Today's lecture

- Examples on z transform, poles-zeros, ROC
- Midterm 1 review
- Filter design using z plane

Announcements

- Midterm 1 this Thursday (10/8 on Webex at 10:10am)
- [Student course evaluations 10/8/20 - 10/16/20]
- I will post Homework 3 soon.

Example

$$X(z) = \frac{3z+5}{z^2+2z+4} = \frac{3z^{-1}+5z^{-2}}{1+2z^{-1}+4z^{-2}} \quad \text{(right-sided signal)}$$

poles

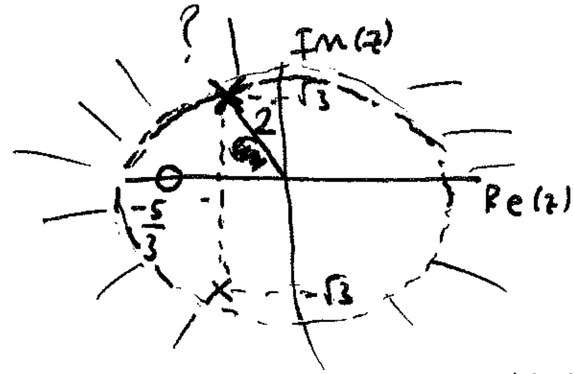
$z^2+2z+4=0$

$(z+1)^2+3=0$

$z+1 = \mp j\sqrt{3}$

$z = -1 \mp j\sqrt{3}$

$$= \underbrace{\frac{3z^{-1}}{1+2z^{-1}+4z^{-2}}}_{\text{previous example}} + \underbrace{\frac{5z^{-2}}{1+2z^{-1}+4z^{-2}}}_{?}$$



Recall that $y[n] \overset{z}{\longleftrightarrow} Y(z)$

$y[n-1] \longleftrightarrow Y(z).z^{-1}$

$$\frac{3z^{-1}}{1+2z^{-1}+4z^{-2}} \overset{z}{\longleftrightarrow} \frac{\sqrt{3}}{2}.2^{n}.\sin\left(\frac{2\pi}{3}n\right).u[n] \quad \text{right-sided}$$

$$\frac{5z^{-2}}{1+2z^{-1}+4z^{-2}} \overset{z}{\longleftrightarrow} \frac{5}{2\sqrt{3}}.2^{n-1}.\sin\left(\frac{2\pi}{3}(n-1)\right).u[n-1]$$

Example

$$X(z) = 3z^{-2}+5z^{-1}-\frac{1}{2}+3z^{3}, \quad \underset{\text{ROC}}{0<|z|<\infty}$$

$x[n] = ?$

$\delta[n] \overset{z}{\longleftrightarrow} 1$

$\delta[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0}$

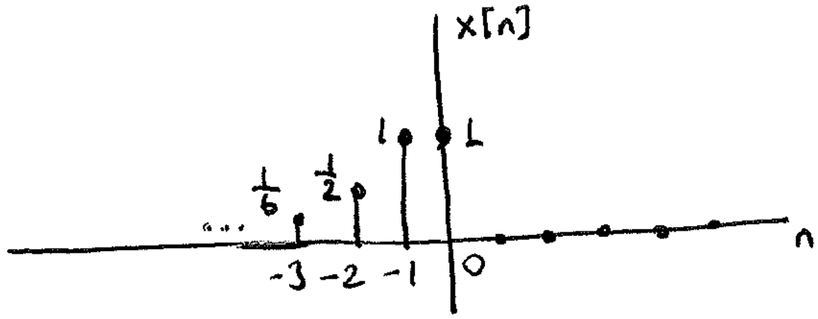
$$x[n] = 3\delta[n-2]+5\delta[n-1]-\frac{1}{2}\delta[n]+3\delta[n+3]$$

Example $X(z) = e^{z}$, $|z| < \infty$ (ROC)

Compute the inverse z transform $x[n]$.

$$X(z) = e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \text{(power series)}$$

$$x[n] = \delta[n] + \delta[n+1] + \frac{1}{2}\delta[n+2] + \frac{1}{6}\delta[n+3] + \cdots$$



(left-sided signal)

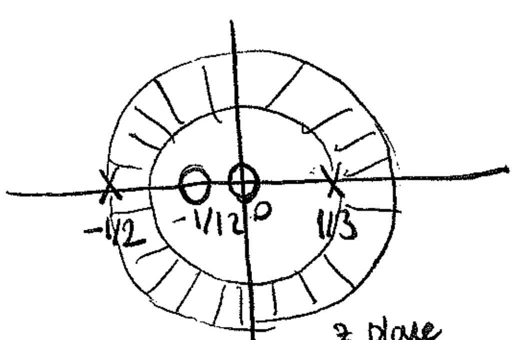
Example
$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} , \quad \text{ROC: } \frac{1}{3} < |z| < \frac{1}{2}$$

$$x[n] = -\left(-\frac{1}{2}\right)^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$$

Determine the z-transform of $x[-n]$. (Time reversal)

$$x[-n] \xleftrightarrow{\text{z-transform}} X\left(\frac{1}{z}\right)$$ (inversion of poles and zeros across unit circle)

$$X(z) = \frac{2z}{\underset{(3z-1)}{2z+1}} + \frac{3z}{\underset{(2z+1)}{3z-1}} = \frac{12z^2 + z}{(2z+1)(3z-1)} = \frac{z(12z+1)}{(2z+1)(3z-1)}$$



pole-zero diagram for $X(z)$

$$x[-n] = -(-2)^n u[n-1] + 3^n u[-n] \xleftrightarrow{\text{z transform}} Y(z)?$$

$$= 2(-2)^{n-1} u[n-1] + 3 \cdot 3^{n-1} u[-n]$$

Recall $\underline{\alpha^n u[n]}$, $\underline{-\alpha^n u[-n-1]} \longleftrightarrow \dfrac{1}{1-\alpha z^{-1}}$ with ROCs $|z| > |\alpha|$ & $|z| < |\alpha|$

$$(-2)^{n-1} u[n-1] \longleftrightarrow z^{-1} \frac{1}{1+2z^{-1}}, \quad |z| > 2$$

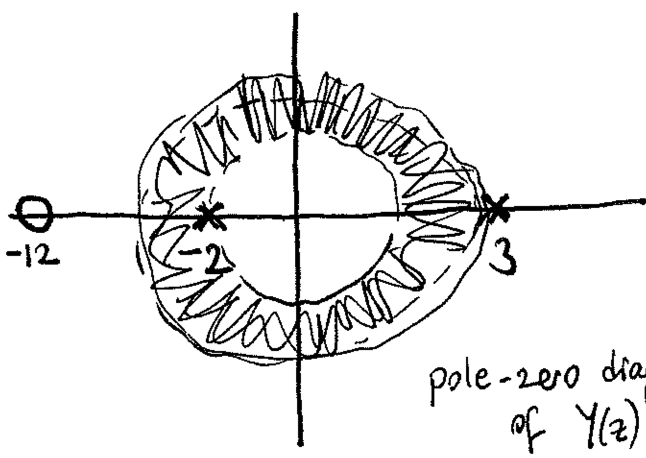
$$-3^{n-1} u[-n] \longleftrightarrow z^{-1} \cdot \frac{1}{1-3z^{-1}}, \quad |z| < 3$$

addition of 1 pole at zero due to time shift

$$Y(z) = 2z^{-1} \cdot \frac{1}{1+2z^{-1}} - 3z^{-1} \cdot \frac{1}{1-3z^{-1}}, \quad 2 < |z| < 3$$

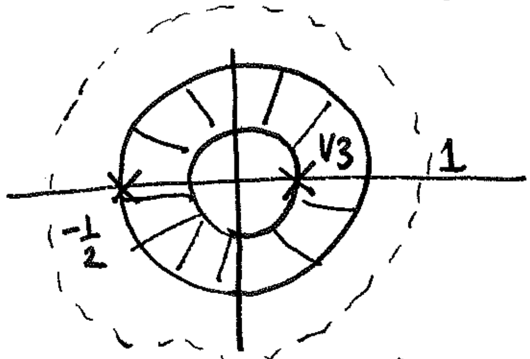
$$= \frac{2}{z+2} - \frac{3}{z-3} = \frac{-z-12}{(z+2)(z-3)}$$

$(z-3)$ $\quad$ $(z+2)$



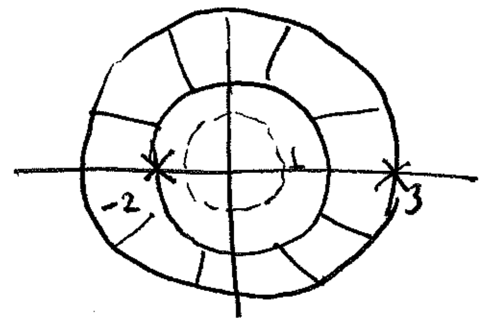
pole-zero diagram of $Y(z)$

For example, let $r_1 = \frac{1}{3}$, $r_2 = \frac{1}{2}$



ROC of $X(z)$

$\frac{1}{3} < |z| < \frac{1}{2}$



ROC of $X\left(\frac{1}{z}\right)$?

$2 < |z| < 3$

<u>Example</u> $x[n] = 2|A| \cdot |p|^n \cos(\theta_p n + \theta_A) \cdot u[n]$, $A = |A| e^{j\theta_A}$

$A = |A| e^{j\angle\theta_A}$, $p = |p| e^{j\angle\theta_p}$

$X(z) = ?$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = 2|A| \sum_{n=0}^{+\infty} \left(\frac{|p|}{z}\right)^n \left(\frac{e^{j(\theta_p n + \theta_A)} + e^{-j(\theta_p n + \theta_A)}}{2} \right)$$

$$= |A| \left(\sum_{n=0}^{+\infty} \left(\frac{|p|}{z} \cdot e^{j\theta_p}\right)^n \cdot e^{j\theta_A} + \left(\frac{|p|}{z} \cdot e^{-j\theta_p}\right)^n e^{-j\theta_A} \right)$$

$$= |A| \left(\frac{1}{1 - \frac{|p|}{z} e^{j\theta_p}} \cdot e^{j\theta_A} + \frac{1}{1 - \frac{|p|}{z} e^{-j\theta_p}} \cdot e^{-j\theta_A} \right)$$

ROC: $|z| > |p|$

$$X(z) = A \cdot \frac{1}{1 - \frac{p}{z}} + A^* \cdot \frac{1}{1 - \frac{p^*}{z}} = \frac{Az}{z - p} + \frac{A^* z}{z - p^*}$$

$(z - p^*) \quad (z - p)$

$$= \frac{\cdots}{(z-p)(z-p^*)}$$