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Temporal error analysis of a BDF2 time-discrete scheme for the incompressible Navier–Stokes equations with variable density

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ABSTRACT

The paper is devoted to the study of the Navier–Stokes equations with variable density by reformulating the original equations. We investigate a BDF2 time-discrete scheme specifically designed for the numerical solution of variable density flows. A thorough unconditional stability analysis of this method is conducted, demonstrating its validity and robustness in handling such complex fluid dynamics problems. Additionally, we establish a rigorous proof of the second-order temporal convergence rate $\mathcal{O}(\tau^2)$ under certain regularity assumptions regarding the smoothness of the solutions, where τ represents the time step size. Finally, we present some numerical experiments to validate the analysis and demonstrate the effectiveness of this scheme.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded and convex domain with the boundary $\Gamma = \partial\Omega$ and $[0, T]$ be the time interval with some $T > 0$. We will consider numerical schemes in this paper for solving the two-dimensional incompressible Navier–Stokes equations with variable density, which are important in several fields of fluid dynamics [1], such as the analysis of highly stratified flows, the investigation of the intricate dynamics at interfaces between fluids of different densities, and the exploration of complex problems related to inertial confinement and astrophysics. The system is governed by the following time-dependent Navier–Stokes system with variable density

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

$$\rho \mathbf{u}_t - \mu \Delta \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \times (0, T], \tag{1.2}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T], \tag{1.3}$$

where the unknown qualities are the density ρ, the velocity of the fluid $\mathbf{u}$ and the pressure p, $\mathbf{f}$ is a given body force and $\mu > 0$ is the viscosity coefficient.

The system (1.1)–(1.3) are supplemented the following initial–boundary conditions:

$$\begin{cases} \rho(x,0) = \rho_0(x), \quad \mathbf{u}(x,0) = \mathbf{u}_0(x), & \text{in } \Omega, \\ \rho(x,t)|_{\Gamma^-} = a(x,t), \quad \mathbf{u}(x,t)|_{\Gamma} = \mathbf{g}(x,t), & \text{on } \Gamma \times (0,T], \end{cases} \tag{1.4}$$

where Γ is the inflow boundary defined by $\Gamma^- = \{x \in \Gamma : \mathbf{g} \cdot \mathbf{n} < 0\}$ and $\mathbf{n}$ is the outward unit normal vector to the boundary Γ. For the reason of simplicity, we adopt the homogeneous Dirichlet boundary condition for the velocity, which is equivalent to setting

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$\mathbf{g} = 0$, this condition ensures that the boundary is impermeable, thereby rendering Γ^- as an empty set. We note that no initial or boundary conditions are required for the pressure p, which acts as a Lagrange multiplier to enforce the incompressibility constraint (1.3). The initial value $\mathbf{u}_0$ satisfies the incompressible conditions $\nabla \cdot \mathbf{u} = 0$ and the initial density ρ_0 is bounded and away from zero, i.e., there exist two constants $\rho_0^{\min} > 0$ and $\rho_0^{\max} > 0$ such that

$$\rho_0^{\min} \le \rho_0(x) \le \rho_0^{\max} \quad \text{in } \Omega,$$

this condition precludes the existence of any vacuum state within the domain Ω, ensuring a physically meaningful initial density configuration.

The mathematical theory of Navier–Stokes equations with variable density was studied in [2–4] for the existence and uniqueness of smooth solutions in two dimensions. The existence and uniqueness of smooth solutions in three dimensions is still an open problem, similar to the Navier–Stokes equations with constant density. Constructing stable and efficient numerical schemes for the system (1.1)–(1.3) poses a significant challenge [5], error analysis has been done only in a few articles, particularly due to the intricate nature of incompressible flows with constant density. Numerical methods are presented and developed for approximating the solutions to (1.1)–(1.3), such as the first-order Euler semi-implicit scheme [6–8], the BDF2 scheme [9,10], the fractional-step methods [11–13], the Gauge–Uzawa methods [5,14], the projection methods [1,15–17], the backward difference method [18], the discontinuous Galerkin method [19–21], and the iteration penalty method [22].

The additional complexity arises from the presence of a transport equation for the density ρ, which not only enforces incompressibility but also guarantees that the mass density remains constant throughout the fluid's motion. Error estimate for the velocity by the assumption of uniform boundedness of numerical density variable in [11]. In the pioneering work presented in [7], the first comprehensive error analysis for the fully discrete finite element method (FEM) applied to the coupled system (1.1)–(1.3) was introduced. This analysis bifurcates the error in the numerical solution into temporal and spatial components. The temporal error estimation is accomplished by the application of discrete maximal L^p-regularity theory for time-dependent Stokes equations [23]. Meanwhile, the spatial error estimate leverages energy techniques grounded on the uniform regularity of solutions derived from semi-discretization in time [24–26]. However, the analysis presented in [7] cannot be readily extended to three dimensions due to the utilization of H^1-conforming finite element solutions for the velocity in the density equation. This necessitates proving the $W^{1,\infty}$-boundedness of the numerical solution for velocity, as a prerequisite for obtaining an accurate error estimate for the density equation. The temporal–spatial error estimates for the coupled system (1.1)–(1.3) was studied in [21], where the key advantage of this approach lies in its capability to enable error analysis in three dimensions, even under more realistic $H^{2+\alpha}$ regularity assumptions on the solution within a convex polyhedron.

Based on the previous discussion, we will develop the BDF2 time-discrete scheme to solve the Navier–Stokes equations with variable density. The main work of this paper is outlined as follows

- Temporal error analysis for numerical algorithms has been explored in [8,13,14,22,27], but these studies have been limited to first-order temporal convergence. The primary objective of this work is to introduce the BDF2 time-discrete scheme for solving variable density flows, the temporal convergence $\mathcal{O}(\tau^2)$ of our BDF2 time discrete scheme is as good as the corresponding schemes for the constant density flows. Additionally, the nonlinear terms are treated by a linearized semi-implicit method, which only requires solving linear systems at each time step. This approach maintains numerical stability while circumventing the computational complexity associated with solving fully nonlinear systems.
- The second goal of this paper is to conduct a rigorous temporal error analysis for the proposed BDF2 time-discrete scheme. The skew-symmetric property, which plays a crucial role in the numerical methods and analysis of incompressible Navier–Stokes equations with constant density, is no longer valid when the ρ is a variable density. The Navier–Stokes equations with variable density are reformulated by introducing $\sigma = \sqrt{\rho}$ to overcome the difficulty in [1]. In addition, deriving the maximum principle for the density and establishing uniform bounds for the numerical solutions is difficult due to the density equation and highly nonlinear terms. To overcome these challenges, we apply the mathematical induction in [7,14,27] for the BDF2 time discrete schemes to derive the regularity estimate for all numerical solutions, which are pivotal for the error analysis.
- The third objective is to present the finite element approximations of the proposed time-discrete scheme and establish the stability results of the finite element approximate scheme. We employ the Taylor–Hood ($P1b - P1$) element for the velocity–pressure pair, and the $P2$ or $P1$ element for the density. Numerical tests are conducted to validate both the stability and accuracy of the BDF2 time discrete scheme. To the best of our knowledge, this is the first work to systematically and comprehensively prove the second-order temporal convergent rate.

The rest of this paper is organized as follows. In the next section, we introduce the notations, numerical scheme, unconditional stability of the BDF2 time discrete algorithm is also presented in this section. We present the main theorem and the details of the proof of the main theorem in Section 3. Numerical results are presented in Section 4 to support the theoretical analysis. The last section is devoted to concluding remarks.

Throughout this paper, we use the symbol C to denote a general positive constant which can be different at different places and is independent of the time step size τ.

2. Mathematical setting and BDF2 time-discrete scheme

For the mathematical setting of this model, we introduce some functional spaces and their associated norms. For $k \in \mathbb{N}^+, 1 \leq p \leq +\infty$, let $W^{k,p}(\Omega)$ denote the standard Sobolev space. The norm in $W^{k,p}(\Omega)$ is denoted by $\|\cdot\|_{W^{k,p}}$. We define $W_0^{k,p}(\Omega)$ to be the subspace of $W^{k,p}(\Omega)$ of functions with zero trace on $\partial\Omega$. When $p=2$, we simply use $H^k(\Omega)$ to denote $W^{k,2}(\Omega)$. The boldface Sobolev spaces $\mathbf{H}^k(\Omega), \mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ are used to denote the vector Sobolev spaces $H^k(\Omega)^2, W^{k,p}(\Omega)^2$ and $L^p(\Omega)^2$, respectively. Let X be a Banach space over a time interval $[0,T]$ with $T>0$. The Bochner space $L^p(0,T;X)$, for $1 \leq p < +\infty$, is defined as the spaces of measurable functions from the interval $[0,T]$ into X such that

$$\int_0^T \|\mathbf{u}(t)\|_X^p dt < +\infty,$$

if $p=+\infty$, the functions in $L^\infty(0,T;X)$ are required to satisfy

$$\sup_{t\in[0,T]} \|\mathbf{u}(t)\|_X < +\infty.$$

Denote

$$\mathbf{V} = \mathbf{H}_0^1(\Omega), \quad \mathbf{V}_0 = \{\mathbf{v} \in \mathbf{V},\ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega\},$$
$$M = L_0^2(\Omega) = \{q \in L^2(\Omega),\ \int_\Omega q\,dx = 0\},$$

the norm in $\mathbf{V}$ is given by

$$\|\mathbf{v}\|_{\mathbf{V}} = (\int_\Omega |\nabla \mathbf{v}|^2 dx)^{\frac{1}{2}} \quad \forall\, \mathbf{v} \in \mathbf{V}.$$

For simplicity, we denote the inner products of both $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ by $(\cdot,\cdot)$, namely,

$$(u,v) = \int_\Omega u(x)v(x)dx \quad \forall\, u,v \in L^2(\Omega),$$
$$(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}(x)\cdot\mathbf{v}(x)dx \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{L}^2(\Omega).$$

By introducing the $\rho = \sigma^2$, we have

$$\sigma(\sigma\mathbf{u})_t = \rho\mathbf{u}_t + \frac{1}{2}\rho_t\mathbf{u} = \rho\mathbf{u}_t - \frac{1}{2}\nabla\cdot(\rho\mathbf{u})\mathbf{u}.$$

We can derive the following equivalent system for (1.1)–(1.3)

$$\sigma_t + \nabla\cdot(\sigma\mathbf{u}) = 0, \tag{2.1}$$

$$\sigma(\sigma\mathbf{u})_t - \mu\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{2}\mathbf{u}\nabla\cdot(\rho\mathbf{u}) + \nabla p = \mathbf{f}, \tag{2.2}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{2.3}$$

The initial and boundary conditions for the equivalent system are

$$\begin{aligned} &\sigma(x,0) = \sigma_0(x) = \sqrt{\rho_0}, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in } \Omega, \\ &\mathbf{u}(x,t) = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.4}$$

which σ_0 satisfies the no vacuum assumption

$$\sqrt{m} \leq \sigma_0(x) \leq \sqrt{M} \quad \forall\, x \in \Omega. \tag{2.5}$$

For some $N \in \mathbb{N}^+$, define the time-step $\tau = T/N$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of [0, T] with $t_n = n\tau$. For any sequence of functions $\{v^n\}_{n=0}^N$, denote

$$D_\tau v^{n+1} = \frac{3v^{n+1} - 4v^v + v^{n-1}}{2\tau},$$

and the linearized extrapolation by

$$\hat{v}^n = 2^n - v^{n-1} \quad \forall\, 1 \leq n \leq N-1,$$

then by the Taylor formula, we have [10]

$$D_\tau v(t_{n+1}) - v(t_{n+1}) = \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} \left(2(t-t_n)_+^2 - (t-t_{n-1})^2\right) v_{ttt} dt, \tag{2.6}$$

$$\hat{v}(t_n) - v(t_{n+1}) = \int_{t_{n-1}}^{t_{n+1}} \left(2(t-t_n)_+ - (t-t_{n-1})\right) v_{tt} dt, \tag{2.7}$$

where $(t-t_n)_+ = \max\{t-t_n, 0\}$, then we can derive

$$D_\tau v(t_{n+1}) - v(t_{n+1}) = \mathcal{O}(\tau^2), \quad \hat{v}(t_{n+1}) - v(t_{n+1}) = \mathcal{O}(\tau^2). \tag{2.8}$$

It is easy to see that

$$
\begin{aligned}
(D_\tau v^{n+1}, v^{n+1}) =& \frac{1}{4\tau}(\|v^{n+1}\|_{L^2}^2 - \|v^n\|_{L^2}^2 + \|\hat{v}^{n+1}\|_{L^2}^2 - \|\hat{v}^n\|_{L^2}^2) \\
&+ \frac{1}{4\tau}\|v^{n+1} - 2v^n + v^{n-1}\|_{L^2}^2 \quad \forall 1 \le n \le N-1.
\end{aligned} \tag{2.9}
$$

We define the divergence-free subspace and curl-free subspace of $\mathbf{L}^q(\Omega)$ as follows [28]:

$$
\begin{aligned}
&\mathbf{L}_\sigma^q(\Omega) = \{\mathbf{w} \in \mathbf{L}^q(\Omega) : \nabla \cdot \mathbf{w} = 0, \quad \mathbf{w} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\}, \\
&\mathbf{L}_\sigma^q(\Omega)^\perp = \left\{\nabla\phi : \phi \in W^{1,q}(\Omega)\right\},
\end{aligned}
$$

For $2 \le q < +\infty$, we introduce the projection operator $\mathbb{P} : \mathbf{L}^q(\Omega) \to \mathbf{L}_\sigma^q(\Omega)$. In terms of the Helmholtz-Weyl decomposition of $\mathbf{L}^q(\Omega)$ [28], for any function $\mathbf{v} \in \mathbf{L}^q(\Omega)$, there has a unique decomposition

$$
\mathbf{v} = \mathbb{P}\mathbf{v} + \nabla\phi,
$$

with $\mathbb{P}\mathbf{v} \in \mathbf{L}_\sigma^q(\Omega)$ and $\nabla\phi \in \mathbf{L}_\sigma^q(\Omega)^\perp$ such that

$$
\|\mathbb{P}\mathbf{v}\|_{L^q} + \|\nabla\phi\|_{L^q} \le C\|\mathbf{v}\|_{L^q}.
$$

In fact, the Helmholtz decomposition on $\mathbf{L}^q(\Omega)$ exists if and only if the Neumann problem [7]

$$
\begin{cases}
\Delta\phi = \nabla \cdot \mathbf{v} & \text{in} \quad \Omega, \\
\nabla\phi \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} & \text{on} \quad \partial\Omega,
\end{cases}
$$

has a unique weak solution $\phi \in W^{1,q}(\Omega) \backslash \mathbb{R}$ satisfying

$$
\|\nabla\phi\|_{L^q(\Omega)} \le C\|\mathbf{v}\|_{L^q(\Omega)}, \quad \forall \mathbf{v} \in \mathbf{L}^q(\Omega).
$$

Then $\mathbb{P}\mathbf{v} = \mathbf{v} - \nabla\phi$ with ϕ given by the above Neumann problem.

In addition, the Stokes operator $\mathbf{A}$ can be defined by $\mathbf{A} = -\mathbb{P}\Delta$ satisfying [29]

$$
\|\mathbf{v}\|_{W^{2,q}} \le C\|\mathbf{A}\mathbf{v}\|_{L^q}.
$$

At the initial time step. we choose $\sigma^0 = \sigma_0$ and $\mathbf{u}^0 = \mathbf{u}_0$. The BDF2 time discrete scheme for the equivalent system (2.1)–(2.3) is described as follow.

BDF2 time-discrete algorithm:

Step 1: For given σ^n and $\hat{\mathbf{u}}^n$, we find σ^{n+1} by

$$
D_\tau \sigma^{n+1} + \nabla \cdot (\sigma^{n+1}\hat{\mathbf{u}}^n) = 0, \tag{2.10}
$$

Step 2: We find $(\mathbf{u}^{n+1}, p^{n+1})$ by

$$
\sigma^{n+1} D_\tau(\sigma^{n+1}\mathbf{u}^{n+1}) - \mu\Delta\mathbf{u}^{n+1} + \rho^{n+1}\hat{\mathbf{u}}^n\nabla\mathbf{u}^{n+1} + \frac{1}{2}\mathbf{u}^{n+1}\nabla \cdot (\rho^{n+1}\hat{\mathbf{u}}^n) + \nabla p^{n+1} = \mathbf{f}^{n+1}, \tag{2.11}
$$

with the incompressibility condition $\nabla \cdot \mathbf{u}^{n+1} = 0$, the boundary condition $\mathbf{u}^{n+1} = 0$ on $\partial\Omega$ and $\rho^{n+1} = (\sigma^{n+1})^2$.

2.1. Existence and uniqueness of the time-discrete solutions

In the following, we prove the existence and uniqueness of the time-discrete solutions of the BDF2 time-discrete scheme (2.10)–(2.11) by the following theorem.

Theorem 2.1. *Let $\sigma, \boldsymbol{u}$ and p be the solutions of the equivalent system (2.1)–(2.3), then for $0 \le n \le N-1$, the time-discrete scheme (2.10)–(2.11) admit the unique solution $\sigma^{n+1}, \boldsymbol{u}^{n+1}$ and p^{n+1} without any condition on the time stepsize τ.*

Proof. The unique solvability of the discrete scheme is established by showing that its corresponding homogeneous linear system has only the trivial solution, similar techniques has been applied in [9,21,30]. If $(\sigma^k, \mathbf{u}^k, p^k)$ are given for $k = 0, 1, 2 \dots, n$, then (2.10) has a unique solution if and only if the corresponding homogeneous linear equation

$$
\frac{3}{2\tau}(\Xi, r) + (\hat{\mathbf{u}}^n \cdot \nabla\Xi, r) = 0, \quad \forall r \in H^1(\Omega), \tag{2.12}
$$

has only zero solution $\Xi = 0$. Setting $r = \Xi$ and using the integration by parts (IBP), $\nabla \cdot \hat{\mathbf{u}}^n = 0$, we have $\|\Xi\|_{L^2} = 0$, which implies $\Xi = 0$. Thus, we prove the unique solvability of (2.10). (2.11) has a unique solution $(\mathbf{u}^{n+1}, p^{n+1})$ if and only if the homogeneous linear equation

$$
\begin{aligned}
&\frac{3}{2\tau}(\sigma^{n+1}\sigma^{n+1}\Theta, \mathbf{v}) + \mu(\nabla\Theta, \nabla\mathbf{v}) + (\rho^{n+1}\hat{\mathbf{u}}^n \cdot \nabla\Theta, \mathbf{v}) \\
&+ \frac{1}{2}(\Theta\nabla \cdot (\rho^{n+1}\hat{\mathbf{u}}^n), \mathbf{v}) - (\Psi, \nabla \cdot \mathbf{v}) = 0, \quad \forall(\mathbf{v}, q) \in (\mathbf{V}, M),
\end{aligned} \tag{2.13}
$$

$$
(\nabla \cdot \Theta, q) = 0. \tag{2.14}
$$

has only zero solution $(\Theta, \Psi) = (\mathbf{0}, 0)$. Setting $(\mathbf{v}, q) = (\Theta, \Psi)$ and using integration by parts (IBP), we have

$$\begin{aligned} \frac{1}{2}\int_\Omega \nabla\cdot(\rho^{n+1}\hat{\mathbf{u}}^n)|\Theta|^2 dx &= -\frac{1}{2}\int_\Omega \nabla|\Theta|^2\cdot(\rho^{n+1}\hat{\mathbf{u}}^n)dx \\ &= -\int_\Omega \rho^{n+1}(\hat{\mathbf{u}}^n\cdot\nabla)\Theta\cdot\Theta dx, \end{aligned} \tag{2.15}$$

Thus

$$\frac{3}{2\tau}\|\sigma^{n+1}\Theta^{n+1}\|_{L^2}^2 + \mu\|\nabla\Theta\|_{L^2}^2 = 0, \tag{2.16}$$

which implies $\Theta = 0$ with the fact $\sigma^{n+1} = \sqrt{\rho^{n+1}} > 0$. Then (2.13) reduces to $(\Psi, \nabla\cdot\mathbf{v}) = 0$, $\forall\mathbf{v}\in\mathbf{V}$, we get $\Psi = 0$. Thus, we prove the unique solvability of (2.11). □

2.2. Unconditional stability results

In this subsection, we will derive the unconditional stability for the BDF2 time-discrete scheme presented in (2.10) and (2.11).

Theorem 2.2. *The BDF2 scheme (2.10)–(2.11) is unconditionally energy stable for any solution $(\mathbf{u}^n, \sigma^n)$ without any time-step restriction, and we have the following bounds*

$$\|\sigma^{n+1}\|_{L^2} \le \|\sigma_0\|_{L^2} \quad \forall 0\le n\le N-1, \tag{2.17}$$

$$\|\sigma^{n+1}\boldsymbol{u}^{n+1}\|_{L^2}^2 \le C\tau\sum_{n=0}^{N-1}\|\boldsymbol{f}^{n+1}\|_{L^2}^2 + \|\sigma^0\boldsymbol{u}^0\|_{L^2}^2 \quad \forall 0\le n\le N-1. \tag{2.18}$$

Proof. Testing (2.10) by $4\tau\sigma^{n+1}$ gives

$$\|\sigma^{n+1}\|_{L^2}^2 - \|\sigma^n\|_{L^2} + \|\hat{\sigma}^{n+1}\|_{L^2}^2 - \|\hat{\sigma}^n\|_{L^2}^2 + \|\sigma^{n+1} - 2\sigma^n + \sigma^{n-1}\|_{L^2}^2 = 0, \tag{2.19}$$

where we use $\nabla\cdot\hat{\mathbf{u}}^n = 0$ in Ω, $\hat{\mathbf{u}}^n = 0$ on $\partial\Omega$, and Gauss's Theorem [31]

$$\int_\Omega \sigma^{n+1}\nabla\cdot(\sigma^{n+1}\hat{\mathbf{u}}^n)dx = \frac{1}{2}\int_\Omega \nabla\cdot(|\sigma^{n+1}|^2\hat{\mathbf{u}}^n)dx = \frac{1}{2}\int_{\partial\Omega}|\sigma^{n+1}|\hat{\mathbf{u}}^n\cdot\mathbf{n}ds = 0. \tag{2.20}$$

Summing (2.19) and we derive the following stability result

$$\|\sigma^{n+1}\|_{L^2} \le \|\sigma_0\|_{L^2} \quad \forall 0\le n\le N-1. \tag{2.21}$$

Taking the inner product of (2.11) with $4\tau\mathbf{u}^{n+1}$ leads to

$$\begin{aligned} &\|\sigma^{n+1}\mathbf{u}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{u}^n\|_{L^2}^2 + \|\widehat{\sigma^{n+1}\mathbf{u}^{n+1}}\|_{L^2}^2 - \|\widehat{\sigma^n\mathbf{u}^n}\|_{L^2}^2 \\ &\quad + \|\sigma^{n+1}\mathbf{u}^{n+1} - 2\sigma^n\mathbf{u}^n + \sigma^{n-1}\mathbf{u}^{n-1}\|_{L^2}^2 + 4\mu\tau\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2 \\ &\le 4\tau\|\mathbf{f}^{n+1}\|_{L^2}\|\nabla\mathbf{u}^{n+1}\|_{L^2} \\ &\le \mu\tau\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2 + C\tau\|\mathbf{f}^{n+1}\|_{L^2}^2, \end{aligned} \tag{2.22}$$

where by applying integration by parts (IBP), we obtain

$$\begin{aligned} \int_\Omega \nabla\cdot(\rho^{n+1}\hat{\mathbf{u}}^n)|\mathbf{u}^{n+1}|^2 dx &= -\int_\Omega \nabla|\mathbf{u}^{n+1}|^2\cdot(\rho^{n+1}\hat{\mathbf{u}}^n)dx \\ &= -2\int_\Omega \rho^{n+1}(\hat{\mathbf{u}}^n\cdot\nabla)\mathbf{u}^{n+1}\cdot\mathbf{u}^{n+1}dx, \end{aligned} \tag{2.23}$$

Noting that Eq. (2.23) indicates that the corresponding nonlinear terms do not contribute to the energy.

Adding up (2.22), we get

$$\begin{aligned} &\|\sigma^{n+1}\mathbf{u}^{n+1}\|_{L^2}^2 + \|\widehat{\sigma^{n+1}\mathbf{u}^{n+1}}\|_{L^2}^2 + \sum_{n=0}^{N-1}\|\sigma^{n+1}\mathbf{u}^{n+1} - 2\sigma^n\mathbf{u}^n + \sigma^{n-1}\mathbf{u}^{n-1}\|_{L^2}^2 + 4\mu\tau\sum_{n=0}^{N-1}\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2 \\ &\le C\tau\sum_{n=0}^{N-1}\|\mathbf{f}^{n+1}\|_{L^2}^2 + \|\sigma^0\mathbf{u}^0\|_{L^2}^2. \end{aligned} \tag{2.24}$$

Thus, we complete the proof of Theorem 2.2. □

2.3. Some known lemmas

Now, we introduce the following two lemmas established in [7,23,27], which are useful tools for our error analysis.

Lemma 2.1. *Assume that $\sigma^n \in H^2(\Omega)\cap W^{1,\infty}(\Omega)$ and $\mathbf{u} \in W^{1,\infty}(\Omega)\cap V_0 \cap H^2(\Omega)$ are given for $0 \le n \le N-1$, then the hyperbolic equation (2.10) has a unique solution $\sigma^{n+1} \in H^2(\Omega)\cap W^{1,\infty}(\Omega)$ which satisfies the maximum principle*

$$\min_{x\in\Omega}\sigma^n(x) \le \sigma^{n+1}(x) \le \max_{x\in\Omega}\sigma^n(x) \quad \forall\, x \in \Omega. \tag{2.25}$$

Lemma 2.2. *For $1 \le n \le N$, let $a^n(x)$ be a function defined on Ω which satisfies*
(1) $k_0 \le a^n(x) \le k_1$ for some positive constants k_0 and k_1,
(2) $\max_{1\le n\le N}\|a^n\|_{W^{1,4}} \le k_2$ for some positive constant k_2,
(3) $\sum_{n=1}^N \|a^n - a^{n-1}\|_{L^\infty} \le k_3$ for some positive constant k_3.
Then the solution $\boldsymbol{v}^n$ to the following problem

$$a^n D_\tau \mathbf{v}^n + \mathbf{A}\mathbf{v}^n = \mathbf{g}^n \quad \text{for } n = 1,\dots,N,$$

satisfies

$$\tau\sum_{n=1}^N(\|D_\tau \mathbf{v}^n\|_{L^q}^p + \|\mathbf{A}\mathbf{v}^n\|_q^p) \le C(\tau^{1-p}\|\mathbf{v}^0\|_{L^q} + \tau\|\mathbf{A}\mathbf{v}^0\|_{L^q}) + C\tau\sum_{n=1}^N\|\mathbf{g}^n\|_{L^q}^p$$

for any $1 < p, q < \infty$, where $C > 0$ is independent of τ and a^n, but may depends on k_0, k_1, k_2, k_3 and T.

The following lemma will be used frequently in the following content.

Lemma 2.3 (*Discrete Gronwall's Inequality [31]*)**.** *Let a_k, b_k and y_k be the nonnegative numbers such that*

$$a_n + \tau\sum_{k=0}^n b_k \le \tau\sum_{k=0}^n \gamma_k a_k + B \quad \text{for } n \ge 1, \tag{2.26}$$

Suppose $\tau\gamma_k \le 1$ and set $\sigma_k = (1-\tau\gamma_k)^{-1}$. Then there holds

$$a_n + \tau\sum_{k=0}^n b_k \le exp(\tau\sum_{k=0}^n \gamma_k\sigma_k)B \quad \text{for } n \ge 1. \tag{2.27}$$

***Remark 2.1** (Discrete Gronwall's Inequality [31]).* If the sum on the right-hand side of (2.26) extends only up to $n-1$, then the estimate (2.27) still holds for all $k \ge 1$ with $\sigma_k = 1$.

In the following, we always assume that the initial density ρ^0 is positive and the system (1.1)–(1.3) has a sufficiently smooth solution which satisfies

$$0 < \min_{x\in\Omega}\rho_0(x) \le \max_{x\in\Omega}\rho_0(x) \le \infty. \tag{2.28}$$

Since we focus on the optimal convergence analysis for the time discrete scheme (2.10)- (2.11), the regularity assumption of the exact solution is needed. Assume that the solutions to the system (2.1)–(2.3) satisfy the following regularities

$$\begin{aligned}
&\sigma \in L^\infty\left(0,T;H^3(\Omega)\right), \sigma_t \in L^\infty\left(0,T;H^1(\Omega)\right)\cap L^2\left(0,T;H^2(\Omega)\right),\\
&\sigma_{tt} \in L^2\left(0,T;L^2(\Omega)\right), \sigma_{ttt} \in L^2\left(0,T;L^2(\Omega)\right),\\
&\mathbf{u} \in L^\infty\left(0,T;\mathbf{H}^3(\Omega)\cap \mathbf{V}_0\right), \mathbf{u}_t \in L^\infty\left(0,T;\mathbf{H}^1(\Omega)\right)\cap L^2\left(0,T;\mathbf{H}^2(\Omega)\right),\\
&\mathbf{u}_{tt} \in L^2\left(0,T;\mathbf{H}^1(\Omega)\right), \mathbf{u}_{ttt} \in L^2\left(0,T;\mathbf{L}^2(\Omega)\right), p \in L^\infty\left(0,T;H^1(\Omega)\cap M\right).
\end{aligned} \tag{2.29}$$

Remark 2.2. The hyperbolic nature of the density Eq. (2.1) inherently complicates error analysis, as it necessitates boundedness of the discrete velocity $\mathbf{u}^n$ in the $W^{1,\infty}$ norm. In order to get this boundedness, we impose the requirement of sufficient smoothness on the exact solutions. As we know that the solution can achieve the H^2 regularity if the initial data is sufficiently smooth in a convex domain such as rectangle. But whether the solution can have the H^3 regularity in a convex polygon domain is still an open problem. We make this regularity assumptions merely to mainly focuses on the error analysis of BDF2 time-discrete scheme of variable density Navier–Stokes equations, the strong regularity conditions have recently been assumed in [7,10,11].

We continue to adopt the positivity and boundedness conditions for ρ, similar to those established in [9,27,32]. Meas$\{x \in \Omega \mid \alpha \le \rho(x,t) \le \beta\}$ is independent of $t \ge 0$ for all $0 \le \alpha \le \beta < +\infty$, then ρ satisfies

$$\min_{x\in\Omega}\rho_0(x) \le \rho(x,t) \le \max_{x\in\Omega}\rho_0(x) \quad \text{for } t \ge 0, \tag{2.30}$$

which implies that

$$\min_{x\in\Omega}\sigma_0(x) \le \sigma(x,t) \le \max_{x\in\Omega}\sigma_0(x) \quad \text{for } t \ge 0. \tag{2.31}$$

3. Error analysis

In this section, we will present the main theorem and provide a comprehensive rigorous convergent analysis for the BDF2 time-discrete algorithm (2.10)–(2.11).

The exact solution $(\mathbf{u}(t_{n+1}), \sigma(t_{n+1}), p(t_{n+1}))$ for the equivalent systems (2.1)–(2.3) satisfies the following equations:

$$D_\tau \sigma(t_{n+1}) + \nabla\sigma(t_{n+1}) \cdot \hat{\mathbf{u}}(t_n) = R_\sigma^{n+1}, \tag{3.1}$$

$$\begin{aligned}&\sigma(t_{n+1}) D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1})) - \mu\Delta\mathbf{u}(t_{n+1}) + \rho(t_{n+1})\hat{\mathbf{u}}(t_n)\cdot\nabla\mathbf{u}(t_{n+1})\\ &+ \frac{1}{2}\mathbf{u}(t_{n+1})\nabla\cdot(\rho(t_{n+1})\hat{\mathbf{u}}(t_n)) + \nabla p(t_{n+1}) = \mathbf{f}(t_{n+1}) + R_{\mathbf{u}}^{n+1},\end{aligned} \tag{3.2}$$

where the truncation error R_σ^{n+1} and $R_{\mathbf{u}}^{n+1}$ satisfy

$$\begin{aligned}R_\sigma^{n+1} &= D_\tau\sigma(t_{n+1}) - \sigma_t(t_{n+1}) + \nabla\sigma(t_{n+1})\cdot\hat{\mathbf{u}}(t_n) - \nabla\sigma(t_{n+1})\cdot\mathbf{u}(t_{n+1}),\\ R_{\mathbf{u}}^{n+1} &= \sigma(t_{n+1})D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1})) - \sigma(t_{n+1})(\sigma\mathbf{u})_t(t_{n+1})\\ &\quad+ \rho(t_{n+1})((\hat{\mathbf{u}}(t_n) - \mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})) + \frac{1}{2}\mathbf{u}(t_{n+1})\nabla\cdot(\rho(t_{n+1})(\hat{\mathbf{u}}(t_n) - \mathbf{u}(t_{n+1}))).\end{aligned} \tag{3.3}$$

Under the regularity assumption (2.29), we can derive

$$\left\|R_\sigma^{n+1}\right\|_{W^{1,\infty}}^2 + \tau\sum_{n=0}^{N-1}\left(\left\|R_\sigma^{n+1}\right\|_{H^2}^2 + \left\|R_\sigma^{n+1}\right\|_{H^1}^2 + \left\|R_u^{n+1}\right\|_{L^2}^2\right) \le C\tau^4. \tag{3.4}$$

Denote

$$e_\sigma^{n+1} = \sigma(t_{n+1}) - \sigma^{n+1},\ \mathbf{e}_{\mathbf{u}}^{n+1} = \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1},\ e_\rho^{n+1} = \rho(t_{n+1}) - \rho^{n+1},\ e_p^{n+1} = p(t_{n+1}) - p^{n+1}.$$

We will now introduce the main theorem in this paper.

Theorem 3.1. *Suppose that the initial data $\sigma_0 \in H^2(\Omega)\cap W^{1,\infty}$, $\mathbf{u}_0 \in W^{1,\infty}\cap H^2(\Omega)$, and σ_0 satisfies the no vacuum assumption (2.5). Let $\sigma, \boldsymbol{u}$ and p be the solutions of the equivalent system (2.1)–(2.3) and satisfy the regularity assumption (2.29), there exists some small $\tau > 0$ such that when $\tau \le \tau_0$, one has*

$$\sqrt{m} \le \sigma^i(x) \le \sqrt{M} \qquad \forall x \in \Omega, \tag{3.5}$$

and the regularity estimates

$$\max_{0\le i\le N}(\|\boldsymbol{u}^i\|_{W^{1,\infty}} + \|\mathbf{A}\boldsymbol{u}^i\|_{L^2} + \|\sigma^i\|_{W^{1,\infty}} + \|\sigma^i\|_{H^2}) \le K. \tag{3.6}$$

Moreover, the following temporal error estimates hold

$$\max_{0\le i\le N}(\|e_\sigma^i\|_{L^2}^2 + \|e_\rho^i\|_{L^2}^2 + \|\boldsymbol{e}_{\mathbf{u}}^i\|_{L^2}^2) \le C\tau^4. \tag{3.7}$$

Firstly we need to derive the error equations by subtracting (2.10)–(2.11) from (3.1)–(3.2)

$$D_\tau e_\sigma^{n+1} + \nabla\sigma(t_{n+1})\cdot\hat{\mathbf{e}}_{\mathbf{u}}^n + \nabla e_\sigma^{n+1}\cdot\hat{\mathbf{u}}^n = R_\sigma^{n+1}, \tag{3.8}$$

$$\sigma^{n+1}D_\tau(\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}) - \mu\Delta\mathbf{e}_{\mathbf{u}}^{n+1} + \nabla e_p^{n+1} + \sum_{i=1}^{10} Y_i^{n+1} = R_{\mathbf{u}}^{n+1}. \tag{3.9}$$

where

$$\begin{aligned}\sum_{i=1}^{10} Y_i^{n+1} =& e_\sigma^{n+1}D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1})) + \sigma^{n+1}e_\sigma^{n+1}D_\tau\mathbf{u}(t_{n+1}) + \sigma^{n+1}D_\tau e_\sigma^{n+1}\mathbf{u}(t_n)\\ &+ \frac{1}{2}\sigma^{n+1}\frac{\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})}{\tau}(e_\sigma^{n+1} - e_\sigma^{n-1}) + e_\rho^{n+1}\hat{\mathbf{u}}(t_n)\cdot\nabla\mathbf{u}(t_{n+1})\\ &+ \rho^{n+1}\hat{\mathbf{e}}_{\mathbf{u}}^n\cdot\nabla\mathbf{u}(t_{n+1}) + \rho^{n+1}\hat{\mathbf{u}}^n\cdot\nabla\mathbf{e}_{\mathbf{u}}^{n+1} + \frac{1}{2}\mathbf{e}_{\mathbf{u}}^{n+1}\nabla\cdot(\rho^{n+1}\hat{\mathbf{u}}^n)\\ &+ \frac{1}{2}\mathbf{u}(t_{n+1})\nabla\cdot(e_\rho^{n+1}\hat{\mathbf{u}}(t_n)) + \frac{1}{2}\mathbf{u}(t_{n+1})\nabla\cdot(\rho^{n+1}\hat{\mathbf{e}}_{\mathbf{u}}^n).\end{aligned} \tag{3.10}$$

In order to complete the proof of Theorem 3.1, we need to prove the maximal principle (3.5), the regularity result (3.6) and the temporal error estimates (3.7) by using the method of mathematical induction.

It is clear that (3.5) and (3.6) hold for $i = 0$. For $0 \le n \le N-1$, we assume that (3.5) and (3.6) are valid for $i = n$, there holds

$$\sqrt{m} \le \sigma^i(x) \le \sqrt{M} \quad \forall x \in \Omega, \quad 0 \le i \le n, \tag{3.11}$$

and the regularity estimates

$$\|\mathbf{u}^n\|_{W^{1,\infty}} + \|\mathbf{A}\mathbf{u}^n\|_{L^2} + \|\sigma^n\|_{W^{1,\infty}} + \|\sigma^n\|_{H^2} \le K. \tag{3.12}$$

According to Lemma 2.1, the hyperbolic equation has a unique solution $\sigma^{n+1} \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ which satisfies

$$\sqrt{m} \leq \min_{x\in\Omega} \sigma^n(x) \leq \sigma^{n+1}(x) \leq \max_{x\in\Omega} \sigma^n(x) \leq \sqrt{M} \quad 0 \leq n \leq N-1. \tag{3.13}$$

To close the mathematical induction, we need to prove that (3.6) and (3.7) are valid for $i = n+1$.

3.1. Estimate of $\|e_\sigma^{n+1}\|_{L^2}, \|\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}$

Lemma 3.1. *Under the assumption (2.29), for sufficiently small τ, there holds*

$$\|e_\sigma^{n+1}\|_{L^2}^2 + \|\hat{e}_\sigma^{n+1}\|_{L^2}^2 + \sum_{n=0}^{N-1} \|e_\sigma^{n+1} - 2e_\sigma^n + e_\sigma^{n-1}\|_{L^2}^2 \leq C\tau^4 + C\tau \sum_{n=0}^{N-1} \|\hat{\sigma}^n \hat{\mathbf{e}}_\mathbf{u}^n\|_{L^2}^2. \tag{3.14}$$

Proof. Testing (3.8) by $4\tau e_\sigma^{n+1}$, we have

$$\begin{aligned} &\|e_\sigma^{n+1}\|_{L^2}^2 - \|e_\sigma^n\|_{L^2}^2 + \|\hat{e}_\sigma^{n+1}\|_{L^2}^2 - \|\hat{e}_\sigma^n\|_{L^2}^2 + \|e_\sigma^{n+1} - 2e_\sigma^n + e_\sigma^{n-1}\|_{L^2}^2 \\ &\quad + 4\tau(\nabla\sigma(t_{n+1})\hat{\mathbf{e}}_\mathbf{u}^n, e_\sigma^{n+1}) + 4\tau(\nabla e_\sigma^{n+1} \cdot \hat{\mathbf{u}}^n, e_\sigma^{n+1}) = 4\tau(R_\sigma^{n+1}, e_\sigma^{n+1}). \end{aligned} \tag{3.15}$$

By using (3.13), the regularity assumption (2.29) and $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, we have

$$4\tau|(\nabla\sigma(t_{n+1})\hat{e}_\mathbf{u}^n, e_\sigma^{n+1})| \leq C\tau\|\hat{e}_\mathbf{u}^n\|_{L^2}^2 + C\tau\|e_\sigma^{n+1}\|_{L^2}^2 \leq C\tau\|\hat{\sigma}^n\hat{e}_\mathbf{u}^n\|_{L^2}^2 + C\tau\|e_\sigma^{n+1}\|_{L^2}^2, \tag{3.16}$$

$$4\tau(R_\sigma^{n+1}, e_\sigma^{n+1}) \leq C\tau\|e_\sigma^{n+1}\|_{L^2}^2 + C\tau\|R_\sigma^{n+1}\|_{L^2}^2. \tag{3.17}$$

By using integration by parts and $\nabla \cdot \hat{\mathbf{u}}^n = 0$, we can derive

$$4\tau(\nabla e_\sigma^{n+1} \cdot \mathbf{u}^n, e_\sigma^{n+1}) = 2\tau \int_\Omega \nabla|e_\sigma^{n+1}|^2 \cdot \hat{\mathbf{u}}^n dx = 0,$$

Substituting these inequalities into (3.15) and summing it, by using truncation error (3.4) and the discrete Gronwall's inequality in Lemma 2.3, we complete the proof of Lemma 3.1. □

Next, we will estimate $\|D_\tau e_\sigma^{n+1}\|_{L^2}^2$ and $\|\nabla e_\sigma^{n+1}\|_{L^2}^2$ by using the following two lemmas, which are parts of the process for estimating $\|\sigma^{n+1}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}$.

Lemma 3.2. *Under the assumption (2.29), for sufficiently small τ, there holds*

$$\frac{\tau}{2}\|D_\tau e_\sigma^{n+1}\|_{L^2}^2 \leq C\tau\|\nabla\hat{\mathbf{e}}_\mathbf{u}^n\|_{L^2}^2 + C\tau\|\nabla e_\sigma^{n+1}\|_{L^2}^2 + C\tau\|R_\mathbf{u}^{n+1}\|_{L^2}^2. \tag{3.18}$$

Proof. Testing (3.8) by $\tau D_\tau e_\sigma^{n+1}$, we get

$$\tau\|D_\tau e_\sigma^{n+1}\|_{L^2}^2 + \tau(\nabla\sigma(t_{n+1})\hat{\mathbf{e}}_\mathbf{u}^n, D_\tau e_\sigma^{n+1}) + \tau(\nabla e_\sigma^{n+1}, \mathbf{u}^n, D_\tau e_\sigma^{n+1}) = \tau(R_\sigma^{n+1}, D_\tau e_\sigma^{n+1}). \tag{3.19}$$

Applications of the Hölder inequality and Young inequality yield

$$\begin{aligned} \tau\|D_\tau e_\sigma^{n+1}\|_{L^2}^2 \leq& \tau\|\nabla\sigma(t_{n+1})\|_{L^4}\|\hat{\mathbf{e}}_\mathbf{u}^n\|_{L^4}\|D_\tau e_\sigma^{n+1}\|_{L^2} + \tau\|\nabla e_\sigma^{n+1}\|_{L^2}\|\hat{\mathbf{u}}^n\|_{L^\infty}\|D_\tau e_\sigma^{n+1}\|_{L^2} \\ &+ \tau\|R_\mathbf{u}^{n+1}\|_{L^2}\|D_\tau e_\sigma^{n+1}\|_{L^2} \\ \leq& \epsilon\tau\|D_\tau e_\sigma^{n+1}\|_{L^2}^2 + C\tau\|\nabla\hat{\mathbf{e}}_\mathbf{u}^n\|_{L^2}^2 + C\tau\|\nabla e_\sigma^{n+1}\|_{L^2}^2 + C\tau\|R_\mathbf{u}^{n+1}\|_{L^2}^2. \end{aligned} \tag{3.20}$$

By choosing sufficiently small ϵ such that $\epsilon\tau \leq \frac{1}{2}$, we complete the proof of Lemma 3.2. □

Lemma 3.3. *Under the assumption (2.29), for sufficiently small τ, there holds*

$$\|\nabla e_\sigma^{n+1}\|_{L^2}^2 + \|\nabla\hat{e}_\sigma^{n+1}\|_{L^2}^2 + \sum_{n=0}^{N-1} \|\nabla e_\sigma^{n+1} - 2\nabla e_\sigma^n + \nabla e_\sigma^{n-1}\|_{L^2}^2 \leq C\tau^4 + C\tau \sum_{n=0}^{N-1} \|\nabla\hat{\mathbf{e}}_\mathbf{u}^n\|_{L^2}^2. \tag{3.21}$$

Proof. Differentiating (3.8) with respect to x_j gives

$$D_j D_\tau e_\sigma^{n+1} + D_j\nabla\sigma(t_{n+1}) \cdot \hat{\mathbf{e}}_\mathbf{u}^n + \nabla\sigma(t_{n+1}) \cdot D_j\hat{\mathbf{e}}_\mathbf{u}^n + D_j\nabla e_\sigma^{n+1} \cdot \hat{\mathbf{u}}^n + \nabla e_\sigma^{n+1} \cdot D_j\hat{\mathbf{u}}^n = D_j R_\sigma^{n+1}. \tag{3.22}$$

Testing (3.22) by $4\tau D_j e_\sigma^{n+1}$ and utilizing Hölder inequality, Young inequality and (3.12), we get

$$\begin{aligned} &\|D_j e_\sigma^{n+1}\|_{L^2}^2 - \|D_j e_\sigma^n\|_{L^2}^2 + \|\widehat{D_j e_\sigma^{n+1}}\|_{L^2}^2 - \|\widehat{D_j e_\sigma^n}\|_{L^2}^2 + \|D_j e_\sigma^{n+1} - 2D_j e_\sigma^n + D_j e_\sigma^{n-1}\|_{L^2} \\ \leq& 4\tau|(D_j\nabla\sigma(t_{n+1}) \cdot \hat{\mathbf{e}}_\mathbf{u}^n, D_j e_\sigma^{n+1})| + 4\tau|(\nabla\sigma(t_{n+1}) \cdot D_j\hat{\mathbf{e}}_\mathbf{u}^n, D_j e_\sigma^{n+1})| + 4\tau|(D_j\nabla e_\sigma^{n+1} \cdot \hat{\mathbf{u}}^n, D_j e_\sigma^{n+1})| \\ &+ 4\tau|(\nabla e_\sigma^{n+1} \cdot D_j\hat{\mathbf{u}}^n, D_j e_\sigma^{n+1})| + 4\tau|(D_j R_\sigma^{n+1}, D_j e_\sigma^{n+1})| \\ \leq& C\tau\|D_j\nabla\sigma(t_{n+1})\|_{L^\infty}\|\hat{\mathbf{e}}_\mathbf{u}^n\|_{L^2}\|D_j e_\sigma^{n+1}\|_{L^2} + C\tau\|\nabla\sigma(t_{n+1})\|_{L^\infty}\|D_j\hat{\mathbf{e}}_\mathbf{u}^n\|_{L^2}\|D_j e_\sigma^{n+1}\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+ C\tau\|\nabla e_\sigma^{n+1}\|_{L^2}\|D_j\hat{\mathbf{u}}^n\|_{L^\infty}\|D_j e_\sigma^{n+1}\|_{L^2} + C\tau\|D_j R_\sigma^{n+1}\|_{L^2}\|D_j e_\sigma^{n+1}\|_{L^2}\\
\le & C\tau\|\nabla e_\sigma^{n+1}\|_{L^2}^2 + C\tau\|\nabla \hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2 + C\tau\|R_\sigma^{n+1}\|_{H^1}^2,
\end{aligned}
\tag{3.23}
$$

which means that

$$
\begin{aligned}
&\|\nabla e_\sigma^{n+1}\|_{L^2}^2 - \|\nabla e_\sigma^n\|_{L^2}^2 + \|\widehat{\nabla e_\sigma^{n+1}}\|_{L^2}^2 - \|\widehat{\nabla e_\sigma^n}\|_{L^2}^2 + \|\nabla e_\sigma^{n+1} - 2\nabla e_\sigma^n + \nabla e_\sigma^{n-1}\|_{L^2}\\
&\le C\tau\|\nabla e_\sigma^{n+1}\|_{L^2}^2 + C\tau\|\nabla \hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2 + C\tau\|R_\sigma^{n+1}\|_{H^1}^2.
\end{aligned}
$$

Summing it and by using (3.4), the discrete Gronwall's inequality in Lemma 2.3, we complete Lemma 3.3. □

Then we will derive the estimate of $\|\sigma^{n+1}e_{\mathbf{u}}^{n+1}\|_{L^2}^2$ in the following.

Lemma 3.4. *Under the assumption (2.29), for sufficiently small τ, there holds*

$$
\|\sigma^{n+1}e_{\mathbf{u}}^{n+1}\|_{L^2}^2 + \|\widehat{\sigma^{n+1}e_{\mathbf{u}}^{n+1}}\|_{L^2}^2 + 4\tau\sum_{n=0}^{N-1}\|\nabla e_{\mathbf{u}}^{n+1}\|_{L^2}^2 \le C\tau^4. \tag{3.24}
$$

Proof. We need to prove Lemma 3.4 by using the method of mathematical induction.

Assuming that

$$
\|\sigma^n\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2 + \|\widehat{\sigma^n\mathbf{e}_{\mathbf{u}}^n}\|_{L^2}^2 + 4\tau\sum_{n=0}^{N-1}\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2 \le C\tau^4. \tag{3.25}
$$

Testing (3.9) by $4\tau\mathbf{e}_{\mathbf{u}}^{n+1}$, we get

$$
\begin{aligned}
&\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2 + \|\widehat{\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}}\|_{L^2}^2 - \|\widehat{\sigma^n\mathbf{e}_{\mathbf{u}}^n}\|_{L^2}^2 + 4\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2\\
&+ \|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1} - 2\sigma^n\mathbf{e}_{\mathbf{u}}^n + \sigma^{n-1}\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2\\
&\le 4\tau\sum_{i=1}^{10}|(Y_i^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1})| + 4\tau(R_{\mathbf{u}}^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1}).
\end{aligned}
\tag{3.26}
$$

By using the Hölder inequality and Young inequality, the right side hand of (3.26) can be estimated as follow

$$
\begin{aligned}
4\tau(R_{\mathbf{u}}^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1}) \le& \epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2 + C\tau\|R_{\mathbf{u}}^{n+1}\|_{L^2}^2,\\
4\tau|(Y_1^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1})| =& 4\tau|(e_\sigma^{n+1}D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1})), \mathbf{e}_{\mathbf{u}}^{n+1})|\\
\le& C\|e_\sigma^{n+1}\|_{L^2}\|D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1}))\|_{L^3}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^6}\\
\le& C\tau\|e_\sigma^{n+1}\|_{L^2}^2 + \epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2,
\end{aligned}
$$

where we use the regularity assumption (2.29) and Lagrange's mean value theorem, there exists $\xi_1 \in (t_n, t_{n+1})$, $\xi_2 \in (t_{n-1}, t_n)$ such that

$$
\begin{aligned}
\|D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1}))\|_{L^3} =& \left\|\frac{3\sigma(t_{n+1})\mathbf{u}(t_{n+1}) - 4\sigma(t_n)\mathbf{u}(t_n) + \sigma(t_{n-1})\mathbf{u}(t_{n-1})}{2\tau}\right\|_{L^3}\\
=& \left\|\frac{3}{2}\frac{\sigma(t_{n+1})\mathbf{u}(t_{n+1}) - \sigma(t_n)\mathbf{u}(t_n)}{\tau} + \frac{\sigma(t_{n-1})\mathbf{u}(t_{n-1}) - \sigma(t_n)\mathbf{u}(t_n)}{2\tau}\right\|_{L^3}\\
\le& C\|(\sigma\mathbf{u})_t(\xi_1)\|_{L^3} + C\|(\sigma\mathbf{u})_t(\xi_2)\|_{L^3}\\
\le& C.
\end{aligned}
\tag{3.27}
$$

Utilizing the same technique in (3.27), we get

$$
\begin{aligned}
4\tau|(Y_2^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1})| =& 4\tau|(\sigma^{n+1}e_\sigma^{n+1}D_\tau\mathbf{u}(t_{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1})|\\
\le& C\tau\|\sigma^{n+1}\|_{L^\infty}\|e_\sigma^{n+1}\|_{L^2}\|D_\tau\mathbf{u}(t_{n+1})\|_{L^3}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^6}\\
\le& C\tau\|e_\sigma^{n+1}\|_{L^2}^2 + \epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2,\\
4\tau|(Y_4^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1})| =& 2\tau|(\sigma^{n+1}\frac{\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})}{\tau}(e_\sigma^{n+1} - e_\sigma^{n-1}), \mathbf{e}_{\mathbf{u}}^{n+1})|\\
\le& 2\tau\|\sigma^{n+1}\|_{L^\infty}\left\|\frac{\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})}{\tau}\right\|_{L^3}\|(e_\sigma^{n+1} - e_\sigma^{n-1})\|_{L^2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^6}\\
\le& C\tau\|e_\sigma^{n+1}\|_{L^2}^2 + C\tau\|e_\sigma^{n-1}\|_{L^2}^2 + \epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2.
\end{aligned}
$$

In terms of (3.18), (3.21) and the regularity assumption (2.29),we get

$$
\begin{aligned}
4\tau|(Y_3^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1})| =& 4\tau|(\sigma^{n+1}D_\tau e_\sigma^{n+1}\mathbf{u}(t_n), \mathbf{e}_{\mathbf{u}}^{n+1})|\\
\le& C\tau\|\sigma^{n+1}\|_{L^\infty}\|D_\tau e_\sigma^{n+1}\|_{L^2}\|\mathbf{u}(t_n)\|_{L^\infty}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}\\
\le& C\tau\|D_\tau e_\sigma^{n+1}\|_{L^2}^2 + \epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2
\end{aligned}
\tag{3.28}
$$

$$\leq \epsilon\tau\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2 + C\tau\|\nabla \hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2 + C\tau^4 + C\tau\|R_{\mathbf{u}}^{n+1}\|_{L^2}^2.$$

By the regularity assumption (2.29) and (3.13), we can derive

$$\begin{aligned}\|e_\rho^{n+1}\|_{L^2} =&\|\rho(t_{n+1})-\rho^{n+1}\|_{L^2}\\ =&\|(\sigma(t_{n+1}))^2-(\sigma^{n+1})^2\|_{L^2}\\ =&\|\sigma(t_{n+1})+\sigma^{n+1}\|_{L^2}\|\sigma(t_{n+1})-\sigma^{n+1}\|_{L^2}\\ \leq&C\|e_\sigma^{n+1}\|_{L^2}.\end{aligned} \tag{3.29}$$

Thus

$$\begin{aligned}4\tau|(Y_5^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})| =&4\tau|(e_\rho^{n+1}\hat{\mathbf{u}}(t_n)\cdot\nabla\mathbf{u}(t_{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1})|\\ \leq&C\tau\|e_\sigma^{n+1}\|_{L^2}\|\hat{\mathbf{u}}(t_n)\|_{L^\infty}\|\nabla\mathbf{u}(t_{n+1})\|_{L^3}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^6}\\ \leq&C\tau\|e_\sigma^{n+1}\|_{L^2}+\epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}.\end{aligned}$$

By using the Hölder inequality and Young inequality, we have

$$\begin{aligned}4\tau|(Y_6^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})| =&4\tau|(\rho^{n+1}\hat{\mathbf{e}}_{\mathbf{u}}^{n}\cdot\nabla\mathbf{u}(t_{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1})|\\ \leq&\tau\|\rho^{n+1}\|_{L^\infty}\|\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}\|\nabla\mathbf{u}(t_{n+1})\|_{L^3}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^6}\\ \leq&C\tau\|\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2+\epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2,\\ 4\tau|(Y_7^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})| =&4\tau|(\rho^{n+1}\hat{\mathbf{u}}^{n}\cdot\nabla\mathbf{e}_{\mathbf{u}}^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})|\\ \leq&\tau\|\rho^{n+1}\|_{L^\infty}\|\hat{\mathbf{u}}^{n}\|_{L^\infty}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}\\ \leq&C\tau\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+\epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2.\end{aligned}$$

By using (3.12), (3.13), (3.29) and integration by parts (IBP), we can derive

$$\begin{aligned}4\tau|(Y_8^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})| =&2\tau|(\mathbf{e}_{\mathbf{u}}^{n+1}\nabla\cdot(\rho^{n+1}\hat{\mathbf{u}}^{n}),\mathbf{e}_{\mathbf{u}}^{n+1})|\\ =&4\tau|(\mathbf{e}_{\mathbf{u}}^{n+1}\rho^{n+1}\cdot\hat{\mathbf{u}}^{n},\nabla\mathbf{e}_{\mathbf{u}}^{n+1})|\\ \leq&C\tau\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+\epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2,\\ 4\tau|(Y_9^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})| =&2\tau|(\mathbf{u}(t_{n+1})\nabla\cdot(e_\rho^{n+1}\hat{\mathbf{u}}(t_n)),\mathbf{e}_{\mathbf{u}}^{n+1})|\\ \leq&2\tau|(\mathbf{u}(t_{n+1})\cdot(e_\rho^{n+1}\hat{\mathbf{u}}(t_n)),\nabla\mathbf{e}_{\mathbf{u}}^{n+1})|\\ \leq&\tau\|\mathbf{u}(t_{n+1})\|_{L^\infty}\|e_\rho^{n+1}\|_{L^2}\|\hat{\mathbf{u}}(t_n)\|_{L^\infty}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}\\ \leq&C\tau\|e_\sigma^{n+1}\|_{L^2}^2+\epsilon\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2,\\ 4\tau|(Y_{10}^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})| =&2\tau|(\mathbf{u}(t_{n+1})\nabla\cdot(\rho^{n+1}\hat{\mathbf{e}}_{\mathbf{u}}^{n}),\mathbf{e}_{\mathbf{u}}^{n+1})|\\ =&2\tau|(\mathbf{u}(t_{n+1})\cdot(\rho^{n+1}\hat{\mathbf{e}}_{\mathbf{u}}^{n}),\nabla\mathbf{e}_{\mathbf{u}}^{n+1})|\\ \leq&\tau\|\mathbf{u}(t_{n+1})\|_{L^\infty}\|\rho^{n+1}\|_{L^\infty}\|\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}\\ \leq&C\tau\|\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2+\epsilon\tau\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^{n+1}\|_{L^2}^2.\end{aligned}$$

Substituting these inequalities into (3.26), by taking sufficiently small ϵ and using (3.14), we can derive

$$\begin{aligned}&\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2-\|\sigma_\sigma^{n}\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2+\|\widehat{\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}}\|_{L^2}^2-\|\widehat{\sigma^{n}\mathbf{e}_{\mathbf{u}}^{n}}\|_{L^2}^2+4\tau\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2\\ &\leq C\tau\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+C\tau\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2+C\tau\|e_\sigma^{n+1}\|_{L^2}^2+C\tau\|e_\sigma^{n-1}\|_{L^2}^2+C\tau\|R_\sigma^{n+1}\|_{H^1}^2+C\tau\|R_{\mathbf{u}}^{n+1}\|_{L^2}^2,\\ &\leq C\tau\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+C\tau\|\hat{\sigma}^{n}\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2+C\tau\|\hat{\sigma}^{n-2}\hat{\mathbf{e}}_{\mathbf{u}}^{n-2}\|_{L^2}^2\\ &\quad+C\tau\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2+C\tau^4+C\tau\|R_\sigma^{n+1}\|_{H^1}^2+C\tau\|R_{\mathbf{u}}^{n+1}\|_{L^2}^2.\end{aligned} \tag{3.30}$$

Summing it and by using the discrete Gronwall's inequality in Lemma 2.3, we can derive (3.24), the proof of Lemma 3.4 is completed. □

Incorporate (3.24) into (3.14) and (3.21), we can infer

$$\|e_\sigma^{n+1}\|_{L^2}^2+\|\hat{e}_\sigma^{n+1}\|_{L^2}^2+\sum_{n=0}^{N-1}\|e_\sigma^{n+1}-2e_\sigma^{n}+e_\sigma^{n-1}\|_{L^2}^2\leq C\tau^4. \tag{3.31}$$

$$\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\nabla\hat{e}_\sigma^{n+1}\|_{L^2}^2+\sum_{n=0}^{N-1}\|\nabla e_\sigma^{n+1}-2\nabla e_\sigma^{n}+\nabla e_\sigma^{n-1}\|_{L^2}^2\leq C\tau^4. \tag{3.32}$$

Furthermore, based on the utilization of (3.29) and (3.31), we have

$$\|e_\rho^{n+1}\|_{L^2}^2\leq C\|e_\sigma^{n+1}\|_{L^2}^2\leq C\tau^4. \tag{3.33}$$

Now, we have proven that the temporal convergence rate stated in (3.7) is valid for $i = n+1$ through the method of mathematical induction.

We start by noting the following inequalities:

$$
\begin{aligned}
&\|\mathbf{u}^{n+1}\|_{H^2} \leq \|\mathbf{u}(t_{n+1})\|_{H^2} + \|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{H^2},\\
&\|\sigma^{n+1}\|_{H^2} \leq \|\sigma(t_{n+1})\|_{H^2} + \|e_{\sigma}^{n+1}\|_{H^2},\\
&\|\mathbf{u}^{n+1}\|_{W^{1,\infty}} \leq \|\mathbf{u}(t_{n+1})\|_{W^{1,\infty}} + \|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{W^{1,\infty}},\\
&\|\sigma^{n+1}\|_{W^{1,\infty}} \leq \|\sigma(t_{n+1})\|_{W^{1,\infty}} + \|e_{\sigma}^{n+1}\|_{W^{1,\infty}}.
\end{aligned} \tag{3.34}
$$

Furthermore, to complete the proof of Theorem 3.1, we need to confirm that the regularity result stated in (3.6) holds for $i = n+1$. It requires to estimate the errors $\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{H^2}$, $\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{W^{1,\infty}}$, $\|e_{\sigma}^{n+1}\|_{H^2}$, and $\|e_{\sigma}^{n+1}\|_{W^{1,\infty}}$.

3.2. Estimate of $\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{H^2}$

Firstly, we rewrite

$$
\sigma^{n+1} D_\tau(\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}) = (\sigma^{n+1})^2 D_\tau \mathbf{e}_{\mathbf{u}}^{n+1} + 2\sigma^{n+1}\frac{(\sigma^{n+1}-\sigma^n)}{\tau}\mathbf{e}_{\mathbf{u}}^{n} + \sigma^{n+1}\frac{\sigma^{n-1}-\sigma^{n+1}}{2\tau}\mathbf{e}_{\mathbf{u}}^{n-1}.
$$

Thus (3.9) can be rewritten as follows

$$
\begin{aligned}
&\rho^{n+1} D_\tau \mathbf{e}_{\mathbf{u}}^{n+1} + 2\sigma^{n+1}\frac{(\sigma^{n+1}-\sigma^n)}{\tau}\mathbf{e}_{\mathbf{u}}^{n} + \sigma^{n+1}\frac{\sigma^{n-1}-\sigma^{n+1}}{2\tau}\mathbf{e}_{\mathbf{u}}^{n-1} - \mu\Delta\mathbf{e}_{\mathbf{u}}^{n+1}\\
&+ \nabla e_p^{n+1} + \sum_{n=0}^{10} Y_i^{n+1} = R_{\mathbf{u}}^{n+1}
\end{aligned} \tag{3.35}
$$

Testing it by $D_\tau \mathbf{e}_{\mathbf{u}}^{n+1}$, we can derive

$$
\begin{aligned}
&\|\sigma^{n+1} D_\tau \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2 + \frac{\mu}{4\tau}(\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2 - \|\nabla \mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2 + \|\widehat{\nabla \mathbf{e}_{\mathbf{u}}^{n+1}}\|_{L^2}^2 - \|\widehat{\nabla \mathbf{e}_{\mathbf{u}}^{n}}\|_{L^2}^2)\\
&\leq C(\|R_{\mathbf{u}}^{n+1}\|_{L^2}^2 + \|2\sigma^{n+1}\frac{(\sigma^{n+1}-\sigma^n)}{\tau}\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2 + \|\sigma^{n+1}\frac{\sigma^{n}-\sigma^{n+1}}{2\tau}\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2 + \sum_{i=1}^{10}\|Y_i^{n+1}\|_{L^2}^2).
\end{aligned} \tag{3.36}
$$

By using (3.13) yields

$$
\|2\sigma^{n+1}\frac{(\sigma^{n+1}-\sigma^n)}{\tau}\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2 + \|\sigma^{n+1}\frac{\sigma^{n}-\sigma^{n+1}}{2\tau}\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2 \leq C\tau^{-2}\|\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2 + C\tau^{-2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2.
$$

Utilizing the same technique in (3.21) and (3.27) yields

$$
\begin{aligned}
\|Y_1^{n+1}\|_{L^2}^2 =&\|e_\sigma^{n+1} D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1}))\|_{L^2}^2\\
\leq& C\|e_\sigma^{n+1}\|_{L^6}\|D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1}))\|_{L^3}\\
\leq& C\tau\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2 + C\tau^4,\\
\|Y_2^{n+1}\|_{L^2}^2 =&\|\sigma^{n+1}e_\sigma^{n+1}D_\tau u(t_{n+1})\|_{L^2}^2\\
\leq& C\|\sigma^{n+1}\|_{L^\infty}\|e_\sigma^{n+1}\|_{L^6}\|D_\tau\mathbf{u}(t_{n+1})\|_{L3}\\
\leq& C\tau\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2 + C\tau^4.
\end{aligned} \tag{3.37}
$$

Based on the utilization of (3.13), (3.18) and (3.21), we get

$$
\begin{aligned}
\|Y_3^{n+1}\|_{L^2}^2 =&\|\sigma^{n+1}D_\tau e_\sigma^{n+1}\mathbf{u}(t_n)\|_{L^2}^2\\
\leq& C\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2 + C\tau^4 + C\|R_{\mathbf{u}}^{n+1}\|_{L^2}^2.
\end{aligned}
$$

Through the application of (3.14) and by the similar technique in (3.37), we can derive

$$
\begin{aligned}
\|Y_4^{n+1}\|_{L^2}^2 =&\frac{1}{4}\|\sigma^{n+1}\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\tau}(e_\sigma^{n+1}-e_\sigma^{n-1})\|_{L^2}^2\\
\leq& C\|e_\sigma^{n+1}\|_{L^2}^2 + C\|e_\sigma^{n-1}\|_{L^2}^2\\
\leq& C\tau^4 + C\tau\sum_{n=0}^{N-1}\|\hat{\sigma}^n\hat{\mathbf{e}}_{\mathbf{u}}^{n}\|_{L^2}^2 + C\tau\sum_{n=0}^{N-1}\|\hat{\sigma}^{n-2}\hat{\mathbf{e}}_{\mathbf{u}}^{n-2}\|_{L^2}^2.
\end{aligned}
$$

By using (3.14) and (3.29), we get

$$
\begin{aligned}
\|Y_5^{n+1}\|_{L^2}^2 =&\|e_\rho^{n+1}\hat{\mathbf{u}}(t_n)\cdot\nabla\mathbf{u}(t_{n+1})\|_{L^2}^2\\
\leq& C\|e_\sigma^{n+1}\|_{L^2}^2
\end{aligned}
$$

$$\leq C\tau^4 + C\tau \sum_{n=0}^{N-1} \|\hat{\sigma}^n \hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2.$$

In terms of (3.13), we have

$$\begin{aligned}\|Y_6^{n+1}\|_{L^2}^2 + \|Y_7^{n+1}\|_{L^2}^2 =& \|\rho^{n+1}\hat{\mathbf{e}}_{\mathbf{u}}^n \cdot \nabla \mathbf{u}(t_{n+1})\|_{L^2}^2 + \|\rho^{n+1}\hat{\mathbf{u}}^n \cdot \nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2 \\ \leq& C\|\hat{\sigma}^n \hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2 + C\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2.\end{aligned}$$

By using (2.10), (3.13) and $\rho^{n+1} = (\sigma^{n+1})^2$, we get

$$\begin{aligned}\|Y_8^{n+1}\|_{L^2}^2 =& \frac{1}{4}\|\mathbf{e}_{\mathbf{u}}^{n+1}\nabla \cdot (\rho^{n+1}\hat{\mathbf{u}}^n)\|_{L^2}^2 \\ =& \frac{1}{4}\|\mathbf{e}_{\mathbf{u}}^{n+1}\nabla \rho^{n+1} \cdot \hat{\mathbf{u}}^n\|_{L^2}^2 \\ =& \frac{1}{4}\|\mathbf{e}_{\mathbf{u}}^{n+1}\nabla (\sigma^{n+1})^2 \cdot \hat{\mathbf{u}}^n\|_{L^2}^2 \\ =& \|\mathbf{e}_{\mathbf{u}}^{n+1}\sigma^{n+1}\nabla \sigma^{n+1} \cdot \hat{\mathbf{u}}^n\|_{L^2}^2 \\ =& \|\mathbf{e}_{\mathbf{u}}^{n+1}\sigma^{n+1} D_\tau \sigma^{n+1}\|_{L^2}^2 \\ \leq& C\tau^{-2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2.\end{aligned}$$

Based on the utilization of (2.29) and (3.21), we get

$$\begin{aligned}\|Y_9^{n+1}\|_{L^2}^2 =& \frac{1}{4}\|\mathbf{u}(t_{n+1})\nabla \cdot (e_\rho^{n+1}\hat{\mathbf{u}}^n)\|_{L^2}^2 \\ \leq& C\|\nabla e_\sigma^{n+1}\|_{L^2}^2 \\ \leq& C\tau \sum_{n=0}^{N-1} \|\nabla \hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2 + C\tau^4, \\ \|Y_{10}^{n+1}\|_{L^2}^2 =& \frac{1}{4}\|\mathbf{u}(t_{n+1})\nabla \cdot (\rho(t_{n+1})\hat{e}_{\mathbf{u}}^n)\|_{L^2}^2 \\ =& \frac{1}{4}\|\mathbf{u}(t_{n+1}\nabla \rho(t_{n+1})\hat{\mathbf{e}}_{\mathbf{u}}^n)\|_{L^2}^2 \\ \leq& C\|\hat{\sigma}^n \hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2.\end{aligned}$$

Substituting these estimates into (3.36) and using (3.25), then multiply it by τ and apply the discrete Gronwall's inequality in Lemma 2.3, we can derive

$$\tau \sum_{n=0}^{N-1} \|\sigma^{n+1} D_\tau \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2 + \frac{1}{4}\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2 + \frac{1}{4}\|\nabla \hat{\mathbf{e}}_{\mathbf{u}}^{n+1}\|_{L^2}^2 \leq C\tau^2. \tag{3.38}$$

Thus, by the regularity of the solution to the Stokes problem, we can derive the following result from (3.35)

$$\begin{aligned}&\tau \sum_{n=0}^{N-1} (\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{H^2}^2 + \|e_p^{n+1}\|_{H^1}^2) \\ \leq& C\tau \sum_{n=0}^{N-1} (\|R_{\mathbf{u}}^{n+1}\|_{L^2}^2 + \|2\sigma^{n+1}\frac{(\sigma^{n+1} - \sigma^n)}{\tau}\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2 + \|\sigma^{n+1}\frac{\sigma^n - \sigma^{n+1}}{2\tau}\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2 \\ &+ \sum_{n=0}^{10} \|Y_i^{n+1}\|_{L^2}^2 + \|\sigma^{n+1} D_\tau \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2) \\ \leq& C\tau^2.\end{aligned} \tag{3.39}$$

Furthermore

$$\|\mathbf{u}^{n+1}\|_{H^2} \leq \|\mathbf{u}(t_{n+1})\|_{H^2} + \|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{H^2} \leq C. \tag{3.40}$$

3.3. Estimate of $\|e_\sigma^{n+1}\|_{H^2}$

Differentiating (3.22) with respect to x_i gives

$$\begin{aligned}&D_{ij} D_\tau e_\sigma^{n+1} + \nabla (D_{ij}\sigma(t_{n+1})) \cdot \hat{\mathbf{e}}_{\mathbf{u}}^n + \nabla D_j \sigma(t_{n+1}) \cdot D_i \hat{\mathbf{e}}_{\mathbf{u}}^n + \nabla D_i \sigma(t_{n+1}) \cdot D_j \hat{\mathbf{e}}_{\mathbf{u}}^n \\ &+ \nabla \sigma(t_{n+1}) \cdot D_{ij}\hat{\mathbf{e}}_{\mathbf{u}}^n + \nabla (D_{ij} e_\sigma^{n+1}) \cdot \hat{\mathbf{u}}^n + \nabla D_j e_\sigma^{n+1} \cdot D_j \hat{\mathbf{u}}^n + \nabla D_i e_\sigma^{n+1} \cdot D_j \hat{\mathbf{u}}^n + \nabla e_\sigma^{n+1} \cdot D_{ij}\hat{\mathbf{u}}^n = D_{ij} R_\sigma^{n+1}.\end{aligned} \tag{3.41}$$

Testing it by $4\tau D_{ij} e_\sigma^{n+1}$, based on the utilization of (2.29) and (IBP) we have

$$\|D_{ij} e_\sigma^{n+1}\|_{L^2}^2 - \|D_{ij} e_\sigma^n\|_{L^2}^2 + \|D_{ij}\hat{e}_\sigma^{n+1}\|_{L^2}^2 - \|D_{ij}\hat{e}_\sigma^n\|_{L^2}^2 + \|D_{ij} e_\sigma^{n+1} - 2D_{ij} e_\sigma^n + D_{ij} e_\sigma^{n-1}\|_{L^2}^2$$

$$\le C\tau(\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2+\|\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{H^2}^2+\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\nabla^2 e_\sigma^{n+1}\|_{L^2}^2+\|R_\sigma^{n+1}\|_{H^2}^2), \tag{3.42}$$

which means that

$$\begin{aligned}&\|\nabla^2 e_\sigma^{n+1}\|_{L^2}^2-\|\nabla^2 e_\sigma^n\|_{L^2}^2+\|\nabla^2\hat{e}_\sigma^{n+1}\|_{L^2}^2-\|\nabla^2\hat{e}_\sigma^n\|_{L^2}^2+\|\nabla^2 e_\sigma^{n+1}-2\nabla^2 e_\sigma^n+\nabla^2 e_\sigma^{n-1}\|_{L^2}^2\\ \le& C\tau(\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2+\|\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{H^2}^2+\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\nabla^2 e_\sigma^{n+1}\|_{L^2}^2+\|R_\sigma^{n+1}\|_{H^2}^2).\end{aligned} \tag{3.43}$$

Summing it and applying the discrete Gronwall's inequality in Lemma 2.3, and by using (3.24), (3.39), (2.29) and (3.32) we can derive

$$\|e_\sigma^{n+1}\|_{H^2}^2+\|\hat{e}_\sigma^{n+1}\|_{H^2}^2\le C\tau^2. \tag{3.44}$$

Furthermore

$$\|\sigma^{n+1}\|_{H^2}\le\|\sigma(t_{n+1})\|_{H^2}+\|e_\sigma^{n+1}\|_{H^2}\le C. \tag{3.45}$$

3.4. Estimate of $\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{W^{1,\infty}}$

In terms of Helmholtz-Weyl decomposition of $\mathbf{L}^q(\Omega)$, we have

$$\mathbb{P}(\rho^{n+1}D_\tau\mathbf{e}_{\mathbf{u}}^{n+1})=\rho^{n+1}D_\tau\mathbf{e}_{\mathbf{u}}^{n+1}-\nabla\phi^{n+1}, \tag{3.46}$$

where $\phi^{n+1}\in W^{1,q}(\Omega)$ with $\int_\Omega\phi^{n+1}dx=0$ is the solution to the following problem

$$\Delta\phi^{n+1}=\nabla\cdot(\rho^{n+1}D_\tau\mathbf{e}_{\mathbf{u}}^{n+1})\quad\text{in}\,\Omega, \tag{3.47}$$

with

$$\nabla\phi^{n+1}\cdot\mathbf{n}=(\rho^{n+1}D_\tau\mathbf{e}_{\mathbf{u}}^{n+1})\cdot\mathbf{n}=0\quad\text{on}\,\Gamma.$$

By the regularity of solution to the elliptic problem, the solution ϕ^{n+1} satisfies

$$\|\nabla\phi^{n+1}\|_{L^4}\le C\|\rho^{n+1}D_\tau\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^4}\le C\|\nabla D_\tau\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2.$$

Thus by using (3.24) we have

$$\begin{aligned}\tau\sum_{n=0}^{N-1}\|\phi^{n+1}\|_{W^{1,4}}^2\le& C\tau\sum_{n=0}^{N-1}\|\nabla D_\tau\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2\\ \le& C\tau^{-1}\sum_{n=0}^{N-1}(\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\|\nabla\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2)\\ \le& C\tau^3.\end{aligned}$$

Applying the projection operator $\mathbb{P}$ to (3.35) and using (3.46), we have

$$\rho^{n+1}D_\tau\mathbf{e}_{\mathbf{u}}^{n+1}+\mu\mathbf{A}\mathbf{e}_{\mathbf{u}}^{n+1}=\nabla\phi^{n+1}-\mathbb{P}(2\sigma^{n+1}\frac{(\sigma^{n+1}-\sigma^n)}{\tau}\mathbf{e}_{\mathbf{u}}^n+\sigma^{n+1}\frac{\sigma^n-\sigma^{n+1}}{2\tau}\mathbf{e}_{\mathbf{u}}^{n-1}+\sum_{n=0}^{10}Y_i^{n+1}-R_{\mathbf{u}}^{n+1}).$$

By the same technique in the proving process of (3.38) and (3.27), we have

$$\begin{aligned}\|2\sigma^{n+1}\frac{(\sigma^{n+1}-\sigma^n)}{\tau}\mathbf{e}_{\mathbf{u}}^n\|_{L^4}^2\le& C\|\sigma^{n+1}\|_{L^\infty}^2\|\frac{\sigma^{n+1}-\sigma^n}{\tau}\|_{L^\infty}^2\|\mathbf{e}_{\mathbf{u}}^n\|_{L^4}^2\le C\tau^{-2}\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2,\\ \|\sigma^{n+1}\frac{\sigma^n-\sigma^{n+1}}{2\tau}\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^4}^2\le& C\tau^{-2}\|\nabla\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2,\\ \|Y_1^{n+1}\|_{L^4}^2=&\|e_\sigma^{n+1}D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1}))\|_{L^4}^2\le C\|\nabla e_\sigma^{n+1}\|_{L^2}^2\\ \|Y_2^{n+1}\|_{L^4}^2=&\|\sigma^{n+1}e_\sigma^{n+1}D_\tau\mathbf{u}(t_{n+1})\|_{L^4}^2\le C\|\nabla e_\sigma^{n+1}\|_{L^2}^2,\\ \|Y_3^{n+1}\|_{L^4}^2=&\|\sigma^{n+1}D_\tau e_\sigma^{n+1}\mathbf{u}(t_n)\|_{L^4}^2\le C\|\nabla D_\tau e_\sigma^{n+1}\|_{L^2}^2\\ =&C\|\frac{3\nabla e_\sigma^{n+1}-4\nabla e_\sigma^n+\nabla e_\sigma^{n-1}}{2\tau}\|_{L^2}^2\\ \le& C\tau^{-2}(\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\nabla e_\sigma^n\|_{L^2}^2+\|\nabla e_\sigma^{n-1}\|_{L^2}^2)\le C\tau^2,\\ \|Y_4^{n+1}\|_{L^4}^2=&\frac{1}{4}\|\sigma^{n+1}\frac{\mathbf{u}(t_{n+1}-\mathbf{u}(t_n))}{\tau}(e_\sigma^{n+1}-e_\sigma^{n-1})\|_{L^4}^2\\ \le& C(\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\nabla e_\sigma^{n-1}\|_{L^2}^2),\\ \|Y_5^{n+1}\|_{L^4}^2=&\|e_\rho^{n+1}\hat{\mathbf{u}}(t_n)\cdot\nabla\mathbf{u}(t_{n+1})\|_{L^4}^2\le C\|e_\rho^{n+1}\|_{L^4}^2\le C\|\nabla e_\sigma^{n+1}\|_{L^2}^2,\\ \|Y_6^{n+1}\|_{L^4}^2=&\|\rho^{n+1}\hat{\mathbf{e}}_{\mathbf{u}}^n\cdot\nabla\mathbf{u}(t_{n+1})\|_{L^4}^2\le C\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}
\|Y_7^{n+1}\|_{L^4}^2 =& \|\rho^{n+1}\hat{\mathbf{u}}^n \cdot \nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^4}^2 \le C\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{H^2}^2,\\
\|Y_8^{n+1}\|_{L^4}^2 =& \frac{1}{4}\|\mathbf{e}_{\mathbf{u}}^{n+1}\nabla\cdot(\rho^{n+1}\hat{\mathbf{u}}^n)\|_{L^4}^2 \le C\tau^{-2}\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2,\\
\|Y_9^{n+1}\|_{L^4}^2 =& \frac{1}{4}\|\mathbf{u}(t_n)\nabla\cdot(e_\rho^{n+1}\hat{\mathbf{u}}^n)\|_{L^4}^2 \le C\|e_\sigma^{n+1}\|_{H^2}^2,\\
\|Y_{10}^{n+1}\|_{L^4}^2 =& \frac{1}{4}\|\mathbf{u}(t_{n+1})\nabla\cdot(\rho(t_{n+1}\hat{\mathbf{e}}_{\mathbf{u}}^n))\|_{L^4}^2 \le C\|\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{H^2}^2.
\end{aligned}$$

In order to apply Lemma 2.2, we need to satisfy the three conditions for ρ in Lemma 2.2.

Firstly, by using (3.32), (3.44) and Agmon inequality, we can infer

$$\|e_\sigma^{n+1}\|_{L^\infty} \le C\|e_\sigma^{n+1}\|_{H^1}^{\frac{1}{2}}\|e_\sigma^{n+1}\|_{H^2}^{\frac{1}{2}} \le C\tau. \tag{3.48}$$

Thus

$$\begin{aligned}
\sum_{n=0}^{N-1}\|\rho^{n+1}-\rho^n\|_{L^\infty} \le& \sum_{n=0}^{N-1}(\|e_\rho^{n+1}-e_\rho^n\|_{L^\infty} + \|\rho(t_{n+1})-\rho^n\|_{L^\infty})\\
\le& C\sum_{n=0}^{N-1}\|e_\sigma^{n+1}-e_\sigma^n\|_{L^\infty} + C \le C.
\end{aligned} \tag{3.49}$$

Secondly, testing (3.22) by $2\tau|D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}$, we have

$$\begin{aligned}
&2\tau(D_j D_\tau e_\sigma^{n+1}, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}) + 2\tau(D_j\nabla\sigma(t_{n+1})\cdot\hat{\mathbf{e}}_{\mathbf{u}}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1})\\
&+2\tau(\nabla\sigma(t_{n+1})\cdot D_j\hat{\mathbf{e}}_{\mathbf{u}}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}) + 2\tau(D_j\nabla e_\sigma^{n+1}\cdot\hat{\mathbf{u}}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1})\\
&+2\tau(\nabla e_\sigma^{n+1}\cdot D_j\hat{\mathbf{u}}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}) = 2\tau(D_j R_\sigma^{n+1}, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}),
\end{aligned}$$

where we use

$$(D_j\nabla e_\sigma^{n+1}\cdot\hat{\mathbf{u}}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}) = -(|D_j e_\sigma^{n+1}|^4, \nabla\cdot\hat{\mathbf{u}}^n) = 0.$$

By the Hölder inequality, we have

$$\begin{aligned}
&3(\|D_j e_\sigma^{n+1}\|_{L^4}^4 - \|D_j e_\sigma^n\|_{L^4}\|D_j e_\sigma^{n+1}\|_{L^4}^3) - (\|D_j e_\sigma^n\|_{L^4}\|D_j e_\sigma^{n+1}\|_{L^4}^3 - \|D_j e_\sigma^{n-1}\|_{L^4}\|D_j e_\sigma^{n+1}\|_{L^4}^3)\\
\le& C\tau(\|D_j\nabla\sigma(t_{n+1})\cdot\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^4} + \|\nabla\sigma(t_{n+1})\cdot D_j\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^4} + \|\nabla e_\sigma^{n+1}\cdot D_j\hat{\mathbf{u}}^n\|_{L^4} + \|D_j R_\sigma^{n+1}\|_{L^4})\|D_j e_\sigma^{n+1}\|_{L^4}^3.
\end{aligned}$$

Summing it, we get

$$\|D_j e_\sigma^{n+1}\|_{L^4} \le C\tau\sum_{n=0}^{N-1}(\|\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{H^2} + \|\nabla e_\sigma^{n+1}\|_{L^4} + \|D_j R_\sigma^{n+1}\|_{L^4}) \tag{3.50}$$

Applying the discrete Gronwall's inequality in Lemma 2.3, (3.39) and (3.4), we get

$$\|\nabla e_\sigma^{n+1}\|_{L^4} \le C\tau^{\frac{3}{2}}. \tag{3.51}$$

So

$$\|\rho^{n+1}\|_{W^{1,4}} \le \|e_\rho^{n+1}\|_{W^{1,4}} + \|\rho(t_{n+1})\|_{W^{1,4}} \le C. \tag{3.52}$$

In terms of (3.13), (3.49) and (3.52), The conditions in Lemma 2.2 are satisfied. Thus we can derive the following result by Lemma 2.2.

$$\begin{aligned}
&\tau\sum_{n=0}^{N-1}(\|D_\tau\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^4}^2 + \|A\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^4}^2)\\
\le& C\tau\sum_{n=0}^{N-1}(\|\nabla\phi^{n+1}\|_{L^4}^2 + \|R_{\mathbf{u}}^{n+1}\|_{L^4}^2 + \|2\sigma^{n+1}\frac{(\sigma^{n+1}-\sigma^n)}{\tau}\mathbf{e}_{\mathbf{u}}^n\|_{L^4}^2\\
&+\|\sigma^{n+1}\frac{\sigma^n-\sigma^{n+1}}{2\tau}\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^4}^2 + \sum_{n=0}^{10}\|Y_i^{n+1}\|_{L^4}^2)\\
\le& C\tau^2.
\end{aligned}$$

Noticing that $W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we can derive

$$\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{W^{1,\infty}} \le C\tau^{\frac{1}{2}}. \tag{3.53}$$

Furthermore

$$\|\mathbf{u}^{n+1}\|_{W^{1,\infty}} \le \|\mathbf{u}(t_{n+1})\|_{W^{1,\infty}} + \|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{W^{1,\infty}} \le C. \tag{3.54}$$

3.5. Estimate of $\|\nabla e_\sigma^{n+1}\|_{L^\infty}$

Testing (3.22) by $\tau|D_j e_\sigma^{n+1}|^{k-1} D_j e_\sigma^{n+1}$ and taking $k \to +\infty$, we have

$$\begin{aligned}\|D_j e_\sigma^{n+1}\|_{L^\infty} - \|D_j e_\sigma^n\|_{L^\infty} \le & C\tau\|\nabla(D_j\sigma(t_{n+1}))\cdot \hat{e}_{\mathbf{u}}^n\|_{L^\infty} + C\tau\|\nabla\sigma(t_{n+1})\cdot(D_j\hat{e}_{\mathbf{u}}^n)\|_{L^\infty} \\ & + C\tau\|\nabla e_\sigma^{n+1}\cdot(D_j\hat{\mathbf{u}}^n)\|_{L^\infty} + C\tau\|\nabla e_\sigma^{n+1}\cdot D_j\hat{\mathbf{u}}^n\|_{L^\infty} + C\tau\|D_j R_\sigma^{n+1}\|_{L^\infty} \\ \le & C\tau(\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^\infty} + \|\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{H^2} + \|R_\sigma^{n+1}\|_{H^2}) + C\tau\|\nabla e_\sigma^{n+1}\|_{L^\infty}.\end{aligned} \tag{3.55}$$

Which means that

$$\|\nabla e_\sigma^{n+1}\|_{L^\infty} - \|\nabla e_\sigma^n\|_{L^\infty} \le C\tau(\|\nabla\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{L^\infty} + \|\hat{\mathbf{e}}_{\mathbf{u}}^n\|_{H^2} + \|R_\sigma^{n+1}\|_{H^2}) + C\tau\|\nabla e_\sigma^{n+1}\|_{L^\infty}. \tag{3.56}$$

Summing it and applying the discrete Gronwall's inequality in Lemma 2.3, and by using (3.4), (3.39) and (3.53), we can get

$$\|\nabla e_\sigma^{n+1}\|_{L^\infty} \le C\tau^{\frac{3}{2}}. \tag{3.57}$$

Furthermore

$$\|\sigma^{n+1}\|_{W^{1,\infty}} \le \|\sigma(t_{n+1})\|_{W^{1,\infty}} + \|\mathbf{e}_\sigma^{n+1}\|_{W^{1,\infty}} \le C. \tag{3.58}$$

Finally, in terms of (3.13), (3.40), (3.45), (3.54) and (3.58), (3.5) and (3.6) are valid for $i = n+1$. The temporal estimate (3.7) in Theorem 3.1 follows from (3.24), (3.31) and (3.33), Thus we complete the proof of Theorem 3.1.

4. Numerical results

4.1. Finite element approximation

In this section, we will give numerical results to confirm the temporal convergence rate $\mathcal{O}(\tau^2)$ derived in Theorem 3.1 by the finite element method to discrete (2.10)–(2.11). We use the MINI (P1b-P1) finite element to approximate the exact solution $(\mathbf{u}, p)$ and the $P2$ or $P1$Lagrange finite element to approximate σ. Start with $\mathbf{u}_h^0 = I_h\mathbf{u}_0$ and $\sigma_h^0 = J_h\sigma_0$, where I_h and J_h both are interpolation operator from $\mathbf{V}$ onto $\mathbf{V}_h$ and $\mathbf{W}$ onto $\mathbf{W}_h$, respectively. Then for $0 \le n \le N-1$, the finite element approximations of (2.10)–(2.11) are described as follows.

Step I: For given $\sigma_h^n \in W_h$ and $\mathbf{u}_h^n \in \mathbf{V}_h$, we find $\sigma_h^{n+1} \in W_h$ by

$$(D_\tau\sigma_h^{n+1}, r_h) + (\nabla\sigma_h^{n+1}\cdot\hat{\mathbf{u}}_h^n, r_h) + \frac{1}{2}(\sigma_h^{n+1}\nabla\cdot\hat{\mathbf{u}}_h^n, r_h) = 0 \quad \forall\ r_h \in W_h, \tag{4.1}$$

where the term $\frac{1}{2}\sigma_h^{n+1}\nabla\cdot\hat{\mathbf{u}}_h^n$ is added to preserve the stability of σ_h^{n+1}.

Step II: We find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times M_h$ by

$$\begin{aligned}&(\sigma_h^{n+1}D_\tau(\sigma_h^{n+1}\mathbf{u}_h^{n+1}), \mathbf{v}_h) + \mu(\nabla\mathbf{u}_h^{n+1}, \nabla\mathbf{v}_h) + (\rho_h^{n+1}(\hat{\mathbf{u}}_h^n\cdot\nabla)\mathbf{u}_h^{n+1}, \mathbf{v}_h) \\ &+\frac{1}{2}(\mathbf{u}_h^{n+1}\nabla\cdot(\rho_h^{n+1}\hat{\mathbf{u}}_h^n), \mathbf{v}_h) - (\nabla\cdot\mathbf{v}_h, p_h^{n+1}) + (\nabla\cdot\mathbf{u}_h^{n+1}, q_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h)\end{aligned} \tag{4.2}$$

for any $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$, where $\rho_h^{n+1} = (\sigma_h^{n+1})^2$.

Taking $r_h = 4\tau\sigma_h^{n+1}$ in (4.1), we have

$$\|\sigma_h^{n+1}\|_{L^2} \le \|\sigma_h^0\|_{L^2} \quad \forall\ 0 \le n \le N. \tag{4.3}$$

where we notice that

$$(\nabla\sigma_h^{n+1}\cdot\hat{\mathbf{u}}_h^n, \sigma_h^{n+1}) = \frac{1}{2}\int_\Omega \nabla|\sigma_h^{n+1}|^2\cdot\hat{\mathbf{u}}_h^n dx = -\frac{1}{2}\int_\Omega |\sigma_h^{n+1}|^2(\nabla\cdot\hat{\mathbf{u}}_h^n)dx.$$

Taking $(\mathbf{v}_h, q_h) = 4\tau(\mathbf{u}_h^{n+1}, p_h^{n+1})$ in (4.2) gives

$$\begin{aligned}&\|\sigma^{n+1}\mathbf{u}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{u}^n\|_{L^2}^2 + \|\widehat{\sigma^{n+1}\mathbf{u}^{n+1}}\|_{L^2}^2 - \|\widehat{\sigma^n\mathbf{u}^n}\|_{L^2}^2 + 4\mu\tau\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2 \\ &\le C\tau\|\mathbf{f}^{n+1}\|_{L^2}^2 + \mu\tau\|\nabla\mathbf{u}_h^{n+1}\|_{L^2}^2,\end{aligned} \tag{4.4}$$

where we use

$$(\rho_h^{n+1}(\hat{\mathbf{u}}_h^n\cdot\nabla)\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) + \frac{1}{2}(\mathbf{u}_h^{n+1}\nabla\cdot(\rho_h^{n+1}\hat{\mathbf{u}}_h^n), \mathbf{u}_h^{n+1}) = 0,$$

Summing up (4.4) yields

$$\begin{aligned}&\|\sigma_h^{n+1}\mathbf{u}_h^{n+1}\|_{L^2}^2 + \|\widehat{\sigma^{n+1}\mathbf{u}^{n+1}}\|_{L^2}^2 + \mu\tau\sum_{m=0}^n\|\nabla\mathbf{u}_h^{m+1}\|_{L^2}^2 \\ &\le C\tau\sum_{m=0}^n\|\mathbf{f}^{m+1}\|_{L^2}^2 + \|\sigma_h^0\mathbf{u}_h^0\|_{L^2}^2 \qquad \forall\ 0 \le n \le N-1.\end{aligned} \tag{4.5}$$

Table 1
Numerical errors and convergence rates with $\tau = h$ using (P2-P1b-P1).

τ	$\|\rho-\rho_h\|_{L^2}$	Rate	$\|\sigma-\sigma_h\|_{L^2}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	Rate	$\|p-p_h\|_{l_2(L^2)}$	Rate
1/4	0.0472916		0.0105447		0.0155503		0.171888	
1/8	0.0131206	1.85	0.00294489	1.84	0.00401961	1.95	0.0391415	2.13
1/16	0.00349222	1.91	0.0007864	1.90	0.00101159	1.99	0.00925388	2.08
1/32	0.0009034	1.95	0.000203684	1.95	0.000253237	2.00	0.00225126	2.04
1/64	0.00022971	1.98	5.18E−05	1.97	6.33E−05	2.00	0.000556639	2.02
1/128	5.79E−05	1.99	1.31E−05	1.99	1.58E−05	2.00	0.000139124	2.00

Table 2
Stability results of the numerical solutions for different τ using (P2-P1b-P1).

τ	$\|\rho_h^N\|_{L^2}$	$\|\sigma_h^N\|_{L^2}$	$\|\sigma_h^N\mathbf{u}_h^N\|_{L^2}$	$\|\mathbf{u}_h^N\|_{L^2}$	$\|\mathbf{u}_h^N\|_{H^1}$
1/4	4.98151	2.22961	0.290312	0.128468	0.0480862
1/8	5.0061	2.23502	0.307142	0.135642	0.0264044
1/16	5.01159	2.23622	0.311317	0.137428	0.0137388
1/32	5.01281	2.23649	0.312342	0.137867	0.00695004
1/64	5.0131	2.23655	0.312596	0.137977	0.00348738
1/128	5.01317	2.23657	0.31266	0.138004	0.00174571

Furthermore, the above inequalities imply the finite element schemes are unconditional stable and the existence and uniqueness of the solution $(\mathbf{u}, \sigma, p)$ to (4.1)–(4.2).

Remark 4.1. In the BDF2 finite element scheme, the nonlinear terms are treated by a linearized semi-implicit method such that we only solve linear systems at each time step, This approach significantly simplifies the computational process while maintaining the accuracy and stability of the algorithm.

4.2. Numerical experiments

In this subsection, we will present the numerical results to verify our theoretical analysis. All programs are implemented using the free finite element software FreeFem++ [33]. For the sake of simplicity, we solve the following coupled system with the artificial functions g and $\mathbf{f}$:

$$\sigma_t + \nabla\cdot(\sigma\mathbf{u}) = g, \tag{4.6}$$

$$\sigma(\sigma\mathbf{u})_t - \mu\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{2}\mathbf{u}\nabla\cdot(\rho\mathbf{u}) + \nabla p = \mathbf{f}, \tag{4.7}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{4.8}$$

in $\Omega\times[0.T]$, where Ω is the unit square:

$$\Omega = \{(x,y)\in\mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}.$$

We set the viscosity coefficient $\mu = 0.1$, the final time $T = 1$. To select the approximate functions g and $\mathbf{f}$, we determine the exact solution $(\sigma, \mathbf{u}, p)$ as follows

$$\begin{aligned}
\mathbf{u} =& (t^3y^2(1-y), t^3x^2(1-x))^T,\\
\sigma =& 2 + x(1-x)\cos(\sin(t)) + y(1-y)\sin(\sin(t)),\\
p =& tx + y - (t+1)/2.
\end{aligned}$$

Denote

$$\begin{aligned}
\|r - r_h\|_{L^2} &= \|r(t_N) - r_h^N\|_{L^2},\\
\|\mathbf{v} - \mathbf{v}_h\|_{L^2} &= \|\mathbf{v}(t_N) - \mathbf{v}_h^N\|_{L^2},\\
\|q - q_h\|_{l_2(L^2)} &= \left(\tau\sum_{n=1}^{N}\|q(t_n) - q_h^n\|_{L^2}^2\right)^{1/2}.
\end{aligned}$$

To evaluate the second-order temporal convergence rate, we conducted a series of numerical experiments utilizing a time step size defined by the relationship $\tau = h$. The time step sizes were systematically selected as $\tau = 1/4, 1/8, \ldots, 1/128$, enabling us to examine the effects of varying temporal resolutions on the convergence characteristics. This range of time steps is particularly crucial for assessing the impact of temporal resolution on solution accuracy. The numerical results obtained are summarized in Table 1 and illustrated graphically in Fig. 1. The stability results of the numerical solutions are provided in Table 2, highlighting the robustness of the approach, where the density is approximated by linear Lagrangian elements $P2$ and the velocity–pressure pair

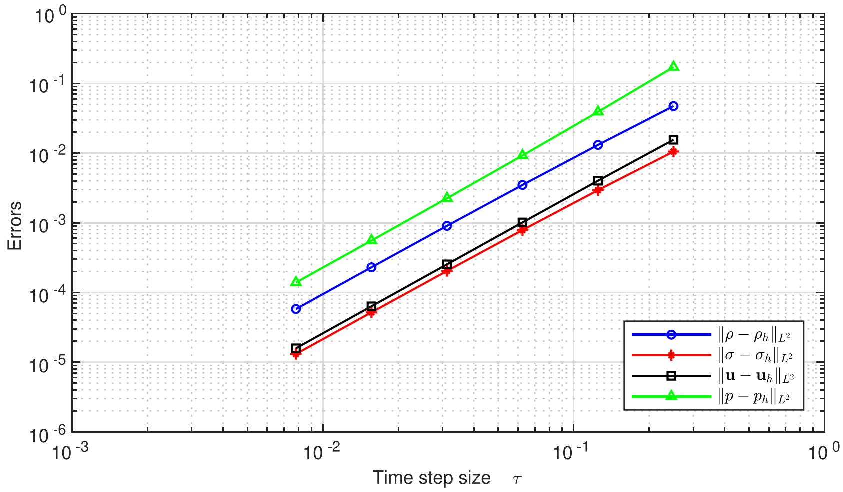

Fig. 1. Convergence history of $(\rho, \sigma, \mathbf{u}, p)$ for different τ.

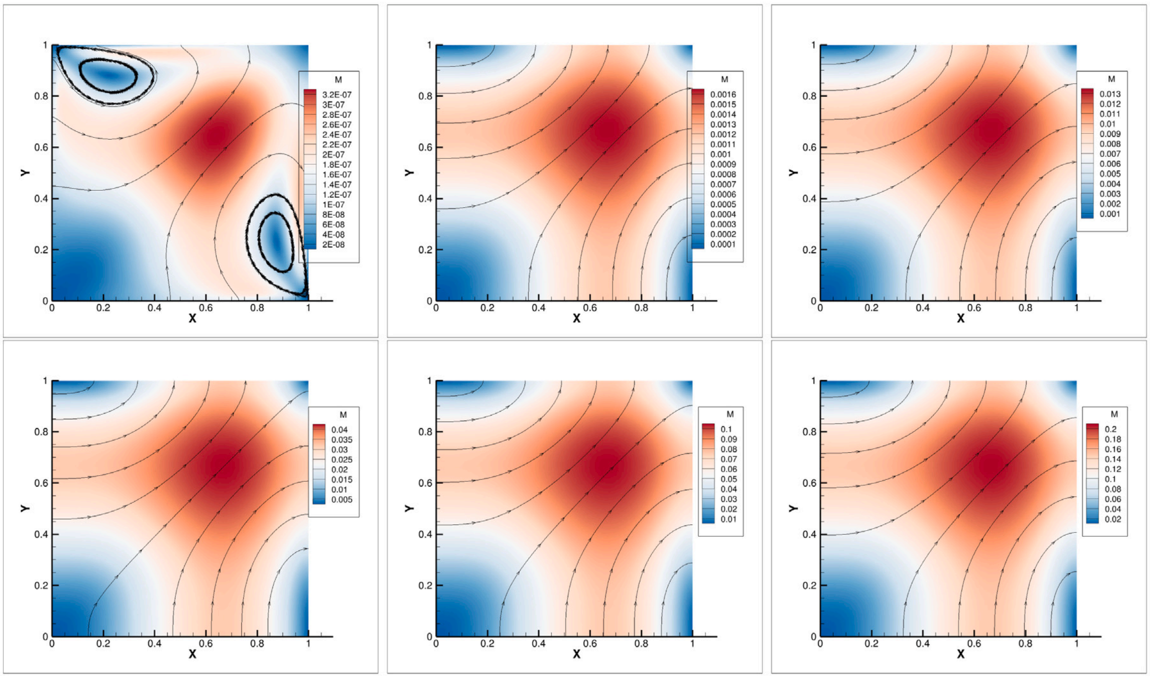

Fig. 2. Numerical solutions of velocity at times t = 0, 0.2, 0.4, 0.6, 0.8, 1.0.

Table 3
Stability results of the numerical solutions for different τ using (P1-P1b-P1).

τ	$\|\rho_h^N\|_{L^2}$	$\|\sigma_h^N\|_{L^2}$	$\|\sigma_h^N \mathbf{u}_h^N\|_{L^2}$	$\|\mathbf{u}_h^N\|_{L^2}$	$\|\mathbf{u}_h^N\|_{H^1}$
1/4	4.93273	2.21863	0.289017	0.12851	0.0481942
1/8	4.99408	2.23232	0.306813	0.135655	0.0264018
1/16	5.00861	2.23555	0.311236	0.137432	0.0137385
1/32	5.01207	2.23633	0.312322	0.137868	0.00694996
1/64	5.01291	2.23651	0.312591	0.137977	0.00348738
1/128	5.01312	2.23656	0.312659	0.138004	0.00174571

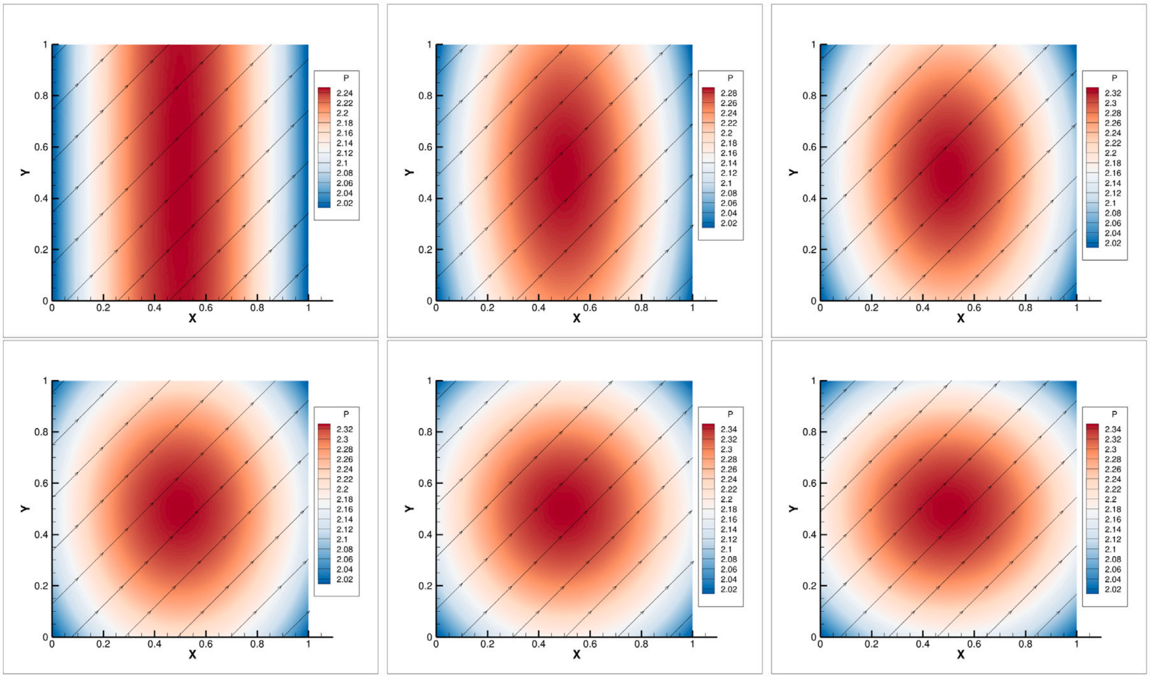

Fig. 3. Numerical solutions of sigma at times t = 0, 0.2, 0.4, 0.6, 0.8, 1.0.

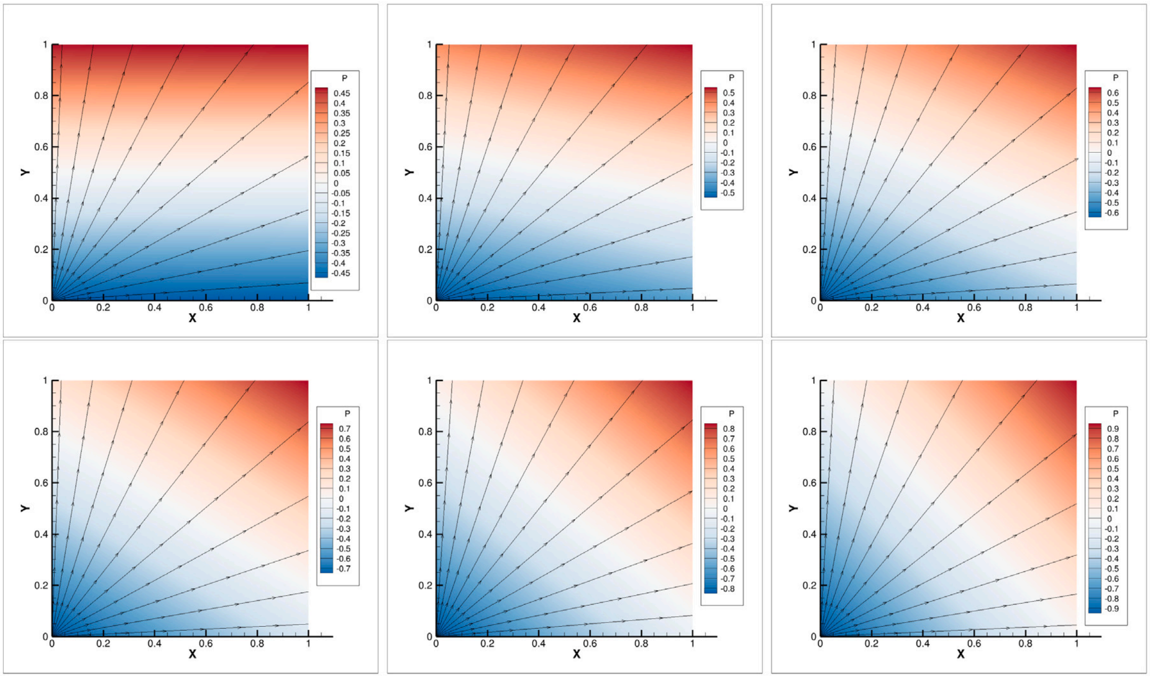

Fig. 4. Numerical solutions of pressure at times t = 0, 0.2, 0.4, 0.6, 0.8, 1.0.

$(\mathbf{u}, p)$ is approximated by the MINI element $(P1b - P1)$. Additionally, numerical stability and convergence rates are summarized in Tables 3 and 4 by using $(P1 - P1b - P1)$ to approximate the density and the velocity–pressure pair.

In our numerical experiences, we computed the errors and convergence rates for the primary variables, namely $(\rho, \sigma, \mathbf{u}, p)$. Remarkably, the computed errors for all considered variables, evaluated in the reported norms, demonstrated consistent second-order convergence. This finding is in strong agreement with our theoretical error estimates, which were derived based on the assumptions

Table 4
Numerical errors and convergence rates with $\tau = h$ using (P1-P1b-P1).

τ	$\|\rho-\rho_h\|_{L^2}$	Rate	$\|\sigma-\sigma_h\|_{L^2}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	Rate	$\|p-p_h\|_{l_2(L^2)}$	Rate
1/4	0.087347		0.0194811		0.0155625		0.181109	
1/8	0.0221283	1.98	0.00493852	1.98	0.00402725	1.95	0.0411726	2.14
1/16	0.00558176	1.99	0.00124807	1.98	0.00101407	1.99	0.00971516	2.08
1/32	0.00140464	1.99	0.000314449	1.99	0.000253896	2.00	0.00236078	2.04
1/64	0.00035239	1.99	7.89E−05	1.99	6.35E−05	2.00	0.000583248	2.02
1/128	8.82E−05	2.00	1.98E−05	2.00	1.59E−05	2.00	0.000145647	2.00

and conditions outlined in our model. Such a concordance reinforces the reliability and validity of the numerical methods employed in this study.

Moreover, the numerical solutions for the variables $(\sigma, \mathbf{u}, p)$ at different times τ are depicted in Figs. 2, 3, and 4. These figures not only provide a visual representation of the results but also illustrate the stability and consistency of the numerical solutions across different temporal resolutions. Each figure clearly shows the expected behavior of the solutions, reflecting the underlying physics of the problem being modeled.

In conclusion, the results obtained from the numerical experiments validate the theoretical analysis presented in Theorem 3.1. They provide compelling evidence of the accuracy and robustness of the BDF2 time-discrete scheme applied in this study. The observed convergence behavior and the stability of the numerical solutions confirm the efficacy of our numerical approach in capturing the dynamics of the modeled phenomena, thereby enhancing our confidence in the results and conclusions drawn from this research.

5. Conclusion

In this paper, we propose a BDF2 time-discrete scheme for solving the incompressible Navier–Stokes equations with variable density, as outlined in Eqs. (1.1) and (1.3). By introducing an equivalent formulation, our numerical scheme is shown to be unconditionally stable, ensuring that stability is maintained regardless of the time step size. Utilizing the discrete maximal L^p regularity of the Stokes problem, we present a rigorous temporal error estimate based on the assumption of sufficiently smooth strong solutions. This leads us to demonstrate that the numerical results align with our theoretical expectations, validating a convergence rate of $\mathcal{O}(\tau^2)$. Moreover, it is important to highlight that the BDF2 time-discrete scheme, along with the finite element approximations introduced in this study, has significant potential for extension to other complex fluid dynamics problems. Specifically, our approach can be adapted to tackle challenges in natural convection problems, magnetohydrodynamics models, and electrohydrodynamic models with variable density. We will provide a thorough convergence analysis and additional numerical results related to these applications in future work.

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Data availability

No data was used for the research described in the article.

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