Geometric analysis of nonlinear differential-algebraic equations via nonlinear control theory

Yahao Chen[a], Witold Respondek[b]

[a] *Bernoulli Institute, University of Groningen, The Netherlands (e-mail: yahao.chen@rug.nl);*
[b] *Normandie Université, INSA-Rouen, LMI, France (e-mail: witold.respondek@insa-rouen.fr).*

Abstract

For nonlinear differential-algebraic equations (DAEs), we define two kinds of equivalences, namely, the external and internal equivalence. Roughly speaking, the word "external" means that we consider a DAE (locally) everywhere and "internal" means that we consider the DAE on its (locally) maximal invariant submanifold (i.e., where its solutions exist) only. First, we revise the geometric reduction method in DAEs solution theory and formulate an implementable algorithm to realize that method. Then a procedure named explicitation with driving variables is proposed to connect nonlinear DAEs with nonlinear control systems and we show that the driving variables of an explicitation system can be reduced under some involutivity conditions. Finally, due to the explicitation procedure, we will use the notion of zero dynamics from nonlinear control theory to derive a nonlinear generalization of the Weierstrass form.

Keywords: geometric methods, nonlinear DAEs, control systems, explicitation, external equivalence, internal equivalence, zero dynamics, invariant submanifolds, Weierstrass form

1. Introduction

Consider a nonlinear differential-algebraic equation (DAE) of the form

$$\Xi : E(x)\dot{x} = F(x), \tag{1}$$

where $x \in X$ is a vector of the generalized states and X is an open subset of $\mathbb{R}^n$ (or an n-dimensional manifold).

$$\begin{tikzcd} TX \arrow[d, "\pi"'] \arrow[dr, "E"] & \\ X \arrow[r, "F"'] & \mathbb{R}^l \end{tikzcd}$$

The maps $E : TX \to \mathbb{R}^l$ and $F : X \to \mathbb{R}^l$ (see the above diagram, where $\pi : TX \to X$ is the canonical projection) are smooth and the word "smooth" will mean throughout this paper $\mathcal{C}^\infty$-smooth . We will denote a DAE of the form (1) by $\Xi_{l,n} = (E, F)$ or, simply, Ξ. Equation (1) is affine with respect to the velocity $\dot{x}$, so sometimes it is called a quasi-linear DAE (see e.g., [1, 2]) and can be considered as an affine Pfaffian system since the rows E^i of E are actually differential 1-forms on X (for linear

Pfaffian systems, see e.g. [3]), so E is a $\mathbb{R}^l$-valued differential 1-form on X. A semi-explicit DAE is of the form

$$\Xi^{SE}: \begin{cases} \dot{x}_1 = F_1(x_1, x_2) \\ 0 = F_2(x_1, x_2), \end{cases} \tag{2}$$

where $x_1 \in X_1$ is a vector of state variables and $x_2 \in X_2$ is a vector of algebraic or free variables (since there are no differential equations for x_2) with X_1 and X_2 being open subsets of $\mathbb{R}^q$ and $\mathbb{R}^{n-q}$, respectively (or q- and $(n-q)$-dimensional manifolds, respectively), the maps $F_1 : X_1 \times X_2 \to TX_1$ and $F_2 : X_1 \times X_2 \to \mathbb{R}^{l-q}$ are smooth. A linear DAE of the form

$$\Delta : E\dot{x} = Hx \tag{3}$$

will be denoted by $\Delta_{l,n} = (E, H)$ or, simply, Δ, where $E \in \mathbb{R}^{l\times n}$ and $H \in \mathbb{R}^{l\times n}$. Both the semi-explicit DAE Ξ^{SE} and the linear DAE Δ can be seen as special cases of DAE Ξ. The motivation of studying DAEs is their frequent presence in modelling of practical systems as electrical circuits [2, 4], chemical processes [5, 6], mechanical systems [7–9], etc.

There are three main results of this paper. The first result concerns analyzing a DAE (locally) everywhere (i.e., externally) or considering the restriction of the DAE to a submanifold (i.e., internally), which corresponds to the external equivalence (see Definition 3.1) and the internal equivalence (see Definition 3.8), respectively. The difference between the two equivalences will be illustrated by their relations with the solutions. In order to analyze the existence of solutions, we use a concept named *locally maximal invariant submanifold* (see Definition 2.2), which is a submanifold where the solutions of DAEs exist and can be constructed via a geometric reduction method shown in Section 2. Note that the geometric reduction method is not new in the theory of nonlinear DAEs, see e.g., [1, 2, 10–12] and the recent papers [13–15]. In the present paper, we will show a practical implementation of this method via an algorithm summarized in Appendix. Note that considering only the restriction of a DAE means that we only care about where and how the solutions of that DAE evolve. However, when a nominal point is not on the maximal invariant submanifold (which is common for practical systems, since an initial point could be anywhere), there are no solutions passing through the point but we still want to steer the solutions to the submanifold and thus we must follow the rules indicated by the "external" form of the DAE, thus considering DAEs everywhere is also important.

The second result of this paper is a nonlinear counterpart of the results of [16], in which we have shown that one can associate a class of linear control systems to any linear DAE (by the procedure of explicitation for linear DAEs). In the present paper, to any nonlinear DAE, by introducing extra variables (called driving variables), we can attach a class of nonlinear control systems. Moreover, we show that the driving variables in this explicitation procedure can be fully reduced under some involutivity conditions which explains when a DAE Ξ is ex-equivalent to a semi-explicit DAE Ξ^{SE}.

It is well-known (see e.g., [17],[18]) that any linear DAE Δ of the form (3) is ex-equivalent (via linear transformations) to the Kronecker canonical form **KCF**. In particular, if Δ is *regular*, i.e., the matrices E and H are square ($l = n$) and $|sE - H| \not\equiv 0$, $\forall s \in \mathbb{C}$, then Δ is ex-equivalent (also via linear transformations) to the Weierstrass form **WF** [19] (see (18) below). The studies on normal forms and canonical forms of DAEs can be found in [17, 19–22] for the linear case and in [15, 23, 24] for the nonlinear case. The last result of this paper is to use such concept as zero dynamics of the nonlinear control theory [25, 26] to derive a nonlinear generalization of the **WF**. In the linear case, canonical forms as the **KCF** and the **WF** are closely related to a geometric concept named the Wong sequences [27] (see Remark 2.6 below). As shown in [21], the relations between the **WF** and the Wong sequences have been built and in [22], the importance of the Wong sequences for the geometric analysis of linear DAEs is reconfirmed, thus a generalization of the Wong sequences for nonlinear DAEs is desired in the present paper.

This paper is organized as follows. In Section 2, we discuss the existence of solutions of DAEs by revising the geometric reduction method. In Section 3, we compare the notions of external equivalence and internal equivalence and discuss the uniqueness of DAEs solutions via the notion of internal regularity. In Section 4, we propose the explicitation (with driving variables) procedure to connect nonlinear DAEs to nonlinear control systems. In Section 5, we show when a nonlinear DAE is externally equivalent to a semi-explicit one and how this problem is related to the explicitation procedure. A nonlinear generalization of the Weierstrass form is given in Section 6. Finally, Section 7 and Section 8 contain proofs and the conclusions, respectively. In Appendix of Section 9, we show a recursive algorithm which implements the geometric reduction method. The following notations will be used throughout the paper: We use $\mathbb{R}^{n\times m}$ to denote the set of real valued matrices with n rows and m columns, $GL\left(n,\mathbb{R}\right)$ to denote the group of nonsingular matrices of $\mathbb{R}^{n\times n}$ and I_n to denote the $n \times n$-identity matrix. For a linear map L, we denote by $\operatorname{rank} L$, $\ker L$ and $\operatorname{Im} L$, the rank, the kernel and the image of L, respectively. Denote by T_xM the tangent space of a submanifold M of $\mathbb{R}^n$ at $x \in M$ and by $\mathcal{C}^k$ the class of functions which are k-times differentiable. For a smooth map $f : X \to \mathbb{R}$, we denote its differentials by $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]$ and for a vector-valued map $f : X \to \mathbb{R}^m$, where $f = [f_1, \dots, f_m]^T$, we denote its differential by $\mathrm{D}f = \begin{bmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_m \end{bmatrix}$. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$.

2. The geometric reduction method revisited

In this section, we revise the geometric reduction method in the DAEs solution theory, other formulations of this method can be consulted in Section 3.4 of [2], Chapter IV of [1] and [13] for DAEs and [14] for DAE control systems. We start from the definition of a solution for a DAE.

Definition 2.1. A solution of a DAE $\Xi_{l,n} = (E, F)$ is a $\mathcal{C}^1$-curve $x : I \to X$ defined on an open interval I such that for all $t \in I$, the curve $x(\cdot)$ satisfies $E(x(t))\dot{x}(t) = F(x(t))$.

Throughout this paper, we will be interested only in solutions of Ξ that are at least $\mathcal{C}^1$. A given point x_0 is called *consistent* (or *admissible*) if there exists at least one solution $x(\cdot)$ of Ξ satisfying $x(t_0) = x_0$ (i.e., $E(x_0)\dot{x}(t_0) = F(x_0)$) for a certain $t_0 \in I$, we will denote by S_c *the consistency set*, i.e., the set of all consistent points.

Definition 2.2 (invariant and locally invariant submanifolds)**.** Consider a DAE $\Xi_{l,n} = (E, F)$ defined on X. A smooth connected embedded submanifold M of X is called *invariant* if for any point $x_0 \in M$, there exists a solution $x : I \to X$ of Ξ such that $x(t_0) = x_0$ for a certain $t_0 \in I$ and $x(t) \in M$ for all $t \in I$. Given a point $x_p \in X$, we will say that M is a *locally invariant submanifold* (around x_p) if there exists an open neighborhood $U \subseteq X$ of x_p such that $M \cap U$ is invariant.

Proposition 2.3. *Consider a DAE $\Xi_{l,n} = (E, F)$ and fix a point x_p. Let M be a smooth connected embedded submanifold containing x_p. If M is a locally invariant submanifold around x_p, then $F(x) \in E(x)T_xM$ for all $x \in M$ around x_p. Conversely, assume that there exists an open neighborhood U of x_p such that, at all $x \in M \cap U$, we have $F(x) \in E(x)T_xM$ and, additionally,* $\dim E(x)T_xM = const.$*, then M is a locally invariant submanifold.*

The proof is given in Section 7.1.

Remark 2.4. Note that the assumption that $\dim E(x)T_xM = const.$ of Proposition 2.3 is not a necessary condition to conclude that M is an invariant submanifold, but it excludes singular points of DAEs and helps to view a DAE as an ordinary differential equation (ODE) defined on the invariant submanifold. Take the following DAE for an example:

$$\Xi_{1,1} : x\dot{x} = x^2,$$

where $x \in X = \mathbb{R}$. Let $M = X$, then clearly, $F(x) = x^2 \in x \cdot T_xX$, at any $x \in M = \mathbb{R}$. We have dim $E(x)T_xM$ equals 1 for $x \neq 0$ and is 0 for $x = 0$, so dim $E(x)T_xM \neq const.$, for all $x \in M$. Nevertheless, for any $x_0 \in M = \mathbb{R}$, there exists a unique solution $x(t)$ satisfying $x(0) = x_0$, namely, $x(t) = e^t x_0$. Therefore $M = \mathbb{R}$ is an invariant submanifold.

A locally invariant submanifold M^* (around x_p) is called locally *maximal*, if there exists a neighborhood U of x_p such that for any other locally invariant submanifold M, we have $M \cap U \subseteq M^* \cap U$. The *geometric reduction method* for DAEs is the following recursive procedure which can be used to construct locally maximal invariant submanifold M^*.

Definition 2.5 (geometric reduction method)**.** Consider a DAE $\Xi_{l,n} = (E, F)$, fix a point $x_p \in X$ and let U_0 be an open connected subset of X containing x_p. Set $M_0 = X$, $M_0^c = U_0$. Suppose that there exist an open neighborhood U_{k-1} of x_p and a sequence of smooth connected embedded submanifolds $M_{k-1}^c \subsetneq \cdots \subsetneq M_0^c$ of U_{k-1} for a certain $k \geq 1$, has been constructed. Define recursively

$$M_k := \left\{ x \in M_{k-1}^c : F(x) \in E(x) T_x M_{k-1}^c \right\}. \tag{4}$$

Then either $x_p \notin M_k$ or $x_p \in M_k$, and in the latter case, assume that there exists a neighborhood U_k of x_p such that $M_k^c = M_k \cap U_k$ is a smooth embedded submanifold (which can always be assumed connected by taking U_k sufficiently small).

Remark 2.6. For a linear DAE $\Delta = (E, H)$ of the form (3), define a sequence of subspaces (one of the Wong sequences [27]) by

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_k = H^{-1} E \mathscr{V}_{k-1}, \quad k \geq 1.$$

If we apply the iterative cons truction of M_k by (4) to the DAE Δ, we get $M_k^c = \mathscr{V}_k$, $\forall k \geq 0$. Thus the sequence of submanifolds M_k can be seen as a nonlinear generalization of the sequence $\mathscr{V}_k$.

The following proposition shows that the geometric reduction method above can be used to construct locally maximal invariant submanifold M^* and to deduce that the consistency set S_c, i.e., the space on which the solutions exist, coincides locally with M^*.

Proposition 2.7. *In the geometric reduction method of Definition 2.5, there always exists $k^* \leq n$ such that either k^* is the smallest integer for which $x_p \notin M_{k^*+1}$ or k^* is the smallest integer such that $x_p \in M_{k^*+1}^c$ and $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$. In the latter case, we assume that $\dim E(x) T_x M_{k^*+1}^c = const.$ in a neighborhood $U^* \subseteq U_{k^*+1}$ of x_p in X and then*

(i) x_p is consistent and $M^* = M_{k^*+1}^c$ is a locally maximal invariant submanifold around x_p.

(ii) M^* coincides locally with the consistency set S_c, i.e., $M^* \cap U = S_c \cap U^*$ (take a smaller U^* if necessary).

The proof is given in Section 7.1. Note that the geometric method can be implemented in practice via an algorithm which we propose in Appendix of the present paper and the results of Proposition 3.3 and Theorem 6.1 below will be based on that algorithm.

3. External equivalence, internal equivalence and internal regularity

Two linear DAEs $E\dot{x} = Hx$ and $\tilde{E}\dot{\tilde{x}} = \tilde{H}\tilde{x}$ are called externally equivalent [16] or strictly equivalent [18], if there exist constant invertible matrices Q and P such that $QEP^{-1} = \tilde{E}$ and $QHP^{-1} = \tilde{H}$. Analogously, we define the external equivalence of two nonlinear DAEs as follows. Given a diffeomorphism $\psi : X \to \tilde{X}$, a differential 1-form Ω and a function h (0-form), we denote their pullbacks [3] by $\psi^*\Omega$ and $\psi^* h$, respectively.

Definition 3.1 (external equivalence)**.** Two DAEs $\Xi_{l,n} = (E, F)$ and $\tilde{\Xi}_{l,n} = (\tilde{E}, \tilde{F})$ defined on X and $\tilde{X}$, respectively, are called externally equivalent, shortly ex-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and $Q : X \to GL(l, \mathbb{R})$ such that

$$\psi^* \tilde{E} = QE \quad \text{and} \quad \psi^* \tilde{F} = QF,$$

that is,

$$\tilde{E}(\psi(x)) = Q(x)E(x) \left(\frac{\partial \psi(x)}{\partial x} \right)^{-1} \quad \text{and} \quad \tilde{F}(\psi(x)) = Q(x)F(x). \tag{5}$$

The ex-equivalence of two DAEs will be denoted by $\Xi \overset{ex}{\sim} \tilde{\Xi}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x_p and $\tilde{U}$ of $\tilde{x}_p$, and $Q(x)$ is defined on U, we will speak about local ex-equivalence.

The following observation relates ex-equivalence with solutions.

Remark 3.2. The ex-equivalence preserves trajectories, i.e., for two DAEs $\Xi \overset{ex}{\sim} \tilde{\Xi}$, if a $\mathcal{C}^1$-curve $x(\cdot)$ is a solution of Ξ passing through $x_0 = x(t_0)$, then $\tilde{x} = \psi \circ x$ is a solution of $\tilde{\Xi}$ passing through $\tilde{x}_0 = \psi(x_0)$; but even if we can smoothly conjugate all trajectories of two DAEs, they are not necessarily ex-equivalent. For example, consider $\Xi_1 = (E_1, F_1)$ and $\Xi_2 = (E_2, F_2)$, where $E_1(x) = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $F_1(x) = \left[\begin{smallmatrix} x_3^2 \\ x_1 \\ x_2 \end{smallmatrix}\right]$, $E_2(x) = \left[\begin{smallmatrix} 0 & x_1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $F_2(x) = \left[\begin{smallmatrix} x_3^2 \\ x_1 \\ x_2 \end{smallmatrix}\right]$. Then for both DAEs Ξ_1 and Ξ_2, the maximal invariant submanifold is $M^* = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_2 = x_3 = 0 \right\}$ and for any $(x_{10}, x_{20}, x_{30}) = (0, 0, x_{30}) \in M^*$, the unique solution of both systems is $x_1(t) = x_2(t) = 0$, $x_3(t) = \frac{x_{30}}{1 - x_{30}t}$. Nevertheless, the DAEs are not ex-equivalent since the distribution $\ker E_1$ is involutive but the distribution $\ker E_2$ is not (clearly, the ex-equivalence of two DAEs preserves the involutivity of $\ker E_1$ and $\ker E_2$ since if $\Xi_1 \overset{ex}{\sim} \Xi_2$, via Q and ψ, then $\ker E_2 = \frac{\partial \psi}{\partial x} \ker E_1$).

Now we use the algorithm presented in the Appendix to implement the geometric reduction method of Definition 3.1 and to show that any DAE Ξ has isomorphic solutions with an "internal" DAE Ξ^* defined on its locally maximal invariant submanifold M^*. In the statement of Proposition 3.3, we refer to the submanifold $M^* = M^*_{k+1}$, the neighborhood $U^* = U^*_{k+1}$ and the DAE $\Xi^*_{r^*, n^*} = (E^*, F^*)$ defined on M^* by the algorithm of the Appendix, where $E^* = E_{k^*+1} : M^* \to \mathbb{R}^{r^* \times n^*}$, $F^* = F_{k^*+1} : M^* \to \mathbb{R}^{r^*}$, $n^* = n_{k^*} = n_{k^*+1}$, $r^* = r_{k^*+1}$ come from Step $k^* + 1$ of the algorithm.

Proposition 3.3 (isomorphic solutions)**.** *Consider a DAE $\Xi_{l,n} = (E, F)$, fix a point $x_p \in X$. Suppose that* **Assumptions 1** and **2** *of the algorithm in Appendix are satisfied. Then M_k^c, for $k = 0, \ldots, k^* + 1$, given by (4) of the geometric reduction method are smooth connected embedded submanifolds and* $\dim E(x)T_xM^* = const.$ *for all $x \in M^* \cap U^*$. Thus by Proposition 2.7, x_p is a consistent point and M^* is a locally maximal invariant submanifold around x_p, given by $M^* = \{x \,|\, \bar{z}_1(x) = 0, \ldots, \bar{z}_{k^*}(x) = 0\}$. Then for the DAE $\Xi^*_{r^*, n^*} = (E^*, F^*)$, defined by the algorithm, given* *on M^* by*

$$\Xi^* : E^*(z^*)\dot{z}^* = F^*(z^*), \tag{6}$$

where $z^* = z_{k^*+1} = z_{k^*}$ are local coordinates on M^*, we have $\operatorname{rank} E^*(z^*) = r^*$, $\forall z^* \in M^*$*, i.e., $E^*(z^*)$ is of full row rank.*

Moreover, the DAE Ξ^* has isomorphic solutions with $\Xi_{l,n}$, i.e., there exists a local diffeomorphism $\Psi : U^* \to \Psi(U^*)$, $\Psi(x) = \hat{z} = (z^*, \bar{z}) = (z^*, \bar{z}_1, \ldots, \bar{z}_{k^*})$, transforming the set of all solutions of $\Xi_{l,n}$ into that of $\hat{\Xi}_{\hat{l},\hat{n}} = (\hat{E}, \hat{F})$, where $\hat{l} = r^* + (n - n^*)$, $\hat{n} = n$, given by

$$\hat{\Xi} : \begin{cases} E^*(z^*)\dot{z}^* = F^*(z^*), \\ \bar{z}_1 = 0, \ldots, \bar{z}_{k^*} = 0. \end{cases} \tag{7}$$

The proof is given in Section 7.2. The analysis of Proposition 3.3 shows clearly the reason behind Remark 3.2: if we assume two DAEs Ξ and $\tilde{\Xi}$ to have corresponding solutions, this assumption only gives the information that the two internal DAE Ξ^* and $\tilde{\Xi}^*$, which have isomorphic solutions with Ξ and $\tilde{\Xi}$, respectively, are ex-equivalent, respectively, i.e., via a diffeomorphism between the submanifolds M^* and $\tilde{M}^*$ and an invertible map Q defined on the invariant submanifold M^*. We do not know, however, whether the diffeomorphism and the map Q can be extended outside the submanifold M^*. In fact, outside the manifolds, the two DAEs may have completely different behaviors or even different size of system matrices. This analysis gives a motivation to introduce the concept of *internal equivalence* of two DAEs (see the formal Definition 3.8), which is defined by the ex-equivalence of two internal DAEs. In Proposition 3.3, the internal DAE Ξ^* is defined with the help of the geometric reduction algorithm. Now we introduce two notions: local restriction and full row rank reduction, which can be used to define the internal DAE Ξ^* of a DAE Ξ (which we call the reduction of local M^*-restriction of Ξ, see Proposition 3.7) without going through the algorithm when the invariant submanifold M^* is a priori given. The local restriction of a DAE to a submanifold N (invariant or not) is defined as follows.

Definition 3.4 (local restriction)**.** Consider a DAE $\Xi_{l,n} = (E, F)$ and a smooth connected embedded submanifold $N \subseteq X$ containing a point x_p. Let $\psi(x) = z = (z_1, z_2)$ be local coordinates on a neighborhood U of x_p such that $N \cap U = \{z_2 = 0\}$ and z_1 are thus coordinates on $N \cap U$. The restriction of Ξ to $N \cap U$, called local N-restriction of Ξ and denoted $\Xi|_N$ is

$$\Xi|_N : \tilde{E}(z_1, 0) \begin{bmatrix} \dot{z}_1 \\ 0 \end{bmatrix} = \tilde{F}(z_1, 0), \tag{8}$$

where $\tilde{E} \circ \psi = E \left(\frac{\partial \psi}{\partial x}\right)^{-1}$, $\tilde{F} \circ \psi = F$.

For any DAE $\Xi_{l,n} = (E, F)$, there may exist some redundant equations (in particular, some trivial algebraic equations $0 = 0$ and some dependent equations). In the linear case, we have defined the full rank reduction of a linear DAE (see Definition 6.4 of [16]). We now generalize this notion of reduction to nonlinear DAEs to get rid of their redundant equations.

Definition 3.5 (reduction)**.** For a DAE $\Xi_{l,n} = (E, F)$, assume $\operatorname{rank} E(x) = const. = q$. Then there exists $Q : X \to GL(l, \mathbb{R})$ such that E_1 of $QE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$ is of full row rank q, denote $QF = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$. Assume that $\operatorname{rank} \mathrm{D}F_2(x) = const. = \hat{l} - q \leq l - q$. Then the full row rank reduction, shortly reduction, of Ξ, denoted by Ξ^{red}, is the DAE

$$\Xi^{red} : \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) \\ \hat{F}_2(x) \end{bmatrix},$$

where $\hat{F}_2 : X \to \mathbb{R}^{\hat{l}-q}$ with $\mathrm{D}\hat{F}_2$ being all independent rows of $\mathrm{D}F_2$.

Remark 3.6. Define $\mathcal{F}_1 := \operatorname{Im} E \subseteq R^l$ and let $\mathcal{F}_2$ be any subspace such that $\mathcal{F}_1 + \mathcal{F}_2 = \mathbb{R}^l$. Then $F_2 : X \to \mathbb{R}^l / \operatorname{Im} E(x) \simeq \mathbb{R}^{l-q}$ is a map satisfies $F = F_1 + F_2$, where $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$. Clearly, since the choice of $Q(x)$ is not unique, the reduction of Ξ is not unique either. Nevertheless, since $Q(x)$ preserves the solutions, each reduction Ξ^{red} has the same solutions as the original DAE Ξ.

For a locally invariant submanifold M, we consider the local M-restriction $\Xi|_M$ of Ξ, and then we construct a reduction of $\Xi|_M$ and denote it by $\Xi|_M^{red}$. Notice that the order matters: to construct $\Xi|_M^{red}$, we first restrict and then reduce while reducing first and then restricting will, in general, not give $\Xi|_M^{red}$ but another DAE $\Xi^{red}|_M$ (which may have redundant equations). Take, for example, the following nonlinear DAE $\Xi : \begin{bmatrix} 1 & 1 \\ x & 0 \\ 0 & 0 \\ e^y & e^y \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x^2 \\ x^3 \\ xy \\ e^y x^2 \end{bmatrix}$ defined on $X = \mathbb{R}^2$. Fix a point $(x_p, y_p) = (1, 0)$, then it is clear that $M^* = \{(x, y) \in \mathbb{R}^2 : x > 0, y = 0\}$ in the locally maximal invariant submanifold around x_p. Set $\psi(x, y) = (z_1, z_2) = (x, y)$ be local coordinates on X. Then the M^*-restriction of Ξ, by Definition 3.4, is $\Xi|_{M^*} : \begin{bmatrix} 1 \\ z_1 \\ 0 \\ 1 \end{bmatrix} \dot{z}_1 = \begin{bmatrix} z_1^2 \\ z_1^3 \\ 0 \\ z_1^2 \end{bmatrix}$ and the reduction of $\Xi|_{M^*}$ is $\Xi|_{M^*}^{red} : q(z_1)\dot{z}_1 = q(z_1)z_1^2$, where $q(z_1)$ can be any non-zero function (illustrating that the reduction is not unique). On the other hand, $\Xi^{red}|_{M^*}$ is $\begin{bmatrix} 1 \\ z_1 \\ 0 \end{bmatrix} \dot{z}_1 = \begin{bmatrix} z_1^2 \\ z_1^3 \\ 0 \end{bmatrix}$, and. clearly, has redundant equations.

Proposition 3.7. *Consider a DAE $\Xi_{l,n} = (E, F)$ and fix a point x_p. Let M be an $\bar{n}$-dimensional locally invariant submanifold of Ξ around x_p. Assume that $\dim E(x)T_xM = const. = \bar{r}$ for all $x \in M$ around x_p. Then a reduction $\Xi|_M^{red}$ of local M-restriction of Ξ is a DAE of the form (1) and the dimensions related to $\Xi|_M^{red}$ are $\bar{r}$ and $\bar{n}$, i.e., $\Xi|_M^{red} = \bar{\Xi}_{\bar{r},\bar{n}}$. Moreover, the matrix $\bar{E}$ of $\bar{\Xi}_{\bar{r},\bar{n}} = (\bar{E}, \bar{F})$ is of full row rank $\bar{r}$.*

Proof. We skip the proof since we have already constructed $\Xi|_M^{red}$ for M being an invariant submanifold, see (21) in the proof of Proposition 2.3, it is clear that $\bar{E} = [\bar{E}_1^1 \ \bar{E}_1^2]$, $\bar{F} = \bar{F}_1$ and $\operatorname{rank} \bar{E} = \bar{r}$. $\square$

The definition of the internal equivalence of two DAEs is given as follows.

Definition 3.8. (internal equivalence) Consider two DAEs $\Xi = (E, F)$ and $\tilde{\Xi} = (\tilde{E}, \tilde{F})$, and fix two points $x_p \in X$ and $\tilde{x}_p \in \tilde{X}$. Let M^* and $\tilde{M}^*$ be two locally maximal invariant submanifolds of Ξ and $\tilde{\Xi}$, around x_p and $\tilde{x}_p$, respectively. Assume that $\dim E(x)T_xM^* = const.$ for $x \in M^*$ around x_p and $\dim \tilde{E}(\tilde{x})T_{\tilde{x}}\tilde{M}^* = const.$ for $\tilde{x} \in \tilde{M}^*$ around $\tilde{x}_p$. Then, Ξ and $\tilde{\Xi}$ are called locally internally equivalent, shortly in-equivalent, if $\Xi|^{red}_{M^*}$ and $\tilde{\Xi}|^{red}_{\tilde{M}^*}$ are ex-equivalent, locally around x_p and $\tilde{x}_p$, respectively. Denote the in-equivalence of two DAEs by $\Xi \overset{in}{\sim} \tilde{\Xi}$.

Remark 3.9. With these constant assumptions of $\dim E(x)T_xM^*$ and $\dim \tilde{E}(\tilde{x})T_{\tilde{x}}\tilde{M}^*$, by Proposition 3.7 applied to M^*, we have $\Xi|^{red}_{M^*} = \Xi^*_{r^*,n^*}$ and $\tilde{\Xi}|^{red}_{\tilde{M}^*} = \tilde{\Xi}^*_{\tilde{r}^*,\tilde{n}^*}$, where $r^* = \dim\ E(x)T_xM^*$, $n^* = \dim\ M^*$ and $\tilde{r}^* = \dim\ \tilde{E}(x)T_{\tilde{x}}\tilde{M}^*$, $\tilde{n}^* = \dim\ \tilde{M}^*$. The dimensions l and n, related to Ξ, and $\tilde{l}$ and $\tilde{n}$ related to $\tilde{\Xi}$ are not required to be the same. However, if Ξ and $\tilde{\Xi}$ are in-equivalent, then by definition, $\Xi|^{red}_{M^*} = \Xi^*_{r^*,n^*}$ and $\tilde{\Xi}|^{red}_{\tilde{M}^*} = \tilde{\Xi}^*_{\tilde{r}^*,\tilde{n}^*}$ are locally ex-equivalent and thus the dimensions related to them have to be the same, i.e., $r^* = \tilde{r}^*$ and $n^* = \tilde{n}^*$.

Now we will study the uniqueness of solutions of DAEs with the help of the notion of internal equivalence (some other results of uniqueness of DAE solutions can be consulted in e.g., [11, 12]). We will say that a solution $x : I \to M^*$ of a DAE Ξ satisfying $x(t_0) = x_0$, where $t_0 \in I$ and $x_0 \in M^*$, is maximal if for any solution $\tilde{x} : \tilde{I} \to M^*$ such that $t_0 \in \tilde{I}$, $\tilde{x}(t_0) = x_0$ and $x(t) = \tilde{x}(t)$, $\forall t \in I \cap \tilde{I}$, we have $\tilde{I} \subseteq I$.

Definition 3.10. (internal regularity) Consider a DAE $\Xi_{l,n} = (E, F)$ and let M^* be a locally maximal invariant submanifold around a point $x_p \in M^*$. Then Ξ is called locally *internally regular* (around x_p) if there exists a neighborhood $U \subseteq X$ of x_p such that for any point $x_0 \in M^* \cap U$, there exists only one maximal solution $x : I \to M^* \cap U$ satisfying $x(t_0) = x_0$ for a certain $t_0 \in I$.

Theorem 3.11. *Consider a DAE $\Xi_{l,n} = (E, F)$ and let M^* be an n^*-dimensional locally maximal invariant submanifold around a point $x_p \in M^*$. Assume that* $\dim\ E(x)T_xM^* = const. = r^*$ *for all $x \in M^*$ around x_p. Then the following conditions are equivalent*

(i) Ξ *is internally regular around x_p.*

(ii) $\dim\ M^* = \dim\ E(x)T_xM^*$, *i.e., $n^* = r^*$, for all $x \in M^*$ around x_p.*

(iii) Ξ *is locally internally equivalent to*

$$\dot{z}^* = f^*(z^*), \tag{9}$$

for $z^* \in M^* \cap U$, where U is a neighborhood of x_p and f^* is a smooth vector field on $M^* \cap U$.

The proof is given in Section 7.2.

Remark 3.12. Theorem 3.11 is a nonlinear generalization of the results on the internal regularity of linear DAEs in [16] (see also [28], where the internal regularity is called autonomy). As stated in

Theorem 6.11 of [16], a linear DAE $\Delta = (E, H)$, given by (3), is internally regular if and only if the maximal invariant subspace $\mathscr{M}^*$ of Δ (i.e., the largest subspace such that $H\mathscr{M}^* \subseteq E\mathscr{M}^*$) satisfies $\dim \mathscr{M}^* = \dim E\mathscr{M}^*$. A nonlinear counterpart of the last condition is (ii) of Theorem 3.11 and thus M^* is a natural nonlinear generalization of $\mathscr{M}^*$. Observe that M^* is the limit of M_k as $\mathscr{V}^*$ is the limit of $\mathscr{V}_k$, defined in Remark 2.6. Moreover, we have shown in [16] that the maximal invariant subspace $\mathscr{M}^* = \mathscr{V}^*$, where $\mathscr{V}^*$ coincides with the limit of the Wong sequence $\mathscr{V}_k$.

Example 3.13. Consider a DAE $\Xi_{6,6} = (E, F)$ with the generalized state $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in X$, where $X = \left\{x \in \mathbb{R}^6 : x_1 \neq x_6, x_6 > 0\right\}$,

$$\begin{bmatrix} -\ln x_6 & x_6(x_3+x_5) & \frac{x_1 x_5 \ln x_6}{x_1 - x_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1-\frac{x_1}{x_6} & 0 \\ 0 & 0 & 0 & 0 & x_5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} (x_1-x_6)(x_3+x_5)-(x_2x_6-x_6^2-x_1)\ln x_6 \\ x_5-x_2+x_6 \\ (1-\frac{x_1}{x_6})(x_6^2-x_6x_2+x_4) \\ x_6+x_5(x_6^2-x_6x_2+x_4) \\ \frac{x_1}{x_6} \\ x_3+x_5 \end{bmatrix}. \tag{10}$$

We consider Ξ around a point $x_p = (0, 1, 0, 0, 0, 1)$ and apply to Ξ the algorithm of the Appendix.

Step 1: We have $\operatorname{rank} E(x) = r_1 = 4$ on $U_1 = X$. Since E is already in the desired form, set $Q_1 = I_6$ to get

$$M_1 = \{x \in X : Q_1 F(x) \in \operatorname{Im} Q_1 E(x)\} = \left\{x \in X : \frac{x_1}{x_6} = 0,\ x_3 + x_5 = 0\right\}.$$

It is clear that $x_p \in M_1$ and $M_1^c = M_1 \cap U_1$ is a locally smooth connected embedded submanifold and $n_1 = \dim M_1^c = 4$. Then choose local new coordinates $\bar{z}_1 = (\bar{x}_1, \bar{x}_3) = (\frac{x_1}{x_6}, x_3 + x_5)$ and keep the remaining coordinates $z_1 = (x_2, x_4, x_5, x_6)$ unchanged. The system in new coordinates, denoted $\hat{\Sigma}$, take the form

$$\hat{\Xi}_1 : \begin{bmatrix} x_5\bar{x}_3 & 0 & \frac{-\bar{x}_1 x_5 \ln x_6}{\bar{x}_1 - 1} & \frac{-\bar{x}_1 \ln x_6}{x_6} & -x_6 \ln x_6 & \frac{\bar{x}_1 x_5 \ln x_6}{\bar{x}_1 - 1} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{x}_1 & 0 & 0 & 1 \\ 0 & 0 & x_5 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{\bar{x}}_1 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} \bar{x}_3 x_6(\bar{x}_1-1)-x_6\ln x_6(x_2-x_6-\bar{x}_1) \\ x_5-x_2+x_6 \\ (1-\bar{x}_1)(x_6^2-x_6x_2+x_4) \\ x_6+x_5(x_6^2-x_6x_2+x_4) \\ \bar{x}_1 \\ \bar{x}_3 \end{bmatrix}.$$

By setting $\bar{z}_1 = (\bar{x}_1, \bar{x}_3) = 0$, we get the reduction of M_1^c-restriction of $\hat{\Xi}_1$ (see Definition 3.4 and 3.5) as

$$\Xi_1 = \hat{\Xi}_1|^{red}_{M_1^c} : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_5 & -1 \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} -x_6\ln x_6(x_2-x_6) \\ x_5-x_2+x_6 \\ (x_6^2-x_6x_2+x_4) \\ x_6+x_5(x_6^2-x_6x_2+x_4) \end{bmatrix}.$$

Step 2: Consider the DAE $\Xi_1 = (E_1, F_1)$. We have $\dim E(x)T_xM_1^c = \operatorname{rank} E_1(z_1) = r_2 = 2$ around x_p (on $W_2 = M_1^c \cap U_2 = M_1^c$, where $U_2 = U_1$). Let $Q_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and define M_2 by

$$M_2 = \{z_1 : Q_1 F_1(\bar{z}_1) \in \operatorname{Im} Q_1 E_1(z_1)\} = \left\{z_1 : x_2 - x_6 = 0,\ x_6^2 - x_6 x_2 + x_4 = 0\right\}.$$

It is clear that $x_p \in M_2$, $M_2^c = M_2 \cap U_2 = M_2$ and $n_2 = \dim M_2^c = 2$. Then choose new coordinates $\bar{z}_2 = (\bar{x}_2, \bar{x}_4) = (x_2 - x_6, x_6^2 - x_6 x_2 + x_4)$ and keep the remaining coordinates $z_2 = (x_5, x_6)$ unchanged, for the system in new coordinates, denoted $\hat{\Xi}_2$, by a similar procedure as in Step 1, we can define the reduction of M_1^c-restriction of $\hat{\Xi}_2$ as

$$\Xi_2 = \hat{\Xi}_2|^{red}_{M_2^c} : \begin{bmatrix} 0 & 0 \\ x_5 & -1 \end{bmatrix} \begin{bmatrix} \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix}.$$

Step 3: For $\Xi_2 = (E_2, F_2)$, we have $\dim E(x)T_xM_2^c = \operatorname{rank} E_2(z_2) = r_2 = 1$ in $W_3 = M_2^c$. By definition, $M_3^c = M_3 = \{z_2 : x_5 = 0\}$. It can be observed that $\dim M_3^c = n_3 = 1$ and by a similar construction as at former steps, we have

$$\Xi_3 = \bar{\Xi}_2|^{red}_{M_3^c} : -\dot{x}_6 = x_6.$$

Step 4: We have $M_4^c = M_3^c$ ($\dim M_4^c = n_4 = n_3 = 1$) and $\dim E(x)T_xM_4^c = r_4 = 1$, thus $k^* = 3$ and the algorithm stops at Step $k^* + 1 = 4$. Therefore, by Proposition 3.3, $M^* = M_4^c = \{x_6 \in \mathbb{R} : x_6 > 0\}$ is locally maximal invariant and $x_p \in M^*$ is a consistent point. Moreover, since $x_6(t) = e^{-t}x_{60}$ is the unique maximal solution of $\Xi^* = \Xi_3$ passing through $x_{60} \in M^*$, we have that $x(t) = \Psi(x_6(t), 0, 0, 0, 0, 0) = (0, e^{-t}x_{60}, 0, 0, 0, e^{-t}x_{60})$ is the unique maximal solution of Ξ passing through $x_0 = \Psi(x_{60}, 0, 0, 0, 0, 0) \in M^*$, where $\Psi(x) = (x_6, \frac{x_1}{x_6}, x_2 - x_6, x_3 + x_5, x_6^2 - x_2x_6 + x_4, x_5)$ is a local diffeomorphism. Hence the DAE Ξ is internally regular around x_p by definition, which illustrates the results of Theorem 3.11 since $\dim M^* = n_4 = \dim E(x)T_xM^* = r_4 = 1$, and Ξ is in-equivalent to the ODE: $\dot{x}_6 = -x_6$.

4. Explicitation with driving variables of nonlinear DAEs

The explicitation (with driving variables) of a DAE Ξ is the following procedure.

- For a DAE $\Xi_{l,n} = (E, F)$, assume that $\operatorname{rank} E(x) = const. = q$ in a neighborhood $U \subseteq X$ of a point $x_p \in X$. Then there exists $Q : U \to GL(l, \mathbb{R})$ such that $Q(x)E(x) = \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right]$, where $E_1 : U \to \mathbb{R}^{q \times n}$, and $\operatorname{rank} E_1(x) = q$. Thus Ξ is, locally on U, ex-equivalent via $Q(x)$ to

$$\begin{cases} E_1(x)\dot{x} = F_1(x), \\ \quad\ 0 \quad\ = F_2(x), \end{cases} \tag{11}$$

  where $Q(x)F(x) = \left[\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right]$, and where $F_1 : U \to \mathbb{R}^q$, $F_2 : U \to \mathbb{R}^{l-q}$.

- The matrix $E_1(x)$ is of full row rank q, so choose its right inverse $E_1^\dagger(x)$, i.e., $E_1E_1^\dagger = I_q$ and set $f(x) = E_1^\dagger(x)F_1(x)$. The collection of all $\dot{x}$ satisfying $E_1(x)\dot{x} = F_1(x)$ of (11) is given by the differential inclusion:

$$\dot{x} \in f(x) + \ker E_1(x) = f(x) + \ker E(x). \tag{12}$$

- Since $\ker E(x)$ is a distribution of constant rank $n - q$, so choose locally $m = n - q$ independent vector fields $g_1, \ldots, g_m$ on X such that $\ker E(x) = \operatorname{span}\{g_1, \ldots, g_m\}(x)$. Then by introducing *driving variables* v_i, $i = 1, \ldots, m$, we can parametrize the affine distribution $f(x) + \ker E_1(x)$ and thus all solutions of (12) are given by all solutions (corresponding to all controls $v_i(t)$) of

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) v_i. \tag{13}$$

- Form a matrix $g(x) = [g_1(x), \ldots, g_m(x)]$. Then, we rewrite equation (13) as $\dot{x} = f(x) + g(x)v$, where $v = (v_1, \ldots, v_m)$, and set $h(x) = F_2(x)$. We claim, see Proposition 4.5 below, that all solutions of DAE (11) (and thus of the original DAE Ξ) are in one-to-one correspondence with all solutions (corresponding to all $\mathcal{C}^0$-controls $v(t)$) of

$$\begin{cases} \dot{x} = f(x) + g(x)v, \\ 0 = h(x). \end{cases} \tag{14}$$

- To (14), we attach the control system $\Sigma = \Sigma_{n,m,p} = (f, g, h)$, given by

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v, \\ y = h(x), \end{cases} \tag{15}$$

  where $n = \dim x$, $m = \dim v$, $p = \dim y$. Clearly, $m = n - q$ and $p = l - q$ (we will use these dimensional relations in the following discussion). In the above way, we attach a control system Σ to a DAE Ξ (actually, a class of control systems, see Proposition 4.2 below).

Definition 4.1. (explicitation with driving variables) Given a DAE $\Xi_{l,n} = (E, F)$ and fix a point $x_p \in X$, assume that $\operatorname{rank} E(x) = const.$ locally around x_p. Then, by a (Q, v)-explicitation, we will call any control system $\Sigma = \Sigma_{n,m,p} = (f, g, h)$ given by (15) with

$$f(x) = E_1^{\dagger} F_1(x), \quad \operatorname{Im} g(x) = \ker E(x), \quad h(x) = F_2(x),$$

where $QE(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$, $QF(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$. The class of all (Q, v)-explicitations will be called shortly the explicitation class. If a particular control system Σ belongs to the explicitation class of Ξ, we will write $\Sigma \in \mathbf{Expl}(\Xi)$.

Notice that a given Ξ has many (Q, v)-explicitations since the construction of $\Sigma \in \mathbf{Expl}(\Xi)$ is not unique: there is a freedom in choosing $Q(x)$, $E_1^{\dagger}(x)$, and $g(x)$. As a consequence of this non-uniqueness of construction, the explicitation Σ of Ξ is a system defined up to a *feedback transformation*, an *output multiplication* and a *generalized output injection* (or, equivalently, a class of systems).

Proposition 4.2. *Assume that a control system $\Sigma_{n,m,p} = (f, g, h)$ is a (Q, v)-explicitation of a DAE $\Xi_{l,n} = (E, F)$ corresponding to a choice of invertible matrix $Q(x)$, right inverse $E_1^{\dagger}(x)$, and matrix $g(x)$. Then a control system $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ is a $(\tilde{Q}, \tilde{v})$- explicitation of Ξ corresponding to a choice of invertible matrix $\tilde{Q}(x)$, right inverse $\tilde{E}_1^{\dagger}(x)$, and matrix $\tilde{g}(x)$ if and only if Σ and $\tilde{\Sigma}$*

are equivalent via a v-feedback transformation of the form $v = \alpha(x) + \beta(x)\tilde{v}$, a generalized output injection $\gamma(x)y = \gamma(x)h(x)$ and an output multiplication $\tilde{y} = \eta(x)y$, which map

$$f \mapsto \tilde{f} = f + \gamma h + g\alpha, \quad g \mapsto \tilde{g} = g\beta, \quad h \mapsto \tilde{h} = \eta h, \tag{16}$$

where α, β and η are smooth matrix-valued functions of appropriate sizes, $\gamma = (\gamma_1, \ldots, \gamma_p)$ is a p-tuple of smooth vector fields on X, and β and η are invertible.

The proof is given in Section 7.3. Since the explicitation of a DAE is a class of control systems, we will propose now an equivalence relation for control systems. An equivalence of two nonlinear control systems is usually defined by state coordinates transformations and feedback transformations (e.g. see [25, 26]), and sometimes output coordinates transformations [29]. In the present paper, we define a more general system equivalence of two control systems as follows.

Definition 4.3. (system equivalence) Consider two control systems $\Sigma_{n,m,p} = (f, g, h)$ and $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ defined on X and $\tilde{X}$, respectively. The systems Σ and $\tilde{\Sigma}$ are called system equivalent, or shortly sys-equivalent, denoted by $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$, if there exist a diffeomorphism $\psi : X \to \tilde{X}$, matrix-valued functions $\alpha : X \to \mathbb{R}^m$, $\gamma : X \to \mathbb{R}^{n\times p}$ and $\beta : X \to GL(m, \mathbb{R})$, and $\eta : X \to GL(p, \mathbb{R})$ such that

$$\tilde{f} \circ \psi = \frac{\partial \psi}{\partial x}\left(f + \gamma h + g\alpha\right), \quad \tilde{g} \circ \psi = \frac{\partial \psi}{\partial x} g\beta, \quad \tilde{h} \circ \psi = \eta h.$$

If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x_p and $\tilde{U}$ of $\tilde{x}_p$, and α, β, γ, η are defined locally on U, we will speak about local sys-equivalence.

Remark 4.4. The above defined sys-equivalence of two nonlinear control systems generalizes the Morse equivalence of two linear control systems (see [16, 30]).

The following proposition shows that solutions of any DAE are in a one-to-one correspondence with solutions of its (Q, v)-explicitation.

Proposition 4.5. *Consider a DAE $\Xi_{l,n} = (E, F)$ and let a control system $\Sigma_{n,m,p} = (f, g, h)$ be a (Q, v)-explicitation of Ξ, i.e., $\Sigma \in \mathbf{Expl}(\Xi)$. Then a $\mathcal{C}^1$-curve $x(\cdot)$ is a solution of Ξ if and only if there exists $v(\cdot) \in \mathcal{C}^0$ such that $(x(\cdot), v(\cdot))$ is a solution of Σ respecting the output constraints $y = 0$, i.e., a solution of (14).*

The proof is given in Section 7.3. The following theorem is a fundamental result of the present paper, which shows that sys-equivalence for explicitation systems (control systems) is a true counterpart of the ex-equivalence for DAEs.

Theorem 4.6. *Consider two DAEs $\Xi_{l,n} = (E, F)$ and $\tilde{\Xi}_{l,n} = (\tilde{E}, \tilde{F})$. Assume that* $\operatorname{rank} E(x)$ *and* $\operatorname{rank} \tilde{E}(\tilde{x})$ *are constant around two points x_p and $\tilde{x}_p$, respectively. Then for any two control systems $\Sigma_{n,m,p} = (f, g, h) \in \mathbf{Expl}(\Xi)$ and $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h}) \in \mathbf{Expl}(\tilde{\Xi})$, we have that locally $\Xi \overset{ex}{\sim} \tilde{\Xi}$ if and only if $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$.*

The proof is given in Section 7.3. In order to show how the explicitation can be useful in the DAEs theory, we discuss below how the analysis of DAEs of Sections 2 and 3 is related to the notion of zero dynamics of nonlinear control theory. For a nonlinear control system $\Sigma_{n,m,p} = (f, g, h)$ and a nominal point x_p, assume $h(x_p) = 0$. Recall its zero dynamics algorithm [25, 26].

Step 1: set $N_1 = h^{-1}(0)$. Step k ($k > 1$): assume for some neighborhood $U_{k-1} \subseteq X$ of x_p, $N^c_{k-1} = N_{k-1} \cap U_{k-1}$ is a smooth embedded and connected submanifold such that $x_p \in N^c_{k-1}$. Set

$$N_k = \left\{x \in N^c_{k-1} : f(x) \in T_x N^c_{k-1} + \mathrm{span}\{g_1(x), \dots, g_m(x)\}\right\}. \tag{17}$$

Remark 4.7. It is shown in [25] that N_k is invariant under feedback transformations. Then assume that a control system $\tilde{\Sigma} = (\tilde{f}, \tilde{g}, \tilde{h})$ is given by applying a *generalized output injection* and an *output multiplication* to Σ, i.e., $\tilde{f} = f + \gamma h$, $\tilde{g} = g$, $\tilde{h} = \eta h$, where $\gamma : X \to \mathbb{R}^{n\times p}$ and $\eta : X \to GL(p, \mathbb{R})$. By $\tilde{N}_0 = \tilde{h}^{-1}(0) = h^{-1}(0)$ (since $\eta(x)$ is invertible) and

$$\begin{aligned}\tilde{N}_k &= \left\{x \in \tilde{N}^c_{k-1} : f(x) + \gamma h(x) \in T_x \tilde{N}^c_{k-1} + \mathrm{span}\{\tilde{g}_1, \dots, \tilde{g}_m\}(x)\right\}\\ &= \left\{x \in \tilde{N}^c_{k-1} : f(x) + 0 \in T_x \tilde{N}^c_{k-1} + \mathrm{span}\{g_1, \dots, g_m\}(x)\right\},\end{aligned}$$

we have $\tilde{N}_k = N_k$ for $k \geq 0$, which means that N_k of the zero dynamics algorithm is invariant under *generalized output injections* and *output multiplications*. The sequence of submanifolds N^c_k of the zero dynamics algorithm is well-defined for the class $\mathbf{Expl}(\Xi)$, i.e., does not depend on the choice of $\Sigma \in \mathbf{Expl}(\Xi)$. Since by Proposition 4.2 any two systems $\Sigma, \Sigma' \in \mathbf{Expl}(\Xi)$ are equivalent via a v-feedback, a generalized output injection, and an output multiplication, then by the argument in item (i) above, we have $\tilde{N}_k = N_k$.

Proposition 4.8. *Consider a DAE $\Xi_{l,n} = (E, F)$ satisfying* $\mathrm{rank}\, E(x) = q = const.$ *around a point x_p and a control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$. Denote $\mathcal{G}(x) = \mathrm{span}\{g_1, \dots, g_m\}(x)$, where g_i, $1 \leq i \leq m$, are the columns of g. The following conditions, for each $k \geq 1$,*

(A1) M^c_k of the geometric reduction method of Section 2 is a smooth embedded connected submanifold and $\dim\, E(x) T_x M^c_{k^*} = const.$ for all $x \in M^c_{k^*}$ around x_p,

(A2) N^c_k of the zero dynamics algorithm above is a smooth embedded connected submanifold and $\dim \mathcal{G}(x) \cap T_x N^c_{k^*} = const.$ for all $x \in N^c_{k^*}$ around x_p (see Proposition 6.1.1 in [25]),

can be concluded from each other (i.e., (A1) implies (A2) and vice versa). Assume that either (A1) or (A2) holds, then the maximal invariant submanifold $M^* = M^c_{k^*}$ of Ξ coincides with the maximal output zeroing submanifold $N^* = N^c_{k^*}$ of Σ. Moreover, Ξ is internally regular (around x_p) if and only if $\mathcal{G}(x_p) \cap T_{x_p} N^* = 0$ (equation (6.4) of [25]).

The proof is given in Section 7.3.

Remark 4.9. By Proposition 4.8, if there exists a unique $u = u(x)$ that renders N^* output zeroing and locally maximal control invariant for a control system $\Sigma \in \mathbf{Expl}(\Xi)$, then the original DAE Ξ is internally regular. Since the zero dynamics do not depend on the choice of explicitation, the internal regularity of Ξ corresponds to the fact that the zero output constraint $y(t) = 0$ of any control system $\Sigma \in \mathbf{Expl}(\Xi)$ can be achieved by a unique control $u(t)$ or, equivalently, the zero dynamics of Σ is a unique vector field on N^*.

The explicitation can be also used to characterize solutions of DAEs which are not necessarily internally regular, that is, the restricted DAE Ξ^*, given by (6, has non-unique maximal solutions (recall that Ξ^* has isomorphic solutions with the original DAE Ξ by Proposition 3.3. We now apply the explicitation method to Ξ^* to have the following results.

Proposition 4.10. *Consider a DAE $\Xi = (E, F)$ and fix a point $x_p \in X$. Assume that the locally maximal invariant submanifold M^* around x_p exists and can be constructed via the algorithm of Appendix. Then the reduction of local M-restriction of Ξ, denoted by $\Xi|^{red}_{M^*}$, coincides with the DAE $\Xi^* : E^*(z^*)\dot{z}^* = F^*(z^*)$ in Proposition 3.3 with $E^*(z^*)$ being of full row rank r^*.*

(i) A curve $z^* : I \to M^*$ is a solution of Ξ^* if and only if it is an integral curve of the affine distribution $\mathcal{A} = f^*(z^*) + \ker E(z^*)$, where $f^* = (E^*)^\dagger F^*$, i.e., $\dot{z}^*(\cdot) \in \mathcal{A}(z^*(\cdot))$.

(ii) Solutions of Ξ^* is in one-to-one correspondence with those of an (Q, v)-explicitation $\Sigma^* \in \mathbf{Expl}(\Xi^*)$ of the form (without outputs)

$$\Sigma^* : z^* = f^*(z^*) + g^*(z^*)v,$$

where $\operatorname{Im} g^* = \ker E$, $g^* = (g_1^*, \ldots, g_{m^*}^*)$ and $v = (v_1, \ldots, v_{m^*})$.

(iii) If $\ker E = \ker E^*$ is involutive, then Ξ^* is ex-equivalent to (or the original DAE Ξ is in-equivalent) to a semi-explicit DAE

$$\dot{z}_1^* = F_1(z_1^*, z_2^*),$$

which can be seen as a control system that is not affined with respect to the control z_2^*.

Proof. We omit the proof since item (i) is clear, and items (ii) and (iii) can be easily deduced by applying, respectively, the results of Proposition 4.5 and that of Theorem 5.3 (see below) to Ξ^*. $\square$

5. Driving variable reducing and semi-explicit DAEs

Now we will show by an example that sometimes we can reduce some of driving variables of a (Q, v)-explicitation.

Example 5.1. Consider a DAE $\Xi = (E, F)$, given by

$$\begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} F_1(x) \\ x_1^2 + x_2^2 - 1 \end{bmatrix},$$

where $F_1 : X \to \mathbb{R}$ is smooth. By rank $E(x) = 1$, the explicitation class $\mathbf{Expl}(\Xi)$ is not empty. A control system $\Sigma \in \mathbf{Expl}(\Xi)$ is:

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix} F_1(x) + \begin{bmatrix} 0 & \cos x_3 \\ 0 & -\sin x_3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \\ \quad y \quad = x_1^2 + x_2^2 - 1, \end{cases}$$

where $\begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \end{bmatrix}^T$ is a right inverse of $E_1(x) = \begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \end{bmatrix}$. Now consider the last equation in the dynamics of Σ, which is $\dot{x}_3 = v_1$. Observe that v_1 acts on $\dot{x}_3$ only, which implies that v_1 is decoupled from the other part of the dynamics. Thus, we may get rid of v_1 and regard x_3 as a new control. Thus the dynamics of Σ become:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sin x_3 F_1(x) \\ -\cos x_3 F_1(x) \end{bmatrix} + \begin{bmatrix} \cos x_3 \\ -\sin x_3 \end{bmatrix} v_2,$$

where x_1 and x_2 are new states, x_3 and v_2 are the new control inputs. By rectifying the vector field $g_2 = \cos x_3 \frac{\partial}{\partial x_1} - \sin x_3 \frac{\partial}{\partial x_2}$, we can reduce v_2 in a similar way. We are, however, not able to reduce v_1 and v_2 simultaneously.

Before giving the main result of this subsection, we formally define what we mean by "reducing" variables of a control system Σ.

Definition 5.2 (driving variable reduction)**.** For a control system $\Sigma_{n,m,p} = (f, g, h)$, let $\mathcal{G}^{red}$ be an involutive sub-distribution of constant rank k of the distribution $\mathcal{G} = \operatorname{span}\{g_1, \ldots, g_m\}$. There exists a feedback transformation and a coordinates change such that, locally, $\mathcal{G}^{red} = \operatorname{span}\left\{\frac{\partial}{\partial x_2^1}, \ldots, \frac{\partial}{\partial x_2^k}\right\}$ and Σ takes the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) + \sum\limits_{i=1}^{m-k} g_1^i(x_1, x_2) v_1^i, \\ \dot{x}_2 = v_2, \\ \ y \ = h(x_1, x_2), \end{cases}$$

where $v_2 = (v_2^1, \ldots, v_2^k)$, we will say that Σ can be $\mathcal{G}^{red}$-reduced to the following control system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) + \sum\limits_{i=1}^{m-k} g_1^i(x_1, x_2) v_1^i, \\ \ y \ = h(x_1, x_2), \end{cases}$$

where x_2 is a new control and the reduced state x_1 is of dimension $n - k$. We say that Σ can be fully reduced if $\mathcal{G}^{red} = \mathcal{G}$.

Now we connect reducing of control systems with semi-explicit DAEs.

Theorem 5.3. *For a DAE $\Xi_{l,n} = (E, F)$, the following statements are equivalent around a point $x_p \in X$:*

(i) $\operatorname{rank} E(x) = const.$ *and the distribution $\mathcal{G} = \ker E(x)$ is involutive.*

(ii) Ξ is locally ex-equivalent to a semi-explicit DAE $\Xi^{SE}: \begin{cases} \dot{x}_1 = F_1(x_1, x_2) \\ 0 = F_2(x_1, x_2) \end{cases}$.

(iii) Any control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$ can be fully reduced.

The proof is given in Section 7.4.

Remark 5.4. (i) Observe that if Ξ is ex-equivalent to Ξ^{SE}, then by rewriting $x_2 = w$ and choosing the output $y = F_2(x_1, w)$, we get the following control system Σ^u with an input w,

$$\Sigma^w : \begin{cases} \dot{x}_1 &= F_1(x_1, w), \\ y &= F_2(x_1, w). \end{cases}$$

The above system Σ^u has the same number of variables as Ξ. Thus Σ^u is an *explicitation without driving variables* of Ξ. So there are two kinds of explicitation for nonlinear DAEs, namely, explicitation with, or without, driving variables.

(ii) A linear DAE $\Delta = (E, H)$, given by (3), has always two kinds of explicitations, since the rank of E is always constant and the distribution $\mathcal{G} = \ker E$ is always involutive. The relations and differences of the two explicitations for linear DAEs are discussed in Chapter 3 of [31] (note that the explicitation without driving variables for linear DAEs is called the (Q, P)-explicitation there).

6. A nonlinear generalization of the Weierstrass form

In this subsection, we will use the explicitation (with driving variables) procedure to transform an internally regular DAE $\Xi_{l,n} = (E, F)$ with $l = n$, into a normal form under the external equivalence. A linear regular DAE is always ex-equivalent (via linear transformations) to the Weierstrass form **WF** [19], given by

$$\mathbf{WF}: \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{z}^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} z \\ z^* \end{bmatrix}, \tag{18}$$

where $N = \operatorname{diag}(N_1, \ldots, N_m)$, and where N_i, $i = 1, \ldots, m$ are nilpotent matrices of index ρ_i, i.e., $N_i^j \neq 0$ for all $j = 1, \ldots, \rho_i - 1$ and $N_i^{\rho_i} = 0$. The following theorem generalizes that result and shows that any internally regular nonlinear DAE (under the assumption that some ranks are constant) is always ex-equivalent to a nonlinear Weierstrass form **NWF** (see (19) below). Note that $\bar{\phi}_k$ in the algorithm of Appendix, defined on $W_k \subseteq M_k^c$, can be considered as maps on $U_0 \subseteq X$ by taking $\bar{\Phi}_k = \bar{\varphi}_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1(x)$. Then for $k \geq 1$, set $H_k = \begin{bmatrix} \bar{\Phi}_1 & \ldots & \bar{\Phi}_k \end{bmatrix}^T$ and H_0 is empty. **Assumption 1** of the algorithm of Appendix says that rank $\tilde{F}_k^2(z_{k-1}) = const.$ for $z_{k-1} \in M_k \cap U_k$. In (A1) below, we replace it by a stronger rank assumption on a neighborhood $U \subseteq X$ of x_p.

Theorem 6.1. *Consider a DAE $\Xi_{l,n}=(E,F)$, assume that* $\operatorname{rank} E(x) = const. = q$ *around a point x_p. Also assume in the geometric reduction algorithm of Appendix that*

(A1) rank $\begin{bmatrix} \mathrm{D}H_{k-1} \\ \mathrm{D}\tilde{F}_k^2 \end{bmatrix}(x) = const.$ *for $1 \le k \le k^*$ (H_0 is absent) and for all x around x_p;*

(A2) $\dim E(x)T_xM_k^c = const.$ *for $x \in M_k^c$ around x_p, $1 \le k \le k^*$;*

(A3) *$l = n$ and* $\dim M^* = \dim E(x)T_xM^*$, *i.e., $r^* = n^*$, for all $x \in M^*$ around x_p.*

Then Ξ is internally regular and there exists a neighborhood U of x_p such that Ξ is locally on U ex-equivalent to the DAE (19), represented by the nonlinear Weierstrass form

$$\mathbf{NWF}: \left[\begin{array}{cccc|c} N_{\rho_1} & 0 & \cdots & 0 & \\ 0 & N_{\rho_2} & \ddots & \vdots & \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & N_{\rho_m} & \\ \hline \multicolumn{4}{c|}{G(z,z^*)} & I \end{array}\right] \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_m \\ \dot{z}^* \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \\ f^*(z,z^*) \end{bmatrix} + \begin{bmatrix} a_1 + b_1\dot{z}^\rho \\ a_2 + b_2\dot{z}^\rho \\ \vdots \\ a_m + b_m\dot{z}^\rho \\ 0 \end{bmatrix}, \tag{19}$$

where $z_i = (z_i^1, \ldots, z_i^{\rho_i})$ and z^* are new coordinates, and $\dot{z}^\rho = (\dot{z}_1^{\rho_1}, \dot{z}_2^{\rho_2}, \ldots, \dot{z}_m^{\rho_m})$. The indices ρ_i, $1 \le i \le m$, with $m = n - q$, satisfy $\rho_1 \le \rho_2 \le \ldots \le \rho_m$.

More specifically, for $1 \le i \le m$, the $\rho_i \times \rho_i$ nilpotent matrices N_{ρ_i} and the ρ_i-dimensional vector-valued functions $a_i + b_i\dot{z}^\rho$ are of the following form

$$N_{\rho_i} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad a_i + b_i\dot{z}^\rho = \begin{bmatrix} 0 \\ a_i^1 + \sum\limits_{l=1}^{m} b_{i,l}^1 \dot{z}_l^{\rho_l} \\ \vdots \\ a_i^{\rho_i - 1} + \sum\limits_{l=1}^{m} b_{i,l}^{\rho_i - 1} \dot{z}_l^{\rho_l} \end{bmatrix},$$

where the functions $a_i^k, b_{i,l}^k$ satisfy $a_i^k|_{M_k^c} = b_{i,l}^k|_{M_k^c} = 0$.

Remark 6.2. (i) Assumption (A2) of Theorem 6.1 is equivalent to **Assumption 1** of the geometric reduction algorithm of Appendix. By Theorem 3.11, we know that (A3) of Theorem 6.1 implies that Ξ is internally regular around x_p.

(ii) A component-wise expression of the above **NWF** is

$$\mathbf{NWF}: \begin{cases} 0 = z_i^1, \quad 1 \le i \le m, \quad 1 \le j \le \rho_i - 1, \\ \dot{z}_i^j = z_i^{j+1} + a_i^j + \sum\limits_{l=1}^{m} b_{i,l}^j \dot{z}_l^{\rho_l}, \\ \dot{z}^* = f^* - G\dot{z}, \end{cases}$$

where $a_i^k, b_{i,l}^k, f^*$ and G depend on (z, z^*).

(iii) The submanifolds M_k^c, $k \ge 1$, of the algorithm are given by

$$M_k^c = \left\{(z,z^*) : z_i^j = 0, \ 1 \le i \le m, \ 1 \le j \le k\right\},$$

the maximal invariant submanifold M^* is given by

$$M^* = \{(z, z^*) : z_i^j = 0,\ 1 \le i \le m, 1 \le j \le \rho_i\}.$$

Therefore, an equivalent condition for $a_i^k|_{M_k^c} = b_{i,l}^k|_{M_k^c} = 0$ is that $a_i^k, b_{i,l}^k \in \mathbf{I}^k$, where $\mathbf{I}^k$ is the ideal generated by z_i^j, $1 \le i \le m$, $1 \le j \le k$ in the ring of smooth functions of z_s^t and z_r.

(iv) We see that all maximal solution $(z(\cdot), z^*(\cdot))$ is unique and of the form $(0, z^*(\cdot))$, where $z^*(\cdot)$ is a solution of the ODE $\dot{z}^* = f^*(0, z^*)$ on M^*, which agrees with the result of Theorem 3.11(iii).

The proof of Theorem 6.1 is given in Section 7.5.

Example 6.3 (continuation of Example 3.13)**.** Consider the DAE $\Xi_{6,6} = (E, F)$ of (10) around the point $x_p = (0, 1, 0, 0, 0, 1)$. A control system $\Sigma_{6,2,2} \in \mathbf{Expl}(\Xi)$ is

$$\Sigma : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_6(x_2 - x_6) - x_1 \\ \frac{x_1}{x_6} - 1 \\ 0 \\ x_5 - x_2 + x_6 \\ x_4 - x_6(x_2 - x_6) \\ -x_6 \end{bmatrix} + \begin{bmatrix} x_6(x_3 + x_5) & \frac{x_1 x_5}{x_6} \\ \ln x_6 & 0 \\ 0 & \frac{x_1}{x_6} - 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & x_5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1}{x_6} \\ x_3 + x_5 \end{bmatrix}.$$

It can be observed from Example 3.13 that the assumptions (A1)-(A3) of Theorem 6.1 are satisfied. Now via the following local changes of coordinates defined on $U = X = \{x \in X : x_6 > 0, x_2 \neq x_6\}$:

$$z_1^1 = \frac{x_1}{x_6}, \quad z_1^2 = x_2 - x_6, \quad z_2^1 = x_3 + x_5, \quad z_2^2 = x_4 - x_2 x_6 + x_6^2, \quad z_2^3 = x_5, \quad z^* = x_6,$$

we can bring Σ into a system of Σ' below, which is of the zero dynamics form (34),

$$\Sigma' : \begin{cases} y_1 = z_1^1 \\ \dot{z}_1^1 = z_1^2 + z_2^1 v_1 \\ \dot{z}_1^2 = z_1^1 + \ln z^* \cdot v_1 - z_2^3 v_2 \\ y_2 = z_2^1 \\ \dot{z}_2^1 = z_2^2 + z_1^1 v_2 \\ \dot{z}_2^2 = z_2^3 + z^*(z_1^1 + \ln z^* \cdot v_1 - z_2^3 v_2) - z_1^2 z_2^3 v_2 \\ \dot{z}_2^3 = z_2^2 + v_2 \\ \dot{z}^* = -z^* + z_2^3 v_2, \end{cases} \Rightarrow \Sigma'' : \begin{cases} y_1 = z_1^1 \\ \dot{z}_1^1 = z_1^2 - \frac{z_1^1 z_2^1}{\ln z^*} + \frac{z_2^1}{\ln z^*} \tilde{v}_1 + \frac{z_2^1 z_2^3}{\ln z^*} \tilde{v}_2 \\ \dot{z}_1^2 = \tilde{v}_1 \\ y_2 = z_2^1 \\ \dot{z}_2^1 = z_2^2 - z_1^1 z_2^2 + z_1^1 \tilde{v}_2 \\ \dot{z}_2^2 = z_2^3 + z^* \tilde{v}_1 + z_1^2 z_2^2 z_2^3 - z_1^2 z_2^3 \tilde{v}_2 \\ \dot{z}_2^3 = \tilde{v}_2 \\ \dot{z}^* = -z^* - z_2^2 z_2^3 + z_2^3 \tilde{v}_2, \end{cases}$$

and then by the feedback transformation

$$\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1}{x_6} \\ x_4 - x_2 x_6 + x_6^2 \end{bmatrix} + \begin{bmatrix} \ln x_6 & -x_5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

the system Σ' is transformed into the system Σ'' above. In order to eliminate $z^* \tilde{v}_1$ in $\dot{z}_2^2 = z_2^3 + z^* \tilde{v}_1 + z_1^2 z_2^2 z_2^3 - z_1^2 z_2^3 \tilde{v}_2$ of Σ'', we define the changes of coordinates

$$\tilde{z}_1^1 = z_1^1, \quad \tilde{z}_1^2 = z_1^2, \quad \tilde{z}_2^1 = z_2^1 - z^* z_1^1, \quad \tilde{z}_2^2 = z_2^2 - z^* z_1^2, \quad \tilde{z}_2^3 = z_2^3.$$

and the output multiplication

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z^* & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then the system Σ'' becomes

$$\tilde{\Sigma}:\begin{cases}\tilde{y}_1=\tilde{z}_1^1\\ \dot{\tilde{z}}_1^1=\tilde{z}_1^2-\frac{\tilde{z}_1^1(\tilde{z}_2^1+\tilde{z}_1^1z^*)}{\ln z^*}+\frac{(\tilde{z}_2^1+\tilde{z}_1^1z^*)}{\ln z^*}\tilde{v}_1+\frac{(\tilde{z}_2^1+\tilde{z}_1^1z^*)\tilde{z}_2^3}{\ln z^*}\tilde{v}_2\\ \dot{\tilde{z}}_1^2=\tilde{v}_1\\ \tilde{y}_2=\tilde{z}_2^1\\ \dot{\tilde{z}}_2^1=\tilde{z}_2^2+\tilde{z}_1^1(\tilde{z}_2^2+\tilde{z}_1^2z^*)(\tilde{z}_2^3-1)+\frac{\tilde{z}_1^1\tilde{z}_2^1z^*}{\ln z^*}-\frac{(\tilde{z}_2^1+\tilde{z}_1^1z^*)z^*}{\ln z^*}\tilde{v}_1-(\tilde{z}_1^1\tilde{z}_2^3+\frac{(\tilde{z}_2^1+\tilde{z}_1^1z^*)\tilde{z}_2^3z^*}{\ln z^*})\tilde{v}_2\\ \dot{\tilde{z}}_2^2=\tilde{z}_2^3+\tilde{z}_1^2z^*\\ \dot{\tilde{z}}_2^3=\tilde{v}_2\\ \dot{z}^*=-z^*-\tilde{z}_2^3(\tilde{z}_2^2+\tilde{z}_1^2z^*)+\tilde{z}_2^3\tilde{v}_2,\end{cases}$$

Now we drop all the tildes in system $\tilde{\Sigma}$ for ease of notation. By setting $y_1=y_2=0$, replacing $v_1=\dot{z}_1^2$, $v_2=z_2^3$, and deleting the equations $\dot{z}_1^2=v_1$, $z_2^3=v_2$, we get the following DAE $\tilde{\Xi}$ from $\tilde{\Sigma}$,

$$\tilde{\Xi}:\left[\begin{array}{cc|ccc|c}0&0&0&0&0&0\\1&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\ \hline 0&0&0&0&z_2^3&1\end{array}\right]\begin{bmatrix}\dot{z}_1^1\\ \dot{z}_1^2\\ \dot{z}_2^1\\ \dot{z}_2^2\\ \dot{z}_2^3\\ \dot{z}^*\end{bmatrix}=\begin{bmatrix}z_1^1\\ z_1^2\\ z_2^1\\ z_2^2\\ z_2^3\\ -z^*-z_2^3(z_2^2+z_1^2z^*)\end{bmatrix}+\begin{bmatrix}0\\ a_1^1+b_{11}^1\dot{z}_1^2+b_{12}^1\dot{z}_2^3\\ 0\\ a_2^1+b_{21}^1\dot{z}_1^2+b_{22}^1\dot{z}_2^3\\ a_2^2\\ 0\end{bmatrix}\tag{20}$$

where $a_1^1=-\frac{z_1^1(z_2^1+z_1^1z^*)}{\ln z^*}$, $b_{11}^1=\frac{(z_2^1+z_1^1z^*)}{\ln z^*}$, $b_{12}^1=\frac{(z_2^1+z_1^1z^*)z_2^3}{\ln z^*}$, $a_2^1=z_1^1(z_2^2+z_1^2z^*)(z_2^3-1)+\frac{z_1^1z_2^1z^*}{\ln z^*}$, $b_{21}^1=-\frac{(z_2^1+z_1^1z^*)z^*}{\ln z^*}$,$b_{22}^1=z_1^1z_2^3+\frac{(z_2^1+z_1^1z^*)z_2^3z^*}{\ln z^*}$, $a_2^2=z_1^2z^*$. It is clear that $\tilde{\Sigma}\in\mathbf{Expl}(\tilde{\Xi})$, thus we have $\Xi\overset{ex}{\sim}\tilde{\Xi}$ since $\Sigma\in\mathbf{Expl}(\Xi)$ and $\Sigma\overset{sys}{\sim}\tilde{\Sigma}$ (see Theorem 4.6). The above DAE $\tilde{\Xi}$ is in the **NWF** of (19) and the sequence of submanifolds M_k^c of the geometric reduction algorithm can be expressed as $M_1^c=\{(z,z^*):z_1^1=z_2^1=0\}$, $M_2^c=\{(z,z^*)\in M_1^c:z_2^1=z_2^2=0\}$ and

$$M^*=M_3^c=\left\{(z,z^*)\in M_2^c:z_2^3=0\right\}.$$

The functions $a_1^1,b_{11}^1,b_{12}^1,a_2^1,b_{21}^1,b_{22}^1\in\mathbf{I}^1$ vanish on M_1^c, and the function $a_2^2\in\mathbf{I}^2$ vanishes on M_2^c.

7. Proofs of the results

7.1. Proofs of Proposition 2.3 and Proposition 2.7

Proof of Proposition 2.3. Suppose that M is a locally invariant submanifold around x_p. By Definition 2.2, there exists a neighborhood U of x_p such that for any point $x_0\in M\cap U$, there exists a solution $x:I\to M\cap U$ satisfying $x(t_0)=x_0$ for a certain $t_0\in I$. Then we have $F(x(t))=E(x(t))\dot{x}(t)\in E(x(t))T_{x(t)}M$, $\forall t\in I$. It follows that $F(x_0)\in E(x_0)T_{x_0}M$ by taking $t=t_0$. Hence $F(x)\in E(x)T_xM$ for all $x\in M\cap U$.

Conversely, suppose that $\dim E(x)T_xM=const.=\bar{r}$ and $F(x)\in E(x)T_xM$ locally for all $x\in M\cap U$. Notice that M is a smooth connected embedded submanifold, thus there exists a neighborhood U_1 of x_p and local coordinates $\psi(x)=z=(z_1,z_2)$ on U_1 such that $M\cap U_1=\{z_2=0\}$, where z_1 are any complementary coordinates, where $\dim z_1 = \bar{n}$, $\dim z_2 = n - \bar{n}$ and $\bar{n} = \dim M$. In the local z-coordinates, the DAE Ξ has the following form

$$E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}\left(\frac{\partial\psi(x)}{\partial x}\right)\dot{x} = F(x) \Rightarrow \begin{bmatrix}\tilde{E}_1(z) & \tilde{E}_2(z)\end{bmatrix}\begin{bmatrix}\dot{z}_1 \\ \dot{z}_2\end{bmatrix} = \tilde{F}(z),$$

where $\tilde{E}_1 : U_1 \to \mathbb{R}^{l\times\bar{n}}$, $\tilde{E}_2 : U_1 \to \mathbb{R}^{l\times(n-\bar{n})}$, $\begin{bmatrix}\tilde{E}_1\circ\psi & \tilde{E}_2\circ\psi\end{bmatrix} = E\left(\frac{\partial\psi}{\partial x}\right)^{-1}$ and $\tilde{F}\circ\psi = F$. By setting $z_2 = 0$, we consider the following DAE defined locally on M (denoted by $\Xi|_M$ and called the local M-restriction of Ξ, see Definition 3.4):

$$\Xi|_M : \tilde{E}_1(z_1, 0)\dot{z}_1 = \tilde{F}(z_1, 0).$$

Then by dim $E(x)T_xM = const. = \bar{r}$ for all $x \in M$ around x_p, there exists a neighborhood $U_2 \subseteq U_1$ of x_p such that $\operatorname{rank}\tilde{E}_1(z_1, 0) = \bar{r}$, $\forall z_1 \in M \cap U_2$. So there exists $Q : M \cap U_2 \to GL\,(l, \mathbb{R})$ such that $\bar{E}_1(z_1)$ of $Q(z_1)\tilde{E}_1(z_1, 0) = \left[\begin{smallmatrix}\bar{E}_1(z_1) \\ 0\end{smallmatrix}\right]$ is of full row rank $\bar{r}$. Rewrite $\bar{E}_1(z_1)\dot{z}_1 = \left[\begin{smallmatrix}\bar{E}_1^1(z_1) & \bar{E}_1^2(z_1)\end{smallmatrix}\right]\left[\begin{smallmatrix}\dot{z}_1^1 \\ \dot{z}_1^2\end{smallmatrix}\right]$, where $z_1 = (z_1^1, z_1^2)$, $\bar{E}_1^1 : M \cap U_2 \to \mathbb{R}^{\bar{r}\times\bar{r}}$ and $\bar{E}_1^2 : M \cap U_2 \to \mathbb{R}^{\bar{r}\times(\bar{n}-\bar{r})}$ and denote $Q(z_1)\tilde{F}(z_1, 0) = \left[\begin{smallmatrix}\bar{F}_1(z_1) \\ \bar{F}_2(z_1)\end{smallmatrix}\right]$. Without loss of generality, we assume that $\bar{E}_1^1(z_1)$ is invertible (if not, we permute the components of z_1 such that the first $\bar{r}$ columns of $\bar{E}_1(z_1)$ are independent). Now by the assumption that $F(x) \in E(x)T_xM$ for all $x \in M$ around x_p, there exists a neighborhood $U_3 \subseteq U_2$ such that $\tilde{F}(z) \in \tilde{E}(z)T_z\psi(M)$ for all $z \in M \cap U_3$, i.e.,

$$\begin{bmatrix}\bar{F}_1(z_1) \\ \bar{F}_2(z_1)\end{bmatrix} \in \operatorname{Im}\begin{bmatrix}\bar{E}_1^1(z_1) & \bar{E}_1^2(z_1) \\ 0 & 0\end{bmatrix}.$$

It follows that $\bar{F}_2(z_1) \equiv 0$ for all $z_1 \in M \cap U_3$. Then consider the following DAE (which is actually a reduction of $\Xi|_M$, denoted by Ξ_M^{red}, see Definition 3.5)

$$\Xi|_M^{red} : \begin{bmatrix}\bar{E}_1^1(z_1) & \bar{E}_1^2(z_1)\end{bmatrix}\begin{bmatrix}\dot{z}_1^1 \\ \dot{z}_1^2\end{bmatrix} = \bar{F}_1(z_1). \tag{21}$$

Note that a $\mathcal{C}^1$-curve $z_1 : I \to M \cap U_3$ is a solution of (21) passing through $z_{10} = (z_{10}^1, z_{10}^2)$ if and only if $x(\cdot) = \psi^{-1}(z_1(\cdot), 0)$ is a solution of Ξ passing through $x_0 = \psi^{-1}(z_{10}, 0)$. Observe that for any initial point $z_{10} \in M \cap U_3$, there always exists a solution $z_1(\cdot)$ of (21) such that $z_1(t_0) = z_{10}$ for a certain $t_0 \in I$ and $z_1(t) \in M \cap U_3$, $\forall t \in I$. Indeed, rewrite the DAE (21) as the following ODE (recall that $\bar{E}_1^1(z_1)$ is invertible):

$$\dot{z}_1^1 = \left(\bar{E}_1^1(z_1)\right)^{-1}\left(\bar{F}_1(z_1) - \bar{E}_1^2(z_1)\dot{z}_1^2\right). \tag{22}$$

It is always possible to parameterize a solution $z_1(\cdot) = (z_1^1(\cdot), z_1^2(\cdot))$ of (22) as follows. Denote $\dot{z}_1^2 = v$, $f(z_1) = (E_1^1)^{-1}\tilde{F}_1(z_1)$ and $g(z_1) = (E_1^1)^{-1}E_1^2(z_1)$, then (22) can be expressed as

$$\begin{cases}\dot{z}_1^1 = f(z_1) + g(z_1)\,v, \\ \dot{z}_1^2 = v,\end{cases} \tag{23}$$

(called a (Q, v)-explicitation of (21), see Definition 4.1), and for any solution $(z_1(\cdot), v(\cdot))$ of (23), with $v \in \mathcal{C}^0$, the curve $z_1(\cdot)$ is a $\mathcal{C}^1$-solution of (21) satisfying $z_1(t_0) = z_{10}$ (see Proposition 4.5). It follows that for any point$x_0 = \psi^{-1}(z_{10}, 0) \in M \cap U_3$, there always exists a solution $x(\cdot) = \psi^{-1}(z_1(\cdot), 0)$ of Ξ such that $x(t_0) = x_0$ for a certain $t_0 \in I$ and that $x(t) \in M \cap U_3$ for all $t \in I$, so M is a locally invariant submanifold of Ξ around x_p by definition. $\square$

Proof of Propostion 2.7. Let k be the smallest integer such that $M_0^c \supsetneq M_1^c \supsetneq \cdots \supsetneq M_k^c$, where M_i^c, $0 \leq i \leq k$ are connected embedded submanifolds, and either $x_p \notin M_{k+1}$ or $x_p \in M_{k+1}$ and $M_{k+1}^c = M_{k+1} \cap U_{k+1}$ is a submanifold (by the recursive procedure assumptions) such that $\dim M_k^c = \dim M_{k+1}^c$. Then $k^* = k$ is the integer whose existence is indicated. The condition $k^* \leq n$ follows from $\dim M_{i-1}^c > \dim M_i^c$, $1 \leq i \leq k^*$.

Claim. If an consistent point $x_c \in S_c \cap U_{k^*}$, then $x_c \in M_{k^*+1}$. Now we prove the latter *Claim* holds. Since x_c is consistent, there exists a solution $(x(t), u(t))$, defined on I, and $t_0 \in I$ such that $x(t_0) = x_c$. It follows that for all $t \in I$,

$$E(x(t))\dot{x}(t) = F(x(t)). \tag{24}$$

So $F(x(t)) \in \operatorname{Im} E(x(t))$, $\forall t \in I$. Thus by equation (4), we have $x(t) \in M_1$, $\forall t \in I$. Suppose that for a certain $i > 1$, we have $x(t) \in M_{i-1}$, $\forall t \in I$. We then have that $\dot{x}(t) \in T_{x(t)}M_{i-1}$, $\forall t \in I$ (note that when restricted to U_{i-1}, the set M_{i-1} is a submanifold). Thus in $U_{k^*} \subseteq U_i$, equation (24) implies $F(x(t)) \in E(x(t))T_{x(t)}M_{i-1}^c$. It follows that $x(t) \in M_i \cap U_{i-1}$, for any $t \in I$, due to (4). By an induction argument, we conclude that $x(t) \in M_{k^*+1} \cap U_{k^*}$, and, in particular, we have $x_c = x(t_0) \in M_{k^*+1} \cap U_{k^*}$.

(i) If $x_p \in M_{k^*+1}$, we have $\dim M_{k^*+1}^c = \dim M_{k^*}^c$ and since $M_{k^*+1}^c \subseteq M_{k^*}^c$, it follows that there exists an open neighborhood U_{k^*+1} such that $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$. By assumption, $M^* = M_{k^*+1}^c \cap U^*$ satisfies $\dim E(x)T_xM^* = const.$ in $U^* \subseteq U_{k^*+1}$. So, using Proposition 2.3, we conclude that M^* is a locally invariant submanifold on U^*. To prove that M^* is maximal in U^*, let M' be any invariant submanifold, then any point $x_0 \in M' \cap U^*$ is consistent, so $x_0 \in S_c \cap U^*$, then by the above *Claim*, $x_0 \in M_{k^*+1} \cap U^* = M^* \cap U^*$ showing that M^* is maximal in U^*.

(ii) We now prove that M^* coincides with the consistent set S_c on U^*. Since $M^* \cap U^*$ is locally invariant, for any point $x_0 \in M^* \cap U^*$, there exist at least one solution $(x(\cdot), u(\cdot))$ on I and $t_0 \in I$ such that $x(t_0) = x_0$, which implies that x_0 is consistent i.e., $x_0 \in S_c$. It follows that $M^* \cap U^* \subseteq S_c \cap U^*$. Conversely, consider any point $x_0 \in S_c \cap U^*$, using again the above *Claim*, we conclude that $x_0 \in M_{k^*+1} \cap U^* = M^* \cap U^*$, which implies $S_c \cap U^* \subseteq M^* \cap U^*$. Therefore, $M^* \cap U^* = S_c \cap U^*$. $\square$

7.2. Proofs of Proposition 3.3 and Theorem 3.11

Proof of Proposition 3.3. At every Step k of the algorithm in Appendix, consider the DAE $\tilde{\Xi}_k = \Xi_{k-1} = (E_{k-1}, F_{k-1})$ and $\hat{\Xi}_k = (\hat{E}_k, \hat{F}_k)$, the latter given by (38). Then we show that the following items are equivalent. (a). $z_{k-1}(\cdot) = \psi_k^{-1}(z_k(\cdot), \bar{z}_k(\cdot))$ is a solution of Ξ_{k-1}; (b). $(z_k(\cdot), \bar{z}_k(\cdot))$ is a solution of $\hat{\Xi}_k$; (c). $\bar{z}_k(\cdot) = 0$ and $z_k(\cdot)$ is a solution of $\Xi_k : E_k(z_k)\dot{z}_k = F_k(z_k)$, where $E_k(z_k) = \hat{E}_k^1(z_k, 0)$, $F_k = \hat{F}_k^1(z_k, 0)$ and where $\hat{E}_k^1$, $\hat{F}_k^1$ are defined in (38). Since $\tilde{\Xi}_k = \Xi_{k-1}$ is locally ex-equivalent to $\hat{\Xi}_k$ via Q_k and ψ_k, we have that item (a) and item (b) above are equivalent (see Remark 3.2). The equivalence of item (b) and item (c) follows from the fact that the solutions exists on M_k^c only and should respect the constraints $\bar{z}_k = 0$.

Then by the equivalence of (c) and (a), we have, at the first step of the algorithm, that $(z_1(\cdot), 0)$ is a solution of $E_1(z_1)\dot{z}_1 = F_1(z_1)$, together with $\bar{z}_1 = 0$, if and only if $z_0(\cdot) = \psi_1^{-1}(z_1(\cdot), 0)$ is a solution of $\Xi_0 = \Xi = (E, F)$. In general, by an induction argument, we can prove that $(z_k(\cdot), 0, \ldots, 0)$ is a solution of $E_k(z_k)\dot{z}_k = F_k(z_k)$, together with $\bar{z}_1 = 0, \ldots, \bar{z}_k = 0$, if and only if $x(\cdot)$ is a solution of Ξ, where $x(\cdot)$ is given by the following iterative formula

$$x(\cdot) = z_0(\cdot) = \psi_1^{-1}(z_1(\cdot), 0),\ z_1(\cdot) = \psi_2^{-1}(z_2(\cdot), 0),\ \ldots,\ z_{k-1}(\cdot) = \psi_k^{-1}(z_k(\cdot), 0). \tag{25}$$

Each diffeomorphism ψ_k is defined on W_k, we extend it to U_k by putting $\Psi_k = (\psi_k, \bar{z}_k, \ldots, \bar{z}_1)$. Now we define the local diffeomorphism $\Psi := \Psi_{k^*} \circ \cdots \circ \Psi_2 \circ \Psi_1 : U_{k^*+1} \to \mathbb{R}^n$ (note that $\Psi_{k^*+1} = \Psi_{k^*}$). To show that the local diffeomorphism $\hat{z} = \Psi(x)$, where $\hat{z} = (z^*, \bar{z})$ transforms solutions of Ξ^u into those of $\hat{\Xi}^{\hat{u}}$, it is enough to observe that Ψ satisfies (25), for $k = k^*+1$. Now we prove that $E^*(z^*)$, for $z^* \in M^*$, is of full row rank. Consider Step k^*+1 of the algorithm, note that the Q_{k^*+1}-transformation ensures that $\tilde{E}_{k^*+1}^1(z_{k^*})$ is of full row rank. By $M_{k^*+1}^c = \left\{ z_{k^*} \in M_{k^*}^c \cap U_{k^*+1} \mid \tilde{F}_{k^*+1}^2(z_{k^*}) = 0 \right\}$ and the fact that $\dim M_{k^*}^c = n_{k^*} = n_{k^*+1} = \dim M_{k^*+1}^c$, we have $\tilde{F}_{k^*+1}^2(z_{k^*}) = 0$, $\forall z_{k^*} \in M_{k^*}^c \cap U_{k^*+1}$. As a consequence, the $\bar{z}_{k^*+1}$-coordinates are not present, so there is no equation $\bar{z}_{k^*+1} = 0$ in (7). Moreover, we have $M_{k^*+1}^c = M_{k^*}^c$ in U_{k^*+1}, implying that $z_{k^*+1} = z_{k^*}$. Finally, it is seen from $E^*(z^*) = E_{k^*+1}(z_{k^*+1}) = \hat{E}_{k^*+1}^1(z_{k^*}) = \tilde{E}_{k^*+1}^1(z_{k^*})$ that $E^*(z^*)$ is of full row rank for all $z^* = z_{k^*+1} \in M^* = M_{k^*+1}^c$. □

Proof of Theorem 3.11. Since M^* is locally invariant around x_p, via a similar construction to that shown in the proof of Proposition 2.3, we can get a DAE $\Xi|_{M^*}^{red}$ of the form (21) (if the maximal invariant submanifold M^* is constructed via the algorithm in Appendix, then $\Xi|_{M^*}^{red}$ coincides with the DAE Ξ^* of (6) from the results of that algorithm). Note that $\Xi|_{M^*}^{red}$ can be seen as an ODE possibly with free variables (see (22) and (23)), and that $\Xi|_{M^*}^{red}$ has isomorphic solutions with Ξ (see Proposition 3.3). Thus Ξ is internally regular around x_p, i.e., there exists only one maximal solution passing through any $x_0 \in M^*$ around x_p if and only if no free variables are present in $\Xi^* = \Xi|_{M^*}^{red}$, i.e., $[\bar{E}_1^1, \bar{E}_1^2]$ of (21) is invertible or, equivalently, $n^* = \dim M^* = \dim E(x)T_xM^* = r^*$ for all $x \in M^*$ around x_p (i.e., E^* of (6) is invertible). Moreover, it is clear that $[\bar{E}_1^1, \bar{E}_1^2]$ is invertible if and only if $\Xi|_{M^*}^{red}$ of (21) (or Ξ^*, given by (6)) is ex-equivalent to an ODE (9) without free variables, where $f^* = [\bar{E}_1^1, \bar{E}_1^2]^{-1}\bar{F}_1$ (or $f^* = (E^*)^{-1}F^*$), that is, Ξ is internally equivalent to (9) around x_p. □

7.3. Proofs of Proposition 4.2, Proposition 4.5, Theorem 4.6 and Proposition 4.8

Proof of Proposition 4.2. If. Throughout the proof below, we may drop the argument x for the functions $f(x), g(x), h(x), \ldots$, for ease of notation. Suppose that Σ and $\tilde{\Sigma}$ are equivalent via transformations given by (16). First, $\operatorname{Im} \tilde{g} = \operatorname{Im} g\beta = \ker E_1 = \ker E$ implies that $\tilde{g}$ is another choice such that $\operatorname{Im} \tilde{g} = \ker E$. Moreover, we have

$$\tilde{\Sigma} : \begin{cases} \dot{x} = \tilde{f} + \tilde{g}\tilde{v} = f + g\alpha + \gamma h + g\beta v = E_1^{\dagger} F_1 + g\alpha + \gamma F_2 + g\beta v, \\ \tilde{y} = \tilde{h} = \eta h, \end{cases}$$

Pre-multiplying the differential part $\dot{x} = E_1^{\dagger} F_1 + g\alpha + \gamma F_2 + g\beta v$ of $\tilde{\Sigma}$ by E_1, we get (note that $\operatorname{Im} g = \ker E_1$)

$$\begin{cases} E_1 \dot{x} = F_1 + E_1 \gamma F_2, \\ \ \ \tilde{y} \ \ = \eta h. \end{cases}$$

Thus $\tilde{\Sigma}$ is an $(I, \tilde{v})$-explicitation of the following DAE:

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1 + E_1 \gamma F_2 \\ \eta F_2 \end{bmatrix}.$$

Since the above DAE can be obtained from Ξ via $\tilde{Q} = Q'Q$, where $Q' = \left[\begin{smallmatrix} I_q & E_1\gamma \\ 0 & \eta \end{smallmatrix}\right]$, it proves that $\tilde{\Sigma}$ is a $(\tilde{Q}, \tilde{v})$-explicitation of Ξ corresponding to the choice of invertible matrix $\tilde{Q} = Q'Q$. Finally, by $E_1 \tilde{f} = F_1 + E_1 \gamma F_2$, we get $\tilde{f} = \tilde{E}_1^{\dagger}(F_1 + \gamma F_2)$ for the choice of right inverse $\tilde{E}_1^{\dagger}$ of E_1.

Only if. Suppose that $\tilde{\Sigma} \in \mathbf{Expl}(\Xi)$ via $\tilde{Q}$, $\tilde{E}_1^{\dagger}$ and $\tilde{g}$. First, by $\operatorname{Im} \tilde{g} = \ker E = \operatorname{Im} g$, there exists an invertible matrix β such that $\tilde{g} = g\beta$. Moreover, since $E_1^{\dagger}$ is a right inverse of E_1 if and only if any solution $\dot{x}$ of $E_1 \dot{x} = w$ is given by $E_1^{\dagger} w$, we have $E_1 E_1^{\dagger} F_1 = F_1$ and $E_1 \tilde{E}_1^{\dagger} F_1 = F_1$. It follows that $E_1(\tilde{E}_1^{\dagger} - E_1^{\dagger})F_1 = 0$, so $(\tilde{E}_1^{\dagger} - E_1^{\dagger})F_1 \in \ker E_1$. Since $\ker E_1 = \operatorname{Im} g$, it follows that $(\tilde{E}_1^{\dagger} - E_1^{\dagger})F_1 = g\alpha$ for a suitable α. Furthermore, since Q is such that E_1 of $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank, any other $\tilde{Q}$, such that $\tilde{E}_1$ of $\tilde{Q}E = \left[\begin{smallmatrix} \tilde{E}_1 \\ 0 \end{smallmatrix}\right]$ is full row rank, must be of the form $\tilde{Q} = Q'Q$, where $Q' = \left[\begin{smallmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{smallmatrix}\right]$. Thus via $\tilde{Q}$, Ξ is ex-equivalent to

$$Q' \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{x} = Q' \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \Rightarrow \begin{bmatrix} Q_1 E_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} Q_1 F_1 + Q_2 F_2 \\ Q_4 F_2 \end{bmatrix}.$$

The equation on the right-hand side of the above can be expressed (using $\tilde{E}_1^{\dagger}$ and $\tilde{g}$) as:

$$\begin{cases} \dot{x} = \tilde{E}_1^{\dagger} F_1 + \tilde{E}_1^{\dagger} Q_1^{-1} Q_2 F_2 + \tilde{g} v = E_1^{\dagger} F_1 + g\alpha + E_1^{\dagger} Q_1^{-1} Q_2 h + g\beta \tilde{v}, \\ 0 = Q_4 F_2 = Q_4 h. \end{cases}$$

Thus the explicitation of Ξ via $\tilde{Q}$, $\tilde{E}_1^{\dagger}$ and $\tilde{g}$ is

$$\tilde{\Sigma} : \begin{cases} \dot{x} = E_1^{\dagger} F_1 + g\alpha + \gamma h + g\beta \tilde{v} = f + \gamma h + g(\alpha + \beta \tilde{v}) = \tilde{f} + \tilde{g}\tilde{v}, \\ \tilde{y} = \eta h = \tilde{h}. \end{cases}$$

where $\gamma = E_1^{\dagger} Q_1^{-1} Q_2$, $\eta = Q_4$. Thus we can see that Σ and $\tilde{\Sigma}$ are equivalent via the transformations of the form (16). $\square$

Proof of Proposition 4.5. Consider the DAE (11) of the (Q, v)-explicitation procedure. Since Q-transformations preserve solutions of Ξ, (11) resulting from a Q-transformation of Ξ has the same solutions as Ξ. Thus we need to prove that (11) and (14) have corresponding solutions for any choices of $E_1^\dagger$ and g. Moreover, the second equation $0 = F_2(x)$ of (11) coincides with $0 = h(x)$ of (14). So we only need to prove that $x(t) \in \mathcal{C}^1$ is a solution of $E_1(x)\dot{x} = F_1(x)$ if and only if there exists $v(t) \in \mathcal{C}^0$ such that $(x(t), v(t))$ is a solution of $\dot{x} = f(x) + g(x)v$ independently of the choice of $E_1^\dagger$, defining $f(x) = E_1^\dagger(x)F_1(x)$, and of the choice of g satisfying $\operatorname{Im} g(x) = \ker E_1(x)$.

If. Suppose that $(x(t), v(t))$ is a solution of $\dot{x} = f(x) + g(x)v$. Then we have $\dot{x}(t) = f(x(t)) + g(x(t))v(t)$. Pre-multiplying the latter equation by $E_1(x(t))$, we get that

$$E_1(x(t))\dot{x}(t) = E_1(x(t))f(x(t)) = E_1(x(t))E_1^\dagger(x(t))F_1(x(t)) = F_1(x(t)),$$

which proves that $(x(t), vd(t))$ is a solution of $E_1(x)\dot{x} = F_1(x)$.

Only if. Suppose that $x(t)$ is a solution of $E_1(x)\dot{x} = F_1(x)$. Rewrite $E_1(x)\dot{x}$ as $\left[\begin{smallmatrix} E_1^1(x) & E_1^2(x) \end{smallmatrix}\right]\left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right]$, where $E_1^1 : U \to \mathbb{R}^{q\times q}$ is smooth and $x = (x_1, x_2)$. Then, by taking a smaller neighborhood U, if necessary, we assume that $E_1^1(x)$ is invertible locally around x_p (if not, we permute the components of x such that the first q columns of $E_1(x)$ is independent). Thus, a choice of right inverse of E_1 is $E_1^\dagger = \left[\begin{smallmatrix} (E_1^1)^{-1} \\ 0 \end{smallmatrix}\right]$. So the maps f and g can be defined as $f := E_1^\dagger F_1 = \left[\begin{smallmatrix} (E_1^1)^{-1}F_1 \\ 0 \end{smallmatrix}\right]$, $g := \left[\begin{smallmatrix} -(E_1^1)^{-1}E_2 \\ I_m \end{smallmatrix}\right]$. Let $v(t) = \dot{x}_2(t)$, then $v \in \mathcal{C}^0$ and it is clear that if $x(t) = ((x_1(t), x_2(t)))$ is a solution of $E_1(x)\dot{x} = F_1(x)$, then $(x(t), v(t))$ solves $\dot{x} = f(x) + g(x)v$ since

$$\begin{bmatrix} E_1^1(x(t)) & E_1^2(x(t)) \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = F_1(x(t)) \Rightarrow \dot{x}_1(t) = (E_1^1)^{-1}F_1(x(t)) - (E_1^1)^{-1}E_1^2(x(t))\dot{x}_2(t).$$

Notice that if we choose another right inverse $\tilde{E}_1^\dagger$ of E_1 and another matrix $\tilde{g}$ such that $\operatorname{Im}\tilde{g} = \ker E_1$, then by Proposition 4.2, we have

$$\dot{x} = \tilde{f}(x) + \tilde{g}(x)\tilde{v} \Leftrightarrow \dot{x} = f(x) + g(x)(\alpha(x) + \beta(x)v).$$

We thus conclude that there exists $\tilde{v}(t) = \alpha(x(t)) + \beta(x(t))v(t) = \alpha(x(t)) + \beta(x(t))\dot{x}_2(t)$ such that $(x(t), \tilde{v}(t))$ solves $\dot{x} = \tilde{f}(x) + \tilde{g}(x)\tilde{v}$. Therefore, Ξ has corresponding solutions with any (Q, v)-explicitation Σ independently of the choice of Q, $E_1^\dagger$ and g. $\square$

Proof of Theorem 4.6. By the assumptions that $\operatorname{rank} E(x) = const. = q$ and $\operatorname{rank}\tilde{E}(\tilde{x}) = const. = \tilde{q}$ around x_p and $\tilde{x}_p$, respectively, we have that Ξ and $\tilde{\Xi}$ are locally ex-equivalent to

$$\Xi' : \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} \quad \text{and} \quad \tilde{\Xi}' : \begin{bmatrix} \tilde{E}_1(\tilde{x}) \\ 0 \end{bmatrix} \dot{\tilde{x}} = \begin{bmatrix} \tilde{F}_1(\tilde{x}) \\ \tilde{F}_2(\tilde{x}) \end{bmatrix},$$

respectively, where $E_1(x)$ and $\tilde{E}_1(\tilde{x})$ are full row rank matrices and their ranks are q and $\tilde{q}$, respectively. By Definition 4.1, we have

$$\begin{aligned} f(x) &= E_1^\dagger(x)F_1(x), \quad \operatorname{Im} g(x) = \ker E_1(x), \quad h(x) = F_2(x), \\ \tilde{f}(\tilde{x}) &= \tilde{E}_1^\dagger(\tilde{x})\tilde{F}_1(\tilde{x}), \quad \operatorname{Im}\tilde{g}(\tilde{x}) = \ker\tilde{E}_1(\tilde{x}), \quad \tilde{h}(\tilde{x}) = \tilde{F}_2(\tilde{x}). \end{aligned} \tag{26}$$

Note that the explicitation system is defined up to a feedback, an output multiplication and a generalized output injection. Any two control systems belonging to $\mathbf{Expl}(\Xi)$ are sys-equivalent to each other and so are any two control systems belonging to $\mathbf{Expl}(\tilde{\Xi})$. Thus the choice of an explicitation system makes no difference for the proof of sys-equivalence. Without loss of generality, we will use $f(x)$, $g(x)$, $h(x)$ and $\tilde{f}(x)$, $\tilde{g}(x)$, $\tilde{h}(x)$ given in (26) for the remaining part of this proof.

If. Suppose $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$ in a neighborhood U of x_p. By Definition 4.3, there exists a diffeomorphism $\tilde{x} = \psi(x)$ and $\beta : U \to GL(m, \mathbb{R})$ such that $\tilde{g} \circ \psi = \frac{\partial \psi}{\partial x} g\beta$, which implies

$$\ker(\tilde{E} \circ \psi) = \operatorname{span}\{\tilde{g}_1, ..., \tilde{g}_m\} \circ \psi = \operatorname{span}\left\{\frac{\partial \psi}{\partial x} g_1, ..., \frac{\partial \psi}{\partial x} g_m\right\} = \frac{\partial \psi}{\partial x} \ker E$$

and $q = \tilde{q}$ (since $\dim \ker \tilde{E} = \tilde{m} = m = \tilde{E}$). We can deduce from the above equation that there exists $Q_1 : U \to GL(q, \mathbb{R})$ such that

$$\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial \psi}{\partial x}\right)^{-1}. \tag{27}$$

Subsequently, by $\tilde{f} \circ \psi = \frac{\partial \psi}{\partial x}(f + \gamma h + g\alpha)$ of Definition 4.3, we have

$$(\tilde{E}_1^{\dagger} \circ \psi)(\tilde{F}_1 \circ \psi) = \frac{\partial \psi}{\partial x}(E_1^{\dagger} F_1 + \gamma F_2 + g\alpha).$$

Pre-multiply the above equation by $\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial \psi}{\partial x}\right)^{-1}$, to obtain

$$\tilde{F}_1 \circ \psi = Q_1 F_1 + Q_1 E_1 \gamma F_2. \tag{28}$$

Then by $\tilde{h} \circ \psi = \eta h$ of Definition 4.3, we immediately get

$$\tilde{F}_2 \circ \psi = \eta F_2. \tag{29}$$

Now combining (27), (28) and (29), we conclude that Ξ' and $\tilde{\Xi}'$ are ex-equivalent via $\tilde{x} = \psi(x)$ and $Q = \left[\begin{smallmatrix} Q_1 & Q_1 E_1 \gamma \\ 0 & \eta \end{smallmatrix}\right]$, which implies that $\Xi \overset{ex}{\sim} \tilde{\Xi}$ (since $\Xi \overset{ex}{\sim} \Xi'$ and $\tilde{\Xi} \overset{ex}{\sim} \tilde{\Xi}'$).

Only if. Suppose that locally $\Xi \overset{ex}{\sim} \tilde{\Xi}$ around x_p. It follows that locally $\Xi' \overset{ex}{\sim} \tilde{\Xi}'$ around x_p. It follows that $q = \tilde{q}$. Assume that they are ex-equivalent via $Q : U \to GL(l, \mathbb{R})$ and $\tilde{x} = \psi(x)$ defined on a neighborhood U of x_p. Let $Q = \left[\begin{smallmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{smallmatrix}\right]$, where Q_1, Q_2, Q_3 and Q_4 are matrix-valued functions of sizes $q \times q$, $q \times m$, $p \times q$ and $p \times p$, respectively. Then by $\left[\begin{smallmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{smallmatrix}\right] \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{E}_1 \circ \psi \\ 0 \end{smallmatrix}\right] \frac{\partial \psi}{\partial x}$, we can deduce that $Q_3 = 0$ and Q_1, Q_4 are invertible matrices. Then we have

$$\begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{E}_1 \circ \psi \\ 0 \end{bmatrix} \frac{\partial \psi}{\partial x}, \qquad \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1 \circ \psi \\ \tilde{F}_2 \circ \psi \end{bmatrix},$$

which implies

$$\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial \psi}{\partial x}\right)^{-1}, \quad \tilde{F}_1 \circ \psi = Q_1 F_1 + Q_2 F_2, \quad \tilde{F}_2 \circ \psi = Q_4 F_2. \tag{30}$$

Thus by $\operatorname{Im} g(x) = \ker E(x) = \ker E_1(x)$ and $\operatorname{Im} \tilde{g}(x) = \ker \tilde{E}(\tilde{x}) = \ker \tilde{E}_1(\tilde{x})$, and using (30), we have

$$\tilde{g} \circ \psi = \frac{\partial \psi}{\partial x} g\beta \tag{31}$$

for some $\beta : U \to GL(m, \mathbb{R})$. Moreover, there exists $\alpha : U \to \mathbb{R}^m$ such that

$$\begin{aligned}\tilde{f} \circ \psi = \tilde{E}_1^{\dagger} \circ \psi \tilde{F}_1 \circ \psi &\overset{(30)}{=} \frac{\partial \psi}{\partial x} E_1^{\dagger} Q_1^{-1} Q_1 F_1 + Q_2 F_2 = \frac{\partial \psi}{\partial x} E_1^{\dagger} Q_1^{-1} \left(Q_1 F_1 + Q_2 F_2 + Q_1 E_1 g\alpha\right) \\ &= \frac{\partial \psi}{\partial x} \left(f + E_1^{\dagger} Q_1^{-1} Q_2 y + g\alpha\right),\end{aligned} \tag{32}$$

In addition, we have

$$\tilde{h} \circ \psi = \tilde{F}_2 \circ \psi \overset{(30)}{=} Q_4 F_2 = Q_4 h. \tag{33}$$

Finally, it can be seen from (31), (32), and (33) that $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$ via $\tilde{x} = \psi(x)$, α, β, $\gamma = E_1^{\dagger} Q_1^{-1} Q_2$ and $\eta = Q_4$. □

Proof of Proposition 4.8. We first show that the sequence of submanifolds M_k^c of the geometric reduction method of the DAE Ξ and the sequence N_k^c of the zero dynamics algorithm of any control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$ locally coincide. Suppose that $\operatorname{rank} E(x) = const. = q$ in a neighborhood U_1 of x_p. Then there always exists an invertible matrix $Q(x)$ defined on U_1 such that $E_1(x)$ of $Q(x)E(x) = \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right]$ is of full row rank q for all $x \in U_1$, denote $Q(x)F(x) = \left[\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right]$. Recall, see Remark 4.7, that N_k of the zero dynamics algorithm are well-defined for the class $\mathbf{Expl}(\Sigma)$ and that N_k are the same for all control systems belonging to $\mathbf{Expl}(\Xi)$. So the choice of an explicitation system makes no difference for N_k. We may choose a control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$, given by $f(x) = E_1^{\dagger}(x) F_1(x)$, $\operatorname{Im} g(x) = \ker E(x)$, $h(x) = F_2(x)$. By the definition of M_1 (see (4)) and $N_1 = h^{-1}(0)$, we have

$$\begin{aligned} M_1^c = M_1 \cap U_1 &= \{x \in U_1 : Q(x)F(x) \in \operatorname{Im} Q(x)E(x)\} = \left\{x \in U_1 : \left(\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right) \in \operatorname{Im} \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right]\right\} \\ &= \{x \in U_1 : F_2(x) = 0\} = \{x \in U_1 : h(x) = 0\} = N_1 \cap U_1 = N_1^c. \end{aligned}$$

For $k > 1$, suppose $M_{k-1}^c = N_{k-1}^c$. Then by (4) and (17), we have

$$\begin{aligned} M_k &= \left\{x \in M_{k-1}^c : Q(x)F(x) \in Q(x)E(x) T_x M_{k-1}^c\right\} = \left\{x \in M_{k-1}^c : \left(\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right) \in \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right] T_x M_{k-1}^c\right\} \\ &= \left\{x \in M_{k-1}^c : F_1(x) \in E_1(x) T_x M_{k-1}^c\right\} = \left\{x \in M_{k-1}^c : f(x) + \ker E_1(x) \subseteq T_x M_{k-1}^c + \ker E_1(x)\right\} \\ &= \left\{x \in N_{k-1}^c : f(x) \in T_x N_{k-1}^c + \mathcal{G}(x)\right\} = N_k, \end{aligned}$$

and thus $M_k^c = N_k^c$. If either one among (A1) and (A2) is satisfied, then by $N_k^c = M_k^c$, we can easily deduce the other one and then both (A1) and (A2) hold. Then by Proposition 2.7, $M^* = M_{k^*}^c$ is a locally maximal invariant submanifold and by Proposition 6.1.1 of [25], $N^* = N_{k^*}^c$ is a local maximal output zeroing submanifold. Moreover, we have locally $M^* = N^*$ (since locally $M_k^c = N_k^c$). Now under the assumption that $\dim E(x)T_xM^* = const.$ for all $x \in M^*$ around x_p, by Theorem 3.11, Ξ is internally regular if and only if $\dim M^* = \dim E(x)T_xM^*$, i.e., $\ker E(x) \cap T_xM^* = 0$, locally $\forall x \in M^*$ around x_p. Thus by $N^* = M^*$ and $\ker E(x) = \mathcal{G}(x)$, we have that Ξ is internally regular (around x_p) if and only if $\mathcal{G}(x_p) \cap T_{x_p}N^* = 0$. $\square$

7.4. Proof of Theorem 5.3

Proof. $(i) \Rightarrow (ii)$: Suppose in a neighborhood U of x_p that $\operatorname{rank} E(x) = q$ and $\mathcal{G}(x) = \ker E(x) = \operatorname{span}\{g_1(x), \dots, g_m(x)\}$ is involutive, where $g_1, \dots, g_m$ are independent vector fields on U and $m = n - q$. Then by the involutivity of $\mathcal{G}$, there exist local coordinates $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) = \psi(x)$, where $\tilde{x}_1 = (\tilde{x}_1^1, \dots, \tilde{x}_1^q)$ and $\tilde{x}_2 = (\tilde{x}_2^1, \dots, \tilde{x}_2^{n-q})$, such that $\operatorname{span}\left\{\mathrm{d}\tilde{x}_1^1, \dots, \mathrm{d}\tilde{x}_1^q\right\} = \operatorname{span}\left\{\mathrm{d}\tilde{x}_1\right\} = \mathcal{G}^\perp$ (Frobenius theorem [3]), where $\mathcal{G}^\perp$ denotes the co-distribution which annihilates $\mathcal{G}$. Note that in the $\tilde{x}$-coordinates, the distribution

$$\ker \tilde{E}(\tilde{x}) = \ker \left(E(x) \left(\frac{\partial \psi(x)}{\partial x} \right)^{-1} \right) = \frac{\partial \psi(x)}{\partial x} \mathcal{G}(x) = \operatorname{span}\{\tilde{g}_1(\tilde{x}), ..., \tilde{g}_m(\tilde{x})\},$$

where $\tilde{g}_i \circ \psi = \frac{\partial \psi}{\partial x} g_i$, $i = 1, ..., m$. Now let $\tilde{g}$ be a matrix whose columns consist of $\tilde{g}_i$, for $i = 1, ..., m$. It follows that $\operatorname{rank} \tilde{g}(\tilde{x}) = m$ around $\tilde{x}_0 = \psi(x_0)$. By $d\tilde{x}_1 = \mathcal{G}^\perp$, we have $\langle \mathrm{d}\tilde{x}_1, \tilde{g}_i \rangle = 0$, for $i = 1, \dots, m$. Thus $\tilde{g}(\tilde{x})$ is of the form $\tilde{g}(\tilde{x}) = \left[\begin{smallmatrix} 0 \\ \tilde{g}_2(\tilde{x}) \end{smallmatrix}\right]$, where $\tilde{g}_2 : \psi(U) \to \mathbb{R}^{m \times m}$. Since $\operatorname{rank} \tilde{g}(\tilde{x}) = m$, it can be seen that $\tilde{g}_2(\tilde{x})$ is an invertible matrix, which implies by $\operatorname{Im} \tilde{g}(\tilde{x}) = \ker \tilde{E}(\tilde{x})$ that $\tilde{E}(\tilde{x})$ has to be of the form $\tilde{E}(\tilde{x}) = \left[\begin{smallmatrix} \tilde{E}_1(\tilde{x}) & 0 \end{smallmatrix}\right]$, where $\tilde{E}_1 : \psi(U) \to \mathbb{R}^{l \times m}$. Thus in the $\tilde{x}$-coordinates, $\tilde{\Xi} = (\tilde{E}, \tilde{F})$ admits the following form:

$$\begin{bmatrix} \tilde{E}_1 (\tilde{x}) & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \tilde{F} (\tilde{x}) .$$

where $\tilde{F} \circ \psi = F$. Now by $\operatorname{rank} E(x) = q$, we get $\operatorname{rank} \begin{bmatrix} \tilde{E}_1 (\tilde{x}) & 0 \end{bmatrix} = \operatorname{rank} E(x) = q$ (the coordinate transformation preserves the rank). Thus there exists $Q : \psi(U) \to GL(l, \mathbb{R})$ such that $Q(\tilde{x})\tilde{E}(\tilde{x}) = Q(\tilde{x}) \begin{bmatrix} \tilde{E}_1 (\tilde{x}) & 0 \end{bmatrix} = \left[\begin{smallmatrix} \tilde{E}_1^1(\tilde{x}) & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $\tilde{E}_1^1 : \psi(U) \to \mathbb{R}^{q \times q}$. Since $Q(\tilde{x})$ preserves the rank of $\tilde{E}(\tilde{x})$, we have $\operatorname{rank} \tilde{E}_1^1 (\tilde{x}) = q$. Therefore, $\tilde{E}_1^1 (\tilde{x})$ is an invertible matrix. Now let $Q'(\tilde{x}) = \left[\begin{smallmatrix} \left(\tilde{E}_1^1(\tilde{x})\right)^{-1} & 0 \\ 0 & I_m \end{smallmatrix}\right] Q(\tilde{x})$ and denote $Q'(\tilde{x})\tilde{F}(\tilde{x}) = \left[\begin{smallmatrix} F_1(\tilde{x}) \\ F_2(\tilde{x}) \end{smallmatrix}\right]$. It is seen that, via $\tilde{x} = \psi(x)$ and Q', Ξ is locally ex-equivalent to $\tilde{\Xi} = (Q'\tilde{E}, Q'\tilde{F})$, where $Q'\tilde{E} \circ \psi = Q'E(\frac{\partial \psi}{\partial x})^{-1} = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Clearly, $\tilde{\Xi}$ is a semi-explicit DAE.

$(ii) \Rightarrow (iii)$: Suppose that Ξ is locally ex-equivalent to Ξ^{SE} of (2) around x_p. Then, any control system $\Sigma \in \mathbf{Expl}(\Xi)$ is locally sys-equivalent to $\Sigma' \in \mathbf{Expl}(\Xi^{SE})$ below (by Theorem 3.11):

$$\Sigma' : \begin{cases} \left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} F_1(x_1, x_2) \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ I_m \end{smallmatrix}\right] v, \\ \quad y \quad = F_2(x_1, x_2). \end{cases}$$

Suppose that $\Sigma \overset{sys}{\sim} \Sigma'$ via $z = (z_1, z_2) = \psi(x)$, α, β and $\gamma = \left[\begin{smallmatrix} \gamma_1 \\ \gamma_2 \end{smallmatrix}\right]$, then

$$\Sigma : \begin{cases} \left[\begin{smallmatrix} \dot{z}_1 \\ \dot{z}_2 \end{smallmatrix}\right] = \dfrac{\partial \psi(x)}{\partial x} \left(\left[\begin{smallmatrix} F_1(x) \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \gamma_1(x) \\ \gamma_2(x) \end{smallmatrix}\right] y + \left[\begin{smallmatrix} 0 \\ I_m \end{smallmatrix}\right] (\alpha(x) + \beta(x)\tilde{v}) \right), \\ \quad \tilde{y} = \eta(x) F_2(x). \end{cases}$$

By Definition 5.2, Σ can be always fully reduced to (by a coordinates change and a feedback transformation)

$$\begin{cases} \dot{x}_1 = F_1(x_1,x_2) + \gamma_1(x_1,x_2)F_2(x_1,x_2), \\ \ y \ = \eta(x_1,x_2)F_2(x_1,x_2), \end{cases}$$

where x_2 is the new control.

$(iii) \Rightarrow (i)$: Suppose (iii) holds. Then $\mathbf{Expl}(\Xi)$ is not empty implies that $E(x)$ has constant rank around x_p. By Definition 5.2, any control system $\Sigma \in \mathbf{Expl}(\Xi)$ can be fully reduced implies $\mathcal{G} = \ker E(x) = \mathrm{span}\,\{g_1, ..., g_m\}$ is involutive. $\square$

7.5. Proof of Theorem 6.1

Claim 7.1. *If assumptions (A1)-(A3) of Theorem 6.1 are satisfied, then the point x_p is a regular point of the zero dynamics algorithm (rank conditions (i), (ii), (iii) of Proposition 6.1.3 of [25] are satisfied) for any control system $\Sigma \in \mathbf{Expl}(\Xi)$. If so, we use Proposition 6.1.5 of [25] with a small modification: there exist local coordinates $(z, z^*) = (z_1, \dots, z_m, z^*)$ such that Σ is the following form*

$$\left\{ \begin{array}{llll} y_1 & = z_1^1 & & \vdots \\ \dot{z}_1^1 & = z_1^2 + \sigma_1^1 v & y_i & = z_i^1, \quad i = 3, \dots, m \\ & \cdots & \dot{z}_i^1 & = z_i^2 + \sum\limits_{s=1}^{i-1} \delta_{i,s}^1 (\alpha_s + \beta_s v) + \sigma_i^1 v \\ \dot{z}_1^{\rho_1 - 1} & = z_1^{\rho_1} + \sigma_1^{\rho_1 - 1} v & & \cdots \\ \dot{z}_1^{\rho_1} & = \alpha_1 + \beta_1 v & \dot{z}_i^{\rho_i - 1} & = z_i^{\rho_i} + \sum\limits_{s=1}^{i-1} \delta_{i,s}^{\rho_i - 1} (\alpha_s + \beta_s v) + \sigma_i^{\rho_i - 1} v \\ y_2 & = z_2^1 & \dot{z}_i^{\rho_i} & = \alpha_i + \beta_i v \\ \dot{z}_2^1 & = z_2^2 + \delta_{2,1}^1 (\alpha_1 + \beta_1 v) + \sigma_2^1 v & & \vdots \\ & \cdots & \dot{z}^* & = f^*(z,z^*) + g^*(z,z^*)v. \\ \dot{z}_2^{\rho_2 - 1} & = z_1^{\rho_2} + \delta_{2,1}^{\rho_2 - 1} (\alpha_1 + \beta_1 v) + \sigma_2^{\rho_2 - 1} v & & \\ \dot{z}_1^{\rho_2} & = \alpha_2 + \beta_2 v & & \end{array} \right. \tag{34}$$

where $\delta_{i,s}^j \equiv 0$ for $1 \le j < \rho_s$, $1 \le s \le i-1$.

Remark 7.2. (i) Note that in (34), $\rho_1 \le \rho_2 \le \dots \le \rho_m$, the matrix $\beta = (\beta_1, \dots, \beta_m)$ is invertible at x_p. The functions σ_k satisfy $\sigma^k|_{N_k} = 0$ for $k = 1, \dots, \rho_i - 1$, where

$$N_k = \{(z, z^*) : z_i^j = 0,\ 1 \le i \le m,\ 1 \le j \le k\}.$$

(ii) There are two differences between system (34) and the zero dynamics form of Proposition 6.1.3 of [25], where the functions $\sigma_1^1, \dots, \sigma_1^{\rho_1 - 1}$ are not present and all the functions $\delta_{i,s}^j$ can be nonzero. However, in (34), $\sigma_1^1, \dots, \sigma_1^{\rho_1 - 1}$ vanish on $N_1, \dots, N_{\rho_1 - 1}$, respectively, but may not outside, and $\delta_{i,s}^j \equiv 0$ for $1 \le j < \rho_s$, $1 \le s \le i-1$.

Proof of Claim 7.1. We will prove that assumptions (A1), (A2), (A3) of Theorem 6.1 correspond to the rank conditions (i), (ii), (iii) of Proposition 6.1.3 in [25]. By the assumption of Theorem 6.1 that $\mathrm{rank}\, E(x) = const.$ around x_p, we have $\mathbf{Expl}(\Xi)$ is not empty. Now, in order to compare the two algorithms (the geometric reduction algorithm of Appendix for Ξ and the zero dynamics algorithm in [25] for $\Sigma \in \mathbf{Expl}(\Xi)$), we use the same notations as in the algorithm of Appendix.

Then for a control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$, we have $f(x) = (\tilde{E}_1^1)^{\dagger}\tilde{F}_1^1(x)$, $\operatorname{Im} g(x) = \ker E(x) = \ker \tilde{\delta}_1^1(x)$, $h(x) = \tilde{F}_1^2(x)$. The zero dynamics algorithm for Σ can be implemented in the following way:

Step 1: by (A1) of Theorem 6.1, we get $\mathrm{D}h(x) = \mathrm{D}\tilde{F}_1^2(x)$ has constant rank $n - n_1$ around x_p (condition (i) of Proposition 6.1.3 in [25]). Thus $h^{-1}(0)$ can be locally expressed as $N_1^c = \{x : H_1(x) = 0\}$, where $H_1 = \psi_1(x) = (\psi_1^1, ..., \psi_1^{n-n_1})$.

Step k ($k > 1$): By the proof of Proposition 4.8, we have $N_{k-1}^c = M_{k-1}^c$, which is

$$N_{k-1}^c = M_{k-1}^c = \{x : H_{k-1}(x) = 0\},$$

where $H_{k-1} = (\psi_0, \dots, \psi_{k-1})$. By the zero dynamic algorithms, N_k can be calculated by all $x \in N_{k-1}^c$ such that

$$L_f H_{k-1}(x) + L_g H_{k-1}(x)u = 0.$$

Then by assumption (A2) of Theorem 6.1, we can deduce that

$$\dim \ker E(x) \cap \ker dH_{k-1} = \dim \operatorname{span}\{g_1, \dots, g_m\} \cap \ker dH_{k-1} = const., \tag{35}$$

for all $x \in M_{k-1}^c$ around x_p. Now by $\dim \ker E(x) = const.$ around x_p (by $\operatorname{rank} E(x) = const.$), we get

$$\dim \operatorname{span}\{g_1, \dots, g_m\} = const. \tag{36}$$

locally around x_p. By (35) and (36), we get $\operatorname{rank} L_g H_{k-1}(x) = const.$ for all $x \in M_{k-1}^c$ around x_p (condition (ii) of Proposition 6.1.3 in [25]).

Since $\operatorname{rank} L_g H_{k-1}(x) = const.$, there exists a basis matrix $R_{k-1}(x)$ of the annihilator of the image of $L_g H_{k-1}(x)$, that is $R_{k-1}(x) L_g H_{k-1}(x) = 0$. Thus N_k^c can be defined by

$$N_k^c = \{x \in U_k : H_{k-1}(x) = 0,\ R_{k-1}(x) L_f H_{k-1}(x) = 0\}.$$

Notice that by the geometric reduction algorithm, we have

$$M_k^c = \{x \in U_k : H_{k-1}(x) = 0, \quad \tilde{F}_k^2(x) = 0\}.$$

By $N_k^c = M_k^c$ and ranks of the differentials of $(H_{k-1}(x), \tilde{F}_k^2(x))$ are constant for all x around x_p (assumption (A1) of Theorem 6.1), it follows that the rank of the differentials of $\left[\begin{smallmatrix} H_{k-1}(x) \\ R_{k-1}(x) L_f H_{k-1}(x) \end{smallmatrix}\right]$ has constant rank around x_p (condition (i) of Proposition 6.1.3 in [25]).

The assumption (A3) of Theorem 6.1 that $\dim E(x) T_x M^* = \dim M^*$ locally around x_p implies

$$\operatorname{span}\{g_1(x_p), \dots, g_m(x_p)\} \cap T_{x_p} N^* = 0.$$

Finally, by $N^* = \{x : H_{k^*}(x) = 0\}$, it follows that the matrix $L_g H_{k^*}(x_p)$ has rank m (condition (iii) of Proposition 6.1.3 in [25]). $\square$

Proof of Theorem 6.1. Observe that by assumption (A3) and Theorem 3.11(iii), we have that Ξ is internally regular. Then by Claim 7.1, we have x_p is a regular point of the zero dynamics algorithm for any control system $\Sigma \in \mathbf{Expl}(\Xi)$. Thus there exist local coordinates (z, z^*) such that Σ is in the form of (34) around x_p. Notice that the matrix $\beta = (\beta_1, \ldots, \beta_m)$ is invertible at x_p and the functions $\sigma_i^k|_{N_k^c} = 0$ for $1 \le i \le m$, $1 \le k \le \rho_i - 1$, which implies $\sigma_i^k \in \mathbf{I}^k$, where $\mathbf{I}^k$ is the ideal generated by z_i^j, $1 \le i \le m$, $1 \le j \le k$ in the ring of smooth functions of z_s^t and z_r. Then for system (34), using the feedback transformation $\tilde{v} = \alpha + \beta v$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$, we get

$$\begin{cases} y_i &= z_i^1, \qquad i = 1, \ldots, m, \\ \dot{z}_i^1 &= z_i^2 + \sum\limits_{s=1}^{i-1} \delta_{i,s}^1 \tilde{v}_s + a_i^1 + b_i^1 \tilde{v}, \\ & \cdots \\ \dot{z}_i^{\rho_i - 1} &= z_i^{\rho_i} + \sum\limits_{s=1}^{i-1} \delta_{i,s}^{\rho_i - 1} \tilde{v}_s + a_i^{\rho_i - 1} + b_i^{\rho_i - 1} \tilde{v}, \\ \dot{z}_i^{\rho_i} &= \tilde{v}_i, \\ \dot{z}^* &= \tilde{f}^*(z, z^*) + \tilde{G}^*(z, z^*) \tilde{v}, \end{cases} \tag{37}$$

where $\tilde{f}^* = f^* - \bar{g}\beta^{-1}\alpha$, $\tilde{G}^* = g^*\beta^{-1}$, and where $a_i^k = -\sigma_i^k \beta^{-1}\alpha$, $b_i^k = \sigma_i^k \beta^{-1}$, for $1 \le i \le m$, $1 \le k \le \rho_i - 1$ and by $\sigma_i^k \in \mathbf{I}^k$, we have $a_i^k, b_{i,l}^k \in \mathbf{I}^k$.

Recall from (34) that the functions $\delta_{i,s}^j \equiv 0$ for $1 \le j < \rho_s$, $1 \le s \le i-1$. Then if the function $\delta_{i,\bar{s}}^j \neq 0$, $j = \rho_{\bar{s}} + k$, for a certain $1 \le \bar{s} \le i-1$ and a certain $0 \le k \le \rho_i - 1 - \rho_{\bar{s}}$, we show that, via suitable changes of coordinates and output multiplications, the nonzero function $\delta_{i,\bar{s}}^{k+\rho_{\bar{s}}}$ can be eliminated. Namely, define the new coordinates (and keep the remaining coordinates unchanged):

$$\tilde{z}_i^{k+1} = z_i^{k+1} - \delta_{i,\bar{s}}^{\rho_{\bar{s}}+k} z_{\bar{s}}^1, \quad \tilde{z}_i^{k+2} = z_i^{k+2} - \delta_{i,\bar{s}}^{\rho_{\bar{s}}+k} z_{\bar{s}}^2, \quad \ldots, \quad \tilde{z}_i^{k+\rho_{\bar{s}}} = z_i^{k+\rho_{\bar{s}}} - \delta_{i,\bar{s}}^{\rho_{\bar{s}}+k} z^{\rho_{\bar{s}}},$$

we have (notice that below $\delta_{\bar{s},s}^1 \equiv 0$ for $1 \le s \le \bar{s} - 1$)

$$\begin{aligned} \dot{\tilde{z}}_i^{k+1} &= z_i^{k+2} + \sum_{s=1}^{i-1} \delta_{i,s}^{k+1} \tilde{v}_s + a_i^{k+1} + b_i^{k+1}\tilde{v} - (\delta_{i,\bar{s}}^{\rho_{\bar{s}}+k})' z_{\bar{s}}^1 - \delta_{i,s}^{\rho_s+k}(z_{\bar{s}}^2 + a_{\bar{s}}^1 + b_{\bar{s}}^1 \tilde{v} + \sum_{s=1}^{\bar{s}-1} \delta_{\bar{s},s}^1 \tilde{v}_s) \\ &= (z_i^{k+2} - \delta_{i,\bar{s}}^{\rho_{\bar{s}}+k} z_{\bar{s}}^2) + (a_i^{k+1} - (\delta_{i,\bar{s}}^{\rho_{\bar{s}}+k})' z_{\bar{s}}^1 - \delta_{i,s}^{\rho_s+k} a_{\bar{s}}^1) + (b_i^{k+1} - \delta_{i,s}^{\rho_s+k} b_{\bar{s}}^1)\tilde{v} + \sum_{s=1}^{i-1} \delta_{i,s}^{k+1} \tilde{v}_s \\ &= \tilde{z}_i^{k+2} + \tilde{a}_i^{k+1} + \tilde{b}_i^{k+1}\tilde{v} + \sum_{s=1}^{i-1} \delta_{i,s}^{k+1} \tilde{v}_s, \end{aligned}$$

where $(\delta_{i,\bar{s}}^{\rho_{\bar{s}}+k})'$ denotes the derivative of $\delta_{i,\bar{s}}^{\rho_{\bar{s}}+k}$ with respect to t, and $\tilde{a}_i^{k+1} = a_i^{k+1} - (\delta_{i,\bar{s}}^{\rho_{\bar{s}}+k})' z_{\bar{s}}^1 - \delta_{i,s}^{\rho_s+k} a_{\bar{s}}^1$, $\tilde{b}_i^{k+1} = b_i^{k+1} - \delta_{i,s}^{\rho_s+k} b_{\bar{s}}^1$ and it is clear $\tilde{a}_i^{k+1}, \tilde{b}_{i,l}^{k+1} \in \mathbf{I}^{k+1}$. Then via similar calculations, we have

$$\dot{\tilde{z}}_i^{k+j} = \tilde{z}_i^{k+j+1} + \tilde{a}_i^{k+j} + \tilde{b}_i^{k+j}\tilde{v} + \sum_{s=1}^{i-1} \delta_{i,s}^{k+j} \tilde{v}_s, \quad 2 \le j \le \rho_{\bar{s}} - 1,$$

for some $\tilde{a}^{k+j}, \tilde{b}^{k+j}_{i,l} \in \mathbf{I}^{k+j}$. Moreover, we have

$$
\begin{aligned}
\dot{\tilde{z}}_i^{k+\rho_{\bar{s}}} &= z_i^{k+\rho_{\bar{s}}+1} + \sum_{s=1}^{i-1} \delta_{i,s}^{k+\rho_{\bar{s}}} \tilde{v}_s + a_i^{k+\rho_{\bar{s}}} + b_i^{k+\rho_{\bar{s}}} \tilde{v} - (\delta_{i,\bar{s}}^{\rho_{\bar{s}}+k})' z_{\bar{s}}^{\rho_{\bar{s}}} - \delta_{i,\bar{s}}^{\rho_{\bar{s}}+k} \tilde{v}_{\bar{s}} \\
&= z_i^{k+\rho_{\bar{s}}+1} + (a_i^{k+\rho_{\bar{s}}} - (\delta_{i,\bar{s}}^{\rho_{\bar{s}}+k})' z_{\bar{s}}^{\rho_{\bar{s}}}) + b_i^{k+1} \tilde{v} + \sum_{s=1}^{i-1} \delta_{i,s}^{k+\rho_{\bar{s}}} \tilde{v}_s - \delta_{i,\bar{s}}^{k+\rho_{\bar{s}}} \tilde{v}_{\bar{s}} \\
&= z_i^{k+\rho_{\bar{s}}+1} + \tilde{a}_i^{k+\rho_{\bar{s}}} + \tilde{b}_i^{k+\rho_{\bar{s}}} \tilde{v} + \sum_{s=1}^{\bar{s}-1} \delta_{i,s}^{k+\rho_{\bar{s}}} \tilde{v}_s + \sum_{s=\bar{s}+1}^{i-1} \delta_{i,\bar{s}}^{k+\rho_{\bar{s}}} \tilde{v}_s,
\end{aligned}
$$

where the functions $\tilde{a}^{k+\rho_{\bar{s}}}, \tilde{b}^{k+\rho_{\bar{s}}}_{i,l} \in \mathbf{I}^{k+\rho_{\bar{s}}}$. Thus in the above formula, the nonzero function $\delta^{k+\rho_{\bar{s}}}_{i,\bar{s}}$ is eliminated. Note that if $k=0$, then the change of coordinate $\tilde{z}_i^1 = z_i^1 - \delta^{\rho_{\bar{s}}}_{i,\bar{s}} z^1_{\bar{s}}$ transforms the first equation $y_i = z_i^1$ of (37) into $y_i = \tilde{z}_i^1 + \delta^{\rho_{\bar{s}}}_{i,\bar{s}} z^1_{\bar{s}}$. We define a new output $\tilde{y}_i = y_i - \delta^{\rho_{\bar{s}}}_{i,\bar{s}} z^1_{\bar{s}} = y_i - \delta^{\rho_{\bar{s}}}_{i,\bar{s}} y_{\bar{s}}$ (which is actually an output multiplication) such that the first equation of (37) becomes $\tilde{y}_i = \tilde{z}_i^1$.

Repeat the above construction to eliminate all the nonzero functions $\delta^j_{i,s}$ for $j \geq \rho_s$, $1 \leq s \leq i-1$. Then system (37) becomes the following control system

$$
\tilde{\Sigma} : \begin{cases}
\tilde{y}_i &= \tilde{z}_i^1, \qquad i = 1, \ldots, m, \\
\dot{\tilde{z}}_i^1 &= \tilde{z}_i^2 + \tilde{a}_i^1 + \tilde{b}_i^1 \tilde{v}, \\
& \cdots \\
\dot{\tilde{z}}_i^{\rho_i - 1} &= \tilde{z}_i^{\rho_i} + \tilde{a}_i^{\rho_i - 1} + \tilde{b}_i^{\rho_i - 1} \tilde{v}, \\
\dot{\tilde{z}}_i^{\rho_i} &= \tilde{v}_i, \\
\dot{z}^* &= \tilde{f}^*(z, z^*) + \tilde{G}^*(z, z^*) \tilde{v}.
\end{cases}
$$

where $a_i^k, b_{i,l}^k \in \mathbf{I}^k$ for $1 \leq k \leq \rho_i - 1$. It is clear that $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$ (we use coordinates changes, feedback transformations and output multiplications to transform Σ into $\tilde{\Sigma}$). Then consider the last row of every subsystem of $\tilde{\Sigma}$, which is $\dot{z}_i^{\rho_i} = \tilde{v}_i$. By deleting this equation in every subsystem and setting $y_i = 0$ for $i = 1, \ldots, m$ and replacing the vector $\tilde{v}$ by $\dot{z}^\rho$, we transform $\tilde{\Sigma}$ into a DAE $\tilde{\Xi}$ below. It is not hard to see that $\tilde{\Sigma} \in \mathbf{Expl}(\tilde{\Xi})$.

$$
\tilde{\Xi} : \begin{cases}
\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}
\begin{bmatrix} \dot{\tilde{z}}_i^1 \\ \dot{\tilde{z}}_i^2 \\ \vdots \\ \dot{\tilde{z}}_i^{\rho_i} \end{bmatrix}
= \begin{bmatrix} \tilde{z}_i^1 \\ \tilde{z}_i^2 \\ \vdots \\ \tilde{z}_i^{\rho_i} \end{bmatrix}
+ \begin{bmatrix} 0 \\ \tilde{a}_i^1 + \tilde{b}_i^1 \dot{\tilde{z}}^\rho \\ \vdots \\ \tilde{a}_i^{\rho_i - 1} + \tilde{b}_i^{\rho_i - 1} \dot{\tilde{z}}^\rho \end{bmatrix}, \quad i = 1, \ldots, m, \\
-\tilde{G}^*(\tilde{z}, z^*) \dot{\tilde{z}}^\rho + \dot{z}^* = \tilde{f}^*(\tilde{z}, z^*).
\end{cases}
$$

Finally, by Theorem 4.6 and $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$, we have that $\Xi \overset{ex}{\sim} \tilde{\Xi}$ and that $\tilde{\Xi}$ is in the **NWF** of (19). $\square$

8. Conclusion

In this paper, for a nonlinear DAE $\Xi = (E, F)$, we first revise the geometric reduction method for the existence of its solutions, and then we define the notions of internal and external equivalence, their differences are discussed by analyzing their relations with solutions. We show that the internal regularity (existence and uniqueness of solutions) of a DAE is equivalent to the fact that the DAE is internally equivalent to an ODE (without free variables) on its maximal invariant submanifold. A procedure named explicitation with driving variables is proposed to connect nonlinear DAEs with nonlinear control systems. We show that the external equivalence for two DAEs is the same as the system equivalence for their explicitation systems. Moreover, we show that Ξ is externally equivalent to a semi-explicit DAE if and only if the distribution defined by $\ker E(x)$ is of constant rank and involutive. If so, the driving variables of a control system $\Sigma \in \mathbf{Expl}(\Xi)$ can be fully reduced. Finally, a nonlinear generalization of the Weierstrass form **WF** is proposed based on a geometric reduction algorithm and the explicitation procedure.

Acknowledgment: The first author is currently supported by Vidi-grant 639.032.733.

References

[1] P. J. Rabier, W. C. Rheinboldt, Theoretical and numerical analysis of differential-algebraic equations, in: P. G. Ciarlet, J. L. Lions (Eds.), Handbook of Numerical Analysis, volume VIII, Elsevier Science, Amsterdam, The Netherlands, 2002, pp. 183–537.

[2] R. Riaza, Differential-Algebraic Systems. Analytical Aspects and Circuit Applications, World Scientific Publishing, Basel, 2008.

[3] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vol. 1, Interscience, 1963.

[4] R. Riaza, DAEs in Circuit Modelling: A Survey, in: A. Ilchmann, T. Reis (Eds.), Surveys in Differential-Algebraic Equations I, Differential-Algebraic Equations Forum, Springer-Verlag, Berlin-Heidelberg, 2013, pp. 97–136.

[5] G. Byrne, P. Ponzi, Differential-algebraic systems, their applications and solutions, Computers Chem. Engng 12 (1988) 377–382.

[6] C. C. Pantelides, D. Gritsis, K. R. Morison, R. W. H. Sargen, The mathematical modeling of transient systems using differential-algebraic equations, Computers Chem. Engng 12 (1988) 449–454.

[7] P. J. Rabier, W. C. Rheinboldt, Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint, volume 68, Society for Industrial and Applied Mathematics, 2000.

[8] P. Betsch, P. Steinmann, A DAE approach to flexible multibody dynamics, Multibody System Dynamics 8 (2002) 365–389.

[9] W. Blajer, K. Kołodziejczyk, Control of underactuated mechanical systems with servo-constraints, Nonlinear Dynamics 50 (2007) 781–791.

[10] S. Reich, On a geometrical interpretation of differential-algebraic equations, Circuits Systems Signal Process. 9 (1990) 367–382.

[11] S. Reich, On an existence and uniqueness theory for nonlinear differential-algebraic equations, Circuits Systems Signal Process. 10 (1991) 343–359.

[12] P. J. Rabier, W. C. Rheinboldt, A geometric treatment of implicit differential-algebraic equations, J. Diff. Eqns. 109 (1994) 110–146.

[13] Y. Chen, S. Trenn, On geometric and differentiation index of nonlinear differential-algebraic equations, in: Proceedings of the MTNS 2020, Cambridge, UK, 2020. In press.

[14] T. Berger, Controlled invariance for nonlinear differential-algebraic systems, Automatica 64 (2016) 226–233.

[15] T. Berger, The zero dynamics form for nonlinear differential-algebraic systems, IEEE Trans. Autom. Control 62 (2017) 4131–4137.

[16] Y. Chen, W. Respondek, Geometric analysis of differential-algebraic equations via linear control theory, SIAM J. Control Optim. 59 (2021) 103–130.

[17] L. Kronecker, Algebraische Reduction der Schaaren bilinearer Formen, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin (1890) 1225–1237.

[18] F. R. Gantmacher, The Theory of Matrices (Vol. II), Chelsea, New York, 1959.

[19] K. Weierstraß, Zur Theorie der bilinearen und quadratischen Formen, Berl. Monatsb. (1868) 310–338.

[20] J. J. Loiseau, K. Özçaldiran, M. Malabre, N. Karcanias, Feedback canonical forms of singular systems, Kybernetika 27 (1991) 289–305.

[21] T. Berger, A. Ilchmann, S. Trenn, The quasi-Weierstraß form for regular matrix pencils, Linear Algebra Appl. 436 (2012) 4052–4069.

[22] T. Berger, S. Trenn, The quasi-Kronecker form for matrix pencils, SIAM J. Matrix Anal. & Appl. 33 (2012) 336–368.

[23] P. Rouchon, M. Fliess, J. Lévine, Kronecker's canonical forms for nonlinear implicit differential systems, in: System Structure and Control, 1992, pp. 248–251.

[24] A. Kumar, P. Daoutidis, Control of Nonlinear Differential Algebraic Equation Systems with Applications to Chemical Processes, volume 397 of *Chapman and Hall/CRC Research Notes in Mathematics*, Chapman and Hall, Boca Raton, FL, 1999.

[25] A. Isidori, Nonlinear Control Systems, Communications and Control Engineering Series, 3rd ed., Springer-Verlag, Berlin, 1995.

[26] H. Nijmeijer, A. J. van der Schaft, Nonlinear Dynamical Control Systems, Springer-Verlag, Berlin-Heidelberg-New York, 1990.

[27] K.-T. Wong, The eigenvalue problem $\lambda Tx + Sx$, J. Diff. Eqns. 16 (1974) 270–280.

[28] T. Berger, T. Reis, Regularization of linear time-invariant differential–algebraic systems, Syst. Control Lett. 78 (2015) 40–46.

[29] R. Marino, W. Respondek, A. Van Der Schaft, Equivalence of nonlinear systems to input-output prime forms, SIAM J. Control Optim. 32 (1994) 387–407.

[30] A. S. Morse, Structural invariants of linear multivariable systems, SIAM J. Cont. 11 (1973) 446–465.

[31] Y. Chen, Geometric Analysis of Differential-Algebraic Equations and Control Systems: Linear, Nonlinear and Linearizable, Ph.D. thesis, Normandie Université, 2019.

9. Appendix

Remark 9.1. (i) The geometric reduction algorithm is a constructive application for Proposition 2.7, but with more assumptions. The **Assumption 1** is made to produce the full row rank matrices $\tilde{E}_k^1$ and the zero-level set $M_k = \left\{ z_{k-1} \in W_k \,|\, \tilde{F}_k^2(z_{k-1}) = 0 \right\}$. The **Assumption 2** assure that $M_k \cap U_k$ is a smooth embedded submanifold and makes it possible to use the components of $\tilde{F}_k^2$ with linearly independent differentials as a part of new local coordinates.

(ii) The integers r_k, n_k of the geometric reduction algorithm, satisfy, for each $k \geq 1$,

$$\begin{cases} l = r_0 \geq r_1 \geq ... \geq r_k \geq ... \geq 0, & n = n_0 \geq n_1 \geq ... \geq n_k \geq ... \geq 0, \\ n_{k-1} \geq r_k, & r_{k-1} - r_k \geq n_{k-1} - n_k. \end{cases}$$

(iii) Note that if Ξ is internally regular around x_p, then we can deduce $n \leq l$ from the dimensional relations in item(ii). Indeed, the internal regularity of Ξ implies, by Theorem 3.11, that $n^* = n_{k^*+1} = r_{k^*+1} = r^*$. Then by $n_{k-1} - n_k \leq r_{k-1} - r_k$, we have $n_{k^*} \leq r_{k^*}$. By an iterative argument, it follows that $n = n_0 \leq r_0 = l$.

Algorithm Geometric reduction algorithm for nonlinear DAEs

Initiatlization: Consider $\Xi_{l,n} = (E, F)$, fix $x_p \in X$ and let $U_0 \subseteq X$ be an open connected subset containing x_p. Set $z_0 = x$, $E_0(z_0) = E(x)$, $F_0(z_0) = F(x)$, $M_0^c = U_0$, $r_0 = l$, $n_0 = n$, and $\Xi_0 = (E_0, F_0)$. Below all sets U_k are open in X and W_k are open in M_{k-1}^c.

Step k: Suppose that we have defined at Step $k-1$: an open neighborhood $U_{k-1} \subseteq X$ of x_p, a smooth embedded connected submanifold M_{k-1}^c of U_{k-1} and a DAE $\Xi_{k-1} = (E_{k-1}, F_{k-1})$ given by smooth matrix-valued maps

$$E_{k-1} : M_{k-1}^c \to \mathbb{R}^{r_{k-1} \times n_{k-1}}, \quad F_{k-1} : M_{k-1}^c \to \mathbb{R}^{r_{k-1}},$$

whose arguments are denoted $z_{k-1} \in M_{k-1}^c$.

1: Rename the maps as $\tilde{E}_k = E_{k-1}$, $\tilde{F}_k = F_{k-1}$ and define $\tilde{\Xi}_k := (\tilde{E}_k, \tilde{F}_k)$.

Assumption 1: There exists an open neighborhood $U_k \subseteq U_{k-1} \subseteq X$ of x_p such that $\operatorname{rank} \tilde{E}_k(z_{k-1}) = \mathit{const.} = r_k$, $\forall z_{k-1} \in W_k = U_k \cap M_{k-1}^c$.

2: Find a smooth map $Q_k : W_k \to GL(r_{k-1}, \mathbb{R})$, such that $\tilde{E}_k^1$ of $Q_k \tilde{E}_k = \begin{bmatrix} \tilde{E}_k^1 \\ 0 \end{bmatrix}$ is of full row rank and denote $Q_k \tilde{F}_k = \begin{bmatrix} \tilde{F}_k^1 \\ \tilde{F}_k^2 \end{bmatrix}$, where $\tilde{E}_k^1 : W_k \to \mathbb{R}^{r_k \times n_{k-1}}$, $\tilde{F}_k^2 : W_k \to \mathbb{R}^{r_{k-1} - r_k}$ (so all the matrices depend on z_{k-1}).

3: Following (4), define $M_k = \left\{ z_{k-1} \in W_k \mid \tilde{F}_k^2(z_{k-1}) = 0 \right\}$.

Assumption 2: $x_p \in M_k$ and $\operatorname{rank} \mathrm{D}\tilde{F}_k^2(z_{k-1}) = \mathit{const.} = n_{k-1} - n_k$ for $z_{k-1} \in M_k \cap U_k$, by taking a smaller U_k (if necessary).

4: By Assumption 2, $M_k \cap U_k$ is a smooth embedded submanifold and by taking again a smaller U_k, we may assume that $M_k^c = M_k \cap U_k$ is connected and choose new coordinates $(z_k, \bar{z}_k) = \psi_k(z_{k-1})$ on W_k, where $\bar{z}_k = (\bar{\varphi}_k^1(z_{k-1}), ..., \bar{\varphi}_k^{n_{k-1}-n_k}(z_{k-1}))$, with $\mathrm{d}\bar{\varphi}_k^1(z_{k-1}), ..., \mathrm{d}\bar{\varphi}_k^{n_{k-1}-n_k}(z_{k-1})$ being all independent rows of $\mathrm{D}\tilde{F}_k^2(z_{k-1})$, and $z_k = (\varphi_k^1(z_{k-1}), ..., \varphi_k^{n_{k-1}}(z_{k-1}))$ are any complementary coordinates such that ψ_k is a local diffeomorphism.

5: Set $\hat{E}_k = Q_k \tilde{E}_k \left(\frac{\partial \bar{\varphi}_k}{\partial z_{k-1}} \right)^{-1}$, $\hat{F}_k = Q_k \tilde{F}_k$. By Definition 3.1, $\tilde{\Xi}_k \overset{ex}{\sim} \hat{\Xi}_k = (\hat{E}_k, \hat{F}_k)$ via Q_k and ψ_k, where

$$\hat{\Xi}_k : \begin{bmatrix} \hat{E}_k^1(z_k, \bar{z}_k) & \bar{E}_k^1(z_k, \bar{z}_k) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_k \\ \dot{\bar{z}}_k \end{bmatrix} = \begin{bmatrix} \hat{F}_k^1(z_k, \bar{z}_k) \\ \hat{F}_k^2(z_k, \bar{z}_k) \end{bmatrix} \tag{38}$$

with $\hat{E}_k^1 : W_k \to \mathbb{R}^{r_k \times n_k}$, $\hat{F}_k^1 \circ \psi_k = \tilde{F}_k^1$, $\hat{F}_k^2 \circ \psi_k = \tilde{F}_k^2$ and $[\hat{E}_k^1 \circ \psi_k \;\; \bar{E}_k^1 \circ \psi_k] = \tilde{E}_k^1 \left(\frac{\partial \psi_k}{\partial z_{k-1}} \right)^{-1}$.

6: Set $\bar{z}_k = 0$ to define the following reduced and restricted DAE on $M_k^c = \{ z_{k-1} \in W_k \mid \bar{z}_k = 0 \}$ by

$$\Xi_k : E_k(z_k)\dot{z}_k = F_k(z_k),$$

where $E_k(z_k) = \hat{E}_k^1(z_k, 0)$, $F_k(z_k) = \hat{F}_k^1(z_k, 0)$ are matrix-valued maps and $E_k : M_k^c \to \mathbb{R}^{r_k \times n_k}$, $F_k : M_k^c \to \mathbb{R}^{r_k}$.

Repeat: Step k for $k = 1, 2, 3, \ldots$, **until** $n_{k+1} = n_k$, set $k^* = k$.

Result: Set $n^* = n_{k^*} = n_{k^*+1}$, $r^* = r_{k^*+1}$, $M^* = M_{k^*+1}^c$, $U^* = U_{k^*+1}$, $z^* = z_{k^*+1} = z_{k^*}$ and $\Xi^* = (E^*, F^*)$ with $E^* = E_{k^*+1}$, $F^* = F_{k^*+1}$.