1 Cointegration and stationarity of financial time series

Before introducing the cointegrating relationships between assets, some key foundamental definitions will be represented in the following section.

1.1 Technical Definitions

- *definition 1: Strictly Stationary*
  Let X_t be a stochastic process and let $F_X(x_{t_1}, ..., x_{t_n})$ be the cumulative distribution function of the joint distribution at X_t at times $t_1, ..., t_n$. Then X_t is said to be strictly stationary if

$$F_X(x_{t_1}, ..., x_{t_n}) \stackrel{d}{=} F_X(x_{t_1+\tau}, ...x_{t_n+\tau}) \tag{1}$$

  for any τ such that $t_{1+\tau}, ..., t_{n+\tau} \in T$, for all $t_1, ..., t_n \in T$, $\tau \in \mathbb{Z}$ and $n \in \mathbb{N}$.

- *definition 2: Covariance Stationary*
  Let X_t be a stochastic process. Then X_t is said to be covariance stationary if, for all $n \in \mathbb{N}$ and for any τ such that $t_{1+\tau}, ..., t_{n+\tau} \in T$, all the joint moments of orders 1 and 2 of $\{X_{t_1}, ..., X_{t_n}\}$ exist, are finite, and equal to the corresponding joint moments of $\{X_{t_1+\tau}, ...X_{t_n+\tau}\}$ i.e.,

$$E(X_t) = E(X_{t+\tau}) \tag{2}$$

$$Var(X_t) = Var(X_{t+\tau}) \tag{3}$$

- *Definition 3: Order of Integration*
  Suppose that a stochastic process X_t has order of integration d, which can be written as $I(d)$. Then d represents the minimum number of difference required to be obtain a covariance stationary process.
  In this case, $I(0)$ represents a stationary process.

1.2 Cointegration Properties

Most of time series analysis such as AR(p), MA(q), solely based on the assumption that stochastic processes are stationary. However, in economic system, fundamental changes can occur gradually or at once due to growth or policy changes, which would violate the stationarity assumption. For instance, Shanghai Composite Index exhibit some dramatic changes corresponding to the financial crisis in 2008(see Figure 1). Moreover, Nelson and Plosser showed that most of the financial or economic time series appear to be $I(1)$. In this case, cointegration is commonly used for measuring non-stationary time series and is defined by Engle and Granger (1987) as follows.

- *definition 4: Cointegration*
  The components of the vector $\mathbf{X}_t$, are said to be cointegrated of order d, b denoted $\mathbf{X}_t \sim CI(d, b)$ if:
  (i) all components of $\mathbf{X}_t$ are $I(d)$;
  (ii) there exists a non-trivial vector α so that $Z_t = \alpha' \mathbf{X}_t \sim I(d - b)$, $b > 0$. The vector α is called the cointegrating vector.

The above definition indicates that two or more non-stationary time series are able to form a stationary time series, which would be useful in terms of analyzing long term relationships between unstable time series. Moreover, compared with correlation, cointegration only measures whether or not the distance between two time series remains stable over time, but correlation measures the probability that two variables move together in each time step. In a nutshell, the main difference is that correlation considers the direction of movements. Hence, we can conclude that if the time series is cointegrated, it must be correlated.

1.3 Cointegration Modelling: Vector Autoregression Model

A VAR model describes the process of a n-dimensional time series $\mathbf{X}_t$ over the same period as a linear combination of only their past values. Technically, a n-dimensional time series $\mathbf{X}_t$ is said to be a VAR(p) model if it can be expressed as follows.

$$\mathbf{X}_t = \boldsymbol{\mu}_t + \boldsymbol{\Phi}_1\mathbf{X}_{t-1} + ... + \boldsymbol{\Phi}_p\mathbf{X}_{t-p} + \boldsymbol{\epsilon}_t \tag{4}$$

$$= \boldsymbol{\mu}_t + \sum_{i=1}^{p} \boldsymbol{\Phi}_i\mathbf{X}_{t-i} + \boldsymbol{\epsilon}_t \tag{5}$$

Where the innovation $\boldsymbol{\epsilon}_t$ is assumed to be Gaussian with zero mean and covariance matrix R, $\boldsymbol{\mu}_t$ is a vector represents the intercept terms. It can be written as $\boldsymbol{\mu}_t = \boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t$, where $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$ are n- dimensional constant vectors.
Define VAR polynomial function as

$$\boldsymbol{\Phi}(Z) = \mathbf{I} - \boldsymbol{\Phi}_1 Z - ... - \boldsymbol{\Phi}_p Z^p \tag{6}$$

$\mathbf{X}_t$ is said to be unit-root stationary if the root $|\boldsymbol{\Phi}(Z)|$ are outside of the unit circle. i.e. $|\boldsymbol{\Phi}(Z)| \neq 0$ for $|Z| \leq 1$. Recall that time series x_t and y_t have cointegrating relationship with order 1 if x_t and y_t are $I(1)$, but the linear combination of them is stationary. Therefore, for the process $\mathbf{X}_t$, if $|\boldsymbol{\Phi}(1)| = 0$, then $\mathbf{X}_t$ must be non-stationary and there may exist some cointegrating relationships between variables. In this case, VAR(p) model is not sufficient to present cointegrating properties. Instead, Vector Error Correlation Model (VECM) is generally used in this case.

1.4 Cointegration Modelling: Vector Error Correlation Model