Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 4530: Digital Signal Processing, Fall 2020

Homework #2 Solution: due Thursday, Oct. 1$^{\text{st}}$, at the beginning of class.

Analytical Problems:

5. (10 points) **Fourier Series and Fourier Transform.**

(a) Consider the signal $x[n] = \{-1, 0, \underline{1}, 2, 4\}$ with Fourier transform $X(\omega) = X_R(\omega) + jX_I(\omega)$. Determine the signal $y[n]$ with the Fourier transform $Y(\omega) = X_I(\omega) + X_R(\omega)e^{j2\omega}$.

The even part of the signal is given as $x_e[n] = \frac{x[n]+x[-n]}{2} = \{\frac{3}{2}, 1, \underline{1}, 1, \frac{3}{2}\}$ and the odd part is given as $x_o[n] = \frac{x[n]-x[-n]}{2} = \{-\frac{5}{2}, -1, \underline{0}, 1, \frac{5}{2}\}$. Then, you can note that

$$x_e[n] \leftrightarrow X_R(\omega)$$
$$x_o[n] \leftrightarrow jX_I(\omega).$$

Since $Y(\omega) = X_I(\omega) + X_R(\omega)e^{j2\omega}$, the signal $y[n]$ is given by

$$\begin{aligned} y[n] &= F^{-1}(X_I(\omega)) + F^{-1}(X_R(\omega)e^{j2\omega}) \\ &= -jx_o[n] + x_e[n+2] = -j\{-\frac{5}{2}, -1, \underline{0}, 1, \frac{5}{2}\} + \{\frac{3}{2}, 1, 1, 1, \underline{\frac{3}{2}}\} \end{aligned}$$

(b) Assume that $x[n]$ is aperiodic and with Fourier transform $X(\omega)$. Now we construct a periodic signal

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-kN].$$

i. What is the period of $y[n]$?

$$N.$$

ii. Determine the Fourier series coefficients of $y[n]$ using $X(\omega)$. Note that

$$x[n-kN] \leftrightarrow X(\omega)e^{-j\omega kN}.$$

Using linearity of FT $Y(\omega) = \sum_{k=-\infty}^{\infty} X(\omega)e^{-j\omega kN}$. The inverse FT is $y(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} X(\omega)e^{-j\omega kN}e^{j\omega t}\mathrm{d}\omega$. Hence the series coefficients are

$$\begin{aligned} a_l &= \frac{1}{N}\int_{-N/2}^{N/2} y(t)e^{-j\frac{2\pi}{N}lt}\mathrm{d}t \\ &= \frac{1}{N}\int_{-N/2}^{N/2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} X(\omega)e^{-j\omega kN}e^{j\omega t}\mathrm{d}\omega\right]e^{-j\frac{2\pi}{N}lt}\mathrm{d}t. \end{aligned}$$

Therefore, reordering the sum and integrals, we have

$$\begin{aligned}
a_l &= \frac{1}{N}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}\int_{-N/2}^{N/2}\left[X(\omega)e^{j\omega t}e^{-j\frac{2\pi}{N}lt}\mathrm{d}t\right]e^{-j\omega kN}\mathrm{d}\omega \\
&= \frac{1}{N}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}X(\omega)\int_{-N/2}^{N/2}\left[e^{jt(\omega-\frac{2\pi}{N}l)}\mathrm{d}t\right]e^{-j\omega kN}\mathrm{d}\omega \\
&= \frac{1}{N}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}X(\omega)\left[2\pi\delta(\omega-\frac{2\pi}{N}l)\right]e^{-j\omega kN}\mathrm{d}\omega \\
&= \frac{1}{N}X(\frac{2\pi}{N}l)
\end{aligned}$$

Alternative solution via DFT. We will later see the connection between Discrete Fourier Transform (DFT) and Discrete time Fourier Series. If you know how DFT is calculated, an alternative solution is:

$$\begin{aligned}
a_l &= \frac{1}{N}\sum_{n=0}^{N-1}y[n]e^{-j\frac{2\pi}{N}ln} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=-\infty}^{\infty}x[n-kN]\right]e^{-j\frac{2\pi}{N}ln} \\
&= \frac{1}{N}\sum_{k=-\infty}^{\infty}\sum_{m=-kN}^{N-1-kN}x[m]e^{-j\frac{2\pi}{N}l(m+kN)}
\end{aligned}$$

However, $\sum_{k=-\infty}^{\infty}\sum_{m=-kN}^{N-1-kN}x[m]e^{-j\omega(m+kN)} = X(\omega)$. You can see this through a change of variables ($m+kN \to m$) Therefore,

$$a_l = \frac{1}{N}X(\frac{2\pi l}{N}).$$

6. (20 points) **Discrete-Time Fourier Transform (DTFT).**

   (a) Compute the discrete-time Fourier transform (DTFT) $X(\omega)$ of $x[n] = \mathrm{sinc(n)}\cdot\mathrm{sinc(n)}$.

   Let $y[n] = \mathrm{sinc(n)}$. Since convolution in the time domain corresponds to multiplication in the frequency domain, from duality, we have the following DTFT pair:

$$x[n] = y[n]\cdot y[n] \overset{DTFT}{\leftrightarrow} X(\omega) = \frac{1}{2\pi}Y(\omega)*Y(\omega),$$

   where $Y(\omega) = \sum_{n=-\infty}^{\infty}y[n]e^{-j\omega n}$.

   Let $Y(\omega) = \begin{cases}\pi, & |\omega| < w_c, \\ 0, & w_c < |\omega| < \pi.\end{cases}$ which is periodic with 2π. Then

$$\begin{aligned}
y[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}Y(\omega)e^{j\omega n}\mathrm{d}w = \frac{1}{2\pi}\int_{-w_c}^{w_c}\pi e^{j\omega n}\mathrm{d}\omega \\
&= \frac{1}{2}\frac{1}{jn}e^{j\omega n}\Big|_{-w_c}^{w_c} = \frac{\sin(w_c n)}{n} = w_c\mathrm{sinc(w_c n)},
\end{aligned}$$

Hence, we can infer that $y[n] = \mathrm{sinc(n)} \overset{\mathrm{DTFT}}{\leftrightarrow} \mathrm{Y}(\omega)$ for $w_c = 1$. Therefore, the DTFT of $x[n]$ over one period is given by

$$X(\omega) = \frac{1}{2\pi} Y(\omega) * Y(\omega) = \begin{cases} \frac{\pi}{2}(2-|\omega|), & |\omega| < 2, \\ 0, & 2 < |\omega| < \pi. \end{cases}$$

(b) Compute $\sum_{n=-\infty}^{\infty} x^2[n]$ for $x[n]$ as given in Part (a).

From Parseval's theorem, we have that $\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \mathrm{d}\omega = \frac{1}{\pi}\int_0^{\pi} |X(\omega)|^2 \mathrm{d}\omega = \frac{1}{\pi}\int_0^{2} \frac{\pi^2}{4}(2-\omega)^2 \mathrm{d}\omega = \frac{\pi}{4}\int_0^2 (2-\omega)^2 \mathrm{d}\omega = \frac{\pi}{4}\int_0^2 (4-4\omega+\omega^2)\mathrm{d}\omega = \frac{\pi}{4}(4\omega - 2\omega^2 + \omega^3/3)\Big|_0^2 = \frac{\pi}{2}(4-2\cdot 2+2^2/3) = \frac{2\pi}{3}$.

(c) Compute and plot the DTFT $X(\omega)$ of the signal $x[n] = x[n+10]$ given as follows

$$x[n] = \begin{cases} 3-|n+2|, & n = -5,\ldots,1 \\ 0, & n = -9,\ldots,-6. \end{cases}$$

which is shown in Figure 1.

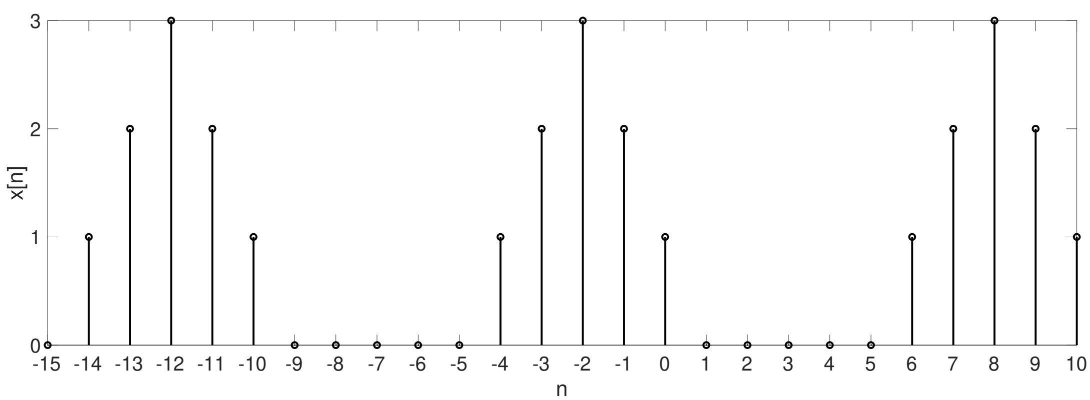


Figure 1: Periodic discrete-time triangular wave.

The signal shown is a periodic triangular signal where the signal over one period is convolution of two rectangular signals. Let $y[n] = \begin{cases} 1, & n = -1,0,1 \\ 0, & \text{else} \end{cases}$. Hence, the signal is given by the following convolution of $y[n]$ and the periodic rectangular pulse train $\sum_{k=-\infty}^{\infty} y[n-kN]$:

$$x[n] = y[n] * \sum_{k=-\infty}^{\infty} y[n-kN],$$

where the period is $N = 10$. Furthermore,

$$\sum_{k=-\infty}^{\infty} y(n-kN) = y[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN].$$

Hence, we have $X(\omega) = Y(\omega)Y(\omega)\mathrm{DTFT}\left(\sum_{k=-\infty}^{\infty} \delta(n-kN)\right)$.

Recall that $\delta(n-kN) \overset{DTFT}{\leftrightarrow} e^{-jkNw}$. Furthermore, $Y(\omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-jwn} = \sum_{n=-1}^{1} e^{-jwn} = e^{jw} + 1 + e^{-jw}$. Therefore,

$$X(\omega) = (e^{j\omega} + 1 + e^{-j\omega})^2 \sum_{k=-\infty}^{\infty} e^{-jkN\omega} = (1 + 2\cos(\omega))^2 \sum_{k=-\infty}^{\infty} e^{-jkN\omega}.$$

Recall that $1 \overset{DTFT}{\leftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$. You can see this by taking the inverse DTFT

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta(\omega) e^{j\omega n} \mathrm{d}\omega = 1.$$

Therefore, we infer that $\sum_{k=-\infty}^{\infty} e^{-j\omega kN} = \sum_{k=-\infty}^{\infty} 1.e^{-j(\omega N)k} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega N - 2\pi k)$. Hence, $X(\omega) = (1 + 2\cos(\omega))^2 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega N - 2\pi k)$. The MATLAB plot (along with the code) is given below.

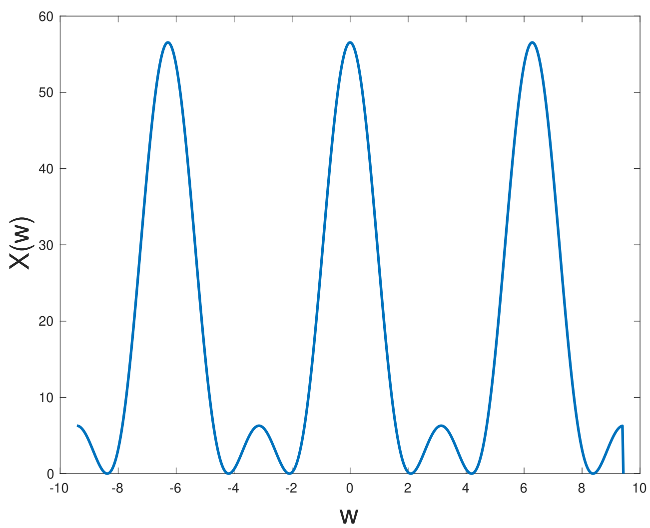


Figure 2: DTFT of $x[n]$.

```
w = -3*pi:pi/100:3*pi;    % Remember the DTFT will be 2pi-periodic

%Generate periodic impulse train
N=10; %period of the impulse train
ws=1/(2*pi/N); % sample frequency
Y=zeros(size(w));
Y(1:1/ws:end)=1;

X = (1+2*cos(w)).^2*2*pi.*Y;

 figure;
 plot(w,X,'linewidth',2)
```

```
xlabel('w','fontsize',20)
ylabel('X(w)','fontsize',20)
```

(d) Determine the range of values of α and β for which the LTI system with input $x[n] = \alpha^n u[n]$ and impulse response $h[n] = \beta^n(u[n]-u[n-3])$ is stable.

For stability of an LTI system, we require that $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$. Thus, $\sum_{n=-\infty}^{\infty} |\beta^n(u[n]-u[n-3])| = \sum_{n=0}^{2} |\beta^n|$ since $u[n]-u[n-3] = 1$ only for $n = 0,1,2$, and $u[n]-u[n-3] = 0$ otherwise. We can observe that $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{2} |\beta^n| = 1+|\beta|+\beta^2$, and hence the system is stable as long as β is finite (i.e. $|\beta| < \infty$).

(e) Consider a linear time-invariant (LTI) system with impulse response $h[n] = \frac{1}{2}e^{-n}u[n]+\frac{1}{2}e^{-3n}u[n]$. Let $y[n]$ be the output for the input $x[n] = e^{-n}u[n]$.

i. Compute the frequency response $H(\omega)$ of the system.
$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}e^{-n}u[n]+\frac{1}{2}e^{-3n}u[n]\right)e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-n}+e^{-3n}\right)e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-(1+j\omega)}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-(3+j\omega)}\right)^n = \frac{1/2}{1-e^{-(1+j\omega)}} + \frac{1/2}{1-e^{-(3+j\omega)}}$

ii. Compute the discrete-time Fourier transform of $y[n]$, i.e. $Y(\omega)$.
The FT is given as $Y(\omega) = X(\omega)H(\omega)$ where $X(\omega) = \sum_{n=-\infty}^{\infty} e^{-n}u[n]e^{-j\omega n} = \sum_{n=0}^{\infty}\left(e^{-(1+j\omega)}\right)^n = \frac{1}{1-e^{-(1+j\omega)}}$. Hence, $Y(\omega)$ is given as

$$Y(\omega) = \frac{1/2}{(1-e^{-(1+j\omega)})^2} + \frac{1/2}{(1-e^{-(3+j\omega)})(1-e^{-(1+j\omega)})}.$$

iii. Compute $y[n]$ using $Y(\omega)$.
We use partial fraction expansion to write $Y(\omega)$ as

$$Y(\omega) = \frac{a}{(1-e^{-(1+j\omega)})^2} + \frac{b}{1-e^{-(1+j\omega)}} + \frac{c}{1-e^{-(3+j\omega)}},$$

where note that $a = 1/2$ and we have the following relation:

$$b\left(1-e^{-(3+j\omega)}\right) + c\left(1-e^{-(1+j\omega)}\right) = 1/2.$$

Rearranging the terms

$$\begin{aligned} b+c-be^{-(3+jw)}-ce^{-(1+j\omega)} &= 1/2 \\ b+c = 1/2, \quad -be^{-(3+j\omega)}-ce^{-(1+j\omega)} &= 0 \\ b+c = 1/2, \quad (-be^{-2}-c)e^{-(1+j\omega)} &= 0 \\ b+c = 1/2, \quad -be^{-2}-c &= 0 \\ b = \frac{1}{2(1-e^{-2})}, \quad c = -\frac{e^{-2}}{2(1-e^{-2})}&. \end{aligned}$$

Hence, $y[n]$ is computed as $y[n] = a(e^{-n}u[n])*(e^{-n}u[n])+be^{-n}u[n]+ce^{-3n}u[n]$. Evaluating the convolution, $(e^{-n}u[n])*(e^{-n}u[n]) = \sum_{k=-\infty}^{\infty} e^{-k}u[k]e^{-(n-k)}u[n-k] = \sum_{k=0}^{n} e^{-k}e^{-(n-k)}u[n] = \sum_{k=0}^{n} e^{-n} = (n+1)e^{-n}u[n]$. Therefore,

$$y[n] = a(n+1)e^{-n}u[n] + be^{-n}u[n] + ce^{-3n}u[n].$$

7. (10 points) **Frequency response of LTI systems.**

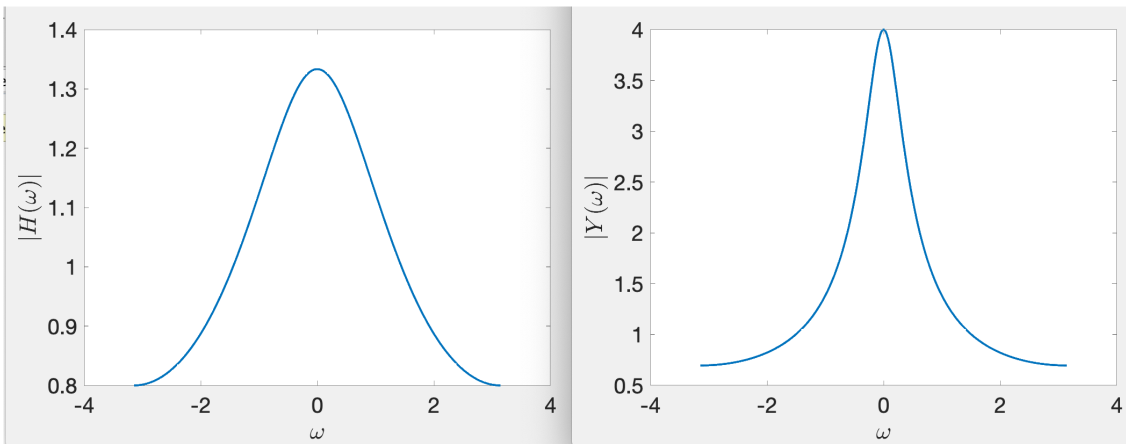
(a) Consider an LTI system with impulse response $h[n] = \left(\frac{1}{4}\right)^n u[n]$.

i. Determine and sketch the magnitude response, i.e., $|H(\omega)|$.

$$H(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{4} e^{-j\omega}\right)^n = \frac{1}{1 - \frac{1}{4} e^{-j\omega}}.$$

ii. Determine and sketch the magnitude response of the LTI system for $x[n] = \cos\left(\frac{3\pi n}{10}\right)$. Hint: eigenfunction.

$$|Y(\omega)| = \frac{1}{\sqrt{(1 - 1/4\cos(\omega))^2 + (1/4\sin(\omega))^2}} = \frac{1}{\sqrt{\frac{17}{16} - \cos(\omega)}}$$



(b) An FIR filter is described by the relation

$$2y[n] = x[n] + x[n-1].$$

i. Is this system LTI?
Yes. This is a moving average system as we have seen in the early lectures.

ii. Let $x_1[n] = \delta[n]$, $x_2[n] = u[n]$ and $x_3[n] = e^{j\omega n}$ be the respective inputs to the above FIR filter. Determine which of these inputs are eigenfunctions of the system.
The first input is not an eigenfunction, i.e., there is no constant c such that

$$x_1[n] = \delta[n], \quad y[n] = \frac{\delta[n] + \delta[n-1]}{2} = \begin{cases} 1/2, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases} \neq c\delta[n]$$

The second input is not an eigenfunction, i.e., there is no constant c such that

$$x_2[n] = u[n], \quad y[n] = \frac{u[n] + u[n-1]}{2} = \begin{cases} 1, & n \geq 1 \\ 1/2, & n = 0 \\ 0, & \text{otherwise} \end{cases} \neq cu[n]$$

The third input is an eigenfunction because it is a complex exponential and we know that complex exponentials are eigenfunctions of LTI systems.

8. (10 points) **z-transform.** Determine the z-transform of the following signals. Do not forget to specify the region of convergence (ROC) for each part.

   (a) $x[n] = \{3, 0, 0, 0, 0, \underline{6}, 1, 4\}$

   $$X(z) = \sum_n x[n] z^{-n} = 3z^5 + 6 + z^{-1} + 4z^{-2}$$

   ROC: $0 < |z| < \infty$.

   (b) $x[n] = (0.1^n + 0.1^{-n})u[n]$

   $$X(z) = \sum_n x[n] z^{-n} = \sum_{n=0}^{\infty} (0.1^n + 0.1^{-n}) z^{-n}$$

   Note that $\sum\limits_{n=0}^{\infty} 0.1^n z^{-n} = \frac{1}{1-0.1z^{-1}}$ where ROC: $|z| > 0.1$ and $\sum\limits_{n=0}^{\infty} 0.1^{-n} z^{-n} = \frac{1}{1-10z^{-1}}$ where ROC: $|z| > 10$.

   Hence $X(z) = \frac{1}{1-0.1z^{-1}} + \frac{1}{1-10z^{-1}} = \frac{2-10.1z^{-1}}{(1-0.1z^{-1})(1-10z^{-1})}$ where ROC: $z > \max(10, 0.1) = 10$.

Readings from textbook: 2.4, 3.1-3.3, 4.1-4.4, 5.1-5.2.
Suggested practice problems from textbook: 2.45, 2.48, 3.14, 3.18, 3.40, 3.42, 4.4, 4.5, 4.9, 4.10, 5.4.