Feedback linearization of nonlinear differential-algebraic control systems

Yahao Chen[a], Witold Respondek[b]

[a] *Bernoulli Institute, University of Groningen, The Netherlands (e-mail: yahao.chen@rug.nl);*
[b] *INSA-Rouen, LMI, 76801 Saint-Etienne-du-Rouvray, France (e-mail: witold.respondek@insa-rouen.fr).*

Abstract

In this paper, we study feedback linearization problems for nonlinear differential-algebraic control systems (DACSs). We consider two kinds of feedback equivalences, namely, the external feedback equivalence, which is defined (locally) on the whole generalized state space, and the internal feedback equivalence, which is defined on the locally maximal controlled invariant submanifold (i.e., on the set where the solutions exist). Necessary and sufficient conditions are given for the locally internal and the locally external feedback linearizability of DACSs with the help of a notion called the explicitation with driving variables, which attaches a class of explicit control systems to a given DACS. We show that the feedback linearizability of a DACS is closely related to the involutivity of the linearizability distributions of the explicitation systems. Finally, we verify our results of feedback linearization of DACSs via both numerical and physical examples.

Keywords: differential-algebraic control systems, external and internal feedback equivalence, feedback linearization, controlled invariant submanifolds, explicitation

---

1. Introduction

Consider a nonlinear differential-algebraic control system (DACS) of the form

$$\Xi^u : E(x)\dot{x} = F(x) + G(x)u, \tag{1}$$

where $x \in X$ is called the generalized state and $(x, \dot{x}) \in TX$, where TX is the tangent bundle of an open subset X in $\mathbb{R}^n$ (or, more general, of an n-dimensional smooth manifold X), the vector of inputs $u \in \mathbb{R}^m$, and where $E : TX \to \mathbb{R}^l$, $F : X \to \mathbb{R}^l$ and $G : X \to \mathbb{R}^{l\times m}$ are smooth maps. The word "smooth" will always mean $\mathcal{C}^\infty$-smooth throughout the paper. We denote a DACS of the form (1) by $\Xi^u_{l,n,m} = (E, F, G)$ or, simply, Ξ^u. A linear DACS is of the form

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{2}$$

where $E, H \in \mathbb{R}^{l\times n}$ and $L \in \mathbb{R}^{l\times m}$ and will be denoted by $\Delta^u_{l,n,m} = (E, H, L)$ or, simply, Δ^u. Linear DACSs have been studied for decades, there is a rich literature devoted to them (see, e.g., the surveys [25, 26] and textbook [14]). In the context of this paper, we will need results about canonical forms

[27],[23] and [10], controllability [4],[13],[15], and geometric subspaces [16],[29]. The motivation of studying linear and nonlinear DACSs is their frequent presence in mathematical models of practical systems e.g., constrained mechanics [31], chemical processes [22], electrical circuits [34], etc.

The map E of a DACS (1) is not necessarily square (i.e., $l \neq n$) nor invertible. As a consequence, some free variables and constrained variables can be implicitly present in the generalized state x (and some constrained control variables may exist in u). We have proposed two normal forms to distinguish the different roles of variables for nonlinear DACSs in [12]. In the case of $E(x) = I_n$, the DACS (1) becomes an explicit control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{3}$$

where $f = F$ and g_i, $1 \leq i \leq m$, being the columns of G, become vector fields on X. The feedback linearization problem for nonlinear explicit control systems (i.e., when there exist a local change of coordinates in the state space and a feedback transformation such that the transformed system has a linear form in the new coordinates) has drawn the attention of researchers for decades (e.g. see survey papers [33],[36] and books [28],[19]). The solution of the feedback linearization problem of explicit control systems was first given in Brockett's paper [5] and developed by Jakubczyk and Respondek [20], Su [35], Hunt et Su [18]. Compared to the explicit control systems, fewer results on the linearization problem of DACSs can be found. Xiaoping [38] transformed a nonlinear DACS into a linear one by state space transformations, Kawaji [21] gave sufficient conditions for the feedback linearization of a special class of DACSs, Jie Wang and Chen Chen [37] considered a semi-explicit differential-algebraic equation (DAE) and linearized the differential part of the DAE. The linearization of semi-explicit DAEs under equivalence of different levels is studies in [8].

In the present paper, our purpose is to find when a given DACS of the form (1) is locally equivalent to a linear completely controllable one (see the definition of the complete controllability of linear DACSs in [4]). In particular, we will consider two kinds of equivalence relations, namely, the external feedback equivalence given in Definition 2.7 and the internal feedback equivalence given in Definition 2.8. Note that the words "external" and "internal", appearing throughout this paper, basically mean that we consider the DACS on an open neighborhood of the generalized state space X and on the *locally maximal controlled invariant submanifold* M^* (see Definition 2.2), respectively. We have discussed in detail the differences and relations of the two equivalence relations for linear DAEs in [10], and for semi-explicit DAEs in [8]. We will use a notion called the *explicitation with driving variables* (see Definition 3.1, firstly proposed in [9] for linear DACSs) to connect nonlinear DACSs with nonlinear explicit control systems. Via the explicitation with driving variables, we can interpret the linearizability of a DACS under internal or external feedback equivalence as that of an explicitation system under system feedback equivalence (see Definition 3.3).

The paper is organized as follows: In Section 2, we define the external and the internal feedback equivalences and discuss their relations with solutions of DACSs. In Section 3, we use the notion of explicitation with driving variables to connect DACSs with explicit control systems. Necessary and sufficient conditions for both the external and the internal feedback linearization problems of DACSs are given in Section 4. We illustrate the results of Section 4 by the two examples in Section 5. The conclusions and perspectives of this paper are given in Section 6 and a technical proof is given in Appendix. We use the following notations in the present paper: We denote by $T_xM \in \mathbb{R}^n$ the tangent space at $x \in M$ of a differentiable submanifold M of $\mathbb{R}^n$. For a smooth map $f : X \to \mathbb{R}$, we denote its differential by $\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathrm{d}x_i = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$. For a map $A : X \to \mathbb{R}^{m\times n}$, $\ker A(x)$, $\operatorname{Im} A(x)$ and $\operatorname{rank} A(x)$ are the kernel, the image and the rank of A at x, respectively. For a full row rank map $R : X \to \mathbb{R}^{r\times n}$, denote $R^\dagger : X \to \mathbb{R}^{n\times r}$ the right inverse of R, i.e., $RR^\dagger = I_r$. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. We assume the reader is familiar with basic notions of differential geometry such as smooth embedded submanifolds, involutive distributions and refer the reader e.g. to the book [24] for the formal definitions of such notions.

2. External and internal feedback equivalence

Definition 2.1 (solutions and admissible set)**.** For a DACS $\Xi^u_{l,n,m} = (E, F, G)$, a curve $(x, u) : I \to X \times \mathbb{R}^m$ defined on an open interval $I \subseteq \mathbb{R}$ with $x(\cdot) \in \mathcal{C}^1$ and $u(\cdot) \in \mathcal{C}^0$, is called a solution of Ξ^u if for all $t \in I$, $E(x(t))\dot{x}(t) = F(x(t)) + G(x(t))u(t)$. We call a point $x_a \in X$ *admissible* if there exists at least one solution $(x(\cdot), u(\cdot))$ such that $x(t_a) = x_a$ for a certain $t_a \in I$. The set of all admissible points will be called the admissible set (or the consistency set) of Ξ^u and denoted by S_a.

A smooth connected embedded submanifold M is called *controlled invariant* if for any point $x_0 \in M$, there exists a solution $(x, u) : I \to M \times \mathbb{R}^m$ such that $x(t_0) = x_0$ for a certain $t_0 \in I$ and $x(t) \in M$ for all $t \in I$. Fix an admissible point $x_a \in X$, a smooth connected embedded submanifold M containing x_a is called *locally controlled invariant* if there exists a neighborhood U of x_a such that $M \cap U$ is controlled invariant.

Definition 2.2 (locally maximal controlled invariant submanifold)**.** A locally controlled invariant submanifold M^*, around an admissible point x_a, is called *maximal* if there exists a neighborhood U of x_a such that for any other locally controlled invariant submanifold M, we have $M \cap U \subseteq M^* \cap U$.

The locally maximal controlled invariant submanifold M^* of a DACS can be construed via the following *geometric reduction method*, which was frequently used (see e.g., [2, 12, 30, 32, 34]) for studying the existence of solutions of DAEs and DACSs.

Definition 2.3 (geometric reduction method [12])**.** Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix a point $x_p \in X$. Let U_0 be a connected subset of X containing x_p. Step 0: Set $M_0 = X$ and

$M_0^c = U_0$. Step k ($k > 0$): Suppose that a sequence of smooth connected embedded submanifolds $M_{k-1}^c \subsetneq \cdots \subsetneq M_0^c$ of U_{k-1} for a certain $k-1$, have been constructed. Define recursively

$$M_k := \left\{ x \in M_{k-1}^c \,|\, F(x) \in E(x) T_x M_{k-1}^c + \operatorname{Im} G(x) \right\}. \tag{4}$$

As long as $x_p \in M_k$, let $M_k^c = M_k \cap U_k$ be a smooth embedded connected submanifold for some neighborhood $U_k \subseteq U_{k-1}$.

Proposition 2.4 ([12])**.** *In the above geometric reduction method, there always exists a smallest k^* such that either k^* is the smallest integer for which $x_p \notin M_{k^*+1}$ or k^* is the smallest integer such that $x_p \in M_{k^*+1}^c$ and $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$. In the latter case, assume that there exists an open neighborhood $U^* \subseteq U_{k^*+1}$ of x_p such that the constant rank condition **(CR)** below is satisfied for all $x \in M_{k^*+1} \cap U^*$, then*

(i) x_p is an admissible point, i.e., $x_p = x_a$ and $M^* = M_{k^*+1}$ is a locally maximal controlled invariant submanifold around x_p;

(ii) M^* coincides locally with the admissible set S_a, i.e., $M^* \cap U^* = S_a \cap U^*$.

By item (ii) of Proposition 2.4, the admissible set S_a locally coincides with M^* on the neighborhood U^* of x_p. So any point $x_0 \in U^* \backslash M^*$ is not admissible and there exist no solutions passing through x_0. Thus to study the solutions of a DACS, it is convenient to consider only the restriction of the DACS to the locally maximal controlled invariant submanifold M^*, which can be defined as follows (see also Remark 3.4(ii) and Theorem 4.4(i) of [12]).

Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix an admissible point $x_a \in X$. Let M^* be the n^*-dimensional maximal controlled invariant submanifold of Ξ^u around x_a. Assume that there exists a neighborhood U of x_a such that for all $x \in M^* \cap U$,

(CR) $\dim E(x) T_x M^* = const. = r^*$ and $E(x) T_x M^* + \operatorname{Im} G(x) = const. = r^* + (m - m^*)$.

Let $\phi : U \to \mathbb{R}^n$ be a local diffeomorphism and $z = \phi(x) = (z_1, z_2)$ be local coordinates on U such that $M^* \cap U = \{z_2 = 0\}$, thus z_1 are local coordinates on $M^* \cap U$. Then in the new z-coordinates, the DACS Ξ^u becomes a system $\tilde{\Xi}^u_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$, given by

$$\begin{bmatrix} \tilde{E}_1(z_1,z_2) & \tilde{E}_2(z_1,z_2) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \tilde{F}(z_1, z_2) + \tilde{G}(z_1, z_2) u,$$

where $\tilde{E}_1 : U \to \mathbb{R}^{l \times n^*}$, $\tilde{E}_2 : U \to \mathbb{R}^{l \times (n-n^*)}$, $\begin{bmatrix} \tilde{E}_1 \circ \phi & \tilde{E}_2 \circ \phi \end{bmatrix} = \tilde{E} \circ \phi = E \cdot \left(\frac{\partial \phi}{\partial x} \right)^{-1}$, $\tilde{F} \circ \phi = F$ and $\tilde{G} \circ \phi = G$. Set $z_2 = 0$ to have the following system (defined on M^*)

$$\begin{bmatrix} \tilde{E}_1(z_1,0) & \tilde{E}_2(z_1,0) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ 0 \end{bmatrix} = \tilde{F}(z_1, 0) + \tilde{G}(z_1, 0) u. \tag{5}$$

By the assumption **(CR)**, there exist a neighborhood $U_1 \subseteq U$ of x_a and $Q : M^* \cap U_1 \to GL(l, \mathbb{R})$ such that $\tilde{E}_1^1(z_1)$ and $\tilde{G}_2(z_1)$ below are of full row rank,

$$Q(z_1) \begin{bmatrix} \tilde{E}_1(z_1,0) & \tilde{F}(z_1,0) & \tilde{G}(z_1,0) \end{bmatrix} = \begin{bmatrix} \tilde{E}_1^1(z_1) & \tilde{F}_1(z_1) & \tilde{G}_1(z_1) \\ 0 & \tilde{F}_2(z_1) & \tilde{G}_2(z_1) \\ 0 & \tilde{F}_3(z_1) & 0 \end{bmatrix},$$

where $\tilde{E}_1^1$, $\tilde{G}_2$ are smooth functions defined on $M^* \cap U_1$ with values in $\mathbb{R}^{r^* \times n^*}$ and $\mathbb{R}^{(m-m^*)\times m}$, respectively, and $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$ and $\tilde{G}_1$ are matrix-valued functions of appropriate sizes. Since $\tilde{G}_2(z_1)$ is of full row rank, we can always assume $\begin{bmatrix} \tilde{G}_1(z_1) \\ \tilde{G}_2(z_1) \end{bmatrix} = \begin{bmatrix} \tilde{G}_1^1(z_1) & \tilde{G}_1^2(z_1) \\ \tilde{G}_2^1(z_1) & \tilde{G}_2^2(z_1) \end{bmatrix}$ such that $\tilde{G}_2^2 : M^* \cap U_1 \to GL(m-m^*, \mathbb{R})$ (if not, we permute the components of u such that $\tilde{G}_2^2(z_1)$ is invertible), where $\tilde{G}_1^1$, $\tilde{G}_1^2$ and $\tilde{G}_2^1$ are of appropriate sizes. Thus, via Q and the following feedback transformation (defined on M^*)

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a(z_1) + b(z_1)u = \begin{bmatrix} 0 \\ \tilde{F}_2(z_1) \end{bmatrix} + \begin{bmatrix} I_{m^*} & 0 \\ \tilde{G}_2^1(z_1) & \tilde{G}_2^2(z_1) \end{bmatrix} u,$$

the DACS (5) is transformed into

$$\begin{bmatrix} \bar{E}_1^1(z_1) \\ 0 \\ 0 \end{bmatrix} \dot{z}_1 = \begin{bmatrix} \bar{F}_1(z_1) \\ 0 \\ \bar{F}_3(z_1) \end{bmatrix} + \begin{bmatrix} \bar{G}_1^1(z_1) & \bar{G}_1^2(z_1) \\ 0 & I_{m-m^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{6}$$

where $\bar{E}_1^1 = \tilde{E}_1^1$, $\bar{F}_3 = \tilde{F}_3$, $\bar{F}_1 = \tilde{F}_1 - \tilde{G}_1^2(\tilde{G}_2^2)^{-1}\tilde{F}_2$, $\bar{G}_1^1 = \tilde{G}_1^1 - \tilde{G}_1^2(\tilde{G}_2^2)^{-1}\tilde{G}_2^1$ and $\bar{G}_1^2 = \tilde{G}_1^2(\tilde{G}_2^2)^{-1}$.

Definition 2.5 (restriction)**.** The local M^*-restriction of Ξ^u, denoted by $\Xi^u|_{M^*}$, is given by

$$\Xi^u|_{M^*} = \Xi^{u^*} : E^*(z^*)\dot{z}^* = F^*(z^*) + G^*(z^*)u^*. \tag{7}$$

where $z^* = z_1$, $u^* = u_1$, $E^* = \bar{E}_1^1 : M^* \to \mathbb{R}^{r^* \times n^*}$, $F^* = \bar{F}_1 : M^* \to \mathbb{R}^{r^*}$ and $G^* = \bar{G}_1^1 : M^* \to \mathbb{R}^{r^* \times m^*}$ come from (6), and where the map E^* is of full row rank r^*.

Remark 2.6. The restriction $\Xi^u|_{M^*}$ is a DACS of the form (1) with associated dimensions r^*, n^*, m^*, i.e., $\Xi^u|_{M^*} = \Xi^{u^*}_{r^*,n^*,m^*}$. It is important to know that Ξ^u and $\Xi^u|_{M^*}$ has isomorphic solutions (see Theorem 4.4(i) in [12]). More specifically, a curve $(x(\cdot), u(\cdot))$ is a solution of Ξ^u passing through a point $x_0 \in X$ if and only if $(z^*(\cdot), u^*(\cdot))$ is a solution of $\Xi^u|_{M^*}$ passing through $z_0^* \in M^*$, where $(z^*(\cdot), 0) = \phi(x(\cdot))$, $(z_0^*, 0) = \phi(x_0)$ and $(u^*(\cdot), 0) = a(z^*(\cdot)) + b(z^*(\cdot))u(\cdot)$.

Now we define the external and the internal feedback equivalences for nonlinear DACSs and compare them by discussing their relations with solutions of DACSs.

Definition 2.7 (external feedback equivalence)**.** Two DACSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively, are called external feedback equivalent, shortly ex-fb-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and smooth functions $Q : X \to GL(l, \mathbb{R})$, $\alpha : X \to \mathbb{R}^m$, $\beta : X \to GL(m, \mathbb{R})$ such that

$$\begin{aligned} \tilde{E}(\psi(x)) &= Q(x)E(x)\left(\tfrac{\partial \psi(x)}{\partial x}\right)^{-1}, \\ \tilde{F}(\psi(x)) &= Q(x)\left(F(x) + G(x)\alpha(x)\right), \\ \tilde{G}(\psi(x)) &= Q(x)G(x)\beta(x). \end{aligned} \tag{8}$$

The ex-fb-equivalence of two DACSs is denoted by $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of a point x_p and $\tilde{U}$ of a point $\tilde{x}_p = \psi(x_p)$, and $Q(x)$, $\alpha(x)$, $\beta(x)$ are defined on U, we will talk about local ex-fb-equivalence.

Definition 2.8 (internal feedback equivalence)**.** Consider two DACSs $\Xi^u = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively. Fix two admissible points $x_a \in X$ and $\tilde{x}_a \in \tilde{X}$. Assume that

(A1) M^* and $\tilde{M}^*$ are locally maximal controlled invariant submanifolds of Ξ^u around x_a and of $\tilde{\Xi}^{\tilde{u}}$ around $\tilde{x}_a$, respectively.

(A2) M^* and $\tilde{M}^*$ satisfy the constant rank condition **(CR)** around x_a and $\tilde{x}_a$, respectively.

Then, Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are called internally feedback equivalent, shortly in-fb-equivalent, if their restrictions $\Xi^u|_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ are ex-fb-equivalent. We will denote the in-fb-equivalence of two DACSs by $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$.

Remark 2.9. The dimensions of two in-fb-equivalent DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are not necessarily the same. However, since $\Xi^u|_{M^*}$ (with dimensions r^*, n^*, m^*) and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ (with dimensions $\tilde{r}^*, \tilde{n}^*, \tilde{m}^*$) are required to be external feedback equivalent, their dimensions have to be the same, i.e., $r^* = \tilde{r}^*$, $n^* = \tilde{n}^*$ and $m^* = \tilde{m}^*$.

Both the ex-fb-equivalence and the in-fb-equivalence preserve solutions of DACSs. Indeed, for two ex-fb-equivalent DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$, the diffeomorphism $\tilde{x} = \psi(x)$ and the feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$ establish a one to one correspondence between solutions (x, u) of Ξ^u and solutions $(\tilde{x}, \tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$, i.e., $\tilde{x} = \psi(x)$ and $u = \alpha(x) + \beta(x)\tilde{u}$. For two in-fb-equivalent DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$, by $\Xi^u|_{M^*} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$, solutions (z^*, u^*) of $\Xi^u|_{M^*}$ and solutions $(\tilde{z}^*, \tilde{u}^*)$ of $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ satisfy $\tilde{z}^* = \psi(z^*)$ and $u^* = \alpha(z^*) + \beta(z^*)\tilde{u}^*$. Then by Remark 2.6, solutions (x, u) of Ξ^u are given by $x = \phi^{-1}(z^*, 0)$ and $u = b^{-1}(z^*)((u^*, 0) - a(z^*))$, and solutions $(\tilde{x}, \tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$ are given by $\tilde{x} = \tilde{\phi}^{-1}(\tilde{z}^*, 0)$ and $\tilde{u} = \tilde{b}^{-1}(\tilde{z}^*)((\tilde{u}^*, 0) - a(\tilde{z}^*))$, where ϕ and $\tilde{\phi}$ are local diffeomorphisms defined on X and $\tilde{X}$, respectively, and a, b and $\tilde{a}, \tilde{b}$ define feedback transformations on M^* and $\tilde{M}^*$, respectively. It is seen that solutions (x, u) are also in a one-to-one correspondence with solutions $(\tilde{x}, \tilde{u})$ if $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$.

Conversely, if solutions of two DACSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are in a one-to-one correspondence via a diffeomorphism and a feedback transformation, then the two DACSs are in-fb-equivalent, however, they are *not* necessarily ex-fb-equivalence. The reason is that solutions of DACSs exist on maximal controlled invariant submanifolds only, by assuming two DACSs have corresponding solutions, we only have the information that the two restrictions $\Xi^u|_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ can be transformed into each other via a Q-transformation and a feedback transformation defined on M^*, and a diffeomorphism between M^* and $\tilde{M}^*$, we do not know, however, if those transformations can be extended outside the submanifolds M^* and $\tilde{M}^*$.

Example 2.10. Consider two DACSs $\Xi^u_{3,3,1} = (E, F, G)$ defined on $X = \mathbb{R}^3$ and $\tilde{\Xi}^{\tilde{u}}_{3,3,1} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on $\tilde{X} = \mathbb{R}^3$, where

$$E(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ F(x) = \begin{bmatrix} x_1^2 \\ e^{x_1} x_2 \\ x_3 \end{bmatrix}, \ G(x) = \begin{bmatrix} e^{x_2} \\ 0 \\ 0 \end{bmatrix}, \ \tilde{E}(\tilde{x}) = \begin{bmatrix} 1 & x_2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \tilde{F}(\tilde{x}) = \begin{bmatrix} \tilde{x}_2 \\ e^{\tilde{x}_1} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}, \ \tilde{G}(\tilde{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

It is seen that $M^* = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_2 = x_3 = 0\}$ and $\tilde{M}^* = \{(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) \in \mathbb{R}^3 \,|\, \tilde{x}_2 = \tilde{x}_3 = 0\}$. The restrictions $\Xi^u|_{M^*} : \dot{x}_1 = x_1^2 + u$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*} : \dot{\tilde{x}}_1 = \tilde{u}$ are ex-fb-equivalent via $Q(x_1) = 1$, $\tilde{x}_1 = \psi(x_1) = x_1$ and $\tilde{u} = x_1^2 + u$. Thus we have $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. It is clear that solutions $((x_1,0),u)$ of Ξ^u and solutions $((\tilde{x}_1,0),\tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$ have a one-to-one correspondence. However, the two DACSs are *not* ex-fb-equivalent since $\operatorname{rank} E(x) \neq \operatorname{rank} \tilde{E}(\tilde{x})$ (the E-matrices of two ex-fb-equivalent DACSs should have the same rank).

Both the external and the internal feedback equivalences play important roles for DACSs. The in-fb-equivalence is convenient when we are only interested in solutions starting from an admissible point. The ex-fb-equivalence is useful when the initial point $x_0 \notin M^*$, i.e., x_0 is not admissible, then there are no solutions passing through x_0 but there may still exist a jump from the inadmissible point x_0 to an admissible point on M^*, see our recent publication [11], where we use external equivalences to study jump solutions of nonlinear DAEs.

3. Explicitation of nonlinear differential-algebraic control systems

We have proposed the notion of explicitation (with driving variables) for linear DACS in [9] (or see Chapter 3 of [7]), we now extend this notion to nonlinear DACSs.

Definition 3.1 (explicitation with driving variables)**.** Given a DACS $\Xi^u_{l,n,m} = (E,F,G)$, fix a point $x_p \in X$. Assume that $\operatorname{rank} E(x) = const. = r$ around x_p. Then locally there exists $Q : X \to GL(l,\mathbb{R})$ such that E_1 below is of full row rank r:

$$Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}, \quad Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \quad Q(x)G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix}.$$

Define locally the maps $f : X \to \mathbb{R}^n$, $g^u : X \to \mathbb{R}^{n\times m}$, $g^v : X \to \mathbb{R}^{n\times s}$, $h : X \to \mathbb{R}^p$, $l^u : X \to \mathbb{R}^{p\times m}$, where $s = n - r$ and $p = l - r$, such that

$$f(x) = E_1^\dagger(x)F_1(x), \quad g^u(x) = E_1^\dagger(x)G_1(x), \quad \operatorname{Im} g^v(x) = \ker E_1(x), \quad h(x) = F_2(x), \quad l^u(x) = G_2(x),$$

where $E_1^\dagger$ is a right inverse of E_1. By a (Q,v)-explicitation, we will call any explicit control system

$$\Sigma^{uv} : \begin{cases} \dot{x} = f(x) + g^u(x)u + g^v(x)v, \\ y = h(x) + l^u(x)u, \end{cases} \tag{9}$$

where $v \in \mathbb{R}^{s\times n}$ is called *the vector of driving variables.* System (9) is denoted by $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ or, simply, Σ^{uv}.

Apparently, in the above definition, the choices of the invertible map Q, the right inverse $E_1^\dagger$ and the map g^v satisfying $\operatorname{Im} g^v = \ker E_1 = \ker E$, are not unique. The following proposition shows that the (Q,v)-explicitation of a given DACS Ξ^u is an explicit control system defined up to a feedback transformation, an output multiplication and a generalized output injection, i.e., a class of control systems. Throughout the class of all (Q,v)-explicitations of Ξ^u will be called the explicitation class. For a particular explicit control system Σ^{uv} belonging to the explicitation class $\mathbf{Expl}(\Xi^u)$ of Ξ^u, we will write $\Sigma^{uv}\in\mathbf{Expl}(\Xi^u)$.

Proposition 3.2. *Assume that an explicit control system $\Sigma^{uv}_{n,m,s,p}=(f,g^u,g^v,h,l^u)$ is a (Q,v)-explicitation of a DACS $\Xi^u=(E,F,G)$ corresponding to the choice of invertible matrix $Q(x)$, right inverse $E_1^{\dagger}(x)$ and matrix $g^v(x)$. Then an explicit control system $\tilde{\Sigma}^{u,\tilde{v}}_{n,m,p}=(\tilde{f},\tilde{g}^u,\tilde{g}^{\tilde{v}},\tilde{h},\tilde{l}^u)$ is a $(\tilde{Q},\tilde{v})$-explicitation of Ξ^u corresponding to the choice of invertible matrix $\tilde{Q}(x)$, right inverse $\tilde{E}_1^{\dagger}(x)$ and matrix $\tilde{g}^{\tilde{v}}(x)$ if and only if Σ^{uv} and $\tilde{\Sigma}^{u,\tilde{v}}$ are equivalent via a v-feedback transformation of the form $v=\alpha^v(x)+\lambda(x)u+\beta^v(x)\tilde{v}$, a generalized output injection $\gamma(x)y=\gamma(x)(h(x)+l^u(x)u)$ and an output multiplication $\tilde{y}=\eta(x)y$, which map*

$$\begin{aligned} f\mapsto\tilde{f}=f+\gamma h+g^v\alpha^v,\qquad g^u\mapsto\tilde{g}^u=g^u+\gamma l^u+g^v\lambda,\\ g^v\mapsto\tilde{g}^{\tilde{v}}=g^v\beta^v,\qquad h\mapsto\tilde{h}=\eta h,\qquad l^u\mapsto\tilde{l}^u=\eta l^u. \end{aligned}\tag{10}$$

where $\alpha^v(x)$, $\beta^v(x)$, $\gamma(x)$, $\lambda(x)$, $\eta(x)$ are smooth matrix-valued functions, and $\beta^v(x)$ and $\eta(x)$ are invertible.

We omit the proof of Proposition 3.2 since it follows the same line as Proposition 2.3 of [9]. Now we will define an equivalence relation for two explicit control systems of the form (9).

Definition 3.3 (system feedback equivalence)**.** Consider two systems $\Sigma^{uv}_{n,m,s,p}=(f,g^u,g^v,h,l^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}_{n,m,s,p}=(\tilde{f},\tilde{g}^{\tilde{u}},\tilde{g}^{\tilde{v}},\tilde{h},\tilde{l}^{\tilde{u}})$ defined on X and $\tilde{X}$, respectively. Then Σ^{uv} and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$ are called system feedback equivalence, shortly sys-fb-equivalent, if there exist a diffeomorphism $\psi:X\to\tilde{X}$, smooth functions $\alpha^u(x)$, $\alpha^v(x)$, $\lambda(x)$ and $\gamma(x)$ with values in $\mathbb{R}^m$, $\mathbb{R}^s$, $\mathbb{R}^{s\times m}$ and $\mathbb{R}^{n\times p}$, respectively, and invertible smooth matrix-valued functions $\beta^u(x)$, $\beta^v(x)$ and $\eta(x)$ with values in $GL(m,\mathbb{R})$, $GL(s,\mathbb{R})$ and $GL(p,\mathbb{R})$, respectively, such that

$$\begin{bmatrix}\tilde{f}\circ\psi & \tilde{g}^{\tilde{u}}\circ\psi & \tilde{g}^{\tilde{v}}\circ\psi\\ \tilde{h}\circ\psi & \tilde{l}^{\tilde{u}}\circ\psi & 0\end{bmatrix}=\begin{bmatrix}\frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial x}\gamma\eta\\ 0 & \eta\end{bmatrix}\begin{bmatrix}f & g^u & g^v\\ h & l^u & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ \alpha^u & \beta^u & 0\\ \alpha^v+\lambda\alpha^u & \lambda\beta^u & \beta^v\end{bmatrix}.\tag{11}$$

The sys-fb-equivalence of two control systems will be denoted by $\Sigma^{uv}\overset{sys-fb}{\sim}\tilde{\Sigma}^{\tilde{u}\tilde{v}}$. If $\psi:U\to\tilde{U}$ is a local diffeomorphism between neighborhoods U of a point x_p and $\tilde{U}$ of a point $\tilde{x}_p=\psi(x_p)$, and α^u, α^v, λ, γ, β^u, β^v, η are defined on U, we will speak about local sys-fb-equivalence.

The two explicit control systems Σ^{uv} and $\tilde{\Sigma}^{u\tilde{v}}$ of Proposition 3.2 are, by definition, system feedback equivalent with ψ being identity, $\alpha^u=0$ and $\beta^u=I_m$. The following observation is crucial and will play an important role for studying the feedback linearization problems of DACSs in Section 4, which points out that the feedback transformations of the explicitation systems of DACSs have a *triangular form* which are different from those of classical control systems (see e.g., [19, 28]).

Remark 3.4. Observe that, in equation (11), there are two kinds of feedback transformations. Namely,

$$u = \alpha^u(x) + \beta^u(x)\tilde{u} \quad \text{and} \quad v = \alpha^v(x) + \lambda(x)u + \beta^v(x)\tilde{v},$$

which can be written together as a feedback transformation of (u, v) with a (lower) *triangular form*:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha^u \\ \alpha^v \end{bmatrix} + \begin{bmatrix} \beta^u & 0 \\ \lambda & \beta^v \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}. \tag{12}$$

It implies that there are two kinds of inputs in the explicit control systems of the form (9), one input (the driving variable v) is more "powerful" than the other input (the original control variable u), since when transforming v, we can use both u and x, but when transforming u, we are *not* allowed to use v. Another difference between u and v is that the input u is injected into the output y via $l^u u$, but the driving variable v is not directly injected into the output y.

The following theorem connects the ex-fb-equivalence of two DACSs with the sys-fb-equivalence of two explicit control systems (explicitations).

Theorem 3.5. *Consider two DACSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively. Assume that $\operatorname{rank} E(x) = const. = r$ in a neighborhood U of a point $x_p \in X$ and $\operatorname{rank} \tilde{E}(\tilde{x}) = r$ in a neighborhood $\tilde{U}$ of a point $\tilde{x}_p \in \tilde{X}$. Then, given any explicit control systems $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u) \in \mathbf{Expl}(\Xi^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}) \in \mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$, we have locally $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ if and only if $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$.*

Proof. By the assumptions that $\operatorname{rank} E(x)$ and $\operatorname{rank} \tilde{E}(x)$ are constant and equal to r around x_p and $\tilde{x}_p$, respectively, there exist invertible matrix-valued functions $Q : U \to GL(l, \mathbb{R})$ and $\tilde{Q} : \tilde{U} \to GL(l, \mathbb{R})$, defined on neighborhoods U of x_p and $\tilde{U}$ of $\tilde{x}_p$, respectively, such that $E'(x) = Q(x)E(x) = \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right]$ and $\tilde{E}'(\tilde{x}) = \tilde{Q}(\tilde{x})\tilde{E}(\tilde{x}) = \left[\begin{smallmatrix} \tilde{E}_1(\tilde{x}) \\ 0 \end{smallmatrix}\right]$, where $E_1 : U \to R^{r\times n}$ and $\tilde{E}_1 : \tilde{U} \to R^{r\times n}$ are of full row rank. We have $\Xi^u \overset{ex-fb}{\sim} \Xi^{u\prime} = (E', F', G')$ and $\tilde{\Xi}^{\tilde{u}} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}\prime} = (\tilde{E}', \tilde{F}', \tilde{G}')$ via $Q(x)$ and $\tilde{Q}(\tilde{x})$, respectively, where

$$\begin{aligned} F'(x) &= Q(x)F(x) = \left[\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right], \quad G'(x) = Q(x)G(x) = \left[\begin{smallmatrix} G_1(x) \\ G_2(x) \end{smallmatrix}\right], \\ \tilde{F}'(\tilde{x}) &= \tilde{Q}(\tilde{x})\tilde{F}(\tilde{x}) = \left[\begin{smallmatrix} \tilde{F}_1(\tilde{x}) \\ \tilde{F}_2(\tilde{x}) \end{smallmatrix}\right], \quad \tilde{G}'(\tilde{x}) = \tilde{Q}(\tilde{x})\tilde{G}(\tilde{x}) = \left[\begin{smallmatrix} \tilde{G}_1(\tilde{x}) \\ \tilde{G}_2(\tilde{x}) \end{smallmatrix}\right]. \end{aligned}$$

In this proof, without loss of generality, we will assume that $\Xi^u = \Xi^{u\prime}$ and $\tilde{\Xi}^{\tilde{u}} = \tilde{\Xi}^{\tilde{u}\prime}$, since $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ if and only if $\Xi^{u\prime} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}\prime}$.

Moreover, choose the maps f, g^u, g^v, h, l^u and $\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}$ such that

$$\begin{aligned} &f(x) = E_1^\dagger(x)F_1(x), \quad g^u(x) = E_1^\dagger(x)G_1(x), \quad h(x) = F_2(x), \quad l^u(x) = G_2(x), \quad \operatorname{Im} g^v(x) = \ker E_1(x) \\ &\tilde{f}(\tilde{x}) = \tilde{E}_1^\dagger(\tilde{x})\tilde{F}_1(\tilde{x}), \quad \tilde{g}^{\tilde{u}}(\tilde{x}) = \tilde{E}_1^\dagger(\tilde{x})\tilde{G}(\tilde{x}), \quad \tilde{h}(\tilde{x}) = \tilde{F}_2(\tilde{x}), \quad \tilde{l}^{\tilde{u}}(\tilde{x}) = \tilde{G}_2(\tilde{x}), \quad \operatorname{Im} \tilde{g}^{\tilde{v}}(\tilde{x}) = \ker \tilde{E}_1(\tilde{x}) \end{aligned} \tag{13}$$

where $E_1^\dagger(x)$ and $\tilde{E}_1^\dagger(\tilde{x})$ are right inverses of $E_1(x)$ and $\tilde{E}_1(\tilde{x})$, respectively. Then by Definition 2.7, $\Sigma^{uv}=(f,g^u,g^v,h,l^u)\in \mathbf{Expl}(\Xi^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}=(\tilde{f},\tilde{g}^{\tilde{u}},\tilde{g}^{\tilde{v}},\tilde{h},\tilde{l}^{\tilde{u}})\in \mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$. It is seen from Proposition 3.2 that any control system in $\mathbf{Expl}(\Xi^u)$ is sys-fb-equivalent to Σ^{uv} and that any control system in $\mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$ is sys-fb-equivalent to $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$. Without loss of generality, in the remaining part of the proof, we use Σ^{uv} and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$ with system matrices given by (13) to represent two explicit control systems in $\mathbf{Expl}(\Xi^u)$ and $\mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$, respectively. Throughout the proof below, we may drop the argument x for the functions $E(x)$, $F(x)$, $G(x)$, ..., for ease of notation.

If. Suppose that locally $\Sigma^{uv}\overset{sys-fb}{\sim}\tilde{\Sigma}^{\tilde{u}\tilde{v}}$. Then there exist a local diffeomorphism $\tilde{x}=\psi(x)$ and matrix-valued functions α^u, α^v, λ, γ, β^u, β^v, η defined on a neighborhood U of x_p such that the system matrices satisfy relations (11) of Definition 3.3.

First, consider $\tilde{g}^{\tilde{v}}\circ\psi=\frac{\partial\psi}{\partial x}g^v\beta^v$. By $\operatorname{Im} g^v=\ker E_1$, $\operatorname{Im}\tilde{g}^{\tilde{v}}=\ker\tilde{E}_1$, we have $\ker\tilde{E}_1\circ\psi=\frac{\partial\psi}{\partial x}\ker E_1$. Thus there exists $Q_1:U\to GL(r,\mathbb{R})$ such that

$$\tilde{E}_1\circ\psi=Q_1E_1\left(\frac{\partial\psi}{\partial x}\right)^{-1}. \tag{14}$$

Then, by (11), the following relation holds:

$$\begin{bmatrix}\tilde{f}\circ\psi & \tilde{g}^{\tilde{u}}\circ\psi\\ \tilde{h}\circ\psi & \tilde{l}^{\tilde{u}}\circ\psi\end{bmatrix}=\begin{bmatrix}\frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial x}\gamma\eta\\ 0 & \eta\end{bmatrix}\begin{bmatrix}f & g^u & g^v\\ h & l^u & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ \alpha^u & \beta^u\\ \alpha^v+\lambda\alpha^u & \lambda\beta^u\end{bmatrix}.$$

Substituting (13) into the above equation, we get

$$\begin{bmatrix}\tilde{E}_1^\dagger\circ\psi\cdot\tilde{F}_1\circ\psi & \tilde{E}_1^\dagger\circ\psi\cdot\tilde{G}_1\circ\psi\\ \tilde{F}_2\circ\psi & \tilde{G}_2\circ\psi\end{bmatrix}\begin{bmatrix}\frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial x}\gamma\eta\\ 0 & \eta\end{bmatrix}\begin{bmatrix}E_1^\dagger F_1 & E_1^\dagger G_1 & g^v\\ F_2 & G_2 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ \alpha^u & \beta^u\\ \alpha^v+\lambda\alpha^u & \lambda\beta^u\end{bmatrix}.$$

Premultiply the above equation by $\begin{bmatrix}\tilde{E}_1\circ\psi & 0\\ 0 & I_p\end{bmatrix}=\begin{bmatrix}Q_1E_1\left(\frac{\partial\psi}{\partial x}\right)^{-1} & 0\\ 0 & I_p\end{bmatrix}$ to get

$$\begin{bmatrix}\tilde{F}_1\circ\psi & \tilde{G}_1\circ\psi\\ \tilde{F}_2\circ\psi & \tilde{G}_2\circ\psi\end{bmatrix}=\begin{bmatrix}Q_1 & Q_1E_1\gamma\eta\\ 0 & \eta\end{bmatrix}\begin{bmatrix}F_1 & G_1\\ F_2 & G_2\end{bmatrix}\begin{bmatrix}I_n & 0\\ \alpha^u & \beta^u\end{bmatrix}. \tag{15}$$

Now from equations (14), (15) and Definition 2.7, it can be seen that $\Xi^u\overset{ex-fb}{\sim}\tilde{\Xi}^{\tilde{u}}$ via $\tilde{x}=\psi(x)$, $Q=\begin{bmatrix}Q_1 & Q_1E_1\gamma\eta\\ 0 & \eta\end{bmatrix}$, α^u and β^u.

Only if. Suppose that $\Xi^u\overset{ex-fb}{\sim}\tilde{\Xi}^{\tilde{u}}$ (in a neighborhood U of x_p). Assume that Ξ^u and $\tilde{\Xi}^u$ are ex-fb-equivalent via an invertible matrix-valued function $Q=\begin{bmatrix}Q_1 & Q_2\\ Q_3 & Q_4\end{bmatrix}$, $\tilde{x}=\psi(x)$, α^u, β^u, where $Q_1:U\to\mathbb{R}^{r\times r}$ and Q_2, Q_3, Q_4 are matrix-valued functions of appropriate sizes.

Then by $QE=\tilde{E}\circ\psi\frac{\partial\psi}{\partial x}\Rightarrow\begin{bmatrix}Q_1 & Q_2\\ Q_3 & Q_4\end{bmatrix}\begin{bmatrix}E_1\\ 0\end{bmatrix}=\begin{bmatrix}\tilde{E}_1\circ\psi\\ 0\end{bmatrix}\frac{\partial\psi}{\partial x}$, we can deduce that

$$\tilde{E}_1\circ\psi=Q_1E_1\left(\frac{\partial\psi}{\partial x}\right)^{-1}. \tag{16}$$

Moreover, we have $Q_3=0$ and Q_1 is invertible (since both E_1 and $\tilde{E}_1$ are of full row rank), which implies that Q_4 is invertible as well (since Q is invertible). Subsequently, by

$$\tilde{F}\circ\psi=Q(F+G\alpha^u)\Rightarrow\begin{bmatrix}\tilde{F}_1\circ\psi\\ \tilde{F}_2\circ\psi\end{bmatrix}=\begin{bmatrix}Q_1 & Q_2\\ 0 & Q_4\end{bmatrix}\left(\begin{bmatrix}F_1\\ F_2\end{bmatrix}+\begin{bmatrix}G_1\\ G_2\end{bmatrix}\alpha^u\right),$$

we have

$$\tilde{F}_1 \circ \psi = Q_1(F_1 + G_1\alpha^u) + Q_2(F_2 + G_2\alpha^u) \tag{17}$$

and

$$\tilde{F}_2 \circ \psi = Q_4(F_2 + G_2\alpha^u). \tag{18}$$

Moreover, by $\tilde{G} \circ \psi = QG\beta^u \Rightarrow \left[\begin{smallmatrix} \tilde{G}_1\circ\psi \\ \tilde{G}_2\circ\psi \end{smallmatrix}\right] = \left[\begin{smallmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{smallmatrix}\right]\left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right]\beta^u$, we have

$$\tilde{G}_1 \circ \psi = Q_1G_1\beta^u + Q_2G_2\beta^u \tag{19}$$

and

$$\tilde{G}_2 \circ \psi = Q_4G_2\beta^u. \tag{20}$$

Recall the system matrices given in equation (13). First, from $\operatorname{Im} g^v = \ker E_1$, $\operatorname{Im} \tilde{g}^{\tilde{v}} \circ \psi = \ker \tilde{E}_1(\tilde{x})$, and equation (16), it is seen that there exists $\beta^v : U \to GL(s, \mathbb{R})$ such that

$$\tilde{g}^{\tilde{v}} \circ \psi = \frac{\partial \psi}{\partial x} g^v \beta^v. \tag{21}$$

Secondly, by equations (16) and (17), we have

$$\begin{aligned} \tilde{f} \circ \psi = \tilde{E}_1^\dagger \circ \psi \tilde{F}_1 \circ \psi &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \left[\begin{smallmatrix} Q_1 & Q_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} F_1 + G_1\alpha^u \\ F_2 + G_2\alpha^u \end{smallmatrix}\right] = \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \left[\begin{smallmatrix} Q_1 & Q_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} F_1 + G_1\alpha^u + E_1 g^v(\lambda\alpha^u + \alpha^v) \\ F_2 + G_2\alpha^u \end{smallmatrix}\right] \\ &= \frac{\partial \psi}{\partial x} \Big(f + g^u\alpha^u + g^v\left(\lambda\alpha^u + \alpha^v\right) + E_1^\dagger Q_1^{-1} Q_2 \left(h + l^u\alpha^u\right)\Big), \end{aligned} \tag{22}$$

where $\alpha^v(x)$ and $\lambda(x)$ are matrix-valued functions of appropriate sizes. Thirdly, by equation (19), we have

$$\begin{aligned} \tilde{g}^{\tilde{u}} \circ \psi = \tilde{E}_1^\dagger \circ \psi \tilde{G}_1 \circ \psi &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \left[\begin{smallmatrix} Q_1 & Q_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} G_1\beta^u \\ G_2\beta^u \end{smallmatrix}\right] = \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \left[\begin{smallmatrix} Q_1 & Q_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} G_1\beta^u + E_1 g^v\lambda \\ G_2\beta^u \end{smallmatrix}\right] \\ &= \frac{\partial \psi}{\partial x} \Big(g^u\beta^u + g^v\lambda + E_1^\dagger Q_1^{-1} Q_2 l^u \beta^u \Big). \end{aligned} \tag{23}$$

Note that we use the equations $E_1 g^v\left(\lambda\alpha^u + \alpha^v\right) = 0$ and $E_1 g^v \lambda = 0$ to deduce (22) and (23). At last, by equations (18) and (20) we have

$$\tilde{h} \circ \psi = \tilde{F}_2 \circ \psi = Q_4(F_2 + G_2\alpha^u) = Q_4\left(h + l^u\alpha^u\right) \tag{24}$$

and

$$\tilde{l}^{\tilde{u}} \circ \psi = \tilde{G}_2 \circ \psi = Q_4G_2\beta^u = Q_4 l^u \beta^u. \tag{25}$$

Finally, it can be seen from (22), (23), (24) and (25), that $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$ via $\tilde{x} = \psi(x)$, α^v, β^v, α^u, β^u, λ, $\gamma = E_1^\dagger Q_1^{-1} Q_2$ and $\eta = Q_4$. $\square$

4. External and internal feedback linearization

In this section, we discuss the problem that when a nonlinear DACS of the form (1) is external or internal feedback equivalent to a linear DACS of the form (2) with complete controllability. First, we review some definitions and criteria for the complete controllability of linear DACSs. The augmented Wong sequences (see e.g., [4, 9, 26]) of a linear DACS $\Delta^u_{l,n,m} = (E, H, L)$, given by (2), are

$$\mathscr{V}_0 := \mathbb{R}^n, \quad \mathscr{V}_{i+1} := H^{-1}(E\mathscr{V}_i + \operatorname{Im} L), \quad i = 0, 1, \ldots; \tag{26}$$

$$\mathscr{W}_0 := 0, \quad \mathscr{W}_{i+1} := E^{-1}(H\mathscr{W}_i + \operatorname{Im} L), \quad i = 0, 1, \ldots. \tag{27}$$

Additionally, recall the following sequence of subspaces (see e.g. [26]):

$$\hat{\mathscr{W}}_1 := \ker E, \quad \hat{\mathscr{W}}_{i+1} := E^{-1}(H\hat{\mathscr{W}}_i + \operatorname{Im} L), \quad i = 1, 2, \ldots. \tag{28}$$

For simplicity of notation, we denote

$$\begin{aligned}
&K_\beta = \operatorname{diag}\{K_{\beta_1}, \ldots, K_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|}, && L_\beta = \operatorname{diag}\{L_{\beta_1}, \ldots, L_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|},\\
&\mathcal{E}_\beta = \operatorname{diag}\{e_{\beta_1}, \ldots, e_{\beta_k}\} \in \mathbb{R}^{|\beta|\times k} && N_\beta = \operatorname{diag}\{N_{\beta_1}, \ldots, N_{\beta_k}\} \in \mathbb{R}^{|\beta|\times|\beta|},\\
&K_{\beta_i} = \left[\begin{smallmatrix} 0 & I_{\beta_i-1}\end{smallmatrix}\right] \in \mathbb{R}^{(\beta_i-1)\times\beta_i}, && L_{\beta_i} = \left[\begin{smallmatrix} I_{\beta_i-1} & 0\end{smallmatrix}\right] \in \mathbb{R}^{(\beta_i-1)\times\beta_i},\\
&e_{\beta_i} = \left[\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right] \in \mathbb{R}^{\beta_i\times 1}, && N_{\beta_i} = \left[\begin{smallmatrix} 0 & 0 \\ I_{\beta_i-1} & 0\end{smallmatrix}\right] \in \mathbb{R}^{\beta_i\times\beta_i},
\end{aligned}$$

where β is a multi-index $\beta = (\beta_1, \ldots, \beta_k)$, and where $|\beta| = \sum\limits_{i=1}^{k} \beta_i$. Definition 2.7 applied to linear systems says that two linear DACSs $\Delta^u_{l,n,m} = (E, H, L)$ and $\tilde{\Delta}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{H}, \tilde{L})$ are ex-fb-equivalent if there exist constant invertible matrices Q, P, S and a matrix R such that $\tilde{E} = QEP^{-1}$, $\tilde{H} = Q(H + LR)P^{-1}$, $\tilde{L} = QLS$.

Definition 4.1 (complete controllability in [4])**.** A linear DACS $\Delta^u_{l,n,m} = (E, H, L)$ is completely controllable if for any $x_0, x_1 \in \mathbb{R}^n$, there exist a solution (x, u) of Δ^u and $t \in \mathbb{R}^+$ such that $x(0) = x_0$ and $x(t) = x_1$.

Lemma 4.2. [4] *For a linear DACS $\Delta^u_{l,n,m} = (E, H, L)$, the following statements are equivalent:*

(i) Δ^u is completely controllable.

(ii) $\operatorname{Im} E + \operatorname{Im} H + \operatorname{Im} L = \operatorname{Im} E + \operatorname{Im} L$ *and* $\operatorname{Im}_{\mathbb{C}} E + \operatorname{Im}_{\mathbb{C}} H + \operatorname{Im}_{\mathbb{C}} L = \operatorname{Im}_{\mathbb{C}}(\lambda E - H) + \operatorname{Im}_{\mathbb{C}} L$, $\forall \lambda \in \mathbb{C}$.

(iii) $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$, where $\mathscr{V}^*$ and $\mathscr{W}^*$ are the limits of the augmented Wong sequences (26) and (27), respectively;

(iv) Δ^u is ex-fb-equivalent (under linear transformations) to

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho & 0 \\ 0 & 0 \\ 0 & I_{m-m^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $\rho = (\rho_1, \ldots, \rho_{m^*})$ and $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_{s^*})$ are multi-indices, and $s^* = n - \operatorname{rank} E$.

Now we will study the feedback linearizability of nonlinear DACSs defined as follows.

Definition 4.3. Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix an admissible point $x_a \in X$. Then Ξ^u is called locally externally (resp. internally) feedback linearizable around x_a if Ξ^u is locally ex-fb-equivalent (resp. in-fb-equivalent) to a linear DACS with complete controllability around x_a.

We consider a nonlinear explicit control system $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$, given by (9). If Σ^{uv} has no outputs, we denote it by $\Sigma^{uv}_{n,m,s} = (f, g^u, g^v)$. Then for $\Sigma^{uv}_{n,m,s} = (f, g^u, g^v)$, define the following two sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, called the *linearizability distributions* of Σ^{uv},

$$\begin{cases} \mathcal{D}_0 & := \{0\}, \\ \mathcal{D}_1 & := \operatorname{span}\{g^u_1, \ldots, g^u_m, g^v_1, \ldots, g^v_s\}, \\ \mathcal{D}_{i+1} & := \mathcal{D}_i + [f, \mathcal{D}_i], \quad i = 1, 2, \ldots, \end{cases} \qquad \begin{cases} & \\ \hat{\mathcal{D}}_1 & := \operatorname{span}\{g^v_1, \ldots, g^v_s\}, \\ \hat{\mathcal{D}}_{i+1} & := \mathcal{D}_i + [f, \hat{\mathcal{D}}_i], \quad i = 1, 2, \ldots. \end{cases} \tag{29}$$

Remark 4.4. (i) The distribution sequences $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy:

$$\mathcal{D}_0 \subsetneq \hat{\mathcal{D}}_1 \subsetneq \mathcal{D}_1 \subsetneq \hat{\mathcal{D}}_2 \subsetneq \mathcal{D}_2 \cdots \subsetneq \hat{\mathcal{D}}_i \subsetneq \mathcal{D}_i \subsetneq \cdots \subsetneq \hat{\mathcal{D}}_{i^*},$$

and either $\hat{\mathcal{D}}_{i^*} = \mathcal{D}_{i^*} = \hat{\mathcal{D}}_{i^*+j} = \mathcal{D}_{i^*+j}$ or $\hat{\mathcal{D}}_{i^*} \subsetneq \mathcal{D}_{i^*} = \hat{\mathcal{D}}_{i^*+j} = \mathcal{D}_{i^*+j}$, where $j \geq 1$ and i^* is the smallest i such that $\mathcal{D}_i = \mathcal{D}_{i+1}$. Note that i^* is not necessarily the smallest i such that $\hat{\mathcal{D}}_i = \hat{\mathcal{D}}_{i+1}$ (as seen in the second case, where $\hat{\mathcal{D}}_{i^*} \subsetneq \hat{\mathcal{D}}_{i^*+1}$). However, $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ always have the same limit.

(ii) For a linear DACS $\Delta^u = (E, H, L)$, denote $\mathscr{W}_i(\Delta^u)$ and $\hat{\mathscr{W}}_i(\Delta^u)$ as the subspaces $\mathscr{W}_i$ of (27) and $\hat{\mathscr{W}}_i$ of Δ^u of (28), respectively. For a linear explicit control system $\Lambda^{uv} = (A, B^u, B^v, C, D^u)$ (of the form (9) but with constant system matrices), denote $\mathcal{W}_i(\Lambda^{uv})$ and $\hat{\mathcal{W}}_i(\Lambda^{uv})$ as the subspaces $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ of Λ^{uv}, respectively, where

$$\mathcal{W}_0 = \{0\}, \quad \mathcal{W}_{i+1} = \left[\begin{smallmatrix} A & B^w \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} \mathcal{W}_i \\ \mathscr{X} \end{smallmatrix}\right] \cap \ker \left[\begin{smallmatrix} C & D^w \end{smallmatrix}\right] \right), \ i = 0, 1, \ldots,$$

$$\hat{\mathcal{W}}_1 = \operatorname{Im} B^v, \quad \hat{\mathcal{W}}_{i+1} = \left[\begin{smallmatrix} A & B^w \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} \hat{\mathcal{W}}_i \\ \mathscr{X} \end{smallmatrix}\right] \cap \ker \left[\begin{smallmatrix} C & D^w \end{smallmatrix}\right] \right), \ i = 1, 2, \ldots,$$

where $w = (u, v)$, $B^w = [B^u, B^v]$ and $D^w = [D^u, 0]$. We have proved in Proposition 2.10 of [9] that if $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$, then

$$\mathscr{W}_i(\Delta^u) = \mathcal{W}_i(\Lambda^{uv}), \ \forall i \geq 0 \quad \text{and} \quad \hat{\mathscr{W}}_i(\Delta^u) = \hat{\mathcal{W}}_i(\Lambda^{uv}), \ \forall i \geq 1.$$

Apparently, $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ are the linear counterparts of $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, respectively, but they are for linear systems with outputs.

Theorem 4.5 (internal feedback linearization)**.** *Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix an admissible point $x_a \in X$. Let M^* be the n^*-dimensional maximal controlled invariant submanifold of Ξ^u around x_a. Assume that the constant rank assumption* (**CR**) *is satisfied for $x \in M^*$ around x_a. Then $\Xi^u|_{M^*}$ is a DACS $\Xi^{u^*}_{r^*,n^*,m^*} = (E^*, F^*, G^*)$ of the form (7) and* $\mathbf{Expl}(\Xi^u|_{M^*})$ *is a class of explicit control systems without outputs. The DACS Ξ^u is locally internally feedback linearizable if and only if for one (and thus any) explicit control system $\Sigma^{u^*v^*} = (f^*, g^{u^*}, g^{v^*}) \in$* $\mathbf{Expl}(\Xi^u|_{M^*})$*, the linearizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ of $\Sigma^{u^*v^*}$ satisfy the following conditions on M^* around x_a :*

(FL1) $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are of constant rank for $1 \leq i \leq n^*$.

(FL2) $\mathcal{D}_{n^*} = \hat{\mathcal{D}}_{n^*} = TM^*$.

(FL3) $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are involutive for $1 \leq i \leq n^* - 1$.

Proof. Since Ξ^u satisfies the condition **(CR)** around x_a, the M^*-restriction $\Xi^u|_{M^*}$ by Definition 2.5 is a DACS $\Xi^u|_{M^*} = \Xi^{u^*}_{r^*,n^*,m^*} = (E^*, F^*, G^*)$ of the form (7) with E^* being of full row rank r^*. It follows by the full row rankness of E^* that the maps $h = F_2$ and $l^{u^*} = G_2$ are absent in the explicitation systems of Ξ^{u^*}, which means that the output $y = h(x) + l^{u^*}(x)u^*$ is absent as well (see Definition 3.1). Thus an explicit control system $\Sigma^{u^*v^*}_{n^*,m^*,s^*} = (f^*, g^{u^*}, g^{v^*}) \in$ **Expl**$(\Xi^u|_{M^*})$ is a control system without outputs of the form

$$\Sigma^{w^*} : \dot{z}^* = f^*(z^*) + g^{u^*}(z^*)u^* + g^{v^*}(z^*)v^*,$$

where $w^* = (u^*, v^*)$, $f^* = (E^*)^\dagger F^*$, $g^{u^*} = (E^*)^\dagger G^*$, $\operatorname{Im} g^{v^*} = \ker E^*$ and $s^* = n^* - r^*$.

Only if. Suppose that Ξ^u is locally internally feedback linearizable, which means that the M^*-restriction $\Xi^u|_{M^*}$, given by (7), is locally ex-fb-equivalent (via $Q(z^*)$, $\tilde{z}^* = \psi(z^*)$ and $u = \alpha(z^*) + \beta(z^*)\tilde{u}^*$ defined on M^*) to a completely controllable linear DACS

$$\Delta^{\tilde{u}^*} : E^*\dot{\tilde{z}}^* = H^*\tilde{z}^* + L^*\tilde{u}^*,$$

where E^*, H^*, L^* are constant matrices of appropriate sizes. A linear explicit control system $\Lambda^{\tilde{w}^*} = (A^*, B^{\tilde{u}^*}, B^{\tilde{v}^*}) \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$, where $\tilde{w}^* = (\tilde{u}^*, \tilde{v}^*)$, is of the form

$$\Lambda^{\tilde{w}^*} : \dot{\tilde{z}}^* = A^*\tilde{z}^* + B^{\tilde{u}^*}\tilde{u}^* + B^{\tilde{v}^*}\tilde{v}^*.$$

Then by Lemma 4.2, the complete controllability of $\Delta^{\tilde{u}^*}$ implies $\hat{\mathscr{W}}_{n^*}(\Delta^{\tilde{u}^*}) = \mathscr{W}_{n^*}(\Delta^{\tilde{u}^*}) = \mathbb{R}^{n^*}$. By Proposition 2.10 of [9] (see also Remark 4.4(ii)), we get $\hat{\mathcal{W}}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{W}_{n^*}(\Lambda^{\tilde{w}^*}) = \hat{\mathscr{W}}_{n^*}(\Delta^{\tilde{u}^*}) = \mathscr{W}_{n^*}(\Delta^{\tilde{u}^*}) = \mathbb{R}^{n^*}$. Since $\Lambda^{\tilde{w}^*}$ is a linear control system without outputs, we have $\hat{\mathcal{D}}_{n^*}(\Lambda^{\tilde{w}^*}) = \hat{\mathcal{W}}_{n^*}(\Lambda^{\tilde{w}^*})$ and $\mathcal{D}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{W}_{n^*}(\Lambda^{\tilde{w}^*})$. Hence, $\hat{\mathcal{D}}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{D}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathbb{R}^{n^*}$. Thus $\Lambda^{\tilde{w}^*}$ satisfies (FL2). Moreover, since $\Lambda^{\tilde{w}^*}$ is a linear control system, it satisfies (FL1) and (FL3) in an obvious way. Notice that the nonlinear system Σ^{w^*} is locally sys-fb-equivalent to $\Lambda^{\tilde{w}^*}$ by Theorem 3.5 as $\Sigma^{w^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$, $\Delta^{\tilde{w}^*} \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$ and $\Xi^u|_{M^*} \overset{ex-fb}{\sim} \Delta^{\tilde{u}^*}$. Since Σ^{w^*} and $\Lambda^{\tilde{w}^*}$ are control systems without outputs, sys-fb-equivalence reduces to feedback equivalence. Thus Σ^{w^*} and $\Lambda^{\tilde{w}^*}$ are locally feedback equivalent (via two kinds of feedback transformations, see Remark 3.4). It is easy to verify by a direct calculation that if $\hat{\mathcal{D}}_i$ and $\mathcal{D}_i$ are involutive, then the two distribution sequences are invariant for the two feedback equivalent control systems Σ^{w^*} and $\Lambda^{\tilde{w}^*}$, i.e., $\psi \circ \hat{\mathcal{D}}_i(\Sigma^{w^*}) = \hat{\mathcal{D}}_i(\Delta^{\tilde{w}^*})$ and $\psi \circ \mathcal{D}_i(\Sigma^{w^*}) = \mathcal{D}_i(\Delta^{\tilde{w}^*})$. So the system Σ^{w^*} being feedback equivalent to $\Lambda^{\tilde{w}^*}$ satisfies conditions (FL1)-(FL3) as well. For any other explicit control system $\hat{\Sigma}^{\hat{w}^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$, we have $\Sigma^{w^*} \overset{sys-fb}{\sim} \hat{\Sigma}^{\hat{w}^*}$ by Proposition 3.2, which means Σ^{w^*} is feedback equivalent (via two kinds of feedback transformations) to $\hat{\Sigma}^{\hat{w}^*}$ as any explicitation system in $\mathbf{Expl}(\Xi^u|_{M^*})$ has no outputs. So any explicitation system $\hat{\Sigma}^{\hat{w}^*}$ also satisfies (FL1)-(FL3) of Theorem 4.5.

If. Suppose that an explicit control system $\Sigma^{u^*v^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$ satisfies (FL1)-(FL3) around x_a. Then we have the following lemma:

Lemma 4.6. *The explicit control system $\Sigma^{w^*} = \Sigma^{u^*v^*}_{n^*,m^*,s^*} = (f^*, g^{u^*}, g^{v^*})$ is locally feedback equivalent, via two kinds of feedback transformations (see Remark 3.4), to the Brunovský canonical form [6] around x_a, which is given by*

$$\Sigma^{\tilde{w}^*}_{Br} = \Sigma^{\tilde{u}^*\tilde{v}^*}_{Br} : \begin{cases} \dot{\xi}_1 = N_\rho^T \xi_1 + \mathcal{E}_\rho \tilde{u}^*, \\ \dot{\xi}_2 = N_{\bar{\rho}}^T \xi_2 + \mathcal{E}_{\bar{\rho}} \tilde{v}^*, \end{cases} \tag{30}$$

where $\rho = (\rho_1, \ldots, \rho_a)$ and $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_b)$ are multi-indices.

The proof of Lemma 4.6 is technical and is put into Appendix. Now we will prove that $\Xi^u|_{M^*}$, given by (7), is locally ex-fb-equivalent to

$$\Delta^{\tilde{u}^*} : \begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho \\ 0 \end{bmatrix} \tilde{u}^*. \tag{31}$$

Notice that by Lemma 4.2, the above linear DACS $\Delta^{\tilde{u}^*}$ is completely controllable. Then it is crucial to observe $\Sigma^{\tilde{w}^*}_{Br} \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$. Since Σ^{w^*} is locally sys-fb-equivalent to $\Sigma^{\tilde{w}^*}_{Br}$ (by Lemma 4.6) and $\Sigma^{w^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$, we have $\Xi^u|_{M^*}$ is locally ex-fb-equivalent to $\Delta^{\tilde{u}^*}$ around x_a by Theorem 3.5. Hence Ξ^u is locally in-fb-equivalent to the complete controllable linear DACS $\Delta^{\tilde{u}^*}$, given by (31), i.e., Ξ^u is internally feedback linearizable by Definition 4.3. □

Theorem 4.7 (external feedback linearization)**.** *Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix an admissible point $x_a \in X$. Then Ξ^u is locally externally feedback linearizable around x_a if and only if there exists a neighborhood $U \subseteq X$ of x_a in which the following conditions are satisfied.*

(EFL1) $\operatorname{rank} E(x)$ *and* $\operatorname{rank}[E(x), G(x)]$ *are constant.*

(EFL2) $F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x)$ *or, equivalently, $M^* = M_0^c = U$.*

(EFL3) For one (and thus any) control system $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|_{M^*})$, which is a system with no outputs on $M^* = U$, the linerizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy (FL1)-(FL3) of Theorem 4.5.

Proof. Only if. Suppose that Ξ^u is locally externally feedback linearizable. By definition, the DACS Ξ^u is locally ex-fb-equivalent, via $z = \psi(x)$, $Q(x)$ and $u = \alpha^u(x) + \beta^u(x)\tilde{u}$, to a linear completely controllable DACS:

$$\Delta^{\tilde{u}} : \tilde{E}\dot{z} = \tilde{H}z + \tilde{L}\tilde{u}. \tag{32}$$

Thus by Definition 2.7, we have

$$Q(x)E(x) = \tilde{E} \cdot \tfrac{\partial \psi(x)}{\partial x}, \quad Q(x)(F(x) + G(x)\alpha^u(x)) = \tilde{H} \cdot \psi(x), \quad Q(x)G(x)\beta^u(x) = \tilde{L}. \tag{33}$$

It is clear that $\Delta^{\tilde{u}}$ satisfies (EFL1). So the system Ξ^u satisfies (EFL1) as well because the ranks of $E(x)$ and $[E(x), G(x)]$ are invariant under ex-fb-equivalence. Moreover, the complete controllability of $\Delta^{\tilde{u}}$ implies $\tilde{H}z \in \operatorname{Im} \tilde{E} + \operatorname{Im} L$ (see Lemma 4.2(ii)). By substituting (33), we get

$$Q(x)(F(x) + G(x)\alpha^u(x)) \in \operatorname{Im} Q(x)E(x) \left(\frac{\partial \psi(x)}{\partial x}\right)^{-1} + \operatorname{Im} Q(x)G(x)\beta^u(x)$$
$$\Rightarrow F(x) + G(x)\alpha^u(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x) \Rightarrow F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x).$$

Thus Ξ^u satisfies (EFL2). Notice that by (EFL2), we have that the locally maximal controlled invariant submanifold M^* around x_a coincides with the neighborhood U. Now consider the restriction $\Delta^{\tilde{u}}|_{M^*} = \Delta^{\tilde{u}}|_U$, which is a linear completely controllable DACS as $\Delta^{\tilde{u}}$. This means that Ξ^u is locally internally feedback linearizable. Thus by Theorem 4.5, Ξ^u satisfies (EFL3) on $M^* = U$.

If. Suppose that in a neighborhood U of x_a, Ξ^u satisfies (EFL1)-(EFL3). Denote $\operatorname{rank} E(x) = r$ and $\operatorname{rank}[E(x), G(x)] = r + \bar{m}^*$ and $m^* = m - \bar{m}^*$. Then, by (EFL1), there exist an invertible $Q(x)$ defined on U and a partition of $u = (u_1, u_2)$ such that

$$Q(x)E(x)\dot{x} = Q(x)F(x) + Q(x)G(x)u \Rightarrow \begin{bmatrix} E^1(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{bmatrix} + \begin{bmatrix} G_1^1(x) & G_1^2(x) \\ G_2^1(x) & G_2^2(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $E_1(x)$ is of full row rank r and $G_2^2(x)$ is a $\bar{m}^* \times \bar{m}^*$ invertible matrix-valued function defined on U. Moreover, by (EFL2), we have $F_3(x) = 0$ for $x \in U$. Now we use the feedback transformation

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2(x) \end{bmatrix} + \begin{bmatrix} I_{m^*} & 0 \\ G_2^1(x) & G_2^2(x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and the system becomes

$$\begin{bmatrix} E^1(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} \tilde{F}_1(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1^1(x) & \tilde{G}_1^2(x) \\ 0 & I_{\bar{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix},$$

where $\tilde{F}_1 = F_1 - G_1^2(G_2^2)^{-1}F_2$, $\tilde{G}_1^1 = G_1^1 - G_1^2(G_2^2)^{-1}G_2^1$ and $\tilde{G}_1^2 = G_1^2(G_2^2)^{-1}$. Premultiply the above equation by $\begin{bmatrix} I_r & -\tilde{G}_1^2(x) & 0 \\ 0 & I_{\bar{m}^*} & 0 \\ 0 & 0 & I_{l-r-\bar{m}^*} \end{bmatrix}$ to get

$$\begin{bmatrix} E^*(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F^*(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} G^*(x) & 0 \\ 0 & I_{\bar{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^* \\ \bar{u}^* \end{bmatrix}, \tag{34}$$

where $E^* = E_1$, $F^* = \tilde{F}_1$, $G^* = \tilde{G}_1^1$, $u^* = \tilde{u}_1$ and $\bar{u}^* = \tilde{u}_2$. Then by Definition 2.5, we have that $\Xi^u|_{M^*} = \Xi^u|_U$ is the following system:

$$\Xi^u|_{M^*} : E^*(x)\dot{x} = F^*(x) + G^*(x)u^*.$$

By Theorem 4.5 and the condition (EFL3), $\Xi^u|_{M^*}$ is locally ex-fb-equivalent (on $M^* = U$) to a linear DACS $\Delta^{\tilde{u}^*}$ of the form (31). It follows from (34) that Ξ^u is locally ex-fb-equivalent to

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho & 0 \\ 0 & 0 \\ 0 & I_{\bar{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^* \\ \bar{u}^* \end{bmatrix},$$

which is completely controllable by Lemma 4.2. Therefore, Ξ^u is locally externally feedback linearizable by Definition 4.3. $\square$

Remark 4.8. (i) By the conditions (EFL1) and (EFL2), the locally maximal controlled invariant submanifold M^* around x_a is a neighborhood U of x_a. So the condition (EFL3) is actually, satisfied if and only if the condition (FL1)-(FL3) are satisfied on $M^* = U$, i.e., locally around x_a.

(ii) The condition (EFL2) and the condition $\hat{\mathcal{D}}_{n^*} = \mathcal{D}_{n^*} = TM^*$ of (FL2) are the nonlinear counterparts of the condition $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$ of Lemma 4.2, which assures that the linearized DACS is completely controllable. The sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ can thus be seen as nonlinear generalizations of the augmented Wong sequence $\mathscr{W}_i$ of (27) and the sequence $\hat{\mathscr{W}_i}$ of (28), respectively.

(iii) If $E(x) = I_n$, a DACS $\Xi^u = (E, F, G)$ becomes an explicit control system of the form (3). Suppose that $G(x) = \left[\, g_1(x) \, \cdots \, g_m(x) \,\right]$ is of constant rank. We have that conditions (EFL1)-(EFL2) of Theorem 4.7 are clearly satisfied and that the condition (EFL3) reduces to the feedback linearizability conditions in the classical sense. Indeed, we have $\Xi^u \in \mathbf{Expl}(\Xi^u|_{M^*}) = \mathbf{Expl}(\Xi^u)$ because Ξ^u with $E(x) = I_n$ is already an explicit system. Thus the driving variable v is absent and the two linearizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy $\hat{\mathcal{D}}_{i+1} = \mathcal{D}_i$ for $i \geq 1$. Hence the conditions (FL1)-(FL3) become (FL1)' $\mathcal{D}_i$ are of constant rank for $1 \leq i \leq n$; (FL2)' $\dim \mathcal{D}_n = n$; (FL3)' $\mathcal{D}_i$ are involutive for $1 \leq i \leq n-1$, which are the feedback linearizability conditions for classical nonlinear control systems, see e.g., [17, 19, 20, 28].

5. Examples

Example 5.1. Consider the following academic example borrowed from [3]:

$$\Xi^u : \begin{bmatrix} x_2 & x_1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_2^2 - x_1^3 + x_3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{35}$$

where $u = (u_1, u_2)$. Fix an admissible point $x_a = (x_{1a}, x_{2a}, x_{3a})$, where $x_{1a} = 1$, $x_{2a} = 0$, $x_{3a} = 0$. Clearly, there exists a neighborhood U ($x_1 \neq 0$ for all $x \in U$) of x_a such that (EFL1) and (EFL2) of Theorem 4.7 are satisfied. Subsequently, via $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$, Ξ^u is ex-fb-equivalent to

$$\begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2^2 - x_1^3 + x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}.$$

Observe that $M^* = U$ and

$$\Xi^u|_{M^*} = \Xi^u|_U : \begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2^2 - x_1^3 + x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u^*,$$

where $u^* = \tilde{u}_1$. Now an explicit control system $\Sigma^{u^*v} \in \mathbf{Expl}(\Xi^u|_{M^*})$ can be taken as

$$\Sigma^{u^*v} : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_2^2 - x_1^3 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2/x_1 \\ 0 \end{bmatrix} u^* + \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \end{bmatrix} v,$$

where v is a driving variable. We calculate the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for the system Σ^{u^*v} to get

$$\hat{\mathcal{D}}_1 = \operatorname{span}\{g^v\}, \quad \mathcal{D}_1 = \operatorname{span}\left\{g^{u^*}, g^v\right\}, \quad \mathcal{D}_2 = \hat{\mathcal{D}}_2 = \operatorname{span}\left\{g^{u^*}, g^v, ad_f g^v\right\},$$

where $g^v = \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \end{bmatrix}$, $g^{u^*} = \begin{bmatrix} 0 \\ 2/x_1 \\ 0 \end{bmatrix}$, $ad_f g^v = \begin{bmatrix} 0 \\ 0 \\ 3x_1^3+2x_2^2+x_1 \end{bmatrix}$. Clearly, the distributions above are of constant rank and $\mathcal{D}_2 = \hat{\mathcal{D}}_2 = T_xU$ for all $x \in U$. Additionally, $[g^{u^*}, g^v] = 0 \in \mathcal{D}_1$ and $\hat{\mathcal{D}}_1$ is of rank one, so the distributions $\hat{\mathcal{D}}_1$, $\mathcal{D}_1$, $\hat{\mathcal{D}}_2$ are all involutive. Thus, the condition (EFL3) of Theorem 4.7 is satisfied. Therefore, system Ξ^u is externally feedback linearizable.

In fact, we can choose $\varphi^{u^*}(x)$ and $\varphi^v(x)$ such that

$$\operatorname{span}\{d\varphi^v\} = \mathcal{D}_1^{\perp}, \qquad \operatorname{span}\left\{d\varphi^v, d\varphi^{u^*}\right\} = \hat{\mathcal{D}}_1^{\perp}.$$

Furthermore, use the following coordinates change and feedback transformation (note that the feedback transformation below has a triangular form as we discussed in (12) and Definition 3.3)

$$\xi = \varphi^{u^*}(x) = x_1 + x_3 - x_{1a}, \quad z_1 = \varphi^v(x) = x_1x_2 - x_{1a}x_{2a}, \quad z_2 = L_f\varphi^v(x) = -(x_1)^3 + (x_2)^2 + x_3,$$
$$\begin{bmatrix} u^* \\ v \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ \frac{2}{3(x_1)^3+x_1+2(x_2)^2} & -\frac{1}{3(x_1)^3+x_1+2(x_2)^2} \end{bmatrix} \begin{bmatrix} \tilde{u}^* \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(x_2)^2-(x_1)^3+x_3}{3(x_1)^3+x_1+2(x_2)^2} \end{bmatrix},$$

the system Σ^{uv} becomes

$$\Lambda^{\tilde{u}^*\tilde{v}} : \begin{cases} \dot{\xi} &= \tilde{u}^*, \\ \dot{z}_1 &= z_2, \\ \dot{z}_2 &= \tilde{v}. \end{cases}$$

Now by Theorem 3.5, $\Xi^u|_{M^*}$ is ex-fb-equivalent to the following linear DACS

$$\Delta^{\tilde{u}^*} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tilde{u}^*,$$

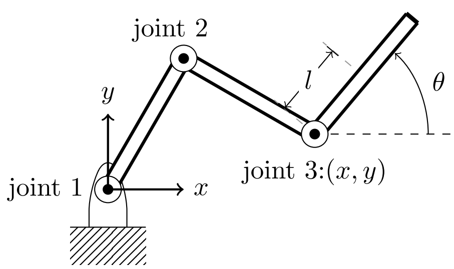
since $\Sigma^{u^*v} \in \mathbf{Expl}(\Xi^u|_{M^*})$, $\Lambda^{\tilde{u}^*\tilde{v}} \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$, and $\Sigma^{u^*v} \overset{sys-fb}{\sim} \Lambda^{\tilde{u}^*\tilde{v}}$. Therefore, the original DACS Ξ^u is ex-fb-equivalent to the following completely controllable linear DACS:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}^* \\ \tilde{u}_2 \end{bmatrix}$$

via

$$Q(x) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \xi \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 - x_{1a}x_{2a} \\ x_1+x_3-x_{1a} \\ -(x_1)^3+(x_2)^2+x_3 \end{bmatrix}, \quad \begin{bmatrix} u^* \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}^* \\ \tilde{u}_2 \end{bmatrix}.$$

Example 5.2. Consider the model of a 3-link manipulator [1] with active joints 1 and 2, and a passive joint 3 (see the following figure). We will call joint 3 a free joint of the manipulator.



The dynamic equations of the manipulator are given by:

$$\begin{cases} m\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta = F_x, \\ m\ddot{y} + ml\cos\theta\ddot{\theta} - ml\dot{\theta}^2\sin\theta = F_y, \\ -ml\sin\theta\ddot{x} + ml\cos\theta\ddot{y} + ml^2\ddot{\theta} = \tau_\theta + F_f, \end{cases} \tag{36}$$

where the mass m and the half length of the free-link l are constants, x and y are the position variables of the free joint, and θ is the angle between the base frame and the link frame, F_x and F_y are the translation force at the free joint in the direction of x and y, respectively, and τ_θ is the torque applied to the free joint (we take $\tau_\theta = 0$ implying that joint 3 is free). We additional consider the friction force F_f caused by the rotation of the free link. We regard (F_x, F_y) as the active control inputs to the system. The friction force F_f is a generalized state variable rather than an active control input since we can not change it arbitrarily. We consider system (36) subjected to the following constraint:

$$x - y = 0. \tag{37}$$

We combine (36) together with (37) as a DACS $\Xi^u_{7,7,2} = (E, F, G)$ of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & -ml\sin\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin\theta_1 & 0 & \cos\theta_1 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} x_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ F_f/l \\ x_1 - y_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

For the DACS Ξ^u, the generalized states $\xi = (x_1, x_2, y_1, y_2, \theta_1, \theta_2, F_f) \in X = \mathbb{R}^6 \times S$ and the vector of control input is (F_x, F_y). Consider Ξ^u around a point $\xi_p = (x_{1p}, x_{2p}, y_{1p}, y_{2p}, \theta_{1p}, \theta_{2p}, F_f) = 0$. The system Ξ^u is *not* externally feedback linearizable since condition (EF2) of Theorem 4.7 is not satisfied. We apply the geometric reduction method of Definition 2.3 to Ξ^u and we get

$$M_0^c = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^6, \quad M_1^c = \{\xi \in M_0^c \,|\, x_1 - y_1 = 0\}, \quad M_2^c = \{\xi \in M_1^c \,|\, x_2 - y_2 = 0\}, \quad M_3^c = M_2^c.$$

Thus by Proposition 2.4, $M^* = M_3^c = M_2^c$ is a locally maximal controlled invariant submanifold and $\xi_p \in M^*$ is an admissible point. Choose new coordinates $z_2 = (\tilde{x}_1, \tilde{x}_2) = (x_1 - y_1, x_2 - y_2)$ and keep the remaining coordinates $z_1 = (y_1, y_2, \theta_1, \theta_2, F_f)$ unchanged, the system represented in the new coordinates is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 & 0 & m \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_1 - \sin\theta_1 & 0 & l & 0 & 0 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \\ \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 + y_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ F_f/l \\ \tilde{x}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

Setting $z_2 = (\tilde{x}_1, \tilde{x}_2) = 0$, we get a DACS of the form (5):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \cos\theta_1 - \sin\theta_1 & 0 & l & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ F_f/l \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

By using $Q(z_1)$ and the feedback transformations defined by

$$Q(z_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \sin\theta_1 & 0 & -\cos\theta_1 & 0 & m \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_f/l \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix},$$

we bring the system into

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ ml\theta_2^2 \sin\theta_1 + \frac{F_f}{l}\sec\theta_1 \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \tan\theta_1 & -\sec\theta_1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

So the local M^*-restriction $\Xi^u|_{M^*} = (E^*, F^*, G^*)$ (see Definition 2.5) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ \frac{F_f}{l}\sec\theta_1 + ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \tan\theta_1 \\ 0 \\ 1 \end{bmatrix} u_1.$$

An explicitation system $\Sigma^{u_1 v} \in \mathbf{Expl}(\Xi^u|_{M^*})$ can be chosen as

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ \frac{F_f\tan\theta_1 + ml^2\theta_2^2}{ml(\cos\theta_1+\sin\theta_1)} \\ \theta_2 \\ \frac{F_f\sec\theta_1 + ml^2\theta_2^2(\sin\theta_1-\cos\theta_1)}{ml^2(\cos\theta_1+\sin\theta_1)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\sec\theta_1}{m(\cos\theta_1+\sin\theta_1)} \\ 0 \\ \frac{\tan\theta_1 - 1}{ml(\cos\theta_1+\sin\theta_1)} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v.$$

Define a new control $u^* := \frac{F_f\tan\theta_1 + ml^2\theta_2^2}{ml(\cos\theta_1+\sin\theta_1)} + \frac{\sec\theta_1}{m(\cos\theta_1+\sin\theta_1)} u_1$. Then the system $\Sigma^{u_1 v}$ under the new control is

$$\Sigma^{u^* v}: \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ 0 \\ \theta_2 \\ \frac{F_f}{ml^2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{l}(\sin\theta_1-\cos\theta_1) \\ 0 \end{bmatrix} u^* + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v.$$

Now calculate the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for the system $\Sigma^{u^* v}$ to get

$$\hat{\mathcal{D}}_1 = \operatorname{span}\{g^v\}, \quad \mathcal{D}_1 = \operatorname{span}\left\{g^{u^*}, g^v\right\}, \quad \hat{\mathcal{D}}_2 = \operatorname{span}\left\{g^{u^*}, g^v, ad_f g^v\right\},$$
$$\mathcal{D}_2 = \operatorname{span}\left\{g^{u^*}, g^v, ad_f g^v, ad_f g^{u^*}\right\}, \quad \mathcal{D}_3 = \hat{\mathcal{D}}_2 = TM^*.$$

where $g^v = \frac{\partial}{\partial F_f}$, $g^{u^*} = \frac{\partial}{\partial y_2} + \frac{1}{l}(\sin\theta_1 - \cos\theta_1)\frac{\partial}{\partial\theta_2}$, $ad_f g^v = \frac{1}{l}\frac{\partial}{\partial\theta_2}$, $ad_f g^{u^*} = \frac{\partial}{\partial y_1} + \frac{1}{l}(\sin\theta_1 - \cos\theta_1)\frac{\partial}{\partial\theta_1}$. Clearly, the distributions above are of constant rank and are all involutive around ξ_p. Thus, conditions (FL1)-(FL3) of Theorem 4.5 are satisfied. Therefore, system Ξ^u is locally internally feedback linearizable around ξ_p. Indeed, choose $\varphi^{u^*}(x)$ and $\varphi^v(x)$ such that

$$\operatorname{span}\{d\varphi^v\} = \mathcal{D}_2^{\perp}, \qquad \operatorname{span}\left\{d\varphi^v, d\varphi^{u^*}\right\} = \hat{\mathcal{D}}_2^{\perp}.$$

Then define the following coordinates change and feedback transformation (which has a triangular form as desired):

$$\tilde{y}_1 = \varphi^v(z_1) = y_1 - l\int a(\theta_1)\mathrm{d}\theta_1, \quad \tilde{y}_2 = L_f\varphi^v(z_1) = y_2 - la(\theta_1)\theta_2,$$
$$\tilde{F}_f = L_f^2\varphi^v(z_1) = -a(\theta_1)F_f - a'(\theta_1)l\theta_2^2, \quad \tilde{\theta}_1 = \varphi^{u^*}(z_1) = \theta_1, \quad \tilde{\theta}_2 = L_f\varphi^{u^*}(z_1) = \theta_2,$$
$$\begin{bmatrix} \tilde{u}^* \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} \frac{1}{l}(\sin\theta_1-\cos\theta_1) & 0 \\ -2a'(\theta_1)(\sin\theta_1-\cos\theta_1)\theta_2 & -a(\theta_1) \end{bmatrix} \begin{bmatrix} u^* \\ v \end{bmatrix} + \begin{bmatrix} \frac{F_f}{ml^2} \\ -3a'(\theta_1)\theta_2 F_f - a''(\theta_1)\theta_2^3 l \end{bmatrix},$$

where $a(\theta_1) = \frac{1}{\sin\theta_1-\cos\theta_1}$, $a'(\theta_1) = \frac{\mathrm{d}a(\theta_1)}{\mathrm{d}\theta_1}$, $a''(\theta_1) = \frac{\mathrm{d}^2a(\theta_1)}{\mathrm{d}\theta_1^2}$. We transform $\Sigma^{u^* v}$ into a linear control system of the Brunovsky form

$$\Lambda^{\tilde{u}^*\tilde{v}}: \dot{\tilde{y}}_1 = \tilde{y}_2, \quad \dot{\tilde{y}}_2 = \tilde{F}_f, \quad \dot{\tilde{F}}_f = \tilde{v}, \quad \dot{\tilde{\theta}}_1 = \tilde{\theta}_2, \quad \dot{\tilde{\theta}}_2 = \tilde{u}^*.$$

Thus by Theorem 3.5, $\Xi^u|_{M^*}$ is locally ex-fb-equivalent to the following linear DACS $\Delta^{\tilde{u}^*}$, since $\Sigma^{u^*v} \overset{sys-fb}{\sim} \Sigma^{u_1 v} \in \mathbf{Expl}(\Xi^u|_{M^*})$, $\Lambda^{\tilde{u}^*\tilde{v}} \in \mathbf{Expl}(\Delta^{\tilde{u}})$, and $\Sigma^{u^*v} \overset{sys-fb}{\sim} \Lambda^{\tilde{u}^*\tilde{v}}$,

$$\Delta^{\tilde{u}^*} : \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \\ \dot{\tilde{F}}_f \\ \dot{\tilde{\theta}}_1 \\ \dot{\tilde{\theta}}_2 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{F}_f \\ \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u}^*.$$

Hence the original DACS Ξ^u is in-fb-equivalent to the completely controllable linear DACS $\Delta^{\tilde{u}^*}$.

6. Conclusions and perspectives

In this paper, we give necessary and sufficient conditions for the problem that when a nonlinear DACS is locally internally or locally externally feedback equivalent to a completely controllable linear DACS. The conditions are based on an explicit control system constructed by the explicitation with driving variables. Some examples are given to illustrate how to externally or internally feedback linearize a nonlinear DACS.

A natural problem for future works is that of when a nonlinear DAE system is ex-fb-equivalent to a linear one which is not necessarily completely controllable. Actually, this problem is more involved than the problem of external feedback linearization with complete controllability. Indeed, since in Theorem 4.7, the maximal controlled invariant submanifold M^* on U is $M^* = U$, it follows that the algebraic constraints are directly governed by some variables of u. Thus the in-fb-equivalence is very close to the ex-fb-equivalence. However, if $M^* \neq U$, then the algebraic constraints may affect the generalized state. Moreover, since the explicitation is defined up to a generalized output injection, it may happen that one system of the explicitation is feedback linearizable but another is not. The general feedback linearizability problem remains open and, in view of the above points, is challenging.

Appendix

Proof of Lemma 4.6. For ease of notation, we drop the index "$*$" for z^*, u^*, v^* and f^* of the system $\Sigma^{u^*v^*}_{n^*,m^*,s^*}$, that is, $\Sigma^{u^*v^*}$ becomes

$$\Sigma^{uv} : \dot{z} = f(z) + g^u(z)u + g^v(z)v.$$

The admissible point x_a for Σ^{uv} in the z-coordinates will be denoted by z_a. We will only show the proof for the case that

$$m^* = s^* = 1, \qquad \operatorname{rank}[g^v(z_a) \;\; g^u(z_a)] = 2.$$

The proof for the general case (i.e., for any m^* and s^*, and for $\operatorname{rank}[g^v(z_a) \;\; g^u(z_a)] = m^* + s^*$) can be done in a similar fashion as that on page 233-238 of [19] for the feedback linearization of nonlinear multi-inputs multi-outputs control systems. We now describe a procedure to construct a change of coordinates $\xi = \psi(z)$ and a feedback transformation:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha^u(z) \\ \alpha^v(z) \end{bmatrix} + \begin{bmatrix} \beta^u(z) & 0 \\ \lambda(z) & \beta^v(z) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \tag{38}$$

to transform Σ^{uv} into its Brunovský canonical form, where $\beta^u, \beta^v, \alpha^u, \lambda, \alpha^v$ are scalar functions, and $\beta^u(z)$ and $\beta^v(z)$ are nonzero around z_a, notice that the designed feedback transformation (38) has a triangular form as in (12).

Consider the two sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, given by (29), for Σ^{uv} and define

$$\rho := \max\left\{ i \in \mathbb{N}^+ \mid \hat{\mathcal{D}}_i \neq \mathcal{D}_i \right\} \text{ and } \bar{\rho} := \max\left\{ i \in \mathbb{N}^+ \mid \mathcal{D}_{i-1} \neq \hat{\mathcal{D}}_i \right\}.$$

By $m^* = s^* = 1$, it is seen that, for each $i \geq 1$,

$$\dim \mathcal{D}_i - \dim \hat{\mathcal{D}}_i = \begin{cases} 0, \text{ if } \mathcal{D}_i = \hat{\mathcal{D}}_i \\ 1, \text{ if } \mathcal{D}_i \neq \hat{\mathcal{D}}_i \end{cases}, \text{ and } \dim \hat{\mathcal{D}}_i - \dim \mathcal{D}_{i-1} = \begin{cases} 0, \text{ if } \hat{\mathcal{D}}_i = \mathcal{D}_{i-1} \\ 1, \text{ if } \hat{\mathcal{D}}_i \neq \mathcal{D}_{i-1} \end{cases}. \tag{39}$$

It follows that $\rho + \bar{\rho} = n^*$. Then only two cases are possible: either $\rho \geq \bar{\rho}$ or $\rho < \bar{\rho}$.

Case 1: If $\rho \geq \bar{\rho}$, then we have

$$\mathcal{D}_0 \subsetneq \hat{\mathcal{D}}_1 \subsetneq \cdots \subsetneq \mathcal{D}_{\bar{\rho}-1} \subsetneq \hat{\mathcal{D}}_{\bar{\rho}} \subsetneq \mathcal{D}_{\bar{\rho}} = \hat{\mathcal{D}}_{\bar{\rho}+1} \subsetneq \mathcal{D}_{\bar{\rho}+1} = \cdots \subsetneq \mathcal{D}_{\rho-1} = \hat{\mathcal{D}}_{\rho} \subsetneq \mathcal{D}_{\rho} = \hat{\mathcal{D}}_{\rho+j} = \mathcal{D}_{\rho+j},\ j > 0.$$

It follows that $\mathcal{D}_\rho = \mathcal{D}_{n^*} = \hat{\mathcal{D}}_{n^*}$ Then by (FL2) of Theorem 4.5, we have $\mathcal{D}_\rho = TM^*$ and thus $\dim \mathcal{D}_\rho = n^*$. By $\hat{\mathcal{D}}_\rho \subsetneq \mathcal{D}_\rho$ and (39), we have $\dim \hat{\mathcal{D}}_\rho = n^* - 1$. Now by the involutivity of $\hat{\mathcal{D}}_\rho^\perp$ (condition (FL3)), we can choose a scalar function $h^u(z)$ such that

$$\operatorname{span}\{dh^u\} = \hat{\mathcal{D}}_\rho^\perp,$$

which implies that for all z around z_a,

$$\begin{aligned} &\left\langle dh^u(z), ad_f^i g^u(z) \right\rangle = 0, \quad 0 \leq i \leq \rho - 2, \quad \left\langle dh^u(z), ad_f^{\rho-1} g^u(z) \right\rangle \neq 0; \\ &\left\langle dh^u(z), ad_f^i g^v(z) \right\rangle = 0, \quad 0 \leq i \leq \rho - 1. \end{aligned} \tag{40}$$

Recall the following result [19][28]:

$$\begin{aligned} &\left\langle dh(z), ad_f^i g(z) \right\rangle = 0,\ 0 \leq i \leq l - 2 \\ &\Rightarrow \left\langle dh(z), ad_f^{l-1} g(z) \right\rangle = (-1)^i \left\langle dL_f^i h(z), ad_f^{l-1-i} g(z) \right\rangle, \quad 0 \leq i \leq l - 1, \end{aligned} \tag{41}$$

310 where $h(z)$ is a scalar function, $f(z)$ and $g(z)$ are vector fields.

It can be deduced from (40) and (41) that for all z around z_a,

$$\begin{aligned} &\left\langle dL_f^i h^u(z), ad_f^j g^u(z) \right\rangle = 0, \quad \left\langle dL_f^i h^u(z), ad_f^{\rho-i-1} g^u(z) \right\rangle \neq 0, \quad 0 \leq i \leq \rho - 2,\ 0 \leq j \leq \rho - i - 2; \\ &\left\langle dL_f^i h^u(z), ad_f^j g^v(z) \right\rangle = 0, \quad 0 \leq i \leq \rho - 1,\ 0 \leq j \leq \rho - i - 1; \end{aligned} \tag{42}$$

By using (42), we have the following table for the expressions of $\left\langle \mathrm{d}L_f^i h^u, ad_f^j g^u \right\rangle$, $0 \le i \le \rho - \bar{\rho}$, $\bar{\rho} - 1 \le j \le \rho - 1$:

$$
\begin{array}{ccccc}
 & ad_f^{\bar{\rho}-1} g^u & ad_f^{\bar{\rho}} g^u & \cdots & ad_f^{\rho-1} g^u \\
\mathrm{d}h^u & 0 & 0 & \cdots & \left\langle \mathrm{d}h^u, ad_f^{\rho-1} g^u \right\rangle \\
\cdots & \cdots & \cdots & * & \\
\mathrm{d}L_f^{\rho-\bar{\rho}-1} h^u & 0 & \left\langle \mathrm{d}L_f^{\rho-\bar{\rho}-1} h^u, ad_f^{\bar{\rho}} g^u \right\rangle & & \\
\mathrm{d}L_f^{\rho-\bar{\rho}} h^u & \left\langle \mathrm{d}L_f^{\rho-\bar{\rho}} h^u, ad_f^{\bar{\rho}-1} g^u \right\rangle & & & ?
\end{array}
$$

Notice that all the anti-diagonal elements of the above table are all nonzero by (42). It follows that the co-distribution

$$\Omega_1 = \mathrm{span}\left\{ dL_f^i h^u, 0 \le i \le \rho - \bar{\rho} \right\}$$

is of dimension $\rho - \bar{\rho} + 1$ around z_a. Notice that $\Omega_1 \subseteq \mathcal{D}_{\bar{\rho}-1}^{\perp}$ since

$$\mathcal{D}_{\bar{\rho}-1} = \mathrm{span}\left\{ g^u, \ldots, ad_f^{\bar{\rho}-2} g^u, g^v, \ldots, ad_f^{\bar{\rho}-2} g^v \right\}$$

and

$$\left\langle \mathrm{d}L_f^i h^u(z), ad_f^j g^u(z) \right\rangle \overset{(42)}{=} 0 \text{ and } \left\langle \mathrm{d}L_f^i h^u(z), ad_f^j g^v(z) \right\rangle \overset{(42)}{=} 0 \text{ for } 0 \le i \le \rho - \bar{\rho},\ 0 \le j \le \bar{\rho} - 2.$$

It is seen that $\dim \mathcal{D}_{\bar{\rho}-1}^{\perp} - \dim \Omega_1 = (n^* - (2\bar{\rho} - 2)) - (\rho - \bar{\rho} + 1) = 1$ and $\Omega_1 \subsetneq \mathcal{D}_{\bar{\rho}-1}^{\perp}$. Then by the involutivity of $\mathcal{D}_{\bar{\rho}-1}$ (condition (FL3)), we can choose a scalar function $h^v(z)$ such that

$$\mathrm{span}\left\{ \mathrm{d}h^v \right\} + \Omega_1 = \mathcal{D}_{\bar{\rho}-1}^{\perp},$$

which implies that for all z around z_a,

$$
\begin{aligned}
&\left\langle dh^v(z), ad_f^i g^u(z) \right\rangle = 0,\ 0 \le i \le \bar{\rho} - 2, \\
&\left\langle dh^v(z), ad_f^i g^u(z) \right\rangle = 0,\ 0 \le i \le \bar{\rho} - 2, \quad \left\langle dh^v(z), ad_f^{\bar{\rho}-1} g^v(z) \right\rangle \neq 0.
\end{aligned} \tag{43}
$$

It can be deduced from (43) and (41) that for all z around z_a,

$$
\begin{aligned}
&\left\langle dL_f^i h^v(z), ad_f^j g^u(z) \right\rangle = 0, \quad 0 \le i \le \bar{\rho} - 2,\ 0 \le j \le \bar{\rho} - i - 2; \\
&\left\langle dL_f^i h^v(z), ad_f^j g^u(z) \right\rangle = 0,\ \left\langle dL_f^i h^v(z), ad_f^{\bar{\rho}-i-1} g^u(z) \right\rangle \neq 0, \quad 0 \le i \le \bar{\rho} - 2,\ 0 \le j \le \bar{\rho} - i - 2.
\end{aligned} \tag{44}
$$

By using (42) and (44), we have the following table:

$$
\begin{array}{cccccccccc}
 & g^v & g^u & \cdots & \cdots & ad_f^{\bar{\rho}-1} g^v & ad_f^{\bar{\rho}-1} g^u & ad_f^{\bar{\rho}} g^u & \cdots & ad_f^{\rho-1} g^u \\
\mathrm{d}h^u & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \left\langle \mathrm{d}h^u, ad_f^{\rho-1} g^u \right\rangle \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * & \\
\mathrm{d}L_f^{\rho-\bar{\rho}-1} h^u & 0 & 0 & \cdots & \cdots & 0 & 0 & \left\langle \mathrm{d}L_f^{\rho-\bar{\rho}-1} h^u, ad_f^{\bar{\rho}} g^u \right\rangle & & \\
\mathrm{d}L_f^{\rho-\bar{\rho}} h^u & 0 & 0 & \cdots & \cdots & 0 & \left\langle \mathrm{d}L_f^{\rho-\bar{\rho}} h^u, ad_f^{\bar{\rho}-1} g^u \right\rangle & & & ? \\
\mathrm{d}h^v & 0 & 0 & \cdots & \cdots & \left\langle \mathrm{d}h^v, ad_f^{\bar{\rho}-1} g^v \right\rangle & ? & & & \\
\cdots & 0 & 0 & \cdots & * & & & & & \\
\cdots & 0 & 0 & * & ? & & & & & \\
\mathrm{d}L_f^{\rho-1} h^u & 0 & L_{g^u} L_f^{\rho-1} h^u & & & & & & & \\
\mathrm{d}L_f^{\bar{\rho}-1} h^v & L_{g^v} L_f^{\bar{\rho}-1} h^v & ? & & & & ? & & & ?
\end{array}
$$

where the $*$ elements are all nonzero. Define

$$\psi(z) := (h^u(z), \ldots, L_f^{\rho-1} h^u(z), h^v(z), \ldots, L_f^{\bar{\rho}-1} h^v(z)), \tag{45}$$

it can be seen that the $(\rho+\bar\rho)\times(\rho+\bar\rho) = n^*\times n^*$ matrix

$$\frac{\partial\psi(z)}{\partial z}\left[\, g^v(z)\ g^u(z)\ \cdots\ \cdots\ ad_f^{\bar\rho-1}g^v(z)\ ad_f^{\bar\rho-1}g^u(z)\ ad_f^{\bar\rho}g^u(z)\ \cdots\ ad_f^{\rho-1}g^u(z)\,\right]$$

is invertible around z_a. Thus the Jacobian matrix $\frac{\partial\psi(z)}{\partial z}$ is invertible around z_a and ψ is a local diffeomorphism. Then set

$$\begin{aligned} \alpha^u(z) = L_f^{\rho}h^u(z), \quad \beta^u(z) = L_{g^u}L_f^{\rho-1}h^u(z), \\ \alpha^v(z) = L_f^{\bar\rho}h^v(z), \quad \beta^v(z) = L_{g^v}L_f^{\bar\rho-1}h^v(z), \quad \lambda(z) = L_{g^u}L_f^{\bar\rho-1}h^v(z). \end{aligned} \tag{46}$$

Note that β^u and $\beta^v(x)$ are nonzero at x_p. It can be seen that $\Sigma^{u^*v^*}$ is mapped, via the coordinates transformations $\xi = (\xi_1,\xi_2) = \psi(z)$ and the feedback transformation (38), into the Brunovsky form $\Sigma_{Br}^{w} = \Sigma_{Br}^{w^*}$ of (30) with indices ρ and $\bar\rho$.

Case 2: If $\rho < \bar\rho$, then we have

$$\mathcal{D}_0 \subsetneq \hat{\mathcal{D}}_1 \subsetneq \cdots \subsetneq \hat{\mathcal{D}}_\rho \subsetneq \mathcal{D}_\rho \subsetneq \hat{\mathcal{D}}_{\rho+1} = \mathcal{D}_{\rho+1} \subsetneq \cdots = \mathcal{D}_{\bar\rho-1} \subsetneq \hat{\mathcal{D}}_{\bar\rho} = \mathcal{D}_{\bar\rho} = \hat{\mathcal{D}}_{\bar\rho+j} = \mathcal{D}_{\bar\rho+j}, \quad j>0.$$

It follows that $\hat{\mathcal{D}}_{\bar\rho} = \mathcal{D}_{\bar\rho} = \hat{\mathcal{D}}_{n^*} = \mathcal{D}_{n^*}$. Then by (FL2) of Theorem 4.5, we have $\hat{\mathcal{D}}_{\bar\rho} = TM^*$ and thus $\dim\hat{\mathcal{D}}_{\bar\rho} = n^*$. By $\mathcal{D}_{\bar\rho-1} \subsetneq \hat{\mathcal{D}}_{\bar\rho}$ and (39), we have $\dim\mathcal{D}_{\bar\rho-1} = n^*-1$. Now by the involutivity of $\mathcal{D}_{\bar\rho}$ (condition (FL1)), we can choose a scalar function $h^v(z)$ such that

$$\operatorname{span}\{dh^v\} = \mathcal{D}_{\bar\rho-1}^{\perp}.$$

Then following a similar proof as in Case 1, we can show that the following distribution

$$\Omega_2 = \operatorname{span}\left\{dL_f^i h^v, 0\le i\le \bar\rho-\rho-1\right\}$$

is of dimension $\rho-\bar\rho$ around z_a and $\Omega_2 \subsetneq \hat{\mathcal{D}}_\rho^{\perp}$. Notice that $\dim\hat{\mathcal{D}}_\rho^{\perp} = n^*-(2\rho-1) = \bar\rho-\rho+1$, we have $\dim\hat{\mathcal{D}}_\rho^{\perp} - \dim\Omega_2 = 1$. Thus by the involutivity of $\hat{\mathcal{D}}_\rho$, we can choose a scalar function $h^u(z)$ such that

$$\operatorname{span}\{\mathrm{d}h^u\} + \Omega_2 = \hat{\mathcal{D}}_\rho^{\perp}.$$

Then, as in Case 1, we construct the follow table

	g^v	g^u	$\cdots$	$\cdots$	$ad_f^{\rho-1}g^v$	$ad_f^{\rho-1}g^u$	$ad_f^{\rho}g^v$	$\cdots$	$ad_f^{\bar\rho-1}g^v$
$\mathrm{d}h^v$	0	0	$\cdots$	$\cdots$	0	0	0	$\cdots$	$\langle \mathrm{d}h^v, ad_f^{\bar\rho-1}g^v\rangle$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	
$\mathrm{d}L_f^{\bar\rho-\rho-1}h^v$	0	0	$\cdots$	$\cdots$	0	0	$\langle \mathrm{d}L_f^{\bar\rho-\rho-1}h^v, ad_f^{\rho}g^v\rangle$		
$\mathrm{d}h^u$	0	0	$\cdots$	$\cdots$	0	$\langle \mathrm{d}h^u, ad_f^{\rho-1}g^u\rangle$			
$\mathrm{d}L_f^{\bar\rho-\rho}h^v$	0	0	$\cdots$	$\cdots$	$\langle \mathrm{d}L_f^{\bar\rho-\rho}h^v, ad_f^{\rho-1}g^v\rangle$				
$\cdots$	0	0	$\cdots$	$*$					
$\cdots$	0	0	$*$						
$\mathrm{d}L_f^{\rho-1}h^u$	0	$L_{g^u}L_f^{\rho-1}h^u$							
$\mathrm{d}L_f^{\bar\rho-1}h^v$	$L_{g^v}L_f^{\bar\rho-1}h^v$								

and show that all the anti-diagonal elements of the table are nonzero around z_a. Finally, we define a diffeomorphism ψ and functions α^u, β^u, α^v, β^v and λ of the same form as in (45) and (46) of Case 1. It is seen that Σ^{uv} is also transformed into the Brunovsky form $\Sigma_{Br}^{w} = \Sigma_{Br}^{w^*}$ of (30) with indices ρ and $\bar\rho$ via $\xi = \psi(z)$ and the feedback transformation (38). $\square$

References

[1] H. Arai, K. Tanie, and N. Shiroma, *Nonholonomic control of a three-DOF planar underactuated manipulator*, IEEE Transactions on Robotics and Automation, 14 (1998), pp. 681–695.

[2] T. Berger, *Controlled invariance for nonlinear differential–algebraic systems*, Automatica, 64 (2016), pp. 226–233.

[3] T. Berger, *The zero dynamics form for nonlinear differential-algebraic systems*, IEEE Transactions on Automatic Control, 62 (2017), pp. 4131–4137.

[4] T. Berger and T. Reis, *Controllability of linear differential-algebraic systems–a survey*, in Surveys in Differential-Algebraic Equations I, Springer, 2013, pp. 1–61.

[5] R. W. Brockett, *Feedback invariants for nonlinear systems*, IFAC Proceedings Volumes, 11 (1978), pp. 1115–1120.

[6] P. Brunovskỳ, *A classification of linear controllable systems*, Kybernetika, 6 (1970), pp. 173–188.

[7] Y. Chen, *Geometric Analysis of Differential-Algebraic Equations and Control Systems: Linear, Nonlinear and Linearizable*, PhD thesis, Normandie Université, 2019.

[8] Y. Chen and W. Respondek, *Internal and external linearization of semi-explicit differential algebraic equations*, IFAC-PapersOnLine, 52 (2019), pp. 292–297.

[9] Y. Chen and W. Respondek, *From Morse triangular form of ODE control systems to feedback canonical form of DAE control systems*. Submitted to publish, preprint available from the website of the authors, 2021.

[10] Y. Chen and W. Respondek, *Geometric analysis of differential-algebraic equations via linear control theory*, SIAM Journal on Control and Optimization, 59 (2021), pp. 103–130.

[11] Y. Chen and S. Trenn, *A singular perturbed system approximation of nonlinear diffrential-algebraic equations*. Accepted by IFAC conference of ADHS2021, preprint available from the website of the authors, 2021.

[12] Y. Chen, S. Trenn, and W. Respondek, *Normal forms and internal regularization of nonlinear differential-algebraic control systems*. Submitted to publish, preprint available from the website of the authors, 2020.

[13] D. Cobb, *Controllability, observability, and duality in singular systems*, IEEE Transactions on Automatic Control, 29 (1984), pp. 1076–1082.

[14] L. Dai, *Singular Control Systems*, vol. 118, Springer, 1989.

[15] H. Frankowska, *On controllability and observability of implicit systems*, Systems & Control Letters, 14 (1990), pp. 219–225.

[16] T. Geerts, *Invariant subspaces and invertibility properties for singular systems: The general case*, Linear algebra and its Applications, 183 (1993), pp. 61–88.

[17] L. Hunt and R. Su, *Linear equivalents of nonlinear time-varying systems*, in Proc. Int. Symposium on Math. Theory of Networks and Systems, Santa Monica, 1981, pp. 119–123.

[18] L. Hunt, R. Su, and G. Meyer, *Global transformations of nonlinear systems*, IEEE Transactions on Automatic Control, 28 (1983), pp. 24–31.

[19] A. Isidori, *Nonlinear Control Systems*, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 3rd ed., 1995.

[20] B. Jakubczyk and W. Respondek, *On linearization of control systems*, Bull. Acad. Polonaise Sci. Ser. Sci. Math., (1980), p. 517522.

[21] S. Kawaji and E. Z. Taha, *Feedback linearization of a class of nonlinear descriptor systems*, in Proceedings of the 33rd IEEE Conference on Decision and Control, vol. 4, IEEE, 1994, pp. 4035–4037.

[22] A. Kumar and P. Daoutidis, *Feedback control of nonlinear differential-algebraic-equation systems*, AIChE Journal, 41 (1995), pp. 619–636.

[23] G. Lebret and J. J. Loiseau, *Proportional and proportional-derivative canonical forms for descriptor systems with outputs*, Automatica, 30 (1994), pp. 847–864.

[24] J. M. Lee, *Introduction to Smooth Manifolds*, Springer, 2001.

[25] F. L. Lewis, *A survey of linear singular systems*, Circuits, Systems and Signal Processing, 5 (1986), pp. 3–36.

[26] F. L. Lewis, *A tutorial on the geometric analysis of linear time-invariant implicit systems*, Automatica, 28 (1992), pp. 119–137.

[27] J. J. Loiseau, K. Özçaldiran, M. Malabre, and N. Karcanias, *Feedback canonical forms of singular systems*, Kybernetika, 27 (1991), pp. 289–305.

[28] H. Nijmeijer and A. Van der Schaft, *Nonlinear Dynamical Control Systems*, vol. 175, Springer, 1990.

[29] K. Özçaldiran, *A geometric characterization of the reachable and the controllable subspaces of descriptor systems*, Circuits, Systems and Signal Processing, 5 (1986), pp. 37–48.

[30] P. J. Rabier and W. C. Rheinboldt, *A geometric treatment of implicit differential-algebraic equations*, Journal of Differential Equations, 109 (1994), pp. 110–146.

[31] P. J. Rabier and W. C. Rheinboldt, *Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint*, vol. 68, Society for Industrial and Applied Mathematics, 2000.

[32] S. Reich, *On an existence and uniqueness theory for nonlinear differential-algebraic equations*, Circuits, Systems and Signal Processing, 10 (1991), pp. 343–359.

[33] W. Respondek, *Geometric methods in linearization of control systems*, Banach Center Publications, 1 (1985), pp. 453–467.

[34] R. Riaza, *Differential-Algebraic Systems: Analytical Aspects and Circuit Applications*, World Scientific, 2008.

[35] R. Su, *On the linear equivalents of nonlinear systems*, Systems Control Letters, 2 (1982), pp. 48 – 52.

[36] I. Tall and W. Respondek, *Feedback equivalence of nonlinear control systems: a survey on formal approach*, in Chaos in Automatic Control, CRC Press, 2005, pp. 156–281.

[37] J. Wang and C. Chen, *Exact linearization of nonlinear differential algebraic systems*, in International Conferences on Info-Tech and Info-Net. Proceedings, vol. 4, Oct 2001, pp. 284–290 vol.4.

[38] L. Xiaoping, *On linearization of nonlinear singular control systems*, in American Control Conference, IEEE, June 1993, pp. 2284–2287.