$|y(n)| = |\sum x \cdot h| \le \sum |x| \cdot |h| \le M \sum |h|$.

DSP C[illegible]b Sheet

Properties of Signal

Memoryless: if output at time n only depend on input at time n

Time Invariance: System behaves same way regardless of what input is applied

Stability: A relaxed system is BIBO stable iff the input x[n] is bounded and y[n] is bounded.
↳ feed a u(n) and check y(n) bound.

LTI System

x[n] → H → h[n] h[n]: impulse function

For 2 Linear system in series, the τ_{eff} is also linear

For 2 TI system in series, the τ_{eff} is also TI

For 2 causal system in series, the τ_{eff} is may not be causal

For 2 LTI system in series, the τ_{eff} is not LTI

For 2 LTI system in series, the τ_{eff} order can be swapped

For 2 TV system in series, τ_{eff} order cannot be swapped

For 2 non-L sys in series, the τ_{eff} is linear

For 2 stable system in series, the τ_{eff} is stable

A total system can be causal but not the individual system

If sys is Parallel, you sum.

Summation Index Shifting

$\sum_{k=1}^{2} a^k \to \sum_{k=0}^{1} a^{k+1}$ — $\sum_{k=1}^{2} a^k \to \sum_{k=2}^{2} a^{k-1}$

$\sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a}$ — $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$

Fourier Series

$x(t) = \sum_{k=-\infty}^{\infty} a^k e^{jkw_o t}$ — $a_k = \frac{1}{T}\int_T x(t)e^{-jkw_o t}$

Time shift: x(t-τ) is —$e^{-jkw_o \tau a_k}$

Time reversal: x(-t) is —a_{-k}

Time Scaling: x(at), a>0 (periodic $\frac{T}{a}$) is —a_k

Conjugation: x*(t) is —a^*_{-k}

Symmetry x(t) real is —$a_k = a^*_{-k}$

Differentiation $\frac{d}{dt}x(t)$ is — $jkw_o a_k$

Integration $\int_{-\infty}^{t} x(t)dt, a_0 = 0$ is — $\frac{a_k}{jkw_0}$

Convolution $\int_T h(\tau) \circledast x(t-\tau)d\tau$ is —$Ta_k b_k$

Multiplication x(t)y(t) is —$\sum_{m=-\infty}^{\infty} a_m b_{k-m}$

Parseval $\frac{1}{T}\int_T |x(t)|^2 dt$ is —$\sum_{k=-\infty}^{\infty} |a_k|^2$

Fourier series only exist if the signal is a continuous

Fourier Transform

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$	$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$
Time Scaling x(at)	$\frac{1}{\lvert a\rvert}X(jw/a)$
Differentiation $\frac{d}{dt}x(t)$	$jwX(jw)$
Integration $\int_{-\infty}^{t} x(t)dt$	$\frac{1}{jw}X(jw) + \pi X(0)\delta(w)$
Multiplication x(t)y(t)	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(ju)Y(jw-ju)du$
Delta $\delta(t)$	1
One 1	$2\pi\delta(w)$
Exponent $e^{jw_o t}$	$2\pi\delta(w-w_0)$
Cosine $\cos(w_0 t)$	$\pi[\delta(w-w_o)+\delta(w+w_0)]$
Sin $\sin(w_0 t)$	$\frac{\pi}{j}[\delta(w-w_o)+\delta(w+w_0)]$
Unit Step u(t)	$\frac{1}{jw} + \pi\delta(w)$
Decaying Step $u(t)e^{-at}, a>0$	$\frac{1}{a+jw}$
Parseval $\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{2\pi} \lvert X(jw)\rvert^2 dw$

A BIBO LTI → Z-trans's ROC contain unit circle

General Notes

Shift, Flip, Scale, break down signal to even and odd to find transform. $x_e(n) = \frac{x(n)+x(-n)}{2}$, $x_o(n) = \frac{x(n)-x(-n)}{2}$

$e^{j\pi} = -1$ $e^{j2\pi} = 1$ $e^{j\frac{\pi}{2}} = j$ $e^{j\frac{3\pi}{2}} = -j$
$e^{-j\pi} = -1$ $e^{-j\pi/2} = -j$ $e^{-j\frac{3\pi}{2}} = j$

If x(n) is real and even, then the $\angle H(W)$ is at 0 if $X(w) \ge 0$ and it would be at π if $X(w) \le 0$

$\int_{-\pi}^{\pi} X(w)dw = 2\pi x(0)$

Frequency Response

Given $y(n) + Ay(n-1) + \cdots = x(n) + Bx(n-1) + \cdots$
$H(Z) = \frac{Y(Z)}{X(Z)}$ $Z = e^{jw}$ Use Z-transform then sub the exponential back in to solve further then.

The impulse response is the inverse transform of H(Z)

Once the frequency response is known, then $y(n) = H(w_0)x(w_0)$

In DTFT, w in the X(w) is a real value.

In Z-transform, w in the X(w) is a complex value.

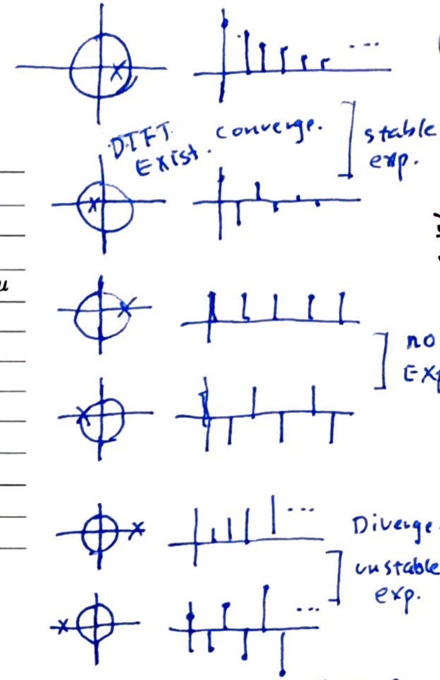
An LTI system only exist if $Y(w)$ is defined at region where X(w) is defined. In other words, if Y(w) is found to be non-zero at w_0 where $X(w_0)$ is zero, then it is not a valid LTI System. → For $H(w_0) \ne 0$, $Y(w_0)$ must $= 0$.
For difference eq, use Z then e^{jw}.

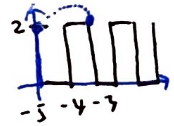
Exponential to Sinusoidal

$cos(x)$	$\frac{e^{jx}+e^{-jx}}{2}$
$sin(x)$	$\frac{e^{jx}-e^{-jx}}{2j}$
e^{jx}	$cos(x) + jsin(x)$

Z-transform.
$x(n) = -u(-1-n)$
↑
LEFT signal.

Pole:
$\alpha \in \mathbb{R}$ → pure exp.
$|\alpha| > 1$ → explode
$|\alpha| < 1$ → converge to 0.
$\alpha \in \mathbb{C}$ → osillatury sig.
$|\alpha| = 1$ → $|a^n| = 1$





$T = 10^{-6}s$ (t_{on})
$N = 2T = 2 \cdot 10^{-6}s$ { total P)

$a_K = \frac{1}{2T}\int_T 2 \cdot e^{-j\frac{2\pi}{2T}Kt} dt = \frac{1}{j\pi K}(1-e^{-j\pi K})$ $K \ne 0$.

$= \frac{1}{2T} X(w)\Big|_{w=\frac{\pi K}{T}}$.

$a_0 = \frac{1}{2T}\int_0^{T} 2\,dt = 1$

$\boxed{b_K = a_K H(\frac{\pi K}{T})}$

a_K need both condition!

$h(n) = \text{sinc}^2(t/T) = \mathcal{F}(\text{sinc}) * \mathcal{F}\{\text{sinc}\}$ $\frac{1}{2\pi}$ $w_c = \frac{1}{T}$

$\text{sinc}\,t/T = \frac{\sin(t/T)}{t/T} \cdot \frac{\pi}{\pi} = \underbrace{\frac{\sin(t/T)}{\pi t}}_{\text{good Form}} \underbrace{T\pi}_{\text{scalar}} \leftrightarrow \begin{cases} T\pi, & |w| \le w_c \\ 0, & \text{else} \end{cases}$

The height of Triangle is $T\pi \cdot T\pi \cdot \frac{1}{2\pi} \cdot \frac{1}{T}$.

$|z|$ must be $\{\mathbb{C}\} - \{0\} - \{\pm\infty\}$.
for finite ~~single~~ value.

$\{ \cdots\cdots \}$
← Both sided signal →

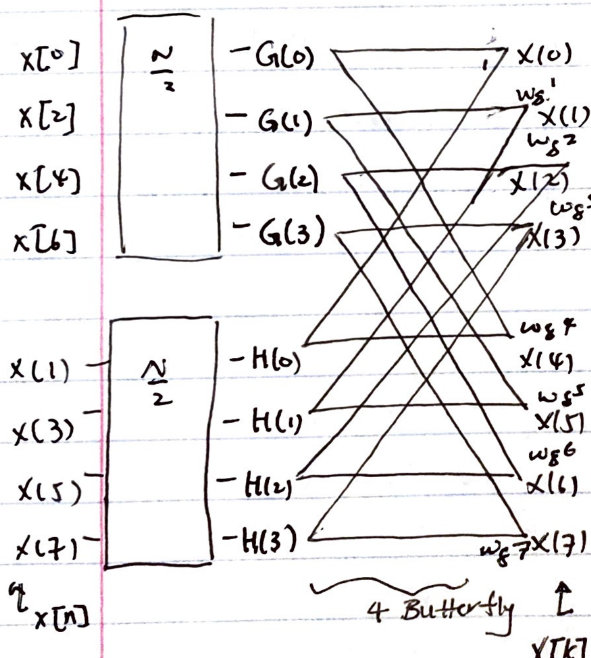
Z-transform

stable : Right sided.

$Z = e^{jw}$ (unit circle).

Transfer Function ↑ └ frequency Resp.

Type	ZERO
1	—
2	$\omega = \pi$
3	$\omega = 0, \pi$
4	$\omega = 0$



$-60\,dB = 20\log(A)$

$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$

$n = 2r$

$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_N^{(2r+1)n}$

$= \sum_{r=0}^{\frac{N}{2}-1} x(2r) (W_{\frac{N}{2}})^{rk}$

$+ W_N^k \, x(2r+1) \, (W_{\frac{N}{2}})^{rk}$

$= \underbrace{G[k]}_{\frac{N}{2}\text{ long}} + W_N^k \underbrace{H[k]}$

down sampling - decimation

$x[n] \rightarrow \boxed{\downarrow M} \rightarrow x(nM) \longleftrightarrow \frac{1}{M} \sum_{i=0}^{M-1} X\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)$

If $M\omega_B < \pi \rightarrow$ no need filter

$M\omega_B > \pi \rightarrow$ need prefilter.

upsampling - interpolation.

$x[n] \rightarrow \boxed{\uparrow L} \rightarrow x\left(\frac{n}{L}\right) \longleftrightarrow X(L\omega)$

$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow y_a(n)$ } eq.
$x(n) \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow y_b(n)$

$x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow y_a(n)$ } eq.
$x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow y_b(n)$

$x[n] \rightarrow \boxed{\uparrow L} \rightarrow H(\min\{\frac{\pi}{M}, \frac{\pi}{L}\}) \rightarrow \boxed{\downarrow M} \rightarrow x_f(n)$

$M > L$ Net Reduction of s. Rate. (need prefilter)

$M < L$ net inc of S.R. (perfect.).

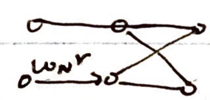
overlap when $\frac{\omega_1}{L} > \frac{2\pi - \omega_1}{L}$

* $W_N^{n(k+N)} = W_N^{nk}$ $\quad W_N^{(r+\frac{N}{2})} = -W_N^r$

$W_N^{kN} = 1 \quad W_N^{\frac{N}{2}(\text{ODD } k)} = -1$

FFT : $O(\underbrace{N}\,\underbrace{\log_2 N})$

\# of multip. Stage



N = 8

1	1	1	1	1	1	1	1
1	W_8^1	W_8^2	W_8^3	W_8^3	-1	$-W_8^1$	$-W_8^2$	$-W_8^3$
1	$-j$	-1	j	j	1	$-j$	-1	j
1	W_8^3	j	W_8^1	W_8^1	-1	$-W_8^3$	$-j$	$-W_8^1$
1	-1	1	-1	-1	1	-1	1	-1
1	$-W_8^1$	W_8^2	$-W_8^3$	$-W_8^3$	-1	W_8^1	$-W_8^2$	W_8^3
1	j	-1	$-j$	$-j$	1	j	-1	$-j$
1	$-W_8^3$	j	$-W_8^1$	$-W_8^1$	-1	W_8^3	$-j$	W_8^1

Even col: Top ½ = BoT ½ ODD col: Top ½ = − Bot ½

sum diff

$$\begin{bmatrix} \begin{pmatrix} I \\ I \end{pmatrix} & \begin{pmatrix} I \\ -I \end{pmatrix} \end{bmatrix} \begin{bmatrix} I & \\ & \underbrace{\begin{bmatrix} \ddots \end{bmatrix}}_{\text{Twiddle}} \end{bmatrix} \underbrace{\begin{bmatrix} F_4 & \\ & F_4 \end{bmatrix}}_{\text{DFT.}} \begin{bmatrix} X_{even} \\ X_{odd} \end{bmatrix}$$

FFT of freq:

$X[2r] = \sum_{n=0}^{\frac{N}{2}-1} [x[n] + x[n + \frac{N}{2}]] W_{\frac{N}{2}}^{nr}$

$X[2r+1] = \sum .. [x(n) - x(n + \frac{N}{2}]) W_N^n W_{\frac{N}{2}}^{nr}$

ZERO LOCATION

if $h(n) = h(N-1-n)$

$H(z) = z^{-(N-1)} H(\frac{1}{z})$

If z_0 is ZERO of Real Linear-phase filter $\{z_0^*, \frac{1}{z_0}, \frac{1}{z_0^*}\}$ also is ZERO.

$\min E = \max_{w \in [0,\pi]} |A(w) - A_d(w)|$

General ZERO exist in group 4

Gen. ZERO at unit circle → 2

ZERO at axis → 2

1 and −1 count.

Least Square Approximation of h[n]

Type I, III : M equations

Type II, IV : $\frac{N}{2} - 1$ equations

Ripple Reduce	Transition band	(less sharp)
Ripple Inc	Transition band	(more sharp)
more points	Transition band	(more sharp)
less points	---- band	(less sharp)

$A(w) = \sum h \cdot \cos(\cdot) + h.$

$\underset{L \times 1}{[A]} = \underset{L \cdot M+1}{[\cos(\cdot)]} \underset{M+1}{[h]}$ Type I/III

$DFT\{\frac{1}{N} X[k]\} = x[-n]$

DFT:

Length N $h[n]$ → ZERO PAD w/ M−1 ZEROS → $\boxed{N+M-1 \text{ DFT}}$ → ⊗ → $\boxed{DFT}$

Length M $x[n]$ → ZERO PAD N−1 ZEROS → $\boxed{N+M-1 \text{ DFT}}$ → ⊗

$x(n)$ → $\boxed{\downarrow M}$ → $E_0(z^M)$ → ⊕

↓ $\boxed{z}$ → $\boxed{\downarrow M}$ → $E_1(z^M)$ → ⊕

↓ $\boxed{z}$ → - - - -

Polyphase decimation $\frac{N}{M}$ tap

$x(n)$ → $\boxed{\downarrow M}$ → $E_0(z)$ → ⊕

↓ $\boxed{z}$ → $\boxed{\downarrow M}$ → $E_1(z)$ → ⊕

↓ - - - Same Rate - -

z

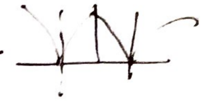
Polyphase Interpolation

$x(n)$ → $\boxed{\uparrow L}$ → $E_0(z^L)$ → ⊕ →.

↳ $\boxed{\uparrow L}$ → $E_1(z^L)$ → $\boxed{z^{-1}}$ ⊕

NO ZERO

FIR Filter Design :

notch filter 

Why we Desire them?

$h_d(n) = \delta[n-n_d] \leftrightarrow H_d(w) e^{-jwn_d}$

$H(w) = A(w) e^{-j(K_1 + K_2 w)}$

Filter Design Process

A → D · Fs scale · Repeat

D → A · T scale · Remove $|F| > F_s/2$

1.) choose desired Freq. Resp.

2.) choose allow-able class of filter Length-N FIR Filter

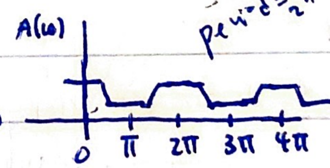
3.) choose measure of quality (How close)

4.) Apply Algo to find best val.

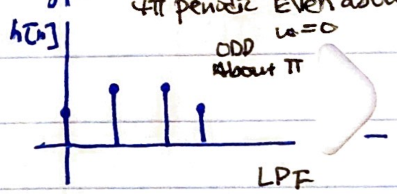
5.) Choose best Realization of filter.

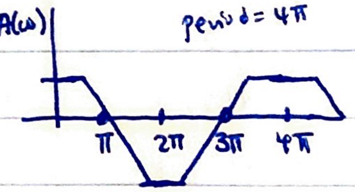
Type of Filter

Type I : N ODD



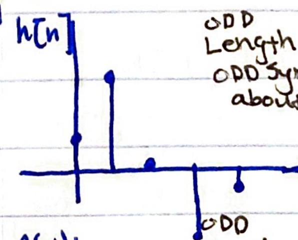
Type II : N Even

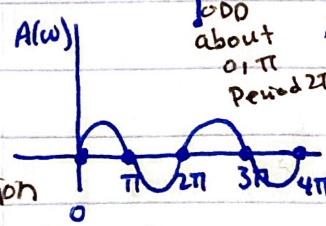




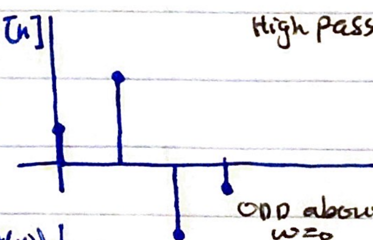
$h(n) = h(N-1-n)$

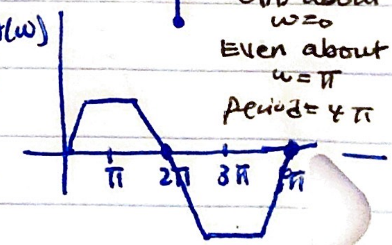
Type III : N ODD





Type IV : N Even





$h(n) = -h(N-1-n)$

- not always stable
- $H_d(\omega)$ should be consistent w/ $Re\{h(n)\} \rightarrow a(n), b(n)$ Real

IIR Filter

Can't do Linear Filter (no Symmetry.)

Low order are sufficient to implement specs.

$$y[n] = \underbrace{\sum_{m=0}^{M} b[m]x[n-m]}_{\text{FIR Filter}} - \underbrace{\sum_{K=1}^{N} a[k]y[n-k]}_{\text{feedback.}}$$

$$Y(z) = \sum_m^M b[m]X(z)z^{-m} - \sum_K^N a[k]Y(z)z^{-K}$$

$$Y(z)(1+\sum_K^N a[k]z^{-K}) = X(z)\sum_m^M b[m]z^{-m}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b[m]z^{-m}}{1+\sum_{K=1}^{N} a[k]z^{-K}}$$

↑ Freq. Resp.
Imp. Resp

Prony's method.

direct design IIR

$h_{des}(n) \leftrightarrow H_{des}(z)$

$H_{des}(z) = \sum_{n=0}^{\infty} h_d(n)z^{-n}$

Set $\boxed{H_{des} = H(z)}$

If: $L = m+n+1$
Then FFT
Else: Least sq

Impulse Response

By Prony's : $\sum_{n=0}^{\infty} h(n)z^{-n} = (\quad)$

$$(h(0)+h_1z^{-1}+\cdots)(1+\sum_{K=1}^{N} a[k]z^{-K}) = \sum_{m=0}^{M} b[m]z^{-m}.$$

no closed form

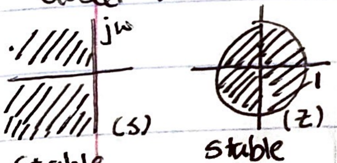
Freq. Response

Sample H_d @ $\frac{2\pi}{N}K$.

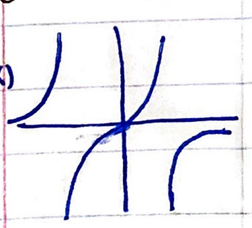
$$H_d[K] = \frac{DFT[b[n]]}{DFT[a[n]]} = \frac{B[K]}{A[K]} \quad \text{where } K=0,\dots,N-1.$$

IIR Design From Analog IIR.

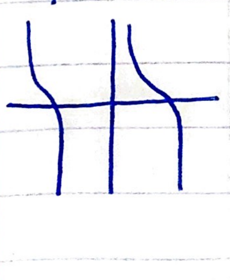
- closed form exist



tan(x)



cot(x)



① Impulse Invariance

$$s \mid z = e^{-sT}$$

poles same as ② but not the zeros.

Good for LPF & BPF b/c of aliasing at HF.

③ Bi-Linear Transformation

$$s = \frac{2}{T}\frac{z-1}{z+1} \qquad z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s} \qquad z = e^{jw}$$

s	z	$w \in [0, 2\pi]$
0	1	0
∞	-1	π
$\frac{2}{T}j$	j	$\pi/2$
$-\frac{2}{T}$	0	N/A

everything depend on sampling frequency!

Analog Freq from $H(s)$ ↓ Digital from $H(z)$ ↓

$$\Omega = \frac{2}{T}\tan\left(\frac{w}{2}\right)$$

Adaptive Filtering.

Update filter with changing $x[n]$ input.

$x[n]$ is the cumulative distribution function (CDF)

Mean: $\mu[n] = E[x[n]] = \int \tau P_n(\tau)$

auto covariance: $c(n, n-k) = E[(x(n)-\mu(n))(x(n-k)-\mu(n+k))]$

autocorr: $r(n, n-k) = E[x(n)x(n-k)]$

White noise

$v[n] : \frac{1}{\sqrt{2\pi}\sigma} \exp(\frac{-t^2}{2\sigma^2})$

$E[v(n)] = 0$

$E[v(n)^2] = \sigma_v^2 \delta[n]$

Wide Sense Stationary Process (need 1-4)

1. $\mu[n] = \mu$
2. $c(n, n-k) = c(n-m, n-m-k) = c(k)$
3. $r(n, n-k) = r(|k|)$
4. finite 2nd moment: $E[|x(n)|^2] < \infty \ \forall n$

Correlation matrix

$R = E[v[n]\, v[n]^T]$

$R_{ij} = E[x[i]\, x[j]] = R(|i-j|) = r(|i-j|)$

$$R = \begin{bmatrix} r(0) & \cdots & r(M-1) \\ \vdots & r(0) & \vdots \\ r(M-1) & \cdots & r(0) \end{bmatrix}$$

Symmetric A Toeplitz matrix. (equal diagonal)

Autoregressive Process

$x[n] \rightarrow \boxed{H(z)} \xrightarrow{} w[n]$ (IIR)

$x[n] = -\sum_{k=1}^{P} a_k x[n-k] + w(n)$

$= x * a$

$H(z) = \frac{B(z)}{A(z)}$

Moving average.

$w[n] \rightarrow \boxed{\ \ } \rightarrow x[n]$ (FIR)

$x[n] = \sum_{l=0}^{K} w[n-l]\, b[l]$

$= w * b$

Yule-Walker Equation

- Can be used on $\boxed{AR}$ process
- Can be used to estimate $\{a_k\}$ and $\{\sigma_v^2\}$: M+1

$$\underbrace{\begin{bmatrix} E[x[n]x[n]] & & \\ & \ddots & \\ & & \end{bmatrix}}_{\substack{R \\ \text{autocorr.}}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{bmatrix}}_{\substack{a \\ \text{coeff.}}} = \underbrace{\begin{bmatrix} r(1) \\ r(2) \\ \vdots \\ r(M) \end{bmatrix}}_{\substack{r \\ \text{corr.}}}$$

Determine AR parameters.

we know R & P.

so Wiener ok.

Wiener Filter.

Design H to drive $y(n)$ to $d[n]$

optimal "linear DT Filter" in MSE sense.

filter observed noisy process WSS signal noise additive

error $e[n] = d[n] - x[n]$

we minimize $J[n] = E[e[n]^2]$.

Function need be convex. 2nd $\partial/\partial h \geq 0$

$\sum_{i=0}^{M} h[i]\, r(|i-k|) = E[d[n]\, x[n-k]] \triangleq p[-k]$ (autocorr. / cross corr.)

$[R][\hat{h}] = [P]$ Wiener Hopf Eq.

($\hat{h}$ unknown)

$E[e^2(n)] = \sigma_d^2 - 2h^T p + h^T p h$

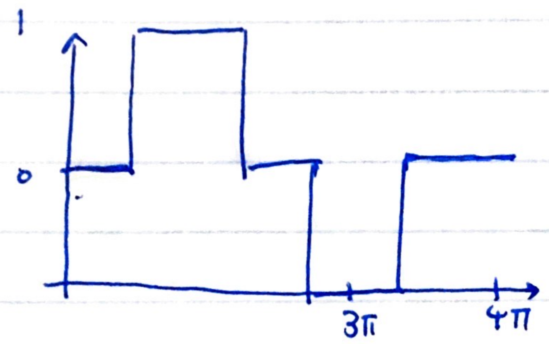
$\hat{h}$ is unique since MSE is a quad f(x)

$\min_{\hat{h}} E[e^2(n)] = \sigma_d^2 - (R^{-1}p)^T p + (R^{-1}p)^T p h$

$= \sigma_d^2 - p^T R^{-1} p$

1.) Linear phase filter

$$A_d(\omega) = \begin{cases} 1 & \omega \in [\pi/2, 3\pi/2], \ \omega \in [5\pi/2, 7\pi/2] \\ 0 & \text{otherwise.} \end{cases}$$



a.) What is the type of filter shown above? $\boxed{\text{High pass}}$ ~~because~~ it ~~allows two specific frequency Ranges~~. ~~to pass~~.

b) The type is. Type IV. filter because period is 4π and even about $\omega = \pi$.

c.) $\frac{2\pi}{N} k$, $k = 0, \dots, 15$. ~~$3\pi/2$ — 4π~~

$2\pi/16 \cdot 0 = 0$

$\frac{2\pi}{16} \cdot 1 = 0$ $\quad \frac{2\pi}{16} \cdot 5 = 1$ $\quad 2\pi/16 \cdot 9 = 1$ $\quad \frac{2\pi}{16} \cdot 14 = 0$

$2\pi/16 \cdot 10 = 1$ $\quad \frac{2\pi}{16} \cdot 15 = 0.$

$2\pi/16 \cdot 2 = 0$ $\quad 2\pi/16 \cdot 6 = 1$ $\quad 2\pi/16 \cdot 11 = 1$

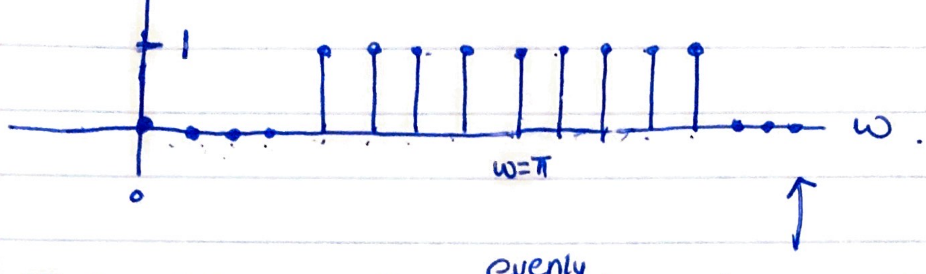
$2\pi/16 \cdot 3 = 0$ $\quad 2\pi/16 \cdot 7 = 1$ $\quad 2\pi/16 \cdot 12 = 1$

$2\pi/16 \cdot 4 = 1$ $\quad \frac{2\pi}{16} \cdot 8 = 1$ $\quad \frac{2\pi}{16} \cdot 13 = 0$

$A_d(\omega) = \{0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0\}$

Pole:
$\alpha \in \mathbb{R} \rightarrow$ pure exp.

$A(\omega) =$



The sequence is evenly symmetric about $\omega = \pi$.
The sequence is odd symmetric about $\omega = 0$.

The Reason is because the Amplitude Resonse only contain cosine term for Type 4.

e.). $h[n] = \text{IDFT}\left\{ j A\left(\frac{2\pi k}{N}\right) * W_N^{-n \cdot k} \right\}$.

$$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} j\, A\left(\frac{2\pi k}{N}\right) W_N^{-nk} \qquad M = \frac{16}{2} - 1 = 7.$$

$$= \frac{1}{16} \sum_{k=0}^{15} j\, A\left(\frac{2\pi k}{N}\right) W_N^{-nk}$$

$$h[n] = \frac{1}{16}\left[\sum_{k=4}^{12} j\, W_N^{-nk}\right]$$

$$h[0] = \frac{1}{16} \sum_{k=4}^{12} j \cdot 1 = j \frac{1}{16}[9] = \frac{9}{16} j$$

$$h[1] = \frac{1}{16} \sum_{k=4}^{12} j\, W_N^{-k} = \frac{j}{16}\left(\frac{W_N^{-4} - W_N^{-12}}{1 - W_N^{-1}}\right)$$

$$h[n] = \begin{cases} \frac{9}{16} j & n = 0. \\ \frac{j}{16}\left(\frac{W_N^{-4n} - W_N^{-12n}}{W_N^{-n}}\right) & 0 < n < 15. \end{cases}$$

$$H\left(\frac{4\pi}{5}\right) = j \cdot A\left(\frac{4\pi}{5}\right) \cdot W_{16}^{-\left(\frac{16}{2}-1\right)k}. \qquad k = 6.$$

$$\left|H\left(\frac{4\pi}{5}\right)\right| = 1$$

$$\angle H\left(\frac{4\pi}{5}\right) = -\left(\frac{16}{2}-1\right)\left(\frac{\pi}{2}\right) = -(7)(\pi/2) = \frac{-7\pi}{2}$$

Pole:

$\alpha \in \mathbb{R} \rightarrow$ pure exp.

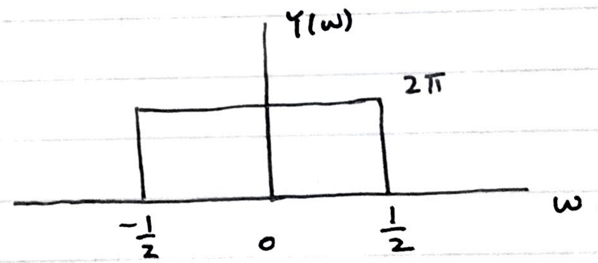
Aiden Chen

12/16/20 ①

2.) Sampling.

$$y(t) = \text{Sinc}(t/2) = \frac{\sin(t/2)}{t/2} = 2\pi \frac{\sin(t/2)}{t\pi}$$

a.) $$Y(\omega) = \begin{cases} 2\pi & |\omega| < \frac{1}{2} \\ 0 & |\omega| > \frac{1}{2} \end{cases}$$

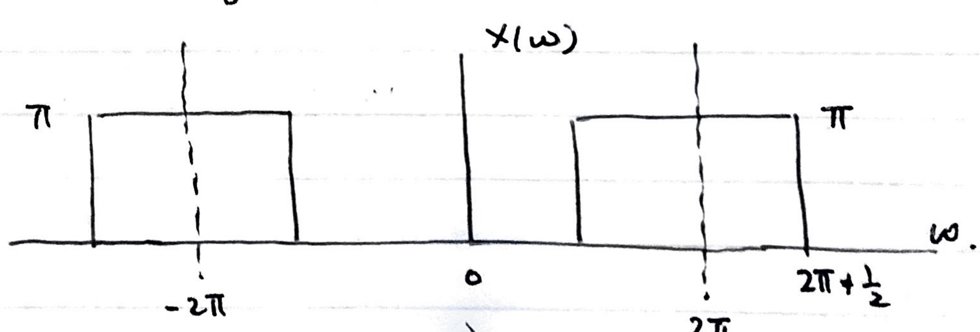


b.) y(t) is bandlimited between $[-\frac{1}{2}, \frac{1}{2}]$

Determine Nyquist Rate and $X(\omega)$

$x(t) = y(t) \cos \omega_c t \qquad \omega_c = 2\pi$

d.)



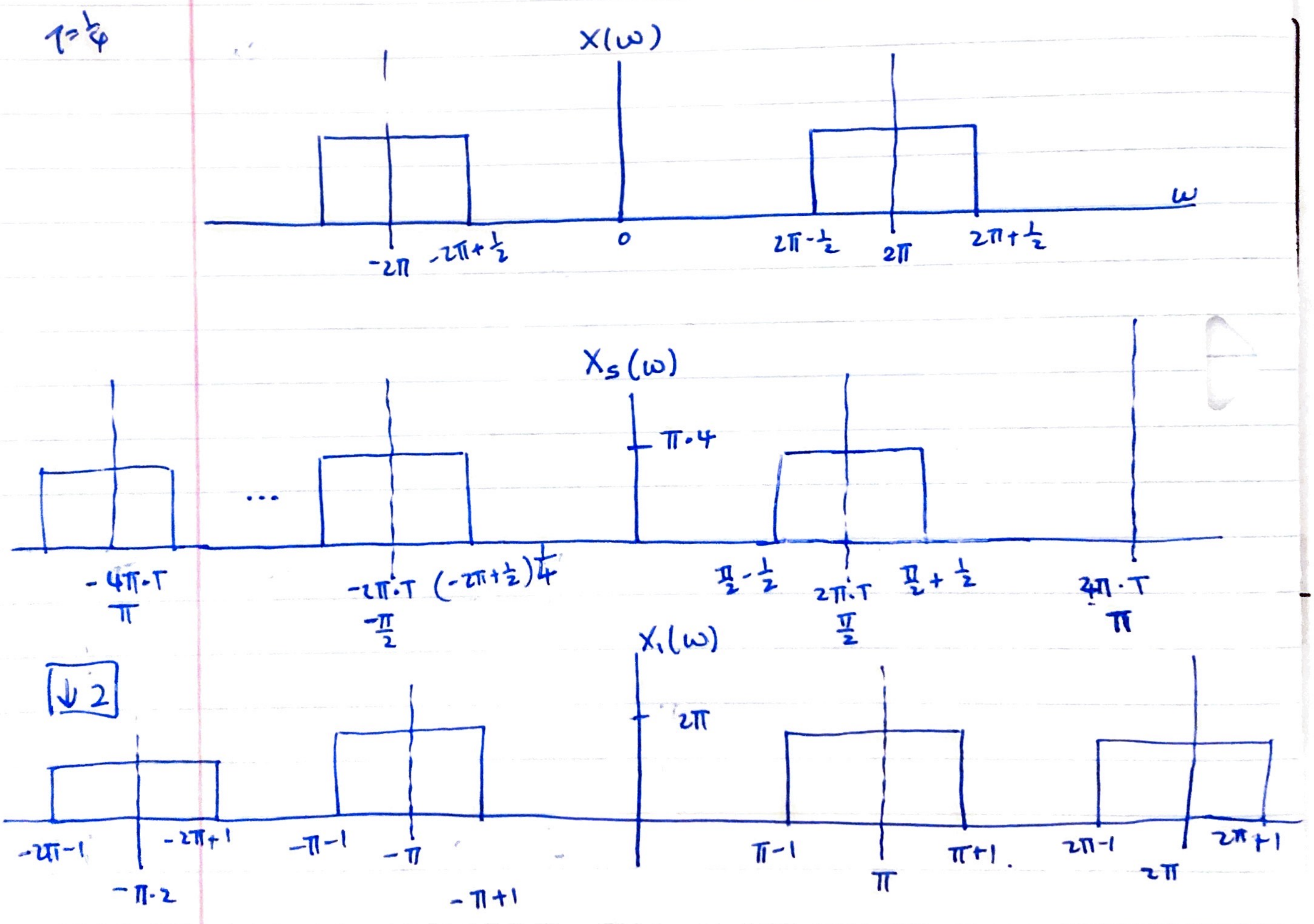
c.) Nyquist Rate is $2\left(2\pi + \frac{1}{2}\right) = 4\pi + 1$

Pole:
$\alpha \in \mathbb{R} \rightarrow$ pure exp.

c.)

$x[t] \rightarrow x[nT] \rightarrow \boxed{\downarrow 2} \rightarrow x_1[n]$

plot $X_1(\omega)$. in freq. transform Range $\omega \in [-4\pi, 4\pi]$



②

(2f.) take N = 64 point DFT. of $x_1[n]$.

$$X_1[k] = \sum_{n=0}^{64-1} x_1[n] W_N^{nk}.$$

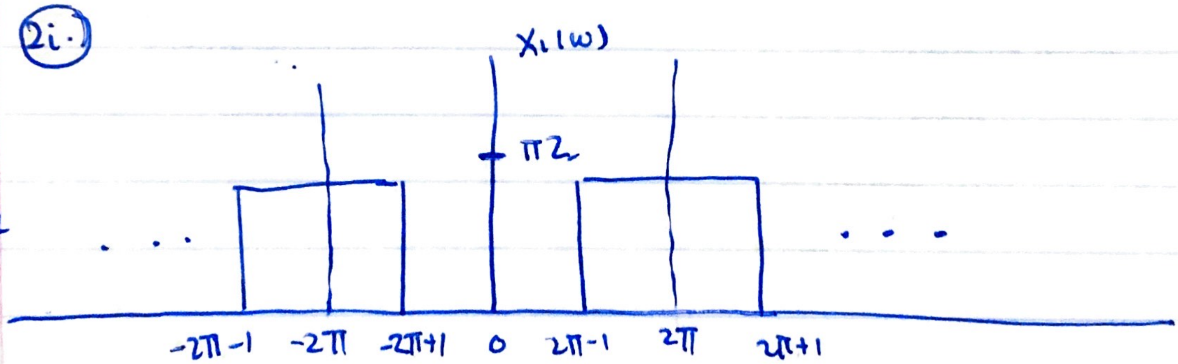
$$X_1[2] = \sum_{n=0}^{64-1} x_1[n] \underbrace{W_{64}^{n2}}$$

$$\boxed{= \sum_{n=0}^{63} x_1[n] W_{64}^{2n}}$$

$$\boxed{= \sum_{n=0}^{63} x_1[n] W_{32}^{n}}$$

(2i.)

$F_s = 2$



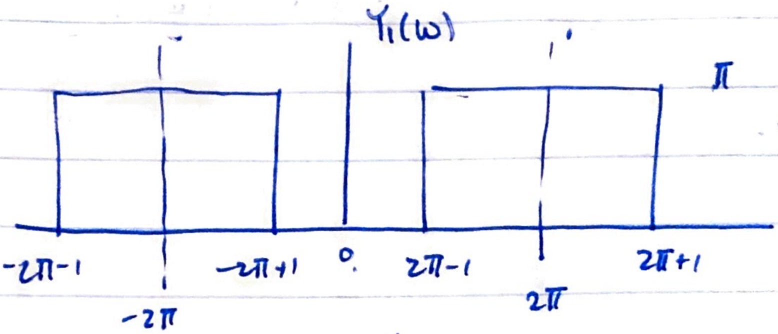
→ $Y_1(\omega)$ is on the back

(2g) It's able to be reconstucted because $\pi + 1 < 2\pi - 1$ in 2f. no aliasing. occured.

Fiter would be $W = 2\pi$, $W_0 = 1$

2h. The output is multiply by $\cos(\omega_c t)$ so that it Resembles $x(t)$.

2i.)



3.) Digital Design.

$|z| > 2\sqrt{2}$.

$$H(z) = \frac{1}{(z^2+8j)(z^2-0.5j)}$$

a.) poles & zero.

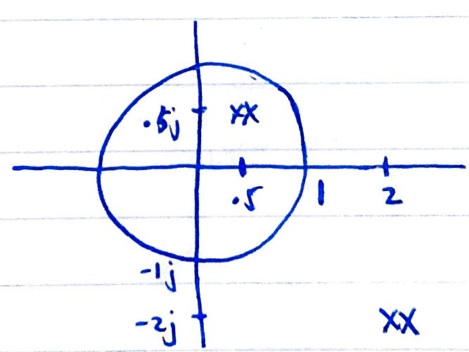
Poles: $z^2 + 8j = 0$. $\qquad z^2 - 0.5j = 0$

$z^2 = -8j \qquad z^2 = 0.5j$

$\boxed{z = 2-2j} \qquad \boxed{z = 0.5+0.5j}$

Zero: None.

b.)



ROC: $|z| > 2\sqrt{2}$

c.) The filter is not stable because the ROC does not contain the unit circle.

It is causal because ROC contain no poles and is outside of all poles.

d.) $\underbrace{\frac{1}{(z^2+8j)}}_{A.} \cdot \underbrace{\frac{1}{(z^2-0.5j)}}_{B}$

A^2-B^2

$(A+B)(A-B)$

A. $\frac{1}{z^2-(-8j)} = \frac{1}{z^2-(2-2j)^2} = \frac{1}{(z-(2-2j))(z+(2-2j))}$

$= \frac{A}{z-(2-2j)} + \frac{B}{z+(2-2j)}$

$A = \left.\frac{1}{z+(2-2j)}\right|_{z=2-2j} = \frac{1}{4-4j}$

$B = \frac{1}{-4+4j}$

B. $\frac{1}{z^2-0.5j} = \frac{1}{z^2-0.5j} = \frac{1}{(z-(0.5+0.5j))(z+(0.5+0.5j))}$

$= \frac{C}{z-(0.5+0.5j)} + \frac{D}{z+(0.5+0.5j)}$

$C = \left.\frac{1}{z+(0.5+0.5j)}\right|_{z=0.5+0.5j} = \frac{1}{1+j}$

$D = \frac{1}{-1-j}$

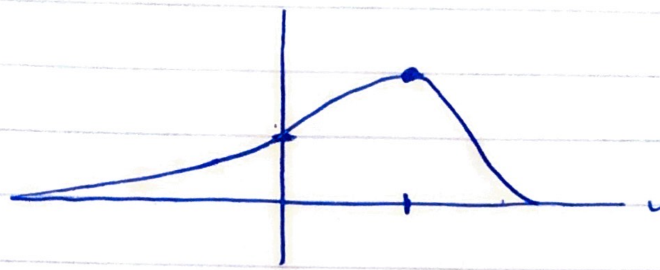
Pole:

(A): $$\frac{A}{z-(2-2j)} + \frac{B}{z+(2-2j)} \longleftrightarrow \frac{1}{4-4j}(2-2j)^{n-1}u(n-1)$$
$$+ \frac{1}{-4+4j}(-2+2j)^{n-1}u(n-1).$$

(B.) $$\frac{C}{z-(\frac{1}{2}+\frac{1}{2}j)} + \frac{D}{z+(\frac{1}{2}+\frac{1}{2}j)} \longleftrightarrow \frac{1}{1+j}\left(\frac{1}{2}+\frac{1}{2}j\right)^{n-1}u(n-1).$$
$$+ \frac{1}{-1-j}\left(-\frac{1}{2}-\frac{1}{2}j\right)^{n-1}u(n-1)$$

$$h[n] = \left[(A) * (B) \right]$$

e.) find $|H(\omega)|$



Low pass filter because at high freq. the magnitude is small (far away from poles.).

f.) $h[-n]$ poles?

$p_1 = 2-2j$ → $p_1 = \frac{1}{2-2j} = \underline{\frac{1}{4} + \frac{1}{4}j}$

$h[n]$ $\qquad$ $h[-n]$

$P_2 = 0.5+0.5j$ → $\frac{1}{0.5+0.5j} = \underline{1-1j}$

$h[n]$ $\qquad$ $h[-n]$

ROC : $\frac{1}{2\sqrt{2}} > |z|$

g.) It is not stable and not causal.

:)

h.) The given filter is not a linear phase filter because the filter is not stable, a Linear phas filter is always 1 stable.

4.) $x[n] = 0.6\,x[n-1] - 0.4\,x[n-2] + v[n]$

$\sigma_v^2 = 1$

$$R^{-1} = \begin{bmatrix} 1 & -0.6 & 0.4 \\ -0.6 & 1.2 & -0.6 \\ 0.4 & -0.6 & 0.1 \end{bmatrix}.$$

a.) $E\left[\, x[n-l]\; x[n] = x[n-l]\left[0.6x[n-1] - 0.4x[n-2] + v[n]\right]\right]$

$$\begin{bmatrix} r_{xx}[0] \\ \vdots \end{bmatrix} = \begin{bmatrix} R \end{bmatrix}\begin{bmatrix} a \end{bmatrix}.$$

$$\begin{bmatrix} \sigma_v^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R^{-1} \end{bmatrix}\begin{bmatrix} r_{xx}[0] \\ r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}.$$

We need to find R to get $r_{xx}[0]$, $r_{xx}[1]$, $r_{xx}[2]$.

$$r_{xx}[0](1) + r_{xx}[1](-0.6) + r_{xx}[2](0.4) = \sigma_v^2$$
$$r_{xx}[0](-0.6) + r_{xx}[1](1.2) + r_{xx}[2](-0.6) = 0.$$
$$r_{xx}[0](0.4) + r_{xx}[1](-0.6) + r_{xx}[2](0.1) = 0.$$

b.) $d[n] = x[n]$

$$\underbrace{\begin{bmatrix} r_{xx}(0) & r_{xx}(1) & r_{xx}(2) \\ r_{xx}(1) & r_{xx}(0) & r_{xx}(1) \\ r_{xx}(2) & r_{xx}(1) & r_{xx}(0) \end{bmatrix}^{-1}}_{R^{-1}} \underbrace{\begin{bmatrix} r_{xx}[0] \\ r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}}_{p[-k]} = \underbrace{\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}}_{\hat{h}}$$

c.) MMSE:

$$E[e^2(n)] = \sigma_v^2 - 2\hat{h}^T p + \hat{h}^T \hat{p} \hat{h}$$

d.) $\underbrace{x[n]}_{AR} + \underbrace{w[n]}_{MA} = y[n]$:

$$-\sum_{k=1}^{p} a_k x[n-k] + v[n] + \sum_{\ell=0}^{k} w[n-\ell] b_\ell = y[n].$$

$$\underbrace{\underbrace{\frac{Y(z)}{X(z)}}_{\sum_{k=0}^{3-1} h[k] z^{-k}}}_{\text{From previous problem.}} = \frac{\sum_{\ell=0}^{q} b_\ell z^{-\ell}}{\underbrace{\sum_{k=0}^{p} a z^{-k}}_{\text{From previous problem.}}}$$

$$\boxed{\sum_{\ell=0}^{3-1} b_\ell z^{-\ell} = \sum_{k=0}^{3-1} a_k z^{-k} \cdot \sum_{k=0}^{3-1} h_k z^{-k}.}$$

To get coefficient of MA component.

5.) $y[n] = Ax[n] + v[n]$, $n = 0, \cdots, N-1$.

$A \sim$ RV, 0-mean, $\sigma_A^2 \neq 0$.
$v \sim$ RV, 0-mean, $\sigma_v^2 \neq 0$.

ⓑ. Find R_y. $\ell = 0, \cdots, N-1$.

$$E[y[n-\ell]\, y[n]] = E[(Ax[n] + v[n])(y[n-\ell])].$$

$$\downarrow$$

$$R_y = E[(Ax[n] + v[n])(Ax[n-\ell] + v[n-\ell])]$$

$$\underline{R_y = \sigma_A^2 R_x + I \cdot \sigma_v^2}, \quad R_x = \begin{bmatrix} E[x[0]x[x-\ell] & \\ \vdots & \ddots \end{bmatrix}$$

$$\underline{\hat{h} = R_y^{-1} \cdot p.}$$

$$p[-k] = E[d[n]\, y[n-\ell]], \quad d[n] = y[n].$$

$$= E[y[n]\, y[n-\ell]].$$

$$= \begin{bmatrix} y[n]\, y[n-0] \\ \vdots \\ y[n]\, y[n-(N-1)] \end{bmatrix} = \begin{bmatrix} R_{yy}[0] \\ \vdots \\ R_{yy}[-(N-1)] \end{bmatrix}$$

$$\hat{h} = R_y^{-1} \circ p.$$

once we have $\hat{h}$, R_y, P, we can find $E[e(n)^2]$:

$$\sigma_d^2 - p^T R_y^{-1} p. = \sigma_v^2 - p^T(\sigma_A^2 R_x + I \cdot \sigma_v^2)^{-1} p$$

6.) $x[n]$ has 100 - 4KHZ information.

a.) $f_s = 8K$.

$x[n]$ now sample to 3kHZ.

by factor of .375, OR 3/8.

$y[n]$ ↑

$$x[n] \rightarrow \boxed{\uparrow 3} \rightarrow H\left(\min\left\{\frac{\pi}{3}, \frac{\pi}{8}\right\}\right) \rightarrow \boxed{\downarrow 8} \rightarrow y[n]$$

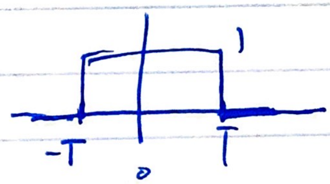
$f_s = 8000HZ$.

interpolation by factor 3

filter to Avoid aliasing.

decimation by factor 8

b.) $z(t) = \begin{cases} 1, & |t| < T \\ 0, & |t| > T. \end{cases}$



$\updownarrow$

$z(\omega) =$

Nyquist Rate is $2 \cdot \left(\frac{1}{2T}\right) = \frac{1}{T}$.

c.) The location of ZERO is different between the impulse invariance and bi-linear transformation. The location of the pole is the same.

d.)

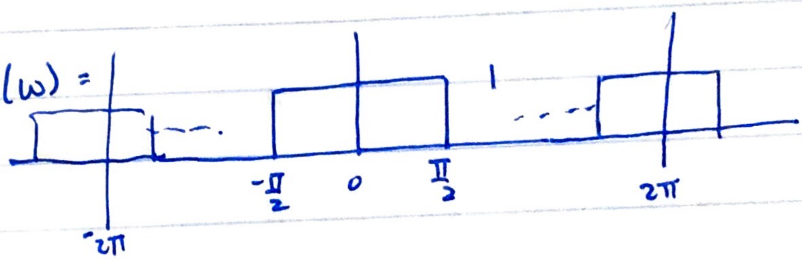
filter a.) FIR, BPF, Length is 5.

filter b.) IIR, BPF.

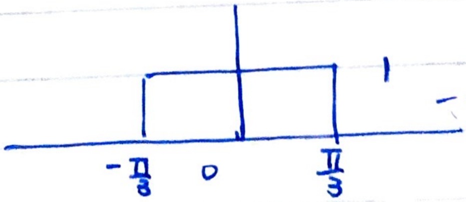
filter c.) IIR, LPF

filter d.) IIR, HPF.

e.) $X(\omega) =$



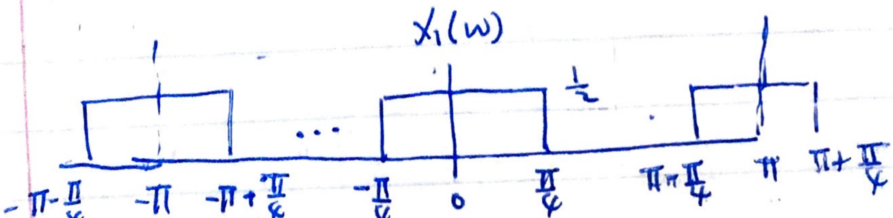
$H(\omega) =$



--- 2π periodic

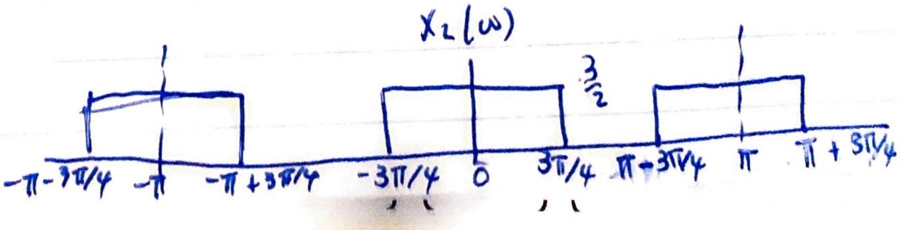
$x[n] \rightarrow \boxed{\uparrow 2} \rightarrow H(z^2) \equiv x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow 2} \rightarrow x_1[n]$

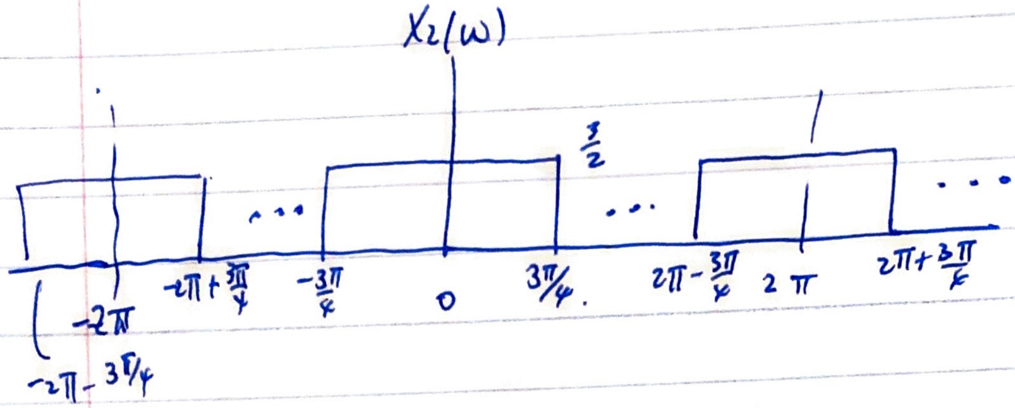
$X_1(\omega)$

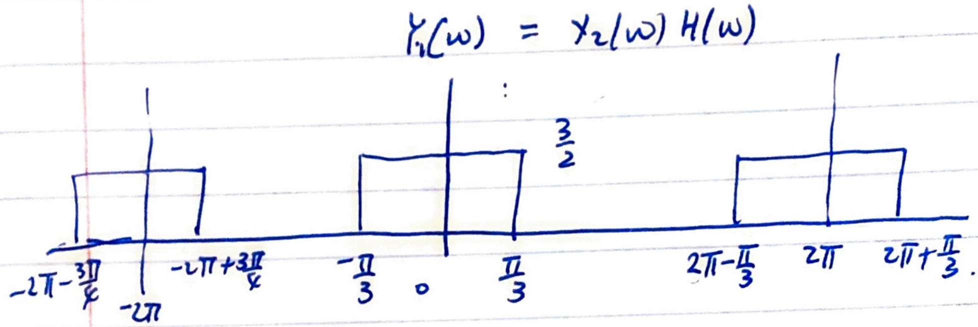


$x_1[n] \rightarrow \boxed{H(z)^3} \rightarrow \boxed{\downarrow 3} \equiv x[n] \rightarrow \boxed{\downarrow 3} \xrightarrow{X_2(\omega)} H(z) \rightarrow y[n]$

$X_2(\omega)$







I would use Sinc interpolation on x[n] because Sinc interpolation is ensuring x[n] can be Reconstructed. fully.

Linear interpolation can be applied on y[n] because linear interpolation can be tune to approximate y[n] to be closer to x[n].