An Introduction of Matrix Lie Algebra

Chenyang Zhao

December 25, 2024

Contents

1 Matrix Groups

1.1 Definition

A *matrix group* is a subset G of $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{R})$ that satisfies the following properties:

- G is closed under matrix multiplication: if $A, B \in G$, then $AB \in G$.
- G is closed under inversion: if $A \in G$, then $A^{-1} \in G$.

Matrix groups naturally inherit the group structure of $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{R})$, making them subgroups of these general linear groups. The closure conditions ensure that matrix groups are well-defined under the group operations of multiplication and inversion.

1.2 Examples

Matrix groups include many important groups encountered in mathematics and physics. Below are some of the key examples:

- **General Linear Group:** $\mathrm{GL}(n, \mathbb{R})$, the group of all invertible $n \times n$ matrices with real entries. This is the most general group of transformations preserving invertibility.
- **Special Linear Group:** $\mathrm{SL}(n, \mathbb{R})$, the group of $n \times n$ real matrices with determinant 1:
$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\}.$$
This group preserves volume under linear transformations.
- **Orthogonal Group:** $\mathrm{O}(n)$, the group of $n \times n$ real matrices A satisfying $A^T A = I$. Orthogonal matrices preserve the Euclidean norm and angles.
- **Special Orthogonal Group:** $\mathrm{SO}(n)$, a subgroup of $\mathrm{O}(n)$ consisting of orthogonal matrices with determinant 1. These matrices represent proper rotations in n-dimensional space.
- **Unitary Group:** $\mathrm{U}(n)$, the group of $n \times n$ complex matrices A satisfying $A^\dagger A = I$, where $A^\dagger$ denotes the conjugate transpose of A. Unitary matrices preserve complex inner products.
- **Special Unitary Group:** $\mathrm{SU}(n)$, a subgroup of $\mathrm{U}(n)$ consisting of unitary matrices with determinant 1. This group is fundamental in quantum mechanics and particle physics.

1.3 Closure Properties

Matrix groups satisfy the following closure properties, which are essential for ensuring their group structure:

- If $A, B \in G$, then $AB \in G$ (closure under multiplication).
- If $A \in G$, then $A^{-1} \in G$ (closure under inversion).

These properties follow directly from the definition of G as a subgroup of $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{R})$, which itself is closed under multiplication and inversion.

1.4 Key Observations and Notes

- Matrix groups often have additional structures, such as being compact, connected, or finite.
- Many matrix groups are defined by polynomial equations, making them algebraic groups. For example, $\mathrm{SL}(n, \mathbb{R})$ is defined by the equation $\det(A) = 1$.
- Matrix groups are central to understanding symmetries in mathematics, physics, and engineering, particularly through their representations.

2 Matrix Lie Groups

2.1 Definition

A *matrix Lie group* is a matrix group that is also a topological group. This means that the group operations, namely matrix multiplication and inversion, are continuous with respect to the topology inherited from $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$.

2.2 Key Examples

Matrix Lie groups include many of the matrix groups discussed earlier. Here are their properties as Lie groups:

- **General Linear Group:** $\mathrm{GL}(n, \mathbb{R})$, consisting of all invertible $n \times n$ matrices with real entries, has dimension n^2.
- **Special Linear Group:** $\mathrm{SL}(n, \mathbb{R})$, the group of $n \times n$ matrices with determinant 1, has dimension $n^2 - 1$.
- **Orthogonal Group:** $\mathrm{O}(n)$, consisting of orthogonal $n \times n$ matrices ($A^T A = I$), has dimension $\frac{n(n-1)}{2}$.
- **Special Orthogonal Group:** $\mathrm{SO}(n)$, a subgroup of $\mathrm{O}(n)$ with $\det(A) = 1$, also has dimension $\frac{n(n-1)}{2}$.
- **Unitary Group:** $\mathrm{U}(n)$, consisting of unitary matrices ($A^\dagger A = I$), has dimension n^2.
- **Special Unitary Group:** $\mathrm{SU}(n)$, a subgroup of $\mathrm{U}(n)$ with $\det(A) = 1$, has dimension $n^2 - 1$.

2.3 Properties of Matrix Lie Groups

Matrix Lie groups satisfy several important properties:

- **Closure under Operations:** Matrix Lie groups are closed under matrix multiplication and inversion.
- **Topology:** A matrix Lie group is a closed subset of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ under the standard topology.
- **Dimension:** A matrix Lie group has a well-defined dimension, which equals the dimension of its Lie algebra (discussed in the next section).

2.4 Theorem: Closed Subgroups Theorem

Theorem 2.1. *A subset $G \subset \mathrm{GL}(n, \mathbb{C})$ or $G \subset \mathrm{GL}(n, \mathbb{R})$ is a matrix Lie group if and only if G is a closed subgroup in the standard topology.*

Corollary 2.1.1. *Matrix Lie groups inherit smooth structures from $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, enabling their study via local approximations such as tangent spaces and exponential maps.*

3 Lie Algebras of Matrix Lie Groups

3.1 Definition

The *Lie algebra* of a matrix Lie group G, denoted by $\mathfrak{g}$, is the set of all $n \times n$ matrices X such that the curve $e^{tX} \in G$ for all $t \in \mathbb{R}$. Formally:

$$\mathfrak{g} = \{X \in M_n(\mathbb{R}) \mid e^{tX} \in G \,\text{for all}\ t \in \mathbb{R}\}.$$

3.2 Exponential Map

The *exponential map*, denoted by exp, provides a connection between the Lie algebra $\mathfrak{g}$ and the Lie group G:

$$\exp(X) = e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}, \quad X \in \mathfrak{g}.$$

- $\exp(0) = I$.
- If $X, Y \in \mathfrak{g}$ and they commute ($XY = YX$), then $\exp(X + Y) = \exp(X)\exp(Y)$.
- The image of $\mathfrak{g}$ under the exponential map lies in G for small t.

3.3 Lie Bracket

The Lie algebra $\mathfrak{g}$ is equipped with an additional operation, the *Lie bracket*, defined as:

$$[X, Y] = XY - YX, \quad \text{for } X, Y \in \mathfrak{g}.$$

- The Lie bracket satisfies bilinearity:

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \quad \text{for } a, b \in \mathbb{R}.$$

- It is antisymmetric: $[X, Y] = -[Y, X]$.
- It satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

3.4 Examples of Lie Algebras

Here are the Lie algebras corresponding to key matrix Lie groups:

- **General Linear Group:** For $G = \mathrm{GL}(n, \mathbb{R})$, the Lie algebra is $\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$, the set of all $n \times n$ real matrices. The Lie bracket is the standard commutator $[X, Y] = XY - YX$.
- **Special Linear Group:** For $G = \mathrm{SL}(n, \mathbb{R})$, the Lie algebra is:

$$\mathfrak{sl}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \mathrm{Tr}(X) = 0\}.$$

- **Orthogonal Group:** For $G = \mathrm{O}(n)$, the Lie algebra is:

$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\},$$

  the set of skew-symmetric matrices.
- **Unitary Group:** For $G = \mathrm{U}(n)$, the Lie algebra is:

$$\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^\dagger = -X\},$$

  where $X^\dagger$ denotes the conjugate transpose.
- **Special Unitary Group:** For $G = \mathrm{SU}(n)$, the Lie algebra is:

$$\mathfrak{su}(n) = \{X \in \mathfrak{u}(n) \mid \mathrm{Tr}(X) = 0\}.$$

3.5 Properties of the Lie Algebra

- **Dimension:** The dimension of $\mathfrak{g}$ equals the dimension of the Lie group G.
- **Local Approximation:** The Lie algebra $\mathfrak{g}$ serves as a local linear approximation of the Lie group G near the identity element.
- **Connection to Subgroups:** Subgroups of G correspond to subalgebras of $\mathfrak{g}$.

3.6 Theorem: Lie Subalgebra Theorem

Theorem 3.1. *Let H be a subgroup of a matrix Lie group G. If H is a matrix Lie group, then its Lie algebra $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, the Lie algebra of G.*

4 Examples of Lie Groups and Their Lie Algebras

This section explores specific examples of matrix Lie groups and their associated Lie algebras. These examples highlight the structure and properties of both Lie groups and their tangent space approximations.

4.1 General Linear Group

- **Group:** The general linear group $\mathrm{GL}(n, \mathbb{R})$ consists of all invertible $n \times n$ matrices with real entries:
$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}.$$
- **Lie Algebra:** The Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ is the space of all $n \times n$ real matrices:
$$\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}).$$
The Lie bracket is the standard commutator:
$$[X, Y] = XY - YX, \quad X, Y \in \mathfrak{gl}(n, \mathbb{R}).$$
- **Dimension:** $\dim(\mathrm{GL}(n, \mathbb{R})) = n^2$.

4.2 Special Linear Group

- **Group:** The special linear group $\mathrm{SL}(n, \mathbb{R})$ consists of $n \times n$ real matrices with determinant 1:
$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\}.$$
- **Lie Algebra:** The Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is the space of $n \times n$ matrices with trace 0:
$$\mathfrak{sl}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \mathrm{Tr}(X) = 0\}.$$
- **Dimension:** $\dim(\mathrm{SL}(n, \mathbb{R})) = n^2 - 1$.

4.3 Orthogonal Group

- **Group:** The orthogonal group $\mathrm{O}(n)$ consists of $n \times n$ real matrices A such that $A^T A = I$:
$$\mathrm{O}(n) = \{A \in M_n(\mathbb{R}) \mid A^T A = I\}.$$
- **Lie Algebra:** The Lie algebra $\mathfrak{so}(n)$ is the space of skew-symmetric matrices:
$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\}.$$
The Lie bracket is given by the commutator $[X, Y] = XY - YX$.
- **Dimension:** $\dim(\mathrm{O}(n)) = \frac{n(n-1)}{2}$.

4.4 Special Orthogonal Group

- **Group:** The special orthogonal group $\mathrm{SO}(n)$ is a subgroup of $\mathrm{O}(n)$ with determinant 1:
$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det(A) = 1\}.$$
- **Lie Algebra:** The Lie algebra is the same as for $\mathrm{O}(n)$:
$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\}.$$
- **Dimension:** $\dim(\mathrm{SO}(n)) = \frac{n(n-1)}{2}$.

4.5 Unitary Group

- **Group:** The unitary group $\mathrm{U}(n)$ consists of $n \times n$ complex matrices A such that $A^\dagger A = I$:
$$\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) \mid A^\dagger A = I\}.$$
- **Lie Algebra:** The Lie algebra $\mathfrak{u}(n)$ is the space of skew-Hermitian matrices:
$$\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^\dagger = -X\}.$$
- **Dimension:** $\dim(\mathrm{U}(n)) = n^2$.

4.6 Special Unitary Group

- **Group:** The special unitary group $\mathrm{SU}(n)$ is a subgroup of $\mathrm{U}(n)$ with determinant 1:
$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det(A) = 1\}.$$
- **Lie Algebra:** The Lie algebra $\mathfrak{su}(n)$ is the space of skew-Hermitian matrices with trace 0:
$$\mathfrak{su}(n) = \{X \in \mathfrak{u}(n) \mid \mathrm{Tr}(X) = 0\}.$$
- **Dimension:** $\dim(\mathrm{SU}(n)) = n^2 - 1$.

4.7 Summary of Dimensions

$$\begin{aligned}
&\dim(\mathrm{GL}(n,\mathbb{R})) = n^2, \quad \dim(\mathrm{SL}(n,\mathbb{R})) = n^2 - 1,\\
&\dim(\mathrm{O}(n)) = \frac{n(n-1)}{2}, \quad \dim(\mathrm{SO}(n)) = \frac{n(n-1)}{2},\\
&\dim(\mathrm{U}(n)) = n^2, \quad \dim(\mathrm{SU}(n)) = n^2 - 1.
\end{aligned}$$

5 Properties of Matrix Lie Groups and Their Lie Algebras

Matrix Lie groups and their Lie algebras exhibit several key properties that connect their algebraic structure with geometry and topology. This section explores these properties in depth.

5.1 Key Properties of Matrix Lie Groups

- **Closed Subsets:** A matrix Lie group is a closed subset of $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$. This follows from the *Closed Subgroups Theorem*, which states that any closed subgroup of a Lie group is itself a Lie group.
- **Smooth Group Operations:** The group operations (matrix multiplication and inversion) in a matrix Lie group are smooth, ensuring the differentiable structure of the group.

- **Dimension:** The dimension of a matrix Lie group corresponds to the number of independent parameters required to describe the group. It is equal to the dimension of its Lie algebra, $\mathfrak{g}$.
- **Local Exponential Map:** The exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism near the identity element. This connects the tangent space (Lie algebra) to the group structure.

5.2 Key Properties of Lie Algebras

- **Tangent Space at Identity:** The Lie algebra $\mathfrak{g}$ is the tangent space of the Lie group G at the identity element, providing a linear approximation to G near the identity.
- **Closure Under Lie Bracket:** The Lie bracket operation $[X, Y] = XY - YX$ is closed within $\mathfrak{g}$, making $\mathfrak{g}$ a Lie algebra.
- **Subalgebras and Subgroups:** Subalgebras of $\mathfrak{g}$ correspond to closed subgroups of G.
- **Dimension and Basis:** The dimension of $\mathfrak{g}$ matches the dimension of G, and $\mathfrak{g}$ admits a basis $\{X_1, X_2, \ldots, X_d\}$, where $d = \dim(\mathfrak{g})$.

5.3 The Exponential Map and Its Properties

The exponential map is a central tool in understanding the relationship between a Lie group G and its Lie algebra $\mathfrak{g}$:

$$\exp : \mathfrak{g} \to G, \quad \exp(X) = e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

- **Surjectivity:** For many matrix Lie groups (e.g., compact groups), the exponential map is surjective, meaning every element of G can be written as $\exp(X)$ for some $X \in \mathfrak{g}$.
- **Local Diffeomorphism:** Near the identity element, exp provides a one-to-one correspondence between $\mathfrak{g}$ and G.
- **One-Parameter Subgroups:** For any $X \in \mathfrak{g}$, the curve e^{tX} (for $t \in \mathbb{R}$) is a one-parameter subgroup of G.
- **Relationship with Lie Bracket:** The Lie bracket $[X, Y]$ in $\mathfrak{g}$ corresponds to the group commutator $ABA^{-1}B^{-1}$ in G under the exponential map.

5.4 Connections Between Lie Groups and Lie Algebras

Theorem 5.1 (Lie Correspondence)**.** *There is a one-to-one correspondence between:*

- *Lie subgroups of G, and*
- *Lie subalgebras of $\mathfrak{g}$.*

Corollary 5.1.1. *If H is a closed subgroup of a matrix Lie group G, then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of the Lie algebra $\mathfrak{g}$ of G.*

6 Structure of Specific Matrix Lie Groups

Matrix Lie groups exhibit rich internal structure, often characterized by their algebraic, topological, and geometric properties. This section delves into the internal organization of specific matrix Lie groups.

6.1 Connected Components

A matrix Lie group G can be decomposed into its connected components. The component containing the identity element is known as the extitidentity component, denoted G^0.

- **Identity Component:** G^0 is a normal subgroup of G, and the quotient G/G^0 is discrete.
- **Example:** For $\mathrm{O}(n)$, the identity component is $\mathrm{SO}(n)$, and $\mathrm{O}(n)/\mathrm{SO}(n)$ consists of two components.

6.2 Compactness

A matrix Lie group G is said to be extitcompact if it is a closed and bounded subset of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$.

- Compactness is preserved under finite products and closed subgroups.
- **Examples:**
  - $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.
  - $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are also compact.

6.3 Centers of Lie Groups

The extitcenter of a matrix Lie group G, denoted $Z(G)$, is the set of elements in G that commute with all other elements of G:

$$Z(G) = \{A \in G \mid AB = BA \text{ for all } B \in G\}.$$

- The center is a closed, normal subgroup of G.
- **Examples:**
  - For $\mathrm{SL}(n, \mathbb{R})$, $Z(\mathrm{SL}(n, \mathbb{R}))$ is trivial for $n > 2$.
  - For $\mathrm{U}(n)$, $Z(\mathrm{U}(n)) = \{e^{i\theta} I \mid \theta \in \mathbb{R}\}$.

6.4 Homomorphisms and Isomorphisms

Homomorphisms between matrix Lie groups respect both the group and topological structure. An extitisomorphism is a bijective homomorphism.

- **Kernel:** The kernel of a homomorphism $\phi : G \to H$ is a normal subgroup of G.
- **Image:** The image $\phi(G)$ is a Lie subgroup of H.
- **Example:** The map $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ is a homomorphism with kernel $\mathrm{SL}(n, \mathbb{R})$.

6.5 Universal Covering Groups

The extituniversal covering group of a connected matrix Lie group G is another Lie group $\tilde{G}$ along with a surjective homomorphism $\pi : \tilde{G} \to G$ satisfying the following:

- $\tilde{G}$ is simply connected.
- The kernel of π is discrete and isomorphic to the fundamental group of G.
- **Example:** The universal covering group of SO(3) is SU(2).

6.6 Invariant Subgroups

An invariant (or normal) subgroup $H \subseteq G$ satisfies $gHg^{-1} \subseteq H$ for all $g \in G$. These subgroups play a key role in understanding the quotient structure of G.

- **Example:** For $\mathrm{SL}(n, \mathbb{R})$, the center $Z(\mathrm{SL}(n, \mathbb{R}))$ is an invariant subgroup.