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Project Report

one-dimensional heat conduction

1. Algorithm framework

In this project, the governing equation of one-dimensional heat conduction problem, the boundary conditions and the initial value can be expressed as

$$\rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \quad on\ \Omega \times (0,T) \tag{1}$$

$$u = 0 \quad on\ \Gamma \times (0,T) \tag{2}$$

$$u|_{t=0} = u_0 = e^x \quad in\ \Omega. \tag{3}$$

Here, Ω :=(0,1) is the 1D domain. The boundary domain is $\Gamma = \{0,1\}$. ρ, c, κ are the density, heat capacity and heat conductivity respectively. $f \ = sin(\pi x)$ is the heat supply per unit volume.

Both explicit and implicit method is applied to solve this problem. The derivation of the two schemes are provided as follows.

Explicit Euler method

$$\begin{aligned}
&\rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \\
&\frac{\partial u}{\partial t} = \frac{\kappa}{\rho c} \frac{\partial^2 u}{\partial x^2} + \frac{f}{\rho c} \\
&\frac{\partial u}{\partial t} \sim \frac{U_i^{n+1} - U_i^n}{\Delta t} \\
&\frac{\partial^2 u}{\partial x^2} \sim \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} \\
&\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{\kappa}{\rho c} \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} + \frac{f}{\rho c} \\
&U_i^{n+1} - U_i^n = \frac{\kappa \Delta t}{\rho c \Delta x^2} (U_{i-1}^n - 2U_i^n + U_{i+1}^n) + \frac{f \Delta t}{\rho c} \\
&U_i^{n+1} = \frac{\kappa \Delta t}{\rho c \Delta x^2} U_{i-1}^n + (1 - 2\frac{\kappa \Delta t}{\rho c \Delta x^2}) U_i^n + \frac{\kappa \Delta t}{\rho c \Delta x^2} U_{i+1}^n + \frac{f \Delta t}{\rho c} \\
&\boldsymbol{A} = \boldsymbol{I} + \frac{\kappa \Delta t}{\rho c \Delta x^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \\
&\boldsymbol{U}^{n+1} = \boldsymbol{A} \boldsymbol{U}^n
\end{aligned}$$

Implicit Euler method

$$
\begin{aligned}
&\rho c\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = f\\
&\frac{\partial u}{\partial t} = \frac{\kappa}{\rho c}\frac{\partial^2 u}{\partial x^2} + \frac{f}{\rho c}\\
&\frac{\partial u}{\partial t} \sim \frac{U_i^{n+1} - U_i^n}{\Delta t}\\
&\frac{\partial^2 u}{\partial x^2} \sim \frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{\Delta x^2}\\
&\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{\kappa}{\rho c}\frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{\Delta x^2} + \frac{f}{\rho c}\\
&U_i^{n+1} - U_i^n = \frac{\kappa\Delta t}{\rho c\Delta x^2}\left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right) + \frac{f\Delta t}{\rho c}\\
&-\frac{\kappa\Delta t}{\rho c\Delta x^2}U_{i+1}^{n+1} + \left(1 + 2\frac{\kappa\Delta t}{\rho c\Delta x^2}\right)U_i^{n+1} - \frac{\kappa\Delta t}{\rho c\Delta x^2}U_{i-1}^{n+1} = U_i^n + \frac{f\Delta t}{\rho c}\\
&\boldsymbol{A} = \boldsymbol{I} + \frac{\kappa\Delta t}{\rho c\Delta x^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}\\
&\boldsymbol{A}\boldsymbol{U}^{n+1} = \boldsymbol{U}^n
\end{aligned}
$$

The convergence condition for explicit scheme is

$$
\frac{\kappa\Delta t}{\rho c\Delta x^2} < \frac{1}{2},
$$

and the implicit scheme always converges.