Feedback linearization of nonlinear differential-algebraic control systems

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Abstract In this paper, we study feedback linearization problems for nonlinear differential-algebraic control systems (DACSs). We consider two kinds of feedback equivalences, namely, external feedback equivalence, which is defined (locally) on the whole generalized state space, and internal feedback equivalence, which is defined on the locally maximal controlled invariant submanifold (i.e., on the set where solutions exist). Necessary and sufficient conditions are given for locally internal and the locally external feedback linearizability of DACSs with the help of a notion called explicitation with driving variables, which attaches a class of ordinary differential equation control systems (ODECSs) to a given DACS. We show that the feedback linearizability of a DACS is closely related to the involutivity of the linearizability distributions for the explicitation systems. Finally, we apply our results of feedback linearization of DACSs to an academical example and a constrained mechanical system.

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1 Introduction

Consider a nonlinear differential-algebraic control system (DACS) of the form

$$\Xi^u : E(x)\dot{x} = F(x) + G(x)u, \qquad (1)$$

where $x \in X$ is called the generalized state and $(x, \dot{x}) \in TX$, where TX is the tangent bundle of an open subset X in $\mathbb{R}^n$ (or, more general, of an n-dimensional smooth manifold X), and $u \in \mathbb{R}^m$ is the vector of inputs, and where $E : TX \to \mathbb{R}^l$, $F : X \to \mathbb{R}^l$ and $G : X \to \mathbb{R}^{l \times m}$ are smooth maps. The word "smooth" will always mean $\mathcal{C}^\infty$-smooth throughout the paper. We denote a DACS of the form (1) by $\Xi^u_{l,n,m} = (E, F, G)$ or, simply, Ξ^u. A linear DACS is of the form

$$\Delta^u : E\dot{x} = Hx + Lu, \qquad (2)$$

where $E, H \in \mathbb{R}^{l \times n}$ and $L \in \mathbb{R}^{l \times m}$. Denote a linear DACS by $\Delta^u_{l,n,m} = (E, H, L)$ or, simply, Δ^u. Linear DACSs have been studied for decades, there is a rich literature devoted to them (see, e.g., the surveys [25, 26] and textbook [14]). In the context of this paper, we will need results about canonical forms [27, 23, 10], controllability [4, 13, 15], and geometric subspaces [16, 29]. The motivation of studying linear and nonlinear DACSs is their frequent presence in mathematical models of practical systems as constrained mechanics [31], chemical processes [22], electrical circuits [34], etc.

The map E of a DACS (1) is not necessarily square (i.e., $l \neq n$) nor invertible. As a consequence, some free variables and constrained variables can be implicitly present in the generalized state x (and also some constrained control variables can exist in the input u). We have proposed two normal forms to distinguish the different roles of variables for nonlinear DACSs in [12].

It is noted that although the free variables of x may perform like an input, we will distinguish them from the real active control variables u. The control u can be changed physically and actively via some actuators while the free variables in x are states coming from unknown constrained forces (e.g., the friction force F_f in Example 5.2 below) or some redundancies of mathematical modeling (e.g., the Lagrange multipliers when modeling constrained mechanical systems [31]). In the case of $E(x) = I_n$, the DACS (1) becomes an ordinary differential equation control system (ODECS)

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \tag{3}$$

where $f = F$ and g_i, $1 \leq i \leq m$, being the columns of G, become vector fields on X. The feedback linearization problem for nonlinear ODECSs (i.e., when there exist a local change of coordinates in the state space and a feedback transformation such that the transformed system has a linear form in the new coordinates) has drawn the attention of researchers for decades (e.g. see survey papers [33,36] and books [28,19]). The solution of the feedback linearization problem of ODECSs was first given in Brockett's paper [5] and developed by Jakubczyk and Respondek [20], Su [35], Hunt et Su [18]. Compared to the ODECSs, fewer results on the linearization problems of DACSs can be found. Xiaoping [38] transformed a nonlinear DACS into a linear one by state space transformations, Kawaji [21] gave sufficient conditions for the feedback linearization of a special class of DACSs, Wang and Chen [37] considered a semi-explicit differential-algebraic equation (DAE) and linearized the differential part of the DAE. The linearization of semi-explicit DAEs under equivalence of different levels is studies in [8].

In the present paper, our purpose is to find when a given DACS of the form (1) is locally equivalent to a linear completely controllable one (see the definition of the complete controllability of linear DACSs in [4]). In particular, we will consider two kinds of equivalence relations, namely, the external feedback equivalence given in Definition 2.7 and the internal feedback equivalence given in Definition 2.8. Note that the words "external" and "internal", appearing throughout this paper, basically mean that we consider the DACS on an open neighborhood of the generalized state space X and on the *locally maximal controlled invariant submanifold M^** (see Definition 2.2), respectively. We have discussed in detail the differences and relations of the two equivalence relations for linear DAEs [10], and for semi-explicit DAEs [8]. We will use a notion called the *explicitation with driving variables* (see Definition 3.1, firstly proposed in [9] for linear DACSs) to connect nonlinear

DACSs with nonlinear ODECSs. Via the explicitation with driving variables, we can interpret the linearizability of a DACS under internal or external feedback equivalence as that of an explicitation system under system feedback equivalence (see Definition 3.3). A previous version of the results in the present paper can be consulted in Chapter 5 of the thesis [7]. In particular, compared to [7], the proof of Theorem 4.5 in the present paper is different since we use a new Lemma 4.6 to simply the proof, and we add an Example 5.2 to illustrate the internal feedback linearization of DACSs.

The paper is organized as follows: In Section 2, we define the external and the internal feedback equivalences and discuss their relations with solutions. In Section 3, we use the notion of explicitation with driving variables to connect DACSs with ODECSs. Necessary and sufficient conditions for both the external and the internal feedback linearization problems of DACSs are given in Section 4. We illustrate the results of Section 4 by the two examples in Section 5. The conclusions and perspectives of this paper are given in Section 6 and a technical proof is given in Appendix.

2 External and internal feedback equivalence

We use the following notations in the present paper: We denote by $T_x M \in \mathbb{R}^n$ the tangent space at $x \in M$ of a differentiable submanifold M of $\mathbb{R}^n$. We use $\mathcal{C}^k$ to denote the class of k-times differentiable functions and $GL(n, \mathbb{R})$ to denote the group of nonsingular matrices of $\mathbb{R}^{n \times n}$. For a smooth map $f : X \to \mathbb{R}$, we denote its differential by $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$. For a map $A : X \to \mathbb{R}^{m \times n}$, $\ker A(x)$, $\text{Im } A(x)$ and $\text{rank } A(x)$ are the kernel, the image and the rank of A at x, respectively. For a full row rank map $R : X \to \mathbb{R}^{r \times n}$, we denote by $R^\dagger : X \to \mathbb{R}^{n \times r}$ the right inverse of R, i.e., $RR^\dagger = I_r$. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. We assume the reader is familiar with basic notions of differential geometry such as smooth embedded submanifolds, involutive distributions and refer the reader e.g. to the book [24] for the formal definitions of such notions.

Definition 2.1 (solutions and admissible set). For a DACS $\Xi_{l,n,m}^u = (E, F, G)$, a curve $(x, u) : I \to X \times \mathbb{R}^m$ defined on an open interval $I \subseteq \mathbb{R}$ with $x(\cdot) \in \mathcal{C}^1$ and $u(\cdot) \in \mathcal{C}^0$, is called a solution of Ξ^u if for all $t \in I$, $E(x(t))\dot{x}(t) = F(x(t)) + G(x(t))u(t)$. We call a point $x_a \in X$ *admissible* if there exists at least one solution $(x(\cdot), u(\cdot))$ such that $x(t_a) = x_a$ for a certain $t_a \in I$. The set of all admissible points will be called the admissible set (or the consistency set) of Ξ^u and denoted by S_a.

A smooth connected embedded submanifold M is called *controlled invariant* if for any point $x_0 \in M$, there exists a solution $(x, u) : I \to M \times \mathbb{R}^m$ such that $x(t_0) = x_0$ for a certain $t_0 \in I$ and $x(t) \in M$, $\forall\, t \in I$. Fix an admissible point $x_a \in X$, a smooth connected embedded submanifold M containing x_a is called *locally controlled invariant* if there exists a neighborhood U of x_a such that $M \cap U$ is controlled invariant.

Definition 2.2 (locally maximal controlled invariant submanifold)**.** A locally controlled invariant submanifold M^*, around an admissible point x_a, is called *maximal* if there exists a neighborhood U of x_a such that for any other locally controlled invariant submanifold M, we have $M \cap U \subseteq M^* \cap U$.

The locally maximal controlled invariant submanifold M^* of a DACS can be constructed via the following *geometric reduction method*, which was frequently used (see e.g., [32, 30, 34, 2, 12]) for studying existence of solutions for DAEs and DACSs.

Definition 2.3 (geometric reduction method [12])**.** For a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point $x_p \in X$. Let U_0 be a connected subset of X containing x_p. Step 0: Set $M_0 = X$ and $M_0^c = U_0$. Step k ($k > 0$): Suppose that a sequence of smooth connected embedded submanifolds $M_{k-1}^c \subsetneq \cdots \subsetneq M_0^c$ of U_{k-1} for a certain $k-1$, have been constructed. Define recursively

$$M_k := \left\{ x \in M_{k-1}^c \,|\, F(x) \in E(x) T_x M_{k-1}^c + \operatorname{Im} G(x) \right\}.$$

As long as $x_p \in M_k$, let $M_k^c = M_k \cap U_k$ be a smooth embedded connected submanifold for some neighborhood $U_k \subseteq U_{k-1}$ of x_p.

Proposition 2.4 ([12])**.** *In the above geometric reduction method, there always exists a smallest k^* such that either k^* is the smallest integer for which $x_p \notin M_{k^*+1}$ or k^* is the smallest integer such that $x_p \in M_{k^*+1}^c$ and $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$. In the latter case, denote $M^* = M_{k^*+1}^c$ and assume that there exists an open neighborhood $U^* \subseteq U_{k^*+1}$ of x_p such that $\dim E(x) T_x M^* = const.$ and $E(x) T_x M^* + \operatorname{Im} G(x) = const.$ for all $x \in M^* \cap U^*$, then*

(i) x_p is an admissible point, i.e., $x_p = x_a$ and M^* is the locally maximal controlled invariant submanifold around x_p;

(ii) M^* coincides locally with the admissible set S_a, i.e., $M^* \cap U^* = S_a \cap U^*$.

By item (ii) of Proposition 2.4, the admissible set S_a locally coincides with M^* on the neighborhood U^* of x_p. So any point $x_0 \in U^* \backslash M^*$ is not admissible and there exist no solutions passing through x_0. Thus to study solutions of a DACS, it is convenient to consider only the restriction of the DACS to its locally

maximal controlled invariant submanifold M^*, which we will define as follows (see also Remark 3.4(iv) and Theorem 4.4(i) of [12]).

Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix an admissible point $x_a \in X$. Let M^* be the n^*-dimensional maximal controlled invariant submanifold of Ξ^u around x_a. Assume that there exists a neighborhood U of x_a such that for all $x \in M^* \cap U$,

(CR) $\dim E(x) T_x M^* = const. = r^*$ and $E(x) T_x M^* + \operatorname{Im} G(x) = const. = r^* + (m - m^*)$.

Let $\psi : U \to \mathbb{R}^n$ be a local diffeomorphism and $z = \psi(x) = (z_1, z_2)$ be local coordinates on U such that $M^* \cap U = \{z_2 = 0\}$, thus z_1 are local coordinates on $M^* \cap U$. Then in the new z-coordinates, the DACS Ξ^u becomes a system $\tilde{\Xi}^u_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$, given by

$$\left[\, \tilde{E}_1(z_1, z_2) \;\; \tilde{E}_2(z_1, z_2) \,\right] \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \tilde{F}(z_1, z_2) + \tilde{G}(z_1, z_2) u,$$

where $\tilde{E}_1 : U \to \mathbb{R}^{l \times n^*}$, $\tilde{E}_2 : U \to \mathbb{R}^{l \times (n - n^*)}$, $\tilde{E} \circ \psi = \left[\, \tilde{E}_1 \circ \psi \;\; \tilde{E}_2 \circ \psi \,\right] = E \cdot \left(\frac{\partial \psi}{\partial x} \right)^{-1}$, $\tilde{F} \circ \psi = F$ and $\tilde{G} \circ \psi = G$. Set $z_2 = 0$ to have the following system (defined on M^*)

$$\left[\, \tilde{E}_1(z_1, 0) \;\; \tilde{E}_2(z_1, 0) \,\right] \begin{bmatrix} \dot{z}_1 \\ 0 \end{bmatrix} = \tilde{F}(z_1, 0) + \tilde{G}(z_1, 0) u. \tag{4}$$

By assumption **(CR)**, there exist a neighborhood $U_1 \subseteq U$ of x_a and $Q : M^* \cap U_1 \to GL(l, \mathbb{R})$ such that $\tilde{E}_1^1(z_1)$ and $\tilde{G}_2(z_1)$ below are of full row rank,

$$Q(z_1) \left[\, \tilde{E}_1(z_1, 0) \;\; \tilde{F}(z_1, 0) \;\; \tilde{G}(z_1, 0) \,\right] = \begin{bmatrix} \tilde{E}_1^1(z_1) & \tilde{F}_1(z_1) & \tilde{G}_1(z_1) \\ 0 & \tilde{F}_2(z_1) & \tilde{G}_2(z_1) \\ 0 & \tilde{F}_3(z_1) & 0 \end{bmatrix},$$

where $\tilde{E}_1^1$, $\tilde{G}_2$ are smooth functions defined on $M^* \cap U_1$ with values in $\mathbb{R}^{r^* \times n^*}$ and $\mathbb{R}^{(m - m^*) \times m}$, respectively, and $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$ and $\tilde{G}_1$ are matrix-valued functions of appropriate sizes. Since $\tilde{G}_2(z_1)$ is of full row rank, we can always assume $\begin{bmatrix} \tilde{G}_1(z_1) \\ \tilde{G}_2(z_1) \end{bmatrix} = \begin{bmatrix} \tilde{G}_1^1(z_1) & \tilde{G}_1^2(z_1) \\ \tilde{G}_2^1(z_1) & \tilde{G}_2^2(z_1) \end{bmatrix}$ with $\tilde{G}_2^2 : M^* \cap U_1 \to GL(m - m^*, \mathbb{R})$ (if not, we permute the components of u such that $\tilde{G}_2^2(z_1)$ is invertible), where $\tilde{G}_1^1$, $\tilde{G}_1^2$ and $\tilde{G}_2^1$ are of appropriate sizes. Thus, via Q and the following feedback transformation (note that a^u, b^u are defined on M^* and $b^u(z_1)$ is invertible),

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a^u(z_1) + b^u(z_1) u = \begin{bmatrix} 0 \\ \tilde{F}_2(z_1) \end{bmatrix} + \begin{bmatrix} I_{m^*} & 0 \\ \tilde{G}_2^1(z_1) & \tilde{G}_2^2(z_1) \end{bmatrix} u,$$

the DACS (4) is transformed into

$$\begin{bmatrix} \bar{E}_1^1(z_1) \\ 0 \\ 0 \end{bmatrix} \dot{z}_1 = \begin{bmatrix} \bar{F}_1(z_1) \\ 0 \\ \bar{F}_3(z_1) \end{bmatrix} + \begin{bmatrix} \bar{G}_1^1(z_1) & \bar{G}_1^2(z_1) \\ 0 & I_{m - m^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{5}$$

where $\bar{E}_1^1 = \tilde{E}_1^1$, $\bar{F}_3 = \tilde{F}_3$, $\bar{F}_1 = \tilde{F}_1 - \tilde{G}_1^2(\tilde{G}_2^2)^{-1}\tilde{F}_2$, $\bar{G}_1^1 = \tilde{G}_1^1 - \tilde{G}_1^2(\tilde{G}_2^2)^{-1}\tilde{G}_2^1$ and $\bar{G}_1^2 = \tilde{G}_1^2(\tilde{G}_2^2)^{-1}$.

Definition 2.5 (restriction)**.** The local M^*-restriction of Ξ^u, denoted by $\Xi^u|_{M^*}$, is given by

$$\Xi^u|_{M^*} = \Xi^{u^*} : E^*(z^*)\dot{z}^* = F^*(z^*) + G^*(z^*)u^*. \qquad (6)$$

where $z^* = z_1$, $u^* = u_1$, $E^* = \bar{E}_1^1 : M^* \to \mathbb{R}^{r^* \times n^*}$, $F^* = \bar{F}_1 : M^* \to \mathbb{R}^{r^*}$ and $G^* = \bar{G}_1^1 : M^* \to \mathbb{R}^{r^* \times m^*}$ come from (5), and where the map E^* is of full row rank r^*.

Remark 2.6. The restriction $\Xi^u|_{M^*}$ is a DACS of the form (1) with associated dimensions r^*, n^*, m^*, i.e., $\Xi^u|_{M^*} = \Xi^{u^*}_{r^*, n^*, m^*}$. It is important to know that Ξ^u and $\Xi^u|_{M^*}$ has isomorphic solutions (see Theorem 4.4(i) of [12]). More specifically, a curve $(x(\cdot), u(\cdot))$ is a solution of Ξ^u passing through a point $x_0 \in X$ if and only if $(z^*(\cdot), u^*(\cdot))$ is a solution of $\Xi^u|_{M^*}$ passing through $z_0^* \in M^*$, where $(z^*(\cdot), 0) = \psi(x(\cdot))$, $(z_0^*, 0) = \psi(x_0)$ and $(u^*(\cdot), 0) = a^u(z^*(\cdot)) + b^u(z^*(\cdot))u(\cdot)$.

Now we define the external and the internal feedback equivalences for nonlinear DACSs and compare them by discussing their relations with solutions.

Definition 2.7 (external feedback equivalence)**.** Two DACSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively, are called externally feedback equivalent, shortly ex-fb-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and smooth functions $Q : X \to GL(l, \mathbb{R})$, $\alpha^u : X \to \mathbb{R}^m$, $\beta^u : X \to GL(m, \mathbb{R})$ such that

$$\begin{aligned}
\tilde{E}(\psi(x)) &= Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}, \\
\tilde{F}(\psi(x)) &= Q(x)\left(F(x) + G(x)\alpha^u(x)\right), \\
\tilde{G}(\psi(x)) &= Q(x)G(x)\beta^u(x).
\end{aligned} \qquad (7)$$

The ex-fb-equivalence of two DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ is denoted by $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of a point x_p and $\tilde{U}$ of a point $\tilde{x}_p = \psi(x_p)$, and $Q(x)$, $\alpha^u(x)$, $\beta^u(x)$ are defined on U, we will talk about local ex-fb-equivalence.

Definition 2.8 (internal feedback equivalence)**.** Consider two DACSs $\Xi^u = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively. Fix two admissible points $x_a \in X$ and $\tilde{x}_a \in \tilde{X}$. Assume that

(A1) M^* and $\tilde{M}^*$ are locally maximal controlled invariant submanifolds of Ξ^u around x_a and of $\tilde{\Xi}^{\tilde{u}}$ around $\tilde{x}_a$, respectively.
(A2) M^* and $\tilde{M}^*$ satisfy the constant rank condition **(CR)** around x_a and $\tilde{x}_a$, respectively.

Then, Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are called internally feedback equivalent, shortly in-fb-equivalent, if their restrictions $\Xi^u|_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ are ex-fb-equivalent. We will denote the in-fb-equivalence of two DACSs by $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$.

Remark 2.9. The dimensions of two in-fb-equivalent DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are not necessarily the same. However, since $\Xi^u|_{M^*} = \Xi^{u^*}_{l^*, n^*, m^*}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*} = \tilde{\Xi}^{\tilde{u}^*}_{\tilde{l}^*, \tilde{n}^*, \tilde{m}^*}$ are required to be external feedback equivalent, their dimensions have to be the same, i.e., $r^* = \tilde{r}^*$, $n^* = \tilde{n}^*$ and $m^* = \tilde{m}^*$.

Both the ex-fb-equivalence and the in-fb-equivalence preserve solutions of DACSs. Indeed, consider two ex-fb-equivalent DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$, the diffeomorphism $\tilde{x} = \psi(x)$ and the feedback transformation $u = \alpha^u(x) + \beta^u(x)\tilde{u}$ (defined on X) establish a one to one correspondence between solutions (x, u) of Ξ^u and solutions $(\tilde{x}, \tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$, i.e., $\tilde{x} = \psi(x)$ and $u = \alpha^u(x) + \beta^u(x)\tilde{u}$. For two in-fb-equivalent DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$, by $\Xi^u|_{M^*} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$, there exist a diffeomorphism $\tilde{z}^* = \psi^*(z^*)$ between M^* and $\tilde{M}^*$, and a feedback transformation $u^* = \alpha^{u^*}(z^*) + \beta^{u^*}(z^*)\tilde{u}^*$ defined on M^* mapping solutions (z^*, u^*) of $\Xi^u|_{M^*}$ into solutions $(\tilde{z}^*, \tilde{u}^*)$ of $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$. Recall from Remark 2.6 that the DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ have isomorphic solutions with their restrictions $\Xi^u|_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}$, respectively. So solutions (x, u) of Ξ^u are also in a one-to-one correspondence with solutions $(\tilde{x}, \tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$ if $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$.

Conversely, if solutions of two DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are in a one-to-one correspondence via a diffeomorphism and a feedback transformation, then the two DACSs are in-fb-equivalent, however, they are *not* necessarily ex-fb-equivalence. The reason is that solutions of DACSs exist on maximal controlled invariant submanifolds only, by assuming two DACSs have corresponding solutions, we only have the information that the two restrictions $\Xi^u|_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ can be transformed into each other via a Q-transformation and a feedback transformation defined on M^*, together with a diffeomorphism between M^* and $\tilde{M}^*$, we do not know, however, if those transformations can be extended outside the submanifolds M^* and $\tilde{M}^*$.

Example 2.10. Consider two DACSs $\Xi^u_{3,3,1} = (E, F, G)$ defined on $X = \mathbb{R}^3$ and $\tilde{\Xi}^{\tilde{u}}_{3,3,1} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on $\tilde{X} = \mathbb{R}^3$, where

$$E(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F(x) = \begin{bmatrix} (x_1)^2 \\ e^{x_1}x_2 \\ x_3 \end{bmatrix}, \quad G(x) = \begin{bmatrix} e^{x_2} \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{E}(\tilde{x}) = \begin{bmatrix} 1 & x_2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tilde{F}(\tilde{x}) = \begin{bmatrix} \tilde{x}_2 \\ e^{\tilde{x}_1}\tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}, \quad \tilde{G}(\tilde{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

It is seen that $M^* = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = x_3 = 0\}$ and $\tilde{M}^* = \{(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{R}^3 \mid \tilde{x}_2 = \tilde{x}_3 = 0\}$. The restrictions $\Xi^u|_{M^*}$: $\dot{x}_1 = (x_1)^2 + u$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$: $\dot{\tilde{x}}_1 = \tilde{u}$ are ex-fb-equivalent via $Q(x_1) = 1$, $\tilde{x}_1 = \psi(x_1) = x_1$ and $\tilde{u} = (x_1)^2 + u$. Thus we have $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. It is clear that solutions $((x_1, 0, 0), u)$ of Ξ^u and solutions

$((\tilde{x}_1, 0, 0), \tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$ have a one-to-one correspondence. However, the two DACSs are *not* ex-fb-equivalent since rank $E(x) \neq$ rank $\tilde{E}(\tilde{x})$ (the matrix-valued functions $E(x)$ and $\tilde{E}(\tilde{x})$ of two ex-fb-equivalent DACSs should have the same rank).

Both the external and the internal feedback equivalences play an important role for DACSs. The internal feedback equivalence is convenient when we are only interested in solutions passing through an admissible point and evolving on M^*. The ex-fb-equivalence is useful when the initial point $x_0 \notin M^*$, i.e., x_0 is not admissible, then there are no solutions passing through x_0 but there may still exist a jump from the inadmissible point x_0 to an admissible one on M^*, see our recent publication [11], where we use external equivalence to study jump solutions of nonlinear DAEs.

3 Explicitation of nonlinear differential-algebraic control systems

We have proposed the notion of explicitation (with driving variables) for linear DACS in [9] (or see Chapter 3 of [7]), we now extend this notion to nonlinear DACSs.

Definition 3.1 (explicitation with driving variables). Given a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point $x_p \in X$. Assume that rank $E(x) = const. = r$ around x_p. Then locally there exists $Q : X \to GL(l, \mathbb{R})$ such that E_1 of $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ is of full row rank r, denote

$$Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \quad Q(x)G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix}.$$

Define locally the maps $f : X \to \mathbb{R}^n$, $g^u : X \to \mathbb{R}^{n \times m}$, $g^v : X \to \mathbb{R}^{n \times s}$, $h : X \to \mathbb{R}^p$, $l^u : X \to \mathbb{R}^{p \times m}$, where $s = n - r$ and $p = l - r$, such that

$$f(x) = E_1^\dagger(x)F_1(x), \quad g^u(x) = E_1^\dagger(x)G_1(x),$$
$$\text{Im } g^v(x) = \ker E_1(x), \quad h(x) = F_2(x), \quad l^u(x) = G_2(x),$$

where $E_1^\dagger$ is a right inverse of E_1. By a (Q, v)-explicitation, we will call any ODECS

$$\Sigma^{uv} : \begin{cases} \dot{x} = f(x) + g^u(x)u + g^v(x)v, \\ y = h(x) + l^u(x)u, \end{cases} \quad (8)$$

where $v \in \mathbb{R}^{s \times n}$ is called *the vector of driving variables*. System (8) is denoted by $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ or, simply, Σ^{uv}.

Apparently, in the above definition, the choices of the invertible map Q, the right inverse $E_1^\dagger$ and the map g^v satisfying Im $g^v = \ker E_1 = \ker E$, are not unique. The following proposition shows that a (Q, v)-explicitation of a given DACS Ξ^u is an ODECS defined up to a feedback transformation, an output multiplication and a generalized output injection, i.e., a

class of control systems. Throughout the class of all (Q, v)-explicitations of Ξ^u will be called the explicitation class. For a particular ODECS Σ^{uv} belonging to the explicitation class $\mathbf{Expl}(\Xi^u)$ of Ξ^u, we will write $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u)$.

Proposition 3.2. *Assume that an ODECS $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ is a (Q, v)-explicitation of a DACS $\Xi^u = (E, F, G)$ corresponding to the choice of invertible matrix $Q(x)$, right inverse $E_1^\dagger(x)$ and matrix $g^v(x)$. We have that an ODECS $\tilde{\Sigma}^{\tilde{u}, \tilde{v}}_{n,m,p} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^u)$ is a $(\tilde{Q}, \tilde{v})$-explicitation of Ξ^u corresponding to the choice of invertible matrix $\tilde{Q}(x)$, right inverse $\tilde{E}_1^\dagger(x)$ and matrix $\tilde{g}^{\tilde{v}}(x)$ if and only if Σ^{uv} and $\tilde{\Sigma}^{\tilde{u}, \tilde{v}}$ are equivalent via a v-feedback transformation of the form $v = \alpha^v(x) + \lambda(x)u + \beta^v(x)\tilde{v}$, a generalized output injection $\gamma(x)y = \gamma(x)(h(x) + l^u(x)u)$ and an output multiplication $\tilde{y} = \eta(x)y$, which map*

$$f \mapsto \tilde{f} = f + \gamma h + g^v \alpha^v, \quad g^u \mapsto \tilde{g}^u = g^u + \gamma l^u + g^v \lambda,$$
$$g^v \mapsto \tilde{g}^{\tilde{v}} = g^v \beta^v, \quad h \mapsto \tilde{h} = \eta h, \quad l^u \mapsto \tilde{l}^u = \eta l^u.$$

where $\alpha^v(x)$, $\beta^v(x)$, $\gamma(x)$, $\lambda(x)$, $\eta(x)$ are smooth matrix-valued functions, and $\beta^v(x)$ and $\eta(x)$ are invertible.

We omit the proof of Proposition 3.2 since it follows the same line as that of Proposition 2.3 in [9]. Now we will define an equivalence relation for two ODECSs of the form (8).

Definition 3.3 (system feedback equivalence). Two ODECSs $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}})$ defined on X and $\tilde{X}$, respectively, are called system feedback equivalent, or shortly sys-fb-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$, smooth functions $\alpha^u(x)$, $\alpha^v(x)$, $\lambda(x)$ and $\gamma(x)$ with values in $\mathbb{R}^m$, $\mathbb{R}^s$, $\mathbb{R}^{s \times m}$ and $\mathbb{R}^{n \times p}$, respectively, and invertible smooth matrix-valued functions $\beta^u(x)$, $\beta^v(x)$ and $\eta(x)$ with values in $GL(m, \mathbb{R})$, $GL(s, \mathbb{R})$ and $GL(p, \mathbb{R})$, respectively, such that

$$\begin{bmatrix} \tilde{f} \circ \psi & \tilde{g}^{\tilde{u}} \circ \psi & \tilde{g}^{\tilde{v}} \circ \psi \\ \tilde{h} \circ \psi & \tilde{l}^{\tilde{u}} \circ \psi & 0 \end{bmatrix} =$$
$$\begin{bmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial x}\gamma \\ 0 & \eta \end{bmatrix} \begin{bmatrix} f & g^u & g^v \\ h & l^u & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \alpha^u & \beta^u & 0 \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u & \beta^v \end{bmatrix}. \quad (9)$$

The sys-fb-equivalence of two control systems will be denoted by $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of a point x_p and $\tilde{U}$ of a point $\tilde{x}_p = \psi(x_p)$, and α^u, α^v, λ, γ, β^u, β^v, η are defined on U, we will speak about local sys-fb-equivalence.

The two ODECSs Σ^{uv} and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$ of Proposition 3.2 are, by definition, system feedback equivalent with ψ

being identity, $\alpha^u = 0$ and $\beta^u = I_m$. The following observation is crucial and will play an important role for studying the feedback linearization problems of DACSs in Section 4, which points out that the feedback transformations of explicitation systems of DACSs have a *triangular form* which are different from those of classical (ODE) control systems (see e.g., [28,19]).

Remark 3.4. Observe that, in (9), there are two kinds of feedback transformations. Namely,

$$u = \alpha^u(x) + \beta^u(x)\tilde{u} \quad \text{and} \quad v = \alpha^v(x) + \lambda(x)u + \beta^v(x)\tilde{v},$$

which can be written together as a feedback transformation of (u, v) with a (lower) *triangular form*:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha^u(x) \\ \alpha^v(x) \end{bmatrix} + \begin{bmatrix} \beta^u(x) & 0 \\ \lambda(x) & \beta^v(x) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}. \tag{10}$$

It implies that there are two kinds of inputs in the ODECSs of the form (8), one input (the driving variable v) is more "powerful" than the other input (the original control variable u), since when transforming v, we can use both u and x, but when transforming u, we are *not* allowed to use v. Another difference between u and v is that the input u is injected into the output y via $l^u u$, but the driving variable v is not directly injected into the output y.

The following theorem connects ex-fb-equivalence of two DACSs with sys-fb-equivalence of two ODECSs (explicitations).

Theorem 3.5. *Consider two DACSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively. Assume that $\operatorname{rank} E(x) = const. = r$ in a neighborhood U of a point $x_p \in X$ and $\operatorname{rank} \tilde{E}(\tilde{x}) = r$ in a neighborhood $\tilde{U}$ of a point $\tilde{x}_p \in \tilde{X}$. Then, given any ODECSs $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u) \in \mathbf{Expl}(\Xi^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}) \in \mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$, we have that locally $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ if and only if $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$.*

Proof. By the assumptions that $\operatorname{rank} E(x)$ and $\operatorname{rank} \tilde{E}(\tilde{x})$ are constant and equal to r around x_p and $\tilde{x}_p$, respectively, there exist invertible matrix-valued functions $Q : U \to GL(l, \mathbb{R})$ and $\tilde{Q} : \tilde{U} \to GL(l, \mathbb{R})$, defined on neighborhoods U of x_p and $\tilde{U}$ of $\tilde{x}_p$, respectively, such that $E'(x) = Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ and $\tilde{E}'(\tilde{x}) = \tilde{Q}(\tilde{x})\tilde{E}(\tilde{x}) = \begin{bmatrix} \tilde{E}_1(\tilde{x}) \\ 0 \end{bmatrix}$, where $E_1 : U \to R^{r \times n}$ and $\tilde{E}_1 : \tilde{U} \to R^{r \times n}$ are of full row rank. We have $\Xi^u \overset{ex-fb}{\sim} \Xi^{u'} = (E', F', G')$ and $\tilde{\Xi}^{\tilde{u}} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}'} = (\tilde{E}', \tilde{F}', \tilde{G}')$ via $Q(x)$ and $\tilde{Q}(\tilde{x})$, respectively, where

$$F'(x) = QF(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, G'(x) = QG(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix},$$
$$\tilde{F}'(\tilde{x}) = \tilde{Q}\tilde{F}(\tilde{x}) = \begin{bmatrix} \tilde{F}_1(\tilde{x}) \\ \tilde{F}_2(\tilde{x}) \end{bmatrix}, \tilde{G}'(\tilde{x}) = \tilde{Q}\tilde{G}(\tilde{x}) = \begin{bmatrix} \tilde{G}_1(\tilde{x}) \\ \tilde{G}_2(\tilde{x}) \end{bmatrix}.$$

In this proof, without loss of generality, we will assume that $\Xi^u = \Xi^{u'}$ and $\tilde{\Xi}^{\tilde{u}} = \tilde{\Xi}^{\tilde{u}'}$, since $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ if and only if $\Xi^{u'} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}'}$.

Moreover, choose maps f, g^u, g^v, h, l^u and $\tilde{f}$, $\tilde{g}^{\tilde{u}}$, $\tilde{g}^{\tilde{v}}$, $\tilde{h}$, $\tilde{l}^{\tilde{u}}$ such that

$$\begin{aligned} f(x) &= E_1^\dagger(x)F_1(x), & \tilde{f}(\tilde{x}) &= \tilde{E}_1^\dagger(\tilde{x})\tilde{F}_1(\tilde{x}), \\ g^u(x) &= E_1^\dagger(x)G_1(x) & \tilde{g}^{\tilde{u}}(\tilde{x}) &= \tilde{E}_1^\dagger(\tilde{x})\tilde{G}_1(\tilde{x}), \\ \operatorname{Im} g^v(x) &= \ker E_1(x), & \operatorname{Im} \tilde{g}^{\tilde{v}}(\tilde{x}) &= \ker \tilde{E}_1(\tilde{x}), \\ h(x) &= F_2(x), & \tilde{h}(\tilde{x}) &= \tilde{F}_2(\tilde{x}), \\ l^u(x) &= G_2(x), & \tilde{l}^{\tilde{u}}(\tilde{x}) &= \tilde{G}_2(\tilde{x}), \end{aligned} \tag{11}$$

where $E_1^\dagger(x)$ and $\tilde{E}_1^\dagger(\tilde{x})$ are right inverses of $E_1(x)$ and $\tilde{E}_1(\tilde{x})$, respectively. Then by Definition 3.1,

$$\Sigma^{uv} = (f, g^u, g^v, h, l^u) \in \mathbf{Expl}(\Xi^u),$$
$$\tilde{\Sigma}^{\tilde{u}\tilde{v}} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}) \in \mathbf{Expl}(\tilde{\Xi}^{\tilde{u}}).$$

It is seen from Proposition 3.2 that any control system in $\mathbf{Expl}(\Xi^u)$ is sys-fb-equivalent to Σ^{uv} and that any control system in $\mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$ is sys-fb-equivalent to $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$. Without loss of generality, in the remaining part of the proof, we use Σ^{uv} and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$ with system matrices given by (11) to represent two ODECSs in $\mathbf{Expl}(\Xi^u)$ and $\mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$, respectively. Throughout the proof below, we may drop the argument x for the functions $E(x)$, $F(x)$, $G(x)$, ..., for ease of notation.

If. Suppose that locally $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$. Then there exist a local diffeomorphism $\tilde{x} = \psi(x)$ and matrix-valued functions α^u, α^v, λ, γ, β^u, β^v, η defined on a neighborhood U of x_p such that the system matrices satisfy relations (9) of Definition 3.3.

First, consider $\tilde{g}^{\tilde{v}} \circ \psi = \frac{\partial \psi}{\partial x} g^v \beta^v$. By $\operatorname{Im} g^v = \ker E_1$, $\operatorname{Im} \tilde{g}^{\tilde{v}} = \ker \tilde{E}_1$, we have $\ker \tilde{E}_1 \circ \psi = \frac{\partial \psi}{\partial x} \ker E_1$. Thus there exists $Q_1 : U \to GL(r, \mathbb{R})$ such that

$$\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial \psi}{\partial x} \right)^{-1}. \tag{12}$$

Then, by (9), the following relation holds:

$$\begin{bmatrix} \tilde{f} \circ \psi & \tilde{g}^{\tilde{u}} \circ \psi \\ \tilde{h} \circ \psi & \tilde{l}^{\tilde{u}} \circ \psi \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial x}\gamma \\ 0 & \eta \end{bmatrix} \begin{bmatrix} f & g^u & g^v \\ h & l^u & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha^u & \beta^u \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u \end{bmatrix}.$$

Substituting (11) into the above equation, we get

$$\begin{bmatrix} \tilde{E}_1^\dagger \circ \psi \cdot \tilde{F}_1 \circ \psi & \tilde{E}_1^\dagger \circ \psi \cdot \tilde{G}_1 \circ \psi \\ \tilde{F}_2 \circ \psi & \tilde{G}_2 \circ \psi \end{bmatrix} =$$
$$\begin{bmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial x}\gamma \\ 0 & \eta \end{bmatrix} \begin{bmatrix} E_1^\dagger F_1 & E_1^\dagger G_1 & g^v \\ F_2 & G_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha^u & \beta^u \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u \end{bmatrix}.$$

Premultiply the above equation by

$$\begin{bmatrix} \tilde{E}_1 \circ \psi & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} Q_1 E_1 \left(\frac{\partial \psi}{\partial x} \right)^{-1} & 0 \\ 0 & I_p \end{bmatrix}$$

to get

$$\begin{bmatrix} \tilde{F}_1 \circ \psi & \tilde{G}_1 \circ \psi \\ \tilde{F}_2 \circ \psi & \tilde{G}_2 \circ \psi \end{bmatrix} = \begin{bmatrix} Q_1 & Q_1 E_1 \gamma \\ 0 & \eta \end{bmatrix} \begin{bmatrix} F_1 & G_1 \\ F_2 & G_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha^u & \beta^u \end{bmatrix}. \qquad (13)$$

Now from equations (12), (13) and Definition 2.7, it can be seen that $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ via the transformations defined by $\tilde{x} = \psi(x)$, $Q = \begin{bmatrix} Q_1 & Q_1 E_1 \gamma \\ 0 & \eta \end{bmatrix}$, α^u and β^u.

Only if. Suppose that $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ (in a neighborhood U of x_p). Assume that Ξ^u and $\tilde{\Xi}^u$ are ex-fb-equivalent via an invertible matrix-valued function $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$, $\tilde{x} = \psi(x)$, α^u, β^u, where $Q_1 : U \to \mathbb{R}^{r \times r}$ and Q_2, Q_3, Q_4 are matrix-valued functions of appropriate sizes. Then by

$$QE = \tilde{E} \circ \psi \frac{\partial \psi}{\partial x} \Rightarrow \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{E}_1 \circ \psi \\ 0 \end{bmatrix} \frac{\partial \psi}{\partial x},$$

we can deduce that

$$\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial \psi}{\partial x} \right)^{-1}. \qquad (14)$$

Moreover, we have $Q_3 = 0$ and Q_1 is invertible (since both E_1 and $\tilde{E}_1$ are of full row rank), which implies that Q_4 is invertible as well (since Q is invertible). Subsequently, by

$$\tilde{F} \circ \psi = Q(F + G\alpha^u) \Rightarrow$$
$$\begin{bmatrix} \tilde{F}_1 \circ \psi \\ \tilde{F}_2 \circ \psi \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \left(\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \alpha^u \right),$$

we have

$$\tilde{F}_1 \circ \psi = Q_1(F_1 + G_1\alpha^u) + Q_2(F_2 + G_2\alpha^u) \qquad (15)$$

and

$$\tilde{F}_2 \circ \psi = Q_4(F_2 + G_2\alpha^u). \qquad (16)$$

Moreover, by

$$\tilde{G} \circ \psi = QG\beta^u \Rightarrow \begin{bmatrix} \tilde{G}_1 \circ \psi \\ \tilde{G}_2 \circ \psi \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \beta^u,$$

we have

$$\tilde{G}_1 \circ \psi = Q_1 G_1 \beta^u + Q_2 G_2 \beta^u \qquad (17)$$

and

$$\tilde{G}_2 \circ \psi = Q_4 G_2 \beta^u. \qquad (18)$$

Recall the system matrices given in (11). First, from $\text{Im } g^v = \ker E_1$, $\text{Im } \tilde{g}^{\tilde{v}} \circ \psi = \ker \tilde{E}_1 \circ \psi$, and equation (14), it is seen that there exists $\beta^v : U \to GL(s, \mathbb{R})$ such that

$$\tilde{g}^{\tilde{v}} \circ \psi = \frac{\partial \psi}{\partial x} g^v \beta^v. \qquad (19)$$

Secondly, by equations (14) and (15), we have

$$\begin{aligned} \tilde{f} \circ \psi &= \tilde{E}_1^\dagger \circ \psi \tilde{F}_1 \circ \psi \\ &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} F_1 + G_1\alpha^u \\ F_2 + G_2\alpha^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} F_1 + G_1\alpha^u + E_1 g^v(\lambda\alpha^u + \alpha^v) \\ F_2 + G_2\alpha^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} (f + g^u\alpha^u + g^v(\lambda\alpha^u + \alpha^v) + \gamma(h + l^u\alpha^u)), \end{aligned}$$
$$(20)$$

where $\gamma = E_1^\dagger Q_1^{-1} Q_2$, and α^v and λ are matrix-valued functions of appropriate sizes. Thirdly, by equation (17), we have

$$\begin{aligned} \tilde{g}^{\tilde{u}} \circ \psi &= \tilde{E}_1^\dagger \circ \psi \tilde{G}_1 \circ \psi \\ &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} G_1 \beta^u \\ G_2 \beta^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} G_1 \beta^u + E_1 g^v \lambda \\ G_2 \beta^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} (g^u \beta^u + g^v \lambda + \gamma l^u \beta^u). \end{aligned}$$
$$(21)$$

Note that we use the equations $E_1 g^v(\lambda\alpha^u + \alpha^v) = 0$ and $E_1 g^v \lambda = 0$ to deduce (20) and (21). At last, by equations (16) and (18) we have

$$\tilde{h} \circ \psi = \tilde{F}_2 \circ \psi = Q_4(F_2 + G_2\alpha^u) = Q_4(h + l^u\alpha^u) \qquad (22)$$

and

$$\tilde{l}^{\tilde{u}} \circ \psi = \tilde{G}_2 \circ \psi = Q_4 G_2 \beta^u = Q_4 l^u \beta^u. \qquad (23)$$

Finally, it can be seen from (20), (21), (22) and (23), that $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$ via $\tilde{x} = \psi(x)$, α^v, β^v, α^u, β^u, λ, $\gamma = E_1^\dagger Q_1^{-1} Q_2$ and $\eta = Q_4$. $\qquad \square$

4 External and internal feedback linearization

In this section, we discuss the problem that when a nonlinear DACS of the form (1) is externally or internally feedback equivalent to a linear DACS of the form (2) with complete controllability. First, we review some definitions and criteria for the complete controllability of linear DACSs. We denote by $A^{-1}\mathscr{B}$, the preimage of a space $\mathscr{B}$ under a linear map A. The augmented Wong sequences (see e.g., [26,4,9]) of a linear DACS $\Delta_{l,n,m}^u = (E, H, L)$, given by (2), are

$$\mathscr{V}_0 := \mathbb{R}^n, \quad \mathscr{V}_{i+1} := H^{-1}(E\mathscr{V}_i + \text{Im } L), \quad i \geq 0; \qquad (24)$$

$$\mathscr{W}_0 := 0, \quad \mathscr{W}_{i+1} := E^{-1}(H\mathscr{W}_i + \text{Im } L), \quad i \geq 0. \qquad (25)$$

Additionally, recall the following sequence of subspaces (see e.g. [26]):

$$\hat{\mathscr{W}}_1 := \ker E, \quad \hat{\mathscr{W}}_{i+1} := E^{-1}(H\hat{\mathscr{W}}_i + \text{Im } L), \ i \geq 1. \quad (26)$$

For simplicity of notation, we denote

$$K_\beta = \mathrm{diag}\{K_{\beta_1}, \ldots, K_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k) \times |\beta|},$$
$$L_\beta = \mathrm{diag}\{L_{\beta_1}, \ldots, L_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k) \times |\beta|},$$
$$\mathcal{E}_\beta = \mathrm{diag}\{e_{\beta_1}, \ldots, e_{\beta_k}\} \in \mathbb{R}^{|\beta| \times k}$$
$$N_\beta = \mathrm{diag}\{N_{\beta_1}, \ldots, N_{\beta_k}\} \in \mathbb{R}^{|\beta| \times |\beta|},$$

where β is a multi-index $\beta = (\beta_1, \ldots, \beta_k)$ and $|\beta| = \sum_{i=1}^{k} \beta_i$, and where

$$K_{\beta_i} = [\, 0 \; I_{\beta_i - 1}\,] \in \mathbb{R}^{(\beta_i - 1) \times \beta_i}, \quad e_{\beta_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\beta_i},$$
$$L_{\beta_i} = [\, I_{\beta_i - 1} \; 0\,] \in \mathbb{R}^{(\beta_i - 1) \times \beta_i}, \quad N_{\beta_i} = \begin{bmatrix} 0 & 0 \\ I_{\beta_i - 1} & 0 \end{bmatrix} \in \mathbb{R}^{\beta_i \times \beta_i}.$$

Definition 2.7 applied to linear systems says that two linear DACSs $\Delta_{l,n,m}^u = (E, H, L)$ and $\tilde{\Delta}_{l,n,m}^{\tilde{u}} = (\tilde{E}, \tilde{H}, \tilde{L})$ are ex-fb-equivalent if there exist constant invertible matrices Q, P, S and a matrix R such that $\tilde{E} = QEP^{-1}$, $\tilde{H} = Q(H + LR)P^{-1}$, $\tilde{L} = QLS$.

Definition 4.1 (complete controllability in [4])**.** A linear DACS $\Delta_{l,n,m}^u = (E, H, L)$ is completely controllable if for any $x_0, x_1 \in \mathbb{R}^n$, there exist a solution (x, u) of Δ^u and $t \in \mathbb{R}^+$ such that $x(0) = x_0$ and $x(t) = x_1$.

Lemma 4.2. [4] *For a linear DACS $\Delta_{l,n,m}^u = (E, H, L)$, the following statements are equivalent:*

(i) *Δ^u is completely controllable.*
(ii) *$\mathrm{Im}\, E + \mathrm{Im}\, H + \mathrm{Im}\, L = \mathrm{Im}\, E + \mathrm{Im}\, L$ and $\mathrm{Im}_{\mathbb{C}} E + \mathrm{Im}_{\mathbb{C}} H + \mathrm{Im}_{\mathbb{C}} L = \mathrm{Im}_{\mathbb{C}}(\lambda E - H) + \mathrm{Im}_{\mathbb{C}} L, \, \forall \lambda \in \mathbb{C}.$*
(iii) *$\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$, where $\mathscr{V}^*$ and $\mathscr{W}^*$ are the limits of the augmented Wong sequences (24) and (25), respectively;*
(iv) *Δ^u is ex-fb-equivalent (under linear transformations) to*

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho & 0 \\ 0 & 0 \\ 0 & I_{m-m^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $\rho = (\rho_1, \ldots, \rho_{m^*})$ and $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_{s^*})$ are multi-indices, and $s^* = n - \mathrm{rank}\, E.$

We define (locally) internal and (locally) external feedback linearizability of nonlinear DACSs as follows.

Definition 4.3. Consider a DACS $\Xi_{l,n,m}^u = (E, F, G)$ and fix an admissible point $x_a \in X$. Then Ξ^u is called locally internally (resp. externally) feedback linearizable around x_a if Ξ^u is locally in-fb-equivalent (resp. ex-fb-equivalent) to a linear DACS with complete controllability around x_a.

We consider an ODECS $\Sigma_{n,m,s,p}^{uv} = (f, g^u, g^v, h, l^u)$, given by (8). If Σ^{uv} has no outputs, we denote it by

$\Sigma_{n,m,s}^{uv} = (f, g^u, g^v)$. Then for $\Sigma_{n,m,s}^{uv} = (f, g^u, g^v)$, define the following two sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, called the *linearizability distributions* of Σ^{uv},

$$\begin{cases} \mathcal{D}_0 & := \{0\}, \\ \mathcal{D}_1 & := \mathrm{span}\,\{g_1^u, \ldots, g_m^u, g_1^v, \ldots, g_s^v\}, \\ \mathcal{D}_{i+1} & := \mathcal{D}_i + [f, \mathcal{D}_i], \quad i = 1, 2, \ldots, \end{cases} \tag{27}$$

$$\begin{cases} \hat{\mathcal{D}}_1 & := \mathrm{span}\,\{g_1^v, \ldots, g_s^v\}, \\ \hat{\mathcal{D}}_{i+1} & := \mathcal{D}_i + [f, \hat{\mathcal{D}}_i], \quad i = 1, 2, \ldots. \end{cases} \tag{28}$$

Remark 4.4. Consider a linear DACS $\Delta^u = (E, H, L)$, denote $\mathscr{W}_i(\Delta^u)$ and $\hat{\mathscr{W}}_i(\Delta^u)$ as the subspaces $\mathscr{W}_i$, given by (25), and $\hat{\mathscr{W}}_i$, given by (26), of Δ^u, respectively. For a linear ODECS $\Lambda^{uv} = (A, B^u, B^v, C, D^u)$ (of the form (8) but with constant system matrices), define the following two sequences of subspaces

$$\mathcal{W}_0 := \{0\},$$
$$\mathcal{W}_{i+1} := [\, A \; B^w\,]\left(\begin{bmatrix} \mathcal{W}_i \\ \mathscr{I} \end{bmatrix} \cap \ker [\, C \; D^w\,]\right), \; i \geq 0,$$

and

$$\hat{\mathcal{W}}_1 := \mathrm{Im}\, B^v,$$
$$\hat{\mathcal{W}}_{i+1} := [\, A \; B^w\,]\left(\begin{bmatrix} \hat{\mathcal{W}}_i \\ \mathscr{I} \end{bmatrix} \cap \ker [\, C \; D^w\,]\right), \; i \geq 1,$$

where $w = (u, v)$, $B^w = [B^u, B^v]$ and $D^w = [D^u, 0]$. We have proved in Proposition 2.10 of [9] that if $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$, then

$$\mathscr{W}_i(\Delta^u) = \mathcal{W}_i(\Lambda^{uv}), \; \forall i \geq 0,$$
$$\hat{\mathscr{W}}_i(\Delta^u) = \hat{\mathcal{W}}_i(\Lambda^{uv}), \; \forall i \geq 1.$$

Apparently, $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ are linear counterparts of $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, respectively, but they are for linear systems with outputs.

Theorem 4.5 (internal feedback linearization)**.** *Consider a DACS $\Xi_{l,n,m}^u = (E, F, G)$, fix an admissible point $x_a \in X$. Let M^* be the n^*-dimensional locally maximal controlled invariant submanifold of Ξ^u around x_a. Assume that the constant rank assumption (**CR**) is satisfied for $x \in M^*$ around x_a. Then $\Xi^u|_{M^*}$ is a DACS $\Xi_{r^*,n^*,m^*}^{u^*} = (E^*, F^*, G^*)$ of the form (6) and its explicitation $\mathbf{Expl}(\Xi^u|_{M^*})$ is a class of ODECSs without outputs. The DACS Ξ^u is locally internally feedback linearizable if and only if for one (and thus any) ODECS $\Sigma^{u^* v^*} = (f^*, g^{u^*}, g^{v^*}) \in \mathbf{Expl}(\Xi^u|_{M^*})$, the linearizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ of $\Sigma^{u^* v^*}$ satisfy the following conditions on M^* around x_a:*

(FL1) *$\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are of constant rank for $1 \leq i \leq n^*$.*
(FL2) *$\mathcal{D}_{n^*} = \hat{\mathcal{D}}_{n^*} = TM^*$.*
(FL3) *$\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are involutive for $1 \leq i \leq n^* - 1$.*

Proof. Since Ξ^u satisfies condition **(CR)** around x_a, its M^*-restriction $\Xi^u|_{M^*}$ by Definition 2.5 is a DACS $\Xi^u|_{M^*} = \Xi^{u^*}_{r^*,n^*,m^*} = (E^*, F^*, G^*)$ of the form (6) with E^* being of full row rank r^*. It follows by the full row rankness of E^* that the maps $h = F_2$ and $l^{u^*} = G_2$ are absent in the explicitation systems of Ξ^{u^*}, which means that the output $y = h(x) + l^{u^*}(x)u^*$ is absent as well (see Definition 3.1). Thus an ODECS $\Sigma^{u^* v^*}_{n^*,m^*,s^*} = (f^*, g^{u^*}, g^{v^*}) \in \mathbf{Expl}(\Xi^u|_{M^*})$ is a control system without outputs, which is in the form

$$\Sigma^{w^*} : \dot{z}^* = f^*(z^*) + g^{u^*}(z^*)u^* + g^{v^*}(z^*)v^*,$$

where $w^* = (u^*, v^*)$, $f^* = (E^*)^\dagger F^*$, $g^{u^*} = (E^*)^\dagger G^*$, $\mathrm{Im}\, g^{v^*} = \ker E^*$ and $s^* = n^* - r^*$.

Only if. Suppose that Ξ^u is locally internally feedback linearizable, which means that its M^*-restriction $\Xi^u|_{M^*}$, given by (6), is locally ex-fb-equivalent to a completely controllable linear DACS

$$\Delta^{\tilde{u}^*} : E^* \dot{\tilde{z}}^* = H^* \tilde{z}^* + L^* \tilde{u}^*,$$

where E^*, H^*, L^* are constant matrices of appropriate sizes. Then a linear ODECS $\Lambda^{\tilde{w}^*} = (A^*, B^{\tilde{u}^*}, B^{\tilde{v}^*}) \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$, where $\tilde{w}^* = (\tilde{u}^*, \tilde{v}^*)$, is of the form

$$\Lambda^{\tilde{w}^*} : \dot{\tilde{z}}^* = A^* \tilde{z}^* + B^{\tilde{u}^*} \tilde{u}^* + B^{\tilde{v}^*} \tilde{v}^*.$$

where $A^* = (E^*)^\dagger H^*$, $B^{\tilde{u}^*} = (E^*)^\dagger L^*$ and $\mathrm{Im}\, B^{\tilde{v}^*} = \ker E^*$. By Lemma 4.2, the complete controllability of $\Delta^{\tilde{u}^*}$ implies $\mathscr{W}_{n^*}(\Delta^{\tilde{u}^*}) = \mathscr{W}_{n^*}(\Delta^{\tilde{u}^*}) = \mathbb{R}^{n^*}$. By Proposition 2.10 of [9] (see also Remark 4.4(ii)), we get

$$\hat{\mathcal{W}}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{W}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathscr{W}_{n^*}(\Delta^{\tilde{u}^*}) = \mathscr{W}_{n^*}(\Delta^{\tilde{u}^*}) = \mathbb{R}^{n^*}.$$

Since $\Lambda^{\tilde{w}^*}$ is a linear control system without outputs, we have $\hat{\mathcal{D}}_{n^*}(\Lambda^{\tilde{w}^*}) = \hat{\mathcal{W}}_{n^*}(\Lambda^{\tilde{w}^*})$, $\mathcal{D}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{W}_{n^*}(\Lambda^{\tilde{w}^*})$. Hence, $\hat{\mathcal{D}}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{D}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathbb{R}^{n^*}$. Thus $\Lambda^{\tilde{w}^*}$ satisfies (FL2). Moreover, since $\Lambda^{\tilde{w}^*}$ is a linear control system, it satisfies (FL1) and (FL3) in an obvious way. Notice that the nonlinear system Σ^{w^*} is locally sys-fb-equivalent to $\Lambda^{\tilde{w}^*}$ by Theorem 3.5 because $\Sigma^{w^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$, $\Delta^{\tilde{w}^*} \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$ and $\Xi^u|_{M^*} \overset{ex-fb}{\sim} \Delta^{\tilde{u}^*}$. Since Σ^{w^*} and $\Lambda^{\tilde{w}^*}$ are control systems without outputs, sys-fb-equivalence reduces to feedback equivalence. Thus Σ^{w^*} and $\Lambda^{\tilde{w}^*}$ are locally feedback equivalent (via $\tilde{z}^* = \psi(z^*)$ and two kinds of feedback transformations defined by $\alpha^{u^*}, \alpha^{v^*}, \beta^{u^*}, \beta^{v^*}, \lambda$, see Remark 3.4). It is easy to verify by a direct calculation that if $\hat{\mathcal{D}}_i$ and $\mathcal{D}_i$ are involutive, then the two distribution sequences are invariant for the two feedback equivalent control systems Σ^{w^*} and $\Lambda^{\tilde{w}^*}$, i.e., $\frac{\partial \psi}{\partial z^*}\hat{\mathcal{D}}_i(\Sigma^{w^*}) = \hat{\mathcal{D}}_i(\Delta^{\tilde{w}^*}) \circ \psi$ and $\frac{\partial \psi}{\partial z^*}\mathcal{D}_i(\Sigma^{w^*}) = \mathcal{D}_i(\Delta^{\tilde{w}^*}) \circ \psi$. So the system Σ^{w^*} being feedback equivalent to $\Lambda^{\tilde{w}^*}$ satisfies conditions (FL1)-(FL3) as well. It is seen from Proposition 3.2 that any

other ODECS $\hat{\Sigma}^{\hat{w}^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$ is sys-fb-equivalent to Σ^{w^*}, which means Σ^{w^*} is feedback equivalent (via two kinds of feedback transformations) to $\hat{\Sigma}^{\hat{w}^*}$ as any explicitation system in $\mathbf{Expl}(\Xi^u|_{M^*})$ has no outputs. So any other explicitation system $\hat{\Sigma}^{\hat{w}^*}$ satisfies (FL1)-(FL3) of Theorem 4.5 as well.

If. Suppose that an ODECS $\Sigma^{u^* v^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$ satisfies (FL1)-(FL3) around x_a. Then the following lemma holds.

Lemma 4.6. *The ODECS*

$$\Sigma^{w^*} = \Sigma^{u^* v^*}_{n^*,m^*,s^*} = (f^*, g^{u^*}, g^{v^*})$$

is locally feedback equivalent, via two kinds of feedback transformations (see Remark 3.4), to the Brunovský canonical form [6] around x_a, which is given by

$$\Sigma^{\tilde{w}^*}_{Br} = \Sigma^{\tilde{u}^* \tilde{v}^*}_{Br} : \begin{cases} \dot{\xi}_1 = N_\rho^T \xi_1 + \mathcal{E}_\rho \tilde{u}^*, \\ \dot{\xi}_2 = N_{\bar{\rho}}^T \xi_2 + \mathcal{E}_{\bar{\rho}} \tilde{v}^*, \end{cases} \quad (29)$$

where $\tilde{w}^* = (\tilde{u}^*, \tilde{v}^*)$, and $\rho = (\rho_1, \ldots, \rho_a)$ and $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_b)$ are multi-indices.

The proof of Lemma 4.6 is technical and is put into Appendix. Now we will prove that the M^*-restriction $\Xi^u|_{M^*}$, given by (6), is locally ex-fb-equivalent to a linear DACS

$$\Delta^{\tilde{u}^*} : \begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho \\ 0 \end{bmatrix} \tilde{u}^*. \quad (30)$$

Notice that by Lemma 4.2, the linear DACS $\Delta^{\tilde{u}^*}$ is completely controllable. Observe that $\Sigma^{\tilde{w}^*}_{Br} \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$, because the ξ_1-subsystems of $\Sigma^{\tilde{w}^*}_{Br}$ and $\Delta^{\tilde{u}^*}$ coincide, $N_{\bar{\rho}}^T = L_{\bar{\rho}}^\dagger K_{\bar{\rho}}$ and $\ker L_{\bar{\rho}} = \mathrm{Im}\, \mathcal{E}_{\bar{\rho}}$. Recall that Σ^{w^*} is locally sys-fb-equivalent to $\Sigma^{\tilde{w}^*}_{Br}$ (by Lemma 4.6) and $\Sigma^{w^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$, it is seen that $\Xi^u|_{M^*}$ is locally ex-fb-equivalent to $\Delta^{\tilde{u}^*}$ around x_a by Theorem 3.5. Hence Ξ^u is locally in-fb-equivalent to the complete controllable linear DACS $\Delta^{\tilde{u}^*}$, i.e., Ξ^u is locally internally feedback linearizable. $\quad\square$

Theorem 4.7 (external feedback linearization). *Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix an admissible point $x_a \in X$. Then Ξ^u is locally externally feedback linearizable around x_a if and only if there exists a neighborhood $U \subseteq X$ of x_a in which the following conditions are satisfied.*

(EFL1) $\mathrm{rank}\, E(x)$ and $\mathrm{rank}\, [E(x), G(x)]$ are constant.

(EFL2) $F(x) \in \mathrm{Im}\, E(x) + \mathrm{Im}\, G(x)$ or, equivalently, the locally maximal invariant submanifold $M^* = M_0^c = U$.

(EFL3) For one (and thus any) control system $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|_{M^*})$, which is a system with no outputs on $M^* = U$, the linerizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy (FL1)-(FL3) of Theorem 4.5.

Proof. Only if. Suppose that Ξ^u is locally externally feedback linearizable. By definition, the DACS Ξ^u is locally ex-fb-equivalent to a linear completely controllable DACS (via $Q(x)$, $z = \psi(x)$ and $u = \alpha^u(x) + \beta^u(x)\tilde{u}$)

$$\Delta^{\tilde{u}} : \tilde{E}\dot{z} = \tilde{H}z + \tilde{L}\tilde{u}. \tag{31}$$

Thus by Definition 2.7, we have

$$\begin{aligned} Q(x)E(x) &= \tilde{E} \cdot \frac{\partial \psi(x)}{\partial x}, \\ Q(x)(F(x) + G(x)\alpha^u(x)) &= \tilde{H} \cdot \psi(x), \\ Q(x)G(x)\beta^u(x) &= \tilde{L}. \end{aligned} \tag{32}$$

It is clear that $\Delta^{\tilde{u}}$ satisfies (EFL1). So the system Ξ^u satisfies (EFL1) as well because the ranks of $E(x)$ and $[E(x), G(x)]$ are invariant under ex-fb-equivalence. The complete controllability of $\Delta^{\tilde{u}}$ implies $\tilde{H}z \in \operatorname{Im}\tilde{E} + \operatorname{Im}\tilde{L}$ (see Lemma 4.2(ii)). By substituting (32), we get

$$Q(F + G\alpha^u)(x) \in \operatorname{Im} QE \left(\frac{\partial \psi}{\partial x} \right)^{-1}(x) + \operatorname{Im} QG\beta^u(x)$$
$$\Rightarrow F(x) + G(x)\alpha^u(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x)$$
$$\Rightarrow F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x).$$

Thus Ξ^u satisfies (EFL2). Notice that by (EFL2), we have that the locally maximal controlled invariant submanifold M^* around x_a coincides with the neighborhood U. Observe that the restriction $\Delta^{\tilde{u}}|_{M^*} = \Delta^{\tilde{u}}|_U$, whose canonical form is given by

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho \\ 0 \end{bmatrix} u^*,$$

is also a linear completely controllable DACS as $\Delta^{\tilde{u}}$. This means that Ξ^u is locally internally feedback linearizable. Thus by Theorem 4.5, the DACS Ξ^u satisfies (EFL3) on $M^* = U$.

If. Suppose that in a neighborhood U of x_a, the DACS Ξ^u satisfies (EFL1)-(EFL3). Denote rank $E(x) = r$, rank $[E(x), G(x)] = r + \tilde{m}^*$ and $m^* = m - \tilde{m}^*$. Then, by (EFL1), there exist an invertible $Q(x)$ defined on U and a partition of $u = (u_1, u_2)$ such that

$$Q(x)E(x)\dot{x} = Q(x)F(x) + Q(x)G(x)u \Rightarrow$$
$$\begin{bmatrix} E_1(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{bmatrix} + \begin{bmatrix} G_1^1(x) & G_1^2(x) \\ G_2^1(x) & G_2^2(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $E_1(x)$ is of full row rank r and $G_2^2(x)$ is a $\tilde{m}^* \times \tilde{m}^*$ invertible matrix-valued function defined on U. Moreover, by (EFL2), we have $F_3(x) = 0$ for $x \in U$. Now we use the feedback transformation

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2(x) \end{bmatrix} + \begin{bmatrix} I_{m^*} & 0 \\ G_2^1(x) & G_2^2(x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and the system becomes

$$\begin{bmatrix} E_1(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} \tilde{F}_1(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1^1(x) & \tilde{G}_1^2(x) \\ 0 & I_{\tilde{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix},$$

where $\tilde{F}_1 = F_1 - G_1^2(G_2^2)^{-1}F_2$, $\tilde{G}_1^1 = G_1^1 - G_1^2(G_2^2)^{-1}G_2^1$ and $\tilde{G}_1^2 = G_1^2(G_2^2)^{-1}$. Premultiply the above equation by $\begin{bmatrix} I_r & -\tilde{G}_1^2(x) & 0 \\ 0 & I_{\tilde{m}^*} & 0 \\ 0 & 0 & I_{l-r-\tilde{m}^*} \end{bmatrix}$ to get

$$\begin{bmatrix} E^*(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F^*(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} G^*(x) & 0 \\ 0 & I_{\tilde{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^* \\ \tilde{u}^* \end{bmatrix}, \tag{33}$$

where $E^* = E_1$, $F^* = \tilde{F}_1$, $G^* = \tilde{G}_1^1$, $u^* = \tilde{u}_1$ and $\tilde{u}^* = \tilde{u}_2$. Then by Definition 2.5, we have that $\Xi^u|_{M^*} = \Xi^u|_U$ is the following system:

$$\Xi^u|_{M^*} : E^*(x)\dot{x} = F^*(x) + G^*(x)u^*.$$

By Theorem 4.5 and condition (EFL3), $\Xi^u|_{M^*}$ is locally ex-fb-equivalent (on $M^* = U$) to a linear DACS $\Delta^{\tilde{u}^*}$ of the form (30). It follows from (33) that Ξ^u is locally on U ex-fb-equivalent to

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho & 0 \\ 0 & 0 \\ 0 & I_{\tilde{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^* \\ \tilde{u}^* \end{bmatrix},$$

which is completely controllable by Lemma 4.2. Therefore, Ξ^u is locally externally feedback linearizable by Definition 4.3. $\qquad\square$

Remark 4.8. (i) By conditions (EFL1) and (EFL2), the locally maximal controlled invariant submanifold M^* around x_a is a neighborhood U of x_a. So condition (EFL3) is actually, satisfied if and only if conditions (FL1)-(FL3) are satisfied on $M^* = U$, i.e., locally around x_a.

(ii) Note that when applying the geometric reduction method of Definition 2.3 to a linear DACS $\Delta^u = (E, H, L)$, we get a sequence of subspaces $\mathscr{V}_i = M_i$, which is actually the augmented Wong sequence $\mathscr{V}_i$ defined by (24). Thus the locally maximal controlled invariant submanifold M^* is a nonlinear generalization of the limit $\mathscr{V}^*$ of $\mathscr{V}_i$. So condition (EFL2) together with condition $\hat{\mathcal{D}}_{n^*} = \mathcal{D}_{n^*} = TM^*$ of (FL2) are the nonlinear counterparts of condition $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$ of Lemma 4.2, which assures that the linearized DACS is completely controllable. The sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ can thus be seen as nonlinear generalizations of the augmented Wong sequence $\mathscr{W}_i$ of (25) and the sequence $\tilde{\mathscr{W}}_i$ of (26), respectively.

(iii) If $E(x) = I_n$, a DACS $\Xi^u = (E, F, G)$ becomes an ODECS of the form (3). Suppose that $G(x) = [g_1(x) \dots g_m(x)]$ is of constant rank. We have that conditions (EFL1)-(EFL2) of Theorem 4.7 are clearly satisfied and that condition (EFL3) reduces to the feedback linearizability conditions in the classical sense. Indeed, we have $\Xi^u \in \mathbf{Expl}(\Xi^u|_{M^*}) = \mathbf{Expl}(\Xi^u)$ because Ξ^u with $E(x) = I_n$ is already an ODECS. Thus the vector of driving variables v is absent and the two linearizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy $\hat{\mathcal{D}}_{i+1} = \mathcal{D}_i$ for

$i \geq 1$. Hence conditions (FL1)-(FL3) become (FL1)' $\mathcal{D}_i$ are of constant rank for $1 \leq i \leq n$; (FL2)' $\dim \mathcal{D}_n = n$; (FL3)' $\mathcal{D}_i$ are involutive for $1 \leq i \leq n-1$, which are the feedback linearizability conditions for classical nonlinear (ODE) control systems, see e.g., [20,17,19,28].

5 Examples

Example 5.1. Consider the following academic example borrowed from [3]. For a DACS Ξ^u, defined on $X = \mathbb{R}^3$, given by

$$\begin{bmatrix} x_2 & x_1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (x_2)^2 - (x_1)^3 + x_3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{34}$$

where $u = (u_1, u_2)$, we fix an admissible point

$$x_a = (x_{1a}, x_{2a}, x_{3a}) = (1, 0, 0) \in X.$$

Clearly, there exists a neighborhood U ($x_1 \neq 0$ for all $x \in U$) of x_a such that conditions (EFL1) and (EFL2) of Theorem 4.7 are satisfied. Subsequently, via $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$, the DACS Ξ^u is ex-fb-equivalent to

$$\begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (x_2)^2 - (x_1)^3 + x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}.$$

Observe that the locally maximal invariant submanifold $M^* = U$ and

$$\Xi^u|_{M^*} = \Xi^u|_U \colon \begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (x_2)^2 - (x_1)^3 + x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u^*,$$

where $u^* = \tilde{u}_1$. Now an ODECS $\Sigma^{u^* v} \in \mathbf{Expl}(\Xi^u|_{M^*})$ can be taken as

$$\Sigma^{u^* v} \colon \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (x_2)^2 - (x_1)^3 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2/x_1 \\ 0 \end{bmatrix} u^* + \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \end{bmatrix} v,$$

where v is a driving variable. We calculate the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for the system $\Sigma^{u^* v}$ to get

$$\hat{\mathcal{D}}_1 = \operatorname{span} \{g^v\}, \qquad \mathcal{D}_1 = \operatorname{span} \{g^{u^*}, g^v\},$$

$$\mathcal{D}_2 = \hat{\mathcal{D}}_2 = \operatorname{span} \{g^{u^*}, g^v, ad_f g^v\},$$

where

$$g^v = \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \end{bmatrix}, \ g^{u^*} = \begin{bmatrix} 0 \\ 2/x_1 \\ 0 \end{bmatrix}, \ ad_f g^v = \begin{bmatrix} 0 \\ 0 \\ 3(x_1)^3 + 2(x_2)^2 + x_1 \end{bmatrix}.$$

Clearly, the distributions above are of constant rank and $\mathcal{D}_2 = \hat{\mathcal{D}}_2 = T_x U$ for all $x \in U$. Additionally, $[g^{u^*}, g^v] = 0 \in \mathcal{D}_1$ and $\hat{\mathcal{D}}_1$ is of rank one, so the distributions $\hat{\mathcal{D}}_1, \mathcal{D}_1, \hat{\mathcal{D}}_2$ are all involutive. Thus, condition

(EFL3) of Theorem 4.7 is satisfied. Therefore, system Ξ^u is externally feedback linearizable.

In fact, we can choose $\varphi^{u^*}(x)$ and $\varphi^v(x)$ such that

$$\operatorname{span} \{d\varphi^v\} = \mathcal{D}_1^\perp, \qquad \operatorname{span} \left\{ d\varphi^v, d\varphi^{u^*} \right\} = \hat{\mathcal{D}}_1^\perp.$$

Furthermore, use the following coordinates change and feedback transformation (note that the feedback transformation below has a triangular form as we discussed in Remark 3.4)

$$\xi = \varphi^{u^*}(x) = x_1 x_2, \quad z_1 = \varphi^v(x) = x_1 + x_3,$$
$$z_2 = L_f \varphi^v(x) = -(x_1)^3 + (x_2)^2 + x_3,$$
$$\begin{bmatrix} \tilde{u}^* \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ \frac{4x_2}{x_1} & -3(x_1)^3 - x_1 - 2(x_2)^2 \end{bmatrix} \begin{bmatrix} u^* \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ (x_2)^2 - (x_1)^3 + x_3 \end{bmatrix},$$

the system Σ^{uv} becomes

$$\Lambda^{\tilde{u}^* \tilde{v}} : \begin{cases} \dot{\xi} = \tilde{u}^*, \\ \dot{z}_1 = z_2, \\ \dot{z}_2 = \tilde{v}. \end{cases}$$

Now by Theorem 3.5, $\Xi^u|_{M^*}$ is ex-fb-equivalent to the following linear DACS

$$\Delta^{\tilde{u}^*} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tilde{u}^*,$$

since $\Sigma^{u^* v} \in \mathbf{Expl}(\Xi^u|_{M^*})$, $\Lambda^{\tilde{u}^* \tilde{v}} \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$, and $\Sigma^{u^* v} \overset{sys-fb}{\sim} \Lambda^{\tilde{u}^* \tilde{v}}$. Therefore, the original DACS Ξ^u is ex-fb-equivalent to the following completely controllable linear DACS:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}^* \\ \tilde{u}_2 \end{bmatrix}$$

via $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and

$$\begin{bmatrix} \xi \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ x_1 + x_3 \\ -(x_1)^3 + (x_2)^2 + x_3 \end{bmatrix}, \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}^* \\ \tilde{u}_2 \end{bmatrix}.$$

Example 5.2. Consider the model of a 3-link manipulator [1] with active joints 1 and 2, and a passive joint 3 (see Figure 1 below). We will call joint 3 a free joint of the manipulator.
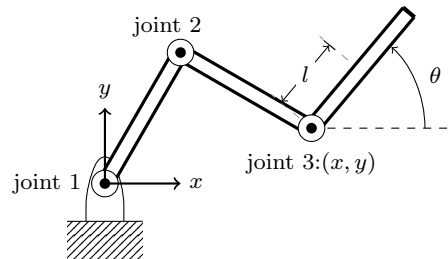

Fig. 1: A 3-link manipulator with a free joint

The dynamic equations of the manipulator are given by:

$$\begin{cases} m\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta = F_x, \\ m\ddot{y} + ml\cos\theta\ddot{\theta} - ml\dot{\theta}^2\sin\theta = F_y, \\ -ml\sin\theta\ddot{x} + ml\cos\theta\ddot{y} + ml^2\ddot{\theta} = \tau_\theta + F_f, \end{cases} \quad (35)$$

where the mass m and the half length of the free-link l are constants, x and y are the position variables of the free joint, and θ is the angle between the base frame and the link frame, F_x and F_y are the translation force at the free joint in the direction of x and y, respectively, and τ_θ is the torque applied to the free joint (we take $\tau_\theta = 0$ implying that joint 3 is free). We additionally consider the friction force F_f caused by the rotation of the free link. We regard (F_x, F_y) as the active control inputs to the system. The friction force F_f is a generalized state variable rather than an active control input since we can not change it arbitrarily. We consider system (35) subjected to the following constraint:

$$x - y = 0. \quad (36)$$

We combine (35) together with (36) as a DACS $\Xi^u_{7,7,2} = (E, F, G)$ of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & -ml\sin\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin\theta_1 & 0 & \cos\theta_1 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} =$$

$$\begin{bmatrix} x_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ \frac{F_f}{ml} \\ x_1 - y_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

For the DACS Ξ^u, the generalized states

$$\xi = (x_1, x_2, y_1, y_2, \theta_1, \theta_2, F_f) \in X = \mathbb{R}^6 \times S$$

and the vector of control inputs is (F_x, F_y). Consider Ξ^u around a point

$$\xi_p = (x_{1p}, x_{2p}, y_{1p}, y_{2p}, \theta_{1p}, \theta_{2p}, F_{fp}) = 0.$$

The system Ξ^u is *not* locally externally feedback linearizable since condition (EF2) of Theorem 4.7 is not satisfied around ξ_p. Now we apply the geometric reduction method of Definition 2.3 to get

$$M_0^c = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^6, \quad M_1^c = \{\xi \in M_0^c \mid x_1 - y_1 = 0\},$$

$$M_2^c = \{\xi \in M_1^c \mid x_2 - y_2 = 0\}, \quad M_3^c = M_2^c.$$

Thus by Proposition 2.4, $M^* = M_3^c = M_2^c$ is the locally maximal controlled invariant submanifold around $x_p \in M^*$ (so x_p is admissible). Choose new coordinates

$\xi_2 = (\tilde{x}_1, \tilde{x}_2) = (x_1 - y_1, x_2 - y_2)$ and keep the remaining coordinates $\xi_1 = (y_1, y_2, \theta_1, \theta_2, F_f)$ unchanged, the system represented in the new coordinates is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 & 0 & m \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 & l & 0 & 0 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \\ \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} =$$

$$\begin{bmatrix} \tilde{x}_2 + y_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ \frac{F_f}{ml} \\ \tilde{x}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

Set $\xi_2 = (\tilde{x}_1, \tilde{x}_2) = 0$ to get a DACS of the form (4)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 & l & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} =$$

$$\begin{bmatrix} y_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ \frac{F_f}{ml} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

By using $Q(\xi_1)$ and the feedback transformations defined on M^* as

$$Q(\xi_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \sin\theta_1 & 0 & -\cos\theta_1 & 0 & m \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_f/l \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix},$$

we bring the system into

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} =$$

$$\begin{bmatrix} y_2 \\ ml\theta_2^2\sin\theta_1 + \frac{F_f}{l}\sec\theta_1 \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \tan\theta_1 & -\sec\theta_1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

So the local M^*-restriction $\Xi^u|_{M^*} = (E^*, F^*, G^*)$ (see Definition 2.5) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} =$$

$$\begin{bmatrix} y_2 \\ \frac{F_f}{l}\sec\theta_1 + ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \tan\theta_1 \\ 0 \\ 1 \end{bmatrix} u_1. \quad (37)$$

An explicitation system $\Sigma^{u_1 v} \in \mathbf{Expl}(\Xi^u|_{M^*})$ can be chosen as

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ \frac{F_f \tan \theta_1 + ml^2 \theta_2^2}{ml(\cos \theta_1 + \sin \theta_1)} \\ \theta_2 \\ \frac{F_f \sec \theta_1 + ml^2 \theta_2^2 (\sin \theta_1 - \cos \theta_1)}{ml^2(\cos \theta_1 + \sin \theta_1)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\sec \theta_1}{m(\cos \theta_1 + \sin \theta_1)} \\ 0 \\ \frac{\tan \theta_1 - 1}{ml(\cos \theta_1 + \sin \theta_1)} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v.$$

Define a new control

$$u^* := \frac{F_f \tan \theta_1 + ml^2 \theta_2^2}{ml(\cos \theta_1 + \sin \theta_1)} + \frac{\sec \theta_1}{m(\cos \theta_1 + \sin \theta_1)} u_1.$$

Then the system $\Sigma^{u_1 v}$ under the new control is $\Sigma^{u^* v} = (f, g^{u^*}, g^v)$:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ 0 \\ \theta_2 \\ \frac{F_f}{ml^2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{l}(\sin \theta_1 - \cos \theta_1) \\ 0 \end{bmatrix} u^* + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v.$$

Now calculate the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for the system $\Sigma^{u^* v}$ to get

$$\hat{\mathcal{D}}_1 = \mathrm{span}\left\{g^v\right\}, \quad \mathcal{D}_1 = \mathrm{span}\left\{g^{u^*}, g^v\right\},$$

$$\hat{\mathcal{D}}_2 = \mathrm{span}\left\{g^{u^*}, g^v, ad_f g^v\right\},$$

$$\mathcal{D}_2 = \mathrm{span}\left\{g^v, g^{u^*}, ad_f g^v, ad_f g^{u^*}\right\},$$

$$\mathcal{D}_3 = \hat{\mathcal{D}}_2 = TM^*.$$

where $g^v = \frac{\partial}{\partial F_f}$, $g^{u^*} = \frac{\partial}{\partial y_2} + \frac{1}{l}(\sin \theta_1 - \cos \theta_1)\frac{\partial}{\partial \theta_2}$, $ad_f g^v = -\frac{1}{ml^2}\frac{\partial}{\partial \theta_2}$, $ad_f g^{u^*} = -\frac{\partial}{\partial y_1} - \frac{1}{l}(\sin \theta_1 - \cos \theta_1)\frac{\partial}{\partial \theta_1} + \frac{1}{l}(\sin \theta_1 + \cos \theta_1)\theta_2 \frac{\partial}{\partial \theta_2}$. Clearly, the distributions above are of constant rank and are all involutive around ξ_p. Thus, conditions (FL1)-(FL3) of Theorem 4.5 are satisfied. Therefore, system Ξ^u is locally internally feedback linearizable around ξ_p. Indeed, choose $\varphi^{u^*}(x)$ and $\varphi^v(x)$ such that

$$\mathrm{span}\left\{d\varphi^v\right\} = \mathcal{D}_2^{\perp}, \quad \mathrm{span}\left\{d\varphi^v, d\varphi^{u^*}\right\} = \hat{\mathcal{D}}_2^{\perp}.$$

Then define the following coordinates change and feedback transformation (which has a triangular form as desired):

$$\tilde{y}_1 = \varphi^v(\xi_1) = y_1 - l \int a(\theta_1) d\theta_1,$$
$$\tilde{y}_2 = L_f \varphi^v(\xi_1) = y_2 - la(\theta_1)\theta_2,$$
$$\tilde{F}_f = L_f^2 \varphi^v(\xi_1) = -a(\theta_1)F_f - a'(\theta_1)l\theta_2^2,$$
$$\tilde{\theta}_1 = \varphi^{u^*}(\xi_1) = \theta_1, \quad \tilde{\theta}_2 = L_f \varphi^{u^*}(\xi_1) = \theta_2,$$
$$\begin{bmatrix} \tilde{u}^* \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} \frac{1}{l}(\sin \theta_1 - \cos \theta_1) & 0 \\ -2a'(\theta_1)(\sin \theta_1 - \cos \theta_1)\theta_2 & -a(\theta_1) \end{bmatrix} \begin{bmatrix} u^* \\ v \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{F_f}{ml^2} \\ -3a'(\theta_1)\theta_2 F_f - a''(\theta_1)\theta_2^3 l \end{bmatrix},$$

where $a(\theta_1) = \frac{1}{\sin \theta_1 - \cos \theta_1}$, $a'(\theta_1) = \frac{da(\theta_1)}{d\theta_1}$, $a''(\theta_1) = \frac{d^2 a(\theta_1)}{d\theta_1^2}$. We transform $\Sigma^{u^* v}$ into a linear control system in the Brunovský form

$$\Lambda^{\tilde{u}^* \tilde{v}} : \dot{\tilde{y}}_1 = \tilde{y}_2, \ \dot{\tilde{y}}_2 = \tilde{F}_f, \ \dot{\tilde{F}}_f = \tilde{v}, \ \dot{\tilde{\theta}}_1 = \tilde{\theta}_2, \ \dot{\tilde{\theta}}_2 = \tilde{u}^*.$$

Thus by Theorem 3.5, the restriction $\Xi^u|_{M^*}$, given by (37), is locally ex-fb-equivalent to the following completely controllable linear DACS $\Delta^{\tilde{u}^*}$,

$$\Delta^{\tilde{u}^*} : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \\ \dot{\tilde{F}}_f \\ \dot{\tilde{\theta}}_1 \\ \dot{\tilde{\theta}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{F}_f \\ \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u}^*.$$

because $\Sigma^{u^* v} \overset{sys-fb}{\sim} \Sigma^{u_1 v} \in \mathbf{Expl}(\Xi^u|_{M^*})$, $\Lambda^{\tilde{u}^* \tilde{v}} \in \mathbf{Expl}(\Delta^{\tilde{u}})$ and $\Sigma^{u^* v} \overset{sys-fb}{\sim} \Lambda^{\tilde{u}^* \tilde{v}}$. Hence the DACS Ξ^u is locally in-fb-equivalent to the linear DACS $\Delta^{\tilde{u}^*}$, i.e., Ξ^u is locally internally feedback linearizable.

6 Conclusions and perspectives

In this paper, we give necessary and sufficient conditions for the problem that when a nonlinear DACS is locally internally or locally externally feedback equivalent to a completely controllable linear DACS. The conditions are based on an ODECS constructed by the explicitation with driving variables. Two examples are given to illustrate how to externally or internally feedback linearize a nonlinear DACS.

A natural problem for future works is that of when a nonlinear DAE system is ex-fb-equivalent to a linear one which is not necessarily completely controllable. Actually, this problem is more involved than the problem of external feedback linearization with complete controllability. Indeed, since in Theorem 4.7, the maximal controlled invariant submanifold M^* on U is $M^* = U$, it follows that the algebraic constraints are directly governed by some variables of u. Thus the in-fb-equivalence is very close to the ex-fb-equivalence. However, if $M^* \neq U$, then the algebraic constraints may affect the generalized state. Moreover, since the explicitation is defined up to a generalized output injection, it may happen that one system of the explicitation is feedback linearizable but another is not. The general feedback linearizability problem remains open and, in view of the above points, is challenging.

Appendix

Proof of Lemma 4.6. For ease of notation, we drop the index "$*$" for z^*, u^*, v^* and f^* of the system $\Sigma^{u^* v^*}_{n^*, m^*, s^*}$,

that is, $\Sigma^{u^*v^*}$ becomes

$$\Sigma^{uv} : \dot{z} = f(z) + g^u(z)u + g^v(z)v.$$

The admissible point x_a in the z-coordinates will be denoted by z_a. We will only show the proof for the case that

$$m^* = s^* = 1, \qquad \mathrm{rank}\,[g^v(z_a) \ \ g^u(z_a)] = 2.$$

The proof for the general case (i.e., for any $m^* \geq 1$ and $s^* \geq 1$, and for $\mathrm{rank}\,[g^v(z_a) \ \ g^u(z_a)] = m^* + s^*$) can be done in a similar fashion as that on page 233-238 of [19] for the feedback linearization of nonlinear multi-inputs multi-outputs control systems. We now describe a procedure to construct a change of coordinates $\xi = \psi(z)$ and a feedback transformation:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha^u(z) \\ \alpha^v(z) \end{bmatrix} + \begin{bmatrix} \beta^u(z) & 0 \\ \lambda(z) & \beta^v(z) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \tag{38}$$

to transform Σ^{uv} into its Brunovský canonical form, where $\beta^u, \beta^v, \alpha^u, \lambda, \alpha^v$ are scalar functions, and $\beta^u(z)$ and $\beta^v(z)$ are nonzero around z_a, notice that the designed feedback transformation (38) has a triangular form as in (10). Note that constructing (38) is equivalent to finding the inverse feedback transformation

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} a^u(z) \\ a^v(z) \end{bmatrix} + \begin{bmatrix} b^u(z) & 0 \\ \tilde{\lambda}(z) & b^v(z) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{39}$$

where

$$a^u = -(\beta^u)^{-1}\alpha^u, \ \ a^v = (\beta^v)^{-1}\lambda(\beta^u)^{-1}\alpha^u - (\beta^v)^{-1}\alpha^v$$
$$b^u = (\beta^u)^{-1}, \ \ b^v = (\beta^u)^{-1}, \ \ \tilde{\lambda} = -(\beta^v)^{-1}\lambda(\beta^u)^{-1}.$$

Below we will search for functions $a^u, a^v, \tilde{\lambda}$, and nonzero functions b^u, b^v to construct (39).

Consider the two sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for Σ^{uv}, given by (27) and (28), respectively, and define

$$\rho := \max\left\{ i \in \mathbb{N}^+ \,|\, \hat{\mathcal{D}}_i \neq \mathcal{D}_i \right\},$$
$$\bar{\rho} := \max\left\{ i \in \mathbb{N}^+ \,|\, \mathcal{D}_{i-1} \neq \hat{\mathcal{D}}_i \right\}.$$

By $m^* = s^* = 1$, it is seen that, for each $i \geq 1$,

$$\dim \mathcal{D}_i - \dim \hat{\mathcal{D}}_i = \begin{cases} 0, & \text{if } \mathcal{D}_i = \hat{\mathcal{D}}_i \\ 1, & \text{if } \mathcal{D}_i \neq \hat{\mathcal{D}}_i \end{cases},$$
$$\dim \hat{\mathcal{D}}_i - \dim \mathcal{D}_{i-1} = \begin{cases} 0, & \text{if } \hat{\mathcal{D}}_i = \mathcal{D}_{i-1} \\ 1, & \text{if } \hat{\mathcal{D}}_i \neq \mathcal{D}_{i-1} \end{cases}. \tag{40}$$

It follows that $\rho + \bar{\rho} = n^*$. Then only two cases are possible: either $\rho \geq \bar{\rho}$ or $\rho < \bar{\rho}$.

Case 1: If $\rho \geq \bar{\rho}$, then we have

$$\mathcal{D}_0 \subsetneq \hat{\mathcal{D}}_1 \subsetneq \cdots \subsetneq \mathcal{D}_{\bar{\rho}-1} \subsetneq \hat{\mathcal{D}}_{\bar{\rho}} \subsetneq \mathcal{D}_{\bar{\rho}} = \hat{\mathcal{D}}_{\bar{\rho}+1} \subsetneq \mathcal{D}_{\bar{\rho}+1} =$$
$$\cdots \subsetneq \mathcal{D}_{\rho-1} = \hat{\mathcal{D}}_\rho \subsetneq \mathcal{D}_\rho = \hat{\mathcal{D}}_{\rho+j} = \mathcal{D}_{\rho+j}, \ j > 0.$$

It follows that $\mathcal{D}_\rho = \mathcal{D}_{n^*} = \hat{\mathcal{D}}_{n^*}$ Then by (FL2) of Theorem 4.5, we have $\mathcal{D}_\rho = TM^*$ and thus $\dim \mathcal{D}_\rho = n^*$. By $\hat{\mathcal{D}}_\rho \subsetneq \mathcal{D}_\rho$ and (40), we have $\dim \hat{\mathcal{D}}_\rho = n^* - 1$. Now by the involutivity of $\hat{\mathcal{D}}_\rho$ (condition (FL3)), we can choose a scalar function $h^u(z)$ such that

$$\mathrm{span}\,\{dh^u\} = \hat{\mathcal{D}}_\rho^\perp,$$

where $\hat{\mathcal{D}}_\rho^\perp$ denotes the annihilator of the distribution $\hat{\mathcal{D}}_\rho$. It follows that for all z around z_a,

$$\begin{cases} \langle dh^u(z), ad_f^i g^u(z) \rangle = 0, \ \ 0 \leq i \leq \rho - 2, \\ \langle dh^u(z), ad_f^{\rho-1} g^u(z) \rangle \neq 0; \\ \langle dh^u(z), ad_f^i g^v(z) \rangle = 0, \ \ 0 \leq i \leq \rho - 1. \end{cases} \tag{41}$$

Recall the following result [19][28]:

$$\begin{aligned} &\langle dh(z), ad_f^i g(z) \rangle = 0, \ 0 \leq i \leq l - 2 \Rightarrow \\ &\langle dh(z), ad_f^{l-1} g(z) \rangle = (-1)^i \langle dL_f^i h(z), ad_f^{l-1-i} g(z) \rangle, \\ &0 \leq i \leq l - 1, \end{aligned} \tag{42}$$

where $h(z)$ is a scalar function, $f(z)$ and $g(z)$ are vector fields.

It can be deduced from (41) and (42) that for all z around z_a,

$$\begin{aligned} &\langle dL_f^i h^u(z), ad_f^j g^u(z) \rangle = 0, \ 0 \leq i \leq \rho - 2, \\ &0 \leq j \leq \rho - i - 2; \\ &\langle dL_f^i h^u(z), ad_f^{\rho-i-1} g^u(z) \rangle \neq 0, \ \ 0 \leq i \leq \rho - 2; \\ &\langle dL_f^i h^u(z), ad_f^j g^v(z) \rangle = 0, \ \ 0 \leq i \leq \rho - 1, \\ &0 \leq j \leq \rho - i - 1; \end{aligned} \tag{43}$$

By using (43), we have the following table for the expressions of $\langle dL_f^i h^u, ad_f^j g^u \rangle$, $0 \leq i \leq \rho - \bar{\rho}$, $\bar{\rho} - 1 \leq j \leq \rho - 1$:

	$ad_f^{\bar{\rho}-1} g^u$	$ad_f^{\bar{\rho}} g^u$	$\cdots$	$ad_f^{\rho-1} g^u$
dh^u	0	0	$\cdots$	$\langle dh^u, ad_f^{\rho-1} g^u \rangle$
$\cdots$	$\cdots$		$\cdots$	$*$
$dL_f^{\rho-\bar{\rho}-1} h^u$	0		$*$	
$dL_f^{\rho-\bar{\rho}} h^u$	$\langle dL_f^{\rho-\bar{\rho}} h^u, ad_f^{\bar{\rho}-1} g^u \rangle$			$?$

Notice that all the anti-diagonal elements of the above table are nonzero by (43). It follows that the co-distribution

$$\Omega_1 = \mathrm{span}\,\left\{ dL_f^i h^u, \ \ 0 \leq i \leq \rho - \bar{\rho} \right\}$$

is of dimension $\rho - \bar{\rho} + 1$ around z_a. Observe that $\Omega_1 \subseteq \mathcal{D}_{\bar{\rho}-1}^\perp$ since for $0 \leq i \leq \rho - \bar{\rho}$, $0 \leq j \leq \bar{\rho} - 2$,

$$\langle dL_f^i h^u(z), ad_f^j g^u(z) \rangle \overset{(43)}{=} 0,$$
$$\langle dL_f^i h^u(z), ad_f^j g^v(z) \rangle \overset{(43)}{=} 0.$$

It is seen that $\dim \mathcal{D}_{\bar\rho-1}^\perp - \dim \Omega_1 = (n^* - (2\bar\rho - 2)) - (\rho - \bar\rho + 1) = 1$ and $\Omega_1 \subsetneq \mathcal{D}_{\bar\rho-1}^\perp$. Then by the involutivity of $\mathcal{D}_{\bar\rho-1}$ (condition (FL3)), we can choose a scalar function $h^v(z)$ such that

$$\operatorname{span}\{dh^v\} + \Omega_1 = \mathcal{D}_{\bar\rho-1}^\perp,$$

which implies that for all z around z_a,

$$\left\langle dh^v(z), ad_f^i g^u(z) \right\rangle = 0, \ 0 \le i \le \bar\rho - 2,$$
$$\left\langle dh^v(z), ad_f^i g^v(z) \right\rangle = 0, \ 0 \le i \le \bar\rho - 2, \tag{44}$$
$$\left\langle dh^v(z), ad_f^{\bar\rho-1} g^v(z) \right\rangle \neq 0.$$

It can be deduced by (44) and (42) that for all z around z_a,

$$\left\langle dL_f^i h^v(z), ad_f^j g^u(z) \right\rangle = 0, \ 0 \le i \le \bar\rho - 2,$$
$$0 \le j \le \bar\rho - i - 2;$$
$$\left\langle dL_f^i h^v(z), ad_f^j g^v(z) \right\rangle = 0, \ 0 \le i \le \bar\rho - 2, \tag{45}$$
$$0 \le j \le \bar\rho - i - 2,$$
$$\left\langle dL_f^i h^v(z), ad_f^{\bar\rho-i-1} g^v(z) \right\rangle \neq 0, \ 0 \le i \le \bar\rho - 2.$$

By using (43) and (45), we can construct the following table:

	g^v	g^u	$\cdots$	$\cdots$	$ad_f^{\bar\rho-1}g^v$	$ad_f^{\bar\rho-1}g^u$	$ad_f^{\bar\rho}g^u$	$\cdots$	$ad_f^{\rho-1}g^u$
dh^u	0	0	$\cdots$	$\cdots$	0	0	0	$\cdots$	$\left\langle dh^u, ad_f^{\rho-1}g^u \right\rangle$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		*
$dL_f^{\rho-\bar\rho-1}h^u$	0	0	$\cdots$	$\cdots$	0	0	$\left\langle dL_f^{\rho-\bar\rho-1}h^u, ad_f^{\bar\rho}g^u \right\rangle$		
$dL_f^{\rho-\bar\rho}h^u$	0	0	$\cdots$	$\cdots$	0	$\left\langle dL_f^{\rho-\bar\rho}h^u, ad_f^{\bar\rho-1}g^u \right\rangle$			?
dh^v	0	0	$\cdots$	$\cdots$	$\left\langle dh^v, ad_f^{\bar\rho-1}g^v \right\rangle$	?			
$\cdots$	0	0	$\cdots$	*					
$\cdots$	0	0	*	?					
$dL_f^{\rho-1}h^u$	0	$L_{g^u}L_f^{\rho-1}h^u$							
$dL_f^{\bar\rho-1}h^v$	$L_{g^v}L_f^{\bar\rho-1}h^v$	?	?			?			

$$\tag{46}$$

Notice that all the anti-diagonal elements of table (46) are nonzero. It follows that the $(\rho+\bar\rho) \times (\rho+\bar\rho) = n^* \times n^*$ matrix

$$\frac{\partial \psi}{\partial z}(z) \left[g^v \ g^u \ \cdots \ \cdots \ ad_f^{\bar\rho-1}g^v \ ad_f^{\bar\rho-1}g^u \ ad_f^{\bar\rho}g^u \ \cdots \ ad_f^{\rho-1}g^u \right](z)$$

is invertible around z_a, where

$$\psi = (h^u, \ldots, L_f^{\rho-1}h^u, h^v, \ldots, L_f^{\bar\rho-1}h^v). \tag{47}$$

Thus the Jacobian matrix $\frac{\partial \psi(z)}{\partial z}$ is invertible around z_a and ψ is a local diffeomorphism. Then set

$$a^u(z) = L_f^\rho h^u(z), \quad b^u(z) = L_{g^u}L_f^{\rho-1}h^u(z),$$
$$a^v(z) = L_f^{\bar\rho}h^v(z), \quad b^v(z) = L_{g^v}L_f^{\bar\rho-1}h^v(z), \tag{48}$$
$$\tilde\lambda(z) = L_{g^u}L_f^{\bar\rho-1}h^v(z).$$

Note that $b^u(z)$ and $b^v(z)$ are nonzero at z_p. It is seen that $\Sigma^{u^*v^*}$ is mapped, via the coordinates transformations $\xi = (\xi_1, \xi_2) = \psi(z)$ and the feedback transformation (39), into the Brunovský form $\Sigma_{Br}^w = \Sigma_{Br}^{w^*}$ of (29) with indices ρ and $\bar\rho$.

Case 2: If $\rho < \bar\rho$, then we have $\mathcal{D}_0 \subsetneq \hat{\mathcal{D}}_1 \subsetneq \cdots \subsetneq \hat{\mathcal{D}}_\rho \subsetneq \mathcal{D}_\rho \subsetneq \hat{\mathcal{D}}_{\rho+1} = \mathcal{D}_{\rho+1} \subsetneq \cdots = \mathcal{D}_{\bar\rho-1} \subsetneq \hat{\mathcal{D}}_{\bar\rho} = $

$\mathcal{D}_{\bar\rho} = \hat{\mathcal{D}}_{\bar\rho+j} = \mathcal{D}_{\bar\rho+j}$, $j > 0$. It follows that $\hat{\mathcal{D}}_{\bar\rho} = \mathcal{D}_{\bar\rho} = \hat{\mathcal{D}}_{n^*} = \mathcal{D}_{n^*}$. Then by (FL2) of Theorem 4.5, we have $\hat{\mathcal{D}}_{\bar\rho} = TM^*$ and thus $\dim \hat{\mathcal{D}}_{\bar\rho} = n^*$. By $\mathcal{D}_{\bar\rho-1} \subsetneq \hat{\mathcal{D}}_{\bar\rho}$ and (40), we have $\dim \mathcal{D}_{\bar\rho-1} = n^* - 1$. Now by the involutivity of $\mathcal{D}_{\bar\rho}$ (condition (FL1)), we can choose a scalar function $h^v(z)$ such that

$$\operatorname{span}\{dh^v\} = \mathcal{D}_{\bar\rho-1}^\perp.$$

Then following a similar proof as in Case 1, we can show that the distribution

$$\Omega_2 = \operatorname{span}\left\{dL_f^i h^v, \ 0 \le i \le \bar\rho - \rho - 1\right\}$$

is of dimension $\rho - \bar\rho$ around z_a and $\Omega_2 \subsetneq \hat{\mathcal{D}}_\rho^\perp$. Notice that $\dim \hat{\mathcal{D}}_\rho^\perp = n^* - (2\rho - 1) = \bar\rho - \rho + 1$, we have $\dim \hat{\mathcal{D}}_\rho^\perp - \dim \Omega_2 = 1$. Thus by the involutivity of $\hat{\mathcal{D}}_\rho$ (condition (FL2)), we can choose a scalar function $h^u(z)$ such that

$$\operatorname{span}\{dh^u\} + \Omega_2 = \hat{\mathcal{D}}_\rho^\perp.$$

Then, similarly as in Case 1, we construct the following table:

	g^v	g^u	$\cdots$	$\cdots$	$ad_f^{\rho-1}g^v$	$ad_f^{\rho-1}g^u$	$ad_f^{\rho}g^v$	$\cdots$	$ad_f^{\bar\rho-1}g^v$
dh^v	0	0	$\cdots$	$\cdots$	0	0	0	$\cdots$	$\left\langle dh^v, ad_f^{\bar\rho-1}g^v \right\rangle$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	*	?
$dL_f^{\bar\rho-\rho-1}h^v$	0	0	$\cdots$	$\cdots$	0	0	$\left\langle dL_f^{\bar\rho-\rho-1}h^v, ad_f^{\rho}g^v \right\rangle$		
dh^u	0	0	$\cdots$	$\cdots$	0	$\left\langle dh^u, ad_f^{\rho-1}g^u \right\rangle$	?		
$dL_f^{\bar\rho-\rho}h^v$	0	0	$\cdots$	$\cdots$	$\left\langle dL_f^{\bar\rho-\rho}h^v, ad_f^{\rho-1}g^v \right\rangle$	?			?
$\cdots$	0	0	$\cdots$	*					
$\cdots$	0	0	*						
$dL_f^{\rho-1}h^u$	0	$L_{g^u}L_f^{\rho-1}h^u$							
$dL_f^{\bar\rho-1}h^v$	$L_{g^v}L_f^{\bar\rho-1}h^v$	?	?			?			?

and show that all the anti-diagonal elements of the table are nonzero around z_a. Finally, we define a diffeomorphism ψ and functions a^u, b^u, a^v, b^v, $\tilde{\lambda}$ in the same form as (47) and (48) of Case 1. It is seen that Σ^{uv} can also be transformed into the Brunovský form $\Sigma^w_{Br} = \Sigma^{w^*}_{Br}$ of (29) with indices ρ and $\bar{\rho}$ via the change of coordinates $\xi = \psi(z)$ and the feedback transformation (39).   □

Conflict of interest

The authors declare that they have no conflict of interest.

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