Chapter III: Electric Potential

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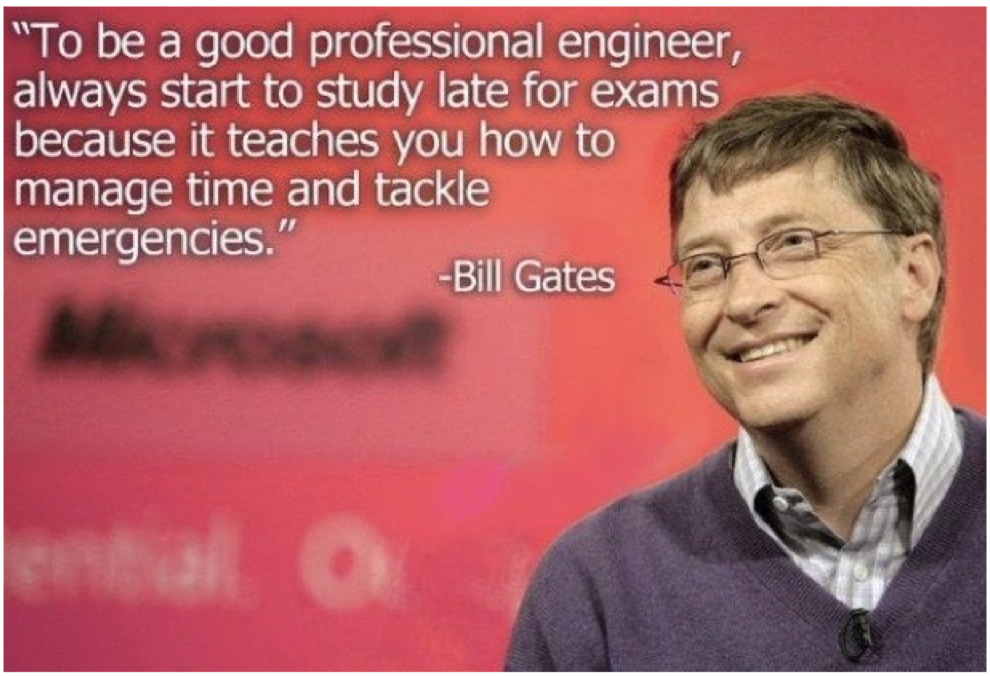

1 Review: Gravitational Potential and Potential Energy.

We will start reviewing gravitational potential energy first and move on to gravitational potential. Note that the main ideas behind these concepts apply also to electric potential and potential energy, so please do not take this review lightly!

1.1 Gravitational Potential Energy

We have seen in Chapter I that:

$$\vec{F}_g = -G\frac{Mm}{r^2}\,\hat{r}$$

where $G = 6.67 \times 10^{-11}\, N \cdot m^2/kg^2$, where $\hat{r}$ points radially outward from COM. Assuming Earth to be spherical, gravitational field $\vec{g}$ is defined as gravitational force per unit mass, is given by:

$$\vec{g} = \frac{\vec{F}_g}{m} = -\frac{GM}{r^2}\,\hat{r}$$

and note that $\vec{g}$ only depends on M, the mass that creates the field, and r, distance from M.

Imagine that we are on a spaceship and trying to move from point A to B, under the influence of the gravitational field.

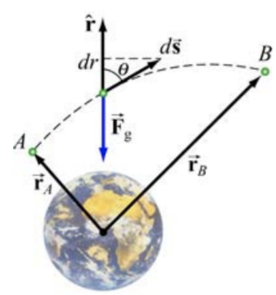

Figure 1: Potential difference.

The work done by gravity to move us (of mass m) could be computed by the line integral:

$$\boxed{W_g = \int \vec{F}_g \cdot d\vec{s} = \int_{r_A}^{r_B} \left(-\frac{GMm}{r^2}\right) dr = \frac{GMm}{r}\bigg|_{r_A}^{r_B} = GMm\left(\frac{1}{r_B} - \frac{1}{r_A}\right)} \tag{1}$$

which confirms what we have already known about work: it is independent of the path taken and only depends on the path taken. Think about what happends when you break the non-linear path between A and B down into small vectors parallel to the path $d\vec{s}$. Because work is done only when the direction of the force applied is parallel to the direction of the path taken, we are able to convert $d\vec{s}$ to dr.

Here is another case: when we are not on the spaceship, but are near the Earth's surface. The gravitational field is pretty much constant, with a magnitude $g = GM/r_E^2 \approx 9.8\,m/s^2$.

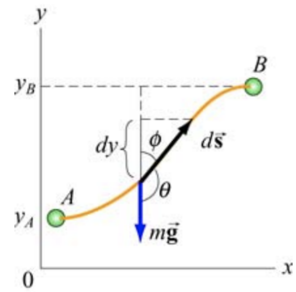

Figure 2: Near Earth's surface.

The work done by gravity moving an object from A to B is given by the following:

$$\boxed{W_g = \int \vec{F}_g \cdot d\vec{s} = \int_A^B mg\cos\theta\, ds = -\int_A^B mg\cos\phi\, ds = \int_{y_A}^{y_B} mg\, dy = -mg(y_B - y_A)} \tag{2}$$

Note: In our case, if the line integral is closed curve, then the integral evaluates to 0; no work is done. Thus we say that the gravitational force is a **conservative** force. More generally, we say a force is conservative if its line integral around a closed loop vanishes:

$$\boxed{\oint \vec{F} \cdot d\vec{s} = 0} \tag{3}$$

We know that work is associated with potential energy. When dealing with a conservative force, it is often convenient to introduce the concept of potential energy U:

$$\boxed{\Delta U = U_B - U_A = -\int_A^B \vec{F} \cdot d\vec{s} = -W} \tag{4}$$

where W is the work done **by** the force **on** the object. Let us rewrite the potential energy:

$$U_g = -\frac{GMm}{r} + U_0$$

Note: Its relationship with equations (4) and (1). I suggest that you try deriving this last expression yourself! I will start you off: rearrange $U_g - U_0 = -\frac{GMm}{r} \rightarrow \ldots$

1.2 Gravitational Potential

Now we are going to derive gravitational potential from ΔU:

$$\boxed{\Delta V_g = \frac{\Delta U_g}{m} = -\int_A^B \frac{\vec{F}_g}{m} \cdot d\vec{s} = -\int_A^B \vec{g} \cdot d\vec{s}} \tag{5}$$

where ΔV_g is the negative work done per unit mass by gravity to move a particle from A to B.

2 Electric Potential and Potential Energy.

2.1 Electric Potential

Now, after we have reviewed gravitation, electrostatics is very similar. It has a distance dependency of r^{-2}, and the electric force is also a **conservative** force. We define the electric potential difference between two points A and B as:

$$\boxed{\Delta V = -\int_A^B \frac{\vec{F}_e}{q_0} \cdot d\vec{s} = -\int_A^B \vec{E} \cdot d\vec{s}} \tag{6}$$

where q_0 is the infinitesimal test charge. ΔV represents the work/unit charge to move the test charge from point A to B, *without changing its kinetic energy*. The units of electric potential is volt $(1\,V = 1\,J/C)$.

2.2 Electric Potential Energy

Please do not confuse the concepts of potential and potential energy! They represent very different things. However,, they do have a correlation:

$$\boxed{\Delta U = q_0 \Delta V} \tag{7}$$

2.3 Electric Potential due to Point Charges

Consider point charge Q, and the electric field it produces $\vec{E} = (Q/4\pi\varepsilon_0 r^2)\hat{r}$.

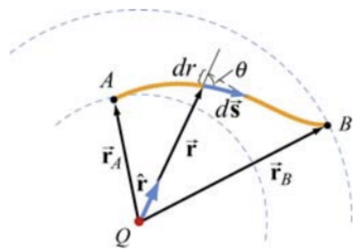

Figure 3: Potential difference due to point charge Q.

Apply $\hat{r} \cdot d\vec{s} = ds\cos\theta = dr$:

$$\boxed{\Delta V = V_B - V_A = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2}\, dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)} \tag{8}$$

Again, note path independence.

When we discuss the concept of "potential" at a given point, it is important that we pick the right reference point. Usually, we want the potential at our reference point to be 0 (standard m.o.), which means the reference point will be infinitely far away:

$$V_P = -\int_\infty^P \vec{E} \cdot d\vec{s}$$

and with this reference, our potential simply becomes:

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon}\frac{Q}{r}} \tag{9}$$

Note: This is a scalar quantity and does not specify direction. This means applying superposition principle has never been easier. Now just add up the potential due to each point charges without worrying about vectors.

2.4 Continuous Charge Distribution

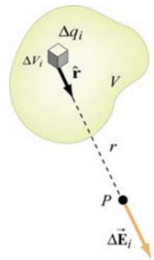

Figure 4: Continuous charge distribution due to bean.

Consider a solid charged bean. The potential at point P can be found by summing over all individual infinitesimal charges dq:

$$dV = \frac{1}{4\pi\varepsilon}\frac{dq}{r} \Rightarrow \boxed{V = \frac{1}{4\pi\varepsilon}\int \frac{dq}{r}} \tag{10}$$

2.5 Deriving the Electric Field from Potential

A simpler but rigorous definition of electric potential is: consider two points separated by a small distance $d\vec{s}$, we will obtain the following differential: $\Delta V = -\vec{E}\cdot d\vec{s}$, and in Cartesian coordinates we have: $\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$ and $d\vec{s} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} \implies \vec{E} = E_x dx + E_y dy + E_z dz$, and:

$$\boxed{E_x = -\frac{\partial V}{\partial x}, E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z}} \tag{11}$$

using the gradient ("del") operator. If our charge distribution has a spherical symmetry, then the resulting electric field can be a function of the radial distance r, and:

$$\boxed{\vec{E} = E_r\hat{r} = -\left(\frac{dV}{dr}\right)\hat{r}} \tag{12}$$

2.6 Gradient and Equipotentials

We have all seen contour maps in one way or another. Here is an example of its usage in geography as topographic map:

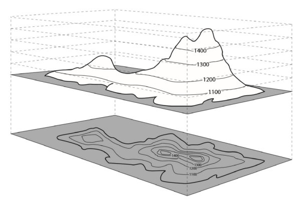

Figure 5: Dissecting a mountain.

Given the contour map in 2D, we can say several things about the structure of the mountain in 3D, e.g. where the peaks are, steepness of the mountain, etc. We can do the same with electricity, illustrated by the blue dotted lines in the figure below:

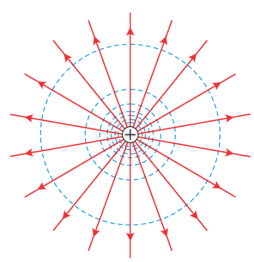

Figure 6: Equipotential surface of a single positive point charge.

Here are the general properties of equipotential surfaces:

- The electric field lines are perpendicular to the equipotentials and point from higher to lower potentials.
- By symmetry, the equipotential surfaces produced by the a point charge form a family of concentric spheres, and for a constant electric field, a family of planes perpendicular to the field lines.
- The tangential component of the electric field along the equipotential surface is 0, o/w non-vanishing work would be done to move the charge along the surface.
- No work is required to move particles along one equipotential surface.

3 Exercises.

3.1 Warm-up

Problem 1: Recall the chart given to you in Chapter 2, and some homework problems you have seen. Now, instead of finding the electric field at a given point P, I will ask you to find the electric potential at P, all the conditions remain the same.

- Uniformly charged line, both finite and infinite.
- Uniformly charged ring.
- Uniformly charged disk.

3.2 More Practice

Problem 2: Find the electric field potential and strength at the center of a hemisphere of radius R with uniform surface charge density σ.

Problem 3: Determine the electric field strength vector if the potential of this field depends on x, y coordinates as: a) $\varphi = a(x^2 + y^2)$; and b) $\varphi = axy$ where a is a constant. Draw the approximate shapes of these fields using lines of force (in the x, y plane).

Problem 4: A charge q is uniformly distributed over the volume of a sphere of radius R. Assuming permittivity to be equal to unity throughout, find the potential

- at the center of the sphere;
- inside the sphere as a function of the distance r from its center.

Problem 5: Two thin parallel threads carry a uniform charge with linear charge densities λ and $-\lambda$, separated by a distance l. Find the potential of the electric field and its magnitude at the distance $r \gg l$ at the angle θ to the vector $\vec{l}$.

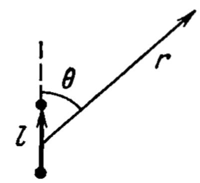

Figure 7: Two parallel lines.

Problem 6: Two coaxial rings, each of radius R, made of thin wire are separated by a small distance l ($l \ll R$) and carry the charges q and $-q$. Find the electric field magnitude and potential at the axis of the system as a function of the x coordinate. Plot the functions and observe what happens when $|x| \gg R$.

Problem 7: Determine the potential $\varphi(x, y, z)$ of an electrostatic field $\vec{E} = ay\hat{\imath} + (ax + bz)\hat{\jmath} + by\hat{k}$, where a and b are constants and $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are unit vectors in Cartesian coordinates.