Report of Project Three

Accelerated Return Securities Based on the Value of the Worst Performing of the S&P 500 Index and the Russell 2000 Index

Product key features

1. The security is a path-dependent European style product because the determinant of the payoff at maturity is the Final Average Index Value of each underlying index, which is the arithmetic average of the closing value of each underlying index on each of the averaging dates.
2. The payoff at maturity is based on the index which performs worse in the whole period and the determinant of performance is the ratio of Final Average Index Value to Initial Index Value.

Data and parameter

The key dates for this product are as following.

Issue Date T_1	4/9/2018
Pricing Date T_0	4/4/2018
Maturity Date T_2	9/11/2023
Averaging Date	6/6/2023-9/6/2023

The Averaging Dates are all business days (65 business days) between the period and so we build the stochastic process of two index levels based on all business days between the Pricing Date and Maturity Date (1362 business days).

Other than initial index level S_0 and R_0 for S&P 500 Index and Russell 2000 Index respectively, time to maturity *T*, there are other parameters we need to consider. In this part, we just clarify what parameters we need, and we will discuss the effect and what choice we made in the last part.

S_0	R_0	T
2614.45	1512.155	5.44

1. Interest rate

The most basic parameter is the different interest rates that will be used in the valuation process. First, the interest rate that we need to use to discount the value on pricing date to issue date is the shortest period of interest that we can get, which is the 3-month treasury bill rates. The rate that is used in the simulation of the index level is the rate from pricing date to maturity date. We can get this rate by interpolation of the rates on the closest dates. Different quote time has slightly different rates due to different settle dates of the bond, here we use the zero-coupon-bond prices quoted on April 4, 2018.

Quoted on April 4		
Maturity Date	Days	Discount
07/05/2018	92	0.994126892
09/19/2018	168	0.989353873
04/04/2022	1461	0.899170341
04/04/2023	1826	0.874767783
04/04/2024	2192	0.850864653

Times	Dates	Days	Interp DF	Interp CC
T_1	2019/3/26	5	N/A	0.02337
T_2	2020/6/22	1986	0.864318327	0.0268

2. Dividend

Since the underlying are two indexes, the most appropriate way we approximate the dividend paying is the continuous dividend yield. We can get the predicted dividend yield of two indexes from Bloomberg, 1.89% and 1.33% respectively and we can adjust the stochastic processes.

3. Sigma

The volatility of the underlying index is the most important parameter in valuation. In the simple stochastic equation of the underlying index, we assume a constant volatility and we get implied volatilities from options prices on Bloomberg. Options with different maturities and different moneyness have different volatilities and it is hard to choose a single value for the whole valuation. We get a table with the many volatilities listed with S&P 500 on the left. We can see that the Russell 2000 Index tends to have a higher volatility than S&P 500 Index.

Maturity / Moneyness	12 Months/%	23 Months/%	34 Months/%	45 Months/%	56 Months/%
100%	19.002/20.179	18.607/21.336	19.104/21.229	19.603/21.565	20.087/22.230
105%	17.046/18.192	17.402/20.283	18.055/20.458	18.767/20.928	19.437/21.657
110%	15.361/17.761	16.252/19.015	16.880/19.669	17.798/20.280	18.680/21.105
115%	14.250/16.839	15.234/18.173	16.065/19.028	17.122/19.741	18.025/20.587

4. Correlation

Since the underlying are two correlated market indexes, we need an estimate of the correlation between two indexes from the historical data. We get 10 years of index level and calculate the correlation of the return of the indexes from 1 year to 10 years.

Year	1	2	3	4	5	6	7	8	9	10
Correlation	0.850633	0.849187	0.866221	0.857348	0.864868	0.874541	0.906597	0.911431	0.915892	0.921799

We can see that the correlation of the two indexes are high and as the time horizon increases, the correlation even increases. We can also use a dynamic conditional correlation model to estimate the correlation between these two indexes, but we think the simple estimation here can capture the correlation.

Model building

According to the features of this security, Monte Carlo Simulation is the most reasonable way to value such a product. We first run a trial with 100,000 times simulation using volatility 17% and 19% and correlation 90% and get the value $971.1045.

Since we need prices on business days, we make $dt = \frac{1}{252}$ and there are 1298 business days between Pricing Date and first Averaging Date,

$$S_{(2)} = S_{(1)} * \exp\left(\left(r - div_1 - 0.5 * sigma_1^2\right) * \frac{1298}{252} + sigma_1 * \sqrt{\frac{1298}{252}} * \phi_1\right)$$

$$R_{(2)} = R_{(1)} * \exp\left(\left(r - div_2 - 0.5 * sigma_2^2\right) * \frac{1298}{252} + sigma_2 * \rho * \sqrt{\frac{1298}{252}} * \phi_1 + sigma_2 * \sqrt{1-\rho^2} * \sqrt{\frac{1298}{252}} * \phi_2\right)$$

where ϕ_1 and ϕ_2 are two independent draws from standard normal distribution.
And for the simulation of the Averaging period, *i* runs from 3 to 66

$$S_{(i)} = S_{(i-1)} + (r - div_1) * S_{(i-1)} * dt + sigma_1 * S_{(i-1)} * dW_{(i-2,1)}$$
$$R_{(i)} = R_{(i-1)} + (r - div_2) * R_{(i-1)} * dt + sigma_2 * R_{(i-1)} * dW_{(i-2,2)}$$

Where $dW_{(i-2,1)}$ and $dW_{(i-2,2)}$ are two draws from the correlated normal distributions with mean zero and the covariance matrix $\Sigma = \begin{bmatrix} dt & \rho * dt \\ \rho * dt & dt \end{bmatrix}$.

And then we calculate the mean of the last 65 simulated index levels and get the ratio of $\frac{S_{mean}}{S_0}$ and $\frac{R_{mean}}{R_0}$.

If $min\left(\frac{S_{mean}}{S_0}, \frac{R_{mean}}{R_0}\right) \geq 1.21$,

$$payoff = \min\left(\left(1000 + 1000 * \left(min\left(\frac{S_{mean}}{S_0}, \frac{R_{mean}}{R_0}\right) - 1.21\right) * 3.34 + 415\right), 2116.4\right)$$

If $1 \leq min\left(\frac{S_{mean}}{S_0}, \frac{R_{mean}}{R_0}\right) < 1.21$,

$$payoff = 1000 + 1000 * \left(min\left(\frac{S_{mean}}{S_0}, \frac{R_{mean}}{R_0}\right) - 1\right) * 1.5 + 100$$

If $0.95 \leq min\left(\frac{S_{mean}}{S_0}, \frac{R_{mean}}{R_0}\right) < 1$,

$$payoff = 1000 + 1000 * \left(min\left(\frac{S_{mean}}{S_0}, \frac{R_{mean}}{R_0}\right) - 0.95\right) * 2$$

If $min\left(\frac{S_{mean}}{S_0}, \frac{R_{mean}}{R_0}\right) < 0.95$,

$$payoff = 1000 * min\left(\frac{S_{mean}}{S_0}, \frac{R_{mean}}{R_0}\right) + 50$$

And for each simulation process, the value at Pricing Date is

$$Value_0^n = payoff * \exp(-r * T)$$

where *n* denotes the nth simulation and $1 \leq n \leq 100000$.
Then the value of the total simulation process is

$$Value = mean(Value_0)$$

Results and Discussion

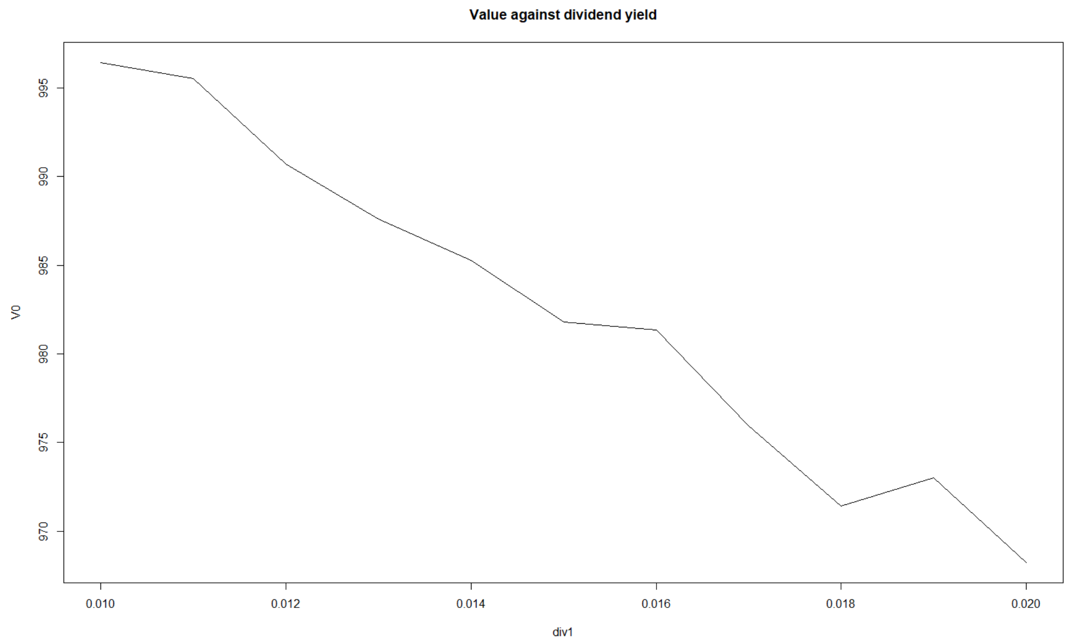
First, using sigma equal to 17% and 19% and correlation 0.9, we get $971.1045. The value is close to close to Morgan Stanley value, so we believe that the model we have built is at least valid. And we tried some other parameters But we still have some concerns.

1. In Monte Carlo Simulation, the convergence of the method is not so fast so for valuing a complicated product, we need a large number of simulations. Here we use 100,000 simulations because we find that the value for each trial is more stable than that for only 10,000 simulations.
2. We assume the risk-free interest rate is constant, but it may be stochastic. This is another possible source of error. In Monte Carlo method, we can adapt the simulation process to include a stochastic process for risk-free rate, i.e. the Vasicek model,

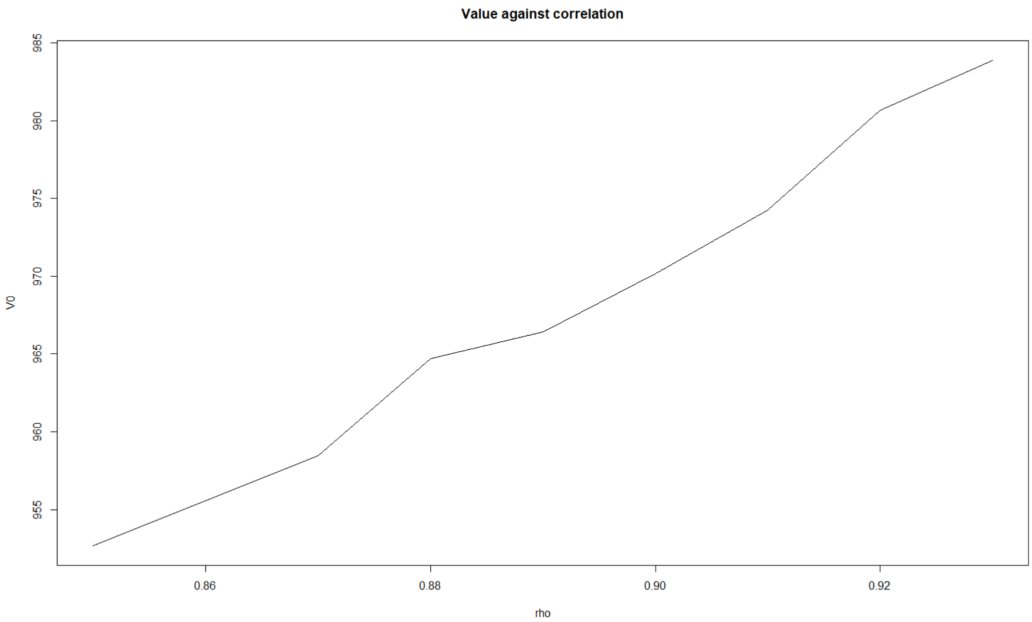
$$dr = \kappa(\bar{r} - r)dt + bdW^r$$

and we can estimate the long-run mean $\bar{r}$ and mean-reverting speed κ for risk-free rates from historical data. Then we need to estimate the correlation between each stochastic process. So we have a more complicated simulation process involving at least three stochastic random factors. In our simple model where we assume the interest rate is constant, the value is sensitive to the risk-free interest rate as well but the effect of the change in interest rate is relatively small.

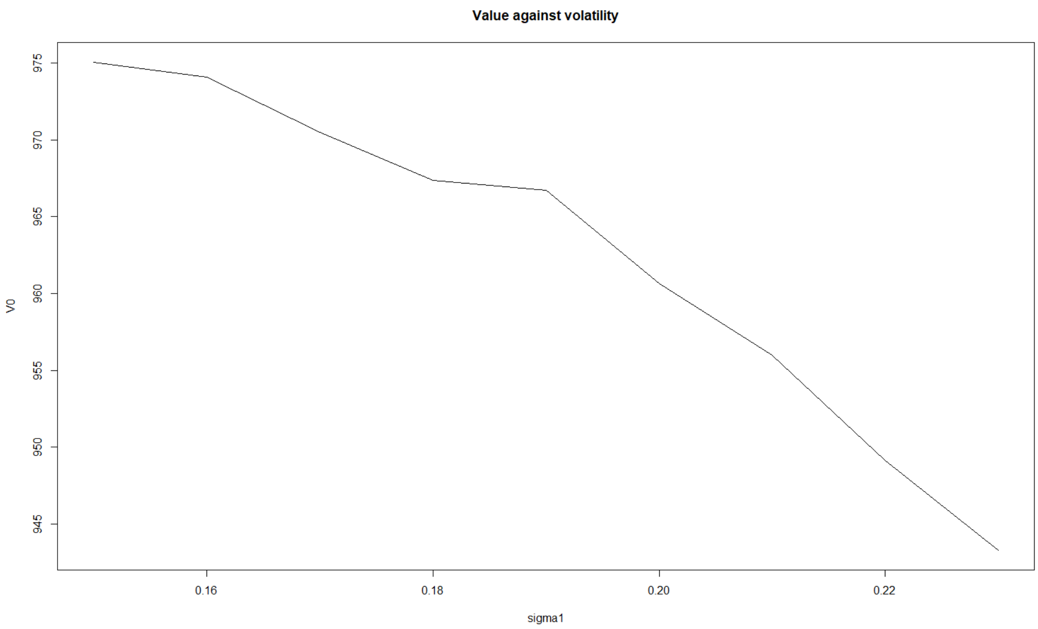
3. The dividend yield appears as an input in the stochastic processes, we tried the yield ranging from 0.018 to 0.02 for S&P 500 Index, the value shows a decreasing trend as dividend yield increases, as what we expect to see. We can also include the stochastic dividend yield in the simulation similar to what we can do for random interest rates to improve the valuation process.



4. If we assume the risk-free rates and volatilities are constant, then the correlation between two indexes matters a lot in valuation. We can see that the value increases as the correlation increases. The correlation here we are using are within the range of the 1 year to 10 years correlation we get from previous estimation. The value is sensitive to correlation as we expect to see.



5. If we assume the constant volatility, we find that higher the volatility of S&P 500 Index, lower the value of product (we find similar patterns if we plot the volatility of Russell 2000 Index). It is hard to choose which volatility to use in the model. We get the implied volatilities of at-the-money options and in-the-money options, ranging from 15% to 23% and get several prices. The graph shows the relationship between sigma and values and it is easy to find that the volatility affects the value dramatically and it is the most important parameter.



Also, we can include a stochastic random volatility model in our Monte Carlo Simulation,

$$d\sigma = \kappa(\sigma_0 - \sigma)dt + s\sigma dW^{\sigma}$$

and we can estimate mean-reverting speed κ and long-run mean of σ_0 for each index and estimate the correlation between each two factors among two index levels and two stochastic volatilities (six correlations). It is much more complicated than what we have done here.

Appendix

Codes of Monte Carlo Simulation in R

```
###define parameters
S0<-2614.45
R0<-1512.155
S<-vector()
S[1]<-S0
R<-vector()
R[1]<-R0
sigma1<-0.17
sigma2<-0.19
div1<-0.0189
div2<-0.0133
r0<-0.02337
r1<-0.0268
rho<-0.9
dt<-1/252
T<-1986/365
covm<-matrix(c(dt,rho*dt,rho*dt,dt),nrow = 2,ncol = 2)
library(MASS)    ###package for drawing multi random variables
###simulation process
nsim<-100000
V<-rep(0,nsim)
for (k in 1:nsim){
  phi1<-rnorm(1)
  phi2<-rnorm(1)
  S[2]<-S[1]*exp((r1-div1-0.5*sigma1^2)*(1298/252)+sigma1*sqrt(1298/252)*phi1)
  R[2]<-R[1]*exp((r1-div2-0.5*sigma2^2)*(1298/252)+sigma2*rho*sqrt(1298/252)*phi1+sigma2*sqrt(1-
rho^2)*sqrt(1298/252)*phi2)
 dW<-mvrnorm(64,mu=c(0,0),Sigma = covm)
  for (i in 3:66){
    S[i]<-S[i-1]+(r1-div1)*S[i-1]*dt+sigma1*S[i-1]*dW[(i-2),1]
    R[i]<-R[i-1]+(r1-div2)*R[i-1]*dt+sigma2*R[i-1]*dW[(i-2),2]
  }
  Smean<-mean(S[2:66])
  Rmean<-mean(R[2:66])
###determining payoff at maturity
  if (min(Smean/S0,Rmean/R0)>=1.21){
    payoff=min((1000+1000*(min(Smean/S0,Rmean/R0)-1.21)*3.34+415),2116.4)
  } else if ((min(Smean/S0,Rmean/R0)<1.21)&(min(Smean/S0,Rmean/R0)>=1)){
    payoff=1000+1000*(min(Smean/S0,Rmean/R0)-1)*1.5+100
  } else if ((min(Smean/S0,Rmean/R0)<1)&(min(Smean/S0,Rmean/R0)>=0.95)){
    payoff=1000+1000*(min(Smean/S0,Rmean/R0)-0.95)*2
  } else if (min(Smean/S0,Rmean/R0)<0.95){
```

```
        payoff=1000*min(Smean/S0,Rmean/R0)+50
    }
    V[k]<-payoff*exp(-r1*T)          ###discount to Pricing Date
}
###calculate the mean
V0<-mean(V)
V0
```