

$$3a.) \quad x(n) = S[n]$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \quad k = 0, \dots, N-1$$

$$= (W_N^o)^N$$

$$= 1 + k.$$

$$3b.) \quad x[n] = S[n-n_0]$$

$$k = 0, \dots, N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$$= W_N^{k \cdot 1} \sum_{n=0}^{N-1} W_N^{kn} \quad k = 0, \dots, N-1.$$

$$= W_N^k \cdot 1 \quad k = 0, \dots, N-1$$

$$= e^{j\frac{2\pi}{N}k} \quad k = 0, \dots, N-1.$$

$$3c.) \quad x[n] = a^n, \quad 0 \leq n \leq N-1.$$

$$x[k] = \sum_{n=0}^{N-1} a^n W_N^{nk} = \sum_{n=0}^{N-1} (a W_N^k)^n$$

$$= \frac{1 - (a W_N^k)^N}{1 - a W_N^k} = \frac{1 - a^N}{1 - a W_N^k} \quad |a W_N^k| < 1 \\ |a| < 1 \quad k = 0, \dots, N-1$$

$$3d.) \quad x(n) = \begin{cases} 1 & 0 \leq n \leq \frac{N}{2}-1 \\ 0 & \frac{N}{2} \leq n \leq N-1. \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} x(n) W_N^{-nk} \quad k = 0, \dots, N-1.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{-nk} + \underbrace{\sum_{n=\frac{N}{2}}^{N-1} x[n] W_N^{-nk}}_{} = 0.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} W_N^{-nk} = \sum_{n=0}^{\frac{N}{2}-1} (W_N^{-k})^n$$

$$= \frac{1 - (W_N^{-k})^{\frac{N}{2}}}{1 - W_N^{-k}}$$

$$\text{For } k=0 \quad X[0] = \sum_{n=0}^{\frac{N}{2}-1} 1 = \frac{N}{2}.$$

$$\text{For } k=\text{even} \quad X[4] = \frac{1 - (W_N)^{\frac{4N}{2}}}{1 - W_N^4} = 0$$

$$\text{For } k=\text{odd}. \quad \frac{1 - (-1)}{1 - e^{-j\frac{\pi}{N}k}} = \frac{2}{e^{-j\frac{\pi}{N}k}(e^{j\frac{\pi}{N}k} - e^{-j\frac{\pi}{N}k})} = \frac{2e^{j\frac{\pi}{N}k}}{2\sin(\frac{\pi}{N}k)} = \frac{e^{j\frac{\pi}{N}k}}{\sin(\frac{\pi}{N}k)}$$

$$X[k] = \begin{cases} \frac{N}{2} & k=0. \\ 0 & k \text{ even} \\ \frac{e^{j\frac{\pi}{N}k}}{\sin(\frac{\pi}{N}k)} & k \text{ odd} \end{cases}$$

$$k = \{0, \dots, N-1\}$$

$$3 e.) \quad X[n] = e^{j \frac{2\pi}{N} k_0 n}$$

$$X[k] = \sum_{n=0}^N w_N^{-k_0 n} w_N^{kn} = \quad k=0, \dots, N-1.$$

$$k=k_0 \Rightarrow \sum_{n=0}^{N-1} 1 = N$$

$$\begin{aligned} k \neq k_0 \Rightarrow X[k] &= \frac{1 - w_N^{(k-k_0)N}}{1 - w_N^{k-k_0}} = \frac{1 - e^{-j2\pi(k-k_0)}}{1 - e^{j\frac{2\pi}{N}(k-k_0)}} \\ &= \frac{e^{-j\pi(k-k_0)} [e^{j\pi(k-k_0)} - e^{-j\pi(k-k_0)}]}{e^{j\frac{\pi}{N}(k-k_0)} [e^{j\frac{\pi}{N}(k-k_0)} - e^{-j\frac{\pi}{N}(k-k_0)}]} \\ &= \frac{e^{-j\pi(k-k_0)(1-\frac{1}{N})}}{\sin(\frac{\pi}{N}(k-k_0))} \frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} \end{aligned}$$

$$X[k] = \begin{cases} N & k = k_0 \\ \frac{e^{-j\pi(k-k_0)(1-\frac{1}{N})}}{\sin(\frac{\pi}{N}(k-k_0))} \frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} & k = \{0, \dots, N-1\} - \{k_0\} \end{cases}$$

$$3f.) x[n] = \cos\left(\frac{2\pi}{N} k_0 n\right), \quad 0 \leq n \leq N-1.$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{-nk} \quad k = \{0, \dots, N-1\}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{2} e^{j\frac{2\pi}{N} k_0 n} + \frac{1}{2} e^{-j\frac{2\pi}{N} k_0 n} \right] W_N^{-nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2} W_N^{-k_0 n} W_N^{-nk} + \frac{1}{2} W_N^{k_0 n} W_N^{-nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2} (W_N^{k-k_0})^n + \frac{1}{2} (W_N^{k+k_0})^n$$

$$X[k]: \quad = \frac{1}{2} \left[\frac{1 - (W_N^{k-k_0})^N}{1 - W_N^{k-k_0}} + \frac{1 - (W_N^{k+k_0})^N}{1 - W_N^{k+k_0}} \right]$$

↓

$$k=k_0 : \quad \sum_{n=0}^{N-1} \frac{1}{2} + \frac{1}{2} W_N^{2k_0 n} = \frac{N}{2} + \frac{1}{2} \frac{1 - (W_N^{2k_0})^N}{1 - W_N^{2k_0}} = \frac{N}{2}$$

$$k=-k_0 \quad \sum_{n=0}^{N-1} \frac{1}{2} W_N^{-2k_0 n} + \frac{1}{2} = 0 + \frac{N}{2} = \frac{N}{2}.$$

$k =$

$$\{0, \dots, N-1\} - \{\pm k_0\}. \quad \frac{1}{2} \left[\frac{1 - (e^{-j\frac{2\pi}{N}(k-k_0)N})}{1 - e^{-j\frac{2\pi}{N}(k-k_0)}} + \frac{1 - (e^{-j\frac{2\pi}{N}(k+k_0)N})}{1 - e^{-j\frac{2\pi}{N}(k+k_0)}} \right]$$

$$= \frac{1}{2} \left[\frac{e^{-j\pi(k-k_0)} (\sin(\pi(k-k_0)))}{e^{j\frac{\pi}{N}(k-k_0)} (\sin(\frac{\pi}{N}(k-k_0)))} + \frac{e^{-j\pi(k+k_0)} \sin(\pi(k+k_0))}{e^{-j\frac{\pi}{N}(k+k_0)} \sin(\frac{\pi}{N}(k+k_0))} \right]$$

$$= \frac{1}{2} e^{-j\pi(k-k_0)(1-\frac{1}{N})} \frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} + \frac{1}{2} e^{-j\pi(k+k_0)(1-\frac{1}{N})} \frac{\sin(\pi(k+k_0))}{\sin(\frac{\pi}{N}(k+k_0))}$$

$$39.) X(n) = \sin\left(\frac{2\pi}{N} k_0 n\right), \quad 0 \leq n \leq N-1.$$

$$X[k] = \sum_{n=0}^{N-1} X(n) W_N^{nk}, \quad \forall k \in \{0, \dots, N-1\}$$

$$= \sum_{n=0}^{N-1} \sin\left(\frac{2\pi}{N} k_0 n\right) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2j} (e^{j \frac{2\pi}{N} k_0 n} - e^{-j \frac{2\pi}{N} k_0 n}) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2j} (W_N^{-k_0 n} - W_N^{+k_0 n}) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2j} (W_N^{(K-K_0)n} - W_N^{(K+K_0)n}).$$

For $K = K_0$

$$= \sum_{n=0}^{N-1} \frac{1}{2j} (W_N^{0K_0 n} - W_N^{2K_0 n})$$

$$= \frac{N}{2j} - 0$$

$$= \frac{1}{2j} \underbrace{\left(1 - W_N^{2K_0 N} \right)}_{=0} \frac{-N}{2j}$$

For $K = -K_0$.

$$= \sum_{n=0}^{N-1} \frac{1}{2j} (W_N^{-2K_0 n} - 1)$$

$$= 0 - \frac{N}{2j}$$

$$= -\frac{N}{2j}$$

$$X[k] = \begin{cases} \frac{N}{2j} & k = k_0 \\ -\frac{N}{2j} & k = -k_0 \\ \frac{1}{2j} \frac{1 - (W_N)^{(k_0 N)}}{1 - (W_N)^{k+k_0}} - \frac{1}{2j} \frac{1 - (W_N)^{+k_0 N}}{1 - (W_N)^{k-k_0}} \end{cases}$$

\downarrow
Simplified more in next page!

$$k = \{0, \dots, N-1\} - \{\pm k_0\}$$

3g)

For $k = \{0, \dots, N-1\} - \{\pm k_0\}$

$$X[k] = \sum_{n=0}^{N-1} \frac{1}{2j} (W_N^{(k-k_0)n} - W_N^{(k+k_0)n})$$

$$= \frac{1}{2j} \left[\frac{1 - W_N^{(k-k_0)N}}{1 - W_N^{k-k_0}} - \frac{1 - W_N^{(k+k_0)N}}{1 - W_N^{k+k_0}} \right]$$

$$= \frac{1}{2j} \left[\frac{1 - e^{-j2\pi(k-k_0)}}{1 - e^{-j\frac{2\pi}{N}(k-k_0)}} - \frac{1 - e^{-j2\pi(k+k_0)}}{1 - e^{-j\frac{2\pi}{N}(k+k_0)}} \right]$$

$$= \frac{1}{2j} \frac{e^{-j\pi(k-k_0)}}{e^{-j\frac{\pi}{N}(k-k_0)}} \frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} - \frac{1}{2j} \frac{e^{-j\pi(k+k_0)}}{e^{-j\frac{\pi}{N}(k+k_0)}} \frac{\sin(\pi(k+k_0))}{\sin(\frac{\pi}{N}(k+k_0))}$$

$$= \frac{1}{2j} \frac{e^{-j\pi(k-k_0)(1-\frac{1}{N})}}{\sin(\frac{\pi}{N}(k-k_0))} \frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} - \frac{1}{2j} \frac{e^{-j\pi(k+k_0)(1-\frac{1}{N})}}{\sin(\frac{\pi}{N}(k+k_0))} \frac{\sin(\pi(k+k_0))}{\sin(\frac{\pi}{N}(k+k_0))}$$

$$3h.) x(n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad 0 \leq n \leq N-1$$

$$x[k] = \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

$$\underset{n=2r}{=} \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_N^{-2rk} + \sum_{r=\frac{N}{2}}^{N-1} x[2r+1] W_N^{-2rk+k}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} W_N^{-2rk} + 0$$

$$x[k] = \sum_{r=0}^{\frac{N}{2}-1} (W_N^{-2k})^r$$

$$\text{for } k=0: x[0] = \sum_{r=0}^{\frac{N}{2}-1} 1 = \frac{N}{2}$$

$$k \neq 0: x[k] = \frac{1 - (W_N^{-2k})^{\frac{N}{2}}}{1 - W_N^{-2k}} = 0$$

$$\text{For odd } k: x[k] = 0$$

$$\text{For even } k: x[k] = 0$$

$$x[k] = \begin{cases} N/2 & k=0 \\ 0 & k \in \{0, \dots, N-1\} - \{0\} \end{cases}$$

HW #4 DSP.

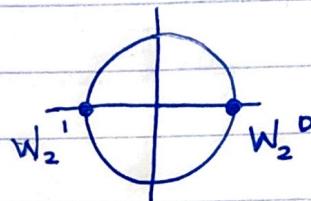
4.) DFT matrix.

$$x[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

a.) Plot $N=2$ in complex plane.

$$W_2^k = e^{-j\frac{2\pi}{2}k} = e^{-jk\pi}$$

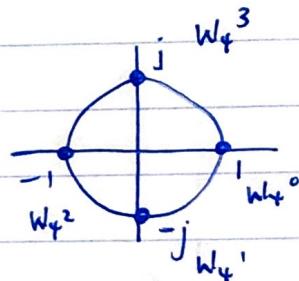
$$k = \{0, 1\}$$



$$N = 4$$

$$W_4^k = \{W_4^0, W_4^1, W_4^2, W_4^3\}$$

$$k = \{0, 1, 2, 3\}$$

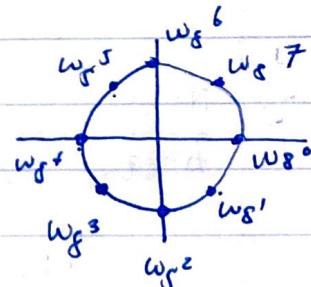


$$N = 8$$

$$k = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$W_8^k = \{W_8^0, W_8^1, W_8^2, \dots, W_8^7\}$$

$$W_8^k = \{1, W_8^1, -j, W_8^3, -1, W_8^5, j, W_8^7\}$$



4b.) Write $N \times N$ DFT matrix for $N=2$.
in terms of W_N 's

$$N=2$$

$$k=\{0, 1\}$$

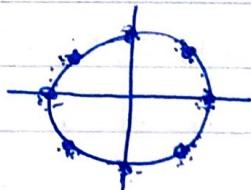
$$\begin{bmatrix} W_2^0 & W_2^0 \\ W_2^{0+1} & W_2^{1+1} \end{bmatrix}$$

$$\begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix}$$

Repeat for W_N 's where $N=8$.

$$N=8$$

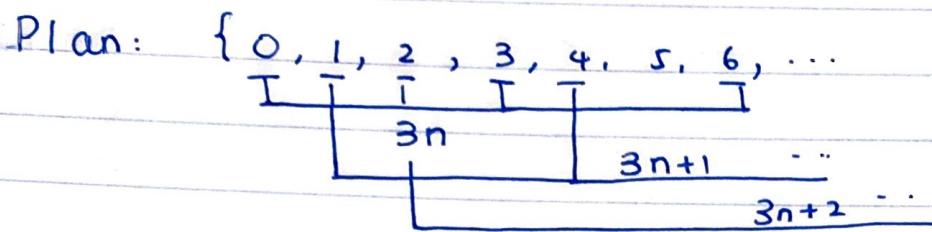
$$k=\{0, \dots, 7\}$$



$n=0$	W_8^0							
$n=1$	W_8^0	W_8^1	W_8^2	W_8^3	W_8^4	W_8^5	W_8^6	W_8^7
$n=2$	W_8^0	W_8^2	W_8^4	W_8^6	W_8^0	W_8^2	W_8^4	W_8^6
$n=3$	W_8^0	W_8^3	W_8^6	W_8^1	W_8^4	W_8^7	W_8^2	W_8^5
$n=4$	W_8^0	W_8^4	W_8^0	W_8^4	W_8^0	W_8^4	W_8^0	W_8^4
$n=5$	W_8^0	W_8^5	W_8^2	W_8^7	W_8^4	W_8^1	W_8^6	W_8^3
$n=6$	W_8^0	W_8^6	W_8^4	W_8^2	W_8^0	W_8^6	W_8^4	W_8^2
$n=7$	W_8^0	W_8^7	W_8^6	W_8^5	W_8^4	W_8^3	W_8^2	W_8^1

5.) Given: 24 point ($N=24$) $x[n]$

3 FFT chip ($K=8$ per chip)



For chip 1, we can have input of $x(n)$

Where index $n: 3n \quad n \in \{0, \dots, 7\}$

For chip 2, we can have input of $x(n)$

Where index $n: 3n+1 \quad n \in \{0, \dots, 7\}$

For chip 3, we can have input of $x(n)$

Where index $n: 3n+2 \quad n \in \{0, \dots, 7\}$.

$$X[k] = \sum_{r=0}^{\frac{N}{3}-1} X[3n] W_N^{3kn} + \sum_{r=\frac{N}{3}}^{\frac{2N}{3}-1} X[3n+1] W_N^{(3n+1)k} + \sum_{r=\frac{2N}{3}}^{N-1} X[3n+2] W_N^{(3n+2)k}$$

$r = 3n$

$$= \sum_{r=0}^{\frac{N}{3}-1} X[3n] + \sum_{r=0}^{\frac{N}{3}-1} X[3n+1] W_N^{3k} + \sum_{r=0}^{\frac{N}{3}-1} X[3n+2] W_N^{2k}$$

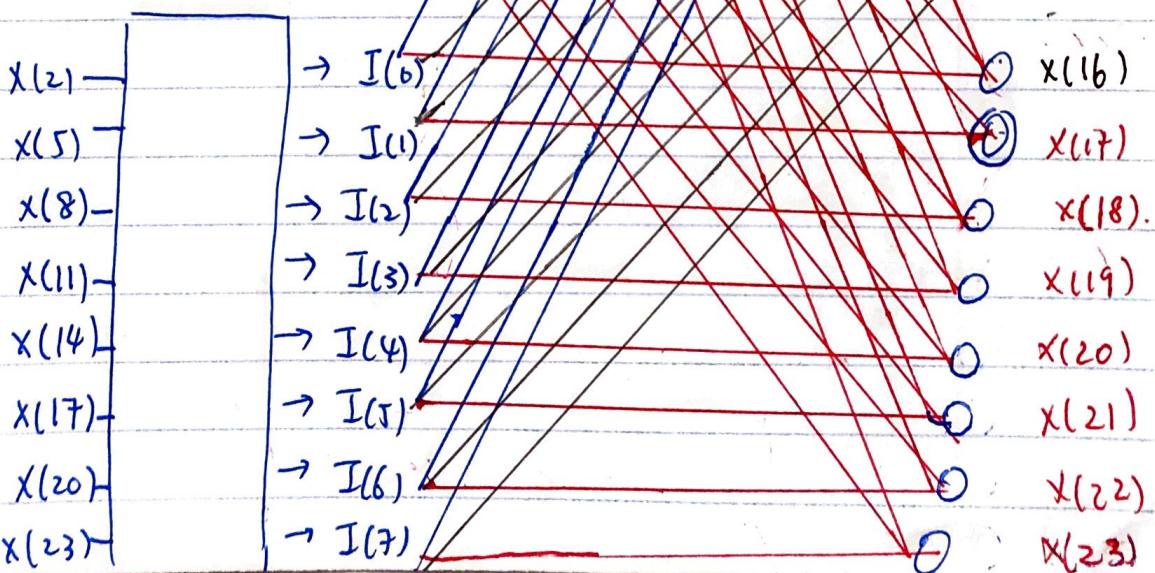
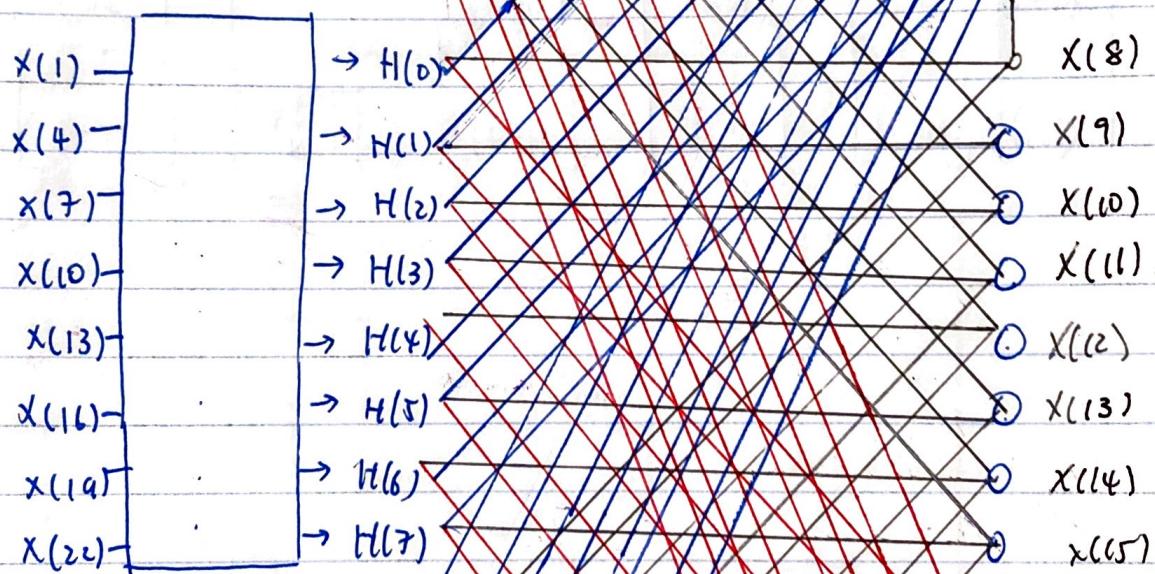
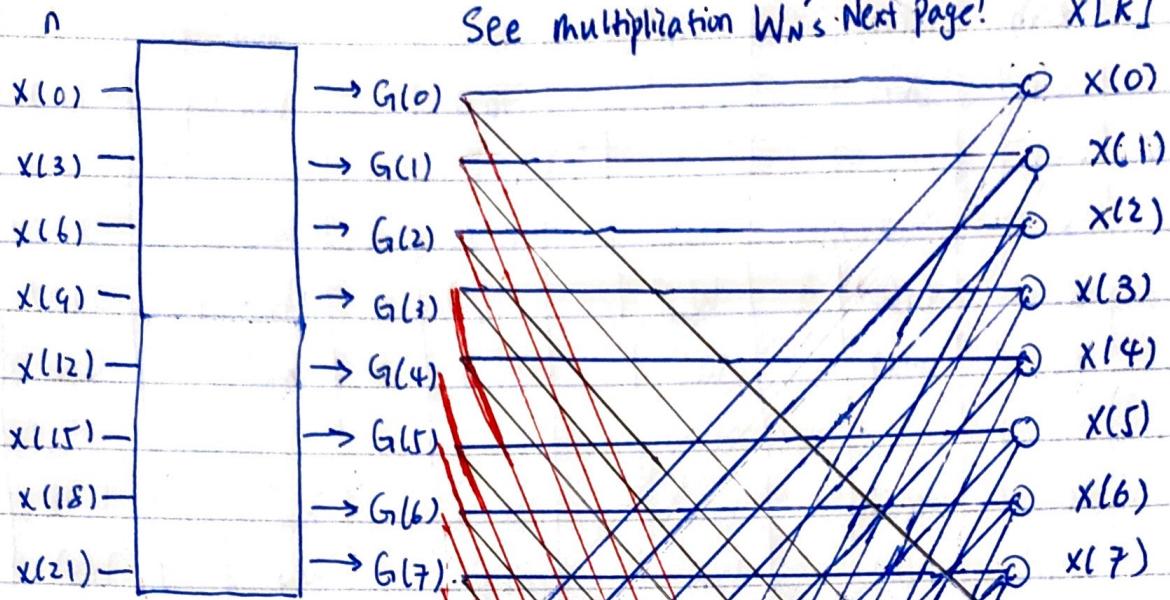
$G[k]$
 $W_N^{3k} H[k]$
 $W_N^{2k} I[k]$

$$X[k+8] = \sum_{n=0}^{\frac{N}{3}-1} X[3n] + \sum_{n=0}^{\frac{N}{3}-1} X[3n+1] W_N^{(k+8)} + \sum_{n=0}^{\frac{N}{3}-1} X[3n+2] W_N^{2(k+8)}$$

For $k \notin \{0, \dots, 7\}$

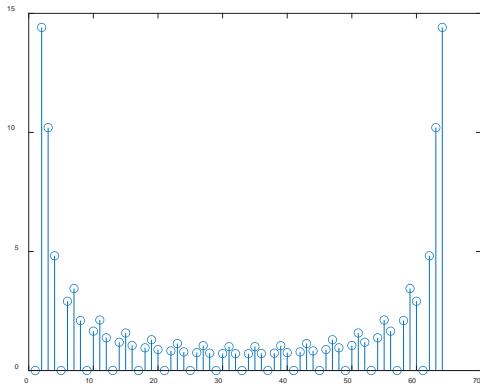
$$X[k+16] = \sum_{n=0}^{\frac{N}{3}-1} X[3n] + \sum_{n=0}^{\frac{N}{3}-1} X[3n+1] W_N^{k+16} + \sum_{n=0}^{\frac{N}{3}-1} X[3n+2] W_N^{2(k+16)}$$

See multiplication W[n]'s Next Page! $X[k]$



$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(7) \\ x(8) \\ x(9) \\ \vdots \\ x(15) \\ x(16) \\ x(17) \\ \vdots \\ x(23) \end{bmatrix} = \begin{bmatrix} G(0) \\ G(1) \\ \vdots \\ G(7) \\ G(0) \\ G(1) \\ \vdots \\ G(7) \\ G(0) \\ G(1) \\ \vdots \\ G(7) \end{bmatrix} + \begin{bmatrix} W_N^0 \\ W_N^1 \\ \vdots \\ W_N^7 \\ W_N^8 \\ W_N^9 \\ \vdots \\ W_N^{15} \\ W_N^{16} \\ W_N^{17} \\ \vdots \\ W_N^{23} \end{bmatrix} \cdot \begin{bmatrix} H(0) \\ H(1) \\ \vdots \\ H(7) \\ H(0) \\ H(1) \\ \vdots \\ H(7) \\ H(0) \\ H(1) \\ \vdots \\ H(7) \end{bmatrix} + \begin{bmatrix} W_N^0 \\ W_N^1 \\ \vdots \\ W_N^7 \\ W_N^8 \\ W_N^9 \\ \vdots \\ W_N^{15} \\ W_N^{16} \\ W_N^{17} \\ \vdots \\ W_N^{23} \end{bmatrix} \cdot \begin{bmatrix} I(0) \\ I(1) \\ \vdots \\ I(7) \\ I(0) \\ I(1) \\ \vdots \\ I(7) \\ I(0) \\ I(1) \\ \vdots \\ I(7) \end{bmatrix}$$

N = 64, Length = 16



Frequency interval between successive samples for the plot in part a is $\frac{2\pi i}{64}$.

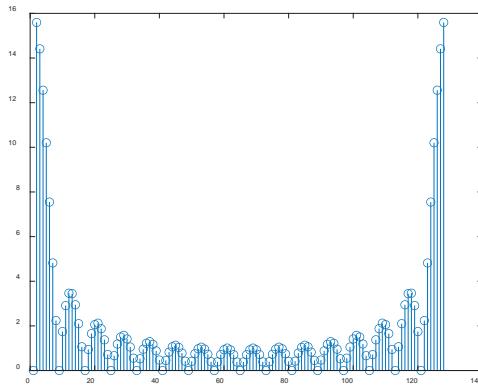
At w = 0, the value is 16

Interval between null samples is $\frac{64}{16}$

The null sample interval is always $\frac{N}{L}$

The one with greater N has a faster sampling rate than the smaller N, the one with higher N look closer to the continuous time signal.

N=128, Length = 16



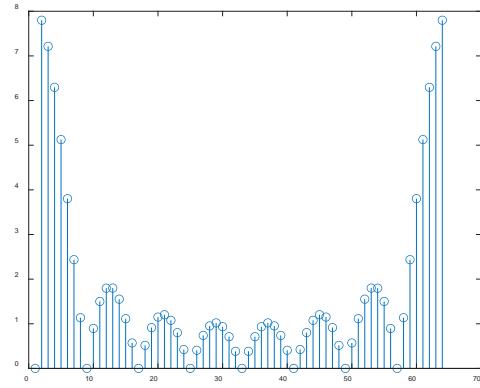
Frequency interval between successive samples for the plot in part a is $\frac{2\pi i}{128}$.

At w = 0, the value is 16

Interval between null samples is $\frac{128}{16}$

The null sample interval is always $\frac{N}{L}$

N = 64, Length = 8



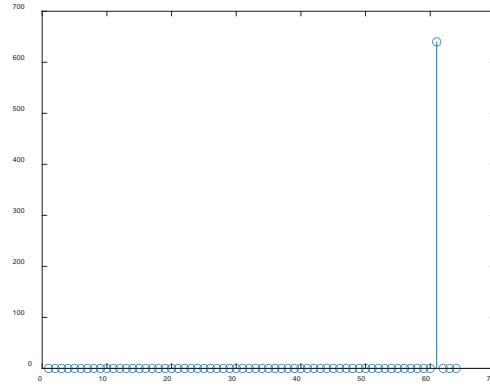
Frequency interval between successive samples for the plot in part a is $\frac{2\pi i}{64}$

At w = 0, the value is 8

Interval between null samples is $\frac{64}{8}$

The null sample interval is always $\frac{N}{L}$

N = 64, Length = 64



Frequency interval between successive samples for the plot in part a is $\frac{2\pi i}{64}$

At w = 0, the value is $\sum_{n=0}^{N-1} x(n) = -8.3489e^{-14} - 4.085e^{i-14}$

Interval between null samples is $\frac{64}{64}$

The null sample interval is always $\frac{N}{L}$