Chapter II: Coulomb's Law

Chenyi Zhu

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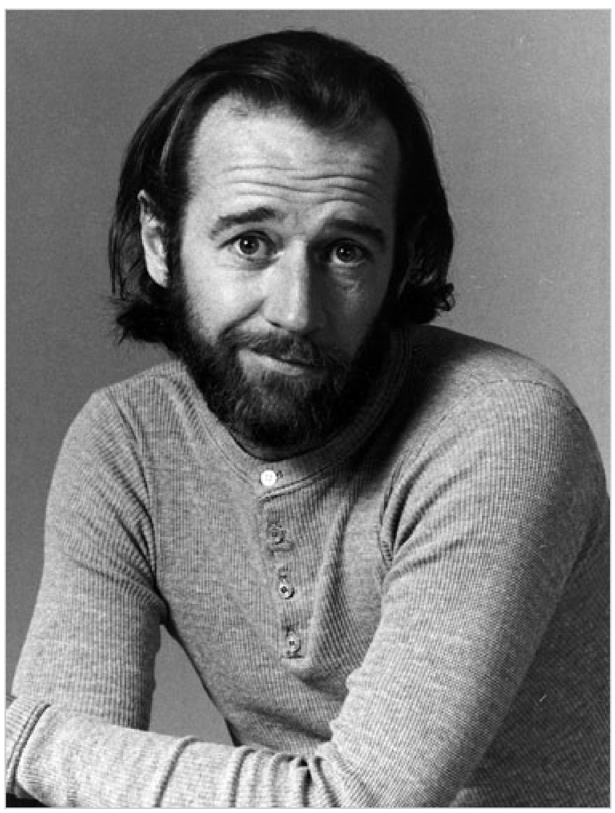
"Electricity is really just organized lightening." - George Carlin



1 Electric Charge, Coulomb's Law, and Superposition.

1.1 Electric Charge

Introducing constant, the magnitude of the charge of an electron:

$$|e^-| = 1.602 \times 10^{-19}\, C$$

Note: The **TOTAL** electric charge of an isolated system is conserved.

1.2 Coulomb's Law

Now we look at Coulomb's law, which defines the electric force as:

$$\boxed{\vec{F} = k_e \frac{q_1 q_2}{r^2}\,\hat{r} \qquad k_e = \frac{1}{4\pi\varepsilon_0} = 8.9875 \times 10^9\, N \cdot m^2/C^2} \tag{1}$$

where r is the distance between two particles and k_e the Coulomb's constant. Bonus points for knowing **permittivity of free space**:

$$\varepsilon_0 = 8.85 \times 10^{-12}\, C^2/N \cdot m^2$$

This will be useful later on when we work with capacitors.

1.3 Superposition Principle

Working with electric forces is as simple as just adding & subtracting vectors. Tip for solving problems that involve a set of charges n and asks for the force an a particular particle j: using superposition principle:

$$\vec{F}_j = \sum_{i=1}^{n} \vec{F}_{i \neq j, j}$$

and remember, this is a sum of vectors, not scalar values!

2 Electric Field.

2.1 Electric Field

The electrostatic force, just like gravitational force, is able to act on an object from a distance (r^2) even if the objects are not touching each other. To explain this phenomenon, we say that one charge creates a field in space which acts on other charges.

While the charge $+Q$ creates a field everywhere in space, its strength varies. To quantify the strength of the field created by the point charge, we propose a test particle with infinitesimal positive charge $+q$, and measure the force it experiences (hypothetically). The electric field is defined as:

$$\boxed{\vec{E} \equiv \lim_{q \to 0} \frac{\vec{F}_e}{q} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}} \tag{2}$$

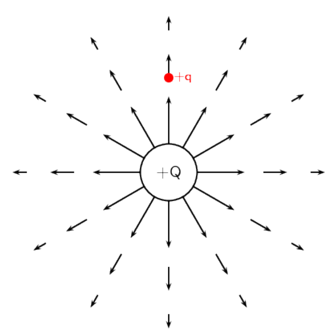


Figure 1: Test charge $+q$ in field produced by $+Q$.

and we say (based on field theory) that charge Q produces an electric field $\vec{E}$ which exerts a force $\vec{F}_e = q\vec{E}$ on q.

Note: we take q to be infinitesimally small so that the field created by our test charge q does not interfere with the “source charge” Q.

If we have a group of source charges q_i, the total electric field is the vector sum of the fields of individual charges.

$$\vec{E}_{total} = \sum_i \vec{E}_i = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r^2} \hat{r}$$

2.2 Electric Field Lines

For positive source charges, the electric field lines point outwards from the point charge, while for negative source charges the electric field lines point inwards to the point charge. Here, three scenarios are presented:

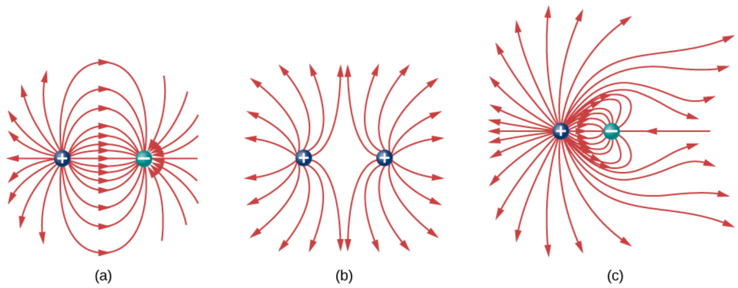


Figure 2: Three possible scenarios of fields' interaction.

In (a), we see two point charge sources of opposite signs and the field line created from this interaction, and we call the pair an **electric dipole**. We can also tell immediately that the two charges are of the same magnitude because they have the same field line density (same number of field lines). In contrast, we observe in (c) that the positive charge is of much higher magnitude than the negative charge, causing the field lines to concentrate closer to the positive charge and producing a stronger field.

2.3 Summary

I. The direction of the electric field vector $\vec{E}$ at a point is tangent to the field lines.

II. The number of lines per unit area through a surface perpendicular to the line is devised to be proportional to the magnitude of the electric field in a given region.

III. The field lines must begin on positive charges (or ad infinitum) and then terminate on negative charges (or ad inifinitum).

IV. The number of lines that originate from a positive charge or terminating on a negative charge must be proportional to the magnitude of the charge.

V. Field lines CANNOT cross each other; otherwise the field would be pointing in two different directions at the same point.

3 Electric Dipole.

3.1 Dipole

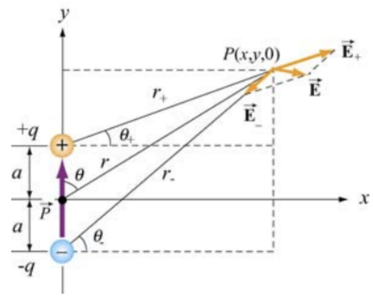


Figure 3: The electric dipole.

As mentioned in the previous section, an electric dipole consists of two equal but opposite charges. Say we have two point charges $+q$ and $-q$ along the y-axis, separated by a distance d. Then the dipole moment vector $\vec{p}$ points from $-q$ to $+q$ is:

$$\boxed{\vec{p} = 2aq\,\hat{j}} \tag{3}$$

where the magnitude of the vector is $p = 2qa$, $q > 0$. For a system of N charges, the overall electric dipole vector is defined as:

$$\vec{p} \equiv \sum_{i=1}^{N} q_i \vec{r}_i$$

where $\vec{r}_i$ is the position vector of charge q_i.

3.2 Electric Field of a Dipole

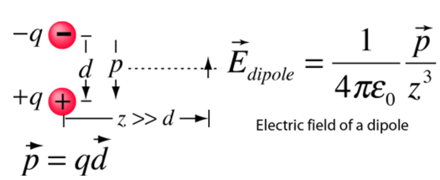


Figure 4: Electric field of a dipole.

Note: The distance between $+q$ and $-q$ is MUCH smaller than the dipole's distance to the test charge.

4 Charge Densities.

So far we have only seen point charges and, at most, a dipole. The electric fields due to these are fairly straightforward to compute with the superposition principle, but what if we encounter a large number of charges distributed (not necessarily evenly) in some region in space? Here we introduce the novel concept of **charge density**. This will come up a lot in problems!

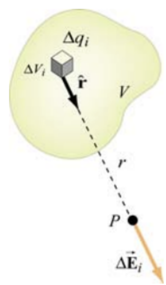


Figure 5: Electric field due to small charge element Δq_i at P.

4.1 Volume Charge Density

Let's find the electric field at point P. Consider small volume element ΔV_i and the small, corresponding charge element Δq_i. Assume the distances among between the charges within ΔV_i is MUCH smaller compared to the distance from ΔV_i to point P (which is r in our case). Make ΔV_i to be infinitesimally small and obtain:

$$\boxed{\rho(\vec{r}) = \lim_{\Delta V_i \to 0} \frac{\Delta q_i}{\Delta V_i} = \frac{dq}{dV}} \tag{4}$$

where the dimension of $\rho(\vec{r})$ is charge/unit volume (C/m^3). To find the total charge of the volume V:

$$Q = \sum_i \Delta q_i = \iiint_V \rho(\vec{r})\, dV$$

Note: $\rho(\vec{r})$ is analogous to mass density $\rho_m(\vec{r})$ we have seen before, given a large number of atoms are densely packed within a volume V:

$$M = \iiint_V \rho_m(\vec{r})\, dV$$

4.2 Surface Charge Density

Now that we got volume over with, the rest are fairly standard. If we have charge spread out across a surface S of area A with a surface charge density σ:

$$\boxed{\sigma(\vec{r}) = \frac{dq}{dA}} \tag{5}$$

where $\sigma(\vec{r})$ has units charge/unit area (C/m^2). Total charge over surface:

$$Q = \iint_S \sigma(\vec{r})\, dA$$

4.3 Line Charge Density

Moving on to the simplest one-dimensional geometry, a line. Suppose we have a certain amount of charge distributed across a line of length l, then the linear charge density λ is:

$$\boxed{\lambda(\vec{r}) = \frac{dq}{dl}} \tag{6}$$

where $\lambda(\vec{r})$ has units charge/unit length (C/m). To obtain the total charge of the line, do:

$$Q = \int_{line} \lambda(\vec{r})\, dl$$

Note: if the charge is uniformly distributed throughout the geometry, then the charge densities (λ, σ, ρ) become uniform as well (otherwise you may end up with step functions depending on your charge density).

4.4 Continuous Charge Distributions

If we take a small charge element dq, its electric field according to Coulomb's law would be:

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2}\, \hat{r}$$

and therefore, all you have to do to find the electric field $\vec{E}$ at a point P is to integrate $\frac{dq}{r^2}\, \hat{r}$. Do your conversion of dq based on the charge density formulae (charge densities are typically given in any problem, or the derivation will be straightforward), and also convert r^2 and $\hat{r}$ to standard Cartesian units of $\hat{x}, \hat{y}, and \hat{z}$ if necessary (remember that it is not always easy to work these problems out in Cartesian coordinates, in which case it would be easiest to keep r as it is).

4.5 General Strategies

I. You should first draw out the direction of the vector and note its magnitude. If we have a symmetric shape, think about whether the vectors can cancel in some way, which will save you a massive amount of computation and time.

II. Start with $d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\,\hat{r}$

III. Rewrite dq and plug back into $d\vec{E}$:

$$dq = \begin{cases} \lambda\, dl & \text{(length)} \\ \sigma\, dA & \text{(area)} \\ \rho\, dV & \text{(volume)} \end{cases} \tag{7}$$

IV. Choose one of the following coordinate systems, whichever suits the problem:

	Cartesian (x, y, z)	Cylindrical (ρ, ϕ, z)	Spherical (r, θ, ϕ)
dl	$dx,\ dy,\ dz$	$d\rho,\ \rho d\phi,\ dz$	$dr,\ r\,d\theta,\ r\sin\theta\, d\phi$
dA	$dx\,dy,\ dy\,dz,\ dz\,dx$	$d\rho\, dz,\ \rho\, d\phi\, dz,\ \rho\, d\phi\, d\rho$	$r\,dr\,d\theta,\ r\sin\theta\, dr\, d\phi,\ r^2\sin\theta\, d\theta\, d\phi$
dV	$dx\,dy\,dz$	$\rho\, d\rho\, d\phi\, dz$	$r^2\sin\theta\, dr\, d\theta\, d\phi$

Figure 6: Coordinate systems.

V. Rewrite $d\vec{E}$ in terms of the chosen coordinate system and integrate.

4.6 Examples

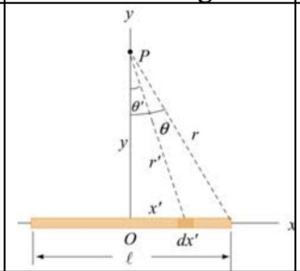
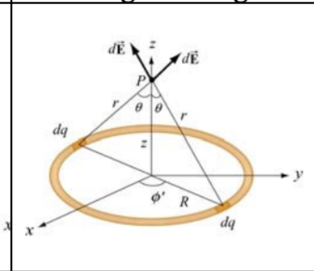
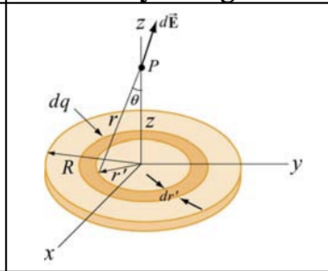
	Line charge	Ring of charge	Uniformly charged disk
Figure			
(2) Express dq in terms of charge density	$dq = \lambda\, dx'$	$dq = \lambda\, d\ell$	$dq = \sigma\, dA$
(3) Write down dE	$dE = k_e \dfrac{\lambda\, dx'}{r'^2}$	$dE = k_e \dfrac{\lambda\, dl}{r^2}$	$dE = k_e \dfrac{\sigma dA}{r^2}$
(4) Rewrite r and the differential element in terms of the appropriate coordinates	dx' $\cos\theta = \dfrac{y}{r'}$ $r' = \sqrt{x'^2 + y^2}$	$d\ell = R\, d\phi'$ $\cos\theta = \dfrac{z}{r}$ $r = \sqrt{R^2 + z^2}$	$dA = 2\pi r'\, dr'$ $\cos\theta = \dfrac{z}{r}$ $r = \sqrt{r'^2 + z^2}$
(5) Apply symmetry argument to identify non-vanishing component(s) of dE	$dE_y = dE\cos\theta$ $= k_e \dfrac{\lambda y dx'}{(x'^2 + y^2)^{3/2}}$	$dE_z = dE\cos\theta$ $= k_e \dfrac{\lambda R z\, d\phi'}{(R^2 + z^2)^{3/2}}$	$dE_z = dE\cos\theta$ $= k_e \dfrac{2\pi\sigma z r'\, dr'}{(r'^2 + z^2)^{3/2}}$
(6) Integrate to get E	$E_y = k_e \lambda y \displaystyle\int_{-\ell/2}^{+\ell/2} \frac{dx}{(x^2+y^2)^{3/2}}$ $= \dfrac{2k_e\lambda}{y} \dfrac{\ell/2}{\sqrt{(\ell/2)^2 + y^2}}$	$E_z = k_e \dfrac{R\lambda z}{(R^2+z^2)^{3/2}} \oint d\phi'$ $= k_e \dfrac{(2\pi R\lambda) z}{(R^2+z^2)^{3/2}}$ $= k_e \dfrac{Qz}{(R^2+z^2)^{3/2}}$	$E_z = 2\pi\sigma k_e z \displaystyle\int_0^R \frac{r'\, dr'}{(r'^2+z^2)^{3/2}}$ $= 2\pi\sigma k_e \left(\dfrac{z}{\lvert z\rvert} - \dfrac{z}{\sqrt{z^2+R^2}} \right)$

Figure 7: Most common examples.

Note: Remember your kinematic equations! There may be questions that involve an object subject to the influence of both the gravitational AND electrostatic forces. Look here for a comprehensive list.

5 Exercises.

5.1 Electric Forces

Problem 1: Calculate the ratio of the electrostatic to gravitational interaction forces between two electrons, or two protons. At what value of $\frac{q}{m}$ would the two forces be equal?

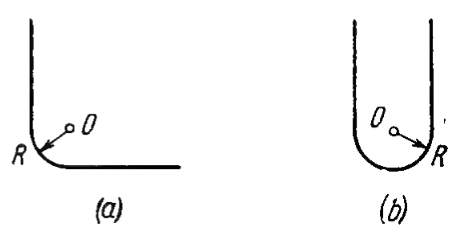
Problem 2: Two positive charges q_1 and q_2 are located at the points with radial vectors $\vec{r}_1$ and $\vec{r}_2$. Find a negative charge q_3 and a radial vector $\vec{r}_3$ of the point at which it has to be placed for the force acting on each of the three charges to be 0.

Problem 3: A system consists of a thin charged wire ring of radius R and a long uniformly charged thread oriented along the axis of the ring, with one of its ends coinciding with the center of the ring. The total charge of the ring is q, and the charge density of the thread if λ. Find the interaction force between the ring and the thread.

5.2 Electric Fields

Problem 4: A thin half-ring of radius R is uniformly charged with a total charge q. Find the electric field at the curvature center of this half-ring.

Problem 5: A thread carrying a uniform charge density λ has configurations shown in Figure 8 a) and b). Assuming the curvature radius R to be considerably less than the length of the thread, find the electric field at the point O.



Problem 6: A sphere of radius r carries a surface charge of density $\sigma = ar$, where a is a constant vector and r the radius vector of a point on the sphere relative to its center. Find the electric field at the center of the sphere.