Phase transition in one and two dimension Ising model

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Introduction to the Ising model

Ising model is a basic and widely-used model of lattice system in statistical mechanics. It can be interpreted as a lattice system with all the sites replaced by atoms. The atoms themselves are affected by the external magnetic field so they can point "up" or "down" (spins). The edges of the nearest atoms' pairs have energy as well because the nearest two atoms interact with each other.
In mathematical physics, we use Hamiltonian to denote the total energy of a system. We assume only the nearest atoms interact with each other. Then the total energy of this lattice system is

$$H = H(\sigma) = -\beta \sum_{<i,j>} \sigma_i \sigma_j - h \sum_i \sigma_i$$

In this formula, σ is a configuration and $< i, j >$ are the nearest pairs on the lattice. σ_i equals 1 when the spin at site "i" is "up" and -1 when the spin is "down". β and h are the parameters and represent the interacting energy between the nearest pairs and the external magnetic energy respectively.

Definition of phase transition

We can define the pressure of the system as $\psi_\Lambda(\beta, h) \overset{\text{def}}{=} \frac{1}{|\Lambda|} \log Z_{\Lambda;\beta,h}$, where Z is the partition function of the system. (The idea of normalization).

Definition 1

The pressure ψ exhibits a first-order phase transition at (β, h) if $h \to \psi(\beta, h)$ fails to be differentiable at that point.

The first part of my project is to use Definition 1 to prove there is no phase transition in 1-dimensional Ising model.

We can also derive Gibbs state (at(β, h)), and then we have Definition 2.

Definition 2

If at least two distinct Gibbs states can be constructed for a pair (β, h), we say that there is a first-order phase transition at (β, h).

The second part of my project is to use Definition 2 to prove there exists a critical inverse temperature β_c for 2-dimensional Ising model such that when $\beta > \beta_c$, a first-order phase transition occurs at $(\beta, 0)$ when h=0, and explain Peierls' argument.

1-dimension Ising model

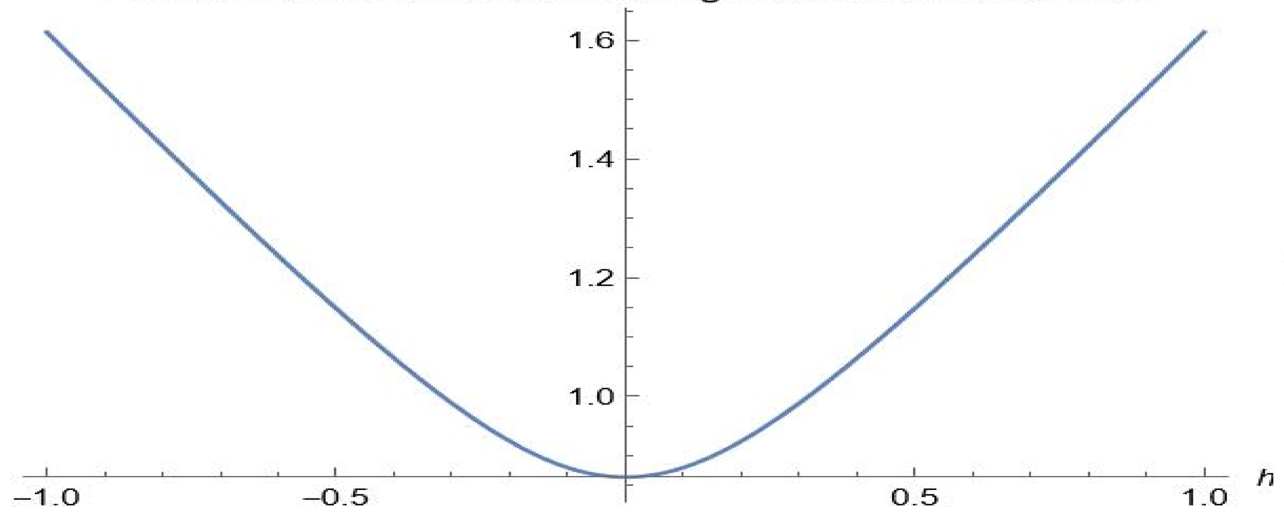
To prove there's no phase transition when d=1, we need to prove that $\psi(\beta, h) \overset{\text{def}}{=} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \psi^{\#}_{\Lambda}(\beta, h)$ is well defined and is independent of the choice of boundary condition and the sequence of $\Lambda \Uparrow \mathbb{Z}^d$.
To prove the existence of the limit, we can first look at the square lattices and find its limit. Then prove the difference of any arbitrary lattice and the square one follows a Cauchy sequence so the limit exists. The way to do it is to part a cube into smaller ones with interactions between the sub-lattices. And we can get it's Cauchy when the square is big enough.
After that we prove the independence of the choices of lattices. Basically we prove for an arbitrary choice of sequence $\Lambda \Uparrow \mathbb{Z}^d$, the limit of the $\psi^{\varnothing}_{\Lambda_n}$ is ψ, hence we use triangular inequality and scale it into three parts as following (the $[\Lambda_n]$ is the smallest covering of the lattices given by the elements of the square lattices): $\left|\psi^{\varnothing}_{\Lambda_n} - \psi\right| \leq \left|\psi^{\varnothing}_{\Lambda_n} - \psi^{\varnothing}_{[\Lambda_n]}\right| + \left|\psi^{\varnothing}_{[\Lambda_n]} - \psi^{\varnothing}_{D_k}\right| + \left|\psi^{\varnothing}_{D_k} - \psi\right|$. Then as in our Analysis course, we find an N large enough for each part and we get when n is large enough, the choice of lattices does not affect the limit.

Because of the independence of boundary condition and the sequence choice, we can choose the torus of n with periodic boundary conditions to find the solution of the 1-d Ising model. This choice can be written as a matrix $\mathbf{Z}^{\text{per}}_{V_n;\beta,h}$. We can find a transfer matrix and use its eigenvector to calculate the expression of the pressure. There are many ways to do it. One is on page 91 of *Statistical Mechanics of Lattice Systems.* The idea is that we can use the eigenvalue of the transfer matrix to construct a diagonal matrix and we can get the n-th power of it as well as its trace, following the limit and the pressure.
The pressure of the 1-dimensional Ising model is given by:

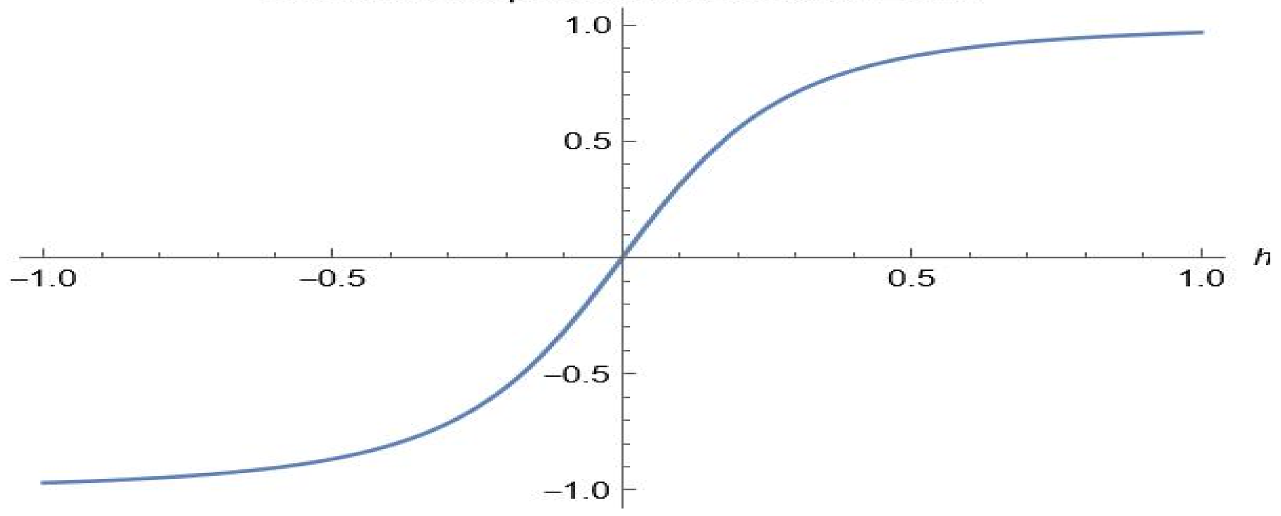
$$\psi(\beta, h) = \log\left\{e^{\beta}\cosh(h) + \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)}\right\}$$

For all $\beta \geq 0$, it is differentiable everywhere with respect to h, therefore the phase transition is missing.

Pressure of one dimensional Ising model when beta = 0.6



Derivative of pressure when beta = 0.6



2-dimensional Ising model

Peierls' argument demonstrates that there is a spontaneous magnetization at sufficiently low temperature and indicates phase transition of Ising model in 2-d.
In this case, spontaneous magnetization is the tendency for the magnetic moments of the lattices to remain in the position after an external magnetic field has been turned off. So one way to investigate this is to let the external magnetic field equals "+1" everywhere and let the border tends to infinite ("move to infinity"). Then the question turns into for a lattie site "0" deep inside, what's the probability that $\sigma_0 = -1$. We need to show at a low temperature, the probability that $\sigma_0 = -1$ is less than 1/2 by an amount which is independent of the lattice size (because without the magnetic field on the boundary, it will be 1/2). Then

$$\text{Prob}(\sigma_0 = -1) = \frac{1}{Z} \sum_{\sigma_0 = -1} e^{-\beta H(\sigma)}$$

2-dimensional Ising model

Since the boundary lattice sites are all "+1", we can consider all the "+1" sites as sea and all the "-1" sites as islands. So that all the "shorelines" correspond with all the adjacent pair of sites $< i, j >$ where $\sigma_i \sigma_j = -1$, and one of the island contains site "0". We draw a shoreline "S" around the "0" site. Let the length of the shoreline be $n(S)$ and Ω_S as the configurations containing "S" as a shoreline. Then we can get:

$$Prob(\Omega_S) = e^{-\beta E n(S)} \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{\beta E \sum_{\langle i,j \rangle \notin S} \sigma_i \sigma_j}$$

Ω_S is the set of all the configurations where the spin of site "0" is "-1".
Then let σ' denote the configuration where all the spins inside S change, we can get:

$$\sum_{\langle i,j \rangle \notin S} \sigma_i \sigma_j = \sum_{\langle i,j \rangle} \sigma'_i \sigma'_j - n(S)$$

Then we substitute this into the first inequality and replace the second configuration by the sum over all configurations. We can get:

$$\text{Prob}(\Omega_S) < e^{-\beta E n(S)} \frac{1}{Z} \sum_{\sigma \in \Omega} e^{-\beta H(\sigma)} = e^{-\beta E n(S)}$$

$$Prob(\sigma_0 = -1) < \sum_{n=4}^{\infty} s(n) e^{-\beta E n}$$

Then we just need to calculate s(n), which is the number of the shorelines of the length n, which can be done by considering the random walk problem. We get $s(n) < \frac{1}{2} n 4^n$. By substituting back and use some techniques in solving the series, we finally get the inequality $\text{Prob}(\sigma_0 = -1) < \frac{1}{2}\left[\frac{4e^{-\beta E}}{(1-4e^{-\beta E})^2}\right]$, which proves that when β is large enough(the temperature is low enough, the RHS is arbitrarily small and spontaneous magnetization(phase transition) is guaranteed at this temperature in 2-d Ising model.

References

[1] Friedli, S. and Velenik, Y. Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction Cambridge University Press, 2017.

[2] Hugo Duminil-Copin 100 Years of the (Critical) Ising Model on the Hypercubic Lattice. arXiv.

[3] Barry A. Cipra. An Introduction to the Ising Model. Mathematical Association of America, 1987.