Question - 1

A is invertible

$\Longleftrightarrow \exists A^{-1}$ such that $A^{-1}A = AA^{-1} = I$

where I is Identity Matrix

a) Prove : $\varphi(x) = \varphi(x') \longrightarrow x = x'$

$\varphi(x) = \varphi(x')$

$\Rightarrow Ax = A \cdot x'$

$\Rightarrow A^{-1}Ax = A^{-1}Ax'$

$\Rightarrow Ix = Ix'$

$\Rightarrow x = x'$

b) Prove : $\forall y \in R^2, \exists x \in R^2$ such that $\varphi(x) = y$

$\forall y \in R^2, \ Iy = y$

$\Leftrightarrow (AA^{-1})y = y$

$\Leftrightarrow A(A^{-1}y) = y$

let $x = A^{-1}y$, we have $Ax = y$

which is equal to $\varphi(x) = y$

Question - 2

ii.) Record landmarks

$$X = \begin{bmatrix} 50 & 100 & 75 & 50 & 100 \\ 35 & 35 & 75 & 115 & 115 \end{bmatrix}$$

$$Y = \begin{bmatrix} 75 & 150 & 115 & 75 & 150 \\ 50 & 50 & 115 & 175 & 175 \end{bmatrix}$$

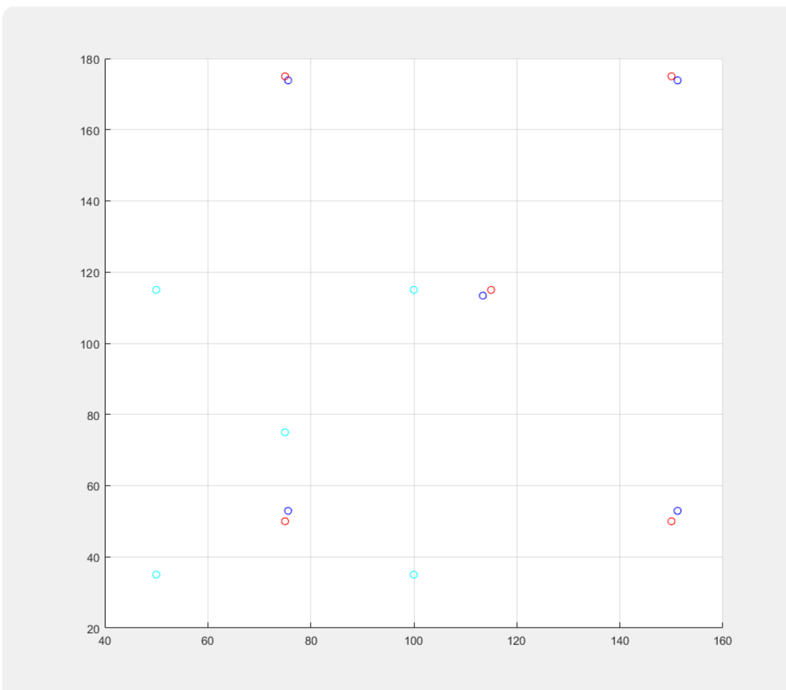
iii.) Find scale factor

$s = 1970 / 1303 \approx 1.5119$

iv.) Transform the landmarks X

<Rounding all elements to 2 decimal places>

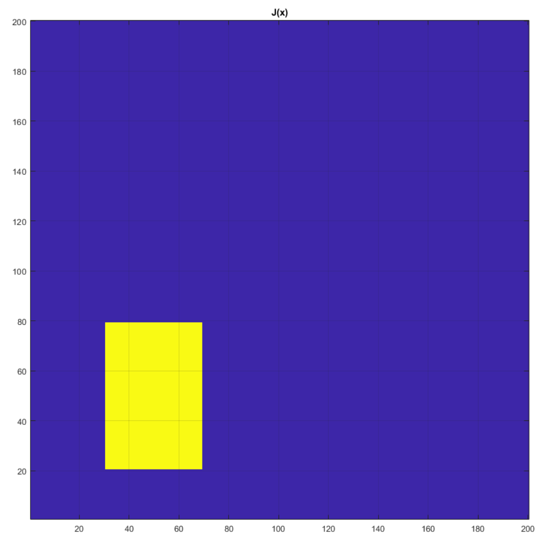
$$sX = \begin{bmatrix} 75.59 & 151.2 & 113.4 & 75.59 & 151.2 \\ 52.92 & 52.92 & 113.4 & 173.9 & 173.9 \end{bmatrix}$$



sX and Y don't match exactly.

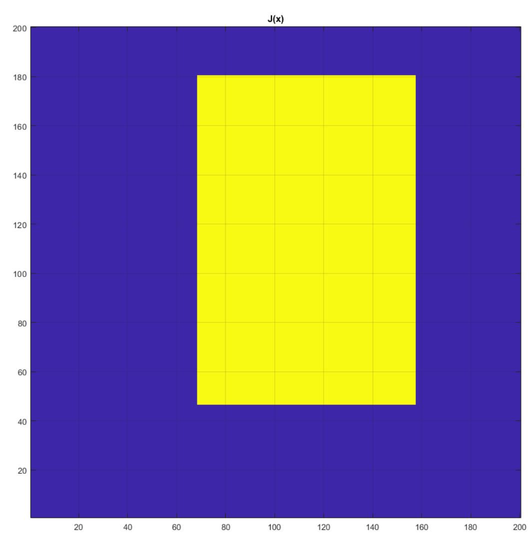
Reason: Y is not exactly proportional to X, which means the rectangles of the landmarks of X/Y, have different aspect ratio, 80 : 50 and 125 : 75.

v.) Transform the image "naively"



The image was transformed in the opposite direction, i.e. $I(x) \Rightarrow I(sx)$, so it didn't match image J.

vi.) Transform the image with the observer equation



Now it matches image J, because when looping through each pixel, it finds value from the original Image, unlike the "naively".

Question - 3

i.) $J(\epsilon) = C[y(x) + \epsilon h(x)]$

$= \int_a^b x\,(y'(x) + \epsilon h'(x))^2\,dx$

$= \int_a^b x\,(y'(x)^2 + 2y'(x)\epsilon h'(x) + \epsilon^2 h'(x)^2)\,dx$

ii.) $\frac{d}{d\epsilon} J(\epsilon) = \frac{d}{d\epsilon} \int_a^b x\,[y'(x)^2 + 2y'(x)\epsilon h'(x) + \epsilon^2 h'(x)^2]\,dx$

$= \int_a^b x\,[2y'(x)h'(x) + 2\epsilon h'(x)^2]\,dx$

$= 2\int_a^b x\,h'(x)\,(y'(x) + \epsilon h'(x))\,dx$

$(\epsilon = 0) \quad = 2\int_a^b x\,y'(x)\cdot h'(x)\,dx$

iii.) If we find a $y(x)$ that optimizes the functional, then for this $y(x)$:

$J(0)$ is the min of $J(\epsilon)$

that is when $\epsilon = 0$, $J(\epsilon) \to J_{min}$

so we have the equation:

$\frac{d}{d\epsilon} J(\epsilon)\Big|_{\epsilon=0} = 0$

$\Rightarrow \int_a^b x\,y'(x)\,h'(x)\,dx = 0$

$x\,y'(x)\,h(x)\Big|_a^b - \int_a^b h(x)[y'(x) + x\,y''(x)]\,dx = 0$

$(h(a) = h(b) = 0)$

$\Rightarrow \int_a^b h(x)\,[y'(x) + x\,y''(x)]\,dx = 0$

according to the Lemma:

$y'(x) + x\,y''(x) = 0$

$\Rightarrow y(x) + x\,y'(x) - \int y'(x)\,dx = C$

Finally, we have $x\,y'(x) = C$

Question - 4

a) i.) $A = \frac{d}{dx} + 1$, $k(x) = e^{-x} u_s(x)$

$Ak(x) = (\frac{d}{dx} + 1) e^{-x} u_s(x)$

$= -e^{-x} u_s(x) + e^{-x} \delta(x) + e^{-x} u_s(x)$

$= e^{-x} \delta(x)$

considering $\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{o.w.} \end{cases}$

$\Rightarrow Ak(x) = \begin{cases} e^{0} \cdot \delta(0) = 1 & , x = 0 \\ 0 & , \text{o.w.} \end{cases}$

$\Rightarrow Ak(x) = \delta(x)$ in this case

ii.) $A = -\frac{d}{dx} + 1$, $k(x) = e^{x} u_s(-x)$

$Ak(x) = (-\frac{d}{dx} + 1) e^{x} u_s(-x)$

$= -[e^{x} u_s(-x) + e^{x}(-\delta(x))] + e^{x} u_s(-x)$

$= e^{x} \cdot \delta(x)$

similar to the first case,

$e^{x} \delta(x) = \delta(x)$

so that $Ak(x) = \delta(x)$

iii.) $A = -\frac{d^2}{dx^2} + 1$, $k(x) = \frac{1}{2} e^{-|x|}$

$Ak(x) = (-\frac{d^2}{dx^2} + 1) \frac{1}{2} e^{-|x|}$

$= (-\frac{d^2}{dx^2} + 1) \frac{1}{2} (e^{-x} u_s(x) + e^{x} u_s(-x))$

$= -\frac{1}{2} \cdot \frac{d^2}{dx^2} (e^{-x} u_s(x) + e^{x} u_s(-x))$

$+ \frac{1}{2} (e^{-x} u_s(x) + e^{x} u_s(-x))$

$= -\frac{1}{2} \frac{d}{dx} [-e^{-x} u_s(x) + \cancel{e^{-x} \delta(x)} + e^{x} u_s(-x) - \cancel{e^{x} \delta(x)}]$

$$+\frac{1}{2}[e^{-x}u_s(x)+e^{x}u_s(-x)]$$

$$=-\frac{1}{2}[\cancel{e^{-x}u_s(x)}-e^{-x}\delta(x)+\cancel{e^{x}u_s(-x)}-e^{x}\delta(x)]$$

$$+\frac{1}{2}[\cancel{e^{-x}u_s(x)}+\cancel{e^{x}u_s(-x)}]$$

$$=\frac{1}{2}e^{-x}\delta(x)+\frac{1}{2}e^{x}\delta(x)$$

$$=\begin{cases}1 & , x=0\\ 0 & , \text{o.w.}\end{cases}$$

So that $Ak(x)=\delta(x)$ in this case

b) i.) $A=\frac{d^2}{dx^2}+2\frac{d}{dx}+1$

$$=\left(\frac{d}{dx}+1\right)^2$$

according to Green's function's property

$$k(x)=[e^{-x}u_s(x)]*[e^{-x}u_s(x)]$$

$$=\int_{-\infty}^{+\infty}e^{-\tau}u_s(\tau)\cdot e^{-x+\tau}u_s(x-\tau)\,d\tau$$

$$=\int_0^x e^{-x}\,d\tau$$

$$=e^{-x}\tau\Big|_{\tau=0}^{\tau=x}$$

$$=x\cdot e^{-x}$$

ii.) $A=-\frac{d^3}{dx^3}-\frac{d^2}{dx^2}+\frac{d}{dx}+1$

$$=\left(\frac{d^2}{dx^2}+2\frac{d}{dx}+1\right)\left(-\frac{d}{dx}+1\right)$$

$$\Rightarrow k(x)=(xe^{-x})*[e^{x}u_s(-x)]$$

$$=\int_{-\infty}^{+\infty}\tau e^{-\tau}e^{x-\tau}u_s(\tau-x)\,d\tau$$

$$=\int_x^{\infty}\tau e^{x}\cdot e^{-2\tau}\,d\tau$$

$$=e^{x}\int_x^{\infty}\tau e^{-2\tau}\,d\tau$$

$$=e^{x}\left[-\frac{1}{2}\tau\cdot e^{-2\tau}\Big|_{\tau=x}^{\tau=+\infty}-\int_x^{\infty}-\frac{1}{2}e^{-2\tau}\,d\tau\right]$$

$$= e^{x}\left[\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2\tau} \Big|_{\tau = x}^{\tau = \infty} \right]$$

$$= e^{x}\left[\frac{1}{2} x e^{-2x} + \frac{1}{4} e^{-2x} \right]$$

$$= \frac{x}{2} e^{-x} + \frac{1}{4} e^{-x}$$

$$= e^{-x}\left(\frac{x}{2} + \frac{1}{4} \right)$$