HW #5. DSP 11/7/20.

5.) Nyquist Rate.

Consider $x(t) \leftrightarrow X(\omega) = \begin{cases} X(\omega) & \omega < |\omega_B| \\ 0 & \text{else.} \end{cases}$

a.) $y(t) = x(t) + x(t-2)$

$x(t-2) \leftrightarrow e^{-j\omega 2} X(\omega)$

↑ phase shift

$x(t) \leftrightarrow X(\omega)$ only affect Amplitude.

$x(t) + x(t-2) \leftrightarrow X(\omega)(1 + e^{-j\omega 2})$

Nyquist Rate $\omega_{nyq} = 2\omega_B$ Rad/s

b.) $y(t) = x(2t)$

$x(2t) \leftrightarrow \frac{1}{|2|} X(\frac{\omega}{2})$ affect Amp & Band. down sampling.

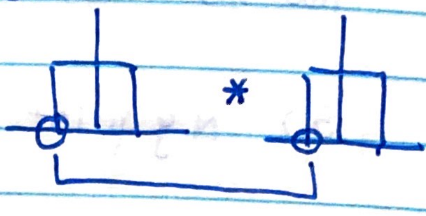
Nyquist Rate $= 2 \cdot 2 \cdot \omega_B = 4\omega_B$. Rad/s.

$\frac{\omega}{2} = \omega_B$

$\omega = 2\omega_B$ (Band-limited freq.)

c.) $y(t) = (x(t))^2 = x(t) \cdot x(t)$

$Y(\omega) = \frac{1}{2\pi} \int X(u)\, X(\omega - u)$



Nyquist Rate : $2 \cdot 2\omega_B = 4\omega_B$ rad/s

d.) $y(t) = x(t) \cos(\omega_0 t)$

$y(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * [\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)]$

$= \frac{1}{2} [X(\omega - \omega_0) + X(\omega + \omega_0)]$

Nyquist Rate = $2 \cdot (\omega_B + \omega_0)$ rad/s

e.) $y(t) = x(t) * \frac{1}{t} \sin(\omega_c t)$

$x_1 = \frac{1}{t} \sin(\omega_c t) \leftrightarrow X_1(w) = \begin{cases} \pi & w < |\omega_c| \\ 0 & \text{else.} \end{cases}$

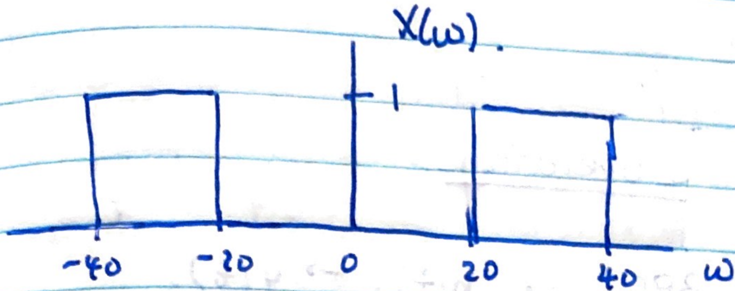
case 1: Nyquist Rate $= 2\omega_c$ Rad/s since X_1 attenuate all information outside of $\omega < |\omega_c|$, $|\omega_c| < \omega_B$

case 2: Nyquist Rate $= 2\omega_B$ Rad/s since $\omega < |\omega_B|$, $|\omega_c| > \omega_B$.

so as a Result. Nyquist Rate = $\min\{2\omega_B, \omega_c \cdot 2\}$

6.) Aliasing. $x(t)$ has CTFT of

$$X(\omega) = \begin{cases} 1 & 20 < |\omega| < 40 \\ 0 & \text{else.} \end{cases}$$



a.) What is the Nyquist Rate?

$$2\omega_B = 2 \cdot 40 = 80. \ \text{rad/s}.$$

b.) What if $x[n] = x(t\,\frac{2\pi}{20})$? $X(\omega/T)$

$T = \frac{2\pi}{20}$

$f = 20/2\pi$

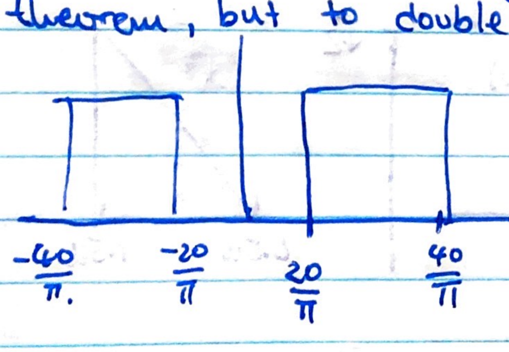
$\omega_s = 20$

$$\boxed{\omega_s < 2\omega_B}$$

there will be aliasing. by the theorem, but to double check $X(\frac{\omega}{T})$

$x(t\,\frac{2\pi}{20}) \Rightarrow$

down sampling.



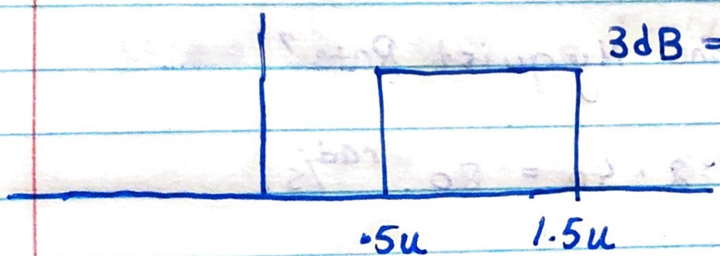
it's actually not aliasing, so the Conclusion here is no aliasing.

7.) Filtering and Sampling.

$x(t)$ is a real signal.
$X(\omega)$ is known
$u > 0$.

We have 2 operations:

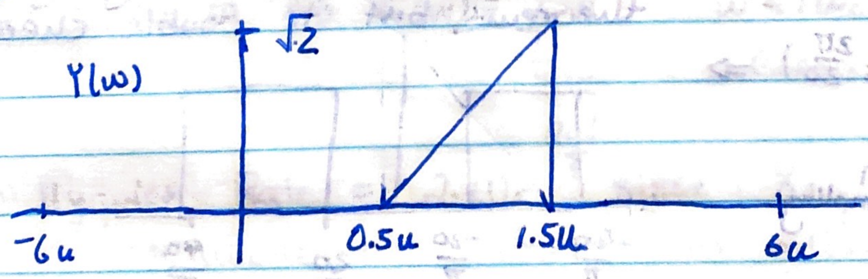
1. $x(t)$ is BPF by $h(t)$. $\rightarrow y(t)$.



2. Sample the Result by $T = \frac{\pi}{2u}$, $y(n)$

$y(t) = x(t) * h(t) \leftrightarrow Y(\omega) = X(W)\, H(W)$.

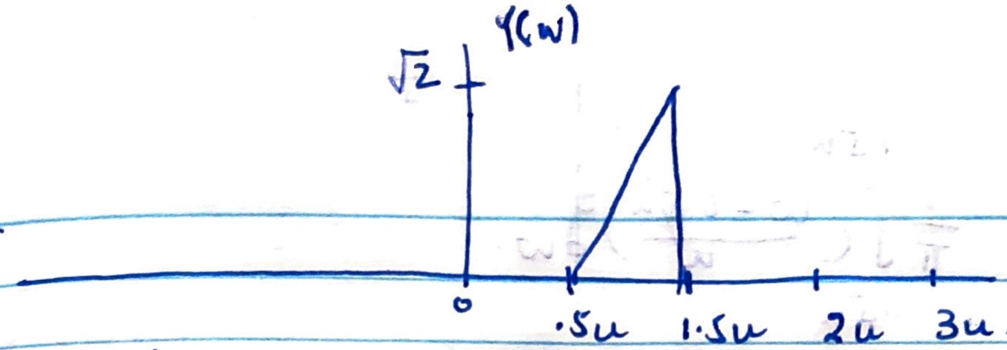
1

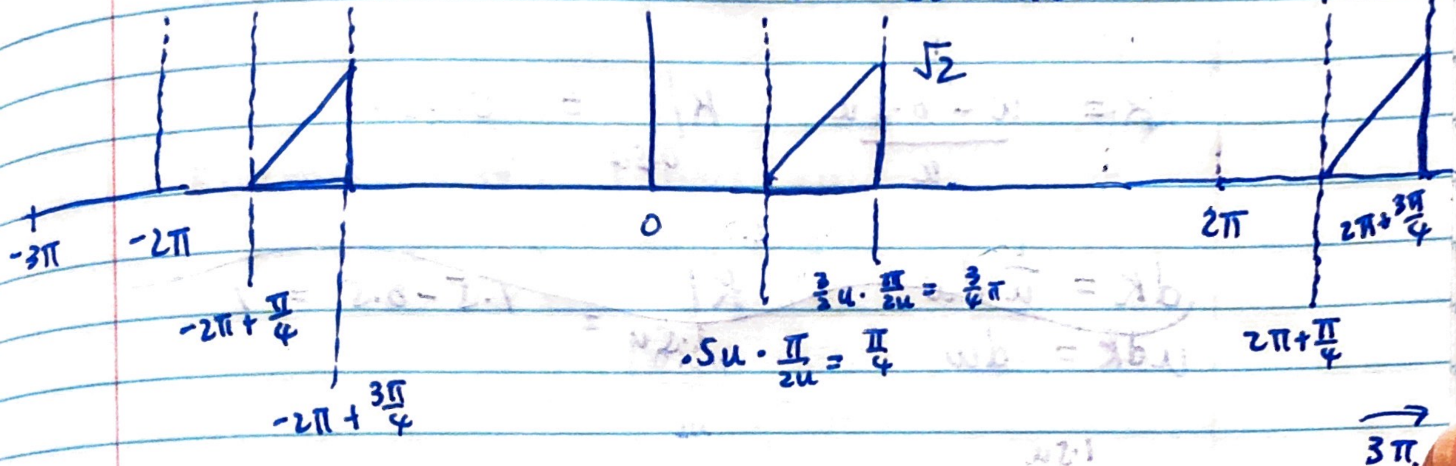


2

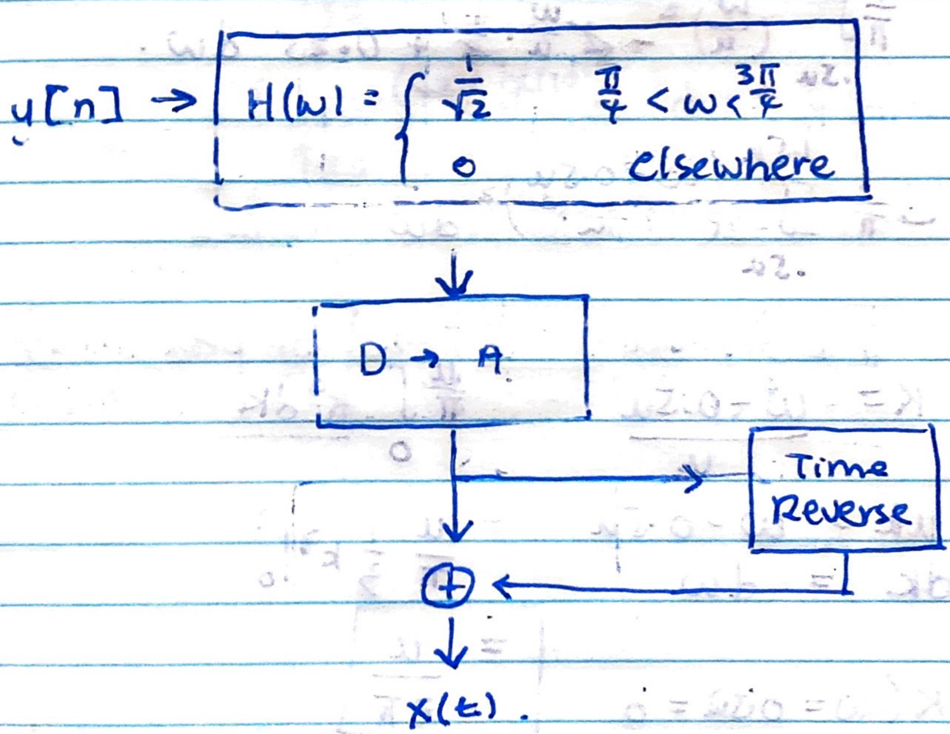
$y[n] = y(t\frac{\pi}{2u}) \leftrightarrow \text{DTFT}\{y(n)\} = \frac{1}{T}\, Y(\frac{\omega}{T})$

2.





c.)



d.) $\int |x(t)|^2 dt = \frac{1}{2\pi}\int |x(\omega)|^2 d\omega$

$$= \frac{2}{2\pi}\int_{+1.5u}^{.5u}\left(\frac{1}{u}(\omega) - 0.5\right)^2 d\omega.$$

$$= \frac{1}{\pi}\int_{+1.5u}^{.5u}\left(\frac{\omega - 0.5u}{u}\right)^2 d\omega$$

$$\frac{1}{\pi}\int_{.5u}^{1.5u} \left(\frac{w-0.5u}{u}\right)^2 dw.$$

$$K = \frac{w-0.5u}{u} \qquad K\Big|_{0.5u} = 0.$$

$$\cancel{dK = \frac{1}{u}\,dw \qquad K\Big|_{1.5u} = 1.5-0.5 = 1}$$

$$u\,dK = dw$$

$$\frac{1}{\pi}\int_{.5u}^{1.5u} \left(\frac{w}{u}\right)^2 - 2\frac{w}{u}\cdot\frac{1}{2} + 0.25\,dw.$$

$$\frac{1}{\pi}\int_{.5u}^{1.5u} \left(\frac{w-0.5u}{u}\right)^2 dw$$

$$K = \frac{w-0.5u}{u} \qquad \frac{u}{\pi}\int_0^1 K^2\,dK$$

$$uK = w-0.5u \qquad = \frac{u}{\pi}\,\frac{1}{3}K^3\Big|_0^1$$

$$u\,dK = dw$$

$$\boxed{= \frac{u}{3\pi}}$$

$$K(w = 0.5u) = 0$$

$$K(w = 1.5u) = 1$$

Energy is $\frac{u}{3\pi}$.

energy of y(t).

$y(t) \leftrightarrow Y(\omega)$

Energy : $\frac{1}{2\pi}\int_{.5u}^{1.5u} |Y(\omega)|^2 \, d\omega.$

$\frac{1}{2\pi}\int_{.5u}^{1.5u} |H(\omega) \cdot X(\omega)|^2 \, d\omega$

$\sqrt{2}$

$\frac{1}{2\pi}\int_{.5u}^{1.5u} (\sqrt{2})^2 |X(\omega)|^2 \, d\omega.$

$\frac{1}{\pi}\int_{.5u}^{1.5u} |X(\omega)|^2 \, d\omega$

$\Rightarrow \boxed{\frac{u}{3\pi}.}$

8.) We know there's a signal $x[n]$

We want $x[n-k]$

$k \neq \mathbb{Z}$

What is the ideal freq. Response of this filter?

$x[n] \longleftrightarrow X(W)$

$x[n-k] \longleftrightarrow e^{-jkw} X(w)$

$h_k[n] \longleftrightarrow \boxed{e^{-jkw} = H_k(w)}$

in a sense:

$x[n] \rightarrow \boxed{h_k(n)} \rightarrow x[n-k]$

9.1 Downsampling + Upsampling.

y_1

$$x[n] \rightarrow \boxed{\uparrow 2} \xrightarrow{x_b(n)} \boxed{\downarrow 2} \rightarrow y_1[n]$$

$L = 2$ $\quad$ $M = 2$

$T \quad \frac{T}{2} \quad 2\cdot\frac{T}{2} \quad T$

$$X_b(n) = x[\tfrac{n}{L}] \longleftrightarrow X_b(\omega) = X(\omega L)$$

$$Y_1(n) = X_b(Mn) \longleftrightarrow Y_1(\omega) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(\frac{\omega L}{M} - \frac{2\pi i}{M}\right)$$

$M=2,\ L=2$

$$= \frac{1}{M}\sum_{i=0}^{M-1} X(\omega - \pi i)$$

$$= \frac{1}{M}\left(X(\omega) + X(\omega - 2\pi)\right)$$

y_2

$$x[n] \rightarrow \boxed{\downarrow 2} \xrightarrow{x_a(n)} \boxed{\uparrow 2} \rightarrow y_2(n)$$

$$x_a[n] = x[Mn] \longleftrightarrow X_a(W) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(\frac{W}{M} - 2\pi i/M\right)$$

$M=2$

$$= \frac{1}{2}\sum_{i=0}^{1} X\left(\frac{W}{2} - \pi i\right)$$

$$= \frac{1}{2}\left(X\left(\frac{W}{2}\right) + X\left(\frac{W}{2} - \pi\right)\right)$$

$$Y_2(n) = X_a[n/L] \longleftrightarrow Y_2(W) = X_a(L\omega)$$

$$= \frac{1}{2}\left(X(\omega) + X(\omega - \pi)\right)$$

$\therefore\ Y_2(\omega) \neq Y_1(\omega).$

now:

$$x(n) \rightarrow \underset{L=2}{\boxed{\uparrow 2}} \xrightarrow[x_a(n)]{} \boxed{H(z)} \xrightarrow[x_b(n)]{} \underset{M=2}{\boxed{\downarrow 2}} \rightarrow y(n)$$

$$x_a(n) = x\left(\frac{n}{L}\right) \text{ for } n = \pm L, \pm 2L, \pm 3L, \ldots$$

$$X_a(w) = X(Lw), \quad L = 2.$$

$$x_b(n) = x_a(n) * h(n) \leftrightarrow X_a(w) \cdot H(\omega)$$

$$X_b(w) = X(L\omega) H(\omega)$$

$$= X(2\omega) H(\omega)$$

$$y(n) = x_b(Mn), \quad M = 2.$$

$$Y(\omega) = \frac{1}{M} \sum_{i=0}^{M-1} X_b\left(\frac{w}{M} - \frac{2\pi i}{M}\right).$$

$$= \frac{1}{2} \sum_{i=0}^{1} X_b\left(\frac{w}{2} - \frac{2\pi i}{2}\right)$$

$$= \frac{1}{2}\left(X_b\left(\frac{w}{2}\right) + X_b\left(\frac{w}{2} - \pi\right)\right)$$

in DTFT, signal is alway 2π periodi

$$= \frac{1}{2}\left(X(\omega) H\left(\frac{w}{2}\right) + X(\omega - 2\pi) H\left(\frac{w}{2} - \pi\right)\right)$$

$$= \frac{1}{2}\left(X(\omega) H\left(\frac{w}{2}\right) + X(\omega) H\left(\frac{w}{2} - \pi\right)\right)$$

$$\frac{Y(w)}{X(w)} = \frac{1}{2}\left(H\left(\frac{w}{2}\right) + \underbrace{H\left(\frac{w}{2} - \pi\right)}_{H\left(\frac{1}{2}(w - 2\pi)\right)}\right)$$

$$z = e^{j\left(\frac{w}{2} - \pi\right)}$$

$$= e^{j\frac{w}{2}} \cdot e^{-j\pi}$$

$$-z = e^{j\frac{w}{2}}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{2}\left(H\left(\frac{z}{2}\right) + H\left(\frac{-z}{2}\right)\right)$$

10.) Polyphase Signal Processing.

consider transfer function $H(z)$.

a.) Find $H_e(z)$ where $h_e(n) = h(2n)$.

$$\cancel{h(n) \rightarrow \boxed{\downarrow 2} \rightarrow h_e(n)}$$

$$\cancel{M = 2.}$$

$$\cancel{h_e(n) = h(2n) \leftrightarrow \frac{1}{2}\sum_{i=0}^{2-1} H\left(\frac{w}{2} - 2\pi i/2\right)}$$

$$\cancel{= \frac{1}{2}\left(H\left(\frac{w}{2}\right) + H\left(\frac{w}{2} - \pi\right)\right)}$$

$$\cancel{H_e(w) = \frac{1}{2}\left(H(z/2) + H(-z/2)\right)}$$

$- \dashrightarrow$

$$h_e(n) = h(2n) \leftrightarrow H_e(z) = \sum_{n=0}^{\infty} h(2n) z^{-n}, \quad |z| \neq 0$$

b.) Find $H_o(z)$ where $h_o(n) = h(2n+1)$

$$H_o(z) = \sum_{n=0}^{\infty} h(2n+1) z^{-n}, \quad |z| \neq 0.$$

c.) $H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n}^{\infty} h(2n) z^{-n} + \sum_{n}^{\infty} h(2n+1) z^{-n}$

$$= \sum_{n=0}^{\infty} h(2n) z^{-2n} + \sum_{n=0}^{\infty} h(2n+1) z^{-(2n+1)}$$

$$= \sum_{n=0}^{\infty} h(2n)(z^2)^{-n} + \sum_{n=0}^{\infty} h(2n+1)(z^2)^{-n} \cdot z^{-1}$$

$$= \sum_{n=0}^{\infty} h(2n)(z^2)^{-n} + z^{-1} \sum_{n=0}^{\infty} h(2n+1)(z^2)^{-n}$$

$$= H_e(z^2) + z^{-1} H_o(z^2),$$

$$|z| > 0$$

d.) $H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$

$$= \sum_{n=0}^{\infty} h_1(Mn)(z^M)^{-n} + \sum_{n=0}^{\infty} h_2(Mn+1)(z^M)^{-n} \cdot z^{-1}$$

$$+ \sum_{n=0}^{\infty} h_3(Mn+2)(z^M)^{-n} z^{-2} + \cdots$$

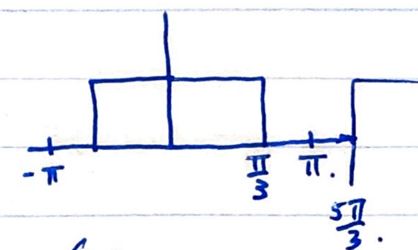
$$= H_1(z^M) + z^{-1} H_2(z^M) + z^{-2} H_3(z^M) + \cdots$$
$$+ z^{-(M-1)} H_M(z^M)$$

$$H_M(z) = \sum_{n=0}^{\infty} h_M(Mn+M-1) z^{-n}.$$

11.) FIR filter Design.

Design an FIR filter w/ linear phase that Approximates the ideal freq. Resp.

$$H_d(w) = \begin{cases} 1, & |w| \leq \frac{\pi}{3} \\ 0, & \pi \geq |w| \geq \frac{\pi}{3}. \end{cases}$$



a.) Determine the coefficient of a 15 tap filter Based on the IDFT method.

$\frac{25\pi}{15}$

$N = 15$

I assume that $h[n]$ is Real and exhibit Symmetry about $M = {}^{N-1}/_2$ Sample

Equally Spaced value for $A_d(w)$

$$w = \frac{2\pi K}{N} = \frac{2\pi K}{15}, \{k = 0, \cdots, N-1\}$$

$$A_d = \left[1 \Big|_{\frac{2\pi K}{15} < \frac{\pi}{3} = \frac{5\pi}{15}}, \; 0 \Big|_{\frac{2\pi K}{15} > \frac{\pi}{3}} \right]$$

$$= \left[\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \frac{2}{15} & \frac{4}{15} & \frac{6}{15} & \frac{8}{15} & \frac{10}{15} & \frac{12}{15} & \frac{14}{15} & \frac{16}{15} & \frac{18}{15} & \frac{20}{15} & \frac{22}{15} & \frac{24}{15} & \frac{26}{25} & \frac{28}{25} \end{matrix} \right]$$

$$h[n] = \frac{1}{15} \sum_{k=0}^{14} A[0] + \sum_{k=1}^{M} 2A[k] \cos \frac{2\pi(n-M)k}{N}$$

$$h[n] = \frac{1}{15}\left[A[0] + \sum_{K=1}^{M} 2A[K]\cos\left(\frac{2\pi(n-M)}{N}K\right)\right]$$

$$M = \frac{N-1}{2} = \frac{15-1}{2} = 7.$$

$$N = 15$$

$$h[14] = h[0] = \frac{1}{15}\left[1 + \sum_{K=1}^{7} 2A[K]\cos\left(\frac{2\pi(0-7)}{15}K\right)\right]$$

$$= \frac{1}{15}\left[1 + \sum_{K=1}^{2} 2\cos\left(\frac{-14\pi}{15}K\right)\right]$$

$$= \frac{1}{15}\left[1 + 2\cos\left(-{}^{14}\!/_{15}\pi\right) + 2\cos\left(\frac{-28\pi}{15}\right)\right]$$

$$h[13] = h[1] = \frac{1}{15}\left[1 + 2\cos\left(\frac{-12\pi}{15}\right) + 2\cos\left(\frac{-24\pi}{15}\right)\right]$$

$$h[12] = h[2] = \frac{1}{15}\left[1 + 2\cos\left(\frac{-10\pi}{15}\right) + 2\cos\left(\frac{-20\pi}{15}\right)\right]$$

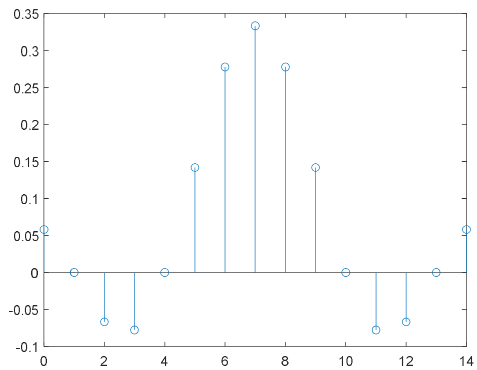
$$h[11] = h[3] = {}^{1}\!/_{15}\left[1 + 2\cos\left(\frac{-8\pi}{15}\right) + 2\cos\left(\frac{-16\pi}{15}\right)\right]$$

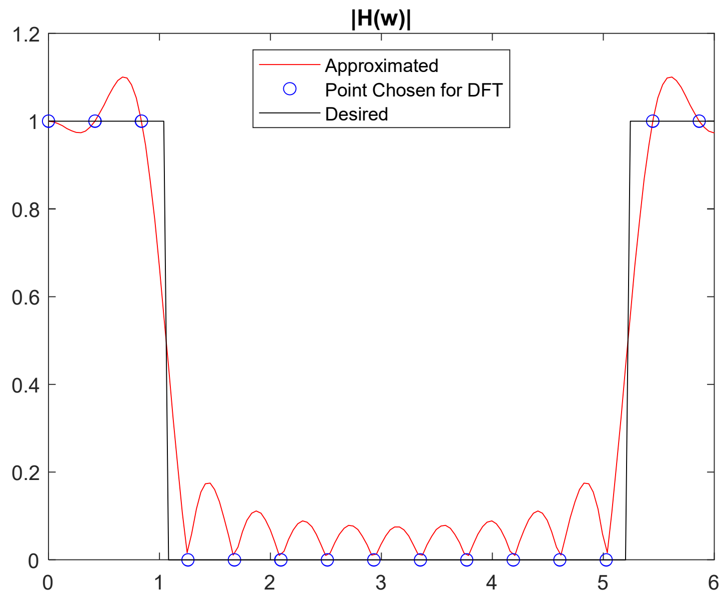
$$h[10] = h[4] = \frac{1}{15}\left[1 + 2\cos\left(\frac{-6\pi}{15}\right) + 2\cos\left(\frac{-12\pi}{15}\right)\right]$$

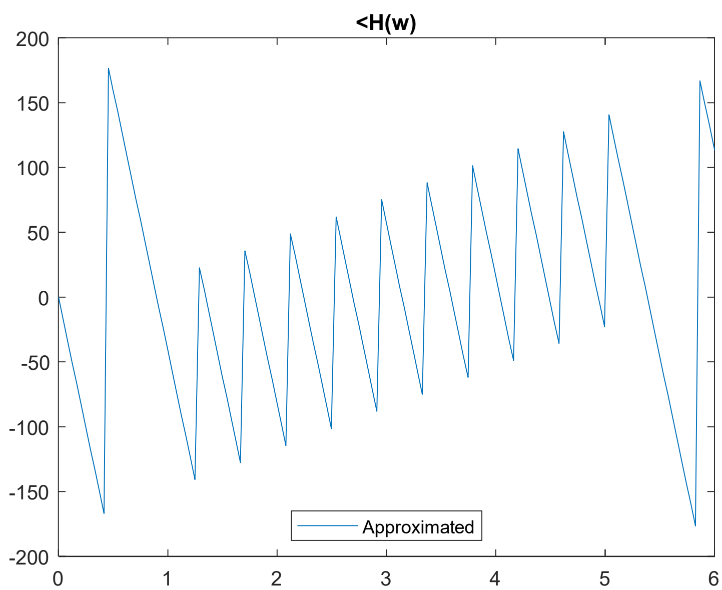
$$h[9] = h[5] = \frac{1}{15}\left[1 + 2\cos\left(\frac{-4\pi}{15}\right) + 2\cos\left(\frac{-8\pi}{15}\right)\right]$$

$$h[8] = h[6] = \frac{1}{15}\left[1 + 2\cos\left(\frac{-2\pi}{15}\right) + 2\cos\left(\frac{-4\pi}{15}\right)\right]$$

$$h[7] = h[7] = \frac{1}{15}\left[1 + 2\cos(0) + 2\cos(0)\right] = \frac{1}{3}.$$


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
2
4
6
8
10
12
14


|H(w)|
Approximated
Point Chosen for DFT
Desired
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6


<H(w)
200
150
100
50
0
-50
-100
-150
-200
Approximated
0
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6