Black hole graphic simulation

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I. MATHEMATICS

A. Schwarzschild metric

The variation of action gives rise to the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(g_{\alpha\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}) - \frac{1}{2}g_{\mu\nu,\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} = 0 \tag{1}$$

The simplest, spherical symmetric metric of a Schwarzschild black hole is given as:

$$\mathrm{d}s^2 = \left(1 - \frac{R_s}{r}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{R_s}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2(\theta)\mathrm{d}\phi^2 \tag{2}$$

The (pseudo-) energy and (pseudo-) angular-momentum conservation can be obtained from setting $\alpha = 0, 3$ in the geodesic eq.(1). These are:

$$\left(1 - \frac{1}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\sigma} = e \tag{3}$$

$$r^2\sin^2(\theta)\frac{\mathrm{d}\phi}{\mathrm{d}\sigma} = l \tag{4}$$

When $\alpha = 2$, the geodesic eq.(1) reads

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}) = r^2\sin(\theta)\cos(\theta)\mathrm{d}\phi^2 \tag{5}$$

Thus, the orbit remains in the plane if taking polar angle $\frac{\theta}{2} = 0$, hence eq.(2) becomes:

$$\mathrm{d}s^2 \longrightarrow \left(1 - \frac{1}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{1}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\phi^2 \tag{6}$$

Without loss of generality, a natural selection of unit 1 is applied as $c = 1$ and the Schwarzschild radius $R_s = 2GM/c^2 = 1$.

1. Photon orbit

The photon orbit is a null curve $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \mathrm{d}s^2 = 0$, Combining the energy and angular moment term with the metric ends up with:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\sigma}\right)^2 = e^2 - \left(\frac{1}{r^2} - \frac{1}{r^3}\right)l^2 \tag{7}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}\sigma^2} = \frac{l^2}{r^3}\left(1 - \frac{3}{2r}\right) \tag{8}$$

which indicates a stable circular orbit of light takes a radius of $r_{circ.} = \dfrac{3R_s}{2}$.

Rewriting these two photon orbit equations with ϕ instead of the affine parameter σ, one gets:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{e^2}{l^2} + (u^3 - u^2) \tag{9}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}\phi^2} = \frac{3}{2}u^2 - u \tag{10}$$

2. Massive particles

For massive particles, $\mathrm{d}s^2 = \mathrm{d}\tau^2$. Replacing σ with τ in the geodesic, energy and angular momentum equations, with the notation $r \to \frac{1}{u}$, one gets:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{r^4e^2}{l^2} - \left(1 - \frac{1}{r}\right)\left(\frac{r^4}{l^2} + r^2\right)$$

or,

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = (u')^2 = \frac{e^2}{l^2} - (1 - u)(l^{-2} + u^2) \tag{11}$$

We see the only difference between the massive particle orbit and photon is an additional l^{-2} term (if we ignore the different form of energy and angular momentum, that essentially changes $e \to e/m$ and $l \to l/m$). This equation derived from the metric resembles an energe conservation equation, where $K = \frac{1}{2}(u')^2$ is the kinetic term and $V = \frac{1}{2}(1 - u)(l^{-2} + u^2)$ is the minus potential. Hence the gradient of the potential w.r.t. u gives rise to the acceleration $u''(\phi)$:

$$u''(\phi) = \nabla_u V = \frac{3}{2}u^2 - u + \frac{1}{2}l^{-2} \tag{12}$$

One can also use the chain rule to derive the same relation.

This second-order differential equation is preferred than the first-order one in terms of the computational efficiency, since square root is computational heavy.

B. Geometry

In the plane of equator [Fig.1], a camera is placed at $\vec{p}_0$ and it traces back light at a direction $\vec{d}$. $\vec{d}$ has a "*seen angle*" ξ with $\vec{p}_0$. Assuming this light is deviated under the

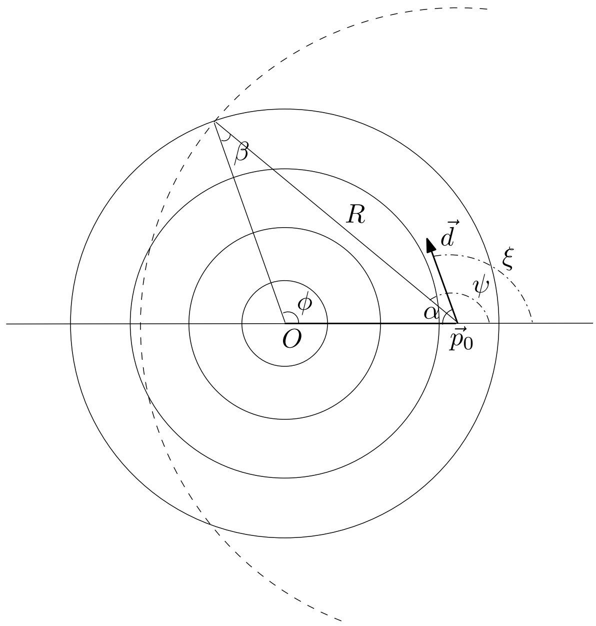

FIG. 1. The black hole sits at the centre O of the polar coordinate. The camera located at $\vec{p}_0$, coinciding with $\phi_0 = 0$, traces rays orienting $\vec{d}$, which is at a seen angle $\xi = \pi - \alpha$ with $\vec{p}_0$. The deviated angle ψ specifies the angle when the ray travels a radial distance R from the camera. The polar angle of this updated location is ϕ, and the angle opposing $\vec{p}_0$ is β.

metric, and travels a radial distance R from the camera. The updated location has a polar angle ϕ, and is with a "*deviate angle*" ψ seen by the camera. Trigonometry tells us that

$$\frac{||\vec{p}_0||}{\sin(\beta)} = \frac{R}{\sin(\phi)} \tag{13}$$

$$\Longrightarrow \psi = \phi + \beta = \phi + \arcsin\left(\frac{R}{||\vec{p}_0||\sin(\phi)}\right) \tag{14}$$

To solve for equation 12, two initial conditions are taken as:

$$u(\phi_0) = \frac{1}{||\vec{p}||} \tag{15}$$

$$u'(\phi_0) = -\frac{1}{u(\phi_0)\tan(\alpha)} \tag{16}$$

where we define the orthonormal basis $\hat{n}$ and $\hat{t}$ as:

$$\hat{n} = \frac{\vec{p}}{||\vec{p}||} \tag{17}$$

$$\hat{t} = \frac{(\hat{n}\times\vec{d})\times\hat{n}}{||(\hat{n}\times\vec{d})\times\hat{n}||} \tag{18}$$

The initial condition of $u'(\phi)$ is calculated geometrically. For a supplementary angle $\alpha = \pi - \xi$, a small

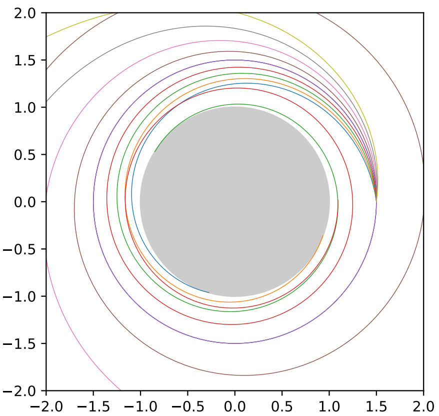

FIG. 2. The equatorial plane view of photon orbits. The photon emits from 1.5 R_s (photon sphere) at different angles and orbits until it runs off the view or falls into the even horizon. Notice the photon sphere at 1.5 R_s

increment of polar angle $\delta\phi$ gives rise to the first order change to u and subsequently a zeroth order u':

$$\begin{aligned}
u(\phi_0+\delta\phi) &= u(\phi_0) + u'(\phi_0)\delta\phi \\
&= u(\phi_0) + \vec{d}\cdot\hat{n} \\
&\Downarrow \\
u'(\phi_0) &= (\vec{d}\cdot\hat{n})/\delta\phi \\
&= \frac{(\vec{d}\cdot\hat{n})}{(\vec{d}\cdot\hat{t})/||\vec{p}_0||} \\
&= \frac{\vec{d}\cdot\hat{n}}{u(\phi_0)\vec{d}\cdot\hat{t}} \\
&= -\frac{1}{u(\phi_0)\tan(\alpha)}
\end{aligned} \tag{19}$$

We can certainly go to a higher order:

$$u(\phi_0+\delta\phi) = u(\phi_0) - \frac{1}{u(\phi_0)\tan(\alpha)}\delta\phi \tag{20}$$

$$u'(\phi_0+\delta\phi) = \frac{1}{2}(\frac{3}{2}u^2(\phi_0) - u(\phi_0))\delta\phi - \frac{1}{u(\phi_0)\tan(\alpha)}$$

In fact, this above first order is used for the code implementation.