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GEOMETRIC ANALYSIS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS VIA LINEAR CONTROL THEORY*

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Abstract. We consider linear differential-algebraic equations (DAEs) of the form $E\dot{x} = Hx$ and the Kronecker canonical form (**KCF**) [L. Kronecker, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, 1890, pp. 1225–1237] of the corresponding matrix pencils $sE - H$. We also consider linear control systems and their Morse canonical form (**MCF**) [A. Morse, *SIAM J. Control*, 11 (1973), pp. 446–465; B. P. Molinari, *Internat. J. Control*, 28 (1978), pp. 493–510]. For a linear DAE, a procedure called explicitation is proposed, which attaches to any linear DAE a linear control system defined up to a coordinates change, a feedback transformation, and an output injection. Then we compare subspaces associated to a DAE in a geometric way with those associated (also in a geometric way) to a control system, namely, we compare the Wong sequences of DAEs and invariant subspaces of control systems. We prove that the **KCF** of linear DAEs and the **MCF** of control systems have a perfect correspondence and that their invariants are related. In this way, we connect the geometric analysis of linear DAEs with the classical geometric linear control theory. Finally, we propose a concept called internal equivalence for DAEs and discuss its relation with internal regularity, i.e., the existence and uniqueness of solutions.

Key words. differential-algebraic equations, implicit systems, control systems, singular systems, Kronecker canonical form, Morse canonical form, invariant subspaces

AMS subject classifications. 15A21, 34H05, 93C05, 93C15

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1. Introduction.

Consider a linear differential-algebraic equation (DAE) of the form

$$\Delta : E\dot{x} = Hx, \tag{1.1}$$

where $x \in \mathscr{X} \cong \mathbb{R}^n$ is called the "generalized" state, $E \in \mathbb{R}^{l\times n}$, and $H \in \mathbb{R}^{l\times n}$. Throughout, a linear DAE of the form (1.1) will be denoted by $\Delta_{l,n} = (E, H)$ or, briefly, Δ, and the corresponding matrix pencil of Δ by $sE-H$, which is a polynomial matrix of degree one. A DAE Δ or a matrix pencil $sE - H$ is called regular if $l = n$ and $|sE - H| \not\equiv 0$.

Terminologies such as "singular," "implicit," and "generalized" are frequently used to describe a DAE due to its difference from an ordinary differential equation (ODE). Since the structure of DAE Δ is totally determined by the corresponding matrix pencil $sE-H$, it is useful to find a simplified form (a normal form or canonical form) for $sE-H$. Under predefined equivalence (see ex-equivalence of Definition 2.1), canonical forms such as the Weierstrass form (**WCF**) [25] for regular matrix pencils and the Kronecker canonical form [11] (for details see **KCF** in the appendix and [8]) for more general matrix pencils have been proposed. Geometric analysis of linear and nonlinear DAEs can be found in [12, 13, 14, 16, 17, 21, 22, 23]. We highlight an important concept called the Wong sequences ($\mathscr{V}_i$ and $\mathscr{W}_i$ of Definition 4.1) for linear

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DAEs, which were first introduced in [26]. Connections between the Wong sequences with the **WCF** and the **KCF** have been recently established in, respectively, [4] and [6, 7]. In particular, invariant properties for the limits of the Wong sequences ($\mathscr{V}^*$ and $\mathscr{W}^*$ in Definition 4.3) were used to obtain a triangular quasi-Kronecker form in [6, 7]. Moreover, the authors of [6, 7] show that some of the Kronecker indices can be calculated via the Wong sequences and the remaining ones can be derived from a modified version of the Wong sequences.

On the other hand, consider a linear time-invariant control system of the following form:

$$\Lambda : \begin{cases} \dot{z} = Az + Bu, \\ y = Cz + Du, \end{cases} \tag{1.2}$$

where $z \in \mathscr{X} = \mathbb{R}^q$ is the system state, $u \in \mathscr{U} = \mathbb{R}^m$ represents the input, and $y \in \mathscr{Y} = \mathbb{R}^p$ is the output. System matrices A, B, C, D above are constant and of appropriate sizes. We also consider the prolongation of Λ of the following form:

$$\mathbf{\Lambda} : \begin{cases} \dot{z} = Az + Bu, \\ \dot{u} = v, \\ y = Cz + Du, \end{cases} \Leftrightarrow \begin{cases} \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}v, \\ y = \mathbf{C}\mathbf{z}, \end{cases} \tag{1.3}$$

where

$$\mathbf{z} = \begin{bmatrix} z \\ u \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} C & D \end{bmatrix}.$$

Denote a control system of the form (1.2) by $\Lambda_{q,m,p} = (A, B, C, D)$ or, simply, Λ and denote the prolonged system (1.3) by $\mathbf{\Lambda}_{n,m,p} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$, or briefly $\mathbf{\Lambda}$, where $n = q + m$. Notice that there is a one-to-one correspondence between $\mathcal{C}^\infty$-solutions of (1.2) and (1.3) (or a one-to-one correspondence between $\mathcal{C}^1$-solutions $(z(t), u(t))$ of (1.2) and $\mathcal{C}^1$-solutions $\mathbf{z}(t)$, given by $\mathcal{C}^0$-controls $v(t)$, of (1.3)). Any control system can be brought (see [20],[19]) into its Morse canonical form (for details, see the **MCF** in the appendix) under the action of a group of transformations consisting of coordinates changes, feedback transformations, and output injections. The **MCF** consists of four decoupled subsystems MCF^1, MCF^2, MCF^3, MCF^4, to which there correspond four sets of structure invariants (the Morse indices ε_i', ρ_i', σ_i', η_i' in the **MCF**).

The first aim of the paper is to find a way to relate linear DAEs with linear control systems and find their geometric connections. In fact, we will show in the next section that to any linear DAE, we can attach a class of linear control systems defined up to a coordinates change, a feedback transformation, and an output injection. The second purpose of the paper is to distinguish two kinds of equivalences in linear DAEs theory, namely, internal equivalence and external equivalence. We will give the formal definition of external equivalence in Definition 2.1. Actually, the external equivalence (also called strict equivalence in [8]) is widely considered in the linear DAEs literature. For example, the **KCF** of a DAE is actually a canonical form under external equivalence, which is simply defined by all linear nonsingular transformations in the whole "generalized" state space of the DAE. However, since solutions of a DAE exist only on a constrained (invariant) subspace, sometimes we only need to perform the analysis on that constrained subspace. This point of view motivates us to introduce the notion of internal equivalence and to find normal forms not on the whole space but only on that constrained subspace.

The paper is organized as follows. In section 2, we introduce the notation, define the external equivalence of two DAEs, and define the Morse equivalence of two control systems. In section 3, we explain how to associate to any DAE a class of control systems. In section 4, we describe geometric relations of DAEs and the attached control systems. In section 5, we show that there exists a perfect correspondence between the **KCF** and the **MCF** and that their invariants have direct relations. In section 6, we introduce the notion of internal equivalence for DAEs and then discuss the internal regularity. Section 7 contains the proofs of our results and section 8 contains the conclusions of this paper. Finally, in the appendix we recall two basic canonical forms: the Kronecker canonical form **KCF** for DAEs and the Morse canonical form **MCF** for control systems.

2. Preliminaries. We use the following notation in the present paper:

$\mathbb{N}$	the set of natural numbers with zero and $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^{n\times m}$	the set of real valued matrices with n rows and m columns
$\mathbb{R}[s]$	the polynomial ring over $\mathbb{R}$ with indeterminate s
$Gl\,(n,\mathbb{R})$	the group of nonsigular matrices of $\mathbb{R}^{n\times n}$
$\operatorname{rank} A$	the rank of a linear map A
$\operatorname{rank}_{\mathbb{R}[s]}(sE-H)$	the rank of a polynomial matrix $sE-H$ over $\mathbb{R}[s]$
$\ker A$	the kernal of a linear map A
$\dim \mathscr{A}$	the dimension of a linear space $\mathscr{A}$
$\operatorname{Im} A$	the image of a linear map A
$\mathscr{A}/\mathscr{B}$	the quotient of a vector space $\mathscr{A}$ by a subspace $\mathscr{B}\subseteq\mathscr{A}$
I_n	the identity matrix of size $n\times n$ for $n\in\mathbb{N}^+$
$0_{n\times m}$	the zero matrix of size $n\times m$ for $n,m\in\mathbb{N}^+$
A^T	the transpose of a matrix A
A^{-1}	the inverse of a matrix A
$A\mathscr{B}$	$\{Ax \,\vert\, x\in\mathscr{B}\}$, the image of $\mathscr{B}$ under a linear map A
$A^{-1}\mathscr{B}$	$\{x \,\vert\, Ax\in\mathscr{B}\}$, the preimage of $\mathscr{B}$ under a linear map A
$A^{-T}\mathscr{B}$	$(A^T)^{-1}\mathscr{B}$
$\mathscr{A}^\perp$	$\{x\in\mathbb{R}^n \,\vert\, \forall a\in\mathscr{A}: x^Ta=0\}$, the orthogonal complement of a subspace $\mathscr{A}\subseteq\mathbb{R}^n$

Consider a DAE $\Delta_{l,n} = (E,H)$, given by (1.1), denoted briefly by Δ, and the corresponding matrix pencil $sE-H$. A *solution*, or *trajectory*, $x(t)$ of Δ is any $\mathcal{C}^1$-differentiable map $x:\mathbb{R}\to\mathscr{X}$ satisfying $E\dot{x}(t) = Hx(t)$. A trajectory starting from a point $x(0) = x^0$ is denoted by $x(t, x^0)$.

DEFINITION 2.1 (external equivalence). *Two DAEs $\Delta_{l,n} = (E,H)$ and $\tilde{\Delta}_{l,n} = (\tilde{E},\tilde{H})$ are called externally equivalent, briefly, ex-equivalent, if there exist $Q\in Gl(l,\mathbb{R})$ and $P\in Gl(n,\mathbb{R})$ such that*

$$\tilde{E} = QEP^{-1} \quad \text{and} \quad \tilde{H} = QHP^{-1}.$$

We denote ex-equivalence of two DAEs as $\Delta \overset{ex}{\sim} \tilde{\Delta}$ and ex-equivalence of the two corresponding matrix pencils as $sE-H \overset{ex}{\sim} s\tilde{E}-\tilde{H}$.

If the "generalized" states of Δ and $\tilde{\Delta}$ are x and $\tilde{x}$, respectively, then $\tilde{x} = Px$ is, clearly, just a coordinate transformation. The following remark points out the relation of the ex-equivalence and solutions of DAEs.

Remark 2.2. Ex-equivalence preserves trajectories; more precisely, if $\Delta \overset{ex}{\sim} \tilde{\Delta}$ via (Q,P), then any trajectory $x(t)$ of Δ satisfying $x(0) = x^0$ is mapped via P into a trajectory $\tilde{x}(t)$ of $\tilde{\Delta}$ passing through $\tilde{x}^0 = Px^0$. Moreover, if $x(t)$ is a trajectory of Δ, then $E\dot{x}(t) - Hx(t) = 0$ and obviously $Q(E\dot{x}(t) - Hx(t)) = 0$, implying that $x(t)$ is also a trajectory of $QE\dot{x} = QHx$. The converse, however, is not true: even if two DAEs have the same trajectories, they are not necessarily ex-equivalent, since the trajectories of DAEs are contained in a subspace $\mathscr{M}^* \subseteq \mathbb{R}^n$ (see Definition 6.1 of section 6).

DEFINITION 2.3 (Morse equivalence and Morse transformation). *Two linear control systems $\Lambda_{q,m,p} = (A, B, C, D)$ and $\tilde{\Lambda}_{q,m,p} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are called Morse equivalent, briefly, M-equivalent, denoted by $\Lambda \overset{M}{\sim} \tilde{\Lambda}$, if there exist $T_s \in Gl(q, \mathbb{R})$, $T_i \in Gl(m, \mathbb{R})$, $T_o \in Gl(p, \mathbb{R})$, $F \in \mathbb{R}^{m\times q}$, $K \in \mathbb{R}^{q\times p}$ such that*

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} T_s & T_sK \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix}. \tag{2.1}$$

Any 5-tuple $M_{tran} = (T_s, T_i, T_o, F, K)$ is called a Morse transformation.

If we consider two control systems without outputs, denoted by $\Lambda_{q,m} = (A, B)$ and $\tilde{\Lambda}_{q,m} = (\tilde{A}, \tilde{B})$, then the Morse equivalence reduces to the feedback equivalence, i.e., the corresponding system matrices satisfy $\tilde{A} = T_s(A+BF)T_s^{-1}$ and $\tilde{B} = T_sBT_i^{-1}$.

3. Implicitation of linear control systems and explicitation of linear DAEs.

It is easy to see that if for a linear control system Λ, given by (1.2), we require the output $y = Cz + Du$ to be identically zero, then Λ can be seen as a DAE. We call such an output zeroing procedure the *implicitation* of a control system, which can be formalized as follows.

DEFINITION 3.1 (implicitation). *For a linear control system $\Lambda_{q,m,p} = (A, B, C, D)$ on $\mathscr{X} = \mathbb{R}^q$ with inputs in $\mathscr{U} = \mathbb{R}^m$ and outputs in $\mathscr{Y} = \mathbb{R}^p$, by setting the output y of Λ to be zero, that is,*

$$\mathrm{Impl}(\Lambda) : \begin{cases} \dot{z} = Az + Bu, \\ 0 = Cz + Du, \end{cases}$$

we define the following DAE Δ^{Impl} with "generalized" states $(z, u) \in \mathbb{R}^{q+m}$:

$$\Delta^{Impl} : \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}. \tag{3.1}$$

We call the procedure of output zeroing above the implicitation procedure, and the DAE given by (3.1) will be called the implicitation of Λ and denoted by $\Delta^{Impl}_{q+p,q+m} = \mathrm{Impl}(\Lambda)$ or, briefly, $\Delta^{Impl} = \mathrm{Impl}(\Lambda)$.

The converse procedure, of associating a control system to a given DAE, is less straightforward, since the variables are expressed implicitly in DAEs. Below we show a way to attach a class of control systems to any given DAE.

- Consider a DAE $\Delta_{l,n} = (E, H)$, given by (1.1). Denote $\operatorname{rank} E = q$, and define $p = l - q$ and $m = n - q$. Choose a map
$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in Gl(n, \mathbb{R}),$$
where $P_1 \in \mathbb{R}^{q\times n}$, $P_2 \in \mathbb{R}^{m\times n}$ such that $\ker P_1 = \ker E$.
- Define coordinates transformation
$$\begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} P_1x \\ P_2x \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} x = Px.$$

Then from $\ker P_1 = \ker E$, we have $EP^{-1} = \begin{bmatrix} E_0 & 0 \end{bmatrix}$, where $E_0 \in \mathbb{R}^{l\times q}$. Moreover, since P is invertible, it follows that $\operatorname{rank} E_0 = \operatorname{rank} E = q$. Thus via P, Δ is ex-equivalent to

$$\begin{bmatrix} E_0 & 0 \end{bmatrix} \begin{bmatrix} \dot z \\ \dot u \end{bmatrix} = H_0 \begin{bmatrix} z \\ u \end{bmatrix},$$

where $H_0 = HP^{-1}$. The variables z are states (dynamical variables, their derivatives $\dot z$ are present) and u are controls (enter statically into the system).

- Since $\operatorname{rank} E_0 = q$, there exists $Q_0 \in Gl(l, \mathbb{R})$ such that $Q_0E_0 = \left[\begin{smallmatrix} E_0^1 \\ 0 \end{smallmatrix}\right]$, where $E_0^1 \in Gl(q, \mathbb{R})$. Thus via (Q_0, P), Δ is ex-equivalent to

$$\begin{bmatrix} E_0^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot z \\ \dot u \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix},$$

where $Q_0H_0 = \left[\begin{smallmatrix} A_0 & B_0 \\ C_0 & D_0 \end{smallmatrix}\right]$, $A_0 \in \mathbb{R}^{q\times q}, B_0 \in \mathbb{R}^{q\times m}, C_0 \in \mathbb{R}^{p\times q}, D_0 \in \mathbb{R}^{p\times m}$.

- Finally, via $Q_1 = \left[\begin{smallmatrix} (E_0^1)^{-1} & 0 \\ 0 & I_p \end{smallmatrix}\right]$, we bring the above DAE into

$$\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot z \\ \dot u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}, \tag{3.2}$$

where $A = (E_0^1)^{-1}A_0$, $B = (E_0^1)^{-1}B_0$, $C = C_0$, $D = D_0$.

- Therefore, the DAE Δ is ex-equivalent (via P and $Q = Q_1Q_0$) to (3.2) and the latter is the control system

$$\Lambda : \begin{cases} \dot z = Az + Bu, \\ y = Cz + Du, \end{cases}$$

together with the constraint $y = 0$, that is, $\Delta \overset{ex}{\sim} \Delta^{Impl} = \operatorname{Impl}(\Lambda)$.

Let us give a few comments on the above construction:

(i) The map $P = \left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]$ defines state variables $z = P_1x$ as coordinates on the state space $\mathscr{X} = \mathbb{R}^n / \ker E$ isomorphic to $\mathbb{R}^q$ and control variables $u = P_2x$ as coordinates on $\mathscr{U} \cong \ker E \cong \mathbb{R}^m$. The output variables y are coordinates on $\mathscr{Y} \cong \mathbb{R}^l/\operatorname{Im} E \cong \mathbb{R}^p$ and define the output map via $y = Cz + Du$.

(ii) Choose other coordinates (z', u') given by $z' = P_1'x$ and $u' = P_2'x$ such that $\ker P_1' = \ker E = \ker P_1$; then

$$\begin{cases} z' = T_s z, \\ u' = F'z + T_i u, \end{cases} \tag{3.3}$$

where $T_s \in Gl(q, \mathbb{R})$ and $F' \in \mathbb{R}^{m\times q}$, $T_i \in Gl(m, \mathbb{R})$. Clearly, $z' = T_sz$ is another set of coordinates on the state space $\mathbb{R}^n/\ker E$ and $u' = F'z + T_iu$ is a *state feedback transformation*.

(iii) The output y takes values in the quotient space $\mathbb{R}^l/\operatorname{Im} E$. Since $y = Cz + Du = 0$, we can add y to the dynamics without changing solutions of the system on the subspace $\{y = 0\}$. Together with a state transformation $z' = T_sz$ and an output transformation $y' = T_oy$, it results in a triangular transformation (*output injection*) of the system

$$\begin{bmatrix} \dot z' \\ y' \end{bmatrix} = \begin{bmatrix} T_s & K' \\ 0 & T_o \end{bmatrix} \begin{bmatrix} \dot z \\ y \end{bmatrix} = \begin{bmatrix} T_s & K' \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}, \tag{3.4}$$

where $K' \in \mathbb{R}^{q\times p}$, $T_o \in Gl(p, \mathbb{R})$.

DEFINITION 3.2 (explicitation). *Given a DAE $\Delta_{l,n} = (E, H)$, there always exist $Q \in Gl(l, \mathbb{R})$ and $P \in Gl(n, \mathbb{R})$ such that $QEP^{-1} = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]$. The control system Λ, given by $\Lambda_{q,m,p} = (A, B, C, D)$, where $QHP^{-1} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, is called the (Q, P)-explicitation of Δ. The class of all (Q, P)-explicitations, corresponding to all $Q \in Gl(l, \mathbb{R})$ and $P \in Gl(n, \mathbb{R})$, will be called the explicitation class of Δ and denoted by* $\mathrm{Expl}(\Delta)$. *If a particular control system Λ belongs to the explicitation class* $\mathrm{Expl}(\Delta)$ *of Δ, we will write $\Lambda \in \mathrm{Expl}(\Delta)$.*

Remark 3.3. The implicitation $\mathrm{Impl}(\Lambda)$ of a given control system Λ is a unique DAE Δ^{Impl}, given by (3.1). The explicitation $\mathrm{Expl}(\Delta)$ of a given DAE Δ is, however, a control system defined up to a coordinates change, a feedback transformation, and an output injection, that is, a class of control systems.

THEOREM 3.4. (i) *Consider a DAE $\Delta = (E, H)$ and a control system $\Lambda = (A, B, C, D)$. Then $\Lambda \in \mathrm{Expl}(\Delta)$ if and only if $\Delta \overset{ex}{\sim} \Delta^{Impl}$, where $\Delta^{Impl} = \mathrm{Impl}(\Lambda)$. More precisely, Λ is the (Q, P)-explicitation of Δ if and only if $\Delta \overset{ex}{\sim} \Delta^{Impl}$ via (Q, P).*

(ii) *Given two DAEs $\Delta = (E, H)$ and $\tilde{\Delta} = (\tilde{E}, \tilde{H})$, choose two control systems $\Lambda \in \mathrm{Expl}(\Delta)$ and $\tilde{\Lambda} \in \mathrm{Expl}(\tilde{\Delta})$. Then $\Delta \overset{ex}{\sim} \tilde{\Delta}$ if and only if $\Lambda \overset{M}{\sim} \tilde{\Lambda}$.*

(iii) *Consider two control systems $\Lambda = (A, B, C, D)$ and $\tilde{\Lambda} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. Then $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ if and only if $\Delta^{Impl} \overset{ex}{\sim} \tilde{\Delta}^{Impl}$, where $\Delta^{Impl} = \mathrm{Impl}(\Lambda)$ and $\tilde{\Delta}^{Impl} = \mathrm{Impl}(\tilde{\Lambda})$.*

The proof is given in section 7.1.

Remark 3.5. Theorem 3.4 describes relations of DAEs and control systems, which we illustrate in Figure 1. We conclude that Morse equivalent control systems (and only such) give, via *implicitation,* ex-equivalent DAEs. Furthermore, *explicitation* is a universal procedure of producing control systems from a DAE and ex-equivalent DAEs produce Morse equivalent control systems.

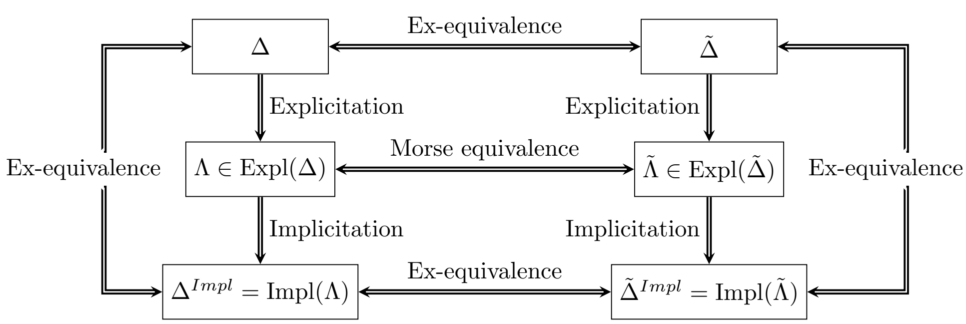

FIG. 1. *Explicitation of DAEs and implicitation of control systems.*

4. Geometric connections between DAEs and control systems. The Wong sequences [26] of a DAE are defined as follows.

DEFINITION 4.1. *For a* DAE $\Delta_{l,n} = (E, H)$*, its Wong sequences are defined by*

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_{i+1} = H^{-1}E\mathscr{V}_i, \quad i \in \mathbb{N}, \tag{4.1}$$

$$\mathscr{W}_0 = \{0\}, \quad \mathscr{W}_{i+1} = E^{-1}H\mathscr{W}_i, \quad i \in \mathbb{N}. \tag{4.2}$$

Remark 4.2. The Wong sequences $\mathscr{V}_i$ and $\mathscr{W}_i$ satisfy

$$\begin{aligned}&\mathscr{V}_0 \supsetneq \mathscr{V}_1 \supsetneq \cdots \supsetneq \mathscr{V}_{k^*} = \mathscr{V}_{k^*+j} = \mathscr{V}^* = H^{-1}E\mathscr{V}^* \supseteq \ker H,\ j \in \mathbb{N},\\ &\mathscr{W}_0 \subseteq \ker E = \mathscr{W}_1 \subsetneq \cdots \subsetneq \mathscr{W}_{l^*} = \mathscr{W}_{l^*+j} = \mathscr{W}^* = E^{-1}H\mathscr{W}^*,\ j \in \mathbb{N}.\end{aligned} \tag{4.3}$$

DEFINITION 4.3. *For a DAE $\Delta_{l,n} = (E, H)$, define $\mathscr{V}^*$ as the largest subspace of $\mathbb{R}^n$ such that $\mathscr{V}^* = H^{-1}E\mathscr{V}^*$ and $\mathscr{W}^*$ as the smallest subspace of $\mathbb{R}^n$ such that $\mathscr{W}^* = E^{-1}H\mathscr{W}^*$.*

Using the same symbols $\mathscr{V}^*$ and $\mathscr{W}^*$ as those for the limits of Wong sequences (see Remark 4.2) is justified by the following.

PROPOSITION 4.4. (i) *For a DAE $\Delta_{l,n} = (E, H)$, the subspaces $\mathscr{V}^*$ and $\mathscr{W}^*$, given by Definition* 4.3*, exist and are given, respectively, by*

$$\mathscr{V}^* = \mathscr{V}_{k^*} \quad \text{and} \quad \mathscr{W}^* = \mathscr{W}_{l^*},$$

where k^* is the smallest integer such that $\mathscr{V}_{k^*} = \mathscr{V}_{k^*+1}$ and l^* is the smallest integer such that $\mathscr{W}_{l^*} = \mathscr{W}_{l^*+1}$.

(ii) *$\mathscr{V}^*$ is also the largest subspace such that $H\mathscr{V}^* \subseteq E\mathscr{V}^*$; however, $\mathscr{W}^*$ is not necessarily the smallest subspace such that $E\mathscr{W}^* \subseteq H\mathscr{W}^*$.*

The proof is given in section 7.2. We now review the notions of invariant subspaces in linear control theory. We consider two cases depending on whether the control system is strictly proper or not (D is zero or not). We will use bold notation for the strictly proper case $D = 0$, since throughout it applies to the prolongation system (1.3), which we denote by bold symbols.

DEFINITION 4.5. *For a control system $\mathbf{\Lambda}_{n,m,p} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$, a subspace $\boldsymbol{\mathcal{V}} \subseteq \mathbb{R}^n$ is called an $(\mathbf{A}, \mathbf{B})$-controlled invariant subspace if $\boldsymbol{\mathcal{V}}$ satisfies*

$$\mathbf{A}\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{V}} + \operatorname{Im}\mathbf{B}$$

and a subspace $\boldsymbol{\mathcal{W}} \subseteq \mathbb{R}^n$ is called a $(\mathbf{C}, \mathbf{A})$-conditioned invariant subspace if $\boldsymbol{\mathcal{W}}$ satisfies

$$\mathbf{A}(\boldsymbol{\mathcal{W}} \cap \ker \mathbf{C}) \subseteq \boldsymbol{\mathcal{W}}.$$

Denote by $\boldsymbol{\mathcal{V}}^*$ the largest $(\mathbf{A}, \mathbf{B})$-controlled invariant subspace contained in $\ker \mathbf{C}$ *and by $\boldsymbol{\mathcal{W}}^*$ the smallest $(\mathbf{C}, \mathbf{A})$-conditioned invariant subspace containing* $\operatorname{Im}\mathbf{B}$.

The following fundamental lemma shows that $\boldsymbol{\mathcal{V}}^*$, $\boldsymbol{\mathcal{W}}^*$ exist and they can be calculated via the sequences of subspaces $\boldsymbol{\mathcal{V}}_i$, $\boldsymbol{\mathcal{W}}_i$ given below.

LEMMA 4.6 (see [27], [1]). *Initialize $\boldsymbol{\mathcal{V}}_0 = \mathbb{R}^n$ and, for $i \in \mathbb{N}$, define inductively*

$$\boldsymbol{\mathcal{V}}_{i+1} = \ker \mathbf{C} \cap \mathbf{A}^{-1}(\boldsymbol{\mathcal{V}}_i + \operatorname{Im}\mathbf{B}). \tag{4.4}$$

Initialize $\boldsymbol{\mathcal{W}}_0 = 0$ and, for $i \in \mathbb{N}$, define inductively

$$\boldsymbol{\mathcal{W}}_{i+1} = \mathbf{A}(\boldsymbol{\mathcal{W}}_i \cap \ker \mathbf{C}) + \operatorname{Im}\mathbf{B}. \tag{4.5}$$

Then there exist $\mathbf{k}^* \leq n$ and $\mathbf{l}^* \leq n$ such that

$$\begin{aligned}&\boldsymbol{\mathcal{V}}_0 \supseteq \ker \mathbf{C} = \boldsymbol{\mathcal{V}}_1 \supsetneq \cdots \supsetneq \boldsymbol{\mathcal{V}}_{\mathbf{k}^*} = \boldsymbol{\mathcal{V}}_{\mathbf{k}^*+j} = \boldsymbol{\mathcal{V}}^* = \ker \mathbf{C} \cap \mathbf{A}^{-1}(\boldsymbol{\mathcal{V}}^* + \operatorname{Im}\mathbf{B}), \quad j \in \mathbb{N},\\ &\boldsymbol{\mathcal{W}}_0 \subseteq \operatorname{Im}\mathbf{B} = \boldsymbol{\mathcal{W}}_1 \subsetneq \cdots \subsetneq \boldsymbol{\mathcal{W}}_{\mathbf{l}^*} = \boldsymbol{\mathcal{W}}_{\mathbf{l}^*+j} = \boldsymbol{\mathcal{W}}^* = \mathbf{A}(\boldsymbol{\mathcal{W}}^* \cap \ker \mathbf{C}) + \operatorname{Im}\mathbf{B},\ j \in \mathbb{N}.\end{aligned}$$

Note that $\mathbf{k}^*$ and $\mathbf{l}^*$ of Lemma 4.6 and k^* and l^* of Remark 4.2 are, in general, not the same (except for some cases described later (see Theorem 4.10) in which they coincide). It is well-known (see, e.g., [28], [27], [1]) that $\boldsymbol{\mathcal{V}}$ is an $(\mathbf{A},\mathbf{B})$-controlled invariant subspace if and only if there exists $\mathbf{F}\in\mathbb{R}^{m\times n}$ such that $(\mathbf{A}+\mathbf{B}\mathbf{F})\boldsymbol{\mathcal{V}}\subseteq\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\mathcal{W}}$ is a $(\mathbf{C},\mathbf{A})$-conditioned invariant subspace if and only if there exists $\mathbf{K}\in\mathbb{R}^{n\times p}$ such that $(\mathbf{A}+\mathbf{K}\mathbf{C})\boldsymbol{\mathcal{W}}\subseteq\boldsymbol{\mathcal{W}}$. For a control system which is not strictly proper (D is not zero), following Definitions 1–4 of [19], we use a generalization of that characterization of invariant subspaces.

Definition 4.7. *For $\Lambda_{q,m,p}=(A,B,C,D)$, a subspace $\mathcal{V}\subseteq\mathbb{R}^q$ is called a null-output (A,B)-controlled invariant subspace if there exists $F\in\mathbb{R}^{m\times q}$ such that*

$$(A+BF)\mathcal{V}\subseteq\mathcal{V}\quad\text{and}\quad(C+DF)\mathcal{V}=0,$$

and for any such $\mathcal{V}$, the subspace $\mathcal{U}\subseteq\mathbb{R}^m$ given by

$$\mathcal{U}=(B^{-1}\mathcal{V})\cap\ker D$$

is called a null-output (A,B)-controlled invariant input subspace. Denote by $\mathcal{V}^*$ (resp. $\mathcal{U}^*$) the largest null-output (A,B)-controlled invariant subspace (resp., input subspace).

A subspace $\mathcal{W}\subseteq\mathbb{R}^q$ is called an unknown-input (C,A)-conditioned invariant subspace if there exists $K\in\mathbb{R}^{q\times p}$ such that

$$(A+KC)\mathcal{W}+(B+KD)\mathscr{U}=\mathcal{W},$$

and for any such $\mathcal{W}$, the subspace $\mathcal{Y}\subseteq\mathbb{R}^p$ given by

$$\mathcal{Y}=C\mathcal{W}+D\mathscr{U},$$

where $\mathscr{U}=\mathbb{R}^m$, is called an unknown-input (C,A)-conditioned invariant output subspace. Denote by $\mathcal{W}^*$ (resp., $\mathcal{Y}^*$) the smallest unknown-input (C,A)-conditioned invariant subspace (resp., output subspace).

The following lemma shows that $\mathcal{V}^*$, $\mathcal{U}^*$, $\mathcal{W}^*$, $\mathcal{Y}^*$ exist and provides a calculable algorithm to find them.

Lemma 4.8 (see [18]). *Initialize $\mathcal{V}_0=\mathbb{R}^q$, and for $i\in\mathbb{N}$, define inductively*

$$\mathcal{V}_{i+1}=\begin{bmatrix}A\\C\end{bmatrix}^{-1}\left(\begin{bmatrix}I\\0\end{bmatrix}\mathcal{V}_i+\operatorname{Im}\begin{bmatrix}B\\D\end{bmatrix}\right)\tag{4.6}$$

and $\mathcal{U}_i\subseteq\mathscr{U}$, where $\mathscr{U}=\mathbb{R}^m$, for $i\in\mathbb{N}$ are given by

$$\mathcal{U}_i=\begin{bmatrix}B\\D\end{bmatrix}^{-1}\begin{bmatrix}\mathcal{V}_i\\0\end{bmatrix}.\tag{4.7}$$

Then $\mathcal{V}^*=\mathcal{V}_q$ and $\mathcal{U}^*=\mathcal{U}_q$.

Initialize $\mathcal{W}_0=\{0\}$, and for $i\in\mathbb{N}$, define inductively

$$\mathcal{W}_{i+1}=\begin{bmatrix}A & B\end{bmatrix}\left(\begin{bmatrix}\mathcal{W}_i\\\mathscr{U}\end{bmatrix}\cap\ker\begin{bmatrix}C & D\end{bmatrix}\right)\tag{4.8}$$

and $\mathcal{Y}_i\subseteq\mathscr{Y}$, where $\mathscr{Y}=\mathbb{R}^p$, for $i\in\mathbb{N}$ are given by

$$\mathcal{Y}_i=\begin{bmatrix}C & D\end{bmatrix}\begin{bmatrix}\mathcal{W}_i\\\mathscr{U}\end{bmatrix}.\tag{4.9}$$

Then $\mathcal{W}^*=\mathcal{W}_q$ and $\mathcal{Y}^*=\mathcal{Y}_q$.

Remark 4.9. (i) Lemma 4.8 generalizes the results of Lemma 4.6 and, if $D=0$, Lemma 4.8 reduces to Lemma 4.6. Note that if D is invertible, then $\mathcal{V}^*=\mathcal{V}_i=\mathbb{R}^q$ and $\mathcal{W}^*=\mathcal{W}_i=0$ for all $i\geq 0$.

(ii) Even if Λ is not strictly proper (if $D \neq 0$), the prolonged system $\mathbf{\Lambda}$ always is and thus throughout we will use $\mathcal{V}^*$, $\mathcal{U}^*$, $\mathcal{W}^*$, and $\mathcal{Y}^*$ for Λ, and $\boldsymbol{\mathcal{V}}^*$ and $\boldsymbol{\mathcal{W}}^*$ for $\mathbf{\Lambda}$.

Throughout the paper, for ease of notation, we will write $\mathscr{V}_i(\Delta)$ to indicate that $\mathscr{V}_i$ is calculated for Δ, similarly for $\mathcal{V}_i(\Lambda)$, $\boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda})$, and all other subspaces.

THEOREM 4.10 (geometric subspaces relations). *Given a DAE* $\Delta_{l,n} = (E, H)$*, a* (Q, P)*-explicitation* $\Lambda = (A, B, C, D) \in \mathrm{Expl}(\Delta)$*, and the prolongation* $\mathbf{\Lambda} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ *of* Λ*, consider the limits of the Wong sequences* $\mathscr{V}^*$ *and* $\mathscr{W}^*$ *of* Δ *and of* $\Delta^{Impl} = \mathrm{Impl}(\Lambda)$*, given by Definition* 4.3*, the invariant subspaces* $\mathcal{V}^*$ *and* $\mathcal{W}^*$ *of* Λ*, given by Definition* 4.7*, and the invariant subspaces* $\boldsymbol{\mathcal{V}}^*$ *and* $\boldsymbol{\mathcal{W}}^*$ *of* $\mathbf{\Lambda}$*, given by Definition* 4.5*. Then the following holds:*

(i) $P\mathscr{V}^*(\Delta) = \mathscr{V}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}^*(\mathbf{\Lambda}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}^*(\Lambda) \\ 0 \end{bmatrix}$,

(ii) $P\mathscr{W}^*(\Delta) = \mathscr{W}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{W}}^*(\mathbf{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}^*(\Lambda) \\ 0 \end{bmatrix}$.

The proof is given in section 7.3.

Remark 4.11. The limits $\mathscr{V}^*$ and $\mathscr{W}^*$ of the Wong sequences coincide for Δ and $\tilde{\Delta}$ that are ex-equivalent via (P, Q), where $P = I_n$ and Q is arbitrary, and do not depend on Q. On the other hand, the system Λ, being a (Q, P)-explicitation of Δ, depends on both P and Q (and so does its prolongation $\mathbf{\Lambda}$) but the invariant subspaces $\mathcal{V}^*(\Lambda)$ and $\mathcal{W}^*(\Lambda)$ depend on P only.

5. Relations between the Kronecker invariants and the Morse invariants.

In this section, we discuss relations of the Kronecker invariants and the Morse invariants (see the appendix). An early result discussing these two sets of invariants goes back to [10], where it is observed that the controllability indices of the pair (A, B) and the Kronecker column indices of the matrix pencil $sE - H$, where $E = [I, 0]$ and $H = [A, B]$, coincide, which can be seen as a special case of the result in this section. Also in [15], it is shown that the Morse indices of the triple (A, B, C) have direct relations with the Kronecker indices of the matrix pencil (called the restricted matrix pencil; see [9]) $N(sI - A)K$, where the rows of N span the annihilator of $\mathrm{Im}\, B$ and the colunms of K span $\ker C$.

It is known (see the appendix) that any DAE can be transformed into its **KCF** which is completely determined by the Kronecker invariants $\varepsilon_1, \ldots, \varepsilon_a$, $\rho_1, \ldots, \rho_b$, $\sigma_1, \ldots, \sigma_c$, $\eta_1, \ldots, \eta_d$, the numbers a, b, c, d of blocks, and the $(\lambda_{\rho_1}, \ldots, \lambda_{\rho_b})$-structure (by the latter we mean the eigenvalues, together with the dimensions $\rho_1, \ldots, \rho_b$ of the corresponding blocks). The Kronecker invariants (except for ρ_i's and the corresponding eigenvalues λ_{ρ_i}'s) can be computed using the Wong sequences as follows. For a DAE $\Delta = (E, H)$, consider the Wong sequences $\mathscr{V}_i$ and $\mathscr{W}_i$ of Definition 4.1, and define $\mathscr{K}_i = \mathscr{W}_i \cap \mathscr{V}^*$ and $\hat{\mathscr{K}}_i = (E\mathscr{V}_{i-1})^{\perp} \cap (H\mathscr{W}^*)^{\perp}$ for $i \in \mathbb{N}^+$.

LEMMA 5.1 (see [6], [7]). *For the* ***KCF*** *of* Δ*, we have*

(i) $a = \dim(\mathscr{K}_1)$*,* $d = \dim(\hat{\mathscr{K}}_1)$*, and*

$$\begin{cases} \varepsilon_j = 0 & for \quad 1 \leq j \leq a - \omega_0, \\ \varepsilon_j = i & for \quad a - \omega_{i-1} + 1 \leq j \leq a - \omega_i, \end{cases} \tag{5.1}$$

$$\begin{cases} \eta_j = 0 & for \quad 1 \leq j \leq d - \hat{\omega}_0, \\ \eta_j = i & for \quad d - \hat{\omega}_{i-1} + 1 \leq j \leq d - \hat{\omega}_i, \end{cases} \tag{5.2}$$

where $\omega_i = \dim(\mathscr{K}_{i+2}) - \dim(\mathscr{K}_{i+1})$ *and* $\hat{\omega}_i = \dim(\hat{\mathscr{K}}_{i+2}) - \dim(\hat{\mathscr{K}}_{i+1})$*,* $i \in \mathbb{N}$*.*

(ii) *Define an integer* ν *by*

$$\nu = \min\{i \in \mathbb{N} \,|\, \mathscr{V}^* + \mathscr{W}_i = \mathscr{V}^* + \mathscr{W}_{i+1}\}. \tag{5.3}$$

Then either $\nu = 0$*, implying that the nilpotent part* $N(s)$ *is absent, or* $\nu > 0$*, in which case* $c = \pi_0$ *and*

$$\sigma_j = i \quad for \quad c - \pi_{i-1} + 1 \le j \le c - \pi_i, \quad i = 1, 2, \ldots, \nu, \tag{5.4}$$

where $\pi_i = \dim(\mathscr{W}_{i+1} + \mathscr{V}^*) - \dim(\mathscr{W}_i + \mathscr{V}^*)$ *for* $i = 0, 1, 2, \ldots, \nu$ *(in the case of* $\pi_{i-1} = \pi_i$*, the respective index range is empty).*

Any control system $\Lambda = (A, B, C, D)$ can be transformed via a Morse transformation into its Morse canonical form **MCF**, which is determined by the Morse indices $\varepsilon'_1, \ldots, \varepsilon'_{a'}, \rho'_1, \ldots, \rho'_{b'}, \sigma'_1, \ldots, \sigma'_{c'}, \eta'_1, \ldots, \eta'_{d'}$, the $(\lambda_{\rho'_1}, .., \lambda_{\rho'_{b'}})$-structure and the numbers $a', b', c', d' \in \mathbb{N}$ of blocks. The following results can be deduced from the results on the Morse indices in [20], [19]. For $\Lambda = (A, B, C, D)$, consider the subspaces $\mathcal{V}_i$, $\mathcal{W}_i$, $\mathcal{U}_i$, $\mathcal{Y}_i$ as in Lemma 4.8, and define $\mathcal{R}_i = \mathcal{W}_i \cap \mathcal{V}^*$ and $\hat{\mathcal{R}}_i = (\mathcal{V}_i)^{\perp} \cap (\mathcal{W}^*)^{\perp}$ for $i \in \mathbb{N}$.

LEMMA 5.2. *For the **MCF** of* Λ*, we have*

(i) $a' = \dim(\mathcal{U}^*)$*,* $d' = \dim(\mathcal{Y}^*)$*, and*

$$\begin{cases} \varepsilon'_j = 0 & for \qquad 1 \le j \le a' - \omega'_0, \\ \varepsilon'_j = i & for \quad a' - \omega'_{i-1} + 1 \le j \le a' - \omega'_i, \end{cases} \tag{5.5}$$

$$\begin{cases} \eta'_j = 0 & for \qquad 1 \le j \le d' - \hat{\omega}'_0, \\ \eta'_j = i & for \quad d' - \hat{\omega}'_{i-1} + 1 \le j \le d' - \hat{\omega}', \end{cases} \tag{5.6}$$

where $\omega'_i = \dim(\mathcal{R}_{i+1}) - \dim(\mathcal{R}_i)$ *and* $\hat{\omega}'_i = \dim(\hat{\mathcal{R}}_{i+1}) - \dim(\hat{\mathcal{R}}_i)$*,* $i \in \mathbb{N}$*.*

(ii) *Define an integer* ν' *by*

$$\nu' = \min\{i \in \mathbb{N} \,|\, \mathcal{V}^* + \mathcal{W}_i = \mathcal{V}^* + \mathcal{W}_{i+1}\}.$$

Then $c' = \dim(\mathscr{U}) - \dim(\mathcal{U}^*)$*,* $\delta = c' - \pi'_0$ *and*

$$\begin{cases} \sigma'_j = 0 & for \quad 1 \le j \le \delta, \\ \sigma'_j = i & for \quad c' - \pi'_{i-1} + 1 \le j \le c' - \pi'_i, \quad i = 1, 2, \ldots, \nu', \end{cases} \tag{5.7}$$

where $\pi'_i = \dim(\mathcal{W}_{i+1} + \mathcal{V}^*) - \dim(\mathcal{W}_i + \mathcal{V}^*)$ *for* $i = 0, 1, 2, \ldots, \nu'$ *(in case of* $\pi'_{i-1} = \pi'_i$ *the respective index range is empty).*

Note that for $\Lambda = (A, B, C, D)$, the integer δ in (5.7) is $\delta = \operatorname{rank} D$. If $\delta = 0$, then $c' = \pi'_0$ and the first row of (5.7) is absent, which implies that all $\sigma'_j \neq 0$. Formal similarities between the statements of Lemmata 5.1 and 5.2 suggest possible relations between the Kronecker and the Morse invariants.

THEOREM 5.3 (invariants relations). *For a DAE* $\Delta_{l,n} = (E, H)$*, consider its Kronecker invariants*

$$(\varepsilon_1, \ldots, \varepsilon_a),\ (\rho_1, \ldots, \rho_b),\ (\sigma_1, \ldots, \sigma_c),\ (\eta_1, \ldots, \eta_d),\ (\lambda_{\rho_1}, \ldots, \lambda_{\rho_b}) \text{ with } a, b, c, d \in \mathbb{N},$$

*of the **KCF**, and for a control system* $\Lambda_{q,m,p} = (A, B, C, D) \in \mathrm{Expl}(\Delta)$*, consider its Morse invariants*

$$(\varepsilon'_1, \ldots, \varepsilon'_{a'}),\ (\rho'_1, \ldots, \rho'_{b'}),\ (\sigma'_1, \ldots, \sigma'_{c'}),\ (\eta'_1, \ldots, \eta'_{d'}),\ (\lambda_{\rho'_1}, .., \lambda_{\rho'_{b'}}) \text{ with } a', b', c', d' \in \mathbb{N},$$

*of the **MCF**. Then the following holds:*

(i) $a = a'$, $\varepsilon_1 = \varepsilon'_1, \dots, \varepsilon_a = \varepsilon'_{a'}$, *and* $d = d'$, $\eta_1 = \eta'_1, \dots, \eta_d = \eta'_{d'}$.

(ii) $N(s)$ *of the* ***KCF*** *is present if and only if the subsystem* MCF^3 *of the* ***MCF*** *is present. Moreover, if they are present, then their invariants satisfy*

$$c = c', \quad \sigma_1 = \sigma'_1 + 1, \dots, \sigma_c = \sigma'_{c'} + 1.$$

(iii) *The invariant factors of* $J(s)$ *in the* ***KCF*** *of* Δ *coincide with those of* MCF^2 *in the* ***MCF*** *of* Λ. *Furthermore, the corresponding invariants satisfy*

$$b = b', \quad \rho_1 = \rho'_1, \dots, \rho_b = \rho'_{b'}, \quad \lambda_{\rho_1} = \lambda_{\rho'_1}, \dots, \lambda_{\rho_b} = \lambda_{\rho'_{b'}}.$$

The proof is given in section 7.4. Notice that in item (ii) of Theorem 5.3, the invariants σ_i and σ'_i do not coincide but differ by one; the reason is that the nilpotent indices[1] $\sigma_1, \dots, \sigma_c$ of $N(s)$ cannot be zero (the minimum nilpotent index is 1 and if σ_i is 1, then $N(s)$ contains the 1×1 matrix pencil $0 \cdot s - 1$), but the controllability and observability indices $\sigma'_1, \dots, \sigma'_{c'}$ of MCF^3 can be zero (if $\sigma'_i = 0$, then the output y^3 of MCF^3 contains the static relation $y^3_i = u^3_i$). It is easy to see from Theorem 5.3 that, given a DAE, there exists a perfect correspondence between the **KCF** of the DAE and the **MCF** of its explicitation systems. More specifically, the four parts of the **KCF** correspond to the four subsystems of the **MCF**: the bidiagonal pencil $L(s)$ to the controllable but unobservable part MCF^1, the Jordan pencil $J(s)$ to the uncontrollable and unobservable part MCF^2, the nilpotent pencil $N(s)$ to the prime part MCF^3, and the pencil $L^p(s)$ to the observable but uncontrollable part MCF^4.

Now we describe connections between the quasi-Weierstrass form for a regular DAE [4] and the Morse normal form [19] for its explicitation system; we also show that the notion of consistency and differential projectors [24] in DAE theory could also be calculated through some objects from control systems.

Remark 5.4. It is shown in [4] that, for a regular DAE $\Delta_{n,n} = (E, H)$, set $\hat{P} = [\hat{V}\ \hat{W}]^{-1}$ and $\hat{Q} = [E\hat{V}\ E\hat{W}]^{-1}$, where $\hat{V}$ and $\hat{W}$ are full column rank matrices such that $\operatorname{Im}\hat{V} = \mathscr{V}^*(\Delta)$ and $\operatorname{Im}\hat{W} = \mathscr{W}^*(\Delta)$, and then via $(\hat{Q}, \hat{P})$, Δ is ex-equivalent to its quasi-Weierstrass form:

$$\hat{Q}(sE - H)\hat{P}^{-1} = \begin{bmatrix} sI_{n_1} - J & 0 \\ 0 & sN - I_{n_2} \end{bmatrix}$$

for some matrix J and where N is a nilpotent matrix, $\dim \mathscr{V}^* = n_1$, and $\dim \mathscr{W}^* = n_2$. Then define (see Definition 2.4 of [24])

$$\Pi_{(E,H)} := \hat{P}^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \hat{P}, \qquad \Pi^{\mathrm{diff}}_{(E,H)} := \hat{Q}^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \hat{P},$$

where $\Pi_{(E,H)}$ and $\Pi^{\mathrm{diff}}_{(E,H)}$ are called the consistency and differential projectors of Δ, respectively. Now let $\Lambda_{n,m,m} = (A, B, C, D)$ be a (Q, P)-explicitation of Δ, which implies that

$$Q\,(sE - H)\,P^{-1} = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix}.$$

[1] Note that for a regular DAE, the index of the DAE is usually defined by the nilpotent index of $N(s)$ in its Weierstrass form (see, e.g., [5]), which is clearly the maximal value of the indices σ_i using the notation of the present paper.

Set $T_s = [V\ W]^{-1}$, $T_i = I_m$, $T_o = I_p$, where V and W are full column rank matrices such that $\operatorname{Im} V = \mathcal{V}^*(\Lambda)$ and $\operatorname{Im} W = \mathcal{W}^*(\Lambda)$, respectively; then there exists F and K, such that via T_s, T_i, T_o, F, K, Λ is M-equivalent to its Morse normal form:

$$\begin{bmatrix} T_s & T_sK \\ 0 & T_o \end{bmatrix} \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A^2 & 0 & 0 \\ 0 & sI_{n_2-m} - A^3 & -B^3 \\ 0 & -C^3 & D^3 \end{bmatrix},$$

where (A^3, B^3, C^3, D^3) is prime. It is clear that the two blocks of the quasi-Weierstrass form have a one-to-one correspondence with those of the Morse normal form, i.e., $sI_{n_1} - J$ with $sI_{n_1} - A^2$ and $sN - I_{n_2}$ with $\left[\begin{smallmatrix} sI_{n_2-m}-A^3 & -B^3 \\ -C^3 & D^3 \end{smallmatrix}\right]$. We can also deduce that

$$\hat{Q} \cong \begin{bmatrix} T_s & T_sK \\ 0 & T_o \end{bmatrix} Q \text{ and } \hat{P}^{-1} = P^{-1} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix},$$

where "$\cong$" stands for equality, up to a rows permutation. The dimensions $n_1 = \sum_{i=1}^{b} \rho_i = \sum_{i=1}^{b'} \rho_i'$ and $n_2 = \sum_{i=1}^{c} \sigma_i = c' + \sum_{i=1}^{c'} \sigma_i'$, where $\rho_1, \dots, \rho_b$ and $\sigma_1, \dots, \sigma_c$ are the **KCF** invariants of Δ and $\rho_1', \dots, \rho_{b'}'$ and $\sigma_1', \dots, \sigma_{c'}'$ are the **MCF** invariants of Λ. Since the projectors $\Pi_{(E,H)}$ and $\Pi_{(E,H)}^{\mathrm{diff}}$ depend only on $\hat{Q}$ and $\hat{P}$ constructed with the help of the subspaces $\mathscr{V}^*$ and $\mathscr{W}^*$ of Δ, we conclude that they can also be calculated via the Morse transformations constructed with the help of the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$ of $\Lambda \in \operatorname{Expl}(\Delta)$.

6. Internal equivalence and regularity of DAEs. An important difference between DAEs and ODEs is that DAEs are not always solvable and solutions of DAEs exist on a subspace of the "generalized" state space only, due to the presence of algebraic constrains. In the following, we show that the existence and uniqueness of solutions of DAEs can be clearly explained using the *explicitation* procedure and the notion of internal equivalence (see Definition 6.8 below).

Definition 6.1. *A linear subspace $\mathscr{M}$ of $\mathbb{R}^n$ is called an invariant subspace of $\Delta_{l,n} = (E, H)$ if for any $x^0 \in \mathscr{M}$, there exists a solution $x(t, x^0)$ of Δ such that $x(0, x^0) = x^0$ and $x(t, x^0) \in \mathscr{M}$ for all $t \in \mathbb{R}$. An invariant subspace $\mathscr{M}^*$ of $\Delta_{l,n} = (E, H)$ is called the maximal invariant subspace if for any other invariant subspace $\mathscr{M}$ of $\mathbb{R}^n$, we have $\mathscr{M} \subseteq \mathscr{M}^*$.*

Remark 6.2. (i) Note that due to the existence of free variables among the "generalized" states, solutions of Δ are not unique. Thus it is possible that one solution of Δ starting at $x^0 \in \mathscr{M}$ stays in $\mathscr{M}$ but other solutions starting at x^0 may escape from $\mathscr{M}$ (either immediately or after a finite time).

(ii) Our notion of the largest invariant subspace $\mathscr{M}^*$ coincides with the notion of consistency space in the linear DAEs literature (see, e.g., [24]), which is the subspace where the solutions of the DAE exist.

It is clear that the sum $\mathscr{M}_1 + \mathscr{M}_2$ of two invariant subspaces of Δ is also invariant. Therefore, $\mathscr{M}^*$ exists and is, actually, the sum of all invariant subspaces. Given an invariant subspace $\mathscr{M}$ of $\Delta_{l,n}$, for any $x^0 \in \mathscr{M}$, there exists a solution $x(t)$ such that $x(0) = x^0$. It is natural to restrict Δ to $\mathscr{M}$ (keeping only those solutions that stay in $\mathscr{M}$ and eliminating all others), in particular, to the largest invariant subspace $\mathscr{M}^*$. Moreover, we would like the restriction to be as simple as possible. We achieve the above goals by introducing, respectively, the notion of *restriction* and that of *reduction*. We will define the restriction of a DAE Δ to a linear subspace $\mathscr{R}$ (invariant or not) as follows.

DEFINITION 6.3 (restriction). *Consider a linear DAE $\Delta_{l,n} = (E, H)$. Let $\mathscr{R}$ be a subspace of $\mathbb{R}^n$. The restriction of Δ to $\mathscr{R}$, called $\mathscr{R}$-restriction of Δ and denoted $\Delta|_{\mathscr{R}}$, is a linear DAE $\Delta|_{\mathscr{R}} = (E|_{\mathscr{R}}, H|_{\mathscr{R}})$, where $E|_{\mathscr{R}}$ and $H|_{\mathscr{R}}$ are, respectively, the restrictions of the linear maps[2] E and H to the linear subspace $\mathscr{R}$.*

Throughout, we consider general DAEs $\Delta_{l,n} = (E, H)$ with no assumptions on the ranks of E and H. In particular, if the map $[E\ H]$ is no full row rank, then $\Delta_{l,n}$ contains redundant equations. But even if we assume that $[E\ H]$ is of full row rank, then this property, in general, is no longer true for the restricted map $[E|_{\mathscr{R}}\ H_{\mathscr{R}}]$, which may contain redundant equations. To get rid of redundant equations (in particular, of trivial algebraic equations $0 = 0$), we propose the notion of full row rank reduction.

DEFINITION 6.4 (reduction). *For a DAE $\Delta_{l,n} = (E, H)$ on $\mathscr{X} \cong \mathbb{R}^n$, assume $\operatorname{rank}[E\ H] = l^* \leq l$. Then there exists $Q \in Gl(l, \mathbb{R})$ such that*

$$Q \begin{bmatrix} E & H \end{bmatrix} = \begin{bmatrix} E^{red} & H^{red} \\ 0 & 0 \end{bmatrix},$$

where $\operatorname{rank}[E^{red}\ H^{red}] = l^*$ and the full row rank reduction, briefly, reduction, of $\Delta_{l,n}$, denoted by Δ^{red}, is a DAE $\Delta^{red}_{l^*,n} = \Delta^{red} = (E^{red}, H^{red})$ on $\mathscr{X} \cong \mathbb{R}^n$.

Remark 6.5. Clearly, the choice of Q is not unique and thus the reduction of Δ is not unique either. Nevertheless, since Q preserves the solutions, each reduction Δ^{red} has the same solutions as the original DAE Δ.

For an invariant subspace $\mathscr{M}$, we consider the $\mathscr{M}$-restriction $\Delta|_{\mathscr{M}}$ of Δ, and then we construct a reduction of $\Delta|_{\mathscr{M}}$ and denote it by $\Delta|^{red}_{\mathscr{M}} = (E|^{red}_{\mathscr{M}}, H|^{red}_{\mathscr{M}})$. Notice that the order matters: to construct $\Delta|^{red}_{\mathscr{M}}$, we first restrict and then reduce, while reducing first and then restricting will, in general, give not $\Delta|^{red}_{\mathscr{M}}$ (which does not have redundant equations) but another DAE $\Delta^{red}|_{\mathscr{M}}$ (which may still have redundant equations). As an example, consider a DAE $\Delta : \left[\begin{smallmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 0 \\ 2 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 1 \\ 2 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ and an invariant subspace $\mathscr{M} = \mathscr{X}_1 = \operatorname{span}\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\}$ of Δ; then $\Delta|^{red}_{\mathscr{M}}$ is $q\dot{x}_1 = qx_1$ (where $q \neq 0$ can be any indicating that the reduction is not unique) and thus has no redundant equations, while a DAE $\Delta^{red}|_{\mathscr{M}} : \left[\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix}\right]\dot{x}_1 = \left[\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix}\right]x_1$ has clearly redundant equations.

PROPOSITION 6.6. *Consider a linear DAE $\Delta_{l,n} = (E, H)$. Let $\mathscr{M}$ be a subspace of $\mathbb{R}^n$. The following statements are equivalent:*

(i) *$\mathscr{M}$ is an invariant subspace of $\Delta_{l,n}$.*
(ii) *$H\mathscr{M} \subseteq E\mathscr{M}$.*
(iii) *For a (and thus any) reduction $\Delta|^{red}_{\mathscr{M}} = (E|^{red}_{\mathscr{M}}, H|^{red}_{\mathscr{M}})$ of $\Delta|_{\mathscr{M}}$, the map $E|^{red}_{\mathscr{M}}$ is of full row rank, i.e., $\operatorname{rank} E|^{red}_{\mathscr{M}} = \operatorname{rank}[E|^{red}_{\mathscr{M}}\ H|^{red}_{\mathscr{M}}]$.*

Proof. (i)⇔(ii) Theorem 4 of [3], for $B = 0$, implies that $\mathscr{M}$ is an invariant subspace if and only if $H\mathscr{M} \subseteq E\mathscr{M}$.

(ii)⇔(iii) For $\Delta_{l,n} = (E, H)$, choose a full column rank matrix $P_1 \in \mathbb{R}^{n \times n_1}$ such that $\operatorname{Im} P_1 = E\mathscr{M}$, where $n_1 = \dim \mathscr{M}$. Find any $P_2 \in \mathbb{R}^{n \times n_2}$ such that the matrix $[P_1\ P_2]$ is invertible, where $n_2 = n - n_1$. Choose new coordinates $z = Px$, where $P = [P_1\ P_2]^{-1}$; then we have

[2]For an r-dimensional subspace $\mathscr{R} \subseteq \mathbb{R}^n$, choose a basis $q_1, \ldots, q_n$ of $\mathbb{R}^n$ such that $q_1, \ldots, q_r$ is a basis of $\mathscr{R}$. The matrix $E \in \mathbb{R}^{l \times n}$, representing a linear map in that basis, is $E = [E_1\ E_2]$, where $E_1 \in \mathbb{R}^{l \times k}$, and then the restriction of E to $\mathscr{R}$ is $E|_{\mathscr{R}} = E_1$.

$$\Delta : EP^{-1}P\dot{x} = HP^{-1}Px \Rightarrow [E_1\ E_2]\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = [H_1\ H_2]\begin{bmatrix}z_1\\ z_2\end{bmatrix},$$

where $E_1 = EP_1$, $E_2 = EP_2$, $H_1 = HP_1$, $H_2 = HP_2$, and $z = (z_1, z_2)$. Now by Definition 6.3, the $\mathscr{M}$-restriction of Δ is

$$\Delta|_{\mathscr{M}} : E_1\dot{z}_1 = H_1 z_1.$$

Find $Q \in Gl(l, \mathbb{R})$ such that $QE_1 = [\begin{smallmatrix}\tilde{E}_1\\ 0\end{smallmatrix}]$, where $\tilde{E}_1$ is of full row rank, then denote $QH_1 = [\begin{smallmatrix}\tilde{H}_1\\ \bar{H}_1\end{smallmatrix}]$. By $H\mathscr{M} \subseteq E\mathscr{M}$, we can deduce that $\bar{H}_1 = 0$ (since $QH\mathscr{M} \subseteq QE\mathscr{M} \Rightarrow \mathrm{Im}[\begin{smallmatrix}\tilde{H}_1\\ \bar{H}_1\end{smallmatrix}] \subseteq \mathrm{Im}[\begin{smallmatrix}\tilde{E}_1\\ 0\end{smallmatrix}]$). Thus a reduction of $\Delta|_{\mathscr{M}}$, according to Definition 6.4, is $\Delta|_{\mathscr{M}}^{red} = (E|_{\mathscr{M}}^{red}, H|_{\mathscr{M}}^{red}) = (\tilde{E}_1, \tilde{H}_1)$. Clearly $E|_{\mathscr{M}}^{red}$ is of full row rank. $\square$

Define $\Lambda|_{(\mathcal{V}^*,\mathcal{U}^*)}$ as the control system $\Lambda = (A, B, C, D)$ restricted to $\mathcal{V}^*$ (which is well-defined because $\mathcal{V}^*$ can be made invariant by a suitable feedback) and with controls u restricted to $\mathcal{U}^* = (B^{-1}\mathcal{V}^*) \cap \ker D$. The output $y = Cx + Du$ of Λ becomes $y = 0$ and $\Lambda|_{(\mathcal{V}^*,\mathcal{U}^*)}^{red}$ is, by its construction, the system $\Lambda|_{(\mathcal{V}^*,\mathcal{U}^*)}$ without the redundant trivial output constraint $y = 0$.

PROPOSITION 6.7. *For a DAE $\Delta_{l,n} = (E, H)$, consider its maximal invariant subspace $\mathscr{M}^*$ and the subspace $\mathscr{V}^*$ in Definition* 4.3. *Then we have as follows:*

(i) $\mathscr{M}^* = \mathscr{V}^*$.

(ii) *Let $\Lambda \in \mathrm{Expl}(\Delta)$ and $\Lambda^* \in \mathrm{Expl}(\Delta|_{\mathscr{M}^*}^{red})$. Then $\Lambda|_{(\mathcal{V}^*,\mathcal{U}^*)}^{red}$ and Λ^* are explicit control systems without outputs, i.e., the* ***MCF*** *of the two control systems has no MCF^3 and MCF^4 parts, and $\Lambda|_{(\mathcal{V}^*,\mathcal{U}^*)}^{red}$ is feedback equivalent to Λ^*.*

The proof is given in section 7.5. By Theorem 4.10, we can relate $\mathscr{M}^*$ of Δ with the corresponding space of the prolongation $\boldsymbol{\Lambda}$ of $\Lambda \in \mathrm{Expl}(\Delta)$, where Λ is a (Q, P)-explicitation. Namely, we have $P\mathscr{M}^* = \boldsymbol{\mathcal{V}}^*(\boldsymbol{\Lambda})$. Using the reduction of $\mathscr{M}^*$-restriction and the ex-equivalence of DAEs, we define the internal equivalence of two DAEs as follows.

DEFINITION 6.8. *For two DAEs $\Delta_{l,n} = (E, H)$ and $\tilde{\Delta}_{\tilde{l},\tilde{n}} = (\tilde{E}, \tilde{H})$, let $\mathscr{M}^*$ and $\tilde{\mathscr{M}}^*$ be the maximal invariant subspaces of Δ and $\tilde{\Delta}$, respectively. Then Δ and $\tilde{\Delta}$ are called internally equivalent, briefly, in-equivalent, if $\Delta|_{\mathscr{M}^*}^{red}$ and $\tilde{\Delta}|_{\tilde{\mathscr{M}}^*}^{red}$ are ex-equivalent and we will denote the in-equivalence of two DAEs as $\Delta \overset{in}{\sim} \tilde{\Delta}$.*

Any $\Lambda^* \in \mathrm{Expl}(\Delta|_{\mathscr{M}^*}^{red})$ is an explicit system without outputs (see Proposition 6.7(ii)) and denote the dimensions of its state space and input space by n^* and m^*, respectively, and its corresponding matrices by A^*, B^* and thus $\Lambda^*_{n^*,m^*} = (A^*, B^*)$.

THEOREM 6.9. *Let $\mathscr{M}^*$ and $\tilde{\mathscr{M}}^*$ be the maximal invariant subspaces of Δ and $\tilde{\Delta}$, respectively. Consider two control systems:*

$$\Lambda^* = (A^*, B^*) \in \mathrm{Expl}\left(\Delta|_{\mathscr{M}^*}^{red}\right), \quad \tilde{\Lambda}^* = (\tilde{A}^*, \tilde{B}^*) \in \mathrm{Expl}\left(\tilde{\Delta}|_{\tilde{\mathscr{M}}^*}^{red}\right).$$

Then the following is equivalent:

(i) $\Delta \overset{in}{\sim} \tilde{\Delta}$;

(ii) *Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent;*

(iii) *Δ and $\tilde{\Delta}$ have isomorphic trajectories, i.e., there exists a linear and invertible map $S : \mathscr{M}^* \to \tilde{\mathscr{M}}^*$ transforming any trajectory $x(t, x^0)$, where $x^0 \in \mathscr{M}^*$ of $\Delta|_{\mathscr{M}^*}^{red}$, into a trajectory $\tilde{x}(t, \tilde{x}^0)$, $\tilde{x}^0 \in \tilde{\mathscr{M}}^*$, of $\tilde{\Delta}|_{\tilde{\mathscr{M}}^*}^{red}$, where $\tilde{x}^0 = Sx^0$, and vice versa.*

The proof is given in section 7.6. In most of the DAEs literature, regularity of DAEs is frequently studied and various definitions are proposed. From the point of view of the existence and uniqueness of solutions, we propose the following definition of internal regularity of DAEs.

DEFINITION 6.10. *Δ is internally regular if for any point $x^0 \in \mathscr{M}^*$, there exists only one solution $x(t)$ of Δ such that $x(0) = x^0$.*

Recall that $\operatorname{rank}_{\mathbb{R}[s]}(sE - H)$ denotes the rank of the polynomial matrix $sE - H$ over the ring $\mathbb{R}[s]$.

THEOREM 6.11 (internal regularity). *Consider a DAE $\Delta_{l,n} = (E, H)$ and denote $\operatorname{rank} E = q$. Then the following statements are equivalent:*

(i) *Δ is internally regular.*
(ii) *Any $\Lambda^* \in \operatorname{Expl}(\Delta|^{red}_{\mathscr{M}^*})$ has no inputs.*
(iii) *The **MCF** of $\Lambda \in \operatorname{Expl}(\Delta)$ has no MCF^1 part.*
(iv) $\operatorname{rank} E = \dim E\mathscr{M}^*$.
(v) $\operatorname{rank}_{\mathbb{R}[s]}(sE - H) = q$.
(vi) *The **MCF** of $\Lambda^* \in \operatorname{Expl}(\Delta|^{red}_{\mathscr{M}^*})$ has the MCF^2 part only.*

The proof is given in section 7.7.

Remark 6.12. (i) The above definition of internal regularity is actually equivalent to the definition of an autonomous DAE in [2]. Both of them mean that the DAE is not underdetermined (there is no $L(s)$ in the **KCF** of $sE - H$).

(ii) Our notion of internal regularity does not imply that the matrices E and H are square, since the presence of the overdetermined part KCF^4 (or $L^p(s)$) is allowed for $\Delta = (E, H)$.

(iii) If E and H are square ($l = n$), then Δ (equivalently, $sE - H$) is internally regular if and only if $|sE - H| \not\equiv 0$. It means that for the case of square matrices, the classical notion of regularity and internal regularity coincide.

7. Proofs of the results.

7.1. Proof of Theorem 3.4.

Proof. (i) This result can be easily deduced from Definitions 3.1 and 3.2 and the explicitation procedure.

(ii) Consider two control systems

$$\Lambda = (A, B, C, D) \in \operatorname{Expl}(\Delta) \quad \text{and} \quad \tilde{\Lambda} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \operatorname{Expl}(\tilde{\Delta}).$$

Then by (i) of Theorem 3.4, there exist invertible matrices $Q, \tilde{Q}, P, \tilde{P}$ of appropriate sizes such that

$$Q\,(sE - H)\,P^{-1} = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix}, \qquad \tilde{Q}\left(s\tilde{E} - \tilde{H}\right)\tilde{P}^{-1} = \begin{bmatrix} sI - \tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}. \tag{7.1}$$

"If." Suppose $\Lambda \overset{M}{\sim} \tilde{\Lambda}$; then there exist Morse transformation matrices T_s, T_i, T_o, F, K such that

$$\begin{bmatrix} T_s & T_sK \\ 0 & T_o \end{bmatrix} \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} = \begin{bmatrix} sI - \tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}. \tag{7.2}$$

By (7.2), we have

$$\begin{bmatrix} T_s & T_sK \\ 0 & T_o \end{bmatrix} Q \left(Q^{-1} \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix} P \right) P^{-1} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} = \tilde{Q} \left(\tilde{Q}^{-1} \begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix} \tilde{P} \right) \tilde{P}^{-1}.$$

Substitute (7.1) into the above equation to have

$$\tilde{Q}^{-1} \begin{bmatrix} T_s & T_sK \\ 0 & T_0 \end{bmatrix} Q\,(sE-H)\,P^{-1} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} \tilde{P} = s\tilde{E} - \tilde{H}.$$

Thus $\Delta \overset{ex}{\sim} \tilde{\Delta}$ via $(\bar{Q}, \bar{P})$, where

$$\bar{Q} = \tilde{Q}^{-1} \begin{bmatrix} T_s & T_sK \\ 0 & T_0 \end{bmatrix} Q \quad \text{and} \quad \bar{P}^{-1} = P^{-1} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} \tilde{P}.$$

"Only if." Suppose $\Delta \overset{ex}{\sim} \tilde{\Delta}$; then there exist invertible matrices $\bar{Q}$ and $\bar{P}$ of appropriate sizes such that $\bar{Q}\,(sE-H)\,\bar{P}^{-1} = s\tilde{E} - \tilde{H}$, which implies that

$$\bar{Q}Q^{-1}\left(Q\,(sE-H)\,P^{-1}\right)P\bar{P}^{-1} = \tilde{Q}^{-1}\left(\tilde{Q}\left(s\tilde{E}-\tilde{H}\right)\tilde{P}^{-1}\right)\tilde{P}$$

$$\overset{(7.1)}{\Rightarrow} \tilde{Q}\bar{Q}Q^{-1} \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix} P\bar{P}^{-1}\tilde{P}^{-1} = \begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}.$$

Denote $\tilde{Q}\bar{Q}Q^{-1} = \left[\begin{smallmatrix} Q^1 & Q^2 \\ Q^3 & Q^4 \end{smallmatrix}\right]$ and $P\bar{P}^{-1}\tilde{P}^{-1} = \left[\begin{smallmatrix} P^1 & P^2 \\ P^3 & P^4 \end{smallmatrix}\right]$, where Q^i and P^i, for $i = 1,2,3,4$, are matrices of suitable sizes. Then we get

$$\begin{bmatrix} Q^1 & Q^2 \\ Q^3 & Q^4 \end{bmatrix} \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} P^1 & P^2 \\ P^3 & P^4 \end{bmatrix} = \begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}.$$

Now by the invertibility of $\tilde{Q}\bar{Q}Q^{-1}$ and $P\bar{P}^{-1}\tilde{P}^{-1}$, we get $\left[\begin{smallmatrix} Q^1 & Q^2 \\ Q^3 & Q^4 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} P^1 & P^2 \\ P^3 & P^4 \end{smallmatrix}\right]$ are invertible. By a direct calculation, we get $Q^3 = 0$, $P^2 = 0$, $Q^1 = (P^1)^{-1}$, thus Q^4 and P^4 are invertible as well. Therefore, $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ via the Morse transformation

$$M_{tran} = \left(Q^1, (P^4)^{-1}, Q^4, P^3Q^1, (Q^1)^{-1}Q^2\right).$$

(iii) Given two control systems $\Lambda = (A,B,C,D)$ and $\tilde{\Lambda} = (\tilde{A},\tilde{B},\tilde{C},\tilde{D})$, the corresponding matrix pencils of $\Delta^{Impl} = \mathrm{Impl}(\Lambda)$ and $\tilde{\Delta}^{Impl} = \mathrm{Impl}(\tilde{\Lambda})$, by Definition 3.1, are $\left[\begin{smallmatrix} sI-A & -B \\ -C & -D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{smallmatrix}\right]$, respectively.

"If." Suppose $\Delta^{Impl} \overset{ex}{\sim} \tilde{\Delta}^{Impl}$, that is, there exist invertible matrices Q and P such that

$$Q \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix} P^{-1} = \begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}. \tag{7.3}$$

Denote $Q = \left[\begin{smallmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{smallmatrix}\right]$ and $P = \left[\begin{smallmatrix} P_1 & P_2 \\ P_3 & P_4 \end{smallmatrix}\right]$ with matrices Q_i and P_i, for $i = 1,2,3,4$, of suitable dimensions. Then by (7.3), we get $Q_3 = 0$, $P_2 = 0$, $Q_1 = (P_1)^{-1}$. Since Q and P are invertible, we can conclude that Q_4 and P_4 are invertible as well. Therefore, $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ via the Morse transformation $M_{tran} = \left(Q_1, (P_4)^{-1}, Q_4, P_3Q_1, (Q_1)^{-1}Q_2\right)$.

"Only if." Suppose $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ via a Morse transformation $M_{tran} = (T_s, T_i, T_o, F, K)$ (see (2.1)); then we have $\Delta^{Impl} \overset{ex}{\sim} \tilde{\Delta}^{Impl}$ via (Q,P), where $Q = \left[\begin{smallmatrix} T_s & T_sK \\ 0 & T_o \end{smallmatrix}\right]$ and $P^{-1} = \left[\begin{smallmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{smallmatrix}\right]$. ∎

7.2. Proof of Proposition 4.4.

Proof. (i) It can be observed from (4.1) that $\mathscr{V}_i$ is nonincreasing. By a dimensional argument, the sequence $\mathscr{V}_i$ gets stabilized at $i = k^* \leq n$ and it can be directly seen that $\mathscr{V}_{k^*} = H^{-1}E\mathscr{V}_{k^*}$. We now prove by induction that it is the largest. Choose any other subspace $\hat{\mathscr{V}}$ such that $\hat{\mathscr{V}} = H^{-1}E\hat{\mathscr{V}}$ and consider (4.1). For $i = 0$, $\hat{\mathscr{V}} \subseteq \mathscr{V}_0$. Suppose $\hat{\mathscr{V}} \subseteq \mathscr{V}_i$, then $H^{-1}E\hat{\mathscr{V}} \subseteq H^{-1}E\mathscr{V}_i$ (since taking the image and preimage preserves inclusion), thus $\hat{\mathscr{V}} = H^{-1}E\hat{\mathscr{V}} \subseteq H^{-1}E\mathscr{V}_i = \mathscr{V}_{i+1}$. Therefore, $\hat{\mathscr{V}} \subseteq \mathscr{V}_i$ for $i \in \mathbb{N}$, i.e., $\hat{\mathscr{V}} \subseteq \mathscr{V}_{k^*}$, it follows that $\mathscr{V}_{k^*}$ is the largest subspace of $\mathbb{R}^n$ such that $\mathscr{V}_{k^*} = H^{-1}E\mathscr{V}_{k^*}$.

Now consider (4.2), observe that the sequence $\mathscr{W}_i$ is nondecreasing, and by a dimensional argument, $\mathscr{W}_i$ gets stabilized at $i = l^* \leq n$. It can be directly seen that $\mathscr{W}_{l^*} = E^{-1}H\mathscr{W}_{l^*}$. We then prove that any other subspace $\hat{\mathscr{W}}$ such that $\hat{\mathscr{W}} = E^{-1}H\hat{\mathscr{W}}$ contains $\mathscr{W}^*$. For $i = 0$, $\mathscr{W}_0 \subseteq \hat{\mathscr{W}}$; if $\mathscr{W}_i \subseteq \hat{\mathscr{W}}$, then $E^{-1}H\mathscr{W}_i \subseteq E^{-1}H\hat{\mathscr{W}}$, so $\mathscr{W}_{i+1} = E^{-1}H\mathscr{W}_i \subseteq E^{-1}H\hat{\mathscr{W}} = \hat{\mathscr{W}}$, that is, $\mathscr{W}_i \subseteq \hat{\mathscr{W}}$ for $i \in \mathbb{N}$, which gives $\mathscr{W}_{l^*} \subseteq \hat{\mathscr{W}}$ and $\mathscr{W}_{l^*}$ is the smallest subspace of $\mathbb{R}^n$ such that $\mathscr{W}_{l^*} = E^{-1}H\mathscr{W}_{l^*}$.

(ii) By Definition 4.3, $\mathscr{V}^*$ satisfies $\mathscr{V}^* = H^{-1}E\mathscr{V}^*$, thus it is seen that $H\mathscr{V}^* \subseteq E\mathscr{V}^*$. We then prove, by induction, that $\mathscr{V}^*$ is the largest satisfying that property. Choose any other subspace $\hat{\mathscr{V}}$ which satisfies $H\hat{\mathscr{V}} \subseteq E\hat{\mathscr{V}}$, and consider (4.1), for $i = 0$, so $\hat{\mathscr{V}} \subseteq \mathscr{V}_0$. Suppose $\hat{\mathscr{V}} \subseteq \mathscr{V}_i$, then $\hat{\mathscr{V}} \subseteq H^{-1}E\hat{\mathscr{V}} \subseteq H^{-1}E\mathscr{V}_i = \mathscr{V}_{i+1}$, thus $\hat{\mathscr{V}} \subseteq H^{-1}E\mathscr{V}_i = \mathscr{V}_{i+1}$, therefore $\hat{\mathscr{V}} \subseteq \mathscr{V}_i$ for $i \in \mathbb{N}$, i.e., $\hat{\mathscr{V}} \subseteq \mathscr{V}_{k^*}$, which implies $\mathscr{V}^* = \mathscr{V}_{k^*}$ is the largest subspace such that $H\mathscr{V}^* \subseteq E\mathscr{V}^*$

Obviously, $\{0\}$ is the smallest subspace satisfying $H\{0\} \subseteq E\{0\}$, but $\mathscr{W}^*$ is not always $\{0\}$, so we prove that $\mathscr{W}^*$ is not necessarily the smallest subspace such that $E\mathscr{W}^* \subseteq H\mathscr{W}^*$. ◻

7.3. Proof of Theorem 4.10.

Proof. Observe that, by Definitions 2.1 and 4.1, if two DAEs Δ and $\tilde{\Delta}$ are ex-equivalent via (Q,P), then direct calculations of the Wong sequences of Δ and $\tilde{\Delta}$ give that $\mathscr{V}_i(\tilde{\Delta}) = P\mathscr{V}_i(\Delta)$ and $\mathscr{W}_i(\tilde{\Delta}) = P\mathscr{W}_i(\Delta)$. As Λ is a (Q,P)-explicitation of Δ, by Theorem 3.4(i), we have $\Delta \overset{ex}{\sim} \Delta^{Impl}$ via (Q,P), where $\Delta^{Impl} = \mathrm{Impl}(\Lambda)$. Thus we have

$$\mathscr{V}_i\left(\Delta^{Impl}\right) = P\mathscr{V}_i(\Delta), \quad \mathscr{W}_i\left(\Delta^{Impl}\right) = P\mathscr{W}_i(\Delta). \tag{7.4}$$

Notice that

$$\Delta^{Impl}_{l,n} = \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right), \quad \mathbf{\Lambda}_{n,m,p} = (\mathbf{A},\mathbf{B},\mathbf{C}) = \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \begin{bmatrix} C & D \end{bmatrix}\right),$$

where $m = n - q$ and $p = l - q$. The proof of (i) will be done in three steps.

Step 1. First we show that for $i \in \mathbb{N}$,

$$\mathscr{V}_i\left(\Delta^{Impl}\right) = \boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda}). \tag{7.5}$$

Calculate $\boldsymbol{\mathcal{V}}_{i+1}(\mathbf{\Lambda})$ using (4.4), to get

$$\boldsymbol{\mathcal{V}}_{i+1}(\mathbf{\Lambda}) = \ker\begin{bmatrix} C & D \end{bmatrix} \cap \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^{-1}\left(\boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda}) + \mathrm{Im}\begin{bmatrix} 0 \\ I_m \end{bmatrix}\right). \tag{7.6}$$

Equation (7.6) can be written as

$$\boldsymbol{\mathcal{V}}_{i+1}(\mathbf{\Lambda}) = \left\{\tilde{v} \mid \begin{bmatrix} A & B \end{bmatrix}\tilde{v} \in \begin{bmatrix} I_q & 0 \end{bmatrix}\boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda}),\ \ \begin{bmatrix} C & D \end{bmatrix}\tilde{v} = 0\right\}$$

or, equivalently,

$$\boldsymbol{\mathcal{V}}_{i+1}(\boldsymbol{\Lambda}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}). \tag{7.7}$$

Now, observe that the inductive formula (7.7) for $\boldsymbol{\mathcal{V}}_{i+1}(\boldsymbol{\Lambda})$ coincides with the inductive formula (4.1) for the Wong sequence $\mathscr{V}_{i+1}(\Delta^{Impl})$. Since $\mathscr{V}_0(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}_0(\boldsymbol{\Lambda}) = \mathbb{R}^n$, we conclude that $\mathscr{V}_i(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda})$ for all $i \in \mathbb{N}$.

Step 2. We then prove that for $i \in \mathbb{N}$,

$$\mathscr{V}_{i+1}(\Delta^{Impl}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_i(\Lambda) \\ 0 \end{bmatrix}. \tag{7.8}$$

By calculating $\mathcal{V}_{i+1}(\Lambda)$ via (4.6), we get

$$\mathcal{V}_{i+1}(\Lambda) = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_i(\Lambda) + \operatorname{Im} \begin{bmatrix} B \\ D \end{bmatrix} \right).$$

We can rewrite the above equation as

$$\mathcal{V}_{i+1}(\Lambda) = \begin{bmatrix} I_q & 0_{q\times m} & 0 \end{bmatrix} \ker \begin{bmatrix} A & B & \bar{V}_i \\ C & D & 0 \end{bmatrix}, \tag{7.9}$$

where $\bar{V}_i$ is a matrix with independent columns such that $\operatorname{Im} \bar{V}_i = \mathcal{V}_i(\Lambda)$.

From basic knowledge of linear algebra, for two matrices $M \in \mathbb{R}^{l\times n}$ and $N \in \mathbb{R}^{l\times m}$, the preimage $M^{-1}\operatorname{Im}N = [I_n, 0] \ker [M, N]$. With this formula, calculate $\mathscr{V}_{i+1}(\Delta^{Impl})$ via (4.1), to get

(7.10)

$$\mathscr{V}_{i+1}\left(\Delta^{Impl}\right) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_q & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \ker \begin{bmatrix} A & B & \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} V_i \\ C & D & \end{bmatrix},$$

where V_i is a matrix with independent columns such that $\operatorname{Im} V_i = \mathscr{V}_i(\Delta)$.

In order to show that (7.8) holds, we will first prove inductively that for all $i \in \mathbb{N}$,

$$\begin{bmatrix} \mathcal{V}_i(\Lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}_i\left(\Delta^{Impl}\right). \tag{7.11}$$

For $i = 0$, $\left[\begin{smallmatrix} \mathcal{V}_0(\Lambda) \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{R}^q \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]\mathscr{V}_0(\Delta^{Impl})$. Suppose that for $i = k \in \mathbb{N}$, (7.11) holds or, equivalently, $\operatorname{Im}\left[\begin{smallmatrix} \bar{V}_k \\ 0 \end{smallmatrix}\right] = \operatorname{Im}\left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]V_k$. Then we have

$$\begin{bmatrix} \mathcal{V}_{k+1}(\Lambda) \\ 0 \end{bmatrix} \overset{(7.9)}{=} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_q & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \ker \begin{bmatrix} A & B & \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} V_k \\ C & D & \end{bmatrix} \overset{(7.10)}{=} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}_{k+1}\left(\Delta^{Impl}\right).$$

Therefore, (7.11) holds for all $i \in \mathbb{N}$.

Consequently, we have for $i \in \mathbb{N}$,

$$\mathscr{V}_{i+1}\left(\Delta^{Impl}\right) \overset{(4.1)}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}_i\left(\Delta^{Impl}\right) \overset{(7.11)}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_i(\Lambda) \\ 0 \end{bmatrix}.$$

Step 3. Finally, since $\mathscr{V}^*$ and $\boldsymbol{\mathcal{V}}^*$ are the limits of the sequences $\mathscr{V}_i$ and $\boldsymbol{\mathcal{V}}_i$, respectively, it follows from (7.5) that $\mathscr{V}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}^*(\boldsymbol{\Lambda})$. Since $\mathscr{V}^*$ and $\mathcal{V}^*$ are the limits of $\mathscr{V}_i$ and $\mathcal{V}_i$, respectively, it follows from (7.8) that $\mathscr{V}^*(\Delta^{Impl}) = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} \mathcal{V}^*(\Lambda) \\ 0 \end{smallmatrix}\right]$. Thus by (7.4), we have $P\mathscr{V}^*(\Delta) = \mathscr{V}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}^*(\boldsymbol{\Lambda}) = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} \mathcal{V}^*(\Lambda) \\ 0 \end{smallmatrix}\right]$.

The proof of (ii) will be done in three steps.

Step 1. First, we show that for $i \in \mathbb{N}$,

$$\mathscr{W}_i\left(\Delta^{Impl}\right) = \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}). \tag{7.12}$$

Calculate $\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda})$ by (4.5), as

$$\begin{aligned}
\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}) &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}) \cap \ker \begin{bmatrix} C & D \end{bmatrix} + \operatorname{Im} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \right) \\
&= \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}) \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}) \right) \cap \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \ker \begin{bmatrix} C & D \end{bmatrix} \right).
\end{aligned}$$

Observe that $\left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \ker \left[\begin{smallmatrix} C & D \end{smallmatrix}\right] = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]^{-1} \operatorname{Im} \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then we have

$$\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}). \tag{7.13}$$

Notice that the inductive formula (7.13) for $\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda})$ coincides with the inductive formula (4.2) for the Wong sequence $\mathscr{W}_{i+1}(\Delta^{Impl})$. Since $\mathscr{W}_0(\Delta^{Impl}) = \boldsymbol{\mathcal{W}}_0(\boldsymbol{\Lambda}) = \{0\}$, we deduce that $\mathscr{W}_i(\Delta^{Impl}) = \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda})$ for $i \in \mathbb{N}$.

Step 2. Subsequently, we will prove that for $i \in \mathbb{N}$,

$$\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{bmatrix}. \tag{7.14}$$

Considering (4.8) for Λ, we have

$$\begin{aligned}
\begin{bmatrix} \mathcal{W}_{i+1}(\Lambda) \\ 0 \end{bmatrix} &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \mathcal{W}_i(\Lambda) \\ \mathbb{R}^m \end{bmatrix} \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right) \\
&= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{bmatrix} \right) \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right),
\end{aligned}$$

which implies that

$$\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_{i+1}(\Lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{bmatrix} \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right) + \operatorname{Im} \begin{bmatrix} 0 \\ I_m \end{bmatrix}. \tag{7.15}$$

Observe that the inductive formula (7.15) for $\left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} \mathcal{W}_{i+1}(\Lambda) \\ 0 \end{smallmatrix}\right]$ coincides with the inductive formula (4.5) for $\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda})$. Since $\boldsymbol{\mathcal{W}}_1(\Lambda) = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} \mathcal{W}_0(\Lambda) \\ 0 \end{smallmatrix}\right] = \operatorname{Im} \left[\begin{smallmatrix} 0 \\ I_m \end{smallmatrix}\right]$, we have $\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}) = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{smallmatrix}\right]$ for all $i \in \mathbb{N}$.

Step 3. Equation (7.12) and the fact that $\mathscr{W}^*$ and $\boldsymbol{\mathcal{W}}^*$ are the limits of $\mathscr{W}_i$ and $\boldsymbol{\mathcal{W}}_i$, respectively, yield $\mathscr{W}^*(\Delta) = \boldsymbol{\mathcal{W}}^*(\boldsymbol{\Lambda})$. Equation (7.14) and the fact that $\mathcal{W}^*$ and $\boldsymbol{\mathcal{W}}^*$ are the limits of $\mathcal{W}_i$ and $\boldsymbol{\mathcal{W}}_i$, respectively, yield $\boldsymbol{\mathcal{W}}^*(\boldsymbol{\Lambda}) = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} \mathcal{W}^*(\Lambda) \\ 0 \end{smallmatrix}\right]$. Thus using (7.4), we prove (ii) of Theorem 4.10. ∎

7.4. Proof of Theorem 5.3. Note that the Kronecker invariants are invariant under ex-equivalence. By $\Delta \overset{ex}{\sim} \Delta^{Impl}$, in our proof we can work with the Kronecker invariants of Δ^{Impl} instead of those of Δ. In what follows, we will use the following two lemmata. Denote by $\mathbb{F}(\mathcal{V}_i(\Lambda))$ the class of maps $F : \mathbb{R}^q \to \mathbb{R}^m$ satisfying $(A + BF)\mathcal{V}_{i+1}(\Lambda) \subset \mathcal{V}_i(\Lambda)$ and $(C + DF)\mathcal{V}_{i+1}(\Lambda) = 0$.

Lemma 7.1. *Given $\Delta_{l,n}=(E,H)$, its (Q,P)-explicitation $\Lambda=(A,B,C,D)\in \mathrm{Expl}(\Delta)$, and $\Delta^{Impl}=\mathrm{Impl}(\Lambda)$, consider the Wong sequences $\mathscr{V}_i$, $\mathscr{W}_i$ of both Δ and Δ^{Impl}, given by Definition 4.1, and the subspaces $\mathcal{V}_i$, $\mathcal{W}_i$ of Λ, given by Lemma 4.6. Then for $i\in\mathbb{N}$, we have*

$$\mathscr{V}_{i+1}(\Delta^{Impl})=P\mathscr{V}_{i+1}(\Delta)=\begin{bmatrix}\mathcal{V}_{i+1}(\Lambda)\\F_i\mathcal{V}_{i+1}(\Lambda)\end{bmatrix}+\begin{bmatrix}0\\\mathcal{U}_i(\Lambda)\end{bmatrix}, \tag{7.16}$$

where $F_i\in\mathbb{F}(\mathcal{V}_i(\Lambda))$ and

$$\mathscr{W}_{i+1}(\Delta^{Impl})=P\mathscr{W}_{i+1}(\Delta)=\begin{bmatrix}\mathcal{W}_i(\Lambda)\\ *\end{bmatrix}+\begin{bmatrix}0\\\mathscr{U}(\Lambda)\end{bmatrix}. \tag{7.17}$$

Lemma 7.2 (see [6], [19]). *For $\Delta=(E,H)$ and its dual $\Delta^d=(E^T,H^T)$, consider the subspaces $\mathscr{V}_i$ and $\mathscr{W}_i$, then $\mathscr{W}_{i+1}(\Delta^d)=(E\mathscr{V}_i(\Delta))^\perp$ and $\mathscr{V}_i(\Delta^d)=(H\mathscr{W}_i(\Delta))^\perp$. For $\Lambda=(A,B,C,D)\in\mathrm{Expl}(\Delta)$ and the dual Λ^d of Λ, given by $\Lambda^d=(A^T,C^T,B^T,D^T)$, consider the subspaces $\mathcal{V}_i$ and $\mathcal{W}_i$. Then $\mathcal{W}_i(\Lambda^d)=(\mathcal{V}_i)^\perp$ and $\mathcal{V}_i(\Lambda^d)=(\mathcal{W}_i(\Lambda))^\perp$.*

The proof of Lemma 7.1 is given after the proof of Theorem 5.3.

Proof of Theorem 5.3. (i) Recall Lemma 5.1(i) for Δ^{Impl} and Lemma 5.2(i) for Λ. For $i\in\mathbb{N}^+$, it holds that

$$\begin{aligned}\mathscr{K}_i\left(\Delta^{Impl}\right)&=\mathscr{W}_i\left(\Delta^{Impl}\right)\cap\mathscr{V}^*\left(\Delta^{Impl}\right)\\&\overset{\text{Lemma 7.1}}{=}\left(\begin{bmatrix}\mathcal{W}_{i-1}(\Lambda)\\ *\end{bmatrix}+\begin{bmatrix}0\\\mathscr{U}(\Lambda)\end{bmatrix}\right)\cap\left(\begin{bmatrix}\mathcal{V}^*(\Lambda)\\F^*\mathcal{V}^*(\Lambda)\end{bmatrix}+\begin{bmatrix}0\\\mathcal{U}^*(\Lambda)\end{bmatrix}\right)\\&=\begin{bmatrix}\mathcal{W}_{i-1}(\Lambda)\cap\mathcal{V}^*(\Lambda)\\F^*\left(\mathcal{W}_{i-1}(\Lambda)\cap\mathcal{V}^*(\Lambda)\right)\end{bmatrix}+\begin{bmatrix}0\\\mathcal{U}^*(\Lambda)\end{bmatrix}\end{aligned} \tag{7.18}$$

for a suitable $F^*\in\mathbb{F}(\mathcal{V}^*(\Lambda))$. Then we have

$$a\overset{\text{Lemma 5.1(i)}}{=}\dim\left(\mathscr{K}_1\left(\Delta^{Impl}\right)\right)\overset{(7.18)}{=}\dim\left(\begin{bmatrix}0\\\mathcal{U}^*(\Lambda)\end{bmatrix}\right)=\dim\left(\mathcal{U}^*(\Lambda)\right)\overset{\text{Lemma 5.2(i)}}{=}a'.$$

Moreover, it is seen that for $i\in\mathbb{N}$,

$$\begin{aligned}\omega_i&\overset{\text{Lemma 5.1(i)}}{=}\dim\left(\mathscr{K}_{i+2}\left(\Delta^{Impl}\right)\right)-\dim\left(\mathscr{K}_{i+1}\left(\Delta^{Impl}\right)\right)\\&\overset{(7.18)}{=}\dim\left(\mathcal{W}_{i+1}(\Lambda)\cap\mathcal{V}^*(\Lambda)\right)-\dim\left(\mathcal{W}_i(\Lambda)\cap\mathcal{V}^*(\Lambda)\right)\\&=\dim\left(\mathcal{R}_{i+1}(\Lambda)\right)-\dim\left(\mathcal{R}_i(\Lambda)\right)\overset{\text{Lemma 5.2(i)}}{=}\omega_i'.\end{aligned}$$

Now consider (5.1) and (5.5) and it is sufficient to show

$$\begin{cases}\varepsilon_j=\varepsilon_j'=0 & \text{for} \quad 1\le j\le a-\omega_0=a'-\omega_0',\\ \varepsilon_j=\varepsilon_j'=i & \text{for} \quad a'-\omega_{i-1}'+1=a-\omega_{i-1}+1\le j\le a-\omega_i=a'-\omega'.\end{cases}$$

The statement that $d=d'$, $\eta_i=\eta_i'$ can be proved in a similar way using dual objects. It is not hard to see that for $i\in\mathbb{N}^+$,

$$\begin{aligned}\hat{\mathscr{K}_i}\left(\Delta^{Impl}\right)&=\left(E\mathscr{V}_{i-1}\left(\Delta^{Impl}\right)\right)^\perp\cap\left(H\mathscr{W}^*\left(\Delta^{Impl}\right)\right)^\perp\\&\overset{\text{Lemma 7.2}}{=}\mathscr{W}_i(\left(\Delta^{Impl}\right)^d)\cap\mathscr{V}^*(\left(\Delta^{Impl}\right)^d)\\&\overset{\text{Lemma 7.1}}{=}\begin{bmatrix}\mathcal{W}_{i-1}(\Lambda^d)\cap\mathcal{V}^*(\Lambda^d)\\ *\end{bmatrix}+\begin{bmatrix}0\\\mathcal{U}^*(\Lambda^d)\end{bmatrix},\end{aligned}$$

where $(\Delta^{Impl})^d$ is the dual system of Δ^{Impl}, which coincides with $\mathrm{Impl}(\Lambda^d)$. It follows that

$$d \overset{\text{Lemma 5.1(i)}}{=} \dim\left(\hat{\mathscr{K}}_1\left(\Delta^{Impl}\right)\right) = \dim\left(\begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda^d) \end{bmatrix}\right) = \dim\left(\mathcal{Y}^*(\Lambda)\right) \overset{\text{Lemma 5.2(i)}}{=} d'.$$

We can also see that for $i \in \mathbb{N}$,

$$\begin{aligned} \hat{\omega}_i &= \dim\left(\hat{\mathscr{K}}_{i+2}\left(\Delta^{Impl}\right)\right) - \dim\left(\hat{\mathscr{K}}_{i+1}\left(\Delta^{Impl}\right)\right) \\ &= \dim\left(\mathcal{W}_{i+1}(\Lambda^d) \cap \mathcal{V}^*(\Lambda^d)\right) - \dim\left(\mathcal{W}_i(\Lambda^d) \cap \mathcal{V}^*(\Lambda^d)\right) \\ &\overset{\text{Lemma7.2}}{=} \dim\left((\mathcal{V}_{i+1})^\perp \cap (\mathcal{W}^*)^\perp\right) - \dim\left((\mathcal{V}_i)^\perp \cap (\mathcal{W}^*)^\perp\right) \\ &= \dim(\hat{\mathcal{R}}_{i+1}(\Lambda)) - \dim(\hat{\mathcal{R}}_i(\Lambda)) = \hat{\omega}'_i. \end{aligned}$$

Now it is sufficient to show that

$$\begin{cases} \eta_j = \eta'_j = 0 & \text{for} \qquad 1 \le j \le d - \hat{\omega}_0 = h - \hat{\omega}'_0, \\ \eta_j = \eta'_j = i & \text{for} \quad h - \omega'_{i-1} + 1 = d - \hat{\omega}_{i-1} + 1 \le j \le d - \hat{\omega}_i = h - \hat{\omega}'. \end{cases}$$

(ii) Recall Lemma 5.1(ii) for Δ^{Impl} and Lemma 5.2(ii) for Λ. We have for all $i \in \mathbb{N}^+$,

$$\begin{aligned} \mathscr{V}^*\left(\Delta^{Impl}\right) + \mathscr{W}_i\left(\Delta^{Impl}\right) &\overset{\text{Lemma 7.1}}{=} \begin{bmatrix} \mathcal{V}^*(\Lambda) \\ F^* * \mathcal{V}^*(\Lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}_i(\Lambda) \end{bmatrix} + \begin{bmatrix} \mathcal{W}_{i-1}(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{V}^*(\Lambda) + \mathcal{W}_{i-1}(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix}. \end{aligned}$$

If $\nu = 0$, then we have the following result by (5.3):

$$\begin{aligned} &\mathscr{V}^*\left(\Delta^{Impl}\right) + \mathscr{W}_0\left(\Delta^{Impl}\right) = \mathscr{V}^*\left(\Delta^{Impl}\right) + \mathscr{W}_1\left(\Delta^{Impl}\right) \\ &\quad \Rightarrow \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ F^*\mathcal{V}^*(\Lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix}\right) = \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix}\right) \Rightarrow \mathscr{U}(\Lambda) = \mathcal{U}^*(\Lambda). \end{aligned}$$

It follows that $c' = \dim(\mathscr{U}(\Lambda)) - \dim(\mathcal{U}^*(\Lambda)) = 0$. Therefore, in this case, the MCF^3-part of **MCF** is absent. As a consequence, if $N(s)$ of **KCF** is absent, then MCF^3 of **MCF** is absent as well. If $\nu > 0$, from (5.3) we get

$$\begin{aligned} \nu &= \min\left\{ i \in \mathbb{N}^+ \,\middle|\, \begin{bmatrix} \mathcal{V}^*(\Lambda) + \mathcal{W}_{i-1}(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} = \begin{bmatrix} \mathcal{V}^*(\Lambda) + \mathcal{W}_i(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \right\} \\ &= \min\left\{ i \in \mathbb{N}^+ \,|\mathcal{V}^*(\Lambda) + \mathcal{W}_{i-1}(\Lambda) = \mathcal{V}^*(\Lambda) + \mathcal{W}_i(\Lambda) \right\} = \nu' + 1. \end{aligned}$$

We have

$$\begin{aligned} c = \pi_0 &= \dim\left(\mathscr{V}^*\left(\Delta^{Impl}\right) + \mathscr{W}_1\left(\Delta^{Impl}\right)\right) - \dim\left(\mathscr{V}^*\left(\Delta^{Impl}\right) + \mathscr{W}_0\left(\Delta^{Impl}\right)\right) \\ &\overset{\text{Lemma 7.1}}{=} \dim\left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix}\right) - \dim\left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}(\Lambda) \end{bmatrix}\right) \\ &= \dim(\mathscr{U}(\Lambda)) - \dim(\mathcal{U}(\Lambda)) = c'. \end{aligned}$$

We also have for $i \in \mathbb{N}^+$,

$$\begin{aligned}\pi_i &= \dim\left(\mathscr{V}^*\left(\Delta^{Impl}\right)+\mathscr{W}_{i+1}\left(\Delta^{Impl}\right)\right)-\dim\left(\mathscr{V}^*\left(\Delta^{Impl}\right)+\mathscr{W}_{i}\left(\Delta^{Impl}\right)\right)\\ &= \dim\left(\begin{bmatrix}\mathcal{V}^*(\Lambda)+\mathcal{W}_i(\Lambda)\\ *\end{bmatrix}+\begin{bmatrix}0\\ \mathscr{U}(\Lambda)\end{bmatrix}\right)-\dim\left(\begin{bmatrix}\mathcal{V}^*(\Lambda)+\mathcal{W}_{i-1}(\Lambda)\\ *\end{bmatrix}+\begin{bmatrix}0\\ \mathscr{U}(\Lambda)\end{bmatrix}\right)\\ &= \dim\left(\mathcal{W}_i(\Lambda)+\mathcal{V}^*(\Lambda)\right)-\dim\left(\mathcal{W}_{i-1}(\Lambda)+\mathcal{V}^*(\Lambda)\right)=\pi'_{i-1}.\end{aligned}$$

Now substituting $c=c'$, $\pi_i=\pi'_{i-1}$, and $\nu=\nu'+1$ into (5.4), we can rewrite (5.4) as

$$\begin{cases}\sigma_j=0 \ \text{ for } \ 1\le j\le c-\pi_1=c'-\pi'_0=\delta,\\ \sigma_j=i \ \text{ for } \ c'-\pi'_{i-2}+1=c-\pi_{i-1}+1\le j\le c-\pi_i=c'-\pi'_{i-1},\ i=2,\dots,\nu'+1.\end{cases}$$

Replacing i by $i-1$, we get

$$\sigma_j=i-1 \quad \mathit{for} \quad c'-\pi'_{i-1}+1\le j\le c'-\pi'_i, \quad i=1,2,\dots,\nu'.$$

Finally, comparing the above expression of σ_j with that for σ'_j of (5.7), it is not hard to see that $\sigma_j+1=\sigma'_j$ for $j=1,\dots,c$.

(iii) We only show that the invariant factors of MCF^2 of Λ coincide with the invariant factors of the real Jordan pencil $J(s)$ of Δ^{Impl}; then the equalities $d=d'$, $\eta_1=\eta'_1,\dots,\eta_d=\eta'_{d'}$, and $\lambda_{\rho_1}=\lambda_{\rho'_1},\dots,\lambda_{\rho_b}=\lambda_{\rho'_{b'}}$ are immediately satisfied. First, let two subspaces $\mathscr{X}_2\subseteq\mathscr{V}^*(\Delta^{Impl})$ and $\mathscr{Z}_2\subseteq\mathcal{V}^*(\Lambda)$ be such that

$$\mathscr{X}_2\oplus\left(\mathscr{V}^*\left(\Delta^{Impl}\right)\cap\mathscr{W}^*\left(\Delta^{Impl}\right)\right)=\mathscr{V}^*\left(\Delta^{Impl}\right), \quad \mathscr{Z}_2\oplus\left(\mathcal{V}^*(\Lambda)\cap\mathcal{W}^*(\Lambda)\right)=\mathcal{V}^*(\Lambda).$$

The above construction gives $\Delta^{Impl}|\mathscr{X}_2\cong KCF^2$ and $\Lambda|\mathscr{Z}_2\cong MCF^2$, where KCF^2 corresponds to the Jordan pencil $J(s)$. Use Lemma 7.1 to conclude that

$$\mathscr{X}_2\oplus\left(\mathscr{V}^*(\Delta^{Impl})\cap\mathscr{W}^*(\Delta^{Impl})\right)=\mathscr{V}^*(\Delta^{Impl})$$

implies

$$\begin{aligned}&\mathscr{X}_2\oplus\left(\left(\begin{bmatrix}\mathcal{W}^*(\Lambda)\\ *\end{bmatrix}+\begin{bmatrix}0\\ \mathscr{U}(\Lambda)\end{bmatrix}\right)\cap\left(\begin{bmatrix}\mathcal{V}^*(\Lambda)\\ F^*\mathcal{V}^*(\Lambda)\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{U}^*(\Lambda)\end{bmatrix}\right)\right)=\left(\begin{bmatrix}\mathcal{V}^*(\Lambda)\\ F^*\mathcal{V}^*(\Lambda)\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{U}^*(\Lambda)\end{bmatrix}\right)\\ &\quad\Rightarrow\mathscr{X}_2\oplus\left(\begin{bmatrix}\mathcal{W}^*(\Lambda)\cap\mathcal{V}^*(\Lambda)\\ F'\left(\mathcal{W}^*(\Lambda)\cap\mathcal{V}^*(\Lambda)\right)\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{U}^*(\Lambda)\end{bmatrix}\right)=\left(\begin{bmatrix}\mathcal{V}^*(\Lambda)\\ F^*\mathcal{V}^*(\Lambda)\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{U}^*(\Lambda)\end{bmatrix}\right),\end{aligned}$$

where $F\in\mathbb{F}(\mathcal{V}^*(\Lambda))$, $F'\in\mathbb{F}(\mathcal{W}^*(\Lambda)\cap\mathcal{V}^*(\Lambda))$. Since $\mathscr{Z}_2\oplus(\mathcal{V}^*(\Lambda)\cap\mathcal{W}^*(\Lambda))=\mathcal{V}^*(\Lambda)$, we have $\mathscr{X}_2=\left[\begin{smallmatrix}\mathscr{Z}_2\\ F''\mathscr{Z}_2\end{smallmatrix}\right]$, where $F''\in\mathbb{F}(\mathscr{Z}_2)$. Then, it follows that

$$\begin{aligned}\left.\begin{bmatrix}sI-A & -B\\ -C & -D\end{bmatrix}\right|_{\mathscr{X}_2}&=\begin{bmatrix}sI-A & -B\\ -C & -D\end{bmatrix}\begin{bmatrix}\mathscr{Z}_2\\ F''\mathscr{Z}_2\end{bmatrix}=\begin{bmatrix}\left(sI-(A+BF'')\right)\mathscr{Z}_2\\ (C+DF'')\mathscr{Z}_2\end{bmatrix}\\ &=\begin{bmatrix}\left(sI-(A+BF'')\right)\mathscr{Z}_2\\ 0\end{bmatrix}.\end{aligned}$$

Now it is known from Lemma 4.1 of [20] that $(A+BF'')|\mathscr{Z}_2$ does not depend on the choice of F''. Thus the invariant factors of $(sI-(A+BF''))\,\mathscr{Z}_2$ coincide with the invariant factors of MCF^2 for Λ. Finally, from the above equation, it is easy to see that the invariant factors of $J(s)$ in **KCF** of Δ coincide with those of MCF^2 of Λ. $\square$

Proof of Lemma 7.1. We first show that (7.16) holds. Let independent vectors $v_1=\left[\begin{smallmatrix}v_1^1\\ v_1^2\end{smallmatrix}\right],\dots,v_\alpha=\left[\begin{smallmatrix}v_\alpha^1\\ v_\alpha^2\end{smallmatrix}\right]\in\mathbb{R}^n$ form a basis of

$$P\mathscr{V}_{i+1}(\Delta)\overset{(7.4)}{=}\mathscr{V}_{i+1}\left(\Delta^{Impl}\right)\overset{(7.8)}{=}\begin{bmatrix}A & B\\ C & D\end{bmatrix}^{-1}\begin{bmatrix}\mathcal{V}_i(\Lambda)\\ 0\end{bmatrix},$$

where $v_j^1 \in \mathbb{R}^q, v_j^2 \in \mathbb{R}^m, j = 1,2,\ldots,\alpha$ (implying that $\dim(\mathscr{V}_{i+1}(\Delta^{Impl})) = \alpha$). Now without loss of generality, assume $v_j^1 \neq 0$ for $j = 1,\ldots,\kappa$ and $v_j^1 = 0$ for $j = \kappa+1,\ldots,\alpha$, where $\kappa < \alpha$ is the number of nonzero vectors v_j^1. Then from (7.11), it can be deduced that v_j^1 for $j = 1,\ldots,\kappa$ form a basis of $\mathcal{V}_{i+1}(\Lambda)$. Moreover, from (7.8), it is not hard to see that v_j^2 for $j = \kappa+1,\ldots,\alpha$ form a basis of $\mathcal{U}_i(\Lambda)$. Let $F_i \in \mathbb{R}^{m\times\kappa}$ be such that $F_i v_j^1 = v_j^2$ for $j = 1,\ldots,\kappa$ (such F_i exists); then $v_1,\ldots,v_\alpha$ form a basis of $\left[\begin{smallmatrix}\mathcal{V}_{i+1}(\Lambda)\\ F_i\mathcal{V}_{i+1}(\Lambda)\end{smallmatrix}\right] + \left[\begin{smallmatrix}0\\ \mathcal{U}_i(\Lambda)\end{smallmatrix}\right]$. Therefore, $\left[\begin{smallmatrix}\mathcal{V}_{i+1}(\Lambda)\\ F_i\mathcal{V}_{i+1}(\Lambda)\end{smallmatrix}\right] + \left[\begin{smallmatrix}0\\ \mathcal{U}_i(\Lambda)\end{smallmatrix}\right] = \left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix}\mathcal{V}_i(\Lambda)\\ 0\end{smallmatrix}\right]$, because both spaces have the same basis $v_1,\ldots,v_\alpha$. We now prove that for any choice of F_i, we have $F_i \in \mathbb{F}(\mathcal{V}_i(\Lambda))$. Premultiply the above equation by $\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$ on the left to obtain

$$\begin{bmatrix}(A+BF_i)\mathcal{V}_{i+1}(\Lambda)\\(C+DF_i)\mathcal{V}_{i+1}(\Lambda)\end{bmatrix} + \begin{bmatrix}B\mathcal{U}_i(\Lambda)\\ D\mathcal{U}_i(\Lambda)\end{bmatrix} \subseteq \begin{bmatrix}\mathcal{V}_i(\Lambda)\\ 0\end{bmatrix}.$$

Moreover, we get $\left[\begin{smallmatrix}B\mathcal{U}_i(\Lambda)\\ D\mathcal{U}_i(\Lambda)\end{smallmatrix}\right] \subseteq \left[\begin{smallmatrix}\mathcal{V}_i(\Lambda)\\ 0\end{smallmatrix}\right]$ by (4.7). Thus it is easy to see that $(A+BF_i)\mathcal{V}_{i+1}(\Lambda) \subseteq \mathcal{V}_i$ and $(C+DF_i)\mathcal{V}_{i+1}(\Lambda) = 0$.

Subsequently, we show that (7.17) holds. By (7.12) and (7.14), it follows that for $i \in \mathbb{N}$,

$$\mathscr{W}_{i+1}(\Delta^{Impl}) = \begin{bmatrix}I_q & 0\\ 0 & 0\end{bmatrix}^{-1}\begin{bmatrix}\mathcal{W}_i(\Lambda)\\ 0\end{bmatrix}. \tag{7.19}$$

Then by (7.4), we have $\mathscr{W}_{i+1}(\Delta^{Impl}) = P\mathscr{W}_{i+1}(\Delta)$ and we complete the proof of (7.17) by calculating explicitly the right-hand side of (7.19). ∎

7.5. Proof of Proposition 6.7.

Proof. (i) By Proposition 6.6, $\mathscr{M}$ is an invariant subspace if and only if $H\mathscr{M} \subseteq E\mathscr{M}$. Therefore, $\mathscr{M}^*$ is the largest subspace such that $H\mathscr{M}^* \subseteq E\mathscr{M}^*$; then by Proposition 4.4(ii), we have $\mathscr{M}^* = \mathscr{V}^*$.

(ii) By Proposition 6.6, for $\Delta|^{red}_{\mathscr{M}^*} = (E|^{red}_{\mathscr{M}^*}, H|^{red}_{\mathscr{M}^*})$, the matrix $E|^{red}_{\mathscr{M}^*}$ is of full row rank. Thus from the explicitation procedure, it is straightforward to see that $\Lambda^* \in \mathrm{Expl}(\Delta|^{red}_{\mathscr{M}^*})$ is a control system without outputs. Note that, by the definitions of reduction and restriction, if two DAEs $\Delta \overset{ex}{\sim} \tilde{\Delta}$, then $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$. Denote the four parts of the **KCF** of Δ as KCF^k, $k = 1,\ldots,4$, and the corresponding matrix pencil of each part is

$$L(s) \text{ for } KCF^1, \quad J(s) \text{ for } KCF^2, \quad N(s) \text{ for } KCF^3, \quad L^p(s) \text{ for } KCF^4.$$

By $\Delta \overset{ex}{\sim} \mathbf{KCF}$, we have

$$\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \mathbf{KCF}|^{red}_{\tilde{\mathscr{M}}^*} = \left(KCF^1, KCF^2\right). \tag{7.20}$$

Moreover, it is clear that if two control systems $\Lambda \overset{M}{\sim} \tilde{\Lambda}$, then $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)} \overset{M}{\sim} \tilde{\Lambda}|^{red}_{(\tilde{\mathcal{V}}^*,\tilde{\mathcal{U}}^*)}$. Since Λ is always M-equivalent to its **MCF**, we have

$$\Lambda\Big|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)} \overset{M}{\sim} \mathbf{MCF}\Big|^{red}_{(\tilde{\mathcal{V}}^*,\tilde{\mathcal{U}}^*)} = \left(MCF^1, MCF^2\right). \tag{7.21}$$

It is seen that $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)}$ is a control system without outputs. From the one-to-one correspondence of the **KCF** and **MCF** discussed in section 5, it is straightforward to see that $\left(MCF^1, MCF^2\right) \in \mathrm{Expl}(KCF^1, KCF^2)$. Now combining the last observation with the relations of (7.20) and (7.21), and using the results of Theorem 3.4, we can deduce that $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)} \in \mathrm{Expl}(\Delta|^{red}_{\mathscr{M}^*})$. Since $\Lambda^* \in \mathrm{Expl}(\Delta|^{red}_{\mathscr{M}^*})$, by Theorem 3.4(ii) we have $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)} \overset{M}{\sim} \Lambda^*$. Finally, since Λ^* and $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)}$ are two control systems without outputs, their Morse equivalence reduces to their feedback equivalence. ∎

7.6. Proof of Theorem 6.9.

Proof. (i)⇔(ii) By Definition 6.8, we have $\Delta \overset{in}{\sim} \tilde{\Delta}$ if and only if $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \tilde{\Delta}|^{red}_{\mathscr{M}^*}$. Consider $\Lambda^* \in \mathrm{Expl}(\Delta|^{red}_{\mathscr{M}^*})$ and $\tilde{\Lambda}^* \in \mathrm{Expl}(\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*})$; then by Theorem 3.4(ii), it follows that $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \tilde{\Delta}|^{red}_{\mathscr{M}^*}$ if and only if $\Lambda^* \overset{M}{\sim} \tilde{\Lambda}^*$. By Proposition 6.7(ii), Λ^* and $\tilde{\Lambda}^*$ are two control systems without outputs, which implies that their Morse equivalence reduces to their feedback equivalence.

(ii)⇔(iii) We first prove that two DAEs $\Delta^* = \mathrm{Impl}(\Lambda^*)$ and $\tilde{\Delta}^* = \mathrm{Impl}(\tilde{\Lambda}^*)$ have isomorphic trajectories if and only if Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent. Let $(z(t), u(t))$ and $(\tilde{z}(t), \tilde{u}(t))$ denote trajectories of Δ^* and $\tilde{\Delta}^*$, respectively. Suppose Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent; then there exist matrices $T_s \in Gl(n^*, \mathbb{R})$, $T_i \in Gl(m^*, \mathbb{R})$, $F \in \mathbb{R}^{m^* \times n^*}$ such that $\tilde{A}^* = T_s(A^* + B^*F)T_s^{-1}$, $\tilde{B}^* = T_sBT_i^{-1}$. Since Λ^* has no output, its implicitation (see Definition 3.1) is

$$\Delta^* : \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A^* & B^* \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}.$$

For $\tilde{\Lambda}^*$, its implicitation is

$$\tilde{\Delta}^* : \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{u}} \end{bmatrix} = \begin{bmatrix} \tilde{A}^* & \tilde{B}^* \end{bmatrix} \begin{bmatrix} \tilde{z} \\ \tilde{u} \end{bmatrix} \Rightarrow \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{u}} \end{bmatrix} = T_s \begin{bmatrix} A^* & B^* \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} \begin{bmatrix} \tilde{z} \\ \tilde{u} \end{bmatrix}.$$

It can be seen that any trajectory $(z(t), u(t))$ of Δ^* satisfying $z(0) = z^0$ and $u(0) = u^0$ is mapped via $T = \left[\begin{smallmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{smallmatrix}\right]^{-1}$ into a trajectory $(\tilde{z}(t), \tilde{u}(t))$ of $\tilde{\Delta}^*$ passing through $\left[\begin{smallmatrix} \tilde{z}^0 \\ \tilde{u}^0 \end{smallmatrix}\right] = T\left[\begin{smallmatrix} z^0 \\ u^0 \end{smallmatrix}\right]$.

Conversely, suppose that there exists an invertible matrix $T = \left[\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} \tilde{z}(t) \\ \tilde{u}(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right]\left[\begin{smallmatrix} z(t) \\ u(t) \end{smallmatrix}\right]$. It follows that $(\tilde{z}(t), \tilde{u}(t))$, being a solution of $\tilde{\Delta}^*$, satisfies

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{pmatrix} \dot{\tilde{z}}(t) \\ \dot{\tilde{u}}(t) \end{pmatrix} = \begin{bmatrix} \tilde{A}^* & \tilde{B}^* \end{bmatrix} \begin{pmatrix} \tilde{z}(t) \\ \tilde{u}(t) \end{pmatrix},$$

which implies

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{pmatrix} \dot{z}(t) \\ \dot{u}(t) \end{pmatrix} = \begin{bmatrix} \tilde{A}^* & \tilde{B}^* \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}.$$

Since $(z(t), u(t))$ satisfies $\dot{z}(t) = A^*z(t) + B^*u(t)$, it follows that

$$\begin{aligned} &T_1\dot{z}(t) + T_2\dot{u}(t) = (\tilde{A}^*T_1 + \tilde{B}^*T_3)z(t) + (\tilde{A}^*T_2 + \tilde{B}^*T_4)u(t) \\ &\quad \Rightarrow T_1(A^*z(t) + B^*u(t)) + T_2\dot{u}(t) = (\tilde{A}^*T_1 + \tilde{B}^*T_3)z(t) + (\tilde{A}^*T_2 + \tilde{B}^*T_4)u(t). \end{aligned} \tag{7.22}$$

Notice that (7.22) is satisfied for any solution $(z(t), u(t))$ of Δ^*.

Set $u(t) \equiv 0$ and let $(z(t, z^0), 0)$ (where $z^0 \neq 0$) be a solution of Δ^* (obviously, such a solution always exists). By substituting this solution into (7.22) and considering it for $t = 0$, we have $T_1 A^* z^0 = (\tilde{A}^* T_1 + \tilde{B}^* T_3) z^0$, where $z^0 = z(0)$ can be taken arbitrary, which implies $A^* = T_1^{-1}(\tilde{A}^* + \tilde{B}^*(T_3 T_1^{-1}))T_1$.

Fix $z(0) = z^0 = 0$ and set $u(t) = u^i(t) = [0, \ldots, t, \ldots, 0]^T$, where t is in the ith row. Evaluating at $t = 0$, we have $z(0) = 0$, $u(0) = 0$, and $\dot{u}^i(0) = [0, \ldots, 1, \ldots, 0]^T$, and thus by (7.22) we have $T_2 \dot{u}^i(0) = 0$. So taking controls, $u^1(t), \ldots, u^{m^*}(t)$ of that form, we conclude that $T_2 = 0$. Now it is easy to see from (7.22) that $B^* = T_1^{-1} \tilde{B}^* T_4$. Thus Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent via $T_s = T_1$, $T_i = T_4^{-1}$, and $F = T_3 T_1^{-1}$. Therefore, any trajectory of Δ^* is transformed via T into a trajectory of $\tilde{\Delta}^*$ if and only if Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent.

By Theorem 3.4(i), we have

$$\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \Delta^* = \mathrm{Impl}(\Lambda^*) \quad \text{and} \quad \tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*} \overset{ex}{\sim} \tilde{\Delta}^* = \mathrm{Impl}(\tilde{\Lambda}^*)$$

(since $\Lambda^* \in \mathrm{Expl}(\Delta|^{red}_{\mathscr{M}^*})$ and $\tilde{\Lambda}^* \in \mathrm{Expl}(\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*})$). Moreover, by Remark 2.2, there exist matrices $P \in Gl(n^*, \mathbb{R})$ and $\tilde{P} \in Gl(n^*, \mathbb{R})$ such that any trajectory of $\Delta|^{red}_{\mathscr{M}^*}$ is mapped via P into the corresponding trajectory of Δ^* and any trajectory of $\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$ is mapped via $\tilde{P}$ into the corresponding trajectory of $\tilde{\Delta}^*$. Now we can conclude that the linear and invertible map $S = PT\tilde{P}^{-1}$ sends any trajectory of $\Delta|^{red}_{\mathscr{M}^*}$ into the corresponding trajectory of $\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$ if and only if Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent. $\square$

7.7. Proof of Theorem 6.11.

Proof. (i)$\Leftrightarrow$(ii) Consider a DAE $\Delta^* = \mathrm{Impl}(\Lambda^*)$. We have $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \Delta^*$ implied by $\Lambda^* \in \mathrm{Expl}(\Delta|^{red}_{\mathscr{M}^*})$ and Theorem 3.4(i). Actually, since Λ^* is defined on $\mathscr{M}^*$, it follows from Definition 6.8 that $\Delta|^{red}_{\mathscr{M}^*} \overset{in}{\sim} \Delta^* = \mathrm{Impl}(\Lambda^*)$. Thus by the equivalence of items (i) and (iii) of Theorem 6.9, the solutions of Δ passing through $x^0 \in \mathscr{M}^*$ are mapped, via a certain linear isomorphism S, into the solutions of Δ^*, which means that Δ is internally regular if and only if Δ^* has only one solution passing through any initial point in $\mathscr{M}^*$. This is true if and only if the input of Λ^* is absent, i.e., Δ^* is an ODE without free variables. Therefore, Δ is internally regular if and only if Λ^* has no inputs.

(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(vi) From the proof of Proposition 6.7(ii), we can see that the input is absent in Λ^* if and only if $\Lambda^* = MCF^2$ of Λ, that is, MCF^1 is absent in the **MCF** of Λ.

(i)$\Leftrightarrow$(iv)$\Leftrightarrow$(v): Using $\mathscr{V}^* = \mathscr{M}^*$ and the **KCF** of Δ, it is straightforward to see this equivalence. $\square$

8. Conclusions. In this paper, we propose a procedure called explicitation for DAEs. The explicitation of a DAE is, simply speaking, attaching to the DAE a class of linear control systems defined up to a coordinates change, a feedback, and an output injection. We prove that the invariant subspaces of the attached control systems have direct relations with the limits of the Wong sequences of the DAE. We show that the Kronecker invariants of the DAE have direct relations with the Morse invariants of the attached control systems, and as a consequence, the Kronecker canonical form **KCF** of the DAE and the Morse canonical from **MCF** of control systems have a perfect correspondence. We also propose a notion called internal equivalence for DAEs and show that the internal equivalence is useful when analyzing the existence and uniqueness of solutions (internal regularity).

Appendix A.

Kronecker canonical form KCF (see [11], [8], [6]). For any matrix pencil $sE - H \in \mathbb{R}^{l \times n}[s]$, there exist matrices $Q \in Gl(l, \mathbb{R})$, $P \in Gl(n, \mathbb{R})$ and integers

$\varepsilon_1,\ldots,\varepsilon_a \in \mathbb{N}, \rho_1,\ldots,\rho_b \in \mathbb{N}^+, \sigma_1,\ldots,\sigma_c \in \mathbb{N}^+, \eta_1,\ldots,\eta_d \in \mathbb{N}$ with $a,b,c,d \in \mathbb{N}$ such that

$$Q(sE-H)P^{-1}$$
$$= \mathrm{diag}\left(L_{\varepsilon_1}(s),\ldots,L_{\varepsilon_a}(s), J_{\rho_1}(s),\ldots,J_{\rho_b}(s), N_{\sigma_1}(s),\ldots,N_{\sigma_c}(s), L^p_{\eta_1}(s),\ldots,L^p_{\eta_d}(s)\right),$$

where (omitting, for simplicity, the index i of $\varepsilon_i,\rho_i,\sigma_i,\eta_i$) the bidiagonal pencils $L_\varepsilon(s) \in \mathbb{R}^{\varepsilon\times(\varepsilon+1)}[s]$, the real Jordan pencils $J_\rho(s) \in \mathbb{R}^{\rho\times\rho}[s]$, the nilpotent pencils $N_\sigma(s) \in \mathbb{R}^{\sigma\times\sigma}[s]$, and the pencils $L^p_\eta(s) \in \mathbb{R}^{\eta\times(\eta+1)}[s]$ have the following form:

$$L_\varepsilon(s) = \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \quad N_\sigma(s) = \begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}, \quad L^p_\eta(s) = \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix},$$

$$J_\rho(s) = \begin{bmatrix} s-\lambda_\rho & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s-\lambda_\rho \end{bmatrix} \text{ or } J_\rho(s) = \begin{bmatrix} S-\Lambda_\rho & -I & & \\ & \ddots & \ddots & \\ & & \ddots & -I \\ & & & S-\Lambda_\rho \end{bmatrix}, \quad S-\Lambda_\rho = \begin{bmatrix} s-\phi_\rho & -\varphi_\rho \\ \varphi_\rho & s-\phi_\rho \end{bmatrix},$$

where λ_ρ, φ_ρ, $\phi_\rho \in \mathbb{R}$. The integers ε_i, ρ_i, σ_i, η_i are called, respectively, Kronecker column (minimal) indices, the degrees of the finite elementary divisors, the degrees of the infinite elementary divisors, and Kronecker row (minimal) indices. In addition, λ_ρ and $\varphi_\rho + i\phi_\rho$ are the corresponding eigenvalues of $J(s)$. These indices and eigenvalues are invariant under external equivalence of Definition 2.1. Notice that the above **KCF** coincides with the one used in [6], which is slightly different from the ones presented in [11] and [8]. More specifically, the invariants ϵ_i, η_i of the **KCF** of [6] and of this paper are allowed to be zero (L_{ϵ_i} is a zero column if $\epsilon_i = 0$ and $L^p_{\eta_i}$ is a zero row if $\eta_i = 0$). In the **KCF** of [11], the trivial case $0_{l\times n}$ is not included, while in [8], there is an extra zero block in the first entry of the quasi-diagonal form (34) (see page 39, Volume II, of [8]).

Morse canonical form MCF (see [20], [19]). Any control system $\Lambda = (A,B,C,D)$ is Morse equivalent to the Morse canonical form **MCF** shown below:

$$\mathbf{MCF}: \begin{cases} MCF^1: \ \dot{z}^1 = A^1z^1 + B^1u^1, \\ MCF^2: \ \dot{z}^2 = A^2z^2, \\ MCF^3: \ \dot{z}^3 = A^3z^3 + B^3u^3, \quad y^3 = C^3z^3 + D^3u^3, \\ MCF^4: \ \dot{z}^4 = A^4z^4, \quad\quad\quad\quad\ \ y^4 = C^4z^4. \end{cases}$$

If a control system $\Lambda = (A,B,C,D)$ is in the **MCF**, then the matrices A,B,C,D, together with all invariants, are thus given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cccc|cc} A^1 & 0 & 0 & 0 & B^1 & 0 \\ 0 & A^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & A^3 & 0 & 0 & B^3 \\ 0 & 0 & 0 & A^4 & 0 & 0 \\ \hline 0 & 0 & C^3 & 0 & 0 & D^3 \\ 0 & 0 & 0 & C^4 & 0 & 0 \end{array}\right],$$

(i) with $A^1 = \mathrm{diag}\{A^1_{\varepsilon'_1},\ldots,A^1_{\varepsilon'_{a'}}\}$, $B^1 = \mathrm{diag}\{B^1_{\varepsilon'_1},\ldots,B^1_{\varepsilon'_{a'}}\}$, where (throughout we omit, for simplicity, the index i of $\varepsilon'_i,\rho'_i,\sigma'_i,\eta'_i$)

$$A^1_{\varepsilon'} = \begin{bmatrix} 0 & I_{\varepsilon'-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\varepsilon'\times\varepsilon'}, \quad B^1_{\varepsilon'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\varepsilon'\times 1},$$

and the integers $\varepsilon'_1, \dots, \varepsilon'_{a'} \in \mathbb{N}$ are the controllability indices of (A^1, B^1);

(ii) $A^2 = \operatorname{diag}\{A^2_{\rho'_1}, \dots, A^2_{\rho'_{b'}}\}$, where $A^2_{\rho'}$ is given by

$$A^2_{\rho'} = \begin{bmatrix} \lambda_{\rho'} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_{\rho'} \end{bmatrix} \quad \text{or} \quad A^2_{\rho'} = \begin{bmatrix} \Lambda_{\rho'} & I & & \\ & \ddots & \ddots & \\ & & \ddots & I \\ & & & \Lambda_{\rho'} \end{bmatrix}, \quad \Lambda_{\rho'} = \begin{bmatrix} s-\phi_{\rho'} & -\varphi_{\rho'} \\ \varphi_{\rho'} & s-\phi_{\rho'} \end{bmatrix},$$

where $\lambda_{\rho'}, \varphi_{\rho'}, \phi_{\rho'} \in \mathbb{R}$;

(iii) the 4-tuple (A^3, B^3, C^3, D^3) is controllable and observable (prime), that is,

$$\begin{bmatrix} A^3 & B^3 \\ C^3 & D^3 \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A}^3 & \hat{B}^3 & 0 \\ \hline \hat{C}^3 & 0 & 0 \\ 0 & 0 & I_\delta \end{array}\right], \tag{A.1}$$

where $\left[\begin{smallmatrix} \hat{A}^3 & \hat{B}^3 \\ \hat{C}^3 & 0 \end{smallmatrix}\right]$ is square and invertible and $\delta = \operatorname{rank} D^3 \in \mathbb{N}$, and the matrices

$$\hat{A}^3 = \operatorname{diag}\{\hat{A}^3_{\sigma'_{\delta+1}}, \dots, \hat{A}^3_{\sigma'_{c'}}\},\ \hat{B}^3 = \operatorname{diag}\{\hat{B}^3_{\sigma'_{\delta+1}}, \dots, \hat{B}^3_{\sigma'_{c'}}\},\ \hat{C}^3 = \operatorname{diag}\{\hat{C}^3_{\sigma'_{\delta+1}}, \dots, \hat{C}^3_{\sigma'_{c'}}\},$$

where

$$\hat{A}^3_{\sigma'} = \begin{bmatrix} 0 & I_{\sigma'-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\sigma' \times \sigma'}, \quad \hat{B}^3_{\sigma'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\sigma' \times 1}, \quad \hat{C}^3_{\sigma'} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1 \times \sigma'};$$

the integers $\sigma'_1 = \cdots = \sigma'_\delta = 0$ and $\sigma'_{\delta+1}, \dots, \sigma'_{c'} \in \mathbb{N}^+$ are the controllability indices of the pair $(\hat{A}^3, \hat{B}^3)$ and they are equal to the observability indices of the pair $(\hat{C}^3, \hat{A}^3)$;

(iv) $A^4 = \operatorname{diag}\{A^4_{\eta'_1}, \dots, A^4_{\eta'_{d'}}\}$, $C^4 = \operatorname{diag}\{C^4_{\eta'_1}, \dots, C^4_{\eta'_{d'}}\}$, where

$$A^4_{\eta'} = \begin{bmatrix} 0 & I_{\eta'-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\eta' \times \eta'}, \qquad C^4_{\eta'} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1 \times \eta'}.$$

The integers $\eta'_1, \dots, \eta'_{d'} \in \mathbb{N}$ are the observability indices of the pair (C^4, A^4).

We call the integers ε'_i, ρ'_i, σ'_i, η'_i the Morse indices of control systems; together with a', b', c', d', δ and $\lambda_{\rho'} \in \mathbb{R}$ or $\lambda_{\rho'} = \varphi_{\rho'} + j\phi_{\rho'} \in \mathbb{C}$, with ρ' taking all values ρ'_i, where $j = \sqrt{-1}$, they are all invariant under Morse equivalence.

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