Computers and Mathematics with Applications 128 (2022) 96–107

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

www.elsevier.com/locate/camwa

# Optimal $L^2$ error analysis of first-order Euler linearized finite element scheme for the 2D magnetohydrodynamics system with variable density

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ARTICLE INFO

*Keywords:*
Incompressible magnetohydrodynamics equations
Variable density
Spatial error analysis

ABSTRACT

In this paper, a first-order Euler semi-implicit finite element scheme is proposed for solving the two-dimensional incompressible magnetohydrodynamics flows with variable density numerically. The proposed FEM algorithm is unconditionally stable at the full discrete level, which is a key issue in designing efficient algorithms for the multi-physical field problems. When the time step size $\tau$ and the mesh size $h$ both are sufficiently small, optimal spatial error estimates in $L^2$-norm are presented for the density, velocity and magnetic field by using energy techniques based on uniform regularities of solutions to the time-discrete scheme and the method of mathematical induction. Finally, numerical results are given to confirm the theoretical analysis.

## 1. Introduction

Let $\Omega \subset \mathbf{R}^2$ be a bounded and convex domain with the boundary $\Gamma = \partial\Omega$ and $[0,T]$ be the time interval with some $T > 0$. Denote $Q_T = \Omega \times (0,T]$. In this paper, we will consider the two-dimensional (2D) incompressible magnetohydrodynamics system with variable density (VD-MHD) which describes the motions of several conducting incompressible immiscible fluids without surface tension in presence of a magnetic field, such as in modeling of astrophysics, geophysics, plasma physics and liquid metals of an aluminum electrolysis cell. We refer to [4,8,15,26] for the understanding of the physical and industrial backgrounds of the MHD system. The VD-MHD system is governed by the following nonlinear coupled system in a non-dimensional form:

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \text{in } Q_T, \tag{1.1}$$

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \frac{1}{Re}\Delta\mathbf{u} + S\mathbf{b}\times curl\,\mathbf{b} + \nabla p = \mathbf{f}, \quad \text{in } Q_T, \tag{1.2}$$

$$\mathbf{b}_t + \frac{1}{Rm}\mathbf{curl}\,(curl\,\mathbf{b}) - \mathbf{curl}\,(\mathbf{u}\times\mathbf{b}) = 0, \quad \text{in } Q_T, \tag{1.3}$$

$$\nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \quad \text{in } Q_T, \tag{1.4}$$

where the unknowns $\rho$, $\mathbf{u}$, $p$ and $\mathbf{b}$ denote the density, velocity, pressure, magnetic field, respectively, and $\mathbf{f}$ is a given body force. The physical parameters are the Reynolds number $Re$, the magnetic Reynolds number $Rm$, the coupling number $S$ given by (cf. [8])

$$Re = \frac{1}{\nu}, \quad Rm = \frac{1}{\sigma\mu}, \quad S = \frac{1}{\mu},$$

where $\nu$ is the kinematic viscosity, $\mu$ is the magnetic permeability and $\sigma$ is the electric conductivity. In addition, the cross product and differential operators in (1.1)-(1.4) are given by

$$\mathbf{u}\times\mathbf{b} = u_1b_2 - u_2b_1, \qquad \mathbf{b}\times r = (rb_2, -rb_1),$$
$$curl\,\mathbf{b} = \frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y}, \qquad \mathbf{curl}\ r = \left(\frac{\partial r}{\partial y}, -\frac{r}{\partial x}\right),$$

where $\mathbf{u} = (u_1, u_2)$, $\mathbf{b} = (b_1, b_2)$ and $r$ is a scalar function.

We consider the VD-MHD system (1.1)-(1.4) by supplementing the following initial and boundary conditions:

$$\begin{cases} \rho(x,0) = \rho_0(x), \quad \rho(x,t)|_{\Gamma_{in}} = a(x,t), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{u}(x,t)|_\Gamma = \mathbf{g}, \\ \mathbf{b}(x,0) = \mathbf{b}_0(x), \quad \mathbf{b}(x,t)\cdot\mathbf{n}|_\Gamma = 0 \quad curl\,\mathbf{b}(x,t)\times\mathbf{n}|_\Gamma = 0, \end{cases} \tag{1.5}$$

where $\mathbf{n}$ is the outward unit normal vector to the boundary $\Gamma$ and $\Gamma_{in}$ is the inflow boundary defined by $\Gamma_{in} = \{x \in \Gamma : \mathbf{g}\cdot\mathbf{n} < 0\}$. For the reason of simplicity, we consider the homogeneous Dirichlet boundary condition for the velocity, i.e., $\mathbf{g} = 0$, which implies that the boundary is impermeable and $\Gamma_{in} = \emptyset$. The initial values $\mathbf{u}_0$ and $\mathbf{b}_0$ satisfy the incompressible conditions:

$$\nabla\cdot\mathbf{u}_0 = 0 \qquad \text{and} \qquad \nabla\cdot\mathbf{b}_0 = 0,$$

and the initial density $\rho_0$ is bounded and away from zero, i.e. there exist two constants $\rho_0^{min} > 0$ and $\rho_0^{max} > 0$ such that

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https://doi.org/10.1016/j.camwa.2022.10.013
Received 29 March 2022; Received in revised form 19 September 2022; Accepted 17 October 2022

$$\rho_0^{min} \le \rho_0(x) \le \rho_0^{max} \qquad \text{in } \Omega, \tag{1.6}$$

which means that there is no vacuum state in $\Omega$.

If the density $\rho$ is a constant, the VD-MHD system (1.1)-(1.4) reduce to the classical MHD system with constant density. There have been many works on numerical methods for the MHD system with constant density. Prohl [28] proposed some decoupling and coupling semi-implicit finite element algorithms based on the Lagrange multiplier. In [12], He studied a Euler semi-implicit coupling scheme by using the Lagrange finite element to approximate the magnetic field, where optimal error estimates in $L^2$-norm were proved by using the technique of negative norm. Gao and Qiu [7] discussed Euler semi-implicit scheme by using the Nédélec edge element to approximate the magnetic field. Other numerical algorithms can be found in [25,33] for decoupling semi-implicit schemes, in [24,34] for second-order Crank-Nicolson schemes, in [2,30] for first-order Euler projection schemes, in [3,31,32] for the fractional-step algorithms, in [17] for the Lagrange finite element method in the non-convex domain.

Compared to the MHD system with constant density, few numerical methods were studied for the VD-MHD system. In [8], Gerbeau, Le Bris and Lelièvre discussed the numerical approximation for the two-fluid VD-MHD system, where the arbitrary Lagrange-Euler (ALE) method was used to track the interface of two fluids. The Euler semi-implicit finite element scheme based on weak ALE formulation was proposed and was proved to be stable in [8], i.e., the numerical solutions satisfy the discrete energy inequalities. By the implicit-explicit linearized method, a first-order Euler semi-implicit scheme was proposed in [19]. For the finite element discretization of Euler semi-implicit scheme proposed in [19], the following constraint condition on finite element spaces is needed:

$$\mathbf{V}_h \cdot \mathbf{V}_h \subset X_h, \tag{1.7}$$

where $\mathbf{V}_h$ and $X_h$ is finite element spaces of the velocity and density, respectively. Condition (1.7) is used to preserve the unconditional stability of numerical algorithm, and is needed in constructing of finite element algorithms for other variable density flows, such as the Navier-Stokes equations with variable density [5,18] and with mass diffusion [9–11,20]. To avoid the constraint condition (1.7), in [22], Li and Cui studied a new Euler semi-implicit scheme based on an equivalent system of (1.1)-(1.4) by introducing a new variable quantity $\sigma = \sqrt{\rho}$, where the finite element algorithm is unconditionally stable without the condition (1.7). Theoretically, the first-order temporal convergence rate $\mathcal{O}(\tau)$ in $L^2$-norm was proved for the density, velocity, pressure and magnetic field under some reasonable regularity assumptions in [22]. A similar method has been used for the Navier-Stokes equations with variable density in [21,23].

This paper is a continuation of the work proposed in [22] and focuses on the spatial error analysis of the new Euler semi-implicit schemes in [22]. We use the piecewise quadratic Lagrange element ($P_2$) to discretize the equation (2.3), the mini element ($P_1b - P_1$) to discretize the Navier-Stokes type equations (2.4) and the piecewise linear Lagrange element ($P_1$) to discretize the Maxwell type equations (2.5). We prove that the finite element fully discrete scheme is unconditionally stable and the discrete energy inequalities are satisfied. Furthermore, the second-order convergence rate $\mathcal{O}(h^2)$ in $L^2$-norm is proved without any time step condition. Alougth the choice of finite element spaces still satisfies the condition (1.7), we remark that the theoretical analysis in this paper can be extended to the case of $(P_2, P_2, P_1, P_1)$-elements.

This paper is organized as follows. In Section 2, we present some notations and recall the equivalent system to (1.1)-(1.4). In Section 3, the Euler semi-implicit time-discrete scheme is given and we recall the temporal error estimates which were proved in [22]. In Section 4, we make the proof of the spatial convergence rate $\mathcal{O}(h^2)$ in $L^2$-norm. In Section 5, we present the main result of this paper. In Section 6, numerical results are provided to confirm the theoretical analysis. Conclusions are drawn in Section 7.

## 2. Preliminaries

### 2.1. Some notations

For $k \in \mathbb{N}^+$ and $1 \le p \le +\infty$, let $L^p$ and $W^{k,p}(\Omega)$ denote the standard Lebesgue and Sobolev spaces, respectively. The norms in $L^p$ and $W^{k,p}(\Omega)$ are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$ which are defined by the classical sense (cf. [1]). Let $W_0^{k,p}(\Omega)$ be the subspace of $W^{k,p}(\Omega)$ of functions with zero trace on $\Gamma$. If $p=2$, then $W^{k,2}(\Omega)$ becomes the Hilbert space $H^k(\Omega)$. The boldface spaces $\mathbf{H}^k(\Omega), \mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ are used to denote the vector-value spaces $H^k(\Omega)^2, W^{k,p}(\Omega)^2$ and $L^p(\Omega)^2$, respectively. In addition, we use $(\cdot,\cdot)$ to denote the $L^2$ or $\mathbf{L}^2$ inner product.

Introduce the following function spaces:

$$\begin{aligned}
&X = H^1(\Omega), \quad \mathbf{V} = \mathbf{H}_0^1(\Omega), \quad \mathbf{V}_0 = \{\mathbf{u} \in \mathbf{V},\ \nabla\cdot\mathbf{u} = 0\},\\
&\mathbf{W} = \{\mathbf{b} \in \mathbf{H}^1(\Omega),\ \mathbf{b}\cdot\mathbf{n}|_\Gamma = 0\}, \quad \mathbf{W}_0 = \{\mathbf{b} \in \mathbf{W},\ \nabla\cdot\mathbf{b} = 0\},\\
&\mathbf{H}(\mathrm{div}\,,\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega),\ \nabla\cdot\mathbf{u} \in L^2(\Omega)\},\\
&M = L_0^2(\Omega) = \{p \in L^2(\Omega),\ \int_\Omega p\,dx = 0\}.
\end{aligned}$$

The spaces $\mathbf{V}$ and $\mathbf{W}$ are equipped with norms

$$\|\mathbf{v}\|_V = \left(\int_\Omega |\nabla\mathbf{v}|^2 dx\right)^{1/2} \quad \text{and} \quad \|\mathbf{w}\|_W = \left(\int_\Omega (|curl\mathbf{w}|^2 + |\nabla\cdot\mathbf{w}|^2) dx\right)^{1/2}.$$

It is well-known that there exists some $C>0$ such that (cf. [8])

$$\|\mathbf{w}\|_{H^1} \le C\|\mathbf{w}\|_W. \tag{2.1}$$

Furthermore, by the Poincaré's and Sobolev's inequalities (cf. [1]), one has

$$\|\mathbf{v}\|_{L^p} \le C\|\mathbf{v}\|_V, \quad \|\mathbf{w}\|_{L^p} \le C\|\mathbf{w}\|_W, \quad 1 \le p < \infty. \tag{2.2}$$

### 2.2. An equivalent system

In this subsection, we recall the following equivalent form of the VD-MHD system (1.1)-(1.4) derived in [22]:

$$\sigma_t + \nabla\cdot(\sigma\mathbf{u}) = 0, \tag{2.3}$$

$$\sigma(\sigma\mathbf{u})_t - \frac{1}{Re}\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\mathbf{u}}{2}\nabla\cdot(\rho\mathbf{u}) + S\mathbf{b}\times curl\mathbf{b} + \nabla p = \mathbf{f}, \tag{2.4}$$

$$\mathbf{b}_t + \frac{1}{Rm}\mathbf{curl}\,(curl\mathbf{b}) - \mathbf{curl}\,(\mathbf{u}\times\mathbf{b}) = 0, \tag{2.5}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0 \tag{2.6}$$

with the initial and boundary conditions

$$\begin{cases} \sigma(x,0) = \sigma_0(x) = \sqrt{\rho_0(x)}, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{b}(x,0) = \mathbf{b}_0(x),\\ \mathbf{u}(x,t)|_\Gamma = 0, \quad \mathbf{b}(x,t)\cdot\mathbf{n}|_\Gamma = 0 \quad curl\mathbf{b}(x,t)\times\mathbf{n}|_\Gamma = 0, \end{cases} \tag{2.7}$$

where $\sigma = \sqrt{\rho}$. In addition, $\sigma_0$ satisfies

$$\sigma_0^{min} := \sqrt{\rho_0^{min}} \le \sigma_0(x) \le \sqrt{\rho_0^{max}} := \sigma_0^{max} \qquad \text{in } \Omega. \tag{2.8}$$

Throughout this paper, we make the following assumptions on the prescribed data and the solution to the new VD-MHD system (2.3)-(2.6).

**(A1):** Assume that the prescribed data $f, \mathbf{u}_0$ and $\rho_0$ satisfy

$$\sigma_0 \in H^3(\Omega),\ \mathbf{u}_0 \in \mathbf{V}_0 \cap \mathbf{H}^2(\Omega),\ \mathbf{b}_0 \in \mathbf{W}_0 \cap \mathbf{H}^2(\Omega), \mathbf{f} \in L^2(0,T;\mathbf{H}^1(\Omega)).$$

**(A2):** Assume that the solutions to the VD-MHD (2.3)-(2.6) satisfy the following regularities:

$$\begin{cases} \sigma \in L^\infty(0,T;H^3(\Omega)),\ \sigma_t \in L^2(0,T;H^2(\Omega)),\ \sigma_{tt} \in L^2(0,T;L^2(\Omega)),\\ \mathbf{u} \in \mathbf{L}^\infty(0,T;\mathbf{V}_0 \cap \mathbf{H}^3(\Omega)),\ \mathbf{u}_{tt} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)),\\ \mathbf{u}_t \in \mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^2(0,T;\mathbf{H}^2(\Omega)),\\ \mathbf{b} \in \mathbf{L}^\infty(0,T;\mathbf{W}_0 \cap \mathbf{H}^3(\Omega)),\ \mathbf{b}_{tt} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)),\\ \mathbf{b}_t \in \mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^2(0,T;\mathbf{H}^2(\Omega)). \end{cases} \tag{2.9}$$

**Remark 2.1.** The verification of the regularity assumptions $\mathbf{u}_{tt} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{b}_{tt} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$ should involve an extra compatibility condition on the data at $t=0$ which are not generally satisfied (see such condition for Navier-Stokes equations in [13]). We make these assumptions merely to simplify the presentation.

## 3. Euler time discrete scheme

In this section, we recall the first-order Euler time discrete scheme proposed in [22] for the VD-MHD system (2.3)-(2.6). For some $N \in \mathbb{N}^+$, define the time-step $\tau = T/N$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[0,T]$ with $t_n = n\tau$. For any sequence of functions $\{v^n\}_{n=0}^N$, denote $D_\tau v^{n+1} = (v^{n+1} - v^n)/\tau$ for $0 \le n \le N-1$.

Start with $\sigma^0 = \sigma_0$, $\mathbf{u}^0 = \mathbf{u}_0$ and $\mathbf{b}^0 = \mathbf{b}_0$. The first-order backward Euler time discrete algorithm is stated as follows.

**First-order Euler time discrete algorithm**

**Step I:** For given $\sigma^n$ and $\mathbf{u}^n$ with $\nabla \cdot \mathbf{u}^n = 0$, we solve $\sigma^{n+1}$ by

$$D_\tau \sigma^{n+1} + \nabla \cdot (\sigma^{n+1} \mathbf{u}^n) = 0, \tag{3.1}$$

**Step II:** For given $\mathbf{b}^n$ with $\nabla \cdot \mathbf{b}^n = 0$, we solve $(\mathbf{u}^{n+1}, p^{n+1}, \mathbf{b}^{n+1})$ by

$$\begin{aligned}\sigma^{n+1} D_\tau(\sigma^{n+1}\mathbf{u}^{n+1}) - \frac{1}{Re}\Delta \mathbf{u}^{n+1} + \nabla p^{n+1} + \rho^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} + \frac{\mathbf{u}^{n+1}}{2}\nabla \cdot (\rho^{n+1}\mathbf{u}^n) \\ + S(\mathbf{b}^n \times curl\mathbf{b}^{n+1}) = \mathbf{f}^{n+1}, \quad \nabla \cdot \mathbf{u}^{n+1} = 0\end{aligned} \tag{3.2}$$

and

$$D_\tau \mathbf{b}^{n+1} + \frac{1}{Rm}\mathbf{curl}\,(curl\mathbf{b}^{n+1}) - \mathbf{curl}\,(\mathbf{u}^{n+1} \times \mathbf{b}^n) = 0, \quad \nabla \cdot \mathbf{b}^{n+1} = 0 \tag{3.3}$$

with the boundary condition $\mathbf{u}^{n+1} = 0$, $\mathbf{b}^{n+1} \cdot \mathbf{n} = 0$ and $curl\mathbf{b}^{n+1} \times \mathbf{n} = 0$ on $\partial\Omega$, where $\rho^{n+1} = (\sigma^{n+1})^2$.

The weak variational formulations of the above linearized problems (3.1)-(3.3) can be described as follows. For given $\sigma^n$ and $\mathbf{u}^n$, we find $\sigma^{n+1} \in X$ such that

$$(D_\tau \sigma^{n+1}, r) + (\nabla \sigma^{n+1} \cdot \mathbf{u}^n, r) = 0, \quad \forall\, r \in X \tag{3.4}$$

and, for given $\mathbf{b}^n$, we find $(\mathbf{u}^{n+1}, p^{n+1}, \mathbf{b}^{n+1}) \in \mathbf{V} \times M \times \mathbf{W}$ such that

$$\begin{aligned}(\sigma^{n+1} D_\tau(\sigma^{n+1}\mathbf{u}^{n+1}), \mathbf{v}) - \frac{1}{Re}(\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p^{n+1}) + (\rho^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{v}) \\ + \frac{1}{2}(\mathbf{u}^{n+1}\nabla \cdot (\rho^{n+1}\mathbf{u}^n), \mathbf{v}) + S(\mathbf{b}^n \times curl\mathbf{b}^{n+1}, \mathbf{v}) + (\nabla \cdot \mathbf{u}^{n+1}, q) = (\mathbf{f}^{n+1}, \mathbf{v})\end{aligned} \tag{3.5}$$

and

$$(D_\tau \mathbf{b}^{n+1}, \mathbf{w}) + \frac{1}{Rm}(curl\mathbf{b}^{n+1}, curl\mathbf{w}) - (\mathbf{u}^{n+1} \times \mathbf{b}^n, curl\mathbf{w}) = 0 \tag{3.6}$$

for any $(\mathbf{v}, q, \mathbf{w}) \in \mathbf{V} \times M \times \mathbf{W}$.

**Remark 3.1.** Taking $\mathbf{w} = \nabla\phi$ with $\phi \in H_0^2(\Omega)$, we can deduce from (3.6) that $\nabla \cdot \mathbf{b}^{n+1} = 0$ thanks to $curl\nabla\phi = 0$ and $\nabla \cdot \mathbf{b}^n = 0$.

For $0 \le n \le N$, we denote temporal error functions by

$$e_\rho^n = \rho(t_n) - \rho^n,\ e_\sigma^n = \sigma(t_n) - \sigma^n,\ e_{\mathbf{u}}^n = \mathbf{u}(t_n) - \mathbf{u}^n,\ e_p^n = p(t_n) - p^n,$$
$$e_{\mathbf{b}}^n = \mathbf{b}(t_n) - \mathbf{b}^n.$$

The following regularities of solutions to (3.1)-(3.3) and temporal error estimates are established in [22].

**Theorem 3.1.** *Under the assumptions (A1) and (A2) and for $0 \le n \le N-1$, there exists some $\tau_0 > 0$ such that when $\tau \le \tau_0$, the Euler time discrete scheme (3.1)-(3.3) has a unique solution $(\sigma^{n+1}, \boldsymbol{u}^{n+1}, p^{n+1}, \boldsymbol{b}^{n+1})$ which satisfy the maximum principle*

$$\sigma_0^{min} \le \sigma^{n+1}(x) \le \sigma_0^{max} \quad \text{in } \Omega, \tag{3.7}$$

*and the regularity estimates*

$$\begin{aligned}\max_{0\le n\le N-1}(\|\boldsymbol{u}^{n+1}\|_{W^{1,\infty}} + \|\boldsymbol{u}^{n+1}\|_{H^2} + \|\boldsymbol{b}^{n+1}\|_{H^2} + \|\sigma^{n+1}\|_{W^{1,\infty}} + \|\sigma^{n+1}\|_{H^2}) \\ \le C\end{aligned} \tag{3.8}$$

*and the temporal error estimates*

$$\max_{0\le n\le N-1}(\|e_\rho^{n+1}\|_{L^2} + \|e_\sigma^{n+1}\|_{L^2} + \|e_{\boldsymbol{u}}^{n+1}\|_V + \|e_{\boldsymbol{b}}^{n+1}\|_W) \le C\tau, \tag{3.9}$$

$$\sum_{n=0}^{N-1}\left(\|e_{\boldsymbol{u}}^{n+1} - e_{\boldsymbol{u}}^n\|_V^2 + \|e_\sigma^{n+1} - e_\sigma^n\|_{H^1}^2 + \|e_{\boldsymbol{b}}^{n+1} - e_{\boldsymbol{b}}^n\|_W^2\right) \le C\tau^2, \tag{3.10}$$

$$\tau\sum_{n=0}^{N-1}\left(\|e_{\boldsymbol{u}}^{n+1}\|_{H^2}^2 + \|e_{\boldsymbol{b}}^{n+1}\|_{H^2}^2 + \|e_p^{n+1}\|_{H^1}^2\right) \le C\tau^2, \tag{3.11}$$

*where $C > 0$ is some constant independent of $\tau$ and $n$.*

In order to get the optimal spatial error estimate, we need to improve the regularities of $(\sigma^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, \mathbf{b}^{n+1})$ in the following lemma.

**Lemma 3.1.** *Under the conditions in Theorem 3.1 and for $0 \le n \le N-1$, the solutions $(\sigma^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{b}^{n+1})$ satisfy*

$$\tau\sum_{n=0}^{N-1}(\|D_\tau \boldsymbol{u}^{n+1}\|_{H^2}^2 + \|D_\tau \boldsymbol{b}^{n+1}\|_{H^2}^2 + \|\boldsymbol{u}^{n+1}\|_{H^3}^2 + \|p^{n+1}\|_{H^2}^2) \le C, \tag{3.12}$$

$$\max_{0\le n\le N-1}(\|D_\tau \boldsymbol{u}^{n+1}\|_{H^1} + \|D_\tau \boldsymbol{b}^{n+1}\|_{H^1} + \|D_\tau \sigma^{n+1}\|_{H^2} + \|\sigma^{n+1}\|_{H^3}) \le C, \tag{3.13}$$

*where $C > 0$ is some constant independent of $\tau$ and $n$.*

**Proof.** The proof of Lemma 3.1 is given in Appendix A. □

## 4. Finite element approximations and error analysis

In this section, we present the finite element approximations of the time discrete algorithm (3.1)-(3.3) based on a post-processed velocity proposed in [16]. Then we establish the unconditional stability of the finite element algorithm and prove the optimal spatial error estimate in $L^2$-norm. The proof is based on the regularities of time discrete solutions and the method of mathematical induction.

### 4.1. Finite element approximations

Since VD-MHD system (2.3)-(2.6) is a strongly nonlinear problem, the inverse inequalities will be used in error analysis. Thus, we require that $\mathcal{T}_h = \{K_j\}_{j=1}^L$ is a quasi-uniform triangular partition of $\Omega$ with the mesh size $h = \max_j\{diam\,K_j\}$. We use the piecewise quadratic element to approximate $\sigma$, the mini element to approximate $(\mathbf{u}, p)$, and the piecewise linear element to approximate $\mathbf{b}$. The finite element spaces of $X$, $\mathbf{V}$, $M$ and $\mathbf{W}$ are denoted by $X_h$, $\mathbf{V}_h$, $M_h$ and $\mathbf{W}_h$, respectively.

We denote Raviart-Thomas finite element space of order 1 by

$$\mathbf{RT}_h = \left\{\mathbf{u}_h \in \mathbf{H}(\mathrm{div}\,,\Omega),\ \mathbf{u}_h|_K \in P_1(K)^2 + xP_1(K),\ \forall K \in \mathcal{T}_h\right\}.$$

Let $\mathbf{P}_{1h}$ be the Raviart-Thomas projection from $\mathbf{H}(\mathrm{div}\,,\Omega)$ onto $\mathbf{RT}_h$ and satisfy (cf. [29])

$$\|\mathbf{u} - \mathbf{P}_{1h}\mathbf{u}\|_{L^2} \le Ch^l\|\mathbf{u}\|_{H^l}, \quad \forall\ \mathbf{u} \in \mathbf{H}^l(\Omega), l = 1, 2.$$

Introduce

$$\mathbf{RT}_{0h} = \left\{\mathbf{u}_h \in \mathbf{RT}_h,\ \nabla \cdot \mathbf{u}_h = 0 \quad \text{and} \quad \mathbf{u}_h \cdot \mathbf{n}|_\Gamma = 0\right\}.$$

Define the $L^2$ projection operator $\mathbf{P}_{0h}$: $L^2(\Omega) \to \mathbf{RT}_{0h}$ by

$(\mathbf{u}-\mathbf{P}_{0h}\mathbf{u},\mathbf{v}_h)=0,\quad \forall\ \mathbf{v}_h\in\mathbf{RT}_{0h},\ \mathbf{u}\in\mathbf{L}^2(\Omega).$

For the time discrete solution $\mathbf{u}^n$ satisfying $\nabla\cdot\mathbf{u}^n=0$ in $\Omega$ and $\mathbf{u}^n\cdot\mathbf{n}=0$ on $\Gamma$, one has

$$\nabla\cdot\mathbf{P}_{1h}\mathbf{u}^n=0 \text{ in } \Omega \quad \text{and} \quad \mathbf{P}_{1h}\mathbf{u}^n\cdot\mathbf{n}=0 \text{ on } \Gamma.$$

Thus, we can see $\mathbf{P}_{1h}\mathbf{u}^n\in\mathbf{RT}_{0h}$. Furthermore, one has (cf. [16])

$$\|\mathbf{u}^n-\mathbf{P}_{0h}\mathbf{u}^n\|_{L^2}\le\|\mathbf{u}^n-\mathbf{P}_{1h}\mathbf{u}^n\|_{L^2}\le Ch^2\|\mathbf{u}^n\|_{H^2}. \tag{4.1}$$

Finally, the following inverse inequality in 2D space is frequently used in our error analysis:

$$\begin{cases}\|\mathbf{v}_h\|_{L^3}\le Ch^{-1/3}\|\mathbf{v}_h\|_{L^2}, & \|\mathbf{v}_h\|_{L^\infty}\le Ch^{-1}\|\mathbf{v}_h\|_{L^2}, \quad \forall\ \mathbf{v}_h\in\mathbf{V}_h,\\ \|r_h\|_{L^3}\le Ch^{-1/3}\|r_h\|_{L^2}, & \|r_h\|_{L^\infty}\le Ch^{-1}\|r_h\|_{L^2}, \quad \forall\ r_h\in X_h.\end{cases} \tag{4.2}$$

For $0\le n\le N-1$, the finite element fully discrete algorithm corresponding to (3.1)-(3.3) is described as follows.

**First-order Euler finite element algorithm**

**Step I:** For given $\sigma_h^n\in X_h$ and $\mathbf{u}_h^n\in\mathbf{V}_h$, we solve $\sigma_h^{n+1}\in X_h$ by

$$(D_\tau\sigma_h^{n+1},r_h)+(\nabla\sigma_h^{n+1}\cdot\mathbf{P}_{0h}\mathbf{u}_h^n,r_h)=0,\quad \forall\ r_h\in X_h. \tag{4.3}$$

**Step II:** For given $\mathbf{u}_h^n\in\mathbf{V}_h$ and $\mathbf{b}_h^n\in\mathbf{W}_h$, we solve $(\mathbf{u}_h^{n+1},p_h^{n+1})\in\mathbf{V}_h\times M_h$ and $\mathbf{b}_h^{n+1}\in\mathbf{W}_h$ such that

$$\begin{aligned}&\sigma_h^{n+1}(\mathbf{D}_\tau(\sigma_h^{n+1}\mathbf{u}_h^{n+1}),\mathbf{v}_h)-\frac{1}{Re}(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}_h)-(\nabla\cdot\mathbf{v}_h,p_h^{n+1})+(\nabla\cdot\mathbf{u}_h^{n+1},q_h)\\&+(\rho_h^{n+1}(\mathbf{u}_h^n\cdot\nabla)\mathbf{u}_h^{n+1},\mathbf{v}_h)+\frac{1}{2}(\mathbf{u}_h^{n+1}\nabla\cdot(\rho_h^{n+1}\mathbf{u}_h^n),\mathbf{v}_h)+S(\mathbf{b}_h^n\times curl\mathbf{b}_h^{n+1},\mathbf{v}_h)\\&=(\mathbf{f}^{n+1},\mathbf{v}_h),\end{aligned} \tag{4.4}$$

for all $(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times\ M_h$, and

$$\begin{aligned}&(D_\tau\mathbf{b}_h^{n+1},\mathbf{w}_h)+\frac{1}{Rm}(curl\mathbf{b}_h^{n+1},curl\mathbf{w}_h)+\frac{1}{Rm}(\nabla\cdot\mathbf{b}_h^{n+1},\nabla\cdot\mathbf{w}_h)\\&-(\mathbf{u}_h^{n+1}\times\mathbf{b}_h^n,curl\mathbf{w}_h)=0,\quad\forall\ \mathbf{w}_h\in\mathbf{W}_h,\end{aligned} \tag{4.5}$$

where $\rho_h^{n+1}=(\sigma_h^{n+1})^2$.

**Remark 4.1.** The proposed finite element algorithm is a semi-implicit linearized scheme and we only need to solve linear systems for $\sigma_h^{n+1}$ and $(\mathbf{u}_h^{n+1},p_h^{n+1},\mathbf{b}_h^{n+1})$ at each time step without using precondition method. The initial values $\sigma_h^0,\mathbf{u}_h^0$ and $\mathbf{b}_h^0$ are defined by $\sigma_h^0=J_h\sigma_0,\mathbf{u}_h^0=\mathbf{I}_h\mathbf{u}_0$ and $\mathbf{b}_h^0=\mathbf{K}_h\mathbf{b}_0$, where $\mathbf{I}_h,J_h$ and $\mathbf{K}_h$ are the interpolation operators onto finite element spaces. Furthermore, there holds

$$\|\sigma^0-\sigma_h^0\|_{L^2}+\|\mathbf{u}^0-\mathbf{u}_h^0\|_{L^2}+\|\mathbf{b}^0-\mathbf{b}_h^0\|_{L^2}\le Ch^2.$$

**Remark 4.2.** The post-processed velocity $\mathbf{P}_{0h}\mathbf{u}_h^n$ is used to preserve the unconditional stability of numerical scheme (4.3).

## *4.2. Stability of the FE algorithm*

Setting $r_h=2\tau\sigma_h^{n+1}$ in (4.3), we have

$$\|\sigma_h^{n+1}\|_{L^2}^2-\|\sigma_h^n\|_{L^2}^2+\|\sigma_h^{n+1}-\sigma_h^n\|_{L^2}^2=0 \tag{4.6}$$

by using

$$\begin{aligned}(\nabla\sigma_h^{n+1}\cdot\mathbf{P}_{0h}\mathbf{u}_h^n,\sigma_h^{n+1})&=\frac{1}{2}\int_\Omega\nabla|\sigma_h^{n+1}|^2\cdot\mathbf{P}_{0h}\mathbf{u}_h^n dx\\&=-\frac{1}{2}\int_\Omega|\sigma_n^{n+1}|^2\nabla\cdot\mathbf{P}_{0h}\mathbf{u}_h^n dx=0.\end{aligned}$$

It is clear that (4.6) gives the following $L^2$-stability:

$$\|\sigma_h^{m+1}\|_{L^2}^2\le\|J_h\sigma_0\|_{L^2}^2\le C\|\sigma_0\|_{L^2}^2,\quad\forall\ 0\le m\le N-1. \tag{4.7}$$

Setting $(\mathbf{v}_h,q_h)=2\tau(\mathbf{u}_h^{n+1},p_h^{n+1})$ and $\mathbf{w}_h=2S\tau\mathbf{b}_h^{n+1}$ in (4.4) and (4.5), respectively, and taking the sum of resulting equations, we have

$$\begin{aligned}&\|\sigma_h^{n+1}\mathbf{u}_h^{n+1}\|_{L^2}^2-\|\sigma_h^n\mathbf{u}_h^n\|_{L^2}^2+\|\sigma_h^{n+1}\mathbf{u}_h^{n+1}-\sigma_h^n\mathbf{u}_h^n\|_{L^2}^2+\frac{2\tau}{Re}\|\nabla\mathbf{u}_h^{n+1}\|_{L^2}^2\\&S(\|\mathbf{b}_h^{n+1}\|_{L^2}^2-\|\mathbf{b}_h^n\|_{L^2}^2+\|\mathbf{b}_h^{n+1}-\mathbf{b}_h^n\|_{L^2}^2)+\frac{2S\tau}{Rm}\|curl\mathbf{b}_h^{n+1}\|_W^2=2\tau(\mathbf{f}^{n+1},\mathbf{u}_h^{n+1}),\end{aligned} \tag{4.8}$$

where we use

$$2(\rho_h^{n+1}(\mathbf{u}_h^n\cdot\nabla)\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1})=(\rho_h^{n+1}\mathbf{u}_h^n,\nabla|\mathbf{u}_h^{n+1}|^2)=-(\mathbf{u}_h^{n+1}\nabla\cdot(\rho_h^{n+1}\mathbf{u}_h^n),\mathbf{u}_h^{n+1})$$

and the following vector formula:

$$(\mathbf{a}\times curl\mathbf{b},\mathbf{c})=(\mathbf{c}\times\mathbf{a},curl\mathbf{b}). \tag{4.9}$$

Summing up (4.8) from 0 to $m$ with $0\le m\le N-1$ gives

$$\begin{aligned}&\|\sigma_h^{m+1}\mathbf{u}_h^{m+1}\|_{L^2}^2+S\|\mathbf{b}_h^{m+1}\|_{L^2}^2+\tau\sum_{n=0}^{m}\Big(\frac{1}{Re}\|\nabla\mathbf{u}_h^{n+1}\|_{L^2}^2+\frac{S}{Rm}\|\mathbf{b}_h^{n+1}\|_W^2\Big)\\&\le\|\sigma_h^0\mathbf{u}_h^0\|_{L^2}^2+S\|\mathbf{b}_h^0\|_{L^2}^2+C\tau\sum_{n=0}^{N-1}\|\mathbf{f}^{n+1}\|_{L^2}^2.\end{aligned} \tag{4.10}$$

Inequalities (4.7) and (4.10) mean that the proposed finite element scheme is unconditionally stable and imply the existence and uniqueness of the solution to the finite element sub-problems (4.3)-(4.5).

## *4.3. Spatial error analysis*

In this subsection, we present a rigorous spatial error analysis in $L^2$-norm for the finite element solution $(\sigma_h^n,\mathbf{u}_h^n,\mathbf{b}_h^n)$ with $1\le n\le N$. To do it, we introduce some classical projection operators $\Pi_h: X\to X_h$, $(\mathbf{R}_h,Q_h):\mathbf{V}\times M\to\mathbf{V}_h\times M_h$ and $\mathbf{Z}_h: W\to W_h$ defined by

$$\begin{aligned}&(\Pi_h\sigma^n-\sigma^n,r_h)=0,\quad\forall\ r_h\in X_h,\\&(\nabla(\mathbf{R}_h\mathbf{u}^n-\mathbf{u}^n),\nabla\mathbf{v}_h)-(\nabla\cdot\mathbf{v}_h,Q_hp^n-p^n)+(\nabla\cdot(\mathbf{R}_h\mathbf{u}^n-\mathbf{u}^n),q_h)=0,\\&\quad\forall\ (\mathbf{v}_h,q_h)\in\mathbf{V}_h\times M_h,\\&(curl(\mathbf{Z}_h\mathbf{b}^n-\mathbf{b}^n),curl\mathbf{w}_h)+(\nabla\cdot(\mathbf{Z}_h\mathbf{b}^n-\mathbf{b}^n),\nabla\cdot\mathbf{w}_h)=0,\ \forall\ \mathbf{w}_h\in\mathbf{W}_h.\end{aligned}$$

For $1\le n\le N$, we denote error functions by

$$\begin{aligned}&\theta^n=\Pi_h\sigma^n-\sigma^n,\ \mathbf{E}^n=\mathbf{R}_h\mathbf{u}^n-\mathbf{u}^n,\ \beta^n=\Pi_h\rho^n-\rho^n,\ \gamma^n=\mathbf{Z}_h\mathbf{b}^n-\mathbf{b}^n,\\&e_{\sigma,h}^n=\Pi_h\sigma^n-\sigma_h^n,\ e_{\rho,h}^n=\Pi_h\rho^n-\rho_h^n,\ e_{\mathbf{u},h}^n=\mathbf{R}_h\mathbf{u}^n-\mathbf{u}_h^n,\ e_{p,h}^n=Q_hp^n-p_h^n,\\&e_{\mathbf{b},h}^n=\mathbf{Z}_h\mathbf{b}^n-\mathbf{b}_h^n,\end{aligned}$$

where $(\sigma_h^n,\mathbf{u}_h^n,p_h^n,\mathbf{b}_h^n)$ and $(\sigma^n,\mathbf{u}^n,p^n,\mathbf{b}^n)$ are solutions to (4.3)-(4.5) and (3.1)-(3.3), respectively. Moreover, according to regularity results (3.8) and (3.12)-(3.13), the following approximations hold:

$$\begin{cases}\|\mathbf{E}^n\|_{L^2}+h\|\nabla\mathbf{E}^n\|_{L^2}\le Ch^2(\|\mathbf{u}^n\|_{H^2}+\|p^n\|_{H^1}),\\ \|\theta^n\|_{L^2}+h\|\theta^n\|_{H^1}\le Ch^3\|\sigma^n\|_{H^3},\\ \|D_\tau\gamma^n\|_{L^2}+\|D_\tau\mathbf{E}^n\|_{L^2}\le Ch^2(\|D_\tau\mathbf{u}^n\|_{H^2}+\|D_\tau\mathbf{b}^n\|_{H^2}),\\ \|D_\tau\theta^n\|_{L^2}\le Ch^2,\\ \|\mathbf{E}^n\|_{L^\infty}+\|\theta^n\|_{L^\infty}\le Ch,\\ \|\mathbf{E}^n\|_{L^4}+h\|\mathbf{E}^n\|_{W^{1,4}}\le Ch^2(\|\mathbf{u}^n\|_{H^3}+\|p^n\|_{H^2}),\\ \|\theta^n\|_{L^4}+h\|\theta^n\|_{W^{1,4}}\le Ch^2\|\sigma^n\|_{H^3}.\end{cases} \tag{4.11}$$

Subtracting (4.3)-(4.5) from (3.4)-(3.6), we get the following error equations:

$$(D_\tau e_{\sigma,h}^{n+1},r_h)+(\nabla\sigma^{n+1}\cdot(\mathbf{u}^n-\mathbf{P}_{0h}\mathbf{u}_h^n),r_h)+(\nabla e_{\sigma,h}^{n+1}\cdot(\mathbf{P}_{0h}\mathbf{u}_h^n-\mathbf{u}^n),r_h)$$
$$+(\nabla e_{\sigma,h}^{n+1}\cdot\mathbf{u}^n,r_h)-(\nabla\theta^{n+1}\cdot(\mathbf{P}_{0h}\mathbf{u}_h^n-\mathbf{u}^n),r_h)-(\nabla\theta^{n+1}\cdot\mathbf{u}^n,r_h)=0,\quad \forall\, r_h\in X_h,\tag{4.12}$$

and

$$\begin{aligned}
&(\sigma_h^{n+1}D_\tau(\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}),\mathbf{v}_h)+\frac{1}{Re}(\nabla e_{\mathbf{u},h}^{n+1},\nabla\mathbf{v}_h)-(\nabla\cdot\mathbf{v}_h,e_{p,h}^{n+1})+(\nabla\cdot e_{\mathbf{u},h}^{n+1},q_h)\\
&=\left(\sigma_h^{n+1}D_\tau(\sigma_h^{n+1}\mathbf{E}^{n+1}),\mathbf{v}_h\right)+\left(\theta^{n+1}D_\tau(\sigma^{n+1}\mathbf{u}^{n+1}),\mathbf{v}_h\right)\\
&\quad+\left(\sigma_h^{n+1}\theta^{n+1}D_\tau\mathbf{u}^{n+1},\mathbf{v}_h\right)-\left(\sigma_h^{n+1}e_{\sigma,h}^{n+1}D_\tau\mathbf{u}^{n+1},\mathbf{v}_h\right)\\
&\quad+\left(\sigma_h^{n+1}D_\tau\theta^{n+1}\mathbf{u}^n,\mathbf{v}_h\right)-\left(\sigma_h^{n+1}D_\tau e_{\sigma,h}^{n+1}\mathbf{u}^n,\mathbf{v}_h\right)-\left(e_{\sigma,h}^{n+1}D_\tau(\sigma^{n+1}\mathbf{u}^{n+1}),\mathbf{v}_h\right)\\
&\quad+\left(\beta^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1},\mathbf{v}_h\right)-\left(e_{\rho,h}^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1},\mathbf{v}_h\right)\\
&\quad+\left(\rho_h^{n+1}(\mathbf{E}^n\cdot\nabla)\mathbf{u}^{n+1},\mathbf{v}_h\right)-\left(\rho_h^{n+1}(e_{\mathbf{u},h}^n\cdot\nabla)\mathbf{u}^{n+1},\mathbf{v}_h\right)\\
&\quad+\left(\rho_h^{n+1}(\mathbf{u}_h^n\cdot\nabla)\mathbf{E}^{n+1},\mathbf{v}_h\right)-\left(\rho_h^{n+1}(\mathbf{u}_h^n\cdot\nabla)e_{\mathbf{u},h}^{n+1},\mathbf{v}_h\right)\\
&\quad+\frac12\left(\mathbf{u}^{n+1}\nabla\cdot(\beta^{n+1}\mathbf{u}^n),\mathbf{v}_h\right)-\frac12\left(\mathbf{u}^{n+1}\nabla\cdot(e_{\rho,h}^{n+1}\mathbf{u}^n),\mathbf{v}_h\right)\\
&\quad+\frac12\left(\mathbf{u}^{n+1}\nabla\cdot(\rho_h^{n+1}\mathbf{E}^n),\mathbf{v}_h\right)-\frac12\left(\mathbf{u}^{n+1}\nabla\cdot(\rho_h^{n+1}e_{\mathbf{u},h}^n),\mathbf{v}_h\right)\\
&\quad-\frac12\left(e_{\mathbf{u},h}^{n+1}\nabla\cdot(\rho_h^{n+1}\mathbf{u}_h^n),\mathbf{v}_h\right)+\frac12\left(\mathbf{E}^{n+1}\nabla\cdot(\rho_h^{n+1}\mathbf{u}_h^n),\mathbf{v}_h\right)\\
&\quad-S\left(e_{\mathbf{b},h}^n\times curl\mathbf{b}^{n+1},\mathbf{v}_h\right)+S\left(\gamma^n\times curl\mathbf{b}^{n+1},\mathbf{v}_h\right)\\
&\quad+S\left(\mathbf{b}_h^n\times curl\gamma^{n+1},\mathbf{v}_h\right)-S\left(\mathbf{b}_h^n\times curl e_{\mathbf{b},h}^{n+1},\mathbf{v}_h\right)
\end{aligned}\tag{4.13}$$
$$=\sum_{i=1}^{23}(J_{i,h}^{n+1},\mathbf{v}_h),\quad\forall\,(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times M_h,$$

where $(J_{i,h}^{n+1},\mathbf{v}_h),(i=1,\cdots,23)$, are defined in order, and

$$\begin{aligned}
&(D_\tau e_{\mathbf{b},h}^{n+1},\mathbf{w}_h)+\frac{1}{Rm}(curl e_{\mathbf{b},h}^{n+1},curl\mathbf{w}_h)+\frac{1}{Rm}(\nabla\cdot e_{\mathbf{b},h}^{n+1},\nabla\cdot\mathbf{w}_h)\\
&=(D_\tau\gamma^{n+1},\mathbf{w}_h)+(\mathbf{u}^{n+1}\times e_{\mathbf{b},h}^n,curl\mathbf{w}_h)-(\mathbf{u}^{n+1}\times\gamma^n,curl\mathbf{w}_h)\\
&\quad+(e_{\mathbf{u},h}^{n+1}\times\mathbf{b}_h^n,curl\mathbf{w}_h)\\
&\quad-(\mathbf{E}^{n+1}\times b_h^n,curl\mathbf{w}_h),\quad\forall\,\mathbf{w}_h\in\mathbf{W}_h.
\end{aligned}\tag{4.14}$$

First, we recall the following two lemmas established in [23].

**Lemma 4.1.** *Under the assumptions **(A1)** and **(A2)**, there exists some small $\tau_1$ such that when $\tau\le\tau_1$, there holds*

$$\|e_{\sigma,h}^{m+1}\|_{L^2}^2+\sum_{n=0}^m\|e_{\sigma,h}^{n+1}-e_{\sigma,h}^n\|_{L^2}^2\le Ch^4+C\tau\sum_{n=0}^m\|\boldsymbol{u}^n-\boldsymbol{P}_{0h}\boldsymbol{u}_h^n\|_{L^2}^2,\tag{4.15}$$

*for all $0\le m\le N-1$.*

**Lemma 4.2.** *Under the assumptions **(A1)** and **(A2)**, if*

$$\|\boldsymbol{u}^n-\boldsymbol{P}_{0h}\boldsymbol{u}_h^n\|_{L^2}\le Ch^{3/2},\quad\forall\,0\le n\le N-1,\tag{4.16}$$

*then we have*

$$\|D_\tau e_{\sigma.h}^{m+1}\|_{L^2}^2\le Ch^2+C\tau h^{-2}\sum_{n=0}^m\|\boldsymbol{u}^n-\boldsymbol{P}_{0h}\boldsymbol{u}_h^n\|_{L^2}^2,\quad\forall\,0\le m\le N-1.\tag{4.17}$$

Based on (4.15) and (4.17), we estimate $e_{\mathbf{u},h}^{n+1}$ and $e_{\mathbf{b},h}^{n+1}$ by using the method of mathematical induction in Theorem 4.1.

**Theorem 4.1.** *Under the assumptions **(A1)** and **(A2)**, there exists $\tau_2$ and $h_2$ such that when $\tau\le\tau_2$ and $h\le h_2$, we have*

$$\|e_{\mathbf{u},h}^m\|_{L^2}^2+\|e_{\mathbf{b},h}^m\|_{L^2}^2+\tau\sum_{i=1}^m\left(\|\nabla e_{\mathbf{u},h}^i\|_{L^2}^2+\|e_{\mathbf{b},h}^i\|_W^2\right)\le C_0^2h^4\tag{4.18}$$

*for $1\le m\le N$, where $C_0>0$ is some constant independent of $\tau$ and $h$.*

**Proof.** We first prove that the error estimate (4.18) is valid for $m=1$. It follows from (4.15) with $m=0$ that

$$\|e_{\sigma,h}^1\|_{L^2}^2\le Ch^4+C\tau\|\mathbf{u}^0-\mathbf{P}_{0h}\mathbf{u}_h^0\|_{L^2}^2\le Ch^4,\tag{4.19}$$

where we noted

$$\|\mathbf{u}^0-\mathbf{P}_{0h}\mathbf{u}_h^0\|_{L^2}^2\le2\|\mathbf{u}^0-\mathbf{P}_{0h}\mathbf{u}^0\|_{L^2}^2+2\|\mathbf{P}_{0h}\mathbf{u}^0-\mathbf{P}_{0h}\mathbf{u}_h^0\|_{L^2}^2\le Ch^4.$$

Combining the inverse inequality (4.2), (4.11) and (4.19) gives

$$\|e_{\sigma,h}^1\|_{L^\infty}\le Ch^{-1}\|e_{\sigma,h}^1\|_{L^2}\le Ch,$$
$$\|\sigma^1-\sigma_h^1\|_{L^\infty}\le\|\sigma^1-\Pi_h\sigma^1\|_{L^\infty}+\|e_{\sigma,h}^1\|_{L^\infty}\le Ch.$$

By noting (3.7) one has

$$\frac12\sigma_0^{min}\le\sigma_0^{min}-Ch\le\sigma_h^1\le\sigma_0^{max}+Ch\le\frac32\sigma_0^{max},\tag{4.20}$$

for sufficiently small $h$ such that $Ch\le\frac12\sigma_0^{min}$.

Setting $\mathbf{w}_h=2\tau e_{\mathbf{b},h}^1$ in (4.14) with $n=0$, and using the Hölder's inequality, Young's inequality, (3.8), (4.11), (4.20), we have

$$\begin{aligned}
&\|e_{\mathbf{b},h}^1\|_{L^2}^2+\frac{2\tau}{Rm}\|e_{\mathbf{b},h}^1\|_W^2\\
&\le2\tau|(D_\tau\gamma^1,e_{\mathbf{b},h}^1)|+2\tau|(\mathbf{u}^1\times e_{\mathbf{b},h}^0,curl e_{\mathbf{b},h}^1)|+2\tau|(\mathbf{u}^1\times\gamma^0,curl e_{\mathbf{b},h}^1)|\\
&\quad+2\tau|(e_{\mathbf{u},h}^1\times\mathbf{b}_h^0,curl e_{\mathbf{b},h}^1)|+2\tau|(\mathbf{E}^1\times\mathbf{b}_h^0,curl e_{\mathbf{b},h}^1)|+\|e_{\mathbf{b},h}^0\|_{L^2}^2\\
&\le\frac{\tau}{Rm}\|e_{\mathbf{b},h}^1\|_W^2+Ch^4+C\tau\|e_{\mathbf{u},h}^1\|_{L^2}^2,
\end{aligned}$$

which leads to

$$\|e_{\mathbf{b},h}^1\|_{L^2}^2+\frac{\tau}{Rm}\|e_{\mathbf{b},h}^1\|_W^2\le Ch^4+\left(\frac{\sigma_0^{min}}{8}\right)^2\|e_{\mathbf{u},h}^1\|_{L^2}^2\tag{4.21}$$

for sufficiently small $\tau$ such that $C\tau\le(\frac{\sigma_0^{min}}{8})^2$.

Setting $(\mathbf{v}_h,q_h)=2\tau(e_{\mathbf{u},h}^1,e_{p,h}^1)$ in (4.13) with $n=0$ and using (4.20), we have

$$\begin{aligned}
\left(\frac{\sigma_0^{min}}{2}\right)^2\|e_{\mathbf{u},h}^1\|_{L^2}^2+\frac{2\tau}{Re}\|\nabla e_{\mathbf{u},h}^1\|_{L^2}^2&\le2\tau\sum_{i=1}^{23}|(J_{i,h}^1,e_{\mathbf{u},h}^1)|+\|\sigma_h^0e_{\mathbf{u},h}^0\|_{L^2}^2\\
&\le Ch^4+2\tau\sum_{i=1}^{23}|(J_{i,h}^1,e_{\mathbf{u},h}^1)|.
\end{aligned}\tag{4.22}$$

By the Hölder's inequality and Young's inequality, we estimate the right-hand side of (4.22) as follows. In terms of (3.8), (3.12), (4.11) and (4.20), one has

$$\begin{aligned}
2\tau|(J_{1,h}^1,e_{\mathbf{u},h}^1)|&\le2|\tau(\rho_h^1D_\tau\mathbf{E}^1,e_{\mathbf{u},h}^1)|+2|(\sigma_h^1\mathbf{E}^0(\sigma_h^1-\sigma_h^0),e_{\mathbf{u},h}^1)|\\
&\le C\tau\|\rho_h^1\|_{L^\infty}\|D_\tau\mathbf{E}^1\|_{L^2}\|e_{\mathbf{u},h}^1\|_{L^2}\\
&\quad+C\|\sigma_h^1\|_{L^\infty}\|\mathbf{E}^0\|_{L^2}\|\sigma_h^1-\sigma_h^0\|_{L^\infty}\|e_{\mathbf{u},h}^1\|_{L^2}\\
&\le Ch^4+\left(\frac{\sigma_0^{min}}{8}\right)^2\|e_{\mathbf{u},h}^1\|_{L^2}^2,\\
2\tau|(J_{2,h}^1,e_{\mathbf{u},h}^1)|&\le2\tau|(\theta^1\sigma^1D_\tau\mathbf{u}^1,e_{\mathbf{u},h}^1)|+2\tau|(\theta^1(D_\tau\sigma^1)\mathbf{u}^0,e_{\mathbf{u},h}^1)|\\
&\le Ch^4+\frac{\tau}{23Re}\|\nabla e_{\mathbf{u},h}^1\|_{L^2}^2.
\end{aligned}$$

It follows from (3.12), (4.11) and (4.20) that

$$\begin{aligned}
2\tau|(J_{3,h}^1,e_{\mathbf{u},h}^1)|&\le C\tau\|\theta^1\|_{L^2}\|D_\tau\mathbf{u}^1\|_{L^3}\|e_{\mathbf{u},h}^1\|_{L^6}\\
&\le Ch^4+\frac{\tau}{23Re}\|\nabla e_{\mathbf{u},h}^1\|_{L^2}^2,\\
2\tau|(J_{4,h}^1,e_{\mathbf{u},h}^1)|&\le C\tau\|e_{\sigma,h}^1\|_{L^2}\|D_\tau\mathbf{u}^1\|_{L^3}\|e_{\mathbf{u},h}^1\|_{L^6}\\
&\le Ch^4+\frac{\tau}{23Re}\|\nabla e_{\mathbf{u},h}^1\|_{L^2}^2,\\
2\tau|(J_{5,h}^1e_{\mathbf{u},h}^1)|&\le C\tau\|\sigma_h^1\|_{L^\infty}\|\mathbf{u}^0\|_{H^2}\|D_\tau\theta^1\|_{L^2}\|e_{\mathbf{u},h}^1\|_{L^2}\\
&\le Ch^4+\left(\frac{\sigma_0^{min}}{8}\right)^2\|e_{\mathbf{u},h}^1\|_{L^2}^2.
\end{aligned}$$

By (3.12) and (4.19), we can estimate the terms on $J^1_{6,h}$ and $J^1_{7h}$ by

$$\begin{aligned}2\tau|(J^1_{6,h},e^1_{\mathbf{u},h})| &\le 2|(\sigma^1_h e^1_{\sigma,h}\mathbf{u}^0,e^1_{\mathbf{u},h})| + 2|(\sigma^1_h(\Pi_h\sigma^0 - J_h\sigma^0)\mathbf{u}^0,e^1_{\mathbf{u},h})|\\ &\le C\|e^1_{\sigma,h}\|_{L^2}\|e^1_{\mathbf{u},h}\|_{L^2} + C(\|\Pi_h\sigma^0-\sigma^0\|_{L^2}\\ &\quad + \|\sigma^0 - J_h\sigma^0\|_{L^2})\|e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \left(\frac{\sigma_0^{min}}{8}\right)^2\|e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{7,h},e^1_{\mathbf{u},h})| &\le 2\tau|(e^1_{\sigma,h}\sigma^1 D_\tau\mathbf{u}^1,e^1_{\mathbf{u},h})| + 2\tau|(e^1_{\sigma,h}\mathbf{u}^0 D_\tau\sigma^1,e^1_{\mathbf{u},h})|\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2}.\end{aligned}$$

From the Sobolev imbedding theorem, (3.8) and (4.11), we have

$$\begin{aligned}2\tau|(J^1_{8,h},e^1_{\mathbf{u},h})| &\le C\tau\|\beta^1\|_{L^2}\|\mathbf{u}^0\|_{L^\infty}\|\nabla\mathbf{u}^1\|_{L^3}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{9,h},e^1_{\mathbf{u},h})| &\le C\tau\|e^1_{\rho,h}\|_{L^2}\|\mathbf{u}^0\|_{L^\infty}\|\nabla\mathbf{u}^1\|_{L^3}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\end{aligned}$$

where we noted

$$\|e^1_{\rho,h}\|_{L^2} \le C\|e^1_{\sigma,h}\|_{L^2} \le Ch^4$$

due to (3.7) and (4.20). By a similar method, we can get

$$\begin{aligned}2\tau|(J^1_{10,h},e^1_{\mathbf{u},h})| &\le C\tau\|\rho^1_h\|_{L^\infty}\|\mathbf{E}^0\|_{L^2}\|\nabla\mathbf{u}^1\|_{L^3}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{11,h},e^1_{\mathbf{u},h})| &\le C\tau\|\rho^1_h\|_{L^\infty}\|e^0_{\mathbf{u},h}\|_{L^2}\|\nabla\mathbf{u}^1\|_{L^3}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2}.\end{aligned}$$

Using the integration by parts, we estimate $2\tau(J^1_{12,h},e^1_{\mathbf{u},h})$ by

$$\begin{aligned}2\tau|(J^1_{12,h},e^1_{\mathbf{u},h})| &\le C\tau\|\rho^1_h\|_{W^{1,3}}\|\mathbf{u}^0_h\|_{L^\infty}\|\mathbf{E}^1\|_{L^2}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\quad + C\tau\|\rho^1_h\|_{L^\infty}\|\mathbf{u}^0_h\|_{W^{1,3}}\|\mathbf{E}^1\|_{L^2}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\quad + C\tau\|\rho^1_h\|_{L^\infty}\|\mathbf{u}^0_h\|_{L^\infty}\|\mathbf{E}^1\|_{L^2}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\end{aligned}$$

where we use

$$\|\rho^1_h\|_{W^{1,3}} \le \|\Pi_h\rho^1\|_{W^{1,3}} + \|e^1_{\rho,h}\|_{W^{1,3}} \le C + Ch^{-\frac{4}{3}}\|e^1_{\sigma,h}\|_{L^2} \le C.$$

By (4.20), we have $\|\rho^1_h\|_{L^\infty} \le C$. Then, one has

$$\begin{aligned}2\tau|(J^1_{13,h},e^1_{\mathbf{u},h})| &\le C\tau\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\|e^1_{\mathbf{u},h}\|_{L^2}\\ &\le \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2} + \left(\frac{\sigma_0^{min}}{8}\right)^2\|e^1_{\mathbf{u},h}\|^2_{L^2}\end{aligned}$$

for sufficiently small $\tau$ such that $C\tau \le (\frac{\sigma_0^{min}}{8})^2$. Using the integration by parts, again, we have

$$\begin{aligned}2\tau|(J^1_{14,h},e^1_{\mathbf{u},h})| &\le C\tau\|\beta^1\|_{L^2}\|\mathbf{u}^0\|_{L^\infty}(\|\nabla\mathbf{u}^1\|_{L^3} + \|\mathbf{u}^1\|_{L^3})\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{15,h},e^1_{\mathbf{u},h})| &\le C\tau\|e^1_{\rho,h}\|_{L^2}\|\mathbf{u}^0\|_{L^\infty}(\|\nabla\mathbf{u}^1\|_{L^3} + \|\mathbf{u}^1\|_{L^3})\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{16,h},e^1_{\mathbf{u},h})| &\le C\tau\|\rho^1_h\|_{L^\infty}\|\mathbf{E}^0\|_{L^2}(\|\nabla\mathbf{u}^1\|_{L^3} + \|\mathbf{u}^1\|_{L^3})\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{17,h},e^1_{\mathbf{u},h})| &\le C\tau\|\rho^1_h\|_{L^\infty}\|e^0_{\mathbf{u},h}\|_{L^2}(\|\nabla\mathbf{u}^1\|_{L^3} + \|\mathbf{u}^1\|_{L^3})\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\end{aligned}$$

and

$$\begin{aligned}2\tau|(J^1_{18,h},e^1_{\mathbf{u},h})| &\le C\tau\|\rho^1_h\|_{W^{1,3}}\|e^1_{\mathbf{u},h}\|_{L^2}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\quad + C\tau\|e^1_{\mathbf{u},h}\|_{L^2}\|\mathbf{u}^0_h\|_{W^{1,3}}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2} + \left(\frac{\sigma_0^{min}}{8}\right)^2\|e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{19,h},e^1_{\mathbf{u},h})| &\le C\tau\|\rho^1_h\|_{W^{1,3}}\|\mathbf{E}^1\|_{L^2}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\quad + C\tau\|\mathbf{u}^0_h\|_{W^{1,3}}\|\mathbf{E}^1\|_{L^2}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2}\end{aligned}$$

for sufficiently small $\tau$ such that $C\tau \le (\frac{\sigma_0^{min}}{8})^2$. According to (3.8), it is easy to show that

$$\begin{aligned}2\tau|(J^1_{20,h},e^1_{\mathbf{u},h})| &\le Ch^4 + \left(\frac{\sigma_0^{min}}{8}\right)^2\|e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{21,h},e^1_{\mathbf{u},h})| &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2},\\ 2\tau|(J^1_{23,h},e^1_{\mathbf{u},h})| &\le \frac{\tau}{5Rm}\|e^1_{\mathbf{b},h}\|^2_{W} + \left(\frac{\sigma_0^{min}}{8}\right)^2\|e^1_{\mathbf{u},h}\|^2_{L^2}\end{aligned}$$

for sufficiently small $\tau$ such that $C\tau \le (\frac{\sigma_0^{min}}{8})^2$. Finally, we use the integration by parts to estimate $2\tau(J^1_{22,h},e^1_{\mathbf{u},h})$ by

$$\begin{aligned}2\tau|(J^1_{22,h},e^1_{\mathbf{u},h})| &\le C\tau\|\nabla\mathbf{b}^0_h\|_{L^3}\|\gamma^1\|_{L^2}\|e^1_{\mathbf{u},h}\|_{L^6} + C\tau\|\mathbf{b}^0_h\|_{L^\infty}\|\gamma^1\|_{L^2}\|\nabla e^1_{\mathbf{u},h}\|_{L^2}\\ &\le Ch^4 + \frac{\tau}{23Re}\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2}.\end{aligned}$$

Combining (4.21)-(4.22) and taking into account the above estimates, we get

$$\|e^1_{\mathbf{u},h}\|^2_{L^2} + \|e^1_{\mathbf{b},h}\|^2_{L^2} + \tau\|e^1_{\mathbf{b},h}\|^2_W + \tau\|\nabla e^1_{\mathbf{u},h}\|^2_{L^2} \le C_1^2h^4, \tag{4.23}$$

where $C_1 > 0$ is independent of $h, \tau$ and $C_0$. Thus, (4.18) is valid for $m = 1$ by taking $C_0 \ge C_1$.

Now, we assume that (4.18) holds for $m \le n$ with $1 \le n \le N-1$, i.e., one has

$$\|e^n_{\mathbf{u},h}\|^2_{L^2} + \|e^n_{\mathbf{b},h}\|^2_{L^2} + \tau\sum_{i=1}^n\left(\|\nabla e^i_{\mathbf{u},h}\|^2_{L^2} + \|e^i_{\mathbf{b},h}\|^2_W\right) \le C_0^2h^4. \tag{4.24}$$

From (4.1), (4.2) and (4.24), we have

$$\begin{aligned}\|\mathbf{P}_{0h}\mathbf{u}^n_h - \mathbf{u}^n\|^2_{L^2} &\le 2\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}^n\|^2_{L^2} + 2\|\mathbf{P}_{0h}\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}^n_h\|^2_{L^2}\\ &\le Ch^4 + 2\|\mathbf{u}^n - \mathbf{u}^n_h\|^2_{L^2}\\ &\le Ch^4 + C\|\mathbf{E}^n\|^2_{L^2} + C\|e^n_{\mathbf{u},h}\|^2_{L^2}\\ &\le Ch^4 + CC_0^2h^4\\ &\le Ch^{\frac{10}{3}}\end{aligned} \tag{4.25}$$

for sufficiently small $h$ such that $(1+C_0^2)h^{\frac{2}{3}} \le 1$, and

$$\begin{aligned}\|\mathbf{P}_{0h}\mathbf{u}^n_h - \mathbf{u}^n\|^2_{L^3} &\le 2\|\mathbf{P}_{0h}\mathbf{u}^n_h - \mathbf{R}_h\mathbf{u}^n\|^2_{L^3} + 2\|\mathbf{R}_h\mathbf{u}^n - \mathbf{u}^n\|^2_{L^3}\\ &\le Ch^{-\frac{2}{3}}\|\mathbf{P}_{0h}\mathbf{u}^n_h - \mathbf{R}_h\mathbf{u}^n\|^2_{L^2}\\ &\quad + C\|\mathbf{R}_h\mathbf{u}^n - \mathbf{u}^n\|_{L^2}\|\nabla(\mathbf{R}_h\mathbf{u}^n - \mathbf{u}^n)\|_{L^2}\\ &\le Ch^{-\frac{2}{3}}(\|\mathbf{E}^n\|^2_{L^2} + \|\mathbf{P}_{0h}\mathbf{u}^n_h - \mathbf{u}^n\|^2_{L^2}) + Ch^3\\ &\le Ch^{\frac{8}{3}}.\end{aligned} \tag{4.26}$$

Furthermore, it follows from (4.2) and (4.24) that

$$\|e^n_{\mathbf{u},h}\|_{L^\infty} \le Ch^{-1}\|e^n_{\mathbf{u},h}\|_{L^2} \le CC_0h \le C,$$

$$\|e^n_{\mathbf{u},h}\|_{L^3} \le Ch^{-1/3}\|e^n_{\mathbf{u},h}\|_{L^2} \le CC_0h^{5/3} \le C$$

for sufficiently small $h$ such that $C_0 h \le 1$, which gives

$$\|\mathbf{u}_h^n\|_{L^\infty} \le \|\mathbf{u}^n\|_{L^\infty} + \|\mathbf{E}^n\|_{L^\infty} + \|e_{\mathbf{u},h}^n\|_{L^\infty} \le C, \tag{4.27}$$

$$\|\nabla \mathbf{u}_h^n\|_{L^3} \le \|\nabla \mathbf{R}_h \mathbf{u}^n\|_{L^3} + \|\nabla e_{\mathbf{u},h}^n\|_{L^3} \le C. \tag{4.28}$$

Similarly, we can get

$$\|\mathbf{b}_h^n\|_{W^{1,3}} + \|\mathbf{b}_h^n\|_{L^\infty} \le C. \tag{4.29}$$

From (4.15) and (4.25), there holds

$$\|e_{\sigma,h}^{n+1}\|_{L^2}^2 + \sum_{i=0}^{n} \|e_{\sigma,h}^{i+1} - e_{\sigma,h}^{i}\|_{L^2}^2 \le Ch^4 + C\tau \sum_{i=0}^{n} \|\mathbf{P}_{0h}\mathbf{u}_h^i - \mathbf{u}^i\|_{L^2}^2 \le Ch^{10/3}. \tag{4.30}$$

By (4.2) and (4.30), we have

$$\begin{aligned} \|\sigma_h^{n+1}\|_{W^{1,3}} &\le \|\Pi_h \sigma^{n+1}\|_{W^{1,3}} + \|e_{\sigma,h}^{n+1}\|_{W^{1,3}} \\ &\le C + Ch^{-\frac{4}{3}} \|e_{\sigma,h}^{n+1}\|_{L^2} \le C, \end{aligned} \tag{4.31}$$

$$\begin{aligned} \|\sigma^{n+1} - \sigma_h^{n+1}\|_{L^\infty} &\le \|\theta^{n+1}\|_{L^\infty} + \|e_{\sigma,h}^{n+1}\|_{L^\infty} \\ &\le Ch + Ch^{-1}\|e_{\sigma,h}^{n+1}\|_{L^2} \le Ch^{1/2}, \end{aligned} \tag{4.32}$$

which with (3.7) leads to

$$\frac{1}{2}\sigma_0^{min} \le \sigma_0^{\min} - Ch^{\frac{1}{2}} \le \sigma_h^{n+1} \le \sigma_0^{\max} + Ch^{\frac{1}{2}} \le \frac{3}{2}\sigma_0^{max} \tag{4.33}$$

for sufficiently small $h$ such that $Ch^{1/2} \le \frac{1}{2}\sigma_0^{min}$.

From (4.16), (4.17) and (4.25), we get

$$\|D_\tau e_{\sigma,h}^{n+1}\|_{L^2}^2 \le Ch^2 + C\tau h^{-2} \sum_{i=0}^{n} \|\mathbf{u}^i - \mathbf{P}_{0h}\mathbf{u}_h^i\|_{L^2}^2 \le Ch^{\frac{4}{3}}, \tag{4.34}$$

$$\|D_\tau e_{\sigma,h}^{n+1}\|_{L^3} \le Ch^{-\frac{1}{3}} \|D_\tau e_{\sigma,h}^{n+1}\|_{L^2} \le Ch^{\frac{1}{3}}. \tag{4.35}$$

To close the mathematical induction, we need to prove that (4.18) is valid for $m \le n+1$ with $1 \le n \le N-1$.

Setting $\mathbf{w}_h = 2S\tau e_{\mathbf{b},h}^{n+1}$ in (4.14), we have

$$\begin{aligned} &S(\|e_{\mathbf{b},h}^{n+1}\|_{L^2}^2 - \|e_{\mathbf{b},h}^{n}\|_{L^2}^2 + \|e_{\mathbf{b},h}^{n+1} - e_{\mathbf{b},h}^{n}\|_{L^2}^2) + \frac{2S\tau}{Rm}\|e_{\mathbf{b},h}^{n+1}\|_W^2 \\ &= 2S\tau(D_\tau \gamma^{n+1}, e_{\mathbf{b},h}^{n+1}) + 2S\tau(\mathbf{u}^{n+1} \times e_{\mathbf{b},h}^{n}, curl e_{\mathbf{b},h}^{n+1}) - 2S\tau(\mathbf{u}^{n+1} \times \gamma^n, curl e_{\mathbf{b},h}^{n+1}) \\ &\quad -2S\tau(\mathbf{E}^{n+1} \times \mathbf{b}_h^n, curl e_{\mathbf{b},h}^{n+1}) + 2S\tau(e_{\mathbf{u},h}^{n+1} \times \mathbf{b}_h^n, curl e_{\mathbf{b},h}^{n+1}). \end{aligned} \tag{4.36}$$

Applying the Hölder's inequality, Young's inequality, regularity results (3.8), (3.13) and the projection errors (4.11), the right-hand side of (4.36) can be estimated by

$$2S\tau(D_\tau \gamma^{n+1}, e_{\mathbf{b},h}^{n+1}) \le C\tau h^4 \|D_\tau \mathbf{b}^{n+1}\|_{H^2}^2 + \frac{S\tau}{4Rm}\|e_{\mathbf{b},h}^{n+1}\|_W^2,$$

$$2S\tau(\mathbf{u}^{n+1} \times e_{\mathbf{b},h}^{n}, curl e_{\mathbf{b},h}^{n+1}) \le C\tau \|e_{\mathbf{b},h}^{n}\|_{L^2}^2 + \frac{S\tau}{4Rm}\|e_{\mathbf{b},h}^{n+1}\|_W^2,$$

$$2S\tau(\mathbf{u}^{n+1} \times \gamma^n, curl e_{\mathbf{b},h}^{n+1}) \le C\tau h^4 + \frac{S\tau}{4Rm}\|e_{\mathbf{b},h}^{n+1}\|_W^2,$$

$$2S\tau(\mathbf{E}^{n+1} \times \mathbf{b}_h^n, curl e_{\mathbf{b},h}^{n+1}) \le C\tau h^4 + \frac{S\tau}{4Rm}\|e_{\mathbf{b},h}^{n+1}\|_W^2.$$

Substituting above estimates into (4.36) gives

$$\begin{aligned} &S(\|e_{\mathbf{b},h}^{n+1}\|_{L^2}^2 - \|e_{\mathbf{b},h}^{n}\|_{L^2}^2 + \|e_{\mathbf{b},h}^{n+1} - e_{\mathbf{b},h}^{n}\|_{L^2}^2) + \frac{S\tau}{Rm}\|e_{\mathbf{b},h}^{n+1}\|_W^2 \\ &\le C\tau h^4 + C\tau h^4 \|D_\tau \mathbf{b}^{n+1}\|_{H^2}^2 + C\tau \|e_{\mathbf{b},h}^{n}\|_{L^2}^2 + 2S\tau(e_{\mathbf{u},h}^{n+1} \times \mathbf{b}_h^n, curl e_{\mathbf{b},h}^{n+1}). \end{aligned} \tag{4.37}$$

Setting $(v_h, q_h) = 2\tau(e_{\mathbf{u},h}^{n+1}, e_{p,h}^{n+1})$ in (4.13), we have

$$\begin{aligned} &\|\sigma_h^{n+1} e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 - \|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 + \frac{2\tau}{Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 \\ &\le 2\tau \sum_{i=1}^{22} (J_{i,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) - 2S\tau \left(\mathbf{b}_h^n \times curl e_{\mathbf{b},h}^{n+1}, e_{\mathbf{u},h}^{n+1}\right). \end{aligned} \tag{4.38}$$

Taking the sum of (4.37) and (4.38) and noting (4.9), we get

$$\begin{aligned} &\|\sigma_h^{n+1} e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 - \|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 + S(\|e_{\mathbf{b},h}^{n+1}\|_{L^2}^2 - \|e_{\mathbf{b},h}^{n}\|_{L^2}^2) \\ &+ \frac{2\tau}{Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + \frac{S\tau}{Rm}\|e_{\mathbf{b},h}^{n+1}\|_W^2 \\ &\le C\tau h^4 + C\tau h^4\|D_\tau \mathbf{b}^{n+1}\|_{H^2}^2 + C\tau\|e_{\mathbf{b},h}^{n}\|_{L^2}^2 + 2\tau \sum_{i=1}^{22}(J_{i,h}^{n+1}, e_{\mathbf{u},h}^{n+1}). \end{aligned} \tag{4.39}$$

By the Hölder's inequality, Young's inequality, regularity results (3.8), (3.12)-(3.13), the projection errors (4.11) and the induction assumptions (4.27)-(4.35), we estimate the right-hand side of (4.39) as follows.

- Estimate of $2\tau(J_{1,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned} 2\tau(J_{1,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le 2\tau|(\rho_h^{n+1} D_\tau \mathbf{E}^{n+1}, e_{\mathbf{u},h}^{n+1})| + 2\tau|(\sigma_h^{n+1}\mathbf{E}^n D_\tau e_{\sigma,h}^{n+1}, e_{\mathbf{u},h}^{n+1})| \\ &\quad +2\tau|(\sigma_h^{n+1}\mathbf{E}^n D_\tau(\Pi_h \sigma^{n+1}), e_{\mathbf{u},h}^{n+1})| \\ &\le C\tau\|\rho_h^{n+1}\|_{L^\infty}\|D_\tau \mathbf{E}^{n+1}\|_{L^2}\|e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &\quad +C\tau\|\sigma_h^{n+1}\|_{L^\infty}\|\mathbf{E}^n\|_{L^2}\|D_\tau e_{\sigma,h}^{n+1}\|_{L^3}\|e_{\mathbf{u},h}^{n+1}\|_{L^6} \\ &\quad +C\tau\|\sigma_h^{n+1}\|_{L^\infty}\|\mathbf{E}^n\|_{L^2}\|D_\tau \Pi_h \sigma^{n+1}\|_{L^3}\|e_{\mathbf{u},h}^{n+1}\|_{L^6} \\ &\le C\tau h^4\|D_\tau \mathbf{u}^{n+1}\|_{H^2}^2 + C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{2,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned} 2\tau(J_{2,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le 2\tau|(\theta^{n+1}\sigma^{n+1} D_\tau \mathbf{u}^{n+1}, e_{\mathbf{u},h}^{n+1})| + 2\tau|(\theta^{n+1}(D_\tau \sigma^{n+1})\mathbf{u}^n, e_{\mathbf{u},h}^{n+1})| \\ &\le C\tau\|\theta^{n+1}\|_{L^2}(\|D_\tau \mathbf{u}^{n+1}\|_{L^3} + \|D_\tau \sigma^{n+1}\|_{L^3})\|e_{\mathbf{u},h}^{n+1}\|_{L^6} \\ &\le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{3,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned} 2\tau(J_{3,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\sigma_h^{n+1}\|_{L^\infty}\|\theta^{n+1}\|_{L^2}\|D_\tau \mathbf{u}^{n+1}\|_{L^3}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &\le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{4,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned} 2\tau(J_{4,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\sigma_h^{n+1}\|_{L^\infty}\|e_{\sigma,h}^{n+1}\|_{L^2}\|D_\tau \mathbf{u}^{n+1}\|_{L^3}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &\le \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{5,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned} 2\tau(J_{5,h}^{n+1} e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\sigma_h^{n+1}\|_{L^\infty}\|\mathbf{u}^n\|_{H^2}\|D_\tau \theta^{n+1}\|_{L^2}\|e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &\le C\tau h^4 + C\tau\|\sigma_h^{n+1} e_{\mathbf{u},h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{6,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned} 2\tau(J_{6,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le 2\tau|(\sigma_h^{n+1} D_\tau e_{\sigma,h}^{n+1}\mathbf{E}^n, e_{\mathbf{u},h}^{n+1})| + 2\tau|(\sigma_h^{n+1} D_\tau e_{\sigma,h}^{n+1} e_{\mathbf{u},h}^n, e_{\mathbf{u},h}^{n+1})| \\ &\quad + 2\tau|(\sigma_h^{n+1}(D_\tau e_{\sigma,h}^{n+1})\mathbf{u}_h^n, e_{\mathbf{u},h}^{n+1})| \\ &\le C\tau\|\sigma_h^{n+1}\|_{L^\infty}\|D_\tau e_{\sigma,h}^{n+1}\|_{L^3}(\|\mathbf{E}^n\|_{L^2} \\ &\quad + \|e_{\mathbf{u},h}^n\|_{L^2})\|e_{\mathbf{u},h}^{n+1}\|_{L^6} \\ &\quad + 2\tau|(\sigma_h^{n+1}(D_\tau e_{\sigma,h}^{n+1})\mathbf{u}_h^n, e_{\mathbf{u},h}^{n+1})| \\ &\le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 \\ &\quad + 2\tau|(\sigma_h^{n+1}(D_\tau e_{\sigma,h}^{n+1})\mathbf{u}_h^n, e_{\mathbf{u},h}^{n+1})|. \end{aligned} \tag{4.40}$$

To estimate the last term of the above inequality, we introduce the piecewise constant finite element space

$$W_h^0 = \left\{q_h \in L^2(\Omega),\ |\ q_h \in P_0(K),\ \forall\ K \in \mathfrak{T}_h\right\}.$$

Let $\pi_h$ be the $L^2$ projection operator from $L^2(\Omega)$ onto $W_h^0$. Then there holds

$$\|q-\pi_h q\|_{L^2} \le Ch\|q\|_{H^1} \quad \text{and} \quad \|\pi_h q\|_{L^2} \le \|q\|_{L^2}. \tag{4.41}$$

By (4.28), one has

$$\|\nabla(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \le \|\nabla \mathbf{u}_h^n\|_{L^3}\|e_{\mathbf{u},h}^{n+1}\|_{L^6} + \|\mathbf{u}_h^n\|_{L^\infty}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \le C\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}.$$

It follows from (4.41) that

$$\|(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}) - \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \le Ch\|\nabla(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \le Ch\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}. \tag{4.42}$$

Thus,

$$\begin{aligned} 2\tau|(\sigma_h^{n+1} D_\tau e_{\sigma,h}^{n+1}\mathbf{u}_h^n, e_{\mathbf{u},h}^{n+1})| &\le 2\tau|(\sigma_h^{n+1} D_\tau e_{\sigma,h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &\quad +2\tau|(\sigma_h^{n+1} D_\tau e_{\sigma,h}^{n+1}, (\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}) - \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &\le 2\tau|(\sigma_h^{n+1} D_\tau e_{\sigma,h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &\quad + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau h^2\|D_\tau e_{\sigma,h}^{n+1}\|_{L^2}^2. \end{aligned} \tag{4.43}$$

Taking $r_h = 2\tau\sigma_h^{n+1}\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})$ in (4.12), we get

$$\begin{aligned} &2\tau|(\sigma_h^{n+1} D_\tau e_{\sigma,h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ \le\ & 2\tau|(\nabla\sigma^{n+1}\cdot(\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n), \sigma_h^{n+1}\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &+2\tau|(\nabla\theta^{n+1}\cdot(\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}_h^n), \sigma_h^{n+1}\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &+2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot(\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n), \sigma_h^{n+1}\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &+2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot\mathbf{u}^n, \sigma^{n+1}\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &+2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot\mathbf{u}^n, \theta^{n+1}\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &+2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot\mathbf{u}^n, e_{\sigma,h}^{n+1}\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ &+2\tau|(\nabla\theta^{n+1}\cdot\mathbf{u}^n, \sigma_h^{n+1}\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ :=\ & 2\tau\sum_{i=1}^{7}|(H_{ih}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))|. \end{aligned} \tag{4.44}$$

We estimate all terms in the right-hand side of (4.44) as follows.

- Estimate of $2\tau|(H_{1h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))|$

$$\begin{aligned} 2\tau|(H_{1h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| &\le C\tau\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}\|\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \\ &\le C\tau\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}\|e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &\le C\tau\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau|(H_{2h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))|$

$$\begin{aligned} 2\tau|(H_{2h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| &\le C\tau\|\nabla\theta^{n+1}\|_{L^\infty}\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}\|\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \\ &\le C\tau\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau|(H_{3h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))|$

$$\begin{aligned} &2\tau|(H_{3h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ \le\ & 2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot(\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n), \sigma_h^{n+1}(\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}) - (\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})))| \\ &+2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot(\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n), \sigma_h^{n+1}\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})| \\ \le\ & C\tau\|\nabla e_{\sigma,h}^{n+1}\|_{L^\infty}\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}\|\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}) - (\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \\ &+C\tau\|\nabla e_{\sigma,h}^{n+1}\|_{L^2}\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^3}\|e_{\mathbf{u},h}^{n+1}\|_{L^6} \\ \le\ & C\tau h^{-1}\|e_{\sigma,h}^{n+1}\|_{L^2}\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &+C\tau h^{-1}\|e_{\sigma,h}^{n+1}\|_{L^2}\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^3}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ \le\ & \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau h^{-2}\|e_{\sigma,h}^{n+1}\|_{L^2}^2\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^3}^2 \\ \le\ & \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau h^4 \end{aligned}$$

where (4.26), (4.30) and (4.41) are used.

- Estimate of $2\tau|(H_{4h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))|$

$$\begin{aligned} &2\tau|(H_{4h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ \le\ & 2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot\mathbf{u}^n, \sigma^{n+1}(\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}) - (\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})))| \\ &+2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot\mathbf{u}^n, \sigma^{n+1}\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})| \\ \le\ & C\tau\|\nabla e_{\sigma,h}^{n+1}\|_{L^2}\|\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}) - (\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \\ &+C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} + C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}\|\mathbf{u}_h^n\|_{W^{1,3}}\|e_{\mathbf{u},h}^{n+1}\|_{L^6} \\ \le\ & C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ \le\ & \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}^2, \end{aligned}$$

where (4.27), (4.28) and (4.41) are used.

- Estimate of $2\tau|(H_{5h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))|$

$$\begin{aligned} 2\tau|(H_{5h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| &\le C\tau\|\nabla e_{\sigma,h}^{n+1}\|_{L^\infty}\|\theta^{n+1}\|_{L^2}\|\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \\ &\le C\tau h\|e_{\sigma,h}^{n+1}\|_{L^2}\|e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &\le C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau h^4, \end{aligned}$$

where (4.27) and (4.11) are used.

- Estimate of $2\tau|(H_{6h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))|$

$$\begin{aligned} &2\tau|(H_{6h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| \\ \le\ & 2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot\mathbf{u}^n, e_{\sigma,h}^{n+1}(\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}) - (\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})))| \\ &+2\tau|(\nabla e_{\sigma,h}^{n+1}\cdot\mathbf{u}^n, e_{\sigma,h}^{n+1}\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})| \\ \le\ & C\tau\|\nabla e_{\sigma,h}^{n+1}\|_{L^\infty}\|e_{\sigma,h}^{n+1}\|_{L^2}\|\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}) - (\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \\ &+C\tau\|e_{\sigma,h}^{n+1}\|_{L^3}\|\nabla e_{\sigma,h}^{n+1}\|_{L^2}\|e_{\mathbf{u},h}^{n+1}\|_{L^6} \\ \le\ & C\tau h^{-\frac{4}{3}}\|e_{\sigma,h}^{n+1}\|_{L^2}^2\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ \le\ & \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau h^4, \end{aligned}$$

where we use (4.27), (4.41) and (4.30).

- Estimate of $2\tau|(H_{7h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))|$

$$\begin{aligned} 2\tau|(H_{7h}^{n+1}, \pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1}))| &\le C\tau\|\nabla\theta^{n+1}\|_{L^2}\|\pi_h(\mathbf{u}_h^n \cdot e_{\mathbf{u},h}^{n+1})\|_{L^2} \\ &\le C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau h^4, \end{aligned}$$

where we use (4.11) and (4.27). Substituting the above estimates for $H_{1h}^{n+1}$ to $H_{7h}^{n+1}$ into (4.44), we get the estimate for the term of $J_{6h}^{n+1}$:

$$\begin{aligned} 2\tau(J_{6,h}^{n+1}, e_{u,h}^{n+1}) &\le C\tau h^4 + C\tau\|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 \\ &\quad + C\tau h^2\|D_\tau e_{\sigma,h}^{n+1}\|_{L^2}^2 \\ &\quad + C\tau\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}^2 + C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}^2 + \frac{5\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 \\ &\le C\tau h^4 + C\tau\|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 \\ &\quad + C\tau\|\mathbf{P}_{0h}\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2}^2 + C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}^2 + \frac{5\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{7,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned} 2\tau(J_{7,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le 2\tau|(e_{\sigma,h}^{n+1}\sigma^{n+1}D_\tau\mathbf{u}^{n+1}, e_{\mathbf{u},h}^{n+1})| + 2\tau|(e_{\sigma,h}^{n+1}\mathbf{u}^n D_\tau\sigma^{n+1}, e_{\mathbf{u},h}^{n+1})| \\ &\le C\tau\|\sigma^{n+1}\|_{L^\infty}\|e_{\sigma,h}^{n+1}\|_{L^2}\|D_\tau\mathbf{u}^{n+1}\|_{L^3}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &\quad +C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}\|\mathbf{u}^n\|_{L^\infty}\|D_\tau\sigma^{n+1}\|_{L^3}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} \\ &\le C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}^2 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{8,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{8,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\beta^{n+1}\|_{L^2}\|\mathbf{u}^n\|_{L^\infty}\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{9,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{9,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|e_{\rho,h}^{n+1}\|_{L^2}\|\mathbf{u}^n\|_{L^\infty}\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{10,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{10,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\rho_h^{n+1}\|_{L^\infty}\|\mathbf{E}^n\|_{L^2}\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{11,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{11,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\rho_h^{n+1}\|_{L^\infty}\|e_{\mathbf{u},h}^{n}\|_{L^2}\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le C\tau\|\sigma_h^n e_{\mathbf{u},h}^{n}\|_{L^2}^2 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{12,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{12,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\rho_h^{n+1}\|_{W^{1,3}}\|\mathbf{E}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\quad + C\tau\|\mathbf{u}_h^n\|_{W^{1,3}}\|\mathbf{E}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\quad + C\tau\|\rho_h^{n+1}\|_{L^\infty}\|\mathbf{u}_h^n\|_{L^\infty}\|\mathbf{E}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2\end{aligned}$$

by using the integration by parts and noting

$$\|\rho_h^{n+1}\|_{W^{1,3}} \le \|\sigma_h^{n+1}\|_{L^6}^2 + \|\sigma_h^{n+1}\|_{L^\infty}\|\nabla\sigma_h^{n+1}\|_{L^3} \le C.$$

- Estimate of $2\tau(J_{13,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{13,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\|e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.\end{aligned}$$

We estimate the terms of $J_{14,h}^{n+1}$ to $J_{19,h}^{n+1}$ by the integration by parts as follows.

- Estimate of $2\tau(J_{14,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{14,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\beta^{n+1}\|_{L^2}\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} + C\tau\|\beta^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{15,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{15,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|e_{\rho,h}^{n+1}\|_{L^2}\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|e_{\mathbf{u},h}^{n+1}\|_{L^2} + C\tau\|e_{\rho,h}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{16,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{16,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\mathbf{E}^n\|_{L^2}\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} + C\tau\|\mathbf{E}^n\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le C\tau h^4 + \frac{\tau}{26Re}\|e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{17,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{17,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|e_{\mathbf{u},h}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{18,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{18,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\rho_h^{n+1}\|_{W^{1,3}}\|e_{\mathbf{u},h}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\quad + C\tau\|\mathbf{u}_h^n\|_{W^{1,3}}\|e_{\mathbf{u},h}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.\end{aligned}$$

- Estimate of $2\tau(J_{19,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{19,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\rho_h^{n+1}\|_{W^{1,3}}\|\mathbf{E}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\quad + C\tau\|\mathbf{u}_h^n\|_{W^{1,3}}\|\mathbf{E}^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2} + C\tau h^4.\end{aligned}$$

- Estimate of $2\tau(J_{20,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$2\tau(J_{20,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) \le C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + \frac{S\tau}{5Rm}\|e_{\mathbf{b},h}^{n}\|_W^2.$$

- Estimate of $2\tau(J_{21,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$2\tau(J_{21,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) \le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2.$$

- Estimate of $2\tau(J_{22,h}^{n+1}, e_{\mathbf{u},h}^{n+1})$

$$\begin{aligned}2\tau(J_{22,h}^{n+1}, e_{\mathbf{u},h}^{n+1}) &\le C\tau\|\mathbf{b}_h^n\|_{W^{1,3}}\|\gamma^{n+1}\|_{L^2}\|e_{\mathbf{u},h}^{n+1}\|_{L^6}\\ &\quad + C\tau\|\mathbf{b}_h^n\|_{L^\infty}\|\gamma^{n+1}\|_{L^2}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}\\ &\le C\tau h^4 + \frac{\tau}{26Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2\end{aligned}$$

by the integration by parts.

Substituting the above estimates into (4.39) and using (4.15), we have

$$\begin{aligned}&\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 - \|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 + S(\|e_{\mathbf{b},h}^{n+1}\|_{L^2}^2 - \|e_{\mathbf{b},h}^{n}\|_{L^2}^2)\\ &+ \frac{\tau}{Re}\|\nabla e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + \frac{4S\tau}{5Rm}\|e_{\mathbf{b},h}^{n+1}\|_W^2\\ &\le C\tau h^4 + C\tau\|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2\\ &\quad + C\tau\|e_{\sigma,h}^{n+1}\|_{L^2}^2 + C\tau\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2\\ &\quad + C\tau\|e_{\mathbf{b},h}^{n}\|_{L^2}^2 + C\tau h^4(\|D_\tau\mathbf{u}^{n+1}\|_{H^2}^2 + \|D_\tau\mathbf{b}^{n+1}\|_{H^2}^2) + \frac{\tau}{5Rm}\|e_{\mathbf{b},h}^{n}\|_W^2\\ &\le C\tau h^4 + C\tau\|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2\\ &\quad + C\tau^2\sum_{i=0}^{n}\|\mathbf{u}^i - \mathbf{P}_{0h}\mathbf{u}_h^i\|_{L^2}^2 + C\tau\|e_{\mathbf{b},h}^{n}\|_{L^2}^2\\ &\quad + C\tau\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2 + C\tau h^4(\|D_\tau\mathbf{u}^{n+1}\|_{H^2}^2\\ &\quad + \|D_\tau\mathbf{b}^{n+1}\|_{H^2}^2) + \frac{\tau}{5Rm}\|e_{\mathbf{b},h}^{n}\|_W^2\\ &\le C\tau h^4 + C\tau\|\sigma_h^n e_{\mathbf{u},h}^n\|_{L^2}^2 + C\tau\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + C\tau h^4(\|D_\tau\mathbf{u}^{n+1}\|_{H^2}^2\\ &\quad + \|D_\tau\mathbf{b}^{n+1}\|_{H^2}^2) + C\tau\|e_{\mathbf{b},h}^{n}\|_{L^2}^2 + \frac{\tau}{5Rm}\|e_{\mathbf{b},h}^{n}\|_W^2.\end{aligned}$$

Taking the sum of the above inequality leads to

$$\begin{aligned}&\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + \|e_{\mathbf{b},h}^{n+1}\|_{L^2}^2 + \tau\sum_{i=0}^{n}\left(\frac{1}{Re}\|\nabla e_{\mathbf{u},h}^{i+1}\|_{L^2}^2 + \frac{S}{5Rm}\|e_{\mathbf{b},h}^{i+1}\|_W^2\right)\\ &\le Ch^4 + C\tau\sum_{i=0}^{n}(\|\sigma_h^{i+1}e_{\mathbf{u},h}^{i+1}\|_{L^2}^2 + \|e_{\mathbf{b},h}^{i}\|_{L^2}^2).\end{aligned}$$

Applying the discrete Gronwall's inequality established in [14], we get

$$\|\sigma_h^{n+1}e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + \|e_{\mathbf{b},h}^{n+1}\|_{L^2}^2 + \tau\sum_{i=0}^{n}\left(\frac{1}{Re}\|\nabla e_{\mathbf{u},h}^{i+1}\|_{L^2}^2 + \frac{S}{5Rm}\|e_{\mathbf{b},h}^{i+1}\|_W^2\right) \le C^2h^4$$

for sufficiently small $\tau$. By noticing (4.33), we conclude that there exists some $C_2 > 0$ which is independent of $h, \tau$ and $C_0$ such that

**Table 1**
$L^2$ errors and convergence rates for $\tau = h^2$.

| $h$ | $\|\rho(T)-\rho_h^N\|_{L^2}$ | rate | $\|\mathbf{u}(T)-\mathbf{u}_h^N\|_{L^2}$ | rate | $\|\mathbf{b}(T)-\mathbf{b}_h^N\|_{L^2}$ | rate |
|---|---|---|---|---|---|---|
| 1/8 | 8.28953e-003 | | 4.21754e-003 | | 1.82123e-004 | |
| 1/16 | 2.10442e-003 | 1.98 | 1.08042e-003 | 1.96 | 4.82405e-005 | 1.92 |
| 1/32 | 5.28559e-004 | 1.99 | 2.72511e-004 | 1.99 | 1.22451e-005 | 1.98 |
| 1/64 | 1.32295e-004 | 2.00 | 6.72053e-005 | 2.02 | 3.07314e-006 | 1.99 |
| 1/128 | 3.30841e-005 | 2.00 | 1.66332e-005 | 2.01 | 7.69031e-007 | 2.00 |

**Table 2**
$L^2$ errors and convergence rates for $\tau = h$.

| $h$ | $\|\rho(T)-\rho_h^N\|_{L^2}$ | rate | $\|\mathbf{u}(T)-\mathbf{u}_h^N\|_{L^2}$ | rate | $\|\mathbf{b}(T)-\mathbf{b}_h^N\|_{L^2}$ | rate |
|---|---|---|---|---|---|---|
| 1/8 | 5.88363e-002 | | 1.15853e-002 | | 1.71949e-004 | |
| 1/16 | 2.97322e-002 | 0.98 | 5.52278e-003 | 1.07 | 4.56673e-005 | 1.91 |
| 1/32 | 1.49247e-002 | 0.99 | 2.75644e-003 | 1.00 | 1.43363e-005 | 1.67 |
| 1/64 | 7.47311e-003 | 1.00 | 1.37372e-003 | 1.00 | 6.24923e-006 | 1.20 |
| 1/128 | 3.73928e-003 | 1.00 | 6.82807e-004 | 1.01 | 3.16227e-006 | 0.98 |

$$\|e_{\mathbf{u},h}^{n+1}\|_{L^2}^2 + \|e_{\mathbf{b},h}^{n+1}\|_{L^2}^2 + \tau \sum_{i=0}^{n} \left( \|\nabla e_{\mathbf{u},h}^{i+1}\|_{L^2}^2 + \|e_{\mathbf{b},h}^{i+1}\|_W^2 \right) \le C_2^2 h^4.$$

Then, the error estimate (4.18) is valid for $m \le n+1$ by taking $C_0 \ge \max\{C_1, C_2\}$. Thus, we close the mathematical induction and finish the proof of Theorem 4.1. □

## 5. Main result

Combining the results in Theorem 3.1, Lemma 4.1 and Theorem 4.1, we present main result in this paper in the following theorem.

**Theorem 5.1.** *Under the assumptions (A1) and (A2), for sufficiently small $\tau$ and $h$, we have*

$$\begin{aligned}&\|\rho(t_n)-\rho_h^n\|_{L^2} + \|\sigma(t_n)-\sigma_h^n\|_{L^2} + \|\boldsymbol{u}(t_n)-\boldsymbol{u}_h^n\|_{L^2} + \|\boldsymbol{b}(t_n)-\boldsymbol{b}_h^n\|_{L^2}\\ &\le \widehat{C}_0(\tau + h^2)\end{aligned} \tag{5.1}$$

*for $1 \le n \le N$, where $\widehat{C}_0 > 0$ is independent of $\tau$ and $h$.*

**Proof.** It follows from (3.9), (4.1), (4.11), (4.15) and (4.18) that

$$\|\sigma(t_n)-\sigma_h^n\|_{L^2} + \|\mathbf{u}(t_n)-\mathbf{u}_h^n\|_{L^2} + \|\mathbf{b}(t_n)-\mathbf{b}_h^n\|_{L^2} \le C_3(\tau + h^2)$$

for some $C_3 > 0$. By noting $\rho_h^n = (\sigma_h^n)^2$ and (4.33), we have

$$\|\rho(t_n)-\rho_h^n\|_{L^2}\| \le \|\sigma(t_n)+\sigma_h^n\|_{L^2}\|\sigma(t_n)-\sigma_h^n\|_{L^2} \le C\|\sigma(t_n)-\sigma_h^n\|_{L^2} \le C_4 h^2.$$

Thus, (5.1) holds for some $\widehat{C}_0 \ge C_3 + C_4$ and we complete the proof of Theorem 5.1. □

**Remark 5.1.** Usually, the optimal estimate for $\sigma$ in $L^2$-norm should be the third-order convergence rate $\mathcal{O}(h^3)$ by using the piecewise quadratic element to approximate $\sigma$. However, finite element solutions of the scalar hyperbolic equation (2.3) have lower-order convergence rates (cf. Remark 3.14 in [27]). On the other hand, by using $P_1b - P_1$ to approximate the velocity field and pressure, the second-order convergence rate $\mathcal{O}(h^2)$ for $\sigma$ should be optimal, which is shown numerically in next section.

**Remark 5.2.** From (4.33), we can see that numerical solutions $\sigma_h^n$ satisfy

$$\frac{1}{2}\sigma_0^{min} \le \sigma_h^n(x) \le \frac{3}{2}\sigma_0^{max} \qquad \text{in } \Omega,\ \forall\ 1 \le n \le N$$

for sufficiently small $\tau$ and $h$, which is important in numerical methods for the variable density flows.

## 6. Numerical results

In this section, numerical results are given to confirm the spatial error estimate (5.1) derived in Theorem 5.1. For the reason of simplicity, we solve the VD-MHD system (2.3)-(2.6) with artificial functions $g$ and $\mathbf{J}$, i.e., we solve the following coupled system:

$$\sigma_t + \nabla\cdot(\sigma\mathbf{u}) = g, \tag{6.1}$$

$$\sigma(\sigma\mathbf{u})_t - \frac{1}{Re}\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\mathbf{u}}{2}\nabla\cdot(\rho\mathbf{u}) + S\mathbf{b}\times curl\mathbf{b} + \nabla p = \mathbf{f}, \tag{6.2}$$

$$\mathbf{b}_t + \frac{1}{Rm}\mathbf{curl}\ (curl\mathbf{b}) - \mathbf{curl}\ (\mathbf{u}\times\mathbf{b}) = \mathbf{J} \tag{6.3}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0 \tag{6.4}$$

in $\Omega\times[0,T]$, where $\Omega$ is the unit square:

$$\Omega = \{(x,y)\in\mathbf{R}^2 :\ 0<x<1,\ 0<y<1\}.$$

We select the appropriate functions $g$, $\mathbf{f}$ and $\mathbf{J}$ such that the exact solution $(\sigma, \mathbf{u}, p, \mathbf{b})$ to (6.1)-(6.4) is given by

$$\begin{aligned}&\sigma(x,y,t) = 2 + x(1-x)\cos(\sin(t)) + y(1-y)\sin(\sin(t)),\\ &\mathbf{u}(x,y,t) = \left(t^3y^2(1-y),\ t^3x^2(1-x)\right)^{\mathrm{T}},\\ &p(x,y,t) = tx + y - (t+1)/2,\\ &\mathbf{b}(x,y,t) = \left(x^2(x-1)^2y(y-1)(2y-1),\ -y^2(y-1)^2x(x-1)(2x-1)\right)^{\mathrm{T}} e^{-t}.\end{aligned}$$

In addition, we take $Re = 1000$, $Rm = 10$, $S = 1$ and the final time $T = 1$.

Firstly, we give numerical results to confirm the $L^2$ error estimate in (5.1):

$$\|\rho(T)-\rho_h^N\|_{L^2} + \|\mathbf{u}(T)-\mathbf{u}_h^N\|_{L^2} + \|\mathbf{b}(T)-\mathbf{b}_h^N\|_{L^2} \le C(\tau + h^2).$$

By taking gradually decreasing mesh sizes $h = 1/8, 1/16, \cdots, 1/128$, we choose different iteration numbers $N = 1/h^2$ and $N = 1/h$ such that $\tau = h^2$ and $\tau = h$, respectively. We present numerical results in Tables 1 and 2. It is clear that the predicted second-order convergence rate $\mathcal{O}(h^2)$ in the case $\tau = h^2$ and first-order convergence rate $\mathcal{O}(h)$ in the case $\tau = h$ are shown, which are in good agreement with the theoretical analysis. In addition, we show numerical solutions $\mathbf{u}_h^N$ and $\mathbf{b}_h^N$ by taking $h = 1/50$ and $N = 1/h^2$ in Fig. 1.

To show that the error estimate for the density in Theorem 5.1 is optimal, we choose $\tau = h^3$. If the optimal convergence rate for $\rho$ is

$$\|\rho(T)-\rho_h^N\|_{L^2} \le C(\tau + h^\alpha) \quad \text{for some } 2 < \alpha \le 3.$$

Then, one has

$$\|\rho(T)-\rho_h^N\|_{L^2} \le Ch^\alpha$$

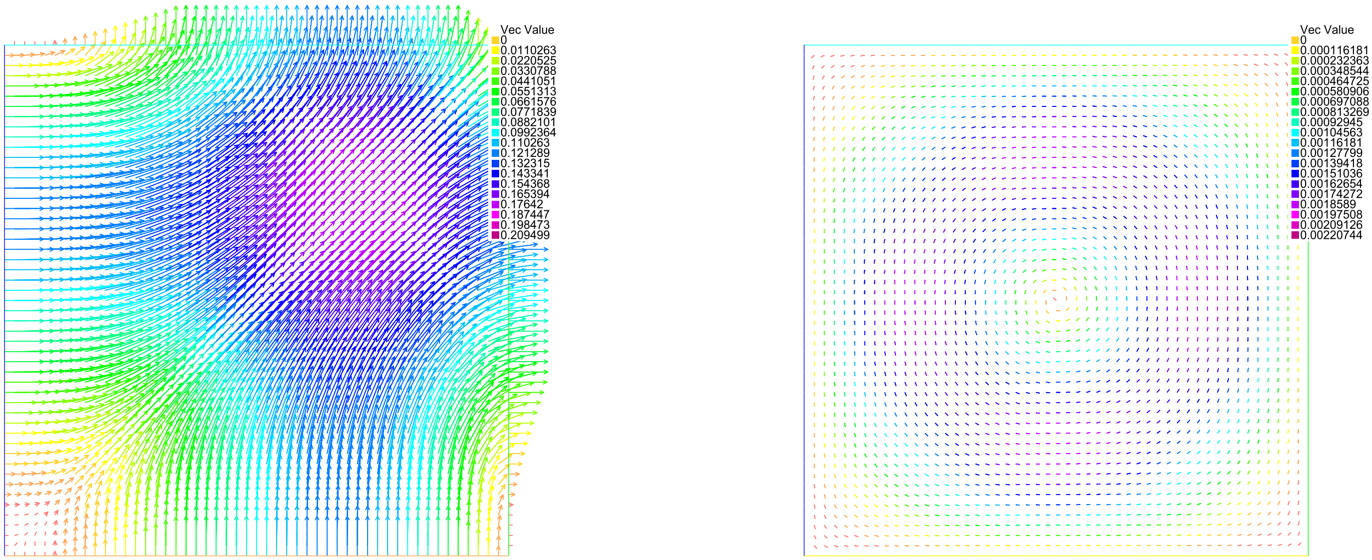


**Fig. 1.** Numerical solutions of velocity and magnetic fields at $t = 1.0$.

**Table 3**
$L^2$ errors and convergence rates for $\tau = h^3$.

| $h$ | $\|\rho(T) - \rho_h^N\|_{L^2}$ | rate | $\|\mathbf{u}(T) - \mathbf{u}_h^N\|_{L^2}$ | rate | $\|\mathbf{b}(T) - \mathbf{b}_h^N\|_{L^2}$ | rate |
|---|---|---|---|---|---|---|
| 1/4 | 1.17566e-002 | | 1.52174e-002 | | 6.00875e-004 | |
| 1/8 | 2.53214e-003 | 2.21 | 3.97172e-003 | 1.94 | 1.83807e-004 | 1.70 |
| 1/16 | 5.87662e-004 | 2.10 | 1.01902e-003 | 1.96 | 4.87629e-005 | 1.91 |
| 1/32 | 1.41798e-004 | 2.05 | 2.56860e-004 | 1.99 | 1.23848e-005 | 1.98 |

under the choice $\tau = h^3$. From numerical errors and convergence rates in Table 3, we can see the second-order convergence rate $\mathcal{O}(h^2)$ which shows that the $L^2$ error estimate for the density in (5.1) is optimal when $P_1 b$ finite element is used to approximate the velocity.

## 7. Conclusions

In this paper, based on the equivalent VD-MHD system (2.3)-(2.6), we proposed a first-order backward Euler finite element scheme for solving (1.1)-(1.4) numerically, where we used the piecewise quadratic element, the mini element and the piecewise linear element to approximate $\sigma$, $(\mathbf{u}, p)$ and $\mathbf{b}$, respectively. It is technical reason for using the piecewise quadratic element to approximate $\sigma$. If the piecewise linear element is used for $\sigma$, Lemma 4.1 does not hold. In addition, we use mini element and the piecewise linear element to approximate $(u, p)$ and $b$ such that there has no additional pollution in error analysis. In the proposed fully discrete scheme, nonlinear terms were treated by a linearized semi-implicit method such that we only solve linear systems at each time step. Furthermore, we proved that the finite element algorithm is unconditionally stable. By a rigorous error analysis, the second-order convergence rate $\mathcal{O}(h^2)$ in $L^2$-norm was derived in Theorem 5.1. Finally, we remark that the theoretical results in this paper can be extended to the case of $(P_2, P_2, P_1, P_1)$ finite elements for the approximation of $(\sigma, \mathbf{u}, p, \mathbf{b})$. In this case, the finite element spaces condition $\mathbf{V}_h \cdot \mathbf{V}_h \subset X_h$ is not needed.

## Data availability

No data was used for the research described in the article.


## Acknowledgements

This work was supported by National Natural Science Foundation of China (No. 11771337).


## Appendix A. Proof of Lemma 3.1

Rewrite (3.2) as the Stokes type problem

$$-\frac{1}{Re}\Delta \mathbf{u}^{n+1} + \nabla p^{n+1} = -\sigma^{n+1} D_\tau(\sigma^{n+1}\mathbf{u}^{n+1}) - \rho^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} \qquad (7.1)$$
$$-\frac{1}{2}\mathbf{u}^{n+1}\nabla \cdot (\rho^{n+1}\mathbf{u}^n) - S(\mathbf{b}^n \times curl\,\mathbf{b}^{n+1}) + \mathbf{f}^{n+1}.$$

By the Hölder's inequality and (3.7)-(3.11), one has

$$\begin{aligned}
&\tau\sum_{n=0}^{N-1} \|\sigma^{n+1} D_\tau(\sigma^{n+1}\mathbf{u}^{n+1})\|_{H^1}^2 \\
&= \frac{1}{\tau}\sum_{n=0}^{N-1} \left\| \sigma^{n+1}\left( -(\sigma^{n+1}e_{\mathbf{u}}^{n+1} - \sigma^n e_{\mathbf{u}}^n) + (\sigma^{n+1} - \sigma^n)\mathbf{u}(t_{n+1}) + \sigma^n \int_{t_n}^{t_{n+1}} \mathbf{u}_t(t)dt \right) \right\|_{H^1}^2 \\
&\le \frac{C}{\tau}\sum_{n=0}^{N-1} \left( \|e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|_{H^1}^2 + \|(\sigma^{n+1} - \sigma^n)e_{\mathbf{u}}^n\|_{H^1}^2 + \|\sigma^{n+1} - \sigma^n\|_{H^1}^2 \right. \\
&\qquad \left. + \tau \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(t)\|_{H^1}^2 dt \right) \\
&\le \frac{C}{\tau}\sum_{n=0}^{N-1} (\|e_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|_{H^1}^2 + \|\sigma^{n+1} - \sigma^n\|_{H^1}^2 \|e_{\mathbf{u}}^n\|_{H^2}^2 + \|e_\sigma^{n+1} - e_\sigma^n\|_{H^1}^2) + C \\
&\le C,
\end{aligned}$$

and

$$\tau\sum_{n=0}^{N-1} \left( \|\rho^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}\|_{H^1}^2 + S\|\mathbf{b}^n \times \mathbf{curl}\ \mathbf{b}^{n+1}\|_{H^1}^2 \right) \le C,$$

and

$$\tau\sum_{n=0}^{N-1} \|\mathbf{u}^{n+1}\nabla \cdot (\rho^{n+1}\mathbf{u}^n)\|_{H^1}^2 = \tau\sum_{n=0}^{N-1} \left\| \mathbf{u}^{n+1}\sigma^{n+1} D_\tau \sigma^{n+1} \right\|_{H^1}^2$$

$$\leq \frac{C}{\tau}\sum_{n=0}^{N-1}\|\sigma^{n+1}-\sigma^{n}\|_{H^1}^2 \leq C$$

by noting (3.1). It follows from the regularity theory of the Stokes problem (cf. [6]) that

$$\tau\sum_{n=0}^{N-1}(\|\mathbf{u}^{n+1}\|_{H^3}^2+\|p^{n+1}\|_{H^2}^2)\leq C. \tag{7.2}$$

The inequalities (3.10) and (3.11) imply that

$$\max_{0\leq n\leq N-1}(\|D_\tau\mathbf{u}^{n+1}\|_{H^1}+\|D_\tau\mathbf{b}^{n+1}\|_{H^1}) + \tau\sum_{n=0}^{N-1}(\|D_\tau\mathbf{u}^{n+1}\|_{H^2}^2+\|D_\tau\mathbf{b}^{n+1}\|_{H^2}^2)\leq C$$

By the same method in [23], there holds

$$\max_{0\leq n\leq N-1}(\|D_\tau\sigma^{n+1}\|_{H^2}+\|\sigma^{n+1}\|_{H^3})\leq C.$$

We complete the proof of Lemma 3.1.

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