Hon Pre-Calc Test Chapter 11 *2017 - 2018*

Show All Work For FULL Credit!!! Circle All Final Answers!!!!

1. Find the volume of the sphere given by the equation:

$$2x^2 + 2y^2 + 2z^2 - 2x - 6y - 4z + 5 = 0$$

$$2(x^2 - x) + 2(y^2 - 3y) + 2(z^2 - 2z) + 5 = 0$$
$$2(x - \frac{1}{2})^2 + 2(y - \frac{3}{2})^2 + 2(z - 1)^2 = 2$$
$$(x - \frac{1}{2})^2 + (y - \frac{3}{2})^2 + (z - 1)^2 = 1 = r^2$$

$\therefore r = 1$, sphere volume $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$

2. Use vectors to determine if the points are collinear or not.
(1, 3, 2), (-1, 2, 5), and (3, 4, -1)
A B C

1. The points A, B and C are collinear if and only if the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel
2. $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel if $\overrightarrow{AB} = c\,\overrightarrow{BC}$

$\overrightarrow{AB}$ = <-2, -1, 3>, $\overrightarrow{BC}$ = <4, 2, -6>
$\overrightarrow{AB} = -2\,\overrightarrow{BC}$, $\therefore$ collinear

3. Find the EXACT area of a triangle with the given vertices: (2,4,0), (-2,-4,0), and (0,0,4)
A B C

$\overrightarrow{AB}$ = <-4, -8, 0>, $\overrightarrow{AC}$ = <-2, -4, 4>

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -4 & -8 & 0 \\ -2 & -4 & 4 \end{vmatrix} = -32i + 16j + 0k$$
= <-32, 16, 0>

Area of triangle $= \frac{1}{2}\,\|\overrightarrow{AB} \times \overrightarrow{AC}\| =$
$\frac{1}{2}\sqrt{(-32)^2 + 16^2 + 0} = \sqrt{1280} = 8\sqrt{5}$

4. Determine if $\vec{u}$ and $\vec{v}$ are parallel, orthogonal, or neither.

a) $\vec{u}$ = <0, 1, 6>
$\vec{v}$ = <1, -6, -1>

$\vec{u} \cdot \vec{v} = 0 - 6 - 6 = -12$ $\therefore$ neither

b) $\vec{u}$ = <-2, 3, -1>
$\vec{v}$ = <2, 1, -1>

$\vec{u} \cdot \vec{v} = -4 + 3 + 1 = 0$ $\therefore$ orthogonal

5. Determine the value of c such that $\|c\vec{u}\| = \sqrt{58}$, where $\vec{u} = 2i + 3j + 4k$

$\vec{u} = <2, 3, 4>$, $\therefore c\,\vec{u} = <2c, 3c, 4c>$

$\therefore \|c\vec{u}\| = \sqrt{4c^2 + 9c^2 + 16c^2} = \sqrt{29c^2} = \sqrt{58}$
$\therefore 29c^2 = 58$

$c^2 = 2$, c = $\pm\sqrt{2}$

6. Find the Exact unit vector orthogonal to $\vec{u}$ and $\vec{v}$ if $\vec{u} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\vec{v} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

vector orthogonal to $\vec{u}$ and $\vec{v}$ is $\vec{u} \times \vec{v}$

$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \langle 2, -2, 0 \rangle$

unit vector $= \frac{\langle 2,-2,0 \rangle}{\sqrt{2^2+(-2)^2}} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \rangle$

6. Given:
A(2, 1, 1), B(2, 3, 1), C(-2, 4, 1), D(-2, 6, 1)

a) Find the area of the parallelogram

$\overrightarrow{AB} = (2, 3, 1) - (2, 1, 1) = \langle 0, 2, 0 \rangle$
$\overrightarrow{AC} = (-2, 4, 1) - (2, 1, 1) = \langle -4, 3, 0 \rangle$

$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \left\| \begin{vmatrix} i & j & k \\ 0 & 2 & 0 \\ -4 & 3 & 0 \end{vmatrix} \right\|$
$= \|\langle 0 - 0, -(0 - 0), (0 - (-8)) \rangle\|$
$= \|\langle 0, 0, 8 \rangle\|$
$= \sqrt{0^2 + 0^2 + 8^2} = 8\ unit^2$

b) Determine if the parallelogram is a rectangle (Show your work)

AB · AC = <0, 2, 0> · <-4, 3, 0> = 6

∴ not rectangle

8. Find the volume of the parallelepiped with the given vertices:

A(0, 0, 0), B(3, 0, 0), C(0, 5, 1), D(3, 5, 1)
E(2, 0, 5), F(5, 0, 5), G(2, 5, 6), H(5, 5, 6)

AB = <3, 0, 0>
AC = <0, 5, 1>
AE = <2, 0, 5>

Volume $= |\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AE})|$
$= \left| \langle 3, 0, 0 \rangle \cdot \begin{vmatrix} i & j & k \\ 0 & 5 & 1 \\ 2 & 0 & 5 \end{vmatrix} \right|$
$= |\langle 3, 0, 0 \rangle \cdot \langle 25, 2, -10 \rangle| = |75|$
$= 75\ unit^3$

9. Find a set of parametric equations of the line that passes through the given points:

(-1, -1, 5), (2, -2, 3)

$\vec{a}$ = (2, -2, 3) – (-1, -1, 5) = <3, -1, -2>
use point (2, -2, 3)
$\therefore \begin{cases} x = 2 + 3t \\ y = -2 - t \\ z = 3 - 2t \end{cases}$

10. Find the general form for the equation of the plane passing through the given points:

(2, 3, -2), (3, 4, 2), (1, -1, 0)
A B C

$\overrightarrow{AB} = (3, 4, 2) - (2, 3, -2) = \langle 1, 1, 4 \rangle$
$\overrightarrow{AC} = (1, -1, 0) - (2, 3, -2) = \langle -1, -4, 2 \rangle$
AB x AC $= \begin{vmatrix} i & j & k \\ 1 & 1 & 4 \\ -1 & -4 & 2 \end{vmatrix} = \langle 18, -6, -3 \rangle$
use point A (2, 3, -2)
∴ general form of plane equation =
$18(x - 2) - 6(y - 3) - 3(z + 2) = 0$
$18x - 6y - 3z - 24 = 0$
$6x - 2y - z - 8 = 0$

11. Find the general form of the equation of the plane which passes through the point (1, -2, 4) and (4, 0, -1) is perpendicular to the *xz* plane.

$\vec{u} = (4,0,-1) - (1,-2,4) = < 3, 2, -5 >$
$\vec{v} = < 0, 1, 0 >$

$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 3 & 2 & -5 \\ 0 & 1 & 0 \end{vmatrix} = <5, 0, 3>$

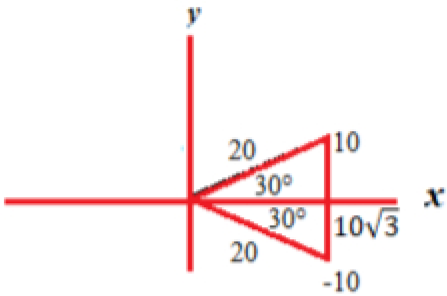
use point (1, -2, 4), ∴ general plane equation
$= 5(x-1) - 0(y+2) + 3(z-4) = 0$
$5x + 3z - 17 = 0$

12. Let M be the plane defined by the equation $6x - 4y + 3z = 12$. Find the general equation for the plane N that is parallel to M and passes through (3, -1, 4)

$\vec{n} = <6, -4, 3>$
use point (3, -1, 4), ∴ general plane equation
$= 6(x-3) - 4(y+1) + 3(z-4) = 0$
$6x - 4y + 3z - 34 = 0$

13. Find the **Exact** component form of $\vec{v}$, $\vec{v}$ lies in the *xy* plane, has magnitude 20, and makes an angle of 30° with the positive *x* axis.

lies in the *xy* plane → $z = 0$
→ only draw *x*, *y* axis



$\therefore \vec{v} = (10\sqrt{3}, \pm 10, 0)$

14. Find a set of parametric equations of the line that passes through (-4, 5, 2) and is perpendicular to $-x + 2y + z = 5$

$\vec{n} = <-1, 2, 1>$
use point (-4, 5, 2), ∴ parametric equations
$= \begin{cases} x = -4 - t \\ y = 5 + 2t \\ z = 2 + t \end{cases}$

15. Find the distance between the point (1, 2, 3) and the plane $2x - y + z = 4$ P

$\vec{n} = <2, -1, 1>$
set Q(0, 0, 4) is on the plane $2x - y + z = 4$
$\overrightarrow{PQ} = (0, 0, 4) - (1, 2, 3) = <-1, -2, 1>$

$$D = \frac{|PQ \cdot n|}{\|n\|} = \frac{|-2+2+1|}{\sqrt{2^2+(-1)^2+1^2}} = \frac{1}{\sqrt{6}}$$

$= \frac{\sqrt{6}}{6}\ units$

16. Given: $x + y - z = 0$

$2x - 5y - z = 1$

a) Find the angle between them.

$\overrightarrow{n_1} = < 1, 1, -1 >, \overrightarrow{n_2} = < 2, -5, -1 >$

$$\cos\theta = \frac{|\overrightarrow{n_1} \cdot \overrightarrow{n_2}|}{\|\overrightarrow{n_1}\|\|\overrightarrow{n_2}\|} = \frac{|2-5+1|}{\sqrt{3} * \sqrt{30}} = \frac{2}{\sqrt{90}}$$

$\therefore \theta \simeq 77.83°$

b) Find parametric equations of their line of intersection.

$$\begin{cases} 2x + 2y - 2z = 0 & - \text{①} \\ 2x - 5y - z = 1 & - \text{②} \end{cases}$$

① − ②: $7y - z = -1 \rightarrow y = \frac{1}{7}(z-1)$

sub into $x + y - z = 0$

$\rightarrow x + \frac{1}{7}(z-1) - z = 0$

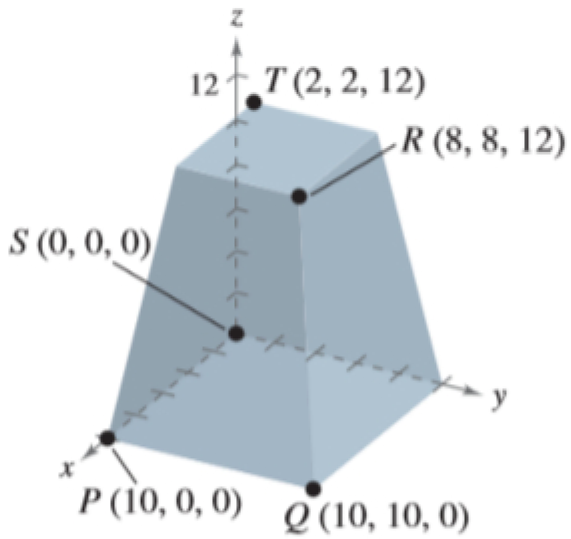
$\rightarrow x = \frac{1}{7}(6z+1)$

let $y = t$, $\therefore t = \frac{1}{7}(z-1) \rightarrow z = 7t + 1$

$x = \frac{1}{7}(6z+1) = \frac{1}{7}[6(7t+1)+1] = 6t + 1$

$$\begin{cases} x = 1 + 6t \\ y = t \\ z = 7t + 1 \end{cases}$$

17. A tractor fuel tank has the shape and dimensions shown in the figure. In fabricating the tank, it is necessary to know the angle between two adjacent sides. Find this angle.



$\overrightarrow{ST}$ = <2, 2, 12>, $\overrightarrow{SP}$ = <10, 0, 0>, $\overrightarrow{n_{TSP}} = \overrightarrow{ST} \times \overrightarrow{SP} = \begin{vmatrix} i & j & k \\ 2 & 2 & 12 \\ 10 & 0 & 0 \end{vmatrix}$ = <0, 120, -20>

$\overrightarrow{QR}$ = <-2, -2, 12>, $\overrightarrow{QP}$ = <0, -10, 0>, $\overrightarrow{n_{PQR}} = \overrightarrow{QR} \times \overrightarrow{QP} = \begin{vmatrix} i & j & k \\ -2 & -2 & 12 \\ 0 & -10 & 0 \end{vmatrix}$ = <120, 0, 20>

$$\cos\theta = \frac{|<0,120,-20> \cdot <120,0,20>|}{\sqrt{0^2+120^2+(-20)^2}\sqrt{120^2+0^2+20^2}} = \frac{|0+0-400|}{14800} = \frac{4}{148} = \frac{1}{37}$$

$\theta \simeq 88.45°$