Solution concepts for linear piecewise affine differential-algebraic equations

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***Abstract*— In this paper, we introduce a definition of solutions for linear piecewise affine differential-algebraic equations (PWA-DAEs). Firstly, to address the conflict between projector-based jump rule and active regions, we propose a concept called state-dependent jump path. Unlike the conventional perspective that treats jumps as discrete-time dynamics, we interpret them as continuous dynamics, parameterized by a virtual time-variable. Secondly, by adapting the hybrid time-domain solution theory for continuous-discrete hybrid systems, we define the concept of jump-flow solutions for PWA-DAEs with the help of Filippov solutions for differential inclusions. Subsequently, we study various boundary behaviors of jump-flow solutions. Finally, we apply the proposed solution concepts in simulating a state-dependent switching circuit.**

I. Introduction

Consider a linear piecewise affine differential-algebraic equation (PWA-DAE) of the form

$$\Delta^{\rm pwa}: E_i\dot{x} = A_i x + b_i,\ x \in \Omega_i \subseteq \mathbb{R}^n,\ i = 1, \dots, N, \quad (1)$$

where $x \in \mathbb{R}^n$ are the state-variables, $E_i, A_i : \mathbb{R}^n \to \mathbb{R}^n$, $b_i \in \mathbb{R}^n$, $N \in \mathbb{N}^+$ is the number of DAE modes, $\{\Omega_i\}$ is the set of active regions, where Ω_i are convex sets satisfying $\bigcup\limits_{i=1}^{N} \Omega_i = \mathbb{R}^n$. In particular, PWA-DAEs can be seen as switched DAE control system (see e.g. [16]) by fixing the switching signal as a state-dependent function and the inputs as constants; switched DAEs have been proved to be powerful tools for modeling various physical systems, including electrical circuits with switching devices [22], [17] and power grids [7].

Solution analysis and control of ordinary differential equation (ODE)-based piecewise linear systems have been well-studied for decades, see e.g., [11], [19] and also [12] for results on closely related switched ODE systems. Moreover, there exist fruitful studies on time-dependent switched DAEs, e.g., in [13], [14], [15], [16], [18]. However, there are far fewer related results on state-dependent switched DAEs and particularly on PWA-DAEs. Typically, the focus has been on studying specific systems rather than establishing a broad solution framework. For example, in [17], the passivity of a state-dependent switched DAE-modelled circuit was discussed, providing insights into a specific application. In [20] and [1], numerical methods and Modelica tools were utilized, respectively, to simulate physical examples involving state-dependent DAEs.

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One challenge in studying PWA-DAE solutions is the absence of a clear definition of state-dependent jumps to ensure consistency during mode changes. A related research area is impulsive systems, particularly state-dependent impulsive systems as reviewed in [26], which can be viewed as special cases of the general hybrid time-domain systems framework proposed in [9]. In this framework, the continuous dynamics (flow) are governed by an ODE (or differential inclusion) in some regions and a (possibly multivalued) jump rule in others; the flows and jumps are generally unrelated. In contrast, a PWA-DAE implicitly defines a consistency space where the flow occurs, while simultaneously implying a projector-based jump rule from an inconsistent initial value to a consistent one. Hence, the jumps in a DAE can be seen as intrinsic jump rules, whereas those in impulsive systems are externally imposed. Filippov solutions for discontinuous DAEs are discussed in [6], [3], but these works primarily focus on semi-linear and index-1 modes, without involving jumps.

In Section II, we revisit certain concepts of linear DAEs. We delve into the issue of state-dependent jumps for PWA-DAEs in Section III-A. The formulation of jump-flow solutions within the hybrid time-domain and the examination of their boundary behaviors are provided in Section III-B. Conclusions and future prospects are given in Section IV.

II. Preliminaries

The following notations will be used throughout the paper. $\mathbb{N}$ and $\mathbb{R}$ are the natural numbers and real numbers, respectively. For a matrix $M \in \mathbb{R}^{n\times m}$, the kernel (null space) of M is denoted by $\ker M$, the image of M is denoted by $\operatorname{im} M$. The identity matrix of size $n \times n$ is denoted by I_n. The image of a set $S \subseteq \mathbb{R}^n$ under M is $MS := \{Mx \in \mathbb{R}^n \mid x \in S\}$ and the pre-image of S under M is $M^{-1}S := \{x \in \mathbb{R}^n \mid Mx \in S\}$.

Each mode of (1) is an affine DAE $E\dot{x} = Ax+b$, denoted by $\Delta = (E, A, b)$. A $\mathcal{C}^1$-curve $x : [0,\infty) \to \mathbb{R}^n$ is called a *$\mathcal{C}^1$-solution* or a flow of Δ if $E\dot{x}(t) = Ax(t) + b$ for all $t \in [0,\infty)$. A point $x_0 \in \mathbb{R}^n$ is called *consistent* if there exists a $\mathcal{C}^1$-solution $x(\cdot)$ starting from x_0, i.e., $x(0) = x_0$. The set of all consistent points is called *consistency space*, denoted by $\mathfrak{C}$. The matrix pair (E, A) is called *regular* if $\det(sE-A)$ is not identically zero. The regularity of (E, A) guarantees the existence and uniqueness of $\mathcal{C}^1$-solutions of Δ. We assume in the following that all matrix pairs (E_i, A_i) of (1) are regular. Any DAE Δ with a regular pair (E, A) can be always transformed, via two constant invertible matrices Q and P, into the (quasi-)Weierstrass form [23], [2] $\tilde{\Delta} =$

(QEP^{-1}, QAP^{-1}, Qb):

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{2}$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix with nilpotency index ν, i.e. $N^{\nu-1} \neq 0$ and $N^\nu = 0$, where $n_1 + n_2 = n$. The index of Δ is defined to be the nilpotency index ν of N, thus we have $N = 0$ for index-1 DAEs. The matrices Q, P can be constructed with the help of the limits $\mathscr{V}^* = \mathscr{V}_n$ and $\mathscr{W}^* := \mathscr{W}_n$ [2] of the Wong sequences [25] of the matrix pencil (E, A), given by,

$$\begin{cases} \mathscr{V}_0 = \mathbb{R}^n, & \mathscr{V}_{k+1} = A^{-1}(E\mathscr{V}_k),\ k \geq 0, \\ \mathscr{W}_0 = \{0\}, & \mathscr{W}_{l+1} = E^{-1}(A\mathscr{W}_l),\ l \geq 0. \end{cases} \tag{3}$$

The consistency projector, the differential selector and the impulse selector of Δ are defined [21], [22], [24], respectively, as follows

$$\Pi := P^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} P, \quad \Pi^{\mathrm{df}} := P^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} Q.$$

and

$$\Pi^{\mathrm{imp}} := P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} Q.$$

With the help of the above definitions, it can easily be concluded that the consistency space of Δ is given by

$$\mathfrak{C} = \operatorname{im} \Pi - \{\Pi^{\mathrm{imp}} b\} = \mathscr{V}^* - \{\Pi^{\mathrm{imp}} b\}$$

and the $\mathcal{C}^1$-solution (flow) starting from a consistent point $x_0^+ \in \mathfrak{C}$ can be expressed by $x(t) = e^{A^{\mathrm{df}} t} x_0^+ + \int_0^t e^{A^{\mathrm{df}}(t-s)} \Pi^{\mathrm{df}} b \, \mathrm{d}s$, where $A^{\mathrm{df}} = \Pi^{\mathrm{df}} A$ is called the flow matrix.

If the initial point $x_0^- \notin \mathfrak{C}$ is not consistent, then a re-initialization procedure is needed to find a consistent point x_0^+ in order to solve the DAE. One approach to achieve the consistent initialization is to introduce a jump (an instant change) from x_0^- to x_0^+; utilizing (2) and following similar arguments as in [21], [13], [22] this jump map is uniquely defined via the projector Π and the selector Π^{imp} by

$$x_0^+ = \Pi x_0^- - \Pi^{\mathrm{imp}} b \in \mathfrak{C}.$$

While this jump rule is also valid for DAE systems of arbitrary index, for non-index-1 DAEs, this jump in general also induces Dirac impulses in the solution [22]; however, here we focus only on the impulse-free part of the solution, the rigorous consideration of the Dirac impulses in the solutions is a topic of future research.

III. Main results

A. State-dependent jumps and jump sliding behavior

The first problem to discuss is the definition of jumps for PWA-DAE (1). Consider an inconsistent initial point $x_0^- \notin \mathfrak{C}_p$ and $x_0^- \in \Omega_p$ for an index $p \in \{1, 2, \ldots, N\}$. If we directly apply the projector Π_p to x_0^- and selector Π_p^{imp} to b_p, we obtain the consistent point $x_0^+ = \Pi_p x_0^- - \Pi_p^{\mathrm{imp}} b_p \in \mathfrak{C}_p$. However, in general, $x_0^+ \notin \Omega_p$, which means that the resulting consistent point violates the active region rule. Therefore, it becomes necessary to introduce a new definition of jumps for PWA-DAE to address this issue.

To generalize the definition of jumps for nonlinear DAEs, a novel approach called the "jump path" is proposed in [4]. We now adapt this notion for PWA-DAEs in the present paper. The key idea is not just to consider the jump map $x_0^- \to x_0^+$, but instead to introduce a jump path $J : [0, a] \to \mathbb{R}^n$, $\tau \mapsto J(\tau)$, with $J(0) = x_0^-$ and $J(a) = x_0^+ \in \mathcal{C}$, such that $\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} \in \ker E$. The latter condition, which requires the jump direction to stay in $\ker E$, is inspired by the impulse-free jump condition $x_0^+ - x_0^- \in \ker E$, meaning that the jump does not cause any Dirac impulse, see, for example, [21], [13], [14] for the distributional solutions theory of DAEs. It can be proved by the results in [4] that for an index-1 linear affine DAE Δ, the jump associated with the jump path is uniquely defined and it coincides with the one defined by the consistency projector, i.e., $J(a) = \Pi x_0^- - \Pi^{\mathrm{imp}} b$.

Define $\mathfrak{C}^{\mathrm{pwa}} := \bigcup_{i=1}^{N} (\mathfrak{C}_i \cap \Omega_i)$ and call it the consistency space for the PWA-DAE Δ^{pwa}. Note that from any point $x_0^+ \in \mathfrak{C}^{\mathrm{pwa}}$, there exists a unique maximal $\mathcal{C}^1$-solution for the corresponding activated DAE mode Δ_p, where p satisfies $x_0^+ \in \Omega_p$.

Definition 1 (State-dependent jump path)**.** Consider a PWA-DAE Δ^{pwa}, an absolutely continuous curve $J : [0, a] \to \mathbb{R}^n$ is called a convergent jump path starting from an initial point $x_0^- \in \mathbb{R}^n$ if $J(0) = x_0^-$, $\forall \tau \in [0, a) : J(\tau) \cap \mathfrak{C}^{\mathrm{pwa}} = \emptyset$, $J(a) \in \mathfrak{C}^{\mathrm{pwa}}$ and

$$\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} \in \mathrm{Cone}\{f_i^{\mathrm{jp}}(J(\tau))\}, \quad J \in \Omega_i \tag{4}$$

where $\mathrm{Cone}\{S\}$ denotes the smallest (closed) cone containing S and

$$f_i^{\mathrm{jp}}(x) := (\Pi_i - I)x - \Pi_i^{\mathrm{imp}} b_i.$$

The change $x_0^- \to x_0^+ := J(a)$ is called a state-dependent jump associated with $J(\tau)$. If $a = \infty$ and $J(\infty) = \lim_{\tau \to \infty} J(\tau)$ does not exist, then $J(\tau)$ is called a divergent jump path.

The motivation behind jump rule (4) is to allow the jumping direction $\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} = \lim_{\epsilon \to 0} \frac{J(\tau+\epsilon) - J(\tau)}{\epsilon}$ to depend on the position $J(\tau)$ of the path and to require any inconsistent point $x \in \Omega_p$ to move towards the consistent initialization $x^+ = \Pi_p x - \Pi_p^{\mathrm{imp}} b_p$ for the active mode Δ_p, i.e., the moving direction is $(\Pi_p x - \Pi_p^{\mathrm{imp}} b_p) - x = f_p^{\mathrm{jp}}(x)$. One should keep in mind that the jump still happens instantaneously, in particular, τ is *not* a real time-variable (it is virtual), but just describes the position on the jump path. It is not necessary to specify how fast the path moves, thus we use inclusion and $\mathrm{Cone}\{f_i^{\mathrm{jp}}\}$ in (4) instead of using $\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} = f_i^{\mathrm{jp}}(J(\tau))$. To solve (4), it is enough to choose any vector $g_i^{\mathrm{jp}}(J, \tau) \in \mathrm{Cone}\{f_i^{\mathrm{jp}}(J)\}$ and solve $\frac{\mathrm{d}J}{\mathrm{d}\tau} = g_i^{\mathrm{jp}}(J, \tau)$. The solutions from different choices of g_i^{jp} are different parametrizations of the *same* curve. The simplest choice is $g_i^{\mathrm{jp}} = f_i^{\mathrm{jp}}$, then $a = \infty$. By applying this choice to a single affine DAE mode

$\Delta = (E, A, b)$, we have

$$J(\tau) = e^{A^{\mathrm{jp}}\tau} x_0^- - \int_0^\tau e^{A^{\mathrm{jp}}(\tau - s)} \Pi^{\mathrm{imp}} b \mathrm{d}s,$$

where $A^{\mathrm{jp}} := \Pi - I$. It follows that $x_0^+ = J(\infty) = \Pi x_0^- - \Pi^{\mathrm{imp}} b$. This means that Definition 1 is a generalization of the projectors-based jump rule for affine DAEs to PWA-DAEs.

Remark 1. (i) If the mode Δ_i is index-1, then $\frac{\mathrm{d}J}{\mathrm{d}\tau} \in \mathrm{Cone}\{f_i^{\mathrm{jp}}(J)\} \subseteq \ker E_i$, thus the defined state-dependent jump does not cause Dirac impulses for index-1 modes and Definition 1 is indeed an adaptation of the impulse-free jump rule [4] to PWA-DAEs.
(ii) For any jump path $J(\tau)$, by a change of variables $\tilde{\tau} = \varphi(\tau)$, where $\varphi : [0, a] \to [\tilde{\tau}_0, \tilde{\tau}_1]$ is a diffeomorphism, we re-parameterize $J(\tau)$ as $\tilde{J}(\tilde{\tau}) = J(\varphi^{-1}(\tilde{\tau})) : [\tilde{\tau}_0, \tilde{\tau}_1] \to \mathbb{R}^n$, it can be seen that $\tilde{J}$ still satisfy (4), i.e., $\frac{\mathrm{d}\tilde{J}}{\mathrm{d}\tilde{\tau}} \in \mathrm{Cone}\{f_i^{\mathrm{jp}}(\tilde{J})\}$, so the above definition is invariant under different parametrizations of the jump path, which means that given any parametrization of a curve starting from x_0^-, we may verify if it is indeed a jump path by directly using Definition 1.

It is worth to note that the defined jump path could be divergent or convergent as a PWA ODE system.

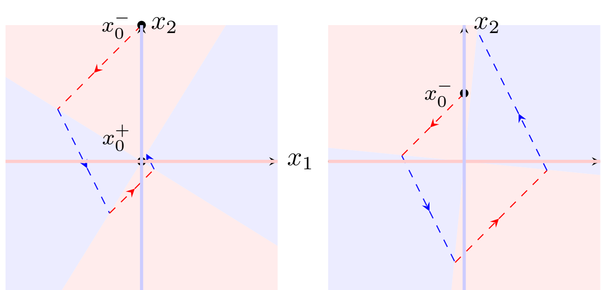
Example 1. Consider a PWA-DAE Δ^{pwa} with two modes with states $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\Delta_1 : \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\Delta_2 : \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

with the active regions [1] given by

$$\Omega_1 = \left\{ x \in \mathbb{R}^2 \;\middle|\; x^2 - y^2 + \gamma xy \leq 0 \right\}, \quad \Omega_2 = \mathbb{R}^2 \setminus \Omega_2.$$

The consistency space $\mathfrak{C}^{\mathrm{pwa}} = \{0\}$ is a single point. In case



(a) $\gamma = 1$ **(b)** $\gamma = 10$

Fig. 1: Red and blue dashed arrows: Jump directions of Δ_1 and Δ_2, Red and blue lines: $\mathfrak{C}_1$ and $\mathfrak{C}_2$, Red and blue regions: Ω_1 and Ω_2

(a) $\gamma = 1$, we have $x_0^+ = 0$ while in case (b) $\gamma = 10$, the jump path is divergent.

[1]Clearly, Ω_1 and Ω_2 are not convex and in order to match our framework it is necessary to further split both regions into two (convex) parts and introduce the additional modes $\Delta_3 = \Delta_1$ and $\Delta_4 = \Delta_2$; however, to avoid unnecessary additional notation, we only consider the system with two modes instead of four.

In the spirit of Filippov solutions for piecewise ODEs [8], [5], [11], we may generalize the rule (4) to the following differential inclusion

$$\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} \in F^{\mathrm{jp}}(J(\tau)),$$

where F^{jp} is a set valued function defined by

$$F^{\mathrm{jp}}(x) := \mathrm{Cone}\{f_i^{\mathrm{jp}}(x), \forall i : x \in \mathrm{clo}(\Omega_i) \neq \emptyset\},$$

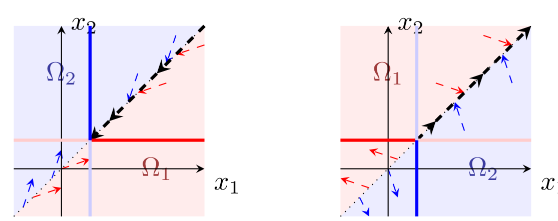
i.e. $F^{\mathrm{jp}}(x)$ is the cone consisting of all convex combination of all possible jump directions.

Jump sliding behavior. Let $S_{pq} := \mathrm{clo}(\Omega_p) \cap \mathrm{clo}(\Omega_q)$ be the common boundary of two neighboring active regions Ω_p and Ω_q. Then for any point $x \in S_{pq} \setminus (\mathfrak{C}_p \cup \mathfrak{C}_q)$ if both the vectors $f_p^{\mathrm{jp}}(x)$ and $f_q^{\mathrm{jp}}(x)$ point towards S_{pq}, then there always exists a convex combination $\alpha f_p^{\mathrm{jp}}(x) + (1-\alpha) f_q^{\mathrm{jp}}(x) \in T_x S_{pq}$ for some $0 \leq \alpha \leq 1$, where $T_x S_{pq}$ is the tangent space of S_{pq} at $x \in S_{pq}$, which means that $F^{\mathrm{jp}}(x) \cap T_x S_{pq} \neq \emptyset$ and the jump path $J(\tau)$ approximates a trajectory sliding on the boundary S_{pq}, which we call the *jump sliding behavior* of Δ^{pwa}.

Example 2. Consider a PWA-DAE Δ^{pwa} with $x = (x_1, x_2) \in \mathbb{R}^2$ and two modes

$$\Delta_1 : \begin{bmatrix} 1 & -\gamma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$\Delta_2 : \begin{bmatrix} -\gamma & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The active regions are $\Omega_1 = \{x \in \mathbb{R}^n \mid \gamma(x_1 - x_2) \geq 0\}$ and $\Omega_2 = \mathbb{R}^2 \setminus \Omega_1$. By a direct calculation, we get $f_1^{\mathrm{jp}}(x) = \begin{bmatrix} -1 & 0 \\ -\frac{1}{\gamma} & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ \frac{1}{\gamma} \end{bmatrix}$ and $f_2^{\mathrm{jp}}(x) = \begin{bmatrix} 0 & -\frac{1}{\gamma} \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} \frac{1}{\gamma} \\ 1 \end{bmatrix}$. The boundary of Ω_1 and Ω_2 is $S_{12} = \left\{(x_1, x_2) \in \mathbb{R}^2 \;\middle|\; x_1 = x_2\right\}$. It can been seen in Fig. 2 that both $f_1^{\mathrm{jp}}(x)$ and $f_2^{\mathrm{jp}}(x)$ point towards S_{12} and there exists $\alpha \in (0, 1)$ s.t. $\alpha f_1^{\mathrm{jp}}(x) + (1-\alpha) f_2^{\mathrm{jp}}(x) \in T_x S_{12}$ whenever $x_1 \geq 1$ and $x_2 \geq 1$. Thus starting from any inconsistent point $x_0^- = \begin{bmatrix} x_{10}^- \\ x_{20}^- \end{bmatrix} \in S_{12} \cap \{x \in \mathbb{R} \mid x_1 \geq 0, x_2 \geq 0\}$, there exists a jump sliding behavior. As seen from Fig 2, the jump sliding behavior $J(\tau)$ converges to $(1, 1)$ (implying that $x_0^+ = (1, 1)$ is the resulting consistent point) if $\gamma > 1$, and $J(\tau)$ diverges if $\gamma < -1$.



(a) $\gamma > 1$ **(b)** $\gamma < -1$

Fig. 2: Red and blue dashed arrows: Jump directions of Δ_1 and Δ_2, Red and blue lines: $\mathfrak{C}_1$ and $\mathfrak{C}_2$, black dashed line with arrows: Jump sliding modes.

B. *PWA-DAE jump-flow solution on hybrid time domain*

Starting from a consistent point $x_0^+ \in \mathfrak{C}^{\mathrm{pwa}}$, there exists a $\mathcal{C}^1$-solution $x(t)$ of the active mode Δ_p, where p satisfies

$x_0^+ \in \Omega_p \cap \mathfrak{C}_p$. It is conceivable that $x(t)$ may exit $\mathfrak{C}^{\text{pwa}}$ at a certain time $t = t_k$, i.e., $x(t_k^-) \notin \mathfrak{C}^{\text{pwa}}$. In such instances, a consistency re-initialization, represented as a jump $x(t_k^-) \to x(t_k^+) \in \mathfrak{C}^{\text{pwa}}$, should be determined following the guidelines outlined in Definition 1. Consequently, a complete trajectory of a PWA-DAE entails a hybrid behavior that incorporates both jump and flow dynamics. Given that these dynamics are characterized using both the real-time variable t and the virtual variable τ, we customize the hybrid time-domain framework proposed in [10], [9] for PWA-DAE solutions.

Definition 2 (PWA-DAE hybrid time domain)**.** A subset

$$\mathcal{E} = \bigcup_j (\{t_j\} \times [\tau_j, \tau_{j+1}]) \cup ([t_j, t_{j+1}] \times \{\tau_{j+1}\}) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$$

is called a PWA-DAE hybrid time domain if it is a union of finite or infinite sequence of indexed intervals $\{t_j\} \times [\tau_j, \tau_{j+1}]$ and $[t_j, t_{j+1}] \times \{\tau_{j+1}\}$, $j = 0, 1, 2, \ldots$, for some ordered sequences $0 \leq \tau_0 \leq \tau_1 \leq \ldots$ and $0 \leq t_0 \leq t_1 \leq \ldots$ in $\mathbb{R}$. In the case of a finite numbers $m+1$ of intervals, the last intervals are allowed to be half-open, i.e., $[\tau_m, \mathcal{T})$ or $[t_m, T)$ with $\mathcal{T}$ and T finite or equal to ∞.

Remark 2. A distinction between Definition 2 and the original definition of hybrid time-domain in [10] is the discrete time-sequence j becomes a continuous virtual time-interval $[\tau_j, \tau_{j+1}]$. This adaptation is necessitated by the nature of the state-dependent jump, which, as previously discussed, embodies an absolutely continuous dynamic. Figure 3 illustrate the typologies of these two distinct definitions.

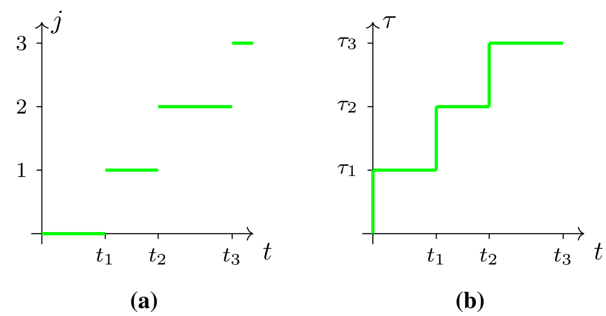


Fig. 3: (a). A hybrid time domains E defined in [10], [9], where E is the union of $[0, t_1] \times \{0\}$, $[t_1, t_2] \times \{1\}$, $[t_2, t_3] \times \{2\}$ and $[t_3, \infty) \times \{3\}$. (b). A PWA-DAE hybrid time domains $\mathcal{E}$ which is the union of $(\{0\} \times [0, \tau_1]) \cup ([0, t_1] \times \{\tau_1\})$, $(\{t_1\} \times [\tau_1, \tau_2]) \cup ([t_1, t_2] \times \{\tau_2\})$, $(\{t_2\} \times [\tau_2, \tau_3]) \cup ([t_2, t_3] \times \{\tau_3\})$.

Definition 3 (PWA-DAE hybrid arc)**.** A function $x : \mathcal{E} \to \mathbb{R}^n$ defined on a PWA-DAE hybrid time-domain is called a PWA-DAE hybrid arc if for each $j = 0, 1, 2, \ldots$, the function $\tau \mapsto x(t_j, \tau)$ by fixing t_j is absolutely continuous on the interval $I_j^\tau := \{\tau \mid (t_j, \tau) \in \mathcal{E}\}$ and the function $t \mapsto x(t, \tau_{j+1})$ by fixing τ_{j+1} is absolutely continuous on the interval $I_j^t := \{t \mid (t, \tau_{j+1}) \in \mathcal{E}\}$.

Now with the help of the above two definitions, we can define the jump-flow solution of a PWA-DAE from any initial point (consistent or not). Recall and define the following jump and flow vector fields

$$f_i^{\text{jp}}(x) = (\Pi_i - I)x - \Pi_i^{\text{imp}} b_i, \quad f_i^{\text{df}}(x) := A_i^{\text{df}} x + \Pi_i^{\text{df}} b_i$$

and define

$$F^{\text{jp}}(x) := \text{Cone}\{f_i^{\text{jp}}(x), \forall i : x \in \text{clo}(\Omega_i) \setminus \mathfrak{C}_i\},$$

$$\begin{aligned} F^{\text{df}}(x) := {} & \text{Conv}\{f_k^{\text{df}}(x), \forall k : x \in \text{clo}(\Omega_k) \cap \mathfrak{C}_k\} \\ & + \text{Cone}\{f_i^{\text{jp}}(x), \forall i : x \in \text{clo}(\Omega_i) \setminus \mathfrak{C}_i\}, \end{aligned}$$

where $\text{Conv}\{S\}$ denotes the (closed) convex hull of S.

Definition 4 (Jump-flow solutions)**.** A PWA-DAE hybrid arc $x : \mathcal{E} \to \mathbb{R}^n$ is a jump-flow solution of Δ^{pwa} starting from an initial point $x_0 \in \mathbb{R}^n$ if $x(0,0) = x_0$ and the following conditions are satisfied:

(Jump Condition) For each $j \in \mathbb{N}$ such that I_j^τ has non empty interior:

$$\begin{aligned} \frac{\mathrm{d}x(t_j, \tau)}{\mathrm{d}\tau} \in F^{\text{jp}}(x(t_j, \tau)) \quad & \text{for almost all } \tau \in I_j^\tau, \\ x(t_j, \tau) \notin \mathfrak{C}^{\text{pwa}} \quad & \text{for all } \tau \in [\min I_j^\tau, \sup I_j^\tau), \end{aligned}$$

(Flow Condition) For each $j \in \mathbb{N}$ such that I_j^t has non empty interior:

$$\begin{aligned} \frac{\mathrm{d}x(t, \tau_{j+1})}{\mathrm{d}t} \in F^{\text{df}}(x(t, \tau_{j+1})) \quad & \text{for almost all } t \in I_j^t, \\ x(t, \tau_{j+1}) \in \mathfrak{C}^{\text{pwa}} \quad & \text{for all } t \in [\min I_j^t, \sup I_j^t), \end{aligned}$$

Remark 3. (i) In contrast to the definitions outlined in [10], [9], the jump condition and flow condition in Definition 4 exhibit a symmetric structure. This symmetry arises from the fact that the jumps considered here are also characterized by absolutely continuous dynamics as the flows. However, it is worth noting that the definitions of F^{jp} and F^{df} are not symmetry, which is because the consideration of the jump-flow sliding behaviors discussed below.

(ii) In solving the differential inclusion within the **(Jump Condition)**, our objective is to identify a specific mapping $G^{\text{jp}} \in F^{\text{jp}}$. Notably, if we were to set $G^{\text{jp}} = \text{Conv}\{f_i^{\text{jp}}(x), \forall i : x \in \text{clo}(\Omega_i) \setminus \mathfrak{C}_i\}$, the jump path defined by $\frac{\mathrm{d}x}{\mathrm{d}\tau} \in G^{\text{jp}}$ would be parameterized over $[0, \infty)$. However, since the jump path in **(Jump Condition)** is required to be parameterized over I_j^τ, we may choose $G^{\text{jp}}(x, \tau) = \text{Conv}\{f_i^{\text{jp}}(x) \left(\frac{\mathrm{d}\varphi_j}{\mathrm{d}\tau}\right)^{-1}, \forall i : x \in \text{clo}(\Omega_i) \setminus \mathfrak{C}_i\}$, where $\varphi_j : [0, \infty) \to I_j^\tau$ represents a change of variables.

Recall that S_{pq} denotes the boundary shared by both Ω_p and Ω_q. For any $x \in S_{pq} \cap \mathfrak{C}_p \cap \mathfrak{C}_q$, meaning x is a consistent point for both Δ_p and Δ_q on the boundary of Ω_p and Ω_q respectively, we have $F^{\text{df}}(x) = \alpha f_p^{\text{df}}(x) + (1-\alpha) f^{\text{df}} q(x)$ for $\alpha \in [0,1]$. If $f_i^{\text{df}}(x)$ and $f_j^{\text{df}}(x)$ point towards S_{pq}, then it is evident that a **flow sliding behavior** will emerge when considering the Filippov solution of the differential inclusion in the **(Flow Condition)**.

A challenge arises when $x \in (S_{pq} \cap \mathfrak{C}_p) \setminus \mathfrak{C}_q$, meaning x is consistent for one mode Δ_q but not for another mode

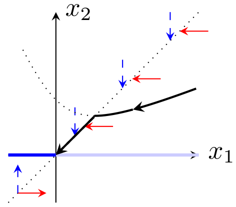


Fig. 4: Red arrows and blue dashed arrows: Flow directions of Δ_1 and jump direction of Δ_2, black line: Jump-flow solutions.

Δ_p. In such cases, the flow rule $\frac{\mathrm{d}x(t,\tau)}{\mathrm{d}t} = f_p^{\mathrm{df}}(x(t,\tau))$ should be followed for Δ_p, while the jump rule $\frac{\mathrm{d}x(t,\tau)}{\mathrm{d}\tau} \in \mathrm{Cone}\{f_q^{\mathrm{jp}}(x(t,\tau))\}$ should be respected for Δ_q. Describing the sliding behavior on $(S_{pq} \cap \mathfrak{C}_p) \setminus \mathfrak{C}_q$ becomes challenging as it involves two dynamics described by different variables, t and τ. The **(Flow Condition)** actually provides a solution with the assistance of the definition of F^{df}.

Jump-flow sliding behavior. In the case that both vector fields $f_p^{\mathrm{df}}(x)$ and $f_q^{\mathrm{jp}}(x)$ point towards $(S_{pq} \cap \mathfrak{C}_p) \setminus \mathfrak{C}_p$, there exists $\beta > 0$ such that

$$F^{\mathrm{df}}(x) \ni f_p^{\mathrm{df}}(x) + \beta f_q^{\mathrm{jp}}(x) \in T_x S_{pq} \tag{5}$$

for $x \in S_{pq}$, the system follows a jump-flow sliding behavior defined by

$$\frac{\mathrm{d}x(t,\tau_{j+1})}{\mathrm{d}t} = f_p^{\mathrm{df}}(x(t,\tau_{j+1})) + \beta f_q^{\mathrm{jp}}(x(t,\tau_{j+1})).$$

Example 3. Consider a PWA-DAE Δ^{pwa} on $\mathbb{R}^2$ with two modes

$$\Delta_1 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\Delta_2 : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Clearly, Δ_1 is an ODE, i.e., an index-0 DAE and Δ_2 is an index-1 DAE. The active regions are

$$\Omega_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1 > x_2 \right\}, \quad \Omega_2 = \mathbb{R}^2 \setminus \Omega_1.$$

Thus $S_{12} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1 = x_2 \right\}$. For each $x \in S_{12} \setminus \{0\}$ in the first quadrant, there exists $\beta > 0$ such that $f_1^{\mathrm{df}}(x) + \beta f_2^{\mathrm{jp}}(x) \in T_x S_{12} = \mathrm{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $f_1^{\mathrm{df}}(x) = \begin{bmatrix} -x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$ and $f_2^{\mathrm{jp}}(x) = \begin{bmatrix} 0 \\ -x_2 \end{bmatrix}$. There exists a jump-flow sliding behavior from x_0 as shown in Fig 4.

Now we discuss the boundary behaviors of PWA-DAE. For any boundary S_{pq} of two neighboring active regions Ω_p and Ω_q, there are basically the following different boundary behaviors possible:
(a) Flow-flow crossing, sliding or repelling if $S_{pq} \cap \mathfrak{C}_p \cap \mathfrak{C}_q \neq \emptyset$, the active vector fields are f_p^{df} and f_q^{df}.
(b) Jump-jump crossing, sliding or repelling if $S_{pq} \setminus (\mathfrak{C}_p \cup \mathfrak{C}_q) \neq \emptyset$, the active vector fields are f_p^{jp} and f_q^{jp}.
(c) Jump-flow crossing, sliding or repelling if $(S_{pq} \cap \mathfrak{C}_p) \setminus \mathfrak{C}_q \neq \emptyset$, the active vector fields are f_p^{df} and f_q^{jp}.
The crossing behaviors happen when the corresponding active vector fields f_p^{df} (or f_p^{jp}) point towards S_{pq} and f_q^{df} (or f_q^{jp}) point away from S_{pq}. The sliding behaviors are present when both f_p^{df} (or f_p^{jp}) and f_q^{df} (or f_q^{jp}) point towards S_{pq}. The repelling behaviors are present when both f_p^{df} (or f_p^{jp}) and f_q^{df} (or f_q^{jp}) point away from S_{pq}, the solution can be continued in a non-unique way, it can leave the boundary into either of the adjacent regions or it can also slide along the boundary.

Example 4. Consider an RLC electric circuit with two switches K_1 and K_2, an inductor L, a capacitor C and two resistors R_1 and R_2. Depending on the situations the two

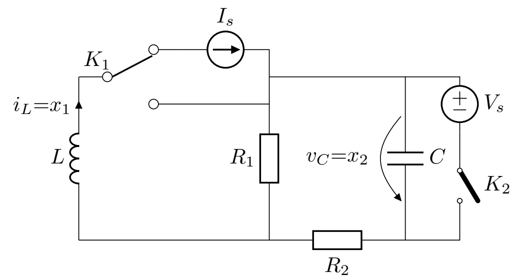


Fig. 5: A switching RLC circuit

switches, the circuit can be modeled by a PWA-DAE Δ^{pwa} via Kirchhoff's law. The states are $x = (x_1, x_2)$, where $x_1 = i_L$ is the current of L and $x_2 = v_c$ is the voltage of C, Δ^{pwa} has four DAE modes Δ_i, $i = 1, 2, 3, 4$.

K_1 \ K_2	Open	Closed
Down	Δ_1	Δ_2
Up	Δ_4	Δ_3

The four modes are, respectively, given by,

$$\Delta_1 : \quad \begin{bmatrix} L & R_2 C \\ \frac{L}{R_1} & -C \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\Delta_2 : \quad \begin{bmatrix} \frac{L}{R_1} & -C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ V_s \end{bmatrix},$$
$$\Delta_3 : \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -V_s \\ I_s \end{bmatrix}.$$
$$\Delta_4 : \quad \begin{bmatrix} L & R_2 C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I_s \end{bmatrix}.$$

We assume for the simplicity of calculations that $L = 1\,A$, $C = 1\,F$, $R_1 = R_2 = 1\,\Omega$, $I_s = 4\,A$ and $V_s = -4\,V$. The active regions are chosen, respectively, as

$$\Omega_1 = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_1 \leq 0, x_2 < 0 \right\},$$
$$\Omega_2 = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_1 < 0, x_2 \geq 0 \right\},$$
$$\Omega_3 = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_1 > 0, x_2 \geq 0 \right\}.$$
$$\Omega_4 = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_1 \geq 0, x_2 < 0 \right\},$$

By calculations, we have $f_1^{\mathrm{df}}(x) = \begin{bmatrix} \frac{-x_1 - x_2}{2} \\ \frac{x_1 - x_2}{2} \end{bmatrix}$, $f_2^{\mathrm{jp}}(x) = \begin{bmatrix} -x_2 - 4 \\ -x_2 - 4 \end{bmatrix}$, $f_3^{\mathrm{jp}}(x) = \begin{bmatrix} -x_1 - 4 \\ -x_2 - 4 \end{bmatrix}$, $f_4^{\mathrm{jp}}(x) = \begin{bmatrix} -x_1 - 4 \\ x_1 + 4 \end{bmatrix}$, these vector fields are drawn below in their active regions.

It can be seen in Figure 6 that there are four boundary behaviors, namely, jump-jump sliding for $x_1 > 0$, $x_2 = 0$; jump-jump crossing for $x_1 = 0$, $x_2 > 0$; jump-flow crossing for $x_1 < 0$, $x_2 = 0$; jump-flow sliding for $x_1 = 0$, $x_2 < 0$.

In Figures 7a and 7b, we draw the jump-flow solution $x(t, \tau) = (x_1(t, \tau), x_2(t, \tau))$ from the initial point $(1, 4.75)$. The solution is defined on $\mathcal{E} = (\{0\} \times [0, \tau_1]) \cup ([0, t_1] \times$

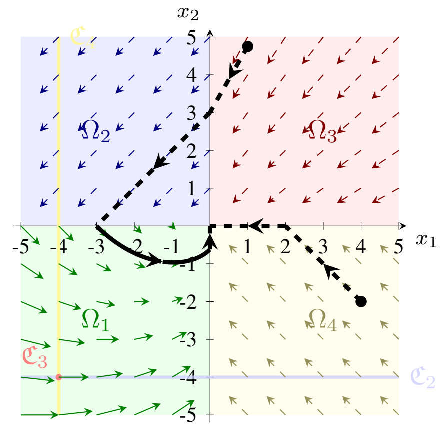
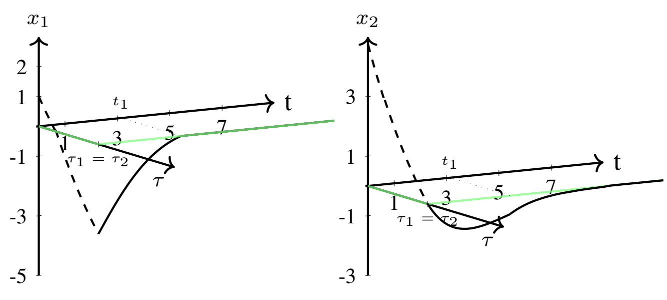


Fig. 6: Jump-flow solutions of the circuit

$\{\tau_1\}) \cup (\{t_1\} \times [\tau_1, \tau_2]) \cup ([t_1, \infty) \times \{\tau_2\})$, where $\tau_1 = \tau_2 = 3.95$ and $t_1 = 3.14$ is the real time that the solution reaches $x_1 = 0$ via the flow. The **(Jump Condition)** on Ω_2 and Ω_3 are chosen as $\frac{\mathrm{d}x}{\mathrm{d}\tau} = f_2^{\mathrm{jp}}(x)$ and $\frac{\mathrm{d}x}{\mathrm{d}\tau} = f_3^{\mathrm{jp}}(x)$, respectively. The solutions for the jump-flow sliding behavior $\frac{\mathrm{d}x}{\mathrm{d}t} = f_1^{\mathrm{df}}(x) + \beta f_4^{\mathrm{jp}}(x)$, $\beta > 0$, are calculated by a MATLAB ODE solver.



(a) The hybrid arc $x_1(t, \tau)$ **(b)** The hybrid arc $x_2(t, \tau)$

Fig. 7: Evolution of solutions in (hybrid) time.

IV. Conclusions and perspectives

In this paper, we present a solution framework for PWA-DAEs. We redefine state-dependent jumps as continuous dynamics in line with the active region rule. Leveraging hybrid time-domain techniques, we establish a well-defined concept of jump-flow solutions, which have various sliding and crossing boundary behaviors. This solution framework offers a foundation for future studies on the stability and stabilization of DAEs under state-dependent switching signals. Furthermore, we aim to explore its applicability in linear complementarity systems.

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