

Stochastic Volatility Models and Simulation

Applied Stochastic Processes (FIN 514)

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Stochastic Volatility (SV) Models

- The price process (martingale):

$$\frac{dS_t}{S_t^\beta} = \sigma_t dW_t = \sigma_t(\rho dZ_t + \rho_* dX_t), \quad \text{for } \rho_* = \sqrt{1 - \rho^2}.$$

BSM-base: $\beta = 1$, normal-base: $\beta = 0$.

For the models except SABR, the base model is BSM (i.e., $\beta = 1$).

- The (stochastic) volatility process may vary:

$$d\sigma_t = a(t, \sigma_t)dt + b(t, \sigma_t)dZ_t,$$

- The correlation between the two Brownian motions:

$$dW_t dZ_t = \rho dt.$$

The correlation explains the *leverage effect*: equity volatility increases as price goes down.

Various SV models

- SABR model [[Hagan et al., 2002](#)]:

$$d\sigma_t = \nu\sigma_t dZ_t.$$

- [Heston \[1993\]](#) model (CIR process):

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dZ_t.$$

- 3/2 model [[Heston, 1997](#), [Lewis, 2000](#)]:

$$dv_t = \kappa v_t(\theta - v_t)dt + \nu v_t^{3/2} dZ_t.$$

- GARCH diffusion model (relatively new):

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t dZ_t.$$

- OU volatility model [[Li and Wu, 2019](#)]:

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu dZ_t.$$

Simulation scheme for the SV models

- Time discretization (Euler/Milstein):
 - Almost no restriction but computationally expensive.
- Conditional MC:
 - Can skip the simulation of price S_t . (Simulate volatility σ_t only)
 - The final price S_T should be expressed by σ_T and $V_T = \int_0^T \sigma_t^2 dt$.
- Exact Simulation:
 - No need for time-discretization: jump from $t = 0$ to T .
 - The conditional MC condition + should be able to sample σ_T and V_T .
 - σ_T follows a well-known distribution and the (conditional) Laplace transform of $V_T | v_T$ should be analytically available (Heston, 3/2, SABR)

$$E(e^{-sV_T} | v_T = x) = f(s, x)$$

If Laplace transform is known, the CDF of $V_T | v_T$ can be obtained by (numerical) inverse-transform although it may be computationally expensive.

- There are other lucky cases too: Normal SABR [[Choi et al., 2019](#)].

Conditional and Exact MC

Conditional MC:

- 1) Simulate path of σ_t for $(0 \leq t \leq T)$
- 2) Obtain σ_T and V_T (trapezoidal / Simpson's rule)

Exact MC:

- 1) Sample v_T from the (well-known) distribution
- 2) Sample V_T from the (numerical) CDF of $V_T | v_T$.

In Common:

- 3) Obtain $E(S_T | v_T, V_T)$ and effective volatility (usually $\rho_* \sqrt{V_T/T}$)
- 4-1) Price sampling: draw normal / log-normal distribution
- 4-2) Option price: Bachelier / BSM option price formula with $S_0 := E(S_T | v_T)$ and $\sigma := \rho_* \sqrt{V_T/T}$.
Then, average over the simulations.

Heston model (conditional MC)

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dZ_t \quad (v_t = \sigma_t^2)$$

Integrating v_t ,

$$v_T - v_0 = \kappa(\theta T - V_T) + \nu \int_0^T \sqrt{v_t} dZ_t$$
$$\int_0^T \sigma_t dZ_t = \frac{1}{\nu} \left(v_T - v_0 - \kappa(T\theta - V_T) \right).$$

S_T is expressed by v_T and V_T (conditional MC possible)!

$$\log \left(\frac{S_T}{S_0} \right) = \frac{\rho}{\nu} (v_T - v_0 - \kappa(\theta - V_T)) - \frac{1}{2} V_T + \rho_* \sqrt{V_T} X_1$$

$$E(S_T \mid v_T, V_T) = S_0 \exp \left(\frac{\rho}{\nu} (v_T - v_0 - \kappa(\theta - V_T)) - \frac{\rho^2}{2} V_T \right)$$

Heston model (exact MC)

- It is known that v_T is distributed as a noncentral chi-square distribution, $\text{NCX2}(\delta, \lambda)$:

$$v_T = \frac{\nu^2(1 - e^{-\kappa T})}{4\kappa} \text{NCX2}(\delta, \lambda) = \frac{e^{-\kappa T/2}}{2\phi(\kappa)} \text{NCX2}(\delta, \lambda),$$

where the degrees of freedom δ and the noncentrality λ are

$$\delta = \frac{4\kappa\theta}{\nu^2}, \quad \lambda = \frac{4v_0\kappa e^{-\kappa T}}{\nu^2(1 - e^{-\kappa T})} = 2v_0e^{-\kappa T/2}\phi(\kappa) \text{ for } \phi(\kappa) = \frac{2\kappa/\nu^2}{\sinh(\kappa T/2)}$$

Standard library is available for drawing NCX2 random number.

- The conditional Laplace transform of V_T is also known [Pitman and Yor, 1982].
- Reference: Broadie and Kaya [2006] and Glasserman and Kim [2011] (improvement)

SABR model (conditional MC)

$$\frac{d\sigma_t}{\sigma_t} = \nu dZ_t \quad \Rightarrow \quad \sigma_T = \sigma_0 \exp \left(-\frac{1}{2} \nu^2 T + \nu Z_T \right)$$

Integrating σ_t ,

$$\nu \int_0^T \sigma_t dZ_t = \sigma_T - \sigma_0 = \sigma_0 \exp \left(-\frac{1}{2} \nu^2 T + \nu Z_T \right) - \sigma_0$$

S_T is expressed by σ_T and V_T (conditional MC possible) !

$$S_T = S_0 + \frac{\rho}{\nu} (\sigma_T - \sigma_0) + \rho_* \sqrt{V_T} X_1$$

$$\log \left(\frac{S_T}{S_0} \right) = \frac{\rho}{\nu} (\sigma_T - \sigma_0) - \frac{1}{2} V_T + \rho_* \sqrt{V_T} X_1$$

SABR Model (exact MC)

- σ_T is distributed by a log-normal distribution. Sampling is trivial.
- The conditional Laplace transform of $1/V_T$ is also known:

$$E\left(e^{-s/V_T} \middle| v_T\right) = \exp\left(-\frac{\phi_x(s)^2 - x^2}{2T}\right)$$

where $\phi_x(s) = \text{acosh}(se^{-x} + \cosh(x))$ and $v_T = \exp(\nu x)$

- From above, we can sample $1/V_T$ and get V_T .
- Reference: [Cai et al. \[2017\]](#)

3/2 model (conditional MC)

$$dv_t = \kappa v_t(\theta - v_t)dt + \nu v_t^{3/2} dZ_t.$$

The change of variable, $x_t = 1/v_t$ yields (a good Itô calculus exercise!)

$$dx_t = -\frac{dv_t}{v_t^2} + \frac{(dv_t)^2}{v_t^3} = (\kappa + \nu^2 - \kappa\theta x_t)dt - \nu\sqrt{x_t} dZ_t.$$

This is same as v_t in Heston model with new parameters:

$$\nu' = -\nu, \quad \kappa' = \kappa\theta, \quad \text{and} \quad \theta' = (\kappa + \nu^2)/(\kappa\theta).$$

We can S_T as a function of V_T and v_T (conditional MC possible)!

$$d\log(x_t) = \left(\frac{\kappa + \nu^2/2}{x_t} - \kappa\theta \right) dt - \frac{\nu}{\sqrt{x_t}} dZ_t$$

$$\int_0^T \frac{1}{\sqrt{x_t}} dZ_t = \frac{1}{\nu} \left(\log\left(\frac{x_0}{x_T}\right) + (\kappa + \nu^2/2)V_T - \kappa\theta T \right),$$

$$\log\left(\frac{S_T}{S_0}\right) = \frac{\rho}{\nu} \left(\log\left(\frac{v_T}{v_0}\right) - \kappa \left(T\theta - \left(1 + \frac{\nu^2}{2\kappa}\right) V_T \right) \right) - \frac{1}{2}V_T + \rho_*\sqrt{V_T} X_1$$

3/2 model (exact MC)

- From Heston model, $1/v_T$ is distributed as a noncentral chi-square distribution, $\text{NCX2}(\delta', \lambda')$ where the degrees of freedom δ' and the noncentrality λ' are

$$\delta' = \frac{4\kappa'\theta'}{\nu^2}, \quad \lambda = \frac{4\kappa' e^{-\kappa'T}}{v_0\nu^2(1 - e^{-\kappa'T})}.$$

Standard library is available for drawing NCX2 random number.

- The conditional Laplace transform of V_T is also known.
- Reference: [Baldeaux \[2012\]](#)

OUSV model (conditional MC). 2019 exam question

Let $U_T = \int_0^T \sigma_t dt$ and $V_T = \int_0^T \sigma_t^2 dt$.

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu dZ_t.$$

$$d\sigma_t^2 = 2\sigma_t d\sigma_t + (d\sigma_t)^2 = (\nu^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\nu\sigma_t dZ_t$$

$$\sigma_T^2 - \sigma_0^2 = \nu^2 T + 2\kappa(\theta U_T - V_T) + 2\nu \int_0^T \sigma_t dZ_t$$

$$\int_0^T \sigma_t dZ_t = \frac{1}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\nu}{2}T - \frac{\kappa\theta}{\nu}U_T + \frac{\kappa}{\nu}V_T$$

S_T is expressed by v_T and V_T (conditional MC possible)!

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \frac{1}{2}V_T \\ &= \frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{1}{2}\right)V_T + \rho_*\sqrt{V_T} X_1\end{aligned}$$

$$S_0 := E(S_T) = S_0 \exp\left(\frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{\rho^2}{2}\right)V_T\right)$$

$$\sigma_{BS} := \rho_*\sqrt{V_T/T}.$$

Project Suggestion: *Almost* Exact MC (by Choi)

- General scheme:
 - Implementing existing paper is OK:
 - Improving Euler / Milstein scheme? or exact simulation?
 - Or try something new (see below):
- Simulation for **GARCH diffusion**:

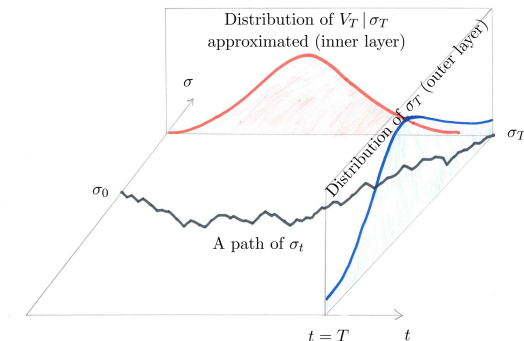
$$dv_t = \kappa(\theta - v_t)dt + \nu v_t dZ_t.$$

- Currently, there is no easy way to solve the SDE.
 - Conditional MC possible? Option pricing with conditional MC?
 - How to express S_T as a function of v_T and V_T (or something else)?
 - Exact simulation possible?
- *Almost* Exact MC (by Choi)

Project Suggestion: *Almost* Exact MC (by Choi)

The illustration of the proposed double layer approximation method:

- 1 The outer layer distribution of σ_T (in blue) is typically known
- 2 The inner layer distribution of $V_T|\sigma_T$ (in red) is approximated as well-known distributions such as log-normal or inverse Gaussian.



Project Suggestion: *Almost* Exact MC (by Choi)

Drawback of the exact simulation methods:

- Inverse of the Laplace transform, $E(e^{-sV_T}|v_T) = f(s, v_T)$, is complicated.
- Drawing random number from the numerical CDF is also slow.

How can we simplify this step with some approximation?

- Approximate $V_T|v_T$ with a well-known distribution by matching the first two moments, $M_1 = E(V_T|v_T)$ and $M_2 = E(V_T^2|v_T)$.
- The RN sampling should be easy from the approximate distribution.

Almost Exact MC: Candidates for distributions

- Log-normal (LN):

$$Y \sim \mu \exp(\sigma Z - \sigma^2/2) \quad \text{for} \quad Z \sim N(0, 1)$$

The parameters (μ, σ) can be obtained from the two moments:

$$\mu = M_1 \quad \text{and} \quad \sigma = \sqrt{\log(M_2/M_1^2)}.$$

- Inverse-Gaussian:

$$f_{\text{IG}}(x | \gamma, \delta) = \frac{\delta}{\sqrt{2\pi x^3}} \exp\left(-\frac{(\gamma x - \delta)^2}{2x}\right) \quad \text{for} \quad \gamma \geq 0, \delta > 0.$$

How to determine (γ, μ) from M_1 and M_2 ?

- The sampling methods for LN and IG are available. See [Michael et al. \[1976\]](#) ([WIKIPEDIA](#)) for IG.

Almost Exact MC: How to obtain M_1 and M_2 ?

Keep in mind for a random variable $X \geq 0$, the MGF and Laplace transform are same:

$$M_X(-s) = E(e^{-sX}) = \int_{x=0}^{\infty} e^{-sx} f_X(x) dx = f(s)$$
$$f(s) = 1 - M_1 s + \frac{1}{2} M_2 s^2 + \dots,$$

where $M_1 = E(V_T|v_T)$ and $M_2 = E(V_T^2|v_T)$.

- Numerical method: [Choudhury and Lucantoni \[1996\]](#)
- Analytic method (from Taylor's expansion or etc):
 - SABR: well-known [Kennedy et al. \[2012\]](#).
 - GARCH: [Barone-Adesi et al. \[2005\]](#)(?)
 - Heston, 3/2, OU?

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