The differentiation index of DAEs vs. the relative degree of control systems

the 91st GAMM Annual Meeting

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Nonlinear DAEs

Consider a nonlinear differential-algebraic equation (DAE):

$$\Xi : E(x)\dot{x} = F(x), \tag{1}$$

- where the generalized state $x \in X$ and X is an open subset of $\mathbb{R}^n$;
- $E : TX \to \mathbb{R}^l$ and $F : X \to \mathbb{R}^l$ are $\mathcal{C}^\infty$-smooth maps.
- The DAE (1) is denoted by $\Xi_{l,n} = (E, F)$.
- Application: constrained mechanics, electrical circuits, chemical processes, etc.

Solutions of nonlinear DAEs

- A *solution* of Ξ is a $\mathcal{C}^1$-curve $x : I \to X$ s.t. $E(x(t))\dot{x}(t) = F(x(t))$, $\forall t \in I$.
- A point x_0 is called called *admissible* if $\exists$ a solution $x(\cdot)$ and $t_0 \in I$ s.t. $x(t_0) = x_0$. The *admissible set* (or consistency set) $S_a := \{x_0 \,|\, \exists x(\cdot), t_0 \in I : x(t_0) = x_0\}$.

Definition (geometric reduction method)

Fix $x_p \in X$. Step 0: Set $M_0 = X$, $M_0^c = U_0$. Step k:

$$M_k := \left\{x \in M_{k-1}^c : F(x) \in E(x)T_xM_{k-1}^c\right\}. \tag{2}$$

Assume: $x_p \in M_k$ and $M_k^c = M_k \cap U_k$ is a smooth connected submanifold.

- Reich (1991); Rabier and Rheinboldt (2002); Berger(2016,2017); Chen and Trenn (2020); Chen, Trenn and Respondek. (2021).
- $M^* = M_{k^*+1}^c = M_{k^*}^c$ and $M^* \cap U^* = S_a \cap U^*$ (Chen, Trenn and Respondek. (2021)).

Differentiation index

Definition (differentiation index)

Let $H(x,\dot{x}) = E(x)\dot{x} - F(x)$, define the k-th order differential array of $H(x,\dot{x}) = 0$ by

$$H_k(x, x', x^{(2)}, \ldots, x^{(k+1)}) = \begin{bmatrix} H \\ \mathrm{D}_x H x' + \mathrm{D}_{x'} H x'' \\ \vdots \\ \frac{\mathrm{d}^k}{\mathrm{d}t^k} H \end{bmatrix} (x, x', w) = 0, \tag{3}$$

The differentiation index ν_d: the least integer k s.t. (3) uniquely determines x' as a function of x, i.e., $x' = v(x)$.

- The vector field v is defined on X or M^*?

Example

Consider the following DAE $\Xi_{3,3} = (E, F)$ around $x_0 = (x_{10}, x_{20}, x_{30}) = (0, 1, 1) \in X$, where $X = \{x \in \mathbb{R}^3 : x_2 > 0\}$,

$$\begin{bmatrix} x_2 & x_1 & 0 \\ 0 & \sin x_3 & x_2 \cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \ln x_2 \\ x_2 x_3^2 \cos x_3 \\ x_1 x_2 \end{bmatrix} \tag{4}$$

- We only differentiate the constraints $0 = x_1 x_2$ to have $(x_1 x_2)' = x_2 \dot{x}_1 + x_1 \dot{x}_2 = \ln x_2 = 0$, $(x_1 x_2)'' = \frac{1}{x_2} x_2' = 0$.
- The vector field $v(x) = \begin{bmatrix} x_2 & x_1 & 0 \\ 0 & \sin x_3 & x_2 \cos x_3 \\ 0 & \frac{1}{x_2} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \ln x_2 \\ x_2 x_3^2 \cos x_3 \\ 0 \end{bmatrix}$.
- Ξ is of differentiation index-2, i.e. $\nu_d = 2$.
- $M_1^c := \{x \in X : x_1 x_2 = 0\}$; $M_2^c := \{x \in M_1^c : \ln x_2 = 0\}$; $M^* = M_3^c = M_2^c$. Solutions exists on M^* only, $v(x)|_{M^*} = \begin{bmatrix} 0 \\ 0 \\ x_3^2 \end{bmatrix}$.
- The solution of Ξ (passing through x_0) is $(x_1(t), x_2(t), x_3(t)) = (0, 1, \frac{1}{1-t})$.

The explicitation of nonlinear DAEs

- For a DAE $\Xi_{l,n} = (E, F)$, assume (on a nbh. U of x_p) $\operatorname{rank} E(x) = const. = q$.
- $\exists Q : U \to GL(l, \mathbb{R})$ s.t. $QE(x)\dot{x} = QF(x)$:

$$\begin{cases} E_1(x)\dot{x} = F_1(x), \\ \quad 0 \quad\;\, = F_2(x). \end{cases} \tag{5}$$

where $E_1(x)$ of $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ is of full row rank and $Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$.

- (crucial) The collection of all $\dot{x}$ satisfying $E_1(x)\dot{x} = F_1(x)$ of (5) is given by the differential inclusion:

$$\dot{x} \in f(x) + \ker E_1(x) = f(x) + \ker E(x). \tag{6}$$

where $f(x) = E_1^{\dagger} F_1(x)$, where $E^{\dagger}$ is a right inverse of E_1.

The explicitation of nonlinear DAEs

- Let $m = n - q$ and $g_1, \ldots, g_m : X \to \mathbb{R}^n$ s.t. $\ker E(x) = \operatorname{span}\{g_1, \ldots, g_m\}(x)$. By introducing driving variables v_i, $i = 1, \ldots, m$, we get

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) v_i. \tag{7}$$

- All solutions of Ξ are in one-to-one correspondence with all solutions (corresponding to all $\mathcal{C}^0$-controls $v(t)$) of

$$\begin{cases} \dot{x} = f(x) + g(x)v, \\ 0 = h(x). \end{cases} \tag{8}$$

where $h(x) = F_2(x)$.

- To (8), we attach the control system $\Sigma = \Sigma_{n,m,p} = (f, g, h)$, given by

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v, \\ y = h(x), \end{cases} \tag{9}$$

where $n = \dim x$, $m = \dim v = n - q$, $p = \dim y = l - q$.

The explicitation of nonlinear DAEs

Definition (explicitation with driving variables)

Given a DAE $\Xi_{l,n} = (E, F)$, by a (Q, v)-explicitation, we will call a control system $\Sigma = \Sigma_{n,m,p} = (f, g, h)$ given by

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v, \\ y = h(x), \end{cases}$$

with

$$f(x) = E_1^{\dagger} F_1(x), \quad \operatorname{Im} g(x) = \ker E(x), \quad h(x) = F_2(x),$$

where $QE(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$, $QF(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$.

- Σ is not uniquely defined since $Q(x)$, $E_1^{\dagger}(x)$ and $g(x)$ are not unique !

The explicitation of nonlinear DAEs

Proposition

Assume: $\Sigma_{n,m,p} = (f, g, h)$, $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ *are two* (Q, v)*-explicitations of a DAE* $\Xi_{l,n} = (E, F)$ *corresponding to different choices of* Q*,* $E_1^\dagger$ *and* g*.*

Then $\exists$ α, γ *and invertible* η, β*, which map*

$$f \mapsto \tilde{f} = f + \gamma h + g\alpha, \quad g \mapsto \tilde{g} = g\beta, \quad h \mapsto \tilde{h} = \eta h.$$

- Σ is defined up to a feedback transformation $v = \alpha + \beta\tilde{v}$, an output injection γy and an output multipulication ηh:

$$\tilde{\Sigma} : \begin{cases} \dot{x} = f(x) + g(x)(\alpha(x) + \beta(x)\tilde{v}) + \gamma(x)h(x) \\ y = \eta(x)h(x) \end{cases}$$

- The (Q, v)-explicitation of Ξ is a class, denoted $\boldsymbol{Expl}(\Xi)$. We write $\Sigma \in \boldsymbol{Expl}(\Xi)$.

External equivalence and system equivalence

Definition (external equivalence)

Two DAEs $\Xi_{l,n} = (E, F)$ and $\tilde{\Xi}_{l,n} = (\tilde{E}, \tilde{F})$ are called externally equivalent, shortly ex-equivalent, if $\exists$ diffeomorphism $\psi : X \to \tilde{X}$ and $Q : X \to GL(l, \mathbb{R})$ s.t.

$$\tilde{E}(\psi(x)) = Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1} \quad \text{and} \quad \tilde{F}(\psi(x)) = Q(x)F(x). \tag{10}$$

Definition (system equivalence)

Two control systems $\Sigma_{n,m,p} = (f, g, h)$ and $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ are called system equivalent, or shortly sys-equivalent if

$$\tilde{f} \circ \psi = \frac{\partial \psi}{\partial x}\left(f + \gamma h + g\alpha\right), \quad \tilde{g} \circ \psi = \frac{\partial \psi}{\partial x} g\beta, \quad \tilde{h} \circ \psi = \eta h.$$

External equivalence and system equivalence

Theorem

Assume: $\operatorname{rank} E(x)$ *and* $\operatorname{rank} \tilde{E}(\tilde{x})$ *are constant around two points* x_p *and* $\tilde{x}_p$*, respectively, for two DAEs* $\Xi_{l,n} = (E, F)$ *and* $\tilde{\Xi}_{l,n} = (\tilde{E}, \tilde{F})$.
Then for two systems $\Sigma_{n,m,p} = (f, g, h) \in \boldsymbol{Expl}(\Xi)$ *and* $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h}) \in \boldsymbol{Expl}(\tilde{\Xi})$*, we have that locally*

$$\Xi \overset{ex}{\sim} \tilde{\Xi} \quad \Leftrightarrow \quad \Sigma \overset{sys}{\sim} \tilde{\Sigma}.$$

Example

$$\Xi = (E, F) \overset{ex}{\sim} \tilde{\Xi} : \begin{cases} \dot{x}_1 &= F_1(x_1) \\ 0 &= x_2^1 \\ \dot{x}_2^1 &= x_2^1 \\ &\vdots \\ \dot{x}_2^{\rho-1} &= x_2^{\rho} \end{cases} \Leftrightarrow \Sigma = (f, g, g) \in \boldsymbol{Expl}(\Xi) \overset{sys}{\sim} \tilde{\Sigma} : \begin{cases} \dot{x}_1 &= F_1(x_1) \\ y &= x_2^1 \\ \dot{x}_2^1 &= x_2^1 \\ &\vdots \\ \dot{x}_2^{\rho-1} &= x_2^{\rho} \\ \dot{x}_2^{\rho} &= v \end{cases}$$

Differentiation index vs. relative degree

Definition (relative degree)

$\Sigma_{n,m,m} = (f, g, h)$ has a (vector) relative degree $\rho = (\rho_1, \dots, \rho_m)$ at a point x_0 if (i) $L_g L_f^k h(x) = 0$, for all $1 \leq j \leq m$, for all $k < \rho_i - 1$ and $1 \leq i \leq m$, and for all x around x_0;
(ii) $D(x_0) = \left(L_{g_j} L_f^{\rho_i - 1} h_i(x_0) \right)_{i,j=1,\dots,m}$ is invertible.

- Any control system $\Sigma = (f, g, h)$ with relative degree $\rho = (\rho_1, \dots, \rho_m)$ is feedback equivalent to the Byrnes-Isidori form

$$
\textbf{B-I}: \begin{cases} \dot{z} &= f_0(z, \xi_1, \dots, \xi_m) + g_0(z, \xi_1, \dots, \xi_m) v \\ y_i &= \xi_i^1, \quad i = 1, \dots, m, \\ \dot{\xi}_i^1 &= \xi_i^2 \\ & \vdots \\ \dot{\xi}_i^{\rho_i - 1} &= \xi_i^{\rho_i} \\ \dot{\xi}_i^{\rho_i} &= v_i \end{cases}
$$

Theorem

Assume: *an explicitaion $\Sigma \in \boldsymbol{Expl}(\Xi)$ has a well-defined relative degree $\rho = (\rho_1, \ldots, \rho_m)$ at x_0, then*

(i) *any $\tilde{\Sigma} \in \boldsymbol{Expl}(\Xi)$ has either the same relative degree ρ with Σ or no well-defined relative degree at x_0;*

(ii) $\nu_d = \max\{\rho_1, \ldots, \rho_m\}$,

(iii) Ξ *is ex-equivalent to*

$$\begin{cases} \dot{z} - g_0(z,\xi)\dot{\xi}^\rho = f_0(z,\xi) \\ N\dot{\xi} = \xi \end{cases}$$

where $N = \operatorname{diag}(N_1, \ldots, N_m)$, and where N_i, $i = 1, \ldots, m$ are nilpotent matrices of index ρ_i.

Continuation of the Example

$$\Xi : \begin{bmatrix} x_2 & x_1 & 0 \\ 0 & \sin x_3 & x_2 \cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \ln x_2 \\ x_2 x_3^2 \cos x_3 \\ x_1 x_2 \end{bmatrix},$$

- A control system

$$\Sigma = (f, g, h) \in \boldsymbol{Expl}(\Xi) : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{\ln x_2}{x_2} \\ 0 \\ x_3^2 \end{bmatrix} + \begin{bmatrix} x_1 \\ -x_2 \\ \tan x_3 \end{bmatrix} v, \quad y = x_1 x_2$$

- The relative degree of Σ at $x_0 = (0, 1, 1)$ is $\rho = 2$.
- $\Xi \overset{ex}{\sim} \begin{cases} \dot{z} - \tan z \dot{\xi}_2 = z^2 \\ 0 = \xi_1 \\ \dot{\xi}_1 = \xi_2 \end{cases} \Leftrightarrow \Sigma \overset{sys}{\sim} \begin{cases} \dot{z} = z^2 + \tan z v \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = v \\ y = \xi_1 \end{cases}$

Summary

- Geometric reduction method and differential array to solve DAEs.
- A universal way to connected nonlinear DAEs with nonlinear control systems: the explicitation with driving variables.
- External equivalence of DAEs and system equivalence of control systems.
- The differentiation index of DAEs and the relative degree of control systems.
- Normal forms (nonlinear generalizations of the Weierstrass form ?).