Black hole graphic simulation

Yuguang Chen

I. MATHEMATICS

A. Schwarzschild metric

The simplest, spherical symmetric metric of a Schwarzschild black hole, with its polar angle $\frac{\theta}{2} = 0$, is given as:

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{R_s}{r}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{R_s}{r}\right)^{-1} \mathrm{d}r^2 \\ &\quad - r^2 \mathrm{d}\theta^2 - r^2 \sin^2(\theta) \mathrm{d}\phi^2 \\ &\longrightarrow \left(1 - \frac{1}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{1}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega \end{aligned} \tag{1}$$

Without loss of generality, a natural selection of unit 1 is applied as $c = 1$ and the Schwarzschild radius $R_s = 2GM/c^2 = 1$. Notice in the convention defined by the metric, $\mathrm{d}s^2$ is always time-like, $\mathrm{d}s^2 \equiv \mathrm{d}\tau^2$.

The (pseudo-) energy and (pseudo-) angular-momentum conservation set up two additional constraints:

$$\left(1 - \frac{1}{r}\right) \frac{\mathrm{d}t}{\mathrm{d}\tau} = e \tag{2}$$

$$r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = l \tag{3}$$

Combining these two constraints with the metric ends up with:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{r^4 e^2}{l^2} - \left(1 - \frac{1}{r}\right)\left(\frac{r^4}{l^2} + r^2\right) \tag{4}$$

In the limit of massless particle – photon –, it's safe to assume $l \to \infty$. With the notation $r \to \frac{1}{u}$, the above first order differential equation becomes

$$\begin{aligned} \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 &= (u')^2 = \frac{e^2}{l^2} - (1 - u)(l^{-2} + u^2) \\ &\longrightarrow u^3 - u^2 \end{aligned} \tag{5}$$

This equation derived from the metric resembles an energe conservation equation, where $K = \frac{1}{2}(u')^2$ is the kinetic term and $V = \frac{1}{2}(u^3 - u^2)$ is the minus potential. Hence the gradient of the potential w.r.t. u gives rise to the acceleration $u''(\phi)$:

$$u''(\phi) = \nabla_u V = -u(1 - \frac{3u}{2}) \tag{6}$$

One can also use the chain rule to derive the same relation.

This second-order differential equation is preferred than the first-order one in terms of the computational efficiency, since square root is computational heavy.

B. Geometry

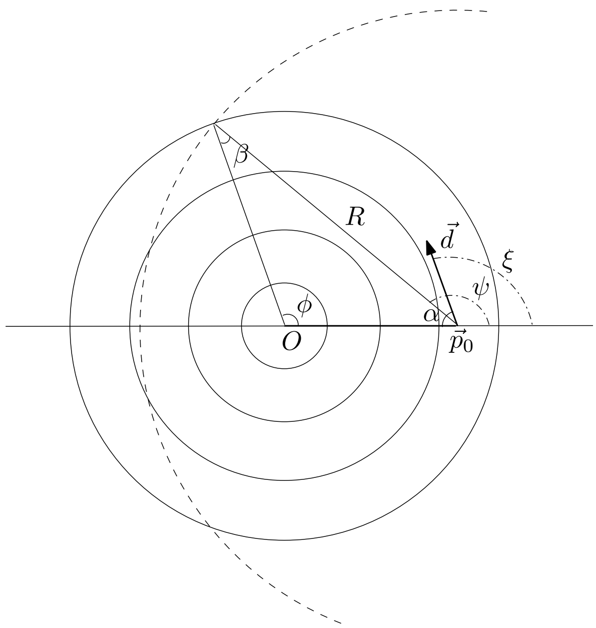

FIG. 1. The black hole sits at the centre O of the polar coordinate. The camera located at $\vec{p}_0$, coinciding with $\phi_0 = 0$, traces rays orienting $\vec{d}$, which is at a seen angle $\xi = \pi - \alpha$ with $\vec{p}_0$. The deviated angle ψ specifies the angle when the ray travels a radial distance R from the camera. The polar angle of this updated location is ϕ, and the angle opposing $\vec{p}_0$ is β.

In the plane of equator [Fig.1], a camera is placed at $\vec{p}_0$ and it traces back light at a direction $\vec{d}$. $\vec{d}$ has a "*seen angle*" ξ with $\vec{p}_0$. Assuming this light is deviated under the metric, and travels a radial distance R from the camera. The updated location has a polar angle ϕ, and is with a "*deviate angle*" ψ seen by the camera. Trigonometry tells us that

$$\frac{||\vec{p}_0||}{\sin(\beta)} = \frac{R}{\sin(\phi)} \tag{7}$$

$$\Longrightarrow \psi = \phi + \beta = \phi + \arcsin\left(\frac{R}{||\vec{p}_0|| \sin(\phi)}\right) \tag{8}$$

To solve for equation 6, two initial conditions are taken as:

$$u(\phi_0) = \frac{1}{||\vec{p}||} \tag{9}$$

$$u'(\phi_0) = -\frac{1}{u(\phi_0)\tan(\alpha)} \tag{10}$$

where we define the orthonormal basis $\hat{n}$ and $\hat{t}$ as:

$$\hat{n} = \frac{\vec{p}}{||\vec{p}||} \tag{11}$$

$$\hat{t} = \frac{(\hat{n}\times\vec{d})\times\hat{n}}{||(\hat{n}\times\vec{d})\times\hat{n}||} \tag{12}$$

The initial condition of $u'(\phi)$ is calculated geometrically. For a supplementary angle $\alpha = \pi - \xi$, a small increment of polar angle $\delta\phi$ gives rise to the first order change to u and subsequently a zeroth order u':

$$\begin{aligned}
u(\phi_0+\delta\phi) &= u(\phi_0) + u'(\phi_0)\delta\phi \\
&= u(\phi_0) + \vec{d}\cdot\hat{n} \\
&\Downarrow \\
u'(\phi_0) &= (\vec{d}\cdot\hat{n})/\delta\phi \\
&= \frac{(\vec{d}\cdot\hat{n})}{(\vec{d}\cdot\hat{t})/||\vec{p}_0||} \\
&= \frac{\vec{d}\cdot\hat{n}}{u(\phi_0)\vec{d}\cdot\hat{t}} \\
&= -\frac{1}{u(\phi_0)\tan(\alpha)}
\end{aligned} \tag{13}$$

We can certainly go to a higher order:

$$u(\phi_0+\delta\phi) = u(\phi_0) - \frac{1}{u(\phi_0)\tan(\alpha)}\delta\phi \tag{14}$$

$$u'(\phi_0+\delta\phi) = \frac{1}{2}(\frac{3}{2}u^2(\phi_0) - u(\phi_0))\delta\phi - \frac{1}{u(\phi_0)\tan(\alpha)}$$

In fact, this above first order is used for the code implementation.