Rings of Invariants

Chenyang Zhao

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Contents

1 Definition and Examples

1.1 Assumptions

- All rings considered are commutative.
- The base field k is $\mathbb{C}$ (any k with characteristic 0 and sufficient roots of unity also works).

1.2 Definition: Ring of Invariants

Let $G \subset GL_n(\mathbb{C})$ be a group acting on $\mathbb{C}[x_1, \ldots, x_n]$, the ring of polynomial functions on $\mathbb{C}^n$. The action of G on $\mathbb{C}[x_1, \ldots, x_n]$ is induced by the action of G on $\mathbb{C}^n$, defined as:

$$\gamma \cdot f(x) = f(\gamma^{-1}x), \quad \forall \gamma \in G.$$

The **ring of invariants** under this action is defined as:

$$\mathbb{C}[x_1, \ldots, x_n]^G = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid \forall \gamma \in G, \gamma \cdot f = f\}.$$

1.3 Example 1: Symmetric Group S_n

- **Group Action:** The symmetric group S_n acts on $\mathbb{C}[x_1, \ldots, x_n]$ by permuting the variables:
$$\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n.$$
- **Ring of Invariants:** The elementary symmetric polynomials generate the ring of invariants:
$$\sigma_1 = x_1 + x_2 + \cdots + x_n, \quad \sigma_2 = \sum_{i<j} x_i x_j, \quad \ldots, \quad \sigma_n = x_1 x_2 \cdots x_n.$$

  Theorem:
$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[\sigma_1, \sigma_2, \ldots, \sigma_n].$$

  The map:
$$\phi : \mathbb{C}[y_1, \ldots, y_n] \to \mathbb{C}[x_1, \ldots, x_n]^{S_n}, \quad y_i \mapsto \sigma_i,$$
  is an isomorphism with no algebraic relations between the σ_i.

1.4 Example 2: Cyclic Group μ_n

- **Group Definition:** The cyclic group of n-th roots of unity, $\mu_n = \{\xi \in \mathbb{C} \mid \xi^n = 1\}$, acts on $\mathbb{C}[x, y]$ as:
$$\xi \cdot (x, y) = (\xi x, \xi^{-1} y).$$
- **Invariants:** Invariant polynomials include:
$$u = x^n, \quad v = y^n, \quad w = xy.$$

  The ring of invariants is:
$$\mathbb{C}[x, y]^{\mu_n} = \mathbb{C}[u, v, w]/(uv - w^n),$$
  where the relation $uv = w^n$ defines the ideal of relations.

1.5 Example 3: Dihedral Group D_n

- **Group Definition:** The dihedral group D_n acts on $\mathbb{C}[x, y]$ by combining rotations and reflections. For instance:

$$r \cdot (x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta), \quad s \cdot (x, y) = (x, -y).$$

- **Invariants:** These actions generate more complex invariant rings, often involving higher-degree polynomials. Detailed computations depend on n.

2 Algebraic Structure

2.1 Finitely Generated Invariant Rings

- **Noether's Theorem:** If $G \subset GL_n(\mathbb{C})$ is a finite group acting on $\mathbb{C}[x_1, \ldots, x_n]$, then the invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$ is a finitely generated $\mathbb{C}$-algebra.
- This result follows from the fact that $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian (Hilbert Basis Theorem).
- **Connection to the Hilbert Basis Theorem:** The invariants form an ideal that is finitely generated, ensuring the entire invariant ring is finitely generated.

2.2 Graded Rings and Homogeneous Components

- A ring R is called graded if it can be written as a direct sum of abelian groups:

$$R = \bigoplus_{n \geq 0} R_n,$$

where $R_n \cdot R_m \subseteq R_{n+m}$.

- **Homogeneous Elements:** Elements in R_n are called homogeneous of degree n.
- **Preservation by Group Actions:** If G acts on R, the grading is preserved, and R^G inherits the grading:

$$R^G = \bigoplus_{n \geq 0} (R^G)_n.$$

2.3 Reynolds Operator

- The Reynolds operator is defined as:

$$\rho(f) = \frac{1}{|G|} \sum_{\gamma \in G} \gamma \cdot f.$$

- **Properties:**
  1. $\rho(f) = f$ if and only if $f \in R^G$.
  2. ρ is linear but not multiplicative.
  3. ρ preserves degrees: if $f \in R_n$, then $\rho(f) \in R_n$.

2.4 Additional Examples

1. **Symmetric Group S_n:** Invariants are generated by elementary symmetric polynomials.
2. **Cyclic Group μ_n:** Invariants involve monomials in x^n, y^n, and xy, subject to a single relation.
3. **Dihedral Group D_n:** Reflection invariants and rotational invariants combine to form the generators.

3 Computational Techniques

3.1 Methods for Finding Generators

Definition 3.1. *The* ***Reynolds operator*** *is a fundamental tool in constructing invariant polynomials. For a finite group G acting on a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, it is defined as:*

$$\rho(f) = \frac{1}{|G|} \sum_{\gamma \in G} \gamma \cdot f, \quad \forall f \in \mathbb{C}[x_1, \ldots, x_n].$$

Remark 3.1. *The Reynolds operator ensures any polynomial f is projected to its invariant component. Specifically:*

- $\rho(f) \in \mathbb{C}[x_1, \ldots, x_n]^G$.
- $\rho(f) = f$ *if and only if* $f \in \mathbb{C}[x_1, \ldots, x_n]^G$.

3.2 Gröbner Bases for Invariants

Definition 3.2. *A* ***Gröbner basis*** *is a set of polynomials that provides a canonical representation of an ideal. For an invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$, a Gröbner basis can simplify computations by systematically generating relations among invariants.*

Theorem 3.1. *Let G be a finite group acting on $\mathbb{C}[x_1, \ldots, x_n]$. The ideal of relations among generators of $\mathbb{C}[x_1, \ldots, x_n]^G$ can be computed using a Gröbner basis.*

3.3 Software Tools

- **Macaulay2:** A computational algebra system ideal for computing Gröbner bases, invariant rings, and polynomial relations.
- **Singular:** Specialized for polynomial computations, including manipulation of ideals and invariant computations.

Example 3.1. *For the cyclic group μ_3 acting on $\mathbb{C}[x, y]$ as $\xi \cdot (x, y) = (\xi x, \xi^{-1} y)$ with $\xi^3 = 1$:*

- *Compute the invariants* $u = x^3, v = y^3, w = xy$.
- *The ring of invariants is:*

$$\mathbb{C}[x, y]^{\mu_3} = \mathbb{C}[u, v, w]/(uv - w^3).$$

Using Macaulay2 or Singular, you can verify the relation $uv = w^3$.

3.4 Explicit Computations

For specific groups:

- **Cyclic Group μ_n:** Compute monomials x^n, y^n, xy and identify relations.
- **Dihedral Group D_n:** Handle both rotational and reflective symmetries.

4 Classical Results

4.1 Emmy Noether's Theorem

Theorem 4.1. *Let $G \subset GL_n(\mathbb{C})$ be a finite group. The invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$ is a finitely generated $\mathbb{C}$-algebra.*

Proof Outline.

- Use the fact that $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian (Hilbert Basis Theorem).
- Construct generators for the invariant ideal.
- The generators form a finite set, ensuring the invariant ring is finitely generated.

□

4.2 Proofs of Noether's Theorem

- **Hilbert Basis Theorem Argument:**
  1. The ideal of invariants is finitely generated since $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian.
  2. Use the Reynolds operator to construct explicit generators.
- **Original Construction by Emmy Noether:**
  1. Use $\mathbb{C}[T]$ and group actions to derive invariant polynomials.
  2. Bound the degree of generators by the group order.

4.3 Applications to Specific Groups

Example 4.1. *For the symmetric group S_n:*

- *Invariants are generated by elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$.*
- *The invariant ring is:*

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[\sigma_1, \ldots, \sigma_n].$$

Example 4.2. *For the cyclic group μ_n:*

- *Invariants include x^n, y^n, xy.*
- *Relations such as $uv = w^n$ define the structure of the invariant ring.*

Example 4.3. *For the dihedral group D_n:*

- *Combine reflection and rotational symmetries.*
- *Explicit computations depend on the value of n.*

5 Connections to Algebraic Geometry

5.1 Invariant Rings as Coordinate Rings of Quotient Varieties

Definition 5.1. *Let G be a finite group acting linearly on $\mathbb{C}^n$. The ring of invariants $\mathbb{C}[x_1, \dots, x_n]^G$ is the coordinate ring of the quotient variety $\mathbb{C}^n/G$. This quotient reflects the geometric orbits of the action of G on $\mathbb{C}^n$.*

Example 5.1. *The action of the cyclic group μ_n on $\mathbb{C}^2$ is defined as:*

$$\xi \cdot (x, y) = (\xi x, \xi^{-1} y), \quad \xi^n = 1.$$

The invariants $u = x^n, v = y^n, w = xy$ generate the ring:

$$\mathbb{C}[x, y]^{\mu_n} = \mathbb{C}[u, v, w]/(uv - w^n),$$

which describes the quotient variety $\mathbb{C}^2/\mu_n$. The equation $uv = w^n$ defines a singular surface known as an A_{n-1}-type singularity.

Remark 5.1. *The quotient $\mathbb{C}^n/G$ may not be smooth, and its singularities encode the symmetries of G. These singularities are central to understanding the geometry of the quotient.*

5.2 Geometry of Quotient Spaces

- Quotient spaces defined by invariant rings often exhibit singularities at points where the group action is not free (e.g., fixed points of G).
- The quotient variety $\mathbb{C}^n/G$ is generally not a manifold but a singular space. Singularities arise at orbits with higher symmetry.
- Techniques from algebraic geometry, such as resolution of singularities, are used to study and smooth these quotient spaces.

Example 5.2. *The binary dihedral group BD_{4n} acts on $\mathbb{C}^2$ by combining rotations and reflections. The invariant ring defines a quotient space with D_{n+2}-type singularities. These singularities reflect the fixed points of the reflections and the geometry of the group action.*

5.3 Singularities from Group Actions

Definition 5.2. *Singularities in quotient varieties correspond to the loci where the stabilizer subgroup of G is non-trivial. These singular points are critical for understanding the geometry of the invariant ring.*

Remark 5.2. *Resolving singularities in quotient varieties often involves techniques such as blow-ups, weighted projective spaces, or toric geometry. The study of these singularities bridges invariant theory and the resolution of singularities in algebraic geometry.*

6 Advanced Topics

6.1 Symmetric Polynomials and Universality

- Symmetric polynomials are foundational in invariant theory and arise naturally in the action of symmetric groups S_n.
- The ring of invariants $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ is generated by the elementary symmetric polynomials:

$$\sigma_1 = x_1 + \cdots + x_n, \quad \sigma_2 = \sum_{i<j} x_i x_j, \quad \ldots, \quad \sigma_n = x_1 x_2 \cdots x_n.$$

- Universality: Symmetric polynomials appear in diverse areas such as Schur functions, symmetric functions, and representation theory of GL_n.

6.2 Modular Invariant Theory

Definition 6.1. *Modular invariant theory studies invariants of group actions over fields with positive characteristic char$(k) > 0$, especially when char$(k) \mid |G|$. The invariants differ significantly from those in characteristic 0 due to new algebraic and combinatorial phenomena.*

Example 6.1. *For $\mathbb{Z}_p$ acting on $\mathbb{F}_p[x_1, \ldots, x_n]$, the invariants include polynomials symmetric under $\mathbb{Z}_p$-rotations. The structure of the ring often requires modular techniques and insights from finite field theory.*

Remark 6.1. *Modular invariant theory introduces new challenges, such as non-finite generation of invariant rings and the presence of inseparable morphisms.*

6.3 Invariant Theory for Non-Finite Groups

- For continuous groups like $GL_n(\mathbb{C})$ or $SL_n(\mathbb{C})$, invariant theory involves differential invariants, algebraic varieties, and representation theory.
- The rings of invariants for non-finite groups often require tools from algebraic geometry, differential equations, and topology.

Example 6.2. *Invariant theory for $SL_2(\mathbb{C})$ acting on binary forms $\mathbb{C}[x, y]_n$ relates to classical geometry and the construction of discriminants.*

6.4 Applications in Modern Fields

- **Physics:** Noether's theorem connects invariants to conserved quantities in classical mechanics, quantum mechanics, and field theory.
- **Statistics:** Algebraic statistics utilizes invariant theory in maximum likelihood estimation and model selection, especially for exponential families and toric models.

Remark 6.2. *These advanced topics demonstrate the versatility of invariant theory, connecting algebra, geometry, and applied sciences.*