Snapshot of the VERY Basic Algebraic Geometry

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Contents

1 Affine Geometry

1.1 Definition of Affine Space

Definition 1.1 (Affine Space). *The n-dimensional affine space over a field k, denoted by $\mathbb{A}^n$, is the set of all ordered n-tuples of elements from k:*

$$\mathbb{A}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in k \text{ for } i = 1, 2, \ldots, n\}.$$

Affine space can be thought of as a "flat" space that generalizes Euclidean space, but without any additional structure such as a metric or designated origin.

Remark 1.1 (Mathematical Meaning of Affine Space). *Affine space provides the stage where algebraic geometry takes place. The points in $\mathbb{A}^n$ are the "coordinates," and algebraic objects like varieties are subsets of $\mathbb{A}^n$ defined by polynomial equations.*

1.2 Polynomial Rings and Their Role

Definition 1.2 (Polynomial Ring). *The polynomial ring in n variables over a field k, denoted $k[x_1, \ldots, x_n]$, consists of all finite sums of monomials of the form:*

$$\sum_{i_1,\ldots,i_n \geq 0} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}, \quad c_{i_1,\ldots,i_n} \in k.$$

Remark 1.2 (Role of Polynomial Rings). *Polynomial rings are the algebraic counterpart of affine space. They encode the "functions" defined on $\mathbb{A}^n$. Studying ideals in these rings allows us to describe and analyze geometric objects like curves and surfaces.*

Example 1.1 (Example of a Polynomial Ring). *In $k[x, y]$, examples of polynomials include $x^2 + y^2$, $3xy + 2y$, and 1. This ring allows us to describe geometric objects in $\mathbb{A}^2$, such as circles ($x^2 + y^2 = 1$) or lines ($x + y = 0$).*

1.3 Affine Varieties

Definition 1.3 (Affine Variety). *An* affine variety *$V \subseteq \mathbb{A}^n$ is the set of common zeros of a collection of polynomials $f_1, f_2, \ldots, f_m \in k[x_1, \ldots, x_n]$:*

$$V(f_1, \ldots, f_m) = \{x \in \mathbb{A}^n \mid f_1(x) = f_2(x) = \cdots = f_m(x) = 0\}.$$

Remark 1.3 (Geometric Interpretation). *Affine varieties represent solutions to systems of polynomial equations. They generalize familiar geometric objects, such as lines, circles, and parabolas, to higher dimensions and more abstract settings.*

Example 1.2 (Circle in $\mathbb{A}^2$). *The unit circle in $\mathbb{A}^2$ over $\mathbb{R}$ is the variety $V(x^2 + y^2 - 1)$:*

$$V(x^2 + y^2 - 1) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

This provides a concrete example of an affine variety defined by a single polynomial equation.

1.4 Coordinate Functions and Morphisms

Definition 1.4 (Coordinate Functions). *For any point $p = (p_1, \ldots, p_n) \in \mathbb{A}^n$, the coordinate functions $x_1, x_2, \ldots, x_n$ are projections:*

$$x_i(p) = p_i \quad \textit{for } i = 1, \ldots, n.$$

These functions generate the polynomial ring $k[x_1, \ldots, x_n]$.

Remark 1.4 (Why Coordinate Functions?). *Coordinate functions allow us to express polynomial equations in terms of the variables $x_1, x_2, \ldots, x_n$. They form the basis for defining and studying affine varieties.*

Definition 1.5 (Morphisms of Affine Varieties). *A* morphism *between affine varieties $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ is a map $\varphi : V \to W$ induced by polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$:*

$$\varphi(p) = (f_1(p), \ldots, f_n(p)) \quad \textit{for all } p \in V.$$

Example 1.3 (Morphisms Between Varieties). *Consider $V = V(x^2 + y^2 - 1) \subset \mathbb{A}^2$ (the unit circle) and $W = \mathbb{A}^1$. The map $\varphi : V \to W$ given by $\varphi(x, y) = x$ projects the circle onto the x-axis.*

1.5 The Zariski Topology

Definition 1.6 (Zariski Closed Sets). *A subset $Z \subseteq \mathbb{A}^n$ is* Zariski closed *if there exists an ideal $I \subseteq k[x_1, \ldots, x_n]$ such that:*

$$Z = V(I) = \{x \in \mathbb{A}^n \mid f(x) = 0 \textit{ for all } f \in I\}.$$

Remark 1.5 (Why the Zariski Topology?). *The Zariski topology provides a natural way to study varieties. Its closed sets correspond to algebraic objects (varieties), making it suitable for algebraic geometry.*

Example 1.4 (Zariski Closed Set in $\mathbb{A}^2$). *In $\mathbb{A}^2$, the variety $V(x^2 + y^2 - 1)$ is a Zariski closed set because it is defined by the polynomial $x^2 + y^2 - 1$.*

Definition 1.7 (Zariski Topology). *The collection of Zariski closed sets defines a topology on $\mathbb{A}^n$, called the* Zariski topology*. Its open sets are the complements of Zariski closed sets.*

2 Ideals in Algebraic Geometry

2.1 Definition of Ideals

Definition 2.1 (Ideal). *An* ideal *I in the ring $k[x_1, \ldots, x_n]$ is a subset such that:*

- $0 \in I$,
- *If $f, g \in I$, then $f + g \in I$ (closure under addition),*
- *If $f \in I$ and $h \in k[x_1, \ldots, x_n]$, then $hf \in I$ (closure under multiplication by ring elements).*

Example 2.1 (Examples of Ideals). *In $k[x, y]$:*

- *$I = \langle x^2 + y^2 - 1 \rangle$: All multiples of $x^2 + y^2 - 1$.*
- *$I = \langle x, y \rangle$: All polynomials divisible by x or y.*

2.2 Operations on Ideals

Definition 2.2 (Sum and Product of Ideals). *For ideals $I, J \subseteq k[x_1, \dots, x_n]$:*

- *The* sum $I + J = \{f + g \mid f \in I, g \in J\}$.
- *The* product $I \cdot J = \{\sum f_i g_i \mid f_i \in I, g_i \in J\}$.

Example 2.2 (Operations on Ideals). *For $I = \langle x \rangle$ and $J = \langle y \rangle$ in $k[x, y]$:*

- *$I + J = \langle x, y \rangle$ contains all linear combinations of x and y.*
- *$I \cdot J = \langle xy \rangle$ contains all multiples of xy.*

2.3 Radical of an Ideal

Definition 2.3 (Radical of an Ideal). *The* radical *of an ideal I, denoted $\sqrt{I}$, is the set of all polynomials $f \in k[x_1, \dots, x_n]$ such that $f^m \in I$ for some $m \geq 1$:*

$$\sqrt{I} = \{f \in k[x_1, \dots, x_n] \mid f^m \in I \text{ for some } m \geq 1\}.$$

Example 2.3 (Radical of an Ideal). *For $I = \langle x^2 \rangle$ in $k[x]$, the radical $\sqrt{I} = \langle x \rangle$ since any $f \in \sqrt{I}$ satisfies $f^2 \in I$.*

2.4 Hilbert's Nullstellensatz

Theorem 2.1 (Hilbert's Nullstellensatz). *If k is an algebraically closed field and $I \subseteq k[x_1, \dots, x_n]$ is an ideal, then the variety $V(I)$ satisfies:*

$$I(V(I)) = \sqrt{I}.$$

Example 2.4 (Nullstellensatz in Action). *For $I = \langle x^2 + y^2 - 1 \rangle$ in $\mathbb{C}[x, y]$, the variety $V(I)$ is the unit circle in $\mathbb{A}^2$. The ideal $I(V(I))$ is $\sqrt{I}$, confirming the theorem.*

2.5 Coordinate Rings

Definition 2.4 (Coordinate Ring). *The* coordinate ring *of a variety $V \subseteq \mathbb{A}^n$ is the quotient ring:*

$$k[V] = k[x_1, \dots, x_n]/I(V),$$

where $I(V)$ is the ideal of all polynomials vanishing on V.

Example 2.5 (Coordinate Ring of a Circle). *For $V = V(x^2+y^2-1)$ in $\mathbb{A}^2$, the coordinate ring is:*

$$k[V] = k[x, y]/\langle x^2 + y^2 - 1 \rangle.$$

3 Projective Geometry

3.1 Definition of Projective Space

Definition 3.1 (Projective Space). *The n-dimensional projective space over a field k, denoted by $\mathbb{P}^n$, is the set of equivalence classes of $\mathbb{A}^{n+1}\setminus\{0\}$ under the equivalence relation:*

$$(x_0, x_1, \dots, x_n) \sim (\lambda x_0, \lambda x_1, \dots, \lambda x_n), \quad \lambda \in k^*.$$

Each equivalence class is written as $[x_0 : x_1 : \cdots : x_n]$.

Remark 3.1 (Geometric Interpretation). *Projective space can be thought of as the space of lines passing through the origin in $\mathbb{A}^{n+1}$. It extends affine space by adding points at infinity to handle intersections more systematically.*

Example 3.1 (Projective Line). *The projective line $\mathbb{P}^1$ over k is the set of equivalence classes $[x_0 : x_1]$, representing all lines through the origin in $\mathbb{A}^2 \setminus \{0\}$. It includes points like $[1 : 0]$ and $[0 : 1]$, as well as affine points $[x : 1]$ for $x \in k$.*

3.2 Homogeneous Coordinates and Polynomials

Definition 3.2 (Homogeneous Polynomial). *A polynomial $f(x_0, \dots, x_n)$ is* homogeneous *of degree d if all terms have the same total degree d:*

$$f(x_0, \dots, x_n) = \sum a_{i_0,\dots,i_n} x_0^{i_0} \cdots x_n^{i_n}, \quad \text{with } i_0 + \cdots + i_n = d.$$

Remark 3.2 (Why Homogeneous Polynomials?). *Homogeneous polynomials define well-behaved varieties in projective space, as they remain invariant under scaling. This ensures that the variety is independent of the choice of representative in the equivalence class.*

Example 3.2 (Circle in Projective Space). *The circle $x^2 + y^2 - 1 = 0$ in affine space $\mathbb{A}^2$ can be extended to projective space $\mathbb{P}^2$ as the homogeneous equation:*

$$x^2 + y^2 - z^2 = 0,$$

where z accounts for points at infinity.

3.3 Projective Varieties

Definition 3.3 (Projective Variety). *A* projective variety *$V \subseteq \mathbb{P}^n$ is the set of common zeros of a collection of homogeneous polynomials $f_1, \dots, f_m \in k[x_0, \dots, x_n]$:*

$$V(f_1, \dots, f_m) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid f_1(x) = \cdots = f_m(x) = 0\}.$$

Example 3.3 (Conic in $\mathbb{P}^2$). *The variety defined by $x^2 + y^2 - z^2 = 0$ in $\mathbb{P}^2$ is a conic. In affine coordinates (where $z \neq 0$), this reduces to the circle $x^2 + y^2 = 1$ in $\mathbb{A}^2$.*

3.4 Relationship Between Affine and Projective Geometry

Remark 3.3 (Affine Charts). *Projective space $\mathbb{P}^n$ can be covered by $n + 1$* affine charts, *where each chart corresponds to setting one coordinate (e.g., x_i) to 1. These charts provide a bridge between affine and projective geometry.*

Example 3.4 (Affine Chart of $\mathbb{P}^2$). *For $\mathbb{P}^2$, the affine chart where $z \neq 0$ corresponds to the affine plane $\mathbb{A}^2$ with coordinates $(x/z, y/z)$.*

4 Dimension and Singularities

4.1 Dimension of a Variety

Definition 4.1 (Dimension)**.** *The* dimension *of an affine or projective variety V, denoted* $\dim(V)$*, is the maximum length of a chain of distinct irreducible subvarieties:*

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k \subseteq V,$$

where each V_i is irreducible. The length of this chain is k.

Remark 4.1 (Geometric Interpretation)**.** *Dimension measures the d̈egrees of freedomïn a variety. For instance:*

- *A point has dimension 0.*
- *A curve has dimension 1.*
- *A surface has dimension 2.*

Example 4.1 (Dimension in $\mathbb{A}^n$)**.**

- *The affine space $\mathbb{A}^n$ has dimension n.*
- *A line in $\mathbb{A}^2$, such as $V(x-y)$, has dimension 1.*
- *A point, such as $V(x, y)$ in $\mathbb{A}^2$, has dimension 0.*

4.2 Tangent Space

Definition 4.2 (Tangent Space)**.** *Let $V \subseteq \mathbb{A}^n$ be an affine variety defined by polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$. At a point $P \in V$, the* tangent space *$T_P V$ is the solution space of the linear system:*

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(P) \cdot x_i = 0, \quad \text{for } j = 1, \ldots, m.$$

Remark 4.2 (Dimension of the Tangent Space)**.** *The dimension of the tangent space at P provides a local approximation to the dimension of V near P. For smooth points,* $\dim(T_P V) = \dim(V)$*.*

Example 4.2 (Tangent Space of a Circle)**.** *Consider the circle $V(x^2 + y^2 - 1)$ in $\mathbb{A}^2$. At the point $P = (1, 0)$, the gradient of $f(x, y) = x^2 + y^2 - 1$ is $\nabla f = (2x, 2y)$. The tangent space at P is the line $2x \cdot \delta x + 2y \cdot \delta y = 0$, which simplifies to $\delta y = 0$.*

4.3 Singular Points

Definition 4.3 (Singular Point)**.** *A point $P \in V$ is a* singular point *if the rank of the Jacobian matrix of the defining polynomials $f_1, \ldots, f_m$ at P is less than* $\dim(V)$*. Otherwise, P is a* smooth point*.*

Example 4.3 (Singular Point of a Curve)**.** *Consider the curve $V(y^2 - x^3) \subseteq \mathbb{A}^2$. The Jacobian matrix is:*

$$\begin{bmatrix} -\frac{\partial}{\partial x}(x^3) & \frac{\partial}{\partial y}(y^2) \end{bmatrix} = \begin{bmatrix} -3x^2 & 2y \end{bmatrix}.$$

At $P = (0, 0)$, the Jacobian matrix is zero, so P is a singular point.

4.4 Applications of Singularity Analysis

Remark 4.3 (Singularities in Geometry). *Singularities play a crucial role in algebraic geometry, as they often indicate points of interest or complexity. For example:*

- *Singular points on a curve may correspond to cusps or intersections.*
- *In higher dimensions, singularities are essential in the study of moduli spaces and birational geometry.*

Example 4.4 (Node and Cusp).

- *The curve* $y^2 - x^2(x+1) = 0$ *has a* node *at* $(0, 0)$.
- *The curve* $y^2 - x^3 = 0$ *has a* cusp *at* $(0, 0)$.

5 Bézout's Theorem

5.1 Intersection of Projective Curves

Definition 5.1 (Intersection of Curves). *Let* C_1 *and* C_2 *be two projective curves in* $\mathbb{P}^2$*, defined by homogeneous polynomials* $f_1(x_0, x_1, x_2)$ *and* $f_2(x_0, x_1, x_2)$*, respectively. Their intersection points are the solutions to:*

$$f_1(x_0, x_1, x_2) = 0 \quad \textit{and} \quad f_2(x_0, x_1, x_2) = 0.$$

5.2 Statement of Bézout's Theorem

Theorem 5.1 (Bézout's Theorem). *Let* C_1 *and* C_2 *be projective curves in* $\mathbb{P}^2$ *defined by homogeneous polynomials of degrees* d_1 *and* d_2*. If* C_1 *and* C_2 *intersect transversally, the number of intersection points (counted with multiplicity) is:*

$$d_1 \cdot d_2.$$

5.3 Examples and Applications

Example 5.1 (Line and Conic in $\mathbb{P}^2$). *Consider a line* L *defined by* $x_0 + x_1 = 0$ *and a conic* C *defined by* $x_0^2 + x_1^2 - x_2^2 = 0$ *in* $\mathbb{P}^2$*. Bézout's theorem predicts:*

$$\deg(L) \cdot \deg(C) = 1 \cdot 2 = 2.$$

Indeed, the line intersects the conic at exactly two points in $\mathbb{P}^2$*.*

Example 5.2 (Two Conics in $\mathbb{P}^2$). *Let* C_1 *and* C_2 *be two conics defined by* $x_0^2 + x_1^2 - x_2^2 = 0$ *and* $x_0 x_1 - x_2^2 = 0$*. Bézout's theorem gives:*

$$\deg(C_1) \cdot \deg(C_2) = 2 \cdot 2 = 4.$$

The two conics intersect at four points (counted with multiplicity).

5.4 Multiplicity of Intersection Points

Definition 5.2 (Multiplicity of an Intersection). *If C_1 and C_2 intersect at a point P in $\mathbb{P}^2$, the* multiplicity *of the intersection at P is determined by the local behavior of the defining equations at P.*

Remark 5.1 (Multiplicity and Tangency). *If C_1 and C_2 are tangent at P, the multiplicity of the intersection at P is greater than 1.*

Example 5.3 (Tangency of a Line and a Conic). *Consider the line L defined by $x_0+x_1 = 0$ and the conic C defined by $(x_0 + x_1)^2 - x_2^2 = 0$. The line is tangent to the conic at $[1 : -1 : 0]$, and the intersection at this point has multiplicity 2.*

5.5 Generalizations of Bézout's Theorem

Remark 5.2 (Higher-Dimensional Varieties). *Bézout's theorem extends to higher dimensional projective varieties. For example, if V_1 and V_2 are varieties in $\mathbb{P}^n$ with dimensions d_1 and d_2, their intersection has a degree determined by the product of their degrees, under suitable transversality assumptions.*

Example 5.4 (Planes in $\mathbb{P}^3$). *Two planes in $\mathbb{P}^3$ intersect in a line. Each plane has degree 1, so their intersection has degree $1 \cdot 1 = 1$, corresponding to the line.*