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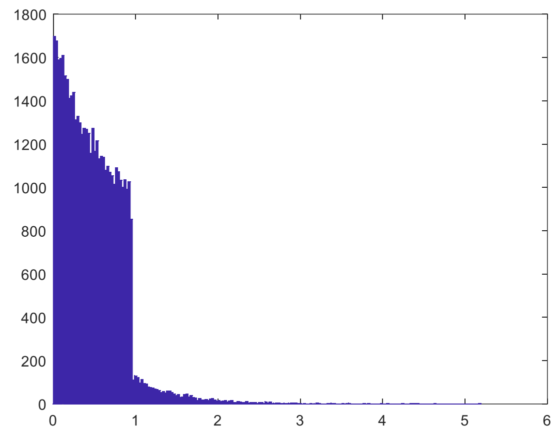
Problem 17.

Generate 5e4 samples and the results is like follows:



The MATLAB code is as follows:

```
samples = [];
sampleSize = 50000;
for i = 1:sampleSize
    u = rand();
    if u <= (3 - exp(-2))/3
        p = find(mnrnd(1,[1/3,2/3]));
        if p == 1
            x = -log(1 - u)/2;
            if x <= 1
                samples = [samples,x];
            end
        elseif p == 2
            x = u;
            samples = [samples,x];
        end

    else
        x = -log(3 - 3*u)/2;
        samples = [samples,x];
    end
end
hist(samples,200);
```

Problem 20:

Loop

Step 1: Compute the derivative of $F(x)$, we get $p(x) = \sum_{i=1}^{n}(p_i(x)\prod_{j=1,j\neq i,}^{n} F_i(x))$;

Step 2: Denote the vector$[S_i]_{i=1}^{n}$, where $S_i = \prod_{j=1,j\neq i,}^{n} F_i(x)$. The vector forms an un-normalized distribution. Sample an integer j from the un-normalized multinomial distribution formed by the vector $[S_i]_{i=1}^{n}$.

Step3 : Use Inverse transform method or acceptance and rejection method to sample x from $p_i(x)$

Until some terminating conditions

Problem 26:

Let $S_k = [X_k, X_{k-1}]$,

Given that $p(X_{k+1}|X_k, X_{k-1}, \dots) = p(X_{k+1}|X_k, X_{k-1})$

We have

$$p(X_{k+1}|X_k, X_{k-1})p(X_k|X_k) = p(X_{k+1}, X_k|X_k, X_{k-1}) = p(S_{k+1}|S_k)$$

Where $\{S_k\}$ is a Markov chain.

Problem 27.

(a). MATLAB code:

```
pi0 = [0.1,0.9,0];
A = [0.3,0.3,0.4;0.1,0.9,0;0.1,0.1,0.8];
%% section a
cur_pi = pi0;
samples = [];
for i = 1:500
    samples = [samples,sampleFunc(cur_pi)];
    cur_pi = cur_pi*A;
end
function samp = sampleFunc(cur_pi)
u = rand();
if u < cur_pi(1)
    samp = 1;
elseif u < (cur_pi(1) + cur_pi(2))
    samp = 2;
else
    samp = 3;
end

end
```

(b). MATLAB code

```
%% section b
proposalDist = [1/3,1/3,1/3];
sampleRej = [];
cur_pi = pi0;
while length(sampleRej) < 500
    c = max(cur_pi)*3;
    x = find(mnrnd(1,proposalDist));
    u = rand();
    if u < cur_pi(x)/(c/3)
        sampleRej = [sampleRej,x];
    end
    cur_pi = cur_pi*A;
end
```

(c). Suppose in the sample path the ordered pair $\langle i, j\rangle, i, j \in \{1,2,3\}$ appears n_{ij} times,

Then the transition probability

$$p(i,j) = \frac{n_{ij}}{\sum_j n_{ij}}$$

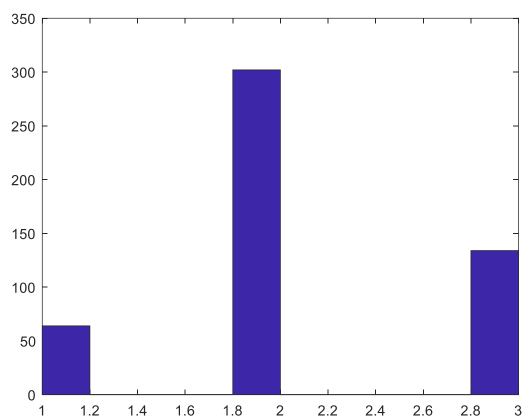
(d).

$$\pi_\infty = \pi_\infty A$$

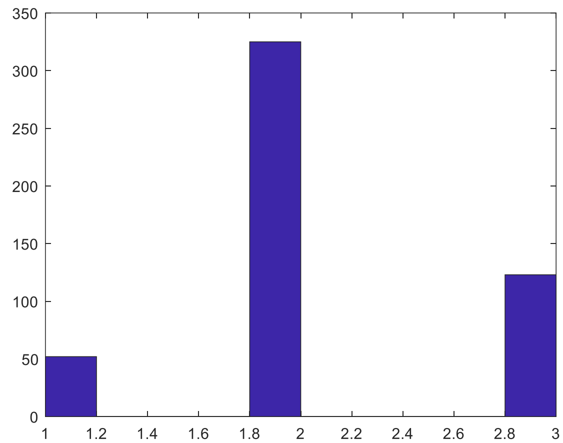
Solve the equation and we get $\pi_\infty = [0.125, 0.625, 0.25]$.

Verification from numerical simulations:

Samples from Inverse CDF:



Verification from Acceptance-Rejection method:



Problem 28

Suppose

$$A = \begin{bmatrix} a & 1-a \\ b & 1-b \end{bmatrix}$$

Let

$$\pi_\infty = \pi_\infty A$$

Then $\pi_\infty = [\frac{b}{b-a+1}, \frac{-(a-1)}{b-a+1}]$.

Consider the eigen decomposition of the matrix $A = UVU^{-1} = \begin{bmatrix} 1 & \frac{a-1}{b} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a-b \end{bmatrix} \begin{bmatrix} \frac{b}{b-a+1} & \frac{1-a}{b-a+1} \\ -\frac{b}{b-a+1} & \frac{b}{b-a+1} \end{bmatrix}$

where $U = \begin{bmatrix} 1 & \frac{a-1}{b} \\ 1 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 \\ 0 & a-b \end{bmatrix}$.

Then we consider

$$A^n = UV^nU^{-1} = \begin{bmatrix} 1 & \frac{a-1}{b} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (a-b)^n \end{bmatrix} \begin{bmatrix} \frac{b}{b-a+1} & \frac{1-a}{b-a+1} \\ -\frac{b}{b-a+1} & \frac{b}{b-a+1} \end{bmatrix}$$

As $n \to \infty$, $(a-b)^n \to 0$, then we have

$$A^n \to \begin{bmatrix} 1 & \frac{a-1}{b} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{b}{b-a+1} & \frac{1-a}{b-a+1} \\ -\frac{b}{b-a+1} & \frac{b}{b-a+1} \end{bmatrix} = \begin{bmatrix} \frac{b}{b-a+1} & \frac{-(a-1)}{b-a+1} \\ \frac{b}{b-a+1} & \frac{-(a-1)}{b-a+1} \end{bmatrix} = \begin{bmatrix} \pi_\infty \\ \pi_\infty \end{bmatrix}$$

For arbitrary initial distribution π_0, as $n \to \infty$

$$\pi_0 A^n \to \pi_0 \begin{bmatrix} \pi_\infty \\ \pi_\infty \end{bmatrix} = \pi_\infty$$

Therefore, the distribution converges to the stationary distribution at the rate of $(a-b)^n$, where $a-b$ is the second largest eigenvalue of A.

Problem 30.

We denote the row vector $J = [0,0,..1,0,0]$, where all entries are 0 except the j^{th} entry is 1.

Then we have $p(X_n = j|X_0 = i) = JP^{n-1}J^T P_{ij}$, where P is the transition matrix.

$$p(X_{n+1} = j|X_n \neq j) = \frac{p(X_{n+1} = j) - p(X_{n+1} = j|X_n = j)p(X_n = j)}{p(X_n \neq j)}$$

$$= \frac{JP^n J^T P_{ij} - P_{jj} JP^{n-1} J^T P_{ij}}{1 - JP^{n-1}J^T P_{ij}} = \frac{J\left(P^{n-1}P_{ij}(P - P_{jj}I)\right)J^T}{J\left(I - P^{n-1}P_{ij}\right)J^T}$$

Also, we have

$$p(X_{n+1} \neq j|X_n \neq j) = 1 - p(X_{n+1} = j|X_n \neq j) = \frac{J\left(I - P^{n-1}P_{ij} - \left(P - P_{jj}I\right)P^{n-1}P_{ij}\right)J^T}{J\left(I - P^{n-1}P_{ij}\right)J^T}$$

$$= \frac{J\left(I - P^{n-1}P_{ij}\left(P + (1 - P_{jj})I\right)\right)J^T}{J\left(I - P^{n-1}P_{ij}\right)J^T}$$

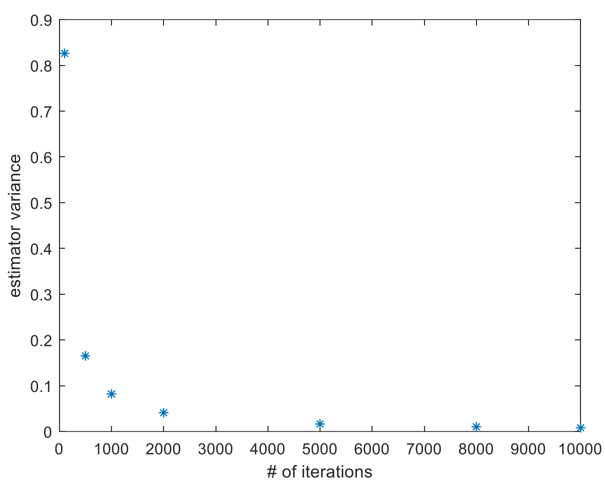
$p(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, \ldots, X_1 \neq j|X_0 = i)$

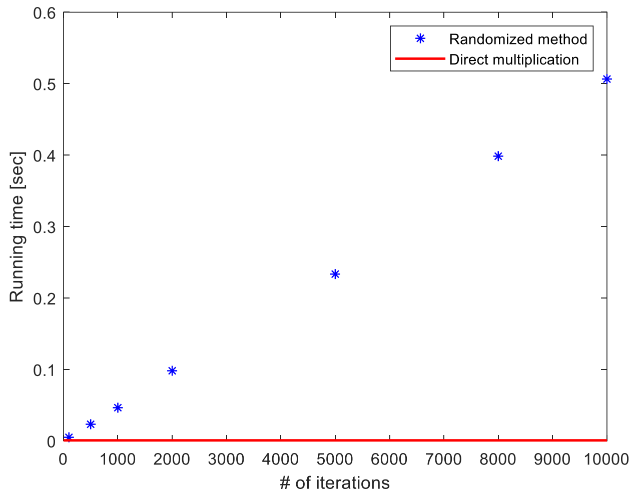
$= p(X_n = j|X_{n-1} \neq j)p(X_{n-1} \neq j|X_{n-2} \neq j) \ldots p(X_1 \neq j|X_0 = i)$

$$= \frac{J\left(P^{n-2}P_{ij}(P - P_{jj}I)\right)J^T}{J\left(I - P^{n-2}P_{ij}\right)J^T} \frac{J\left(I - P^{n-3}P_{ij}\left(P + (1 - P_{jj})I\right)\right)J^T}{J\left(I - P^{n-3}P_{ij}\right)J^T} \ldots \frac{J\left(I - P_{ij}\left(P + (1 - P_{jj})I\right)\right)J^T}{1 - P_{ij}} (1 - P_{ij})$$

$$= J\left(P^{n-2}P_{ij}\left(P - P_{jj}I\right)\right)J^T\left(1 - \frac{JP^{n-3}P_{ij}\left(1 - P_{jj}\right)J^T}{J\left(I - P^{n-2}P_{ij}\right)J^T}\right)\dots(1 - \frac{P_{ij}(1 - P_{jj})}{J\left(1 - PP_{ij}\right)J^T})$$

Problem 33.





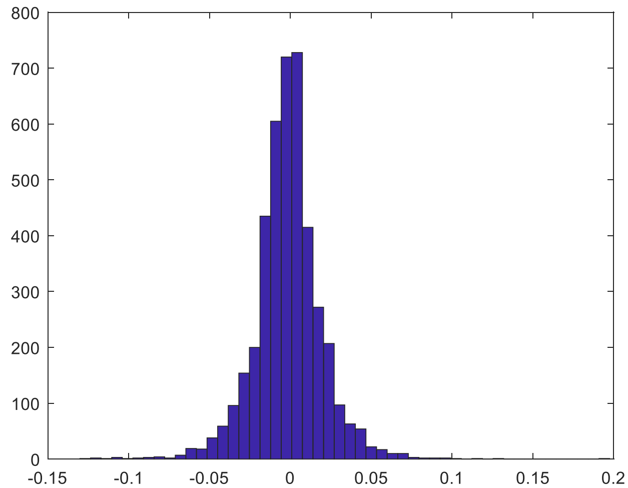
MATLAB code:

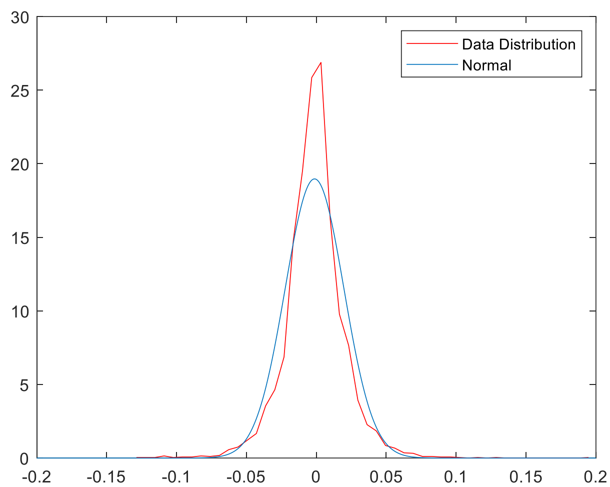
```
clc,clear,close all
N = 1000;
Ms = [100,500,1000,2000,5000,8000,10000] ;
L = 100;
A = rand(N);
pik = rand([N,1]);
pik = pik/sum(pik);
%Use rejection sampler
proposalDist = ones(1,N)/N;
c = max(pik)/(proposalDist(1));
At = A';
piKp1_est = zeros(N,1);
tic
piKp1 = At*pik;
t1 = toc;
mses = [];
vars = [];
times = [];
for M = Ms
    errs = [];
    piKp1_estm = [];
    ts = [];
    for i = 1:L
        num = 0;
        tic
        while (num < M)
            x = find(mnrnd(1,proposalDist));
            u = rand();
            if u < pik(x)/(c/N)
                %accept
                piKp1_est = piKp1_est + At(:,x);
                num = num + 1;
            end
        end
        piKp1_est = piKp1_est/M;
        t = toc;
        ts = [ts,t];
```

```
        piKp1_estm = [piKp1_estm,piKp1_est];
        err = norm(piKp1 - piKp1_est);
        errs = [errs,err];
    end
    times = [times,mean(ts)];
    meanEst = mean(piKp1_estm,2);
    var = trace((piKp1_estm - meanEst)'*(piKp1_estm - meanEst))/L;
    mse = sum(errs)/L;
    mses = [mses,mse];
    vars = [vars,var];
end
figure
plot(Ms,vars,'*');
xlabel('# of iterations')
ylabel('estimator variance')
figure
plt1 = plot(Ms,times,'b*');
hold on
plt2 = line([0,10000],[t1,t1],'LineWidth',1.5','Color','red');
legend([plt1,plt2],{'Randomized method','Direct multiplication'})
xlabel('# of iterations')
ylabel('Running time [sec]')
```

Problem 34.

Collect the daily stock closed price of APPLE INC from Sep 16, 2002 to Sep 16, 2019 and the compute the log return of one day. We have the following histogram:





I used Pearson chi square test.

The continues normal distribution is segmented into 46 bins that have non-zero items in them.

Compute the statistic $\sum_{i=1}^{46} \frac{(E_i - O_i)^2}{E_i} = 1.028 \times 10^5$, which is very large and we can reject the hypothesis that the distribution is normal.