Gathering more grid points near the boundary via grid transformation

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June 15, 2017

Abstract

To gather more grid points near the boundary of the 2D computational domain, grid transformation is applied. Here we only talk about stretching transformations with univariate functions, i.e. $y = y(\eta)$ and $z = z(\zeta)$. The final formula in this document is therefore not valid for general grid transformations $y = y(\eta, \zeta)$ and $z = z(\eta, \zeta)$.

To gather more grid points near the boundaries of a 2D rectangular physical domain, the following stretching transformation which maps the uniform computational grid system (η, ζ) to the non-uniform grid (y, z) is applied

$$y(\eta) = \frac{\tanh(a\eta)}{\tanh(a)}, \; z(\zeta) = \frac{\tanh(a\zeta)}{\tanh(a)}, \tag{1}$$

where $-1 \leqslant \eta \leqslant 1, -1 \leqslant \zeta \leqslant 1$, and a is a parameter controlling the distribution of the grids.

When the univariate functions Eq. (1) are used, the derivatives with respect to (y, z) are given by [1, chapter 4]

$$\begin{bmatrix} \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \frac{1}{|\mathbb{J}|} \begin{bmatrix} \dfrac{\mathrm{d}z}{\mathrm{d}\zeta} & 0 \\ 0 & \dfrac{\mathrm{d}y}{\mathrm{d}\eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\mathrm{d}y}{\mathrm{d}\eta}\right)^{-1} \dfrac{\partial}{\partial \eta} \\ \left(\dfrac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^{-1} \dfrac{\partial}{\partial \zeta} \end{bmatrix}, \tag{2a}$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2} &= \frac{1}{|\mathbb{J}|^2} \left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^2 \frac{\partial^2}{\partial \eta^2} - \frac{1}{|\mathbb{J}|^3} \left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^3 \frac{\partial^2 y}{\partial \eta^2} \frac{\partial}{\partial \eta} \\ &= \left(\frac{\mathrm{d}y}{\mathrm{d}\eta}\right)^{-2} \frac{\partial^2}{\partial \eta^2} - \left(\frac{\mathrm{d}y}{\mathrm{d}\eta}\right)^{-3} \frac{\partial^2 y}{\partial \eta^2} \frac{\partial}{\partial \eta}, \end{aligned} \tag{2b}$$

$$\begin{aligned} \frac{\partial^2}{\partial z^2} &= \frac{1}{|\mathbb{J}|^2} \left(\frac{\mathrm{d}y}{\mathrm{d}\eta}\right)^2 \frac{\partial^2}{\partial \zeta^2} - \frac{1}{|\mathbb{J}|^3} \left(\frac{\mathrm{d}y}{\mathrm{d}\eta}\right)^3 \frac{\partial^2 z}{\partial \zeta^2} \frac{\partial}{\partial \zeta} \\ &= \left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^{-2} \frac{\partial^2}{\partial \zeta^2} - \left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^{-3} \frac{\partial^2 z}{\partial \zeta^2} \frac{\partial}{\partial \zeta}, \end{aligned} \tag{2c}$$

where

$$\mathbb{J} = \begin{bmatrix} \dfrac{\mathrm{d}y}{\mathrm{d}\eta} & 0 \\ 0 & \dfrac{\mathrm{d}z}{\mathrm{d}\zeta} \end{bmatrix} \tag{3}$$

is the Jacobian matrix, the determinant of which is

$$|\mathbb{J}| = \frac{\mathrm{d}y}{\mathrm{d}\eta} \frac{\mathrm{d}z}{\mathrm{d}\zeta}. \tag{4}$$

One should avoid $\mathrm{d}y/\mathrm{d}\eta = 0$ or $\mathrm{d}z/\mathrm{d}\zeta = 0$ (or $\mathrm{d}y/\mathrm{d}\eta = \infty$ or $\mathrm{d}z/\mathrm{d}\zeta = \infty$) otherwise the transformation given by Eq. (2) is singular.

The derivatives $\mathrm{d}y/\mathrm{d}\eta$, $\mathrm{d}^2y/\mathrm{d}\eta^2$, $\mathrm{d}z/\mathrm{d}\zeta$ and $\mathrm{d}^2z/\mathrm{d}\zeta^2$ are calculated by Eq. (1) as

$$\frac{\mathrm{d}y}{\mathrm{d}\eta} = \frac{a\,\mathrm{sech}^2(a\eta)}{\tanh(a)}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}\eta^2} = -\frac{2a^2\mathrm{sech}^2(a\eta)\tanh(a\eta)}{\tanh(a)}, \tag{5}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{a\,\mathrm{sech}^2(a\zeta)}{\tanh(a)}, \quad \frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2} = -\frac{2a^2\mathrm{sech}^2(a\zeta)\tanh(a\zeta)}{\tanh(a)}. \tag{6}$$

In Fig. 1, an example of the grid transformation via Eq. (2) is shown.

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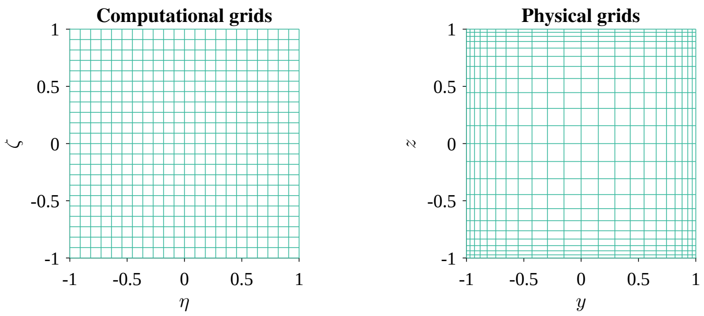

Figure 1. Grid transformation given by Eq. (2), with $a = 1.5$ and 1.6 for y and z respectively. See `testGridTransform.m` for the generation of the figure.

References

[1] T.J. Chung. *Computational Fluid Dynamics.* Cambridge University Press, New York, USA, 2nd edition, 2010. ISBN 9780521769693.