. $h[3]=h[4]=0$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} h[0] & h[4] & h[3] \\ h[1] & h[0] & h[4] \\ h[2] & h[1] & h[0] \\ h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 2 \\ 5/3 \\ 1 \end{bmatrix}$$

(Same as A)

D. $N=7$ circular

$$\sum_{k=0}^6 x[k] h_7[n-k]$$

(zero pad with 4 zeros)

i.e. $h[3]=h[4]=h[5]=h[6]=0$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} h[0] & h[6] & h[5] \\ h[1] & h[0] & h[6] \\ h[2] & h[1] & h[0] \\ h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \\ h[6] & h[5] & h[4] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 2 \\ 5/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Complexity of DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] \underbrace{e^{-j\frac{2\pi}{N}kn}}_{W_N^{kn}}, \quad k=0, \dots, N-1$$

N^2 complex multiplications

$N(N-1)$ complex additions

Fast Fourier Transform (FFT)

* Reduce the number of operations to $N \log N$.

$$N=10^3 \quad \text{DFT} \sim 10^6$$

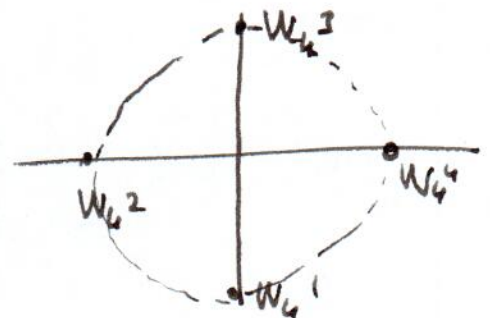
$$\text{FFT} \sim 10^4 \quad (\text{100 times faster than DFT})$$

* How?

- Decomposition into smaller DFTs

- Simplifications: $W_N^{kN} = 1$

$$W_N^{(k_{\text{odd}}) \frac{N}{2}} = -1$$



$N=4$

* Periodicity

$$W_N^{n(k+N)} = W_N^{k(n+N)} = W_N^{kn}$$

* Conjugation

$$W_N^{k(N-n)} = W_N^{-kn} = (W_N^{kn})^* \quad (\text{conjugation})$$

Recall that there are only N unique entries in DFT matrix.

"dfmtx"
(4) (16) (32)

Decimation in time (N even)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$$= \sum_{n \text{ even}} x[n] W_N^{nk} + \sum_{n \text{ odd}} x[n] W_N^{nk}$$

$$= \sum_{l=0}^{N/2-1} x[2l] W_N^{2lk} + \sum_{l=0}^{N/2-1} x[2l+1] W_N^{(2l+1)k}$$

$$= \sum_{l=0}^{N/2-1} x[2l] (W_N^2)^{lk} + \sum_{l=0}^{N/2-1} x[2l+1] (W_N^2)^{lk} \cdot W_N^k$$

$$= \underbrace{\sum_{l=0}^{N/2-1} x[2l] W_{N/2}^{lk}}_{\text{length } N/2 \text{ DFT of even terms}} + W_N^k \underbrace{\sum_{l=0}^{N/2-1} x[2l+1] W_{N/2}^{lk}}_{\text{length } N/2 \text{ DFT of odd terms}}$$

length $N/2$ DFT
of even terms

//

$G[k]$

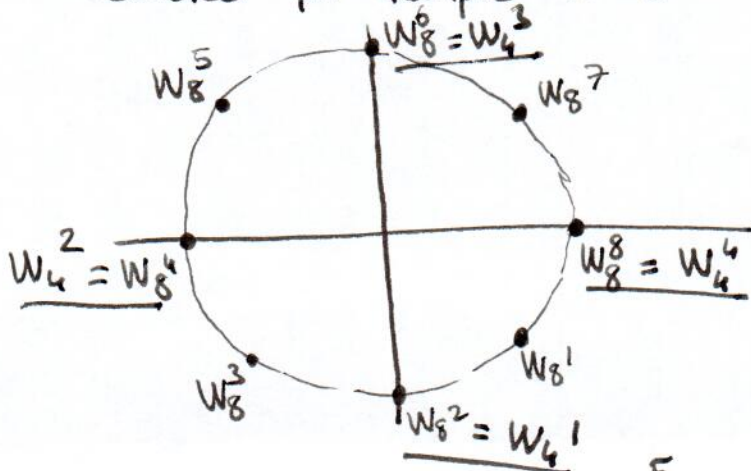
length $N/2$ DFT
of odd terms

//

$H[k]$

$$X[k] = G[k] + W_N^k H[k], \quad k=0, \dots, N-1$$

Consider for example $N=8$



$$G[0] = G[4], \quad H[0] = H[4]$$

$$G[1] = G[5], \quad H[1] = H[5]$$

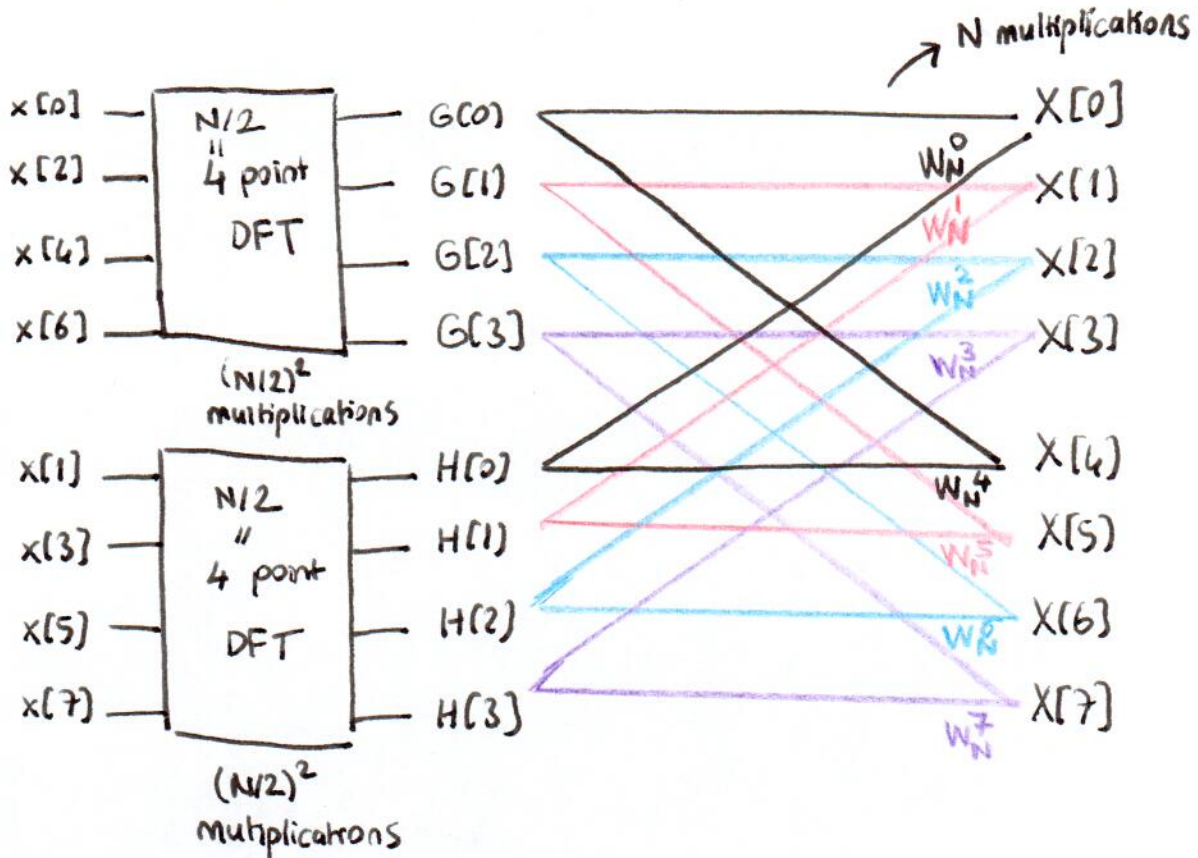
$$G[2] = G[6], \quad H[2] = H[6]$$

$$G[3] = G[7], \quad H[3] = H[7]$$

periodic with $N/2 = 4$

Assume $N=8$ (given)

$$X[k] = G[k] + W_N^k H[k], \quad k=0, \dots, 7$$



Total number of multiplications: $2 \cdot \left(\frac{N}{2}\right)^2 + N \approx \frac{N^2}{2}$

Length 2 DFT

$$\underbrace{x[0], x[1]}_{\text{time domain}} \longrightarrow \underbrace{X[0], X[1]}_{k \text{ domain}}$$

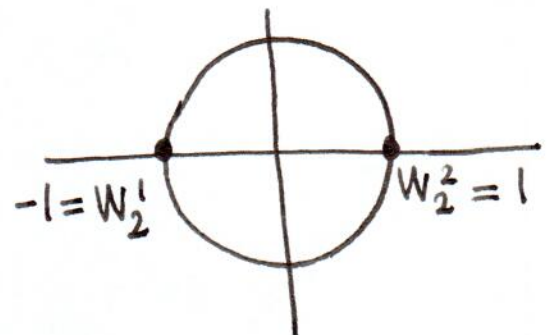
$N=2 \quad W_2^1 = -1, \quad W_2^2 = 1$

$$X[k] = \sum_{n=0}^1 x[n] W_2^{nk}$$

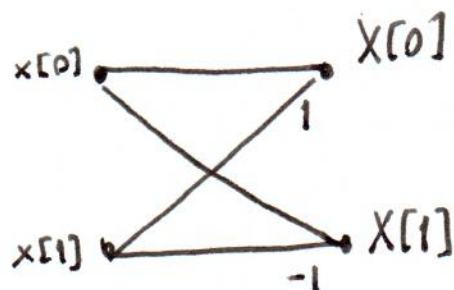
$$X[0] = x[0] + x[1]$$

$$X[1] = x[0] - x[1]$$

(no multiplication needed)



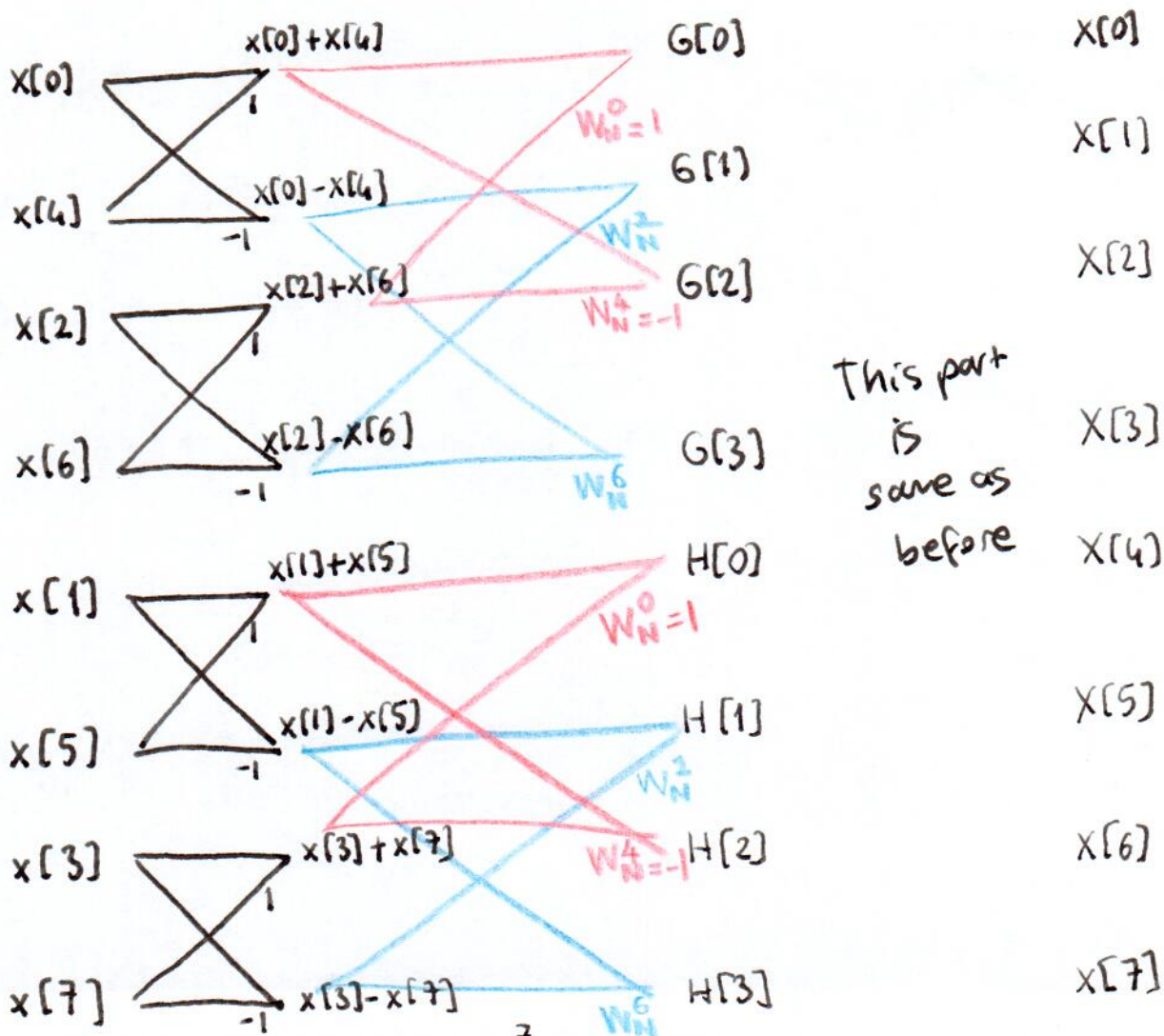
2 point DFT



Recall that $G[k] = \sum_{l=0}^{N/2-1} x[2l] W_{N/2}^{lk}$ (length $N/2$ DFT of even terms)

$H[k] = \sum_{l=0}^{N/2-1} x[2l+1] W_{N/2}^{lk}$ (length $N/2$ DFT of odd terms)

$$X[k] = G[k] + W_N^k H[k], \quad k=0, \dots, N-1$$



Let's verify some branches:

$$G[0] = x[0] + x[2] + x[4] + x[6]$$

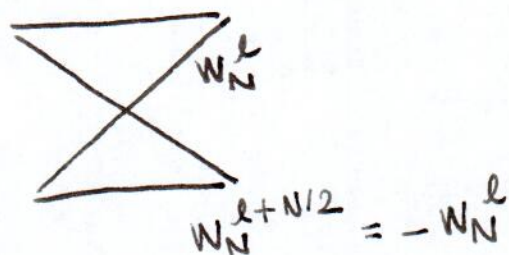
$$G[1] = x[0] + x[2]W_N^2 + x[4]W_N^4 + x[6]W_N^6 \quad (\text{Recall } N=8)$$

$$= (x[0] - x[4]) + W_N^2 (x[2] - x[6])$$

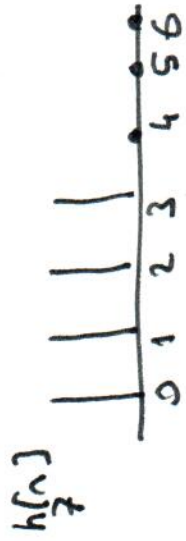
The rest of $G[k]$, $H[k]$, please verify.

Radix-2 FFT

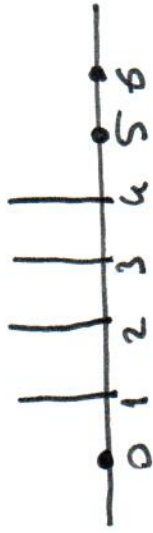
- The number of data points is a power of 2. In the above example $N = 2^3$. 3 is the number of stages.
- Consider a pattern from the figure



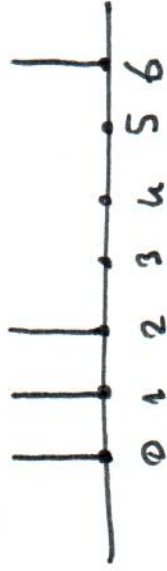
- We divide the number of multiplications we need by 2.
- This reduced complexity approach can provide savings in terms of storage space.



$$h_2[n-1] = \tilde{h}_2[n]$$



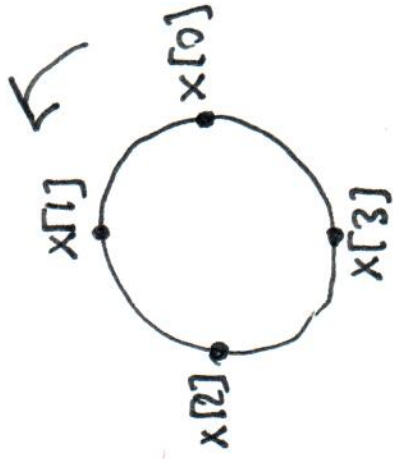
$$h_2[n+1]$$



$$h_2[n-1]$$

$$n=7$$

$$\tilde{h}_2[n] = \tilde{h}_2[0]$$



$$w_8^0 = w_4^0$$

$$w_8^2 = w_4^1$$

$$w_8^4 = w_4^2$$

$$w_8^6 = w_4^3$$

$$w_8^8 = w_4^4$$

$$w_8^{10} = w_4^5$$

$$w_8^{12} = w_4^6$$

$$w_8^{14} = w_4^7$$

$$w_8^{16} = w_4^8$$

$$w_8^{18} = w_4^9$$

$$w_8^{20} = w_4^{10}$$

$$w_8^{22} = w_4^{11}$$

$$w_8^{24} = w_4^{12}$$

$$w_8^{26} = w_4^{13}$$

$$w_8^{28} = w_4^{14}$$

$$w_8^{30} = w_4^{15}$$

$$w_8^{32} = w_4^{16}$$

$$w_8^{34} = w_4^{17}$$

$$w_8^{36} = w_4^{18}$$

$$w_8^{38} = w_4^{19}$$

$$w_8^{40} = w_4^{20}$$

$$w_8^{42} = w_4^{21}$$

$$w_8^{44} = w_4^{22}$$

$$w_8^{46} = w_4^{23}$$

$$w_8^{48} = w_4^{24}$$

$$w_8^{50} = w_4^{25}$$

$$w_8^{52} = w_4^{26}$$

$$w_8^{54} = w_4^{27}$$

$$w_8^{56} = w_4^{28}$$

$$w_8^{58} = w_4^{29}$$

$$w_8^{60} = w_4^{30}$$

$$w_8^{62} = w_4^{31}$$

$$w_8^{64} = w_4^{32}$$

$$w_8^{66} = w_4^{33}$$

$$w_8^{68} = w_4^{34}$$

$$w_8^{70} = w_4^{35}$$

$$w_8^{72} = w_4^{36}$$

$$w_8^{74} = w_4^{37}$$

$$w_8^{76} = w_4^{38}$$

$$w_8^{78} = w_4^{39}$$

$$w_8^{80} = w_4^{40}$$

$$w_8^{82} = w_4^{41}$$

$$w_8^{84} = w_4^{42}$$

$$w_8^{86} = w_4^{43}$$

$$w_8^{88} = w_4^{44}$$

$$w_8^{90} = w_4^{45}$$

$$w_8^{92} = w_4^{46}$$

$$w_8^{94} = w_4^{47}$$

$$w_8^{96} = w_4^{48}$$

$$w_8^{98} = w_4^{49}$$

$$w_8^{100} = w_4^{50}$$

$$w_8^{102} = w_4^{51}$$

$$w_8^{104} = w_4^{52}$$

$$w_8^{106} = w_4^{53}$$

$$w_8^{108} = w_4^{54}$$

$$w_8^{110} = w_4^{55}$$

$$w_8^{112} = w_4^{56}$$

$$w_8^{114} = w_4^{57}$$

$$w_8^{116} = w_4^{58}$$

$$w_8^{118} = w_4^{59}$$

$$w_8^{120} = w_4^{60}$$

$$w_8^{122} = w_4^{61}$$

$$w_8^{124} = w_4^{62}$$

$$w_8^{126} = w_4^{63}$$

$$w_8^{128} = w_4^{64}$$

$$w_8^{130} = w_4^{65}$$

$$w_8^{132} = w_4^{66}$$

$$w_8^{134} = w_4^{67}$$

$$w_8^{136} = w_4^{68}$$

$$w_8^{138} = w_4^{69}$$

$$w_8^{140} = w_4^{70}$$

$$w_8^{142} = w_4^{71}$$

$$w_8^{144} = w_4^{72}$$

$$w_8^{146} = w_4^{73}$$

$$w_8^{148} = w_4^{74}$$

$$w_8^{150} = w_4^{75}$$

$$w_8^{152} = w_4^{76}$$

$$w_8^{154} = w_4^{77}$$

$$w_8^{156} = w_4^{78}$$

$$w_8^{158} = w_4^{79}$$

$$w_8^{160} = w_4^{80}$$

$$w_8^{162} = w_4^{81}$$

$$w_8^{164} = w_4^{82}$$

$$w_8^{166} = w_4^{83}$$

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$$w_8^{206} = w_4^{103}$$

$$w_8^{208} = w_4^{104}$$

$$w_8^{210} = w_4^{105}$$

$$w_8^{212} = w_4^{106}$$

$$w_8^{214} = w_4^{107}$$

$$w_8^{216} = w_4^{108}$$

$$w_8^{218} = w_4^{109}$$

$$w_8^{220} = w_4^{110}$$

$$w_8^{222} = w_4^{111}$$

$$w_8^{224} = w_4^{112}$$

$$w_8^{226} = w_4^{113}$$

$$w_8^{228} = w_4^{114}$$

$$w_8^{230} = w_4^{115}$$

$$w_8^{232} = w_4^{116}$$

$$w_8^{234} = w_4^{117}$$

$$w_8^{236} = w_4^{118}$$

$$w_8^{238} = w_4^{119}$$

$$w_8^{240} = w_4^{120}$$

$$w_8^{242} = w_4^{121}$$

$$w_8^{244} = w_4^{122}$$

$$w_8^{246} = w_4^{123}$$

$$w_8^{248} = w_4^{124}$$

$$w_8^{250} = w_4^{125}$$

$$w_8^{252} = w_4^{126}$$

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$$w_8^{256} = w_4^{128}$$

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$$w_8^{300} = w_4^{150}$$

$$w_8^{302} = w_4^{151}$$

$$w_8^{304} = w_4^{152}$$

$$w_8^{306} = w_4^{153}$$

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$$w_8^{322} = w_4^{161}$$

$$w_8^{324} = w_4^{162}$$

$$w_8^{326} = w_4^{163}$$

$$w_8^{328} = w_4^{164}$$

$$w_8^{330} = w_4^{165}$$

$$w_8^{332} = w_4^{166}$$

$$w_8^{334} = w_4^{167}$$

$$w_8^{336} = w_4^{168}$$

$$w_8^{338} = w_4^{169}$$

$$w_8^{340} = w_4^{170}$$

$$w_8^{342} = w_4^{171}$$

$$w_8^{344} = w_4^{172}$$

$$w_8^{346} = w_4^{173}$$

$$w_8^{348} = w_4^{174}$$

$$w_8^{350} = w_4^{175}$$

$$w_8^{352} = w_4^{176}$$

$$w_8^{354} = w_4^{177}$$

$$w_8^{356} = w_4^{178}$$

$$w_8^{358} = w_4^{179}$$

$$w_8^{360} = w_4^{180}$$

$$w_8^{362} = w_4^{181}$$

$$w_8^{364} = w_4^{182}$$

$$w_8^{366} = w_4^{183}$$

$$w_8^{368} = w_4^{184}$$

$$w_8^{370} = w_4^{185}$$

$$w_8^{372} = w_4^{186}$$

$$w_8^{374} = w_4^{187}$$

$$w_8^{376} = w_4^{188}$$

$$w_8^{378} = w_4^{189}$$

$$w_8^{380} = w_4^{190}$$

$$w_8^{382} = w_4^{191}$$

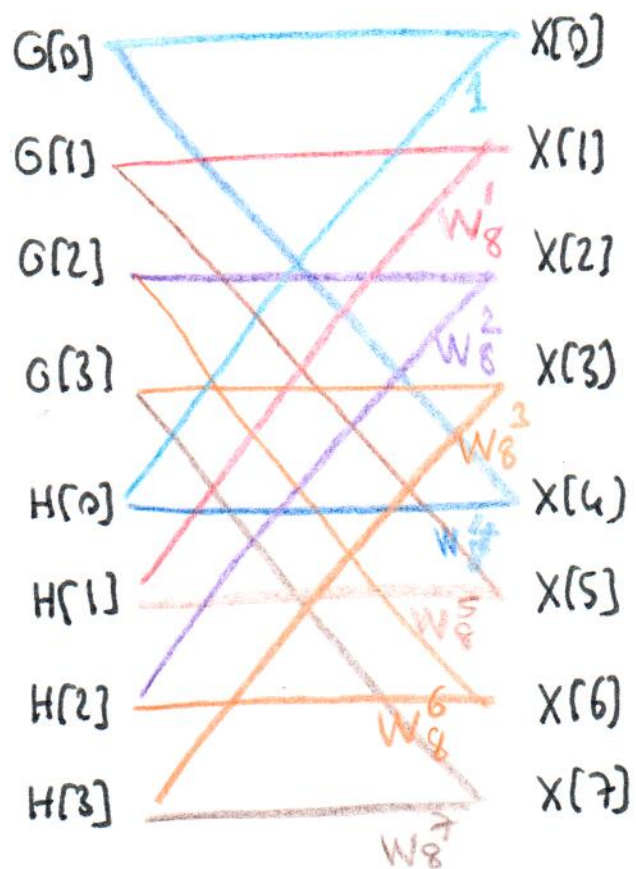
$$w_8^{384} = w_4^{192}$$

$$w_8^{386} = w_4^{193}$$

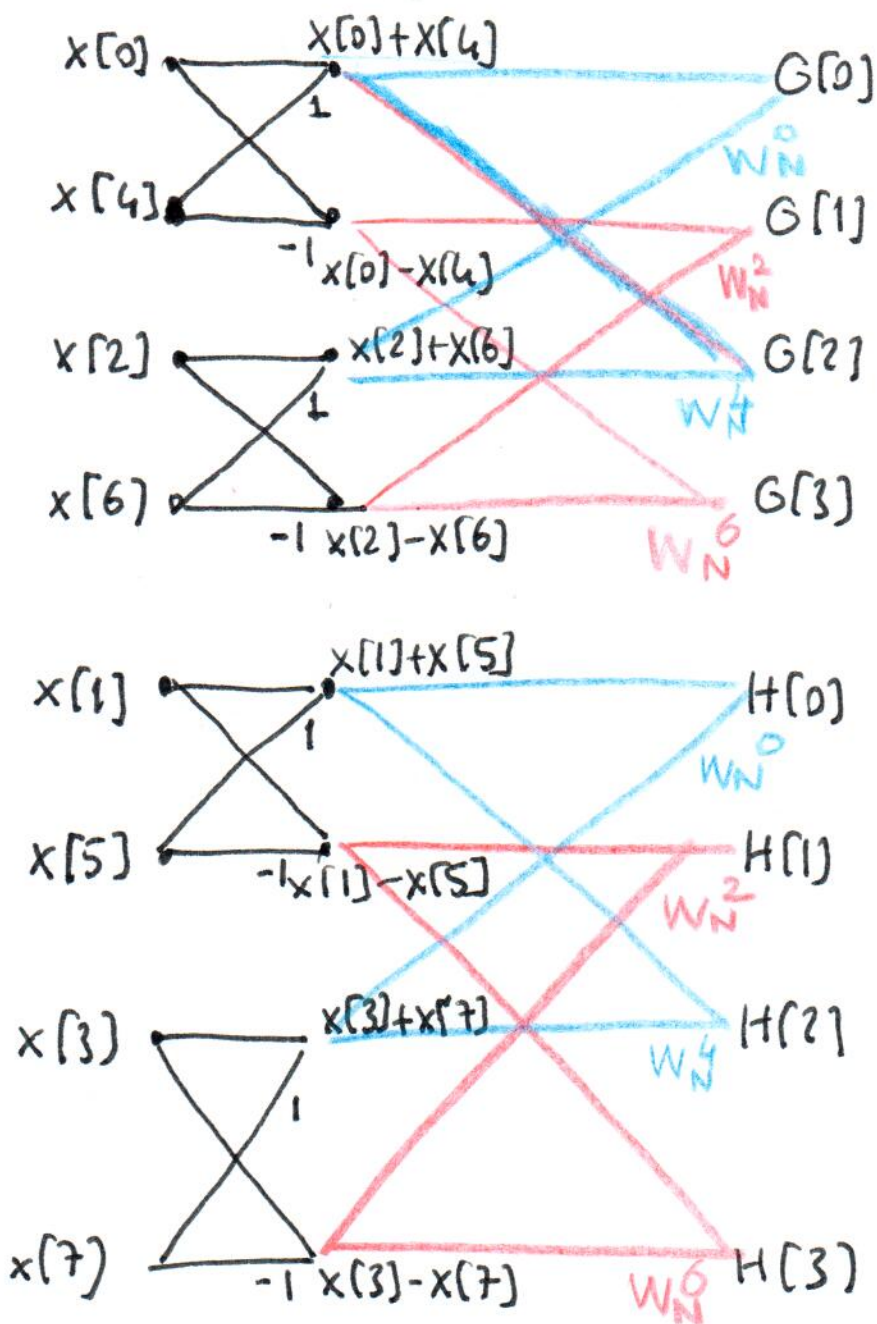
$$w_8^{388} = w_4^{194}$$

$$w_8^{390} = w_4^{195}$$

$$w_8^{392} = w_4^{196}$$



$$X[k] = G[k] + W_N^k H[k], \quad k=0, \dots, N-1$$



$$G[0] = x[0] + x[2] + x[4] + x[6]$$

$$\begin{aligned} G[1] &= x[0] + x[2] W_4^1 + x[4] W_4^2 + x[6] W_4^3 \\ &= x[0] - x[4] + W_4^1 (x[2] + x[6] W_4^2) \\ &= x[0] - x[4] + W_4^1 (x[2] - x[6]) \end{aligned}$$

