Impulse-free jump solutions of nonlinear differential-algebraic equations

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Abstract

In this paper, we propose a novel definition of impulse-free jump solutions for nonlinear differential-algebraic equations (DAEs) of the form $E(x)\dot{x} = F(x)$ with inconsistent initial values, where the term "impulse-free" means that there are no Dirac impulses caused by jumps, i.e., the directions of jumps stay in $\ker E(x)$. We find that the existence and uniqueness of impulse-free jumps are closely related to the notion of geometric index-1 and the involutivity of the distribution defined by $\ker E(x)$. Moreover, a singular perturbed system approximation is proposed for nonlinear DAEs; we show that its solution approximates both the impulse-free jump solutions and the $\mathcal{C}^1$-solutions of nonlinear DAEs. Finally, we show by some examples that our results of impulse-free jumps are useful for the problems like consistent initializations of nonlinear DAEs and transient behavior simulations of electric circuits.

Keywords: nonlinear differential-algebraic equations, discontinues solutions, inconsistent initial values, impulse-free jumps, geometric methods, singular perturbed systems

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1. Introduction

Consider a nonlinear differential-algebraic equation (DAE) in quasi-linear form

$$\Xi : E(x)\dot{x} = F(x), \tag{1}$$

where $x \in X$ is a vector of the generalized states and $(x, \dot{x}) \in TX$, where TX is the tangent bundle of the open subset X in $\mathbb{R}^n$. The maps $E : TX \to \mathbb{R}^l$ and $F : X \to \mathbb{R}^l$ are $\mathcal{C}^\infty$-smooth, and for each $x \in X$, we have $E(x) : \mathbb{R}^n \to \mathbb{R}^l$ is a linear map. We will denote a DAE of the form (1) by $\Xi_{l,n} = (E, F)$ or, simply, Ξ. A linear DAE of the form

$$\Delta : E\dot{x} = Hx \tag{2}$$

will be denoted by $\Delta_{l,n} = (E, H)$ or, simply, Δ, where $E \in \mathbb{R}^{l\times n}$ and $H \in \mathbb{R}^{l\times n}$. A linear DAE is called *regular* if $l = n$ and $sE - H \in \mathbb{R}^{n\times n}[s]\backslash 0$.

Definition 1.1 ($\mathcal{C}^1$-solutions and consistency space)**.** The trajectory $x : \mathcal{I} \to X$ for some open interval $\mathcal{I} \subseteq \mathbb{R}$ is called a $\mathcal{C}^1$-solution of the DAE $\Xi_{l,n} = (E, F)$ if x is continuously differentiable and satisfies $E(x(t))\dot{x}(t) = F(x(t))$ for all $t \in \mathcal{I}$.

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A point $x_c \in X$ is called *consistent* (or *admissible* [25, 26]) if there exists a $\mathcal{C}^1$-solution $x : \mathcal{I} \to X$ and $t_c \in I$ such that $x(t_c) = x_c$. The *consistency space* $S_c \subseteq X$ is the set of all consistent points.

By re-parameterizing the time variable t, we can always assume $I = (0, T)$ for some $T > 0$. We will give a brief review of the existence and uniqueness of $\mathcal{C}^1$-solutions for nonlinear DAEs in Section 2. In the present paper, we will study solutions of nonlinear DAEs of the form (1) with inconsistent initial values, i.e., solutions when the initial point $x_0^- \notin S_c$, where the initial point x_0^- is usually defined (see e.g. [1],[2]) via the right limits of some past trajectories $x(t), t < 0$ (which may or may not governed by (1)). More specifically, x_0^- is an initial point of Ξ if $x_0^- = \lim\limits_{t\to 0^-} x(t) = x(0^-)$. Assume that there exists one $\mathcal{C}^1$-solution $x(t)$ of Ξ for $t \in (0, T)$, then we have $x_0^+ = \lim\limits_{t\to 0^+} x(t) = x(0^+) \in S_c$. Thus if x_0^- is not consistent, then there has to be an "instantaneous" change of values for $x(t)$ at $t = 0$, i.e., a jump $x_0^- \to x_0^+$ to steer the inconsistent point x_0^- towards a consistent one x_0^+.

The jump behaviors in practical DAE systems are not rare phenomenons, e.g., the inconsistent initial values of electric circuits caused by switching devices (see e.g., [3–5]), the discontinues transient dynamics in hybrid/switched systems as power systems [6], multi-body dynamics [7] and battery models [8]. All the jumps which we consider in the present article, are called external/exogenous jumps [9], which are different from the jumps happened at the impasse or singular points discussed in [10],[11],[12]. More specifically, we suppose throughout that once the inconsistent initial point x_0^- jumps to a consistent point $x_0^+ \in S_c$, then we will consider only $\mathcal{C}^1$-solutions starting from x_0^+, that means, there are no jumps in $x(t)$ for $t \in (0, T)$.

For a linear DAE $\Delta = (E, H)$, given by (2), with an inconsistent initial value x_0^-, the jump behavior at $t = 0$ can be described by a vector $e_0 = x(0^+) - x(0^-) = x_0^+ - x_0^-$. To deal with the discontinuity introduced by the jump behavior at $t = 0$, the distributional (generalized function) [1] solutions theory for linear DAEs were established e.g. in [1, 2, 14, 15]. The distributional derivative of the jump of x at $t = 0$ is $(x_0^+ - x_0^-)\delta_0$, where δ_0 is the Dirac impulse at $t = 0$, i.e., taking distributional derivative of a jump results in a Dirac impulse δ_0 whose amplitude is the jump vector e_0 [15]. The distributional restriction of Δ to $t = 0$ can be represented by $E\dot{x}[0] = Hx[0]$, where $x[0] = \sum_{i=0}^{k} \alpha_k \delta_0^{(i)}$ and $\dot{x}[0] = e_0\delta_0 + \sum_{i=0}^{k} \alpha_k \delta_0^{(i+1)}$ for some $k \geq 0$. It can be deduced that there are no Dirac impulses and their derivatives $\delta_0, \cdots, \delta_0^{(k)}$ caused by jumps at $t = 0$ if and only if $E \cdot e_0\delta_0 = 0$, i.e., $e_0 \in \ker E$, and we call a jump satisfies the latter condition an impulse-free jump of the Δ.

The difficulty of studying jump behaviors for DAEs of the form (1) comes from the nonlinearity of the map E, which makes the distributional (generalized function) solution theory a possible non-

[1]Note that there are two terminologies called distribution in our paper, one is a subset of the tangent bundle of a manifold in differential geometry, the other is a generalized function which helps to differentiate functions whose derivatives do not exist in the classical sense.

suitable setting for our problems. As stated in Remake 46.2 of [13], "*This does not mean that discontinuous solutions of quasilinear problems cannot be investigated, but only that their treatment as distribution solutions is inadequate. In other words, discontinuous solutions of general quasilinear problems must, if possible at all, be introduced by a different process which remains to be determined.*" An extension of the notion of impulsive-free jump to nonlinear DAEs was made in Assumption A4 of [16], where it is assumed that a jump vector $e_0 = x_0^+ - x_0^-$ should satisfy the jump rule $e_0 \in \ker E(x_0^+)$. The problem of finding the consistent point x_0^+ for a given inconsistent point x_0^- is call consistent initialization (see e.g., [17, 18]) in the numerical analysis of nonlinear DAEs. In particular, the consistent initialization of nonlinear DAEs can be solved by the function *decic* of MATLAB (see [19]). We will show below by examples that both the jumps defined by the rule of [16] and that calculated by *decic* are not invariant under nonlinear coordinates transformations, meaning that those two consistent initialization methods in [16] and [19] are not coordinate-free. A main contribution of this paper is the coordinate-free impulse-free jump rule introduced in Definition 4.1, under which we show that the existence and uniqueness of impulse-free jumps are closely related to the notion of geometric index (see Definition 3.1) and the involutivity of the distribution defined by $\ker E(x)$.

Some other works of studying jump behaviors of DAEs (see e.g., [9–13]) mainly focused on semi-explicit(called also semi-linear) DAEs of the form

$$\Xi^{SE} : \begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \;\; 0 = f_2(x_1, x_2), \end{cases} \tag{3}$$

i.e., $E(x) = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$; such DAEs are usually related to the models of electric circuits and singular perturbation theory (see e.g., [11], Chapter 11 of [20] and Chapter VIII of [13]). A preliminary result of using singular perturbation theory to study nonlinear DAEs of the form (1) is our conference publication [21], which only considers the case that Ξ is equivalent to a fully decoupled normal form (see the index-1 nonlinear Weirstrass form **(INWF)** in Theorem 4.6) without a formal definition of impulse-free jump. In this paper, we propose a singular perturbed system approximation for nonlinear DAEs (which are not necessarily equivalent to the **(INWF)**) and we show that the solutions of the perturbed systems approximate both the $\mathcal{C}^1$-solutions and the impulse-free jump solutions of the DAEs.

This paper is organized as follows: We give the notations of the paper and some preliminaries on $\mathcal{C}^1$-solutions of DAEs in Section 2. We recall the notion of geometric index-1 and give some characterizations for that notion in Section 3. We introduce the definition of impulse-free jumps for nonlinear DAEs and study the existence and uniqueness of impulse-free jumps in Section 4. Singular perturbed system approximations of nonlinear DAEs are discussed in Section 5. The proofs of the main results are given in Section 6. The conclusions and perspectives of the paper are given in Section 7.

2. Notations and Preliminaries on geometric $\mathcal{C}^1$-solutions theory of DAEs

We denote by $T_xM \subseteq \mathbb{R}^n$ the tangent space at $x \in M$ of a submanifold M of $\mathbb{R}^n$ and by TM we denote the corresponding tangent bundle. By $\mathcal{C}^k$ the class of k-times differentiable functions is denoted. For a smooth map $f : X \to \mathbb{R}$, we denote its differentials by $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$ and for a vector-valued map $f : X \to \mathbb{R}^m$, where $f = [f_1, \ldots, f_m]^T$, we denote its differential by $\mathrm{D}f = \begin{bmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_m \end{bmatrix}$. For a map $A : X \to \mathbb{R}^{n\times n}$, $\ker A(x)$, $\operatorname{Im} A(x)$ and $\operatorname{rank} A(x)$ are the kernel, the image and the rank of A at x, respectively. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. We assume familiarity with basic notions of differential geometry such as smooth embedded submanifolds, involutive distributions and refer the reader e.g. to the book [22] for the formal definitions of such notions.

We now recall some basic notions and results from the geometric analysis of the existence and uniqueness of $\mathcal{C}^1$-solutions for nonlinear DAEs (see e.g., [13, 23–26]).

Definition 2.1 (invariant and locally invariant submanifold)**.** For a DAE $\Xi_{l,n} = (E, F)$, a smooth connected embedded submanifold M is called *invariant* if for any $x_0^+ \in M$, there exists a $\mathcal{C}^1$-solution $x : \mathcal{I} \to X$ such that $x(t_0) = x_0^+$ for some $t_0 \in I$ and $x(t) \in M$, $\forall\, t \in \mathcal{I}$. Fix a point $x_p \in X$, a smooth embedded submanifold M containing x_p is called *locally invariant*, if there exists a neighborhood U of x_p such that $M \cap U$ is invariant.

A locally invariant submanifold M^*, around a point x_p, is called locally *maximal*, if there exists a neighborhood U of x_p such that for any other locally invariant submanifold M, we have $M \cap U \subseteq M^* \cap U$. The following procedure is called the *geometric reduction method* [13, 24, 26], which is used to construct the locally maximal invariant submanifold M^* around a consistent point x_p (see item (i) of Proposition 2.3).

Definition 2.2 (Geometric reduction method)**.** Consider a DAE $\Xi_{l,n}$ and fix a point $x_p \in X$. Let U_0 be a connected subset of X containing x_p. Step 0: $M_0^c = U_0$. Step k: Suppose that a sequence of smooth connected embedded submanifolds $M_{k-1}^c \subsetneq \cdots \subsetneq M_0^c$ of U_{k-1} for a certain $k-1$, have been constructed. Define recursively

$$M_k := \left\{ x \in M_{k-1}^c \,|\, F(x) \in E(x) T_x M_{k-1}^c \right\}. \tag{4}$$

As long as $x_p \in M_k$ let $M_k^c = M_k \cap U_k$ be a smooth embedded connected submanifold for some neighborhood $U_k \subseteq U_{k-1}$.

Proposition 2.3 ([26],[27])**.** *In the above geometric reduction method, there always exists a smallest k such that either $x_p \notin M_k$ or $M_{k+1}^c = M_k^c$ in U_{k+1}. In the latter case denote $k^* = k$ and $M^* = M_{k^*+1}^c$ and assume that there exists an open neighborhood $U^* \subseteq U_{k^*+1}$ of x_p such that $\dim E(x) T_x M^* = const.$ for $x \in M^* \cap U^*$, then* *(i)* x_p *is a consistent point, i.e.,* $x_p = x_c$*, and* M^* *is a locally maximal invariant submanifold around* x_p*;*

(ii) M^* *coincides locally with the consistency space* S_c*, i.e.,* $M^* \cap U^* = S_c \cap U^*$*.*

Notice that by item (ii) of Proposition 2.3, the consistency space S_c locally coincides with M^* on the neighborhood U^* of x_p. So any point $x_0^- \in U^* \backslash M^*$ is not consistent and there exist no $\mathcal{C}^1$-solutions starting from x_0^-. The uniqueness of $\mathcal{C}^1$-solutions is characterized via the following notion of local *internal regularity.* We call a $\mathcal{C}^1$-solution $x : \mathcal{I} \to (U \subseteq) X$ *maximal* (in U) if there is no other solution $\widetilde{x} : \widetilde{\mathcal{I}} \to (U \subseteq) X$ with $\mathcal{I} \subsetneq \widetilde{\mathcal{I}}$ and $x(t) = \widetilde{x}(t)$ for all $t \in \mathcal{I}$.

Definition 2.4 (local internal regularity)**.** Consider a DAE Ξ and let M^* be the locally maximal invariant submanifold around a consistent point $x_c \in M^*$. Then Ξ is called locally *internally regular* (around x_c) if there exists a neighborhood $U \subseteq X$ of x_c such that for any point $x_0^+ \in M^* \cap U$, there exists only one maximal solution $x : \mathcal{I} \to U$ with $t_0 \in \mathcal{I}$ and satisfying $x(t_0) = x_0^+$.

Proposition 2.5 ([25, 26])**.** *Given a DAE* Ξ *and the locally maximal invariant submanifold* M^* *around a consistent point* $x_c \in X$*, suppose that there exists an open neighborhood* U *of* x_c *such that* $\dim E(x)T_xM^* = const.$ *for* $x \in M^* \cap U$*. Then* Ξ *is locally internally regular around* x_c *if and only if*

$$\dim E(x)T_xM^* = \dim M^*, \ \forall x \in M^* \cap U. \tag{5}$$

Two linear DAEs $\Delta = (E, H)$ and $\tilde{\Delta} = (\tilde{E}, \tilde{H})$ are called strictly equivalent or externally equivalent (see [28]) if there exist invertible matrices Q and P such that $\tilde{E} = QEP^{-1}$ and $\tilde{H} = QHP^{-1}$. The same notion can be extended to nonlinear DAEs.

Definition 2.6. (external equivalence) Two DAEs $\Xi_{l,n} = (E, F)$ and $\tilde{\Xi}_{l,n} = (\tilde{E}, \tilde{F})$ defined on X and $\tilde{X}$, respectively, are called externally equivalent, shortly ex-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and $Q : X \to GL(l, \mathbb{R})$ such that

$$\tilde{E}(\psi(x)) = Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1} \quad \text{and} \quad \tilde{F}(\psi(x)) = Q(x)F(x). \tag{6}$$

Fix a point $x_p \in X$, if ψ and Q is defined locally around x_p, we will speak about local ex-equivalence.

Remark 2.7. In the above definition of ex-equivalence, Q combines equations but does not change the $\mathcal{C}^1$-solutions of the DAE; ψ defines new coordinates and maps $\mathcal{C}^1$-solutions to $\mathcal{C}^1$-solutions, i.e., a curve $x : I \to X$ is a $\mathcal{C}^1$-solution of Ξ if and only if $\psi \circ x$ is a $\mathcal{C}^1$-solution of $\tilde{\Xi}$.

3. Geometric index-1 nonlinear DAEs

There are various notions of index for nonlinear DAEs, see our recent paper [27] and the references therein. In the present paper, we will use only the notion of geometric index, which is defined via the sequence of submanifolds $M_0^c \subsetneq \cdots \subsetneq M_k^c$ constructed by the geometric reduction method in Section 2.

Definition 3.1 (geometric index [26, 27])**.** Consider the sequence M_k^c constructed via Definition 2.2 around some consistent point $x_c \in S_c$, then the (local) geometric index, or shortly, the index, of a DAE Ξ is defined by

$$\nu_g := \min\left\{k \geq 0 \,|\, M_{k+1}^c = M_k^c\right\}.$$

Clearly, the geometric index ν_g is the least integer k such that the sequence of submanifolds M_k^c gets stabilized, which is also the smallest number of steps has to be performed in order to construct the maximal invariant submanifold M^* and to solve the DAE.

Remark 3.2. A regular linear DAE $\Delta_{n,n} = (E, H)$ is always ex-equivalent, via two constant invertible matrices Q and P, to the Weierstrass form (**WF**)

$$\tilde{\Delta} = (QEP^{-1}, QHP^{-1}) : \begin{cases} \dot{x}_1 = A_1 x_1, \\ N\dot{x}_2 = x_2, \end{cases} \tag{7}$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and N is a nilpotent matrix. The index ν of Δ is defined by the nilpotency of N, i.e., $N^{\nu-1} \neq 0$ and $N^\nu = 0$ (where $\nu = 0$ means that the x_2-variables vanish, see [29]). The geometric index ν_g is a nonlinear generalization of the index ν of linear DAEs [27]. Indeed, the index ν of Δ can be alternatively defined as: $\nu := \min\left\{k \geq 0 \,|\, \mathscr{V}_{k+1} = \mathscr{V}_k\right\}$, where the sequence $\mathscr{V}_i$ (called the Wong sequence [30]) is a linear counterpart of M_k^c and is given by

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_{i+1} = H^{-1}E\mathscr{V}_i. \tag{8}$$

Now for a DAE $\Xi_{l,n} = (E, F)$ and a consistent point $x_c \in X$, we introduce the following regularity and constant rank conditions:

(RE) $l = n$ and Ξ is locally internally regular;

(CR) there exists a neighborhood U of x_c such that $M_1^c = M_1 \cap U$ constructed by (4) is connected and the following ranks are constant: $\operatorname{rank} E(x) = const. = r$ for $x \in U$; $\dim E(x)T_x M_1^c = const.$ and $\dim \mathrm{D}F_2(x) = const.$ for $x \in M_1^c$, where $F_2 := F \backslash \operatorname{Im} E := Q_2 F$, where $Q_2 : U \to \mathbb{R}^{(n-r)\times n}$ is full row rank and $Q_2 E = 0$.

A linear DAE $\Delta_{l,n} = (E, H)$, given by (2), is regular if and only if $l = n$ and Δ is internally regular (see [28, 29]). So the condition **(RE)** is a nonlinear version of the regularity of linear DAEs. The condition $\operatorname{rank} E(x) = const. = r$ (throughout we denote this rank by r) ensures that there exists $Q : U \to GL(n, \mathbb{R})$ such that $E_1 : U \to \mathbb{R}^{r\times n}$ of $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank r. The assumption $\operatorname{rank} \mathrm{D}F_2(x) = const.$ guarantees that the zero-level set $M_1^c = \{x \in U \,|\, F_2(x) = 0\}$ is a smooth embedded submanifold (by taking a smaller U, we can always assume M_1^c is connected) and the condition $\dim E(x)T_xM_1^c = const.$ excludes singular/impasses points (see [12]) and helps to view the DAE as a differential equation defined on its maximal invariant submanifold [25].

Proposition 3.3 (geometric index-1)**.** *Consider a DAE $\Xi_{l,n} = (E, F)$ and a consistent point $x_c \in S_c$. Assume that the conditions* **(RE)**, **(CR)** *are satisfied in an open neighborhood U of x_c. Then the following statements are equivalent around x_c:*

(i) The DAE Ξ is of geometric index $\nu_g = 1$.

(ii) The locally maximal invariant submanifold $M^* = M_1^c$.

(iii) $\operatorname{rank} E(x) = \dim E(x)T_xM_1^c$ *or, equivalently,* $\ker E(x) \cap T_xM_1^c = 0$, $\forall x \in M_1^c$.

(iv) Let $Z : U \to \mathbb{R}^{n\times(n-r)}$ be any smooth map such that $\operatorname{Im} Z(x) = \ker E(x)$, $\forall x \in U$. Then $A(x) = \mathrm{D}F_2(x) \cdot Z(x)$ is invertible or, equivalently, $B(x) = \left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right]$ is invertible, $\forall x \in M_1^c$.

(v) There exists an open neighborhood $V \subseteq U$ of x_c such that Ξ is locally (on V) ex-equivalent to

$$\begin{bmatrix} I_r & E_2(\xi_1, \xi_2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} F^*(\xi_1, \xi_2) \\ \xi_2 \end{bmatrix}, \tag{9}$$

where $M^* \cap V = \{\xi \in V \mid \xi_2 = 0\}$, $\xi = (\xi_1, \xi_2)$ and ξ_1 is a system of coordinates on $M^* \cap V$.

The proof is given in Section 6.

Remark 3.4. (i) By the constant rank assumption **(CR)**, we only need to check whether the item (iii) or (iv) of Proposition 3.3 holds at the point $x = x_c$ (or at any point x_0^+ of M_1^c) in order to conclude that Ξ is of geometric index-1 or not.

(ii) The map $F_2 = F \backslash \operatorname{Im} E$ in Proposition 3.3(iv) is not uniquely defined. More specifically, we may choose another invertible map $\tilde{Q} : U \to GL(n, \mathbb{R})$ such that $\tilde{E}_1$ of $\tilde{Q}E = \left[\begin{smallmatrix} \tilde{E}_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank. Then $\tilde{F}_2$ of $\tilde{Q}F = \left[\begin{smallmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{smallmatrix}\right]$ is different from F_2, but there alway exists $\bar{Q} : U \to GL(n-r, \mathbb{R})$ such that $\bar{Q}F_2 = \tilde{F}_2$. Then by $F_2(x) = 0$, $\forall x \in M_1^c$, it is seen that the differentials $\mathrm{D}F_2(x) = \mathrm{D}(\bar{Q}F_2(x)) = \sum_{i=1}^{n-r} F_2^i(x)\mathrm{D}\bar{Q}_i(x) + \bar{Q}(x)\mathrm{D}F_2(x) = \bar{Q}(x)\mathrm{D}F_2(x)$, $\forall x \in M_1^c$, where $\bar{Q}_i$ are the columns of $\bar{Q}$ and F_2^i are the rows of F_2. Therefore, item (iv) of Proposition 3.3 still holds even for any other choice of $\tilde{Q}$ since for all $x \in M_1^c$, $\tilde{A}(x) = \mathrm{D}\tilde{F}_2(x) \cdot Z(x) = \bar{Q}(x)\mathrm{D}F_2(x) \cdot Z(x) = \bar{Q}(x)A(x)$ is invertible if and only if $A(x)$ is invertible.

(iii) For a linear regular DAE $\Delta_{n,n} = (E, H)$, consider its index ν and the sequence $\mathscr{V}_i$ of (8). Then the following is equivalent: (i)' $\nu = 1$; (ii)' $\mathscr{V}_1$ is the largest subspace such that $A\mathscr{V}_1 \subseteq E\mathscr{V}_1$; (iii)' $\operatorname{rank} E = \dim E\mathscr{V}_1$ or $\ker E \cap \mathscr{V}_1 = 0$; (iv)' For any invertible matrices Q and P such that $QEP^{-1} = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $r = \operatorname{rank} E$, we have that A_4 of $QAP^{-1} = \left[\begin{smallmatrix} A_1 & A_2 \\ A3 & A_4 \end{smallmatrix}\right]$ is invertible; (v)' Δ is ex-equivalent to the DAE $\dot{\xi}_1 = A^*\xi_1$, $0 = \xi_2$. Observe that item (iv)' is also equivalent to $\operatorname{rank}[E, AZ] = n$, where Z is a full column rank matrix such that $\operatorname{Im} Z = \ker E$, or to $\operatorname{rank}\left[\begin{smallmatrix} E & 0 \\ A & E \end{smallmatrix}\right] = n + \dim\ker E$. The later two conditions are known (see e.g., [31]) to be a characterization of the impulse-freeness of linear DAEs.

4. Impulse-free jump solutions of nonlinear DAEs

We introduce the following definition of an impulse-free jump of a nonlinear DAE.

Definition 4.1 (impulse-free jump)**.** Consider a DAE $\Xi_{l,n} = (E, F)$, let S_c be the consistency space of Ξ, fix an inconsistent initial point $x_0^- \in X/S_c$. An impulse-free jump solution (trajectory), shortly, an IFJ solution, of Ξ starting from x_0^- is a $\mathcal{C}^1$-curve $J : [0, a] \to X$ satisfying

$$J(0) = x_0^- \notin S_c, \quad J(a) = x_0^+ \in S_c, \quad \forall \tau \in [0, a]: \ E(J(\tau))\frac{dJ(\tau)}{d\tau} = 0. \tag{10}$$

A jump $x_0^- \to x_0^+$ associated with an IFJ trajectory $J(\cdot)$ is called an *impulse-free jump* of Ξ.

It is crucial to note that the parametrization variable τ of the differentiable curve $J(\tau)$ is, in general, *not* a time variable (unless we connect it with the time-variable t, see Section 4). In Definition 4.1 only the direction of the tangent vector $\frac{dJ(\tau)}{d\tau}$ is required to stay in $\ker(J(\tau))$ while there are no other requirements on how fast the trajectory $J(\tau)$ should evolve with respect to τ (i.e, the magnitude of $\frac{dJ(\tau)}{d\tau}$). Consequently, even if the curve which we want to parametrize is possibly unique (indicating that there exists a unique impulse-free jump $x_0^- \to x_0^+$), the IFJ trajectory is always non-unique since there are infinitely many parameterizations of a curve. Indeed, by defining $\tilde{\tau} = \varphi(\tau)$ and $\tilde{J}(\varphi(\tau)) = J(\tau)$, where $\varphi : [0, a] \to [0, \tilde{a}]$ is diffeomorphism, we get $J(0) = x_0^-$, $J(\tilde{a}) = x_0^+$ and $E(\tilde{J}(\tilde{\tau}))\frac{d\tilde{J}(\tilde{\tau})}{d\tilde{\tau}} = E(J(\tau))\frac{d\tau}{d\tilde{\tau}}\frac{dJ(\tau)}{d\tau} = \frac{d\tau}{d\tilde{\tau}}E(J(\tau))\frac{dJ(\tau)}{d\tau} = 0$, $\forall \tau \in [0, \tilde{a}]$, which implies that $\tilde{J}(\tilde{\tau})$ is another IFJ trajectory of Ξ. The upper bound a of the domain of $J(\tau)$ is not fixed since it can always be scaled by φ into any $\tilde{\alpha} > 0$, including $\tilde{\alpha} = +\infty$.

Remark 4.2. We can regard the notion of IFJ trajectory as a nonlinear generalization of that of jump vector $e_0 = x_0^+ - x_0^-$ of linear DAEs. The impulse-free jump rule $E \cdot e_0\delta_0 = 0$ of linear DAEs is generalized into $Ee_0u(\tau) = 0$ for some $u : [0, a] \to \mathbb{R}$ with $\int_0^a u(\tau)d\tau = 1$. With other words we can consider the term $\frac{dJ(\tau)}{d\tau}$ in (10) as a linear control system

$$\frac{dJ(\tau)}{d\tau} = e_0u(\tau), \ \forall \tau \in [0, a], \quad e_0 \in \ker E, \quad J(0) = x_0^-, \quad J(a) = x_0^+. \tag{11}$$

In the following example, we compare our jump rule defined by (10) with two existing jump rules: one is

$$x_0^+ - x_0^- \in E(x_0^+) \tag{12}$$

introduced in [16] and another is given by the MATLAB function *decic* [19], which calculates consistent initial values for DAEs via a numerical searching method [18].

Example 4.3. Consider a DAE $\Xi_{2,2} = (E, F)$, given by

$$\Xi : \begin{bmatrix} 1 & 3x_2^2 - 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}. \tag{13}$$

Fix a point $x_p = (x_{1p}, x_{2p}) = (0, 1)$, it is clear that $M^* = M_1^c = \left\{x \in \mathbb{R}^2 \,|\, x_1 = 0, x_2 > \frac{\sqrt{3}}{3}\right\}$ (note that M^* should be connected) and that $\dim E(x)T_xM^* = 1$, $\forall x \in M^* \cap U^*$, where $U^* = \left\{x \in \mathbb{R}^2 \,|\, x_2 > \frac{\sqrt{3}}{3}\right\}$. By Proposition 2.3, $x_p = x_c$ is consistent, and M^* is the locally maximal invariant submanifold (around x_c) and coincides with the consistency space S_c on U^*. The inconsistent initial point which we consider is $x_0^- = (x_{10}^-, x_{20}^-) = (1, 1) \in U^* \backslash M^*$. Firstly, let

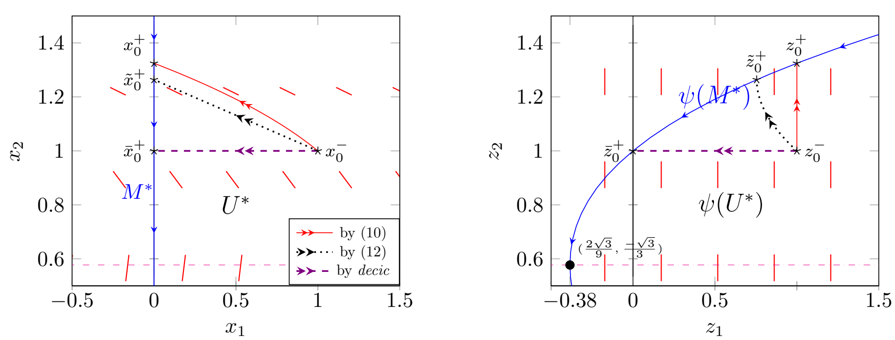
$$\frac{dJ(\tau)}{d\tau} = \begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 1 - 3x_2^2 \\ 1 \end{bmatrix}, \quad J(0) = x_0^-. \tag{14}$$

The solution $J(\tau) = (\tau + 2 - (\tau + 1)^3, \tau + 1))$ of the above ODE on the interval $[0, a]$ with $a \approx 0.3247$ is an IFJ trajectory of Ξ since $J(a) = x_0^+ \approx (0, 1.3247) \in M^* \cap U^*$ and $\left[\begin{smallmatrix} 1-3x_2^2 \\ 1 \end{smallmatrix}\right] \in \ker \left[\begin{smallmatrix} 1 & 3x_2^2-1 \\ 0 & 0 \end{smallmatrix}\right]$. Hence $x_0^- = J(0) \to x_0^+ = J(a)$ is an impulse-free jump in the sense of Definition 4.1. Secondly, we follow the jump rule $\tilde{x}_0^+ - x_0^- \in \ker E(\tilde{x}_0^+)$ of (12) to get three possible jump rules $x_0^- \to \tilde{x}_0^+$ with either $\tilde{x}_0^+ = (0, 0)$, $\tilde{x}_0^+ = (0, \frac{1+\sqrt{7/3}}{2}) \approx (0, 1.2638)$ or $\tilde{x}_0^+ = (0, \frac{1-\sqrt{7/3}}{2}) \approx (0, -0.2638)$, but only the second is contained in U^*.

Thirdly, we calculate the consistent initial point for Ξ by MATLAB using *decic* function, the result is $\bar{x}_0^+ = (0, 1)$. We draw those three different jumps reaching at the consistent points

$$x_0^+ = (0, 1.3247), \quad \tilde{x}_0^+ = (0, 1.2638), \quad \bar{x}_0^+ = (0, 1),$$

in Figure 1(a) below. Now choose new coordinates $z = (z_1, z_2) = (x_1 + x_2^3 - x_2, x_2)$, then the DAE



(a) x-coordinates (b) z-coordinates

Figure 1: The jumps calculated by (10), (12) and MATLAB *decic* function, respectively, shown in different coordinates. The red quiver plot illustrates the direction of $\ker E(x)$ and $\ker \tilde{E}(z)$, the blue solid lines with arrows represent the evolution of $\mathcal{C}^1$ solutions and the dash-dotted magenta lines depict the sets of singular/impasses points.

Ξ is ex-equivalent (on U^*), via the diffeomorphism $\psi(x) = z(x)$, to $\tilde{\Xi} = (\tilde{E}, \tilde{F})$ given by

$$\tilde{\Xi} : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -z_2 \\ z_1 - z_2^3 + z_2 \end{bmatrix}.$$

Note that the DAE $\tilde{\Xi}$ is a degenerate form of the van der Pol oscillator equation, which was a well-studied case (see e.g., [11, 13, 20]) for analyzing discontinue solutions of DAEs via singular perturbation theory. Under the new z-coordinates, the inconsistent initial point is $z_0^- = \psi(x_0^-) = (1, 1)$ and all three jump rules agree on the jump from z_0^- to $z_0^+ \approx (1, 1.3247)$. However, the transformed consistent points are given by, see also Figure 1(b),

$$z_0^+ = \psi(x_0^+) = (1, 1.3247), \quad \tilde{z}_0^+ = \psi(\tilde{x}_0^+) = (0.7547, 1.2638), \quad \bar{z}_0^+ = \psi(\bar{x}_0^+) = (0, 1).$$

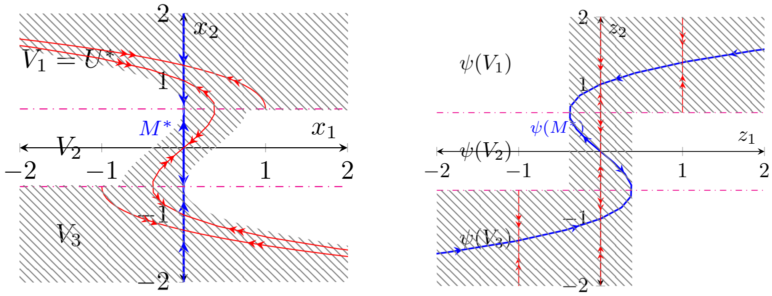
Clearly, $\tilde{z}_0^+$ and $\bar{z}_0^+$ do not coincide with the "correct" value z_0^+, which shows that the jump rule from [16] and MATLAB's *decic* jump rule are not invariant under coordinate transformations.

Remark 4.4. (i) Recall from Remark 2.7 that the ex-equivalence preserves $\mathcal{C}^1$-solutions of DAEs. Now we show that for two ex-equivalent (via Q and ψ) DAEs Ξ and $\tilde{\Xi}$, there exists a one-to-one correspondence between any IFJ trajectory of Ξ and that of $\tilde{\Xi}$. More specifically, any IFJ trajectory $J(\tau)$ of Ξ is mapped via ψ to an IFJ trajectory $\tilde{J}(\tau) = \psi(J(\tau))$ of $\tilde{\Xi}$ (and vice versa) since by (6) and (10), we have

$$Q(J(\tau))E(J(\tau)) \left(\frac{\partial \psi}{\partial x}(J(\tau)) \right)^{-1} \frac{\partial \psi}{\partial x}(J(\tau)) \cdot \frac{dJ(\tau)}{d\tau} = 0 \Rightarrow \tilde{E}(\psi(J(\tau))) \frac{d\psi(J(\tau))}{d\tau} = 0.$$

As a result, the impulse-jump $x_0^- \to x_0^+$ is mapped via ψ to $z_0^- = \psi(x_0^-) \to z_0^+ = \psi(x_0^+)$.

(ii) The DAE $\tilde{\Xi}$ in the new z-coordinates of Example 4.3 is easier for the analysis of impulse-free jumps since the distribution $\mathcal{E} = \ker E$ is rectified into span $\left\{ \frac{\partial}{\partial z_2} \right\}$ such that only z_2-variables are allowed to change. Observe in Figure 1(b) that for any inconsistent initial point $z_0^- = (z_{10}^-, z_{20}^-) \in \psi(U^*)$ such that $z_{10}^- < -\frac{2\sqrt{3}}{9}$, there does not exist an impulse free jump on $\psi(U^*)$ since we can not steer z_0^- into $\psi(M^*)$ on $\psi(U^*)$ without changing z_1-variables. Actually, the two DAE Ξ and $\tilde{\Xi}$ are locally ex-equivalent not only on U^*, but also on the other two connected sets $V_2 = \left\{ x \in \mathbb{R}^2 \,|\, -\frac{\sqrt{3}}{3} < x_2 < \frac{\sqrt{3}}{3} \right\}$ and $V_3 = \left\{ x \in \mathbb{R}^2 \,|\, x_2 < -\frac{\sqrt{3}}{3} \right\}$. Observe that $X = \mathbb{R}^2 = \bigcup_{i=1}^{3} \mathrm{cl}(V_i)$, by an analysis for the inconsistent initial points on V_2 and V_3, we get a semi-global result of the existence of impulse-free jump solutions for almost all points of $\mathbb{R}^2$ (except for the singular set $\left\{ x \in \mathbb{R}^2 \,|\, x_2 = \pm\frac{\sqrt{3}}{3} \right\}$). We draw the results of analysis in Figure 2 below, where the shadow area depicts the set of inconsistent initial points which admits an impulse-free jump. Note that if we allow impulse-free jumps to cross the singular set, then we may find impulse-free jumps for the inconsistent points in the white area in Figure 2, e.g., a inconsistent point $z_0^- = (1, 0)$ on Figure 2(b) can then jump upwards to $z_0^+ \approx (1, 1.3247)$, nevertheless, we may loss the uniqueness of impulse-free jumps, e.g., for any point $(0, z_{20}^-)$ with $0 < z_{20}^- < 1$, it may jump upwards to $(0, 1)$ or downwards to $(0, 0)$ or $(0, -1)$ along z_2-axis.



(a) x-coordinates

(b) z-coordinates

Figure 2: Semi-global impulse-free jump solutions of the DAE of Example 4.3 in different coordinates

In the following discussions, we will focus on impulse-free jumps in a neighborhood of a consistent point x_c to study their existence and uniqueness. Consider the jump rule (10) in Definition 4.1, the collection of all $\frac{dJ(\tau)}{d\tau}$ satisfying $E(J(\tau))\frac{dJ(\tau)}{d\tau} = 0$ is given by the differential inclusion $\frac{dJ(\tau)}{d\tau} \in \ker E(J(\tau))$. Assume that $\operatorname{rank} E(x) = const. = r$, then $\dim \ker E = const. = n - r$, we can choose locally $m = n - r$ independent vector fields $g_1, \ldots, g_m$ such that

$$\operatorname{span}\{g_1, \ldots, g_m\} = \ker E.$$

By introducing extra variables u_i, $i = 1, \ldots, m$, we can parametrize the distribution $\ker E$ and thus solutions of $\frac{dJ(\tau)}{d\tau} \in \ker E(J(\tau))$ staring from a point x_0^- are given by all solutions (corresponding to all $\mathcal{C}^0$-controls $u_i(\tau)$) of

$$\Sigma : \frac{dJ(\tau)}{d\tau} = \sum_{i=1}^{m} g_i(J(\tau))u_i(\tau), \quad x(0) = x_0^-, \tag{15}$$

which is a drift-less control system. So the existence of an IFJ solution of Ξ is equivalent to that of an input $u(\tau)$ such that the solution $J(\tau)$ of Σ staring from x_0^- can reach a consistent point $x_0^+ \in M^*$, such a problem is related to the reachability analysis of control systems.

Remark 4.5. In practical, we may interpret the u-variables in the control system Σ as some unknown forces steering the inconsistent initial value x_0^- into the consistency set S_c of the DAE Ξ The u-variables can be seen as an analogue of the Dirac impulse δ in the distributional solutions of linear DAEs (compare Remark 4.2). Note that we may solve the linear ODE (11) with $u = \delta_0$ in the sense of distribution (generalized function) while it is hard to solve $\Sigma_\delta : \frac{dx}{d\tau} = \sum_{i=1}^{m} g_i(x)\delta$, which is a nonlinear ODE with distributions (generalized function) in coefficients (see some discussions on its solutions in Chapter 3 of [32]).

Let us first recall some notions as integral manifolds, involutivity, invariant distributions from differential geometry and the reachability analysis in nonlinear control theory (see e.g. Chapter 2 of [33] and Chapter 1 of [34]). A distribution $\mathcal{D}$ is said to be *invariant* under a veter field f

if the Lie brackets $[f,g] \in \mathcal{D}$, $\forall g \in \mathcal{D}$. For a DAE $\Xi = (E,F)$, fix a consistent point $x_c \in X$, let $\mathcal{E} = \ker E = \operatorname{span}\{g_1,\dots,g_m\}$ and denote by $\langle g_1,\dots,g_m|\mathcal{E}\rangle$ the smallest invariant distribution under $g_1,\dots g_m$ which contains $\mathcal{E} = \ker E$. Then we introduce the following assumption:

(DS) there exists a neighborhood U of x_c such that the distribution $\mathcal{D} := \langle g_1,\dots,g_m|\mathcal{E}\rangle$ is nonsingular, i.e., $\dim \mathcal{D}(x) = const. = k \geq m$ for all $x \in U$.

Note that if **(DS)** is satisfied, then the distribution $\mathcal{D}$ is involutive (see Lemma 1.8.5 of [34]) and by Frobenius theorem, for any point $x_0^- \in U$, we can find a neighborhood $V \subseteq U$ of x_0^- and a coordinate transformation $z = \Phi(x) = (\phi_1(x),\dots,\phi_n(x))$, such that $\operatorname{span}\{d\phi_1,\dots,d\phi_{n-k}\} = \mathcal{D}^{\perp}$, where $\mathcal{D}^{\perp}$ denotes the co-distribution annihilating $\mathcal{D}$. The integral submanifold of the distribution $\mathcal{D}$ passing through x_0^- is given by

$$N_{x_0^-} = \left\{x \in V \mid \phi_1(x) = \phi_1(x_0^-),\dots,\phi_{n-k}(x) = \phi_{n-k}(x_0^-)\right\}.$$

Note that $N_{x_0^-} \subseteq V$ coincides with the local reachable space $R^V(x_0^-)$ of Σ from x_0^- (see Proposition 3.12 and Proposition 3.15 of [33]). Now we are ready to present our results of the existence and uniqueness of local impulse-free jumps in a neighborhood V of a consistent point $x_c \in X$ for index-1 nonlinear DAEs.

Theorem 4.6. *Consider a DAE $\Xi_{l,n} = (E,F)$ and a consistent point $x_c \in X$. Assume that the conditions* **(RE)**, **(CR)**, **(DS)** *are satisfied in a neighborhood U of x_c. Suppose that Ξ is index-1, implying (by Proposition 3.3) that $M^* = M_1^c \subsetneq U$ is a locally maximal invariant submanifold around x_c. Then for any point $x_0^- \in V \backslash M^*$ satisfying $N_{x_0^-} \cap M^* \neq \emptyset$ in a neighborhood $V \subseteq U$ of x_c, there exists an IFJ trajectory $J(\tau)$ of Ξ with*

$$J(0) = x_0^- \quad \text{and} \quad J(a) = x_0^+ \in M^* \cap N_{x_0^-},$$

where $N_{x_0^-} \subseteq V$ is the integral submanifold of the distribution $\mathcal{D} = \langle g_1,\dots,g_m| \ker E\rangle$. Moreover, the following statements are equivalent around x_c:

(i) The impulse-free jump $x_0^- \to x_0^+$ is unique.

(ii) The distribution defined by $\ker E$ is involutive.

(iii) Ξ is locally on V ex-equivalent to the following index-1 nonlinear Weierstrass form

$$(\mathbf{INWF}): \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} F^*(\xi_1) \\ \xi_2 \end{bmatrix}, \tag{16}$$

where $M^* \cap V = \{\xi \in V \mid \xi_2 = 0\}$, $\xi = (\xi_1,\xi_2)$ and ξ_1 is a system of coordinates on $M^* \cap V$.

The proof is given in Section 6.

Remark 4.7. (i) Theorem 4.6 is a local result for the existence and uniqueness of impulse-free jumps. The neighborhood V of x_c is the set in which Ξ can be normalized (via ex-equivalence) to the simplified form (9) or (16). Note that not all the inconsistent points x_0^- on $V \backslash M^*$ but only the ones satisfying $R^V(x_0^-) \cap M^* \neq \emptyset$ admits an impulse-free jump to $x_0^+ \in N_{x_0^-} \cap M^*$. So the set of points from which there exists an impulse-free jump is a subset of V, which we will call the local *admissible impulse-free jump set* around x_c. Note, however, that the results of Theorem 4.6 can be used to analyze "global" impulse-free jumps in some cases as we have discussed in Remark 4.4(ii), where the shadow areas (the three unconnected sets) in Figure 2 are the admissible sets of impulse-free jumps in the neighborhoods V_1, V_2, V_3, respectively.

(ii) For a DAE Ξ, being index-1 is not a necessary condition for the existence of impulse-free jumps. Take the following DAE for example: $\Xi : \begin{bmatrix} 0 & x & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$, which is of geometric index-2 since $M^* = M_2^c = \{x = y = z = 0\}$. Let $u = u(\tau)$ be an input which stabilizes $\begin{bmatrix} \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x \end{bmatrix} u$, such $u(\tau)$ always exists since the system is controllable (and thus stabilizable). Then for any inconsistent initial point $(x_0^-, y_0^-, z_0^-) \notin M^*$, there exists a solution $J(\tau) = (x(\tau), y(\tau), z(\tau))$ such that $J(0) = (x_0^-, y_0^-, z_0^-)$ and $(x(\mathcal{T}), y(\mathcal{T}), z(\mathcal{T})) = (0, 0, 0)$ for some $\mathcal{T} > 0$. Clearly, $J(\tau)$ is an IFJ trajectory of Ξ and $(x_0^-, y_0^-, z_0^-) \to (0, 0, 0)$ is an impulse-free jump.

For a linear regular DAE $\Delta_{n,n} = (E, H)$, its consistency projector [16, 35] is defined by

$$\Pi_{E,H} := P^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} P,$$

where the dimension n_1 and the matrix P comes from the **(WF)** of Δ, given by (7). We now generalize the above notion of consistency projector to nonlinear DAEs with the help of the **(INWF)**, given by (16).

Definition 4.8 (nonlinear consistency projector)**.** Consider a DAE $\Xi_{l,n} = (E, F)$ and a consistent point $x_c \in X$. Assume that there exists a neighborhood V of x_c such that Ξ is locally (on V) ex-equivalent to the **(INWF)** of (16) via a Q-transformation and a local diffeomorphism ψ. The nonlinear (local) *consistency projector* $\Omega_{E,F} : V \backslash M^* \to V \cap M^*$ of Ξ is then defined by

$$\Omega_{E,F} := \psi^{-1} \circ \pi \circ \psi,$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection $(\xi_1, \xi_2) \mapsto (\xi_1, 0)$.

For a linear DAE Δ, any inconsistent initial value x_0^- of Δ jumps to $x_0^+ = \Pi x_0^-$ and the jump $x_0^- \to x_0^+$ is impulse-free, i.e., $e_0 = x_0^+ - x_0^-$, if and only if $E(I - \Pi) = 0$ (compare Theorem 3.8 of [16]), which actually is equivalent to that Δ is index-1. For a nonlinear DAE, in order that the existence and uniqueness of impulse-free jumps are satisfied, we need both that Ξ is index-1 and that $\ker E$ is involutive, as seen from the following corollary.

Corollary 4.9. *Consider a DAE $\Xi_{l,n} = (E, F)$ and a consistent point $x_c \in X$. Assume that the conditions* **(RE)** *and* **(CR)** *are satisfied in an open neighborhood U of x_c. Then there exists a neighborhood $V \subseteq U$ of x_c such that for any inconsistent initial point $x_0^- \in V/M^*$ satisfying $N_{x_0^-} \cap M^* \neq \emptyset$, there exists a unique impulse-free jump $x_0^- \to x_0^+$ if and only if Ξ is index-1 and $\mathcal{E} = \ker E$ is involutive. Let $\Omega_{E,F}$ be the consistency projector of Ξ defined on V, the unique impulse-free jump is given by*

$$x_0^- \to x_0^+ = \Omega_{E,F}(x_0^-) \in M^* \cap N_{x_0^-}.$$

Proof. "*Only if.*" Suppose that $x_0^- \to x_0^+$ is unique, that is, $N_{x_0^-} \cap M^*$ is a unique point x_0^+ on M^*. It follows that $\dim(M^* \cap N_{x_0^-}) = 0$, which implies that

$$T_{x_0^+} M^* \cap T_{x_0^+} N_{x_0^-} = T_{x_0^+} M^* \cap \ker E(x_0^+) = 0. \tag{17}$$

Thus we have that Ξ is index-1 by Proposition 3.3. Hence by Theorem 4.6, the impulse-free jump is unique implies that $\mathcal{E} = \ker E$ is involutive.

"*If.*" Suppose that the distribution $\mathcal{E} = \ker E$ is involutive, then the condition $\operatorname{rank} E(x) = const.$ of **(CR)** implies that $\mathcal{D} = \langle g_1, \ldots, g_m | \mathcal{E} \rangle = \ker E$ is nonsingular (i.e., **(DI)** holds). Suppose additionally that Ξ is index-1, then by Theorem 4.6, Ξ is ex-equivalent to the **(INWF)**, given by (16), and there exists a unique impulse-free jump

$$x_0^- = \psi^{-1}(\xi_0^-) \to x_0^+ = \psi^{-1}(\xi_0^+) \in M^* \cap N_{x_0^-},$$

where $\xi_0^+ = \pi(\xi_0^-)$ since for the **(INWF)**, only ξ_2-variables are allowed to jump. It follows that $x_0^+ = \psi^{-1} \circ \pi \circ \psi(x_0^-) = \Omega(x_0^-)$. ☐

Example 4.10. We reconsider the DAE Ξ, given by (13), of Example 4.3. It is clear that Ξ is of index-1 and the distribution $\mathcal{E} = \ker E =$ is involutive. We have that Ξ with the initial point $x_0^- = (1, 1)$ is locally ex-equivalent to its **INWF** represented by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -f(\xi_1, 0) \\ \xi_2 \end{bmatrix}, \ \xi_0^- = \psi(x_0^-) = (1, 1), \tag{18}$$

on $V = U^* = \left\{ x \in \mathbb{R}^2 \,|\, x_1 = 0, x_2 > \frac{\sqrt{3}}{3} \right\}$, via $\psi = \xi = (\xi_1, \xi_2) = (x_1 + x_2^3 - x_2, x_1)$ and $Q = \left[\begin{smallmatrix} 1 & f' \\ 0 & 1 \end{smallmatrix}\right]$, where $f(\xi) = f(\xi_1, 0) + f'(\xi)\xi_2$ and

$$f = \frac{1}{3}\left(a + \sqrt{a^2 - \frac{1}{27}}\right)^{-\frac{1}{3}} + \left(a + \sqrt{a^2 - \frac{1}{27}}\right)^{\frac{1}{3}}, \quad a(\xi_1, \xi_2) = \frac{\xi_1 - \xi_2}{2}.$$

Thus the nonlinear (local) consistency projector of Ξ is

$$\Omega = \psi^{-1} \circ \pi \circ \psi = \begin{bmatrix} 0 \\ f(x_1 + x_2^2 - x_2, 0) \end{bmatrix}.$$

Hence $x_0^+ = \Omega(x_0^-) \approx (0, 1.3247)$, which agrees with the result of Example 4.3.

5. Singular perturbed system approximation of nonlinear DAEs

The singular perturbation theory was frequently used (see e.g., [6, 11, 13, 20]) to approximate DAEs of the semi-explicit form (3), the main idea is to regularize a DAE Ξ^{SE} of the form (3) by replacing the algebraic constraint $0 = f_2(x_1, x_2)$ with $\epsilon\dot{x}_2 = f_2(x_1, x_2)$, where ϵ represents some ignored small modeling parameters (e.g, the small inductance of an inductor, see the electric circuits in page 367 of [13]). Then by rescaling time t to τ such that $\frac{d\tau}{dt} = \frac{1}{\epsilon}$, we get a perturbed system in the time-scale τ as shown on the right-hand side of the following equations.

$$\Xi_\epsilon^{SE} : \begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \epsilon\dot{x}_2 = f_2(x_1, x_2). \end{cases} \overset{\epsilon=\frac{dt}{d\tau}}{\Leftrightarrow} \begin{cases} \dfrac{dx_1}{d\tau} = \epsilon f_1(x_1, x_2), \\ \dfrac{dx_2}{d\tau} = f_2(x_1, x_2). \end{cases}$$

Note that there are, in general, two assumptions in the above approximation method of DAEs: (a) $\frac{df_2}{dx_2}$ is invertible (which is actually equivalent to that Ξ^{SE} is index-1); (b) the so-called boundary layer model $\frac{dx_2}{d\tau} = f_2(x_{10}^-, x_2)$ is asymptotically stable uniformly in x_2. Then under the assumptions (a),(b), the well-known Tihkonov's theorem (see e.g., [20] and a similar result in Theorem III.1 of [11]) states that if a unique solution $(x_1(t), x_2(t))$ of Ξ^{SE} starting from a consistent initial point (x_{10}^+, x_{20}^+) exists on the interval $I = [0, \alpha)$, then there exists $\delta \geq 0$ such that a solution $(\bar{x}_1(t,\epsilon), \bar{x}_2(t,\epsilon))$ of Ξ_ϵ^{SE} starting from any point (x_{10}^-, x_{20}^-) with $||x_{10}^+ - x_{10}^-|| + ||x_{20}^+ - x_{20}^-|| < \delta$ satisfy

$$\lim_{\epsilon\to 0} ||x_1(t) - \bar{x}_1(t,\epsilon)|| = 0, \quad \text{and} \quad \lim_{\epsilon\to 0} ||x_2(t) - \bar{x}_2(t,\epsilon)|| = 0,$$

on all closed subintervals of I. In this section, we will propose a singular perturbed system approximation for index-1 nonlinear DAEs Ξ with the help of the results in Proposition 3.3 and Theorem 4.6.

Definition 5.1 (singular perturbed system)**.** Consider a DAE $\Xi_{l,n} = (E, F)$ and fix a consistent point x_c. Assume that there exists a neighborhood V of x_c such that Ξ is locally (on V) ex-equivalent to the DAE (9) (or in particular, the **(INWF)** of (16)) via a Q-transformation and a local diffeomorphism ψ. Define the following singular perturbed system on V:

$$\Xi_\epsilon : \dot{x} = E_W^{-1}(x,\epsilon)F(x) \text{ with } E_W(x,\epsilon) = E(x) + Q^{-1}(x)\begin{bmatrix} 0 & 0 \\ 0 & -\epsilon W^{-1} \end{bmatrix}\frac{\partial\psi(x)}{\partial x}, \tag{19}$$

where $W = \operatorname{diag}\{w_1, \ldots, w_m\} \in GL(m, \mathbb{R})$ is a positive-define weight matrix with $w_i > 0$ for $i = 1, \ldots, m$. Then by rescaling time-scale t to τ, where $\frac{d\tau}{dt} = \frac{1}{\epsilon}$, we define

$$\frac{dx}{d\tau} = \epsilon E_W^{-1}(x,\epsilon)F(x). \tag{20}$$

Remark 5.2. Any linear index-1 regular DAE $\Delta = (E, H)$ of the form (2) is always ex-equivalent, via two constant matrices Q and P, to the **(WF)** of (7) with $N = 0$, i.e., $QEP^{-1} = \left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $QHP^{-1} = \left[\begin{smallmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{smallmatrix}\right]$. Applying the construction of (19) to Δ and setting $W = I_{n_2}$, we get the following singular perturbed system:

$$\Delta_\epsilon : \dot{x} = E(\epsilon)^{-1}Hx = P^{-1}\left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & -\frac{1}{\epsilon}I_{n_2} \end{smallmatrix}\right]QHx = P^{-1}\left[\begin{smallmatrix} A_1 & 0 \\ 0 & -\frac{1}{\epsilon}I_{n_2} \end{smallmatrix}\right]Px,$$

where $E(\epsilon) = Q^{-1}\left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & -\epsilon I_{n_2} \end{smallmatrix}\right] P$. Note that the above perturbed linear system Δ_ϵ is proposed in Section IV of [36] as an ODE approximation of the linear DAE Δ.

The following theorem shows that the solution $\bar{x}(t,\epsilon)$ of the proposed perturbed system Ξ_ϵ of (19) coincides with the $\mathcal{C}^1$-solution $x(t)$ of Ξ when staring from a consistent point x_0^+ and that the solution $\bar{x}(\tau,\epsilon)$ of Ξ_ϵ in the rescaled time-scale τ, given by (20), converges to an impulse-free jump trajectory $J(\tau)$ of Ξ when staring from an inconsistent point x_0^-.

Theorem 5.3. *Consider a DAE $\Xi_{l,n} = (E,F)$ and a consistent point $x_c \in X$. Assume that the conditions* **(RE)**, **(CR)**, **(DS)** *are satisfied in a neighborhood U of x_c. Suppose that Ξ is index-1, implying that there exists a neighborhood $V \subseteq U$ of x_c such that Ξ is locally (on V) ex-equivalent to the DAE (9) via Q and ψ. Fix an inconsistent initial point $x_0^- \in V/M^*$ such that $N_{x_0^-} \cap M^* \neq \emptyset$, if a solution $\bar{x}_W(\tau,\epsilon) : [0,+\infty) \to V$ of the perturbed system (20) starting from x_0^- exists, then there exists an IFJ solution $J(\tau) : [0,+\infty) \to V$ of Ξ starting from $J(0) = x_0^-$ such that*

$$\lim_{\epsilon\to 0} ||\bar{x}_W(\tau,\epsilon) - J(\tau)|| = 0, \quad \forall \tau \in [0,+\infty). \tag{21}$$

The consistent point

$$x_0^+ = J(+\infty) = \lim_{\epsilon\to 0} \bar{x}_W(+\infty,\epsilon) \in M^* \cap V$$

is not unique and depends on the choice of the weight matrix W. Moreover, if we additionally assume that the distribution $\mathcal{E} = \ker E$ is involutive, implying that Ξ is locally (on V) ex-equivalent to the (**INWF**) *of (16), then the consistent point $x_0^+ = J(+\infty)$ is unique, independently of the choice of W (actually $x_0^+ = \Omega_{E,F}(x_0^-)$ by Corollary 4.9).*

Furthermore, the solution $\bar{x}(t) : I \to M^* \cap V$ of the perturbed system (19) starting from any consistent initial point on M^* coincides with the $\mathcal{C}^1$-solution $x(t)$ of Ξ, which does not depend on ϵ and W.

The proof is given is Section 6.

Remark 5.4. The coefficients $w_i > 0$ of the weigh matrix $W = \operatorname{diag}\{w_1,\ldots,w_m\}$ in Theorem 5.3 are parameters indicating the rate of convergence of $J(\tau) \to x_0^+$ as $\tau \to \infty$. As seen from (31) below, the solution of the ξ_2^i-subsystem is $\xi_2^i(\tau) = e^{-w_i\tau}$, so w_i is the rate of convergence for $\xi_2^i(\tau) \to 0$. Recall that an IFJ solution of Ξ can be seen as a solution of a control system $\frac{dJ(\tau)}{d\tau} = \sum_{i=1}^m g_i(J(\tau))u_i(\tau) = g(J(\tau))u(\tau)$ (see (15)), thus the choice of W can be regarded as some particular choices of the inputs u_i, e.g., we have that $g = \left[\begin{smallmatrix} E_2 \\ I_m \end{smallmatrix}\right]$ and $u(\tau) = W\xi_2(\tau)$ for (31), so $u(\tau)$ is a particular feedback which stabilizes the ξ_2-subsystem. As a consequence, the solutions $\bar{x}_W(\tau,\epsilon)$ corresponding to all W-matrices may not approximate all the possible impulse-free jumps, meaning that the set of all $x_0^+ = \lim\limits_{\epsilon\to 0} \bar{x}_W(\infty,\epsilon)$ corresponding to all W-matrices is a subset of $M^* \cap N_{x_0^-}$, the latter is the set of all points which can be jumped into from x_0^-, see Theorem 4.6.

Example 5.5. Consider the electrical circuit shown in Figure 3 below, which consists of a capacitor C, a nonlinear resistor N as the simple circuit discussed in [11–13]. A controlled current source S is additionally connected in parallel with N in order to generate nonlinear terms in $E(x)$ of the DAE model. Note that controlled current sources have been used in [37] for electric circuits analog of mechanical systems under non-holonomic constraints. The relations between the current $i_N = x$ and

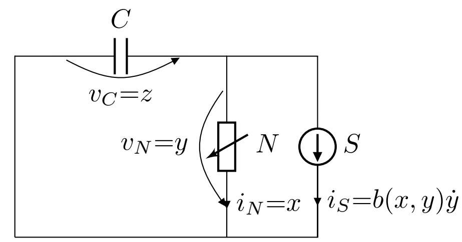


Figure 3: An electric circuit with nonlinear resistor and controlled current source

the voltage $v_n = y$ of the nonlinear resistor N is characterized by the following algebraic equation

$$0 = a(x, y),$$

and the current i_S of S is equal to $b(x, y)\dot{y}$, where $a : \mathbb{R}^2 \to \mathbb{R}$ and $b : \mathbb{R}^2 \to \mathbb{R}$ are smooth maps . Using Kirchoff's law, we model the circuit as a DAE $\Xi_{3,3} = (E, F)$:

$$\begin{bmatrix} 0 & -b(x,y) & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x \\ y+z \\ a(x,y) \end{bmatrix}.$$

Let $\eta = (x, y, z)$ and $\eta_c = (0, 0, 0)$, we consider two different cases, for which the distribution $\mathcal{E} = \ker E$ is involutive in Case 1 but is not in Case 2.

Case 1: Consider $a(x, y) = x - y^2 - 2y$, $b(x, y) = y$, $C = 1$, the conditions **(RE)**, **(CR)** are satisfied on $U = \{\eta \in \mathbb{R}^3 \,|\, y < 1\}$ (note that $\dim E(\eta)T_\eta M_1^c = 0$ for $y = 1$). The locally maximal invariant submanifold M^* (around η_c) is

$$M^* = M_1^c = \left\{\eta \in U \,|\, y + z = x - y^2 - 2y = 0\right\}.$$

Since $\mathcal{E} = \ker E = \mathrm{span}\{\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\}$ is involutive and Ξ is of index-1, the DAE Ξ is locally (on $V = U$) ex-equivalent to the following DAE represented in **(INWF)**:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{y}} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} -2\tilde{z} \\ \tilde{y} \\ \tilde{x} \end{bmatrix}. \tag{22}$$

via

$$Q = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \psi = \tilde{\eta} = (\tilde{z}, \tilde{y}, \tilde{x}) = (-\frac{1}{2}y^2 + z, y + z, x - y^2 - 2y).$$

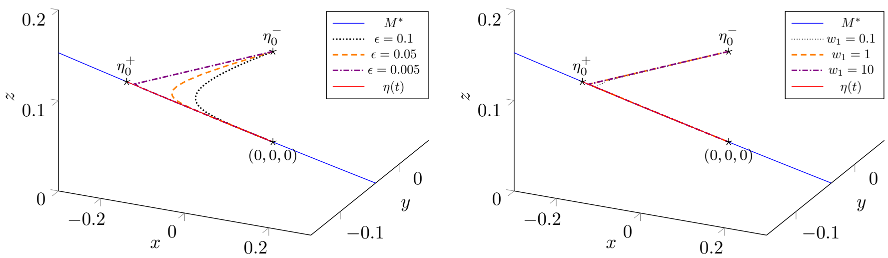
Following (19) of Definition 5.1, we construct a singular perturbed system Ξ_ϵ:

$$Q^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\epsilon}{w_1} & 0 \\ 0 & 0 & -\frac{\epsilon}{w_2} \end{bmatrix} \frac{\partial \psi}{\partial \eta} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x \\ y+z \\ x-y^2-2y \end{bmatrix} \Rightarrow \Xi_\epsilon : \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f_1(\eta,\epsilon,w_1,w_2) \\ f_2(\eta,\epsilon,w_1,w_2) \\ f_3(\eta,\epsilon,w_1,w_2) \end{bmatrix},$$

where $f_1 = -\frac{-w_2x+w_2y(2+y)-2\epsilon(y^2-2z)-2w_1(y+z)}{\epsilon}$, $f_2 = -\frac{w_1y+\epsilon y^2-2\epsilon z+w_1z}{\epsilon+\epsilon y}$, $f_3 = \frac{\epsilon(y^2-2z)-w_1y(y+z)}{\epsilon(1+y)}$. Consider an inconsistent initial point $\eta_0^- = (0,0,0.1) \in V\backslash M^*$, by Corollary 4.9, we get

$$\eta_0^+ = \Omega_{E,F}(\eta_0^-) = \psi^{-1}\circ\pi\circ\psi(\eta_0^-) = (-0.2,-0.1056,0.1056),$$

which defines the unique impulse-free jump $\eta_0^- \to \eta_0^+$ of Ξ. Now we use MATLAB ode45 solver to simulate the solutions $\bar{\eta}_W(t,\epsilon)$ of the perturbed system Ξ_ϵ for different ϵ, w_1 and w_2. First, we fix



(a) Fix $w_1 = 1$, $w_2 = 1$ and change ϵ. (b) Fix $\epsilon = 0.001$, $w_2 = 1$ and change w_1.

Figure 4: The solutions $\bar{\eta}_W(t,\epsilon)$ of Ξ_ϵ with different parameters and the solution $\eta(t)$ of Ξ.

$w_1 = 1$ and $w_2 = 1$, and change ϵ from 0.1 to 0.005, as seen from Figure 4(a) that the solution $\bar{\eta}_W(t,\epsilon)$ of Ξ_ϵ approaches the impulse-free jump $\eta_0^- \to \eta_0^+$ of Ξ closer as the perturbation parameter ϵ gets smaller, which agrees with the result (21) of Theorem 5.3. Then we fix $w_2 = 1$ and $\epsilon = 0.001$, and change w_1 from 0.1 to 10, it is seen from Figure 4(b) that $\bar{\eta}_W(t,\epsilon)$ approaches the same jump $\eta_0^- \to \eta_0^+$ independently from the choice of w_1, which also agree with the results of Theorem 5.3. Note that $\bar{\eta}_W(t,\epsilon)$ coincides the C^1-solution $\eta(t)$ of Ξ on M^*, which converges to $(0,0,0)$ indicating that Ξ is asymptotically stable.

Case 2: Consider $a(x,y) = x - y^3$, $b(x,y) = x$, $C = 1$, then the conditions **(RE)**, **(CR)**, **(DS)** are satisfied on $U = \{\eta \in \mathbb{R}^3 \,|\, x > -1\}$. The locally maximal invariant submanifold M^* (around η_c) is

$$M^* = M_1^c = \left\{\eta \in U \,|\, y + z = x - y^3 = 0\right\}.$$

The distribution $\mathcal{E} = \ker E$ is *not* involutive but the DAE Ξ is index-1. By Proposition 3.3, Ξ is locally (on $V = U$) ex-equivalent to the following DAE of the form (9):

$$\begin{bmatrix} 1 & -\frac{x}{1+x} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{y}} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} \frac{x}{1+x} \\ \tilde{y} \\ \tilde{x} \end{bmatrix}, \tag{23}$$

where $x = \tilde{x}+\tilde{y}(\tilde{y}^2-3\tilde{y}\tilde{z}+3\tilde{z}^2)$. Then we construct the singular perturbed system Ξ_ϵ by Definition 5.1 to get

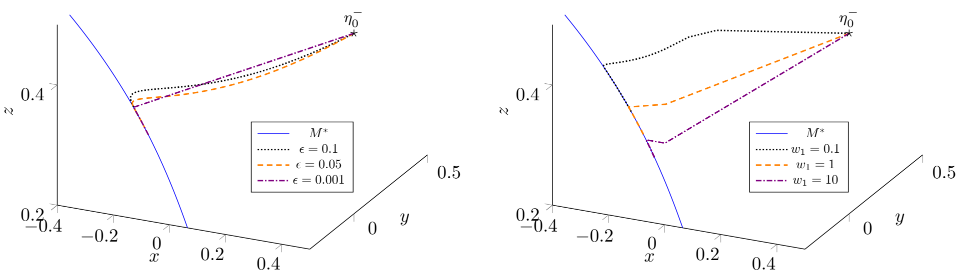
$$\Xi_\epsilon : \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f_1(\eta,\epsilon,w_1,w_2) \\ f_2(\eta,\epsilon,w_1,w_2) \\ f_3(\eta,\epsilon,w_1,w_2) \end{bmatrix},$$

where $f_1 = \frac{w_2(-x^2+(x+1)(y^3-1))-3y^2(\epsilon x+w_1(y+z))}{\epsilon(1+x)}$, $f_2 = -\frac{w_1y+\epsilon x+w_1z}{\epsilon(1+x)}$, $f_3 = \frac{x(\epsilon-w_1(y+z))}{\epsilon(1+x)}$. Consider an inconsistent initial point $\eta_0^- = (0.5,0,0.5) \in V\backslash M^*$, by Theorem 4.6, the impulse-free jump

$\eta_0^- \to \eta_0^+$ is *not* unique and $\eta_0^+ \in M^* \cap N_{\eta_0^-}$, where $N_{\eta_0^-}$ is the integral submanifold of the distribution $\mathcal{D} = \langle g_1, \dots, g_m | \mathcal{E} \rangle$. In this example, $\mathcal{E} = \ker E = \operatorname{span}\{g_1, \dots, g_m\}$, where $g_1 = \frac{\partial}{\partial x}$, $g_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ and thus

$$\mathcal{D} = \operatorname{span}\{g_1, g_2, [g_1, g_2]\} = T_\eta U,$$

it follows that $N_{\eta_0^-} = U$ and that $\eta_0^+ \in M^* \cap N_{\eta_0^-}$ can be any point on M^*. Then we implement a similar simulation for the solution $\bar{\eta}_W(t, \epsilon)$ of Ξ_ϵ as in Case 1 to get Figure 5 below. Figure 5(a)



(a) Fixed $w_1 = 1$, $w_2 = 1$ and variant ϵ.

(b) Fixed $\epsilon = 0.01$, $w_2 = 1$ and variant w_1.

Figure 5: Trajectories $\eta_W(t, \epsilon)$ of the perturbed system Ξ_ϵ with different values of parameters

contains similar messages as Figure 4(a): the solution $\bar{\eta}_W(t, \tau)$ approaches closer to an impulse-free jump of Ξ as $\epsilon \to 0$. Nevertheless, as seen Figure 5(b), the impulse-free jump $\eta_0^- \to \eta_0^+$ approximated by $\bar{\eta}_W(t, \epsilon)$ is not unique and depends on w_1 (and thus on W), which verifies the results of Theorem 5.3 for DAEs with non-involutive $\ker E$. Observe that the ex-equivalent DAE (23) restricted to $\psi(M^*) = \{\tilde{\eta} \,|\, \tilde{x} = \tilde{y} = 0\}$ is $\dot{\tilde{z}} = 0$, so the $\mathcal{C}^1$-solution of Ξ is the initial consistent point $\eta(t) = \eta(0) = \eta_0^+$. Hence the solution $\bar{\eta}_W(t, \tau)$ of the perturbed system Ξ_ϵ on M^* will become a fixed point as $\epsilon \to 0$.

6. Proofs of the results

Proof of Proposition 3.3. Note that our DAE Ξ is square by $l = n$ of **(RE)**. Following (4), we have (notice that $E_1(x)$ is of full row rank r)

$$\begin{aligned} M_1^c = M_1 \cap U &:= \{x \in U \,|\, QF(x) \in \operatorname{Im} QE(x)\} = \left\{ x \in U \,\middle|\, \left[\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right] \in \operatorname{Im} \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right] \right\} \\ &= \{x \in U \,|\, F_2(x) = 0\}. \end{aligned} \tag{24}$$

$(i) \Rightarrow (ii)$: It is a direct consequence of Definition 3.1 and Proposition 2.3.

$(ii) \Leftrightarrow (iii)$: Suppose that $M^* = M_1^c$ is locally maximal invariant. Since Ξ is locally internally regular (the condition **(RE)**), we have that $\dim E(x) T_x M_1^c = \dim M_1^c$, $\forall x \in M_1^c$ by (5). Observe that $F_2 : U \to \mathbb{R}^{n-r}$ and $\operatorname{rank} \mathrm{D}F_2(x) = const. \leq n - r$, $\forall x \in M_1^c$. It follows that $\dim M_1^c =$

$n - \operatorname{rank} \mathrm{D}F_2 \geq n-(n-r) = r = \operatorname{rank} E(x)$. We conclude that $\operatorname{rank} E(x) = \dim E(x) T_x M_1^c$, $\forall x \in M_1^c$ by

$$\operatorname{rank} E(x) = r \leq \dim M_1^c = \dim E(x) T_x M_1^c \leq \operatorname{rank} E(x), \quad \forall x \in M_1^c. \tag{25}$$

Conversely, suppose that $\operatorname{rank} E(x) = \dim E(x) T_x M_1^c$, $\forall x \in M_1^c$, which implies that $\dim E_1(x) T_x M_1^c = \operatorname{rank} E_1(x)$, where E_1 comes from (24). It follows that $F_1(x) \in E_1(x) T_x M_1^c$, $\forall x \in M_1^c$. Observe that $F_2(x) = 0$, $\forall x \in M_1^c$, thus

$$M_2^c = M_2 \cap U = \left\{ x \in M_1^c \;\middle|\; \left[\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right] T_x M_1^c \right\} = M_1^c.$$

Then we conclude that $M^* = M_1^c$ is a locally maximal invariant submanifold by Proposition 2.3. Notice that the inequality $\dim M_1^c = \dim T_x M_1^c \geq \operatorname{rank} E(x)$ always holds for $x \in M_1^c$ (since $\operatorname{rank} \mathrm{D}F_2(x) \leq n-r$), hence $\ker E(x) \cap T_x M_1^c = 0$ if and only if $\operatorname{rank} E(x) = \dim E(x) T_x M_1^c$.

(iii) $\Rightarrow$ *(iv)*: Suppose that item *(iii)* holds. The equivalence of *(ii)* and *(iii)* implies that $\dim M_1^c = \operatorname{rank} E(x) = r$, $\forall x \in M_1^c$. Thus $\operatorname{rank} \mathrm{D}F_2(x) = n - \dim M_1^c = n-r$, i.e., $\mathrm{D}F_2(x)$ is of full row rank for all $x \in M_1^c$. Now by $\ker \mathrm{D}F_2(x) = T_x M_1^c$ and $\ker E(x) \cap T_x M_1^c = 0$, $\forall x \in M_1^c$, it follows that $\operatorname{rank} \mathrm{D}F_2(x) Z(x) = \operatorname{rank} \mathrm{D}F_2(x) = n-r$ and $\operatorname{rank} \left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right] = \operatorname{rank} E_1(x) + \operatorname{rank} \mathrm{D}F_2(x) = n$, $\forall x \in M_1^c$. Hence $\mathrm{D}F_2(x) Z(x)$ and $\left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right]$ are invertible for all $x \in M_1^c$.

(iv) $\Rightarrow$ *(v)*: Suppose that the matrix $A = \mathrm{D}F_2(x) Z(x)$ or $B = \left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right]$ is invertible for all $x \in M_1^c$. It follows that $\mathrm{D}F_2(x)$ is of full row rank, i.e., $\operatorname{rank} \mathrm{D}F_2(x) = n-r = m$, $\forall x \in U$. Let $\xi_2 = F_2$, then there exist a neighborhood $U_1 \subseteq U$ of x_c and a smooth map $\xi_1 : U_1 \to \mathbb{R}^r$ such that $\psi(x) = (\xi_1(x), \xi_2(x))$ is a local diffeomorphism on U_1. Thus Ξ is ex-equivalent (via Q and ψ) to

$$Q(x)E(x) \left(\frac{\partial \psi(x)}{\partial x} \right)^{-1} \frac{\partial \psi(x)}{\partial x} \dot{x} = Q(x)F(x) \Leftrightarrow \begin{bmatrix} E_1^1(\xi) & E_1^2(\xi) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(\xi) \\ \xi_2 \end{bmatrix}$$

where $E_1^1 : U_1 \to \mathbb{R}^{r \times r}$, $[E_1^1 \circ \psi \;\; E_1^2 \circ \psi] = E_1$ and $\tilde{F}_1 \circ \psi = F_1$. Observe that $A(x)$ or $B(x)$ is invertible implies $\ker E(x_c) \cap T_{x_c} M_1^c = 0$ since $\ker E(x_c) = \operatorname{Im} Z(x_c)$ and $\ker \mathrm{D}F_2(x_c) = T_{x_c} M_1^c$. Thus $\operatorname{rank} E_1^1(\psi(x_c)) = \dim E(x_c) T_{x_c} M_1^c = r$, i.e., $E_1^1(\psi(x_c))$ is invertible. Then by the smoothness of E_1^1, there exists a neighborhood $U_2 \subseteq U_1$ such that $E_1^1(\psi(x))$ is invertible $\forall x \in U_2$. Define $Q_1 := \left[\begin{smallmatrix} (E_1^1)^{-1} & 0 \\ 0 & I_m \end{smallmatrix}\right]$, then via the $Q_1 Q$-transformation and the diffeomorphism $\xi = \psi(x)$, Ξ is locally (on $V = U_2$) ex-equivalent to (9), with $E_2 = (E_1^1)^{-1} E_1^2$ and $F^* = (E_1^1)^{-1} \tilde{F}_1$.

(v) $\Rightarrow$ *(i)*: Note that the sequence of submanifolds M_k^c constructed by (4) is invariant under the ex-equivalence, i.e., for two ex-equivalent DAEs Ξ and $\tilde{\Xi}$, the submanifolds M_k^c of Ξ and $\tilde{M}_k^c$ of $\tilde{\Xi}$ satisfies $\tilde{M}_k^c = \psi(M_k^c)$. Thus the geometric index ν_g, which depends only on the sequence M_k^c, is also invariant under the ex-equivalence. By a direct calculation of the submanifolds M_1^c and M_2^c for (9), it is seen that (9) is index-1. Hence, the DAE Ξ, being ex-equivalent to (9), is also index-1. $\square$

Proof of Theorem 4.6. As Ξ is of index-1, there exists a neighborhood $V \subseteq U$ of x_c such that Ξ is locally ex-equivalent (via a diffeomorphism ψ and a Q-transformation) to the DAE (9) on V.

Note that for any point $x_0^- \in V\backslash M^*$, we have that $\xi_0^- = (\xi_{10}^-, \xi_{20}^-) = \psi(x_0^-)$ satisfies $\xi_{20}^- \neq 0$ since $M^* \cap V = \{\xi \in V \,|\, \xi_2 = 0\}$ and that M^* locally coincides with the consistent set S_c on V by Proposition 2.3. Then consider the following control system defined on V with a vector of inputs $u \in \mathcal{C}^0$,

$$\begin{bmatrix} \frac{d\xi_1}{d\tau} \\ \frac{d\xi_2}{d\tau} \end{bmatrix} = \sum_{i=1}^{m} \tilde{g}_i(\xi)u_i = \begin{bmatrix} -E_2(\xi) \\ I_m \end{bmatrix} u, \quad \xi(0) = \xi_0^- = (\xi_{10}^-, \xi_{20}^-), \tag{26}$$

where $\operatorname{span}\{\tilde{g}_1 \circ \psi, \ldots, \tilde{g}_m \circ \psi\} = \ker(\tilde{E} \circ \psi) = \frac{\partial \psi}{\partial x} \ker E$. By the condition **(DS)**, the distribution $\tilde{\mathcal{D}} = \left\langle \tilde{g}_1, \ldots, \tilde{g}_m | \ker \tilde{E} \right\rangle$ is involutive, thus there exist $\tilde{\phi}_i : V \to \mathbb{R}$, $i = 1, \ldots, k$, such that $\operatorname{span}\left\{ d\tilde{\phi}_1, \ldots, d\tilde{\phi}_{n-k} \right\} = \tilde{\mathcal{D}}^{\perp}$. Then let $\tilde{\xi}_1 = (\tilde{\phi}_1, \ldots, \tilde{\phi}_k)$, it is directly seen from (26) that $\operatorname{span}\{d\xi_2\} \cap \tilde{\mathcal{D}}^{\perp} = 0$ and thus $d\tilde{\xi}_1$ and $d\xi_2$ are linearly independent. By taking a smaller V, if necessary, we can choose new local coordinates $\bar{\xi} = (\tilde{\xi}_1, \bar{\xi}_1, \xi_2)$ on V, where $\bar{\xi}_1 = (\tilde{\phi}_{n-k+1}, \ldots, \tilde{\phi}_{n-m})$ is chosen such that $\tilde{\Phi}(\xi) = (\tilde{\phi}_1(\xi), \ldots, \tilde{\phi}_{n-m}(\xi), \xi_2)$ is a local diffeomorphism. Then under the new local $\bar{\xi}$-coordinates, the control system (26) becomes

$$\begin{bmatrix} \frac{d\tilde{\xi}_1}{d\tau} \\ \frac{d\bar{\xi}_1}{d\tau} \\ \frac{d\xi_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{E}_1(\bar{\xi}) \\ I_m \end{bmatrix} u, \quad \bar{\xi}(0) = \tilde{\Phi}(\xi_0^-) = (\tilde{\xi}_{10}^-, \bar{\xi}_{10}^-, \xi_{20}^-) \in V\backslash M^*, \tag{27}$$

where $\bar{E}_1 : V \to \mathbb{R}^{(k-m)\times m}$. Note that by Proposition 3.12 and Proposition 3.15 of [33], system (27) restricted to $N_{\bar{\xi}_0^-} = \left\{ \bar{\xi} \in V \,|\, \tilde{\xi}_1 = \tilde{\xi}_{10}^- \right\}$ is controllable. It follows that for any $\bar{\xi}_0^- = (\tilde{\xi}_{10}^-, \bar{\xi}_{10}^-, \xi_{20}^-) \in V\backslash M^*$ with $N_{\bar{\xi}_0^-} \cap M^* = \left\{ \bar{\xi} \in V \,|\, \tilde{\xi}_1 = \tilde{\xi}_{10}^-, \ \xi_2 = 0 \right\} \neq \emptyset$, there exist $u = u(\tau)$ and $\mathcal{T} > 0$ such that the $\mathcal{C}^1$-solution $\bar{\xi}(\tau)$ of (27) under the input $u = u(\tau)$ satisfies $\bar{\xi}(0) = \bar{\xi}_0^-$ and $\bar{\xi}(\mathcal{T}) = \bar{\xi}_0^+ = (\tilde{\xi}_{10}^+, \bar{\xi}_{10}^+, \xi_{20}^+) \in M^* \cap N_{x_0^-}$, i.e., $\tilde{\xi}_{10}^+ = \tilde{\xi}_{10}^-$, $\xi_{20}^+ = 0$ and $\bar{\xi}_{10}^+$ being arbitrary. Then by Definition 4.1, $\xi(\tau) = \tilde{\Phi}^{-1}(\bar{\xi}(\tau))$ is an IFJ trajectory of (9) with $a = \mathcal{T}$ since $\xi(0) = \xi_0^- = \tilde{\Phi}^{-1}(\bar{\xi}_0^-) \in V\backslash M^*$, $\xi(\mathcal{T}) = \xi_0^+ = \tilde{\Phi}^{-1}(\bar{\xi}_0^+) \in M^* \cap V$ and $\left[\begin{smallmatrix} I & E_2(\xi(\tau)) \\ 0 & 0 \end{smallmatrix}\right] \frac{d\xi(\tau)}{d\tau} = 0$ for $\tau \in [0, \mathcal{T}]$. Since Ξ and (9) are ex-equivalent (via Q and ψ), we conclude that (see Remark 4.4(i)) for any inconsistent initial value $x_0^- = \psi^{-1}(\xi_0^-) \in V/M^*$ satisfying $M^* \cap N_{x_0^-} \neq \emptyset$, there exists an IFJ trajectory $J(\tau) = \psi^{-1}(\xi(\tau))$ satisfying that $J(0) = x_0^-$ and $J(a) = x_0^+ = \psi^{-1}(\xi_0^+) = \psi^{-1} \circ \tilde{\Phi}^{-1}(\bar{\xi}_0^+) \in N_{x_0^-} \cap M^*$.

$(i) \Rightarrow (ii)$: Suppose that for a fixed $x_0^- \in V\backslash M^*$, the impulse-free jump $x_0^- \to x_0^+$ of Ξ is unique. It follows that the impulse-free jump $\xi_0^- \to \xi_0^+$ of (9) is unique and so is the point $\bar{\xi}_0^+ = (\tilde{\xi}_{10}^+, \bar{\xi}_{10}^+, \xi_{20}^-) = \tilde{\Phi}(\xi_0^+)$. Thus $\bar{\xi}_1$-variables is not present in (27) since $\tilde{\xi}_{10}^+ = \tilde{\xi}_{10}^-$ and $\xi_{20}^- = 0$ are fixed but $\bar{\xi}_{10}^+$ is arbitrary. Hence, we have $\dim\ker E = m = k = \dim \mathcal{D}$, which means that the distribution $\ker E(x)$ is involutive.

$(ii) \Rightarrow (iii)$: Suppose that the distribution $\ker E(x)$ is involutive. Choose $Q : U \to GL(n, \mathbb{R})$ such that $E_1 : U \to \mathbb{R}^{r\times n}$ of $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank r and denote $QF = \left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right]$. Because Ξ is index-1, we have that $\left[\begin{smallmatrix} E_1(x_c) \\ \mathrm{D}F_2(x_c) \end{smallmatrix}\right]$ is invertible by Proposition 3.3. Since the distribution $\Xi = \ker E$ is involutive, by Frobenius theorem (see e.g., [22]), there exist a neighborhood $U_1 \subseteq U$ and a smooth map $\xi_1 : U_1 \to \mathbb{R}^r$ such that $\operatorname{span}\left\{\mathrm{d}\xi_1^1, \ldots, \mathrm{d}\xi_1^r\right\} = \mathcal{E}^{\perp}$, where $\mathrm{d}\xi_1^i$ are independent rows of $\mathrm{D}\xi_1$ and $\mathcal{E} = \ker E = \ker E_1$, i.e., $\mathrm{D}\xi_1(x) \ker E_1(x) = 0$, $\forall x \in U_1$. It follows that there exists $Q_1 : U_1 \to GL(r, \mathbb{R})$ such that $\mathrm{D}\xi_1(x) = Q_1(x)E_1(x)$. Set $\xi_2 = F_2$, then we have $\psi(x) = (\xi_1(x), \xi_2(x))$ is a local diffeomorphism on a neighborhood $U_2 \subseteq U_1$ of x_c since

$$\frac{\partial \psi(x_c)}{\partial x} = \begin{bmatrix} \mathrm{D}\xi_1(x_c) \\ \mathrm{D}F_2(x_c) \end{bmatrix} = \begin{bmatrix} Q_1(x_c) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_1(x_c) \\ \mathrm{D}F_2(x_c) \end{bmatrix}$$

is invertible. Define the new local coordinates $\xi = \psi = (\xi_1, \xi_2)$ on U_2, we get

$$\begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \left(\frac{\partial \psi(x)}{\partial x}\right)^{-1} \frac{\partial \psi(x)}{\partial x}\dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} \Leftrightarrow \begin{bmatrix} E_1^1(\xi_1, \xi_2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(\xi_1, \xi_2) \\ \xi_2 \end{bmatrix}, \tag{28}$$

where $E_1^1 : U_2 \to \mathbb{R}^{r \times r}$, $[E_1^1 \circ \psi, E_1^2 \circ \psi] = E_1(\frac{\partial \psi}{\partial x})^{-1}$ with $E_1^2 \equiv 0$, $\tilde{F}_1 \circ \psi = F_1$. Notice that $E_1^2 \equiv 0$ because $\operatorname{Im} E_1^2(x) = E_1(x) \ker \mathrm{D}\xi_1(x) = 0$ and that $E_1^1(x)$ is invertible for $x \in U_2$ since $\operatorname{rank} E(x) = \mathit{const.} = r$, $\forall x \in U_2$. Let $\bar{F}_1 = (E_1^1)^{-1}\tilde{F}_1$, we can always find $\bar{F}_1' : U_2 \to \mathbb{R}^{r \times m}$ such that $\bar{F}_1(\xi_1, \xi_2) = \bar{F}_1(\xi_1, 0) + \bar{F}_1'(\xi_1, \xi_2)\xi_2$. Then via $\tilde{Q} = \begin{bmatrix} (E_1^1)^{-1} & -\bar{F}_1' \\ 0 & I \end{bmatrix}$, the DAE (28) is ex-equivalent to the **(INWF)** with $F^*(\xi_1) = \bar{F}_1(\xi_1, 0)$. Finally, it is seen that Ξ is locally (on $V = U_2$) ex-equivalent to the **(INWF)** via the $\tilde{Q}Q$-transformation and the diffeomorphism ψ.

$(iii) \Rightarrow (i)$: Suppose that Ξ is locally ex-equivalent to (16). Then by a similar analysis as in the beginning of the proof using the **(INWF)** rather than the form (9), we can deduce that the $\bar{\xi}_1$-variables of (27) is absent, which implies that the impulse-free jump $\xi_0^- \to \xi_0^+$ (and thus $x_0^- \to x_0^+$) is unique. $\square$

Proof of Theorem 5.3. Suppose that Ξ is locally (on V) ex-equivalent to (9) via Q and ψ. Consider the following disturbed system for (9)

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} I_r & E_2(\xi_1, \xi_2) \\ 0 & -\epsilon W^{-1} \end{bmatrix}^{-1} \begin{bmatrix} F^*(\xi_1, \xi_2) \\ \xi_2 \end{bmatrix} = \begin{bmatrix} F^*(\xi_1, \xi_2) + \frac{1}{\epsilon} W E_2(\xi_1, \xi_2)\xi_2 \\ -\frac{1}{\epsilon} W \xi_2 \end{bmatrix}, \tag{29}$$

which is ex-equivalent (via $\begin{bmatrix} I & E_2 \\ 0 & -\epsilon W^{-1} \end{bmatrix} Q$ and ψ) to Ξ_ϵ of (19). By rescaling t to τ such that $\frac{d\tau}{dt} = \frac{1}{\epsilon}$, we get

$$\begin{bmatrix} \frac{d\xi_1}{d\tau} \\ \frac{d\xi_2}{d\tau} \end{bmatrix} = \begin{bmatrix} \epsilon F^*(\xi_1, \xi_2) + E_2(\xi_1, \xi_2) W \xi_2 \\ -W\xi_2 \end{bmatrix}. \tag{30}$$

Let $\bar{\xi}_W(\tau, \epsilon) = (\bar{\xi}_1(\tau, \epsilon), \bar{\xi}_2(\tau, \epsilon))$ be the solution of (30) existing on $[0, +\infty)$ and starting from an inconsistent initial point $\bar{\xi}_W(0, \epsilon) = (\xi_{10}^-, \xi_{20}^-) = \psi(x_0^-)$ of (9), i.e., $\xi_{20}^- \neq 0$ (recall that $M^* \cap V = \{\xi \in V \,|\, \xi_2 = 0\}$), and let $\tilde{J}(\tau) = (\xi_1(\tau), \xi_2(\tau)) : [0, +\infty) \to V$ be a solution of

$$\begin{bmatrix} \frac{d\xi_1}{d\tau} \\ \frac{d\xi_2}{d\tau} \end{bmatrix} = \begin{bmatrix} E_2(\xi_1, \xi_2) W \xi_2 \\ -W\xi_2 \end{bmatrix}, \ \xi_1(0) = \xi_{10}^-, \ \xi_2(0) = \xi_{20}^-. \tag{31}$$

It is clear that $\xi_2(\tau) = e^{-W\tau}\xi_{20}^-$ and $\xi_{20}^+ = \xi_2(+\infty) = 0$. Denote $\xi_{10}^+ = \xi_1(+\infty)$, we have that $\tilde{J}(\tau)$ is an IFJ trajectory of (9) starting from (ξ_{10}^-, ξ_{20}^-) since $\tilde{J}(+\infty) = \lim_{\tau\to\infty}(\xi_1(\tau), \xi_2(\tau)) = (\xi_{10}^+, 0) \in M^*\cap V$ is consistent and that $\left[\, I_r \;\; E_2(\bar{J}(\tau)) \,\right] \frac{d\bar{J}(\tau)}{d\tau} = 0$, $\forall \tau \in [0, +\infty)$. Define $\gamma(\tau, \epsilon) := \bar{\xi}_W(\tau, \epsilon) - J(\tau)$, it follows that $\frac{d\gamma(\tau,\epsilon)}{d\tau} = \left[\begin{smallmatrix} \epsilon F^*(\xi_1,\xi_2) \\ 0 \end{smallmatrix}\right]$, and thus $\frac{d\gamma(\tau,\epsilon)}{d\tau} \to 0$ as $\epsilon \to 0$. So

$$\lim_{\epsilon\to 0} ||\bar{\xi}_W(\tau, \epsilon) - \bar{J}(\tau)|| = \lim_{\epsilon\to 0} ||\gamma(\tau, \epsilon)|| = const. = ||\gamma(0, \epsilon)|| = 0.$$

Since the ex-equivalence preserves jump trajectories (see Remark 4.4(ii)), we have that $J(\tau) = \psi^{-1}(\tilde{J}(\tau))$ is an IFJ trajectory of Ξ starting from $x_0^- = \psi^{-1}(\xi_{10}^-, \xi_{20}^-)$ and ending at $x_0^+ = \psi^{-1}(\xi_{10}^+, 0)$. Therefore, by the Lipschitz condition of ψ^{-1}, we have

$$\lim_{\epsilon\to 0} ||\bar{\xi}_W(\tau, \epsilon) - J(\tau)|| = \lim_{\epsilon\to 0} ||\psi^{-1} \circ \bar{\xi}_W(\tau, \epsilon) - \psi^{-1} \circ \bar{J}(\tau)|| \leq \lim_{\epsilon\to 0} K||\bar{\xi}_W(\tau, \epsilon) - \bar{J}(\tau)|| = 0,$$

where K is the Lipschitz constant. Hence if a solution $\bar{x}(\tau, \epsilon) : [0, +\infty) \to V$ of (20) exists, then there exists an IFJ trajectory $J(\tau) = \psi^{-1}(\tilde{J}(\tau))$ such that (21) holds. Note that the consistent point $x_0^+ = \psi^{-1}(\xi_{10}^+, 0)$ depends on the choice of W since $\xi_{10}^+ = \lim_{\tau\to\infty} \xi_1(\tau)$ and $(\xi_1(\tau), \xi_2(\tau))$ is the solution of the ODE (31) which depends on W. If Ξ is locally (on V) ex-equivalent to the **(INWF)** of (16) via Q and ψ, then the matrix $E_2(\xi_1, \xi_2) \equiv 0$ of (31), which implies that $\frac{d\xi_1}{d\tau} = 0$ and $\xi_1(\tau) = const. = \xi_{10}^-$. Therefore, we have $\lim_{\tau\to\infty} \xi_1(\tau) = \xi_{10}^+ = \xi_{10}^-$ is unique and does not depend on the choice of W, so $x_0^+ = \psi^{-1}(\xi_{10}^+, 0) = \psi^{-1} \circ \pi \circ \psi(x_0^-) = \Omega_{E,F}(x_0^-)$ is unique.

Furthermore, let $(\bar{\xi}_1(t, \epsilon), \bar{\xi}_2(t, \epsilon)) : I \to V$ be the solution of (29) starting from any consistent point $(\xi_{10}^+, 0)$. We have $\xi_2(t, \epsilon) = 0$, $\forall t \in I$ (since $\xi_2(0) = 0$ is an equilibrium point of $\dot{\xi}_2 = -\frac{1}{\epsilon}W\xi_2$) and $\xi_1(t, \epsilon)$ solves $\dot{\xi}_1 = F^*(\xi_1, 0)$. Hence both $\bar{\xi}_1(t, \epsilon)$ and $\bar{\xi}_2(t, \epsilon)$ do not depend on ϵ and W, and $(\bar{\xi}_1(t), 0)$ is a $\mathcal{C}^1$-solution of (9). Since that the ex-equivalence preserves also $\mathcal{C}^1$-solutions, it follows that $x(t) = \psi^{-1}(\bar{\xi}_1(t), 0)$ is the solution of both the DAE Ξ and the perturbed system Ξ_ϵ staring from the consistent point $x_0^+ = \psi^{-1}(\xi_{10}^+, 0)$.

$\square$

7. Conclusions and perspectives

In this paper, we propose a novel definition of impulse-free jumps for nonlinear DAEs with inconsistent initial values by regarding the jumps as parametrized curves satisfying an impulse-free condition. We show that the impulse-free jumps under this new definition are invariant under the external equivalence of DAEs. We discuss the notion of geometric index-1 and its characterizations. We show that the existence and uniqueness of the impulse free jumps are closely related to the notion of index-1 and the involutivity of the distribution defined by $\ker E$. We also generalize the consistency projector of linear DAEs to the nonlinear case by proposing a normal form called the index-1 nonlinear Weierstrass form **(INWF)**. At last, we propose a singular perturbation system approximation for nonlinear DAEs, the solutions of the perturbed system not only approximate the impulse-free jumps but also the C^1-solutions of the DAE. Our future research would be extending or applying our results of impulse-free jumps to problems like consistent initializations of switched nonlinear DAEs [16], solving ODE systems with distributional coefficients [32].

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