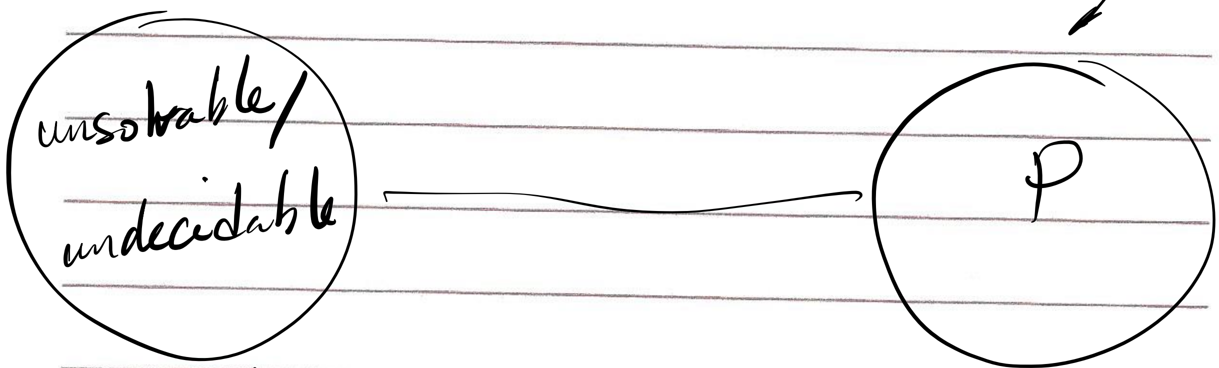
# Computational Tractability

Prob's that can be solved in pol. time




ex. Halting prob.

Plan: Explore the space of computationally hard problems to arrive at a mathematical characterization of a large class of them.

Technique: Compare relative difficulty of different problems.

Loose definition: If problem X is at least as hard as problem Y, it means that if we could solve X, we could also solve Y.

Formal definition:

$Y \leq_p X$ (Y is polynomial time reducible to X)

if Y can be solved using a polynomial number of standard computational steps plus a polynomial number of calls to a blackbox that solves X.

Suppose $Y \leq_p X$, if $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.

Suppose $Y \leq_p X$, if $Y$ cannot be solved in polynomial time, then X cannot be solved in polyn. time.

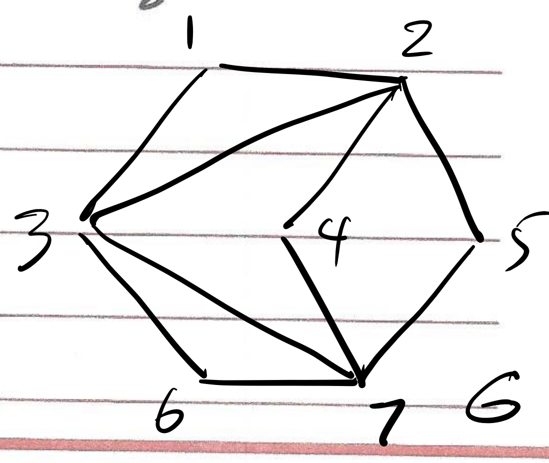
## Independent Set

**Def.** In a graph $G=(V,E)$, we say that a set of nodes $S \subseteq V$ is "independent" if no two nodes in $S$ are joined by an edge.

$\{1,6,5\}$, $\{3,4,5\}$

$\{1,4,5,6\}$

$\{1\}$, $\{1,5\}$




## Independent set problem

- Find the largest independent set in graph G.

(optimization version)

- Given a graph G, and a no. $k$ does G contain an indep set of size at least $k$?

(decision version)

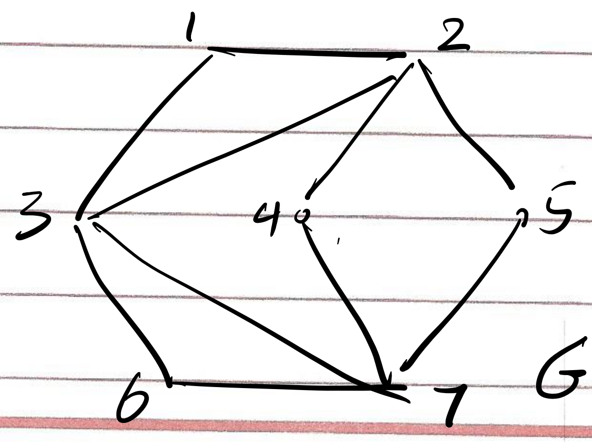
## Vertex Cover

Def. Given a graph $G = (V, E)$, we say that a set of nodes $S \subseteq V$ is a vertex cover if every edge in $E$ has at least one end in $S$.

$\{3, 1, 2, 7, 6\}$

$\{2, 3, 7\}$

$\{1, 2, 3, 4, 5, 6, 7\}$




## Vertex Cover problem

- Find the smallest vertex cover set in G.

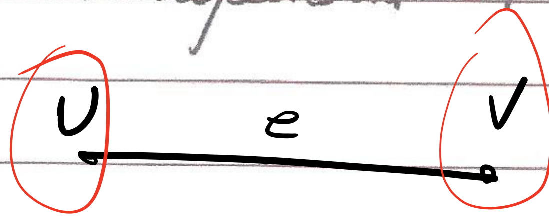
(optimization version)

- Given a graph G and a no. $\underline{k}$, does G contain a vertex cover of size at most k?

(decision version)

**FACT:** let $G=(V,E)$ be a graph, then $S$ is an independent set if and only if its complement $V-S$ is a vertex cover set.

Proof: A) First suppose that $S$ is an independent set




1- $U$ is in $S$ and $V$ is not
$\Rightarrow V-S$ will have $V$ and not $U$.

2- $V$ is in $S$ and $U$ is not
$\Rightarrow V-S$ will have $U$ and not $V$

3- Neither $V$ nor $U$ is in $S$
$\Rightarrow V-S$ will have both $V$ & $U$

B) Suppose that $V-S$ is a vertex Cover

Claim: Indep. set $\leq_p$ vertex cover

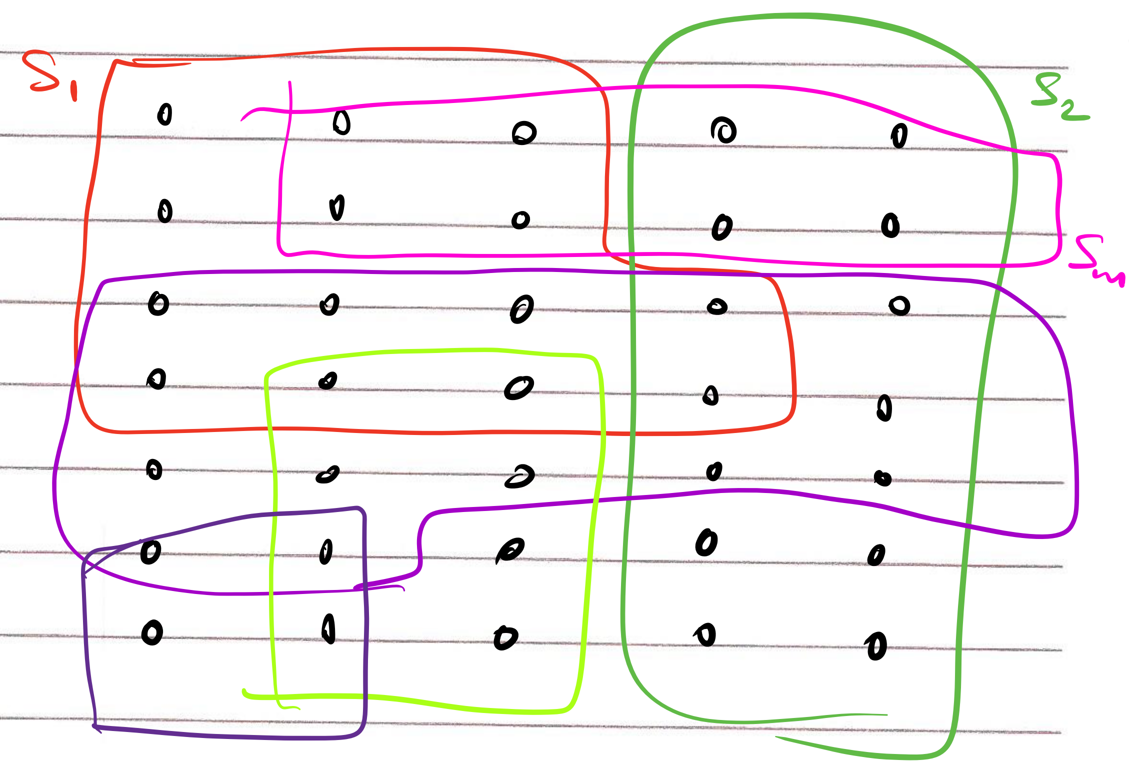
Proof: If we have a blackbox to solve vertex cover, we can decide if G has an independent set of size at least k, by asking the blackbox if G has a vertex cover of size at most n − k.

Claim: Vertex Cover $\leq_p$ Indep set

Proof: If we have a blackbox to solve independent set, we can decide if G has a vertex cover set of size at most k, by asking the blackbox if G has an indep set of size at least n − k.

## Set Cover Problem

Given a set $U$ of $n$ elements, a collection $S_1, S_2, \ldots S_m$ of subsets of $U$, and a number $k$, does there exist a collection of at most $k$ of these sets whose union is equal to all of $U$.



Claim: Vertex Cover $\le_p$ Set Cover

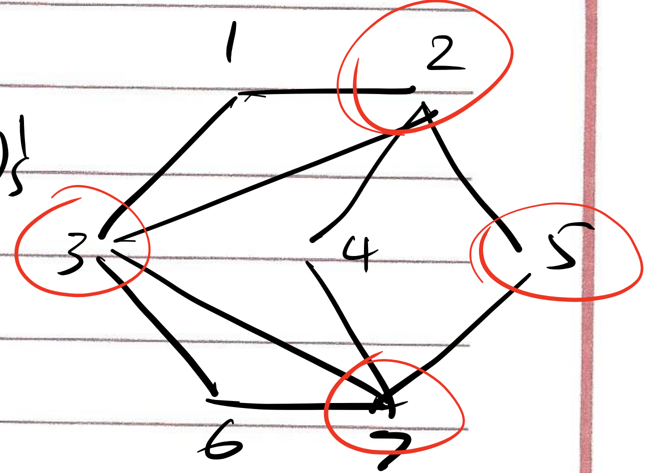
$S_1 = \{(1,2),(1,3)\}$

$S_2 = \{(1,2),(2,3),(2,4),(2,5)\}$

$S_3 = \vdots$

$S_5$

$S_7 = \dots$




G, $\underline{k}$

$\underline{k}$

Need to show that G has a vertex cover of size k, iff the corresponding set cover instance has $\underline{k}$ sets whose union ~~is~~ contains ~~equal to~~ all edges in G.

Proof:

A) If I have a vertex cover set of size $\underline{k}$ in G, I can find a collection of $\underline{k}$ sets whose union contains all edges in G.

B) If I have k sets whose union contains all edges in G, I can find a vertex cover set of size k in G.

Reduction Using Gadgets

- Given $\underline{n}$ Boolean variables $x_1 \ldots x_n$, a clause is a disjunction of terms $t_1 \vee t_2 \vee \ldots t_\ell$ where $t_i \in \{x_1 \ldots x_n, \bar{x}_1 \ldots \bar{x}_n\}$

- A truth assignment for X is an assignment of values $\underline{0}$ or $\underline{1}$ to each $x_i$.

- An assignment satisfies a clause C if it causes $\underline{C}$ to evaluate to 1.

- An assignment satisfies a collection of clauses if

$$C_1 \wedge C_2 \wedge \ldots \quad C_k$$

~~it~~ evaluates to 1.

ex. $(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3)$

$x_1 = 1$ , $x_2 = 1$ , $x_3 = 1$ ✗

$x_1 = 0$ , $x_2 = 0$ , $x_3 = 0$ ✓

$x_1 = 1$ , $x_2 = 0$ , $x_3 = 0$ ✓

Problem Statement: Given a set of clauses $C_1 \ldots C_k$ over a set of variables $X = \{x_1 \ldots x_n\}$ does there exist a satisfying truth assignment?

Satisfiability Problem

(SAT)

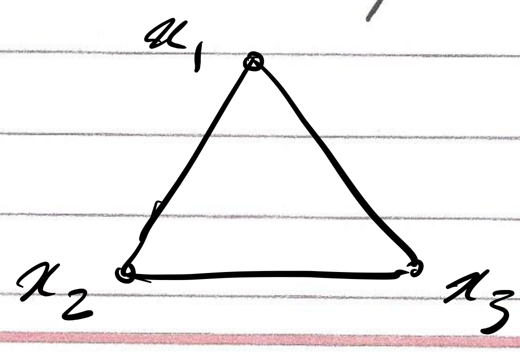
Problem statement: Given a set of clauses $C_1 \ldots C_k$ each of length 3 over a set of variables $X = \{x_1 \ldots x_n\}$ does there exist a satisfying truth assignment?

(3SAT)

Claim: 3SAT $\leq_p$ Independent Set

Plan: Given an instance of 3SAT with $\underline{k}$ clauses, build a graph $G$ that has an indep. set of size $\underline{k}$ iff the 3SAT instance is satisfiable.

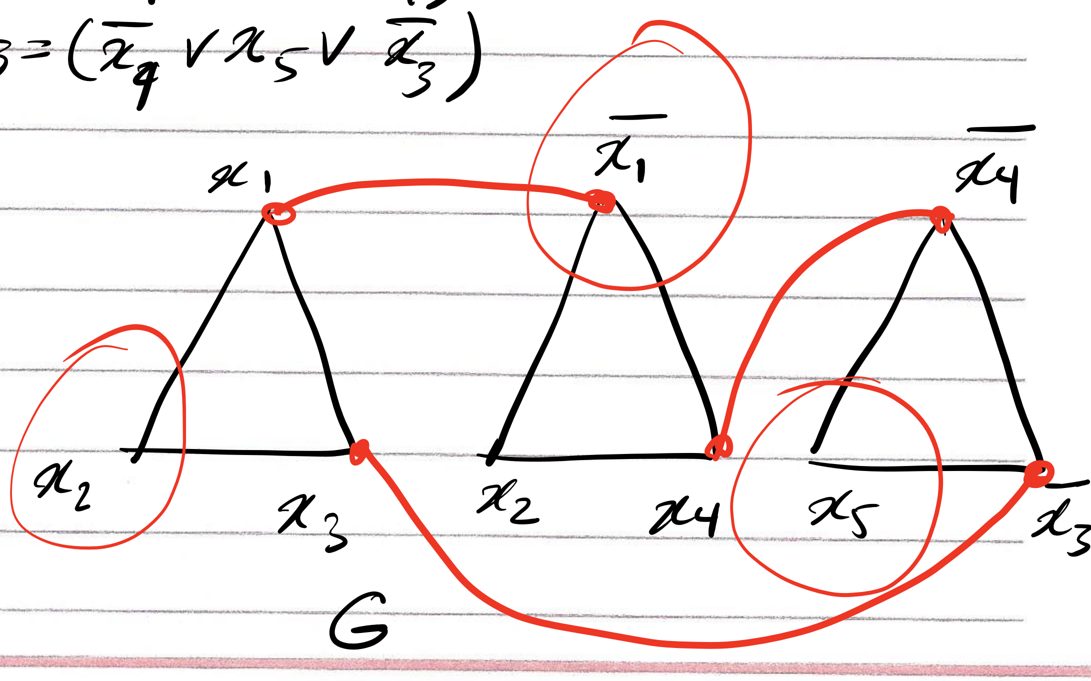
$(x_1 \vee x_2 \vee x_3)$




ex. $C_1 = (x_1 \vee x_2 \vee x_3)$

$C_2 = (\bar{x}_1 \vee x_2 \vee x_4)$

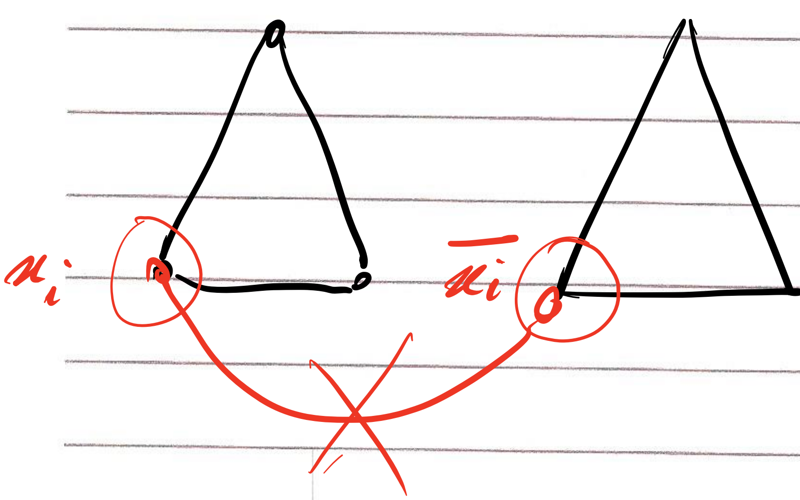
$C_3 = (\bar{x}_4 \vee x_5 \vee \bar{x}_3)$



Claim: The 3-SAT instance is satisfiable iff the graph G has an independent set of size k.

Proof: A) If the 3-SAT instance is satisfiable, then there is at least one node label per triangle that evaluates to 1.

Let S be a set containing one such node from each triangle



B) Suppose G has an independent set $S$ of size at least $k$.

if $x_i$ appears as a label in $S$ then set $x_i$ to $\underline{1}$

if $\overline{x_i}$ appears as a label in $S$ then set $x_i$ to $\underline{0}$

if neither $x_i$ nor $\overline{x_i}$ appear as a label in $S$, then set $x_i$ to either $\underline{0}$ or $\underline{1}$

Efficient Certification

To show efficient certification:

1. Polynomial length certificate

2. Polynomial time certifier

# Efficient certification

## 3-SAT

Certificate $t$ is an assignment of truth values to variables ($x_i$)

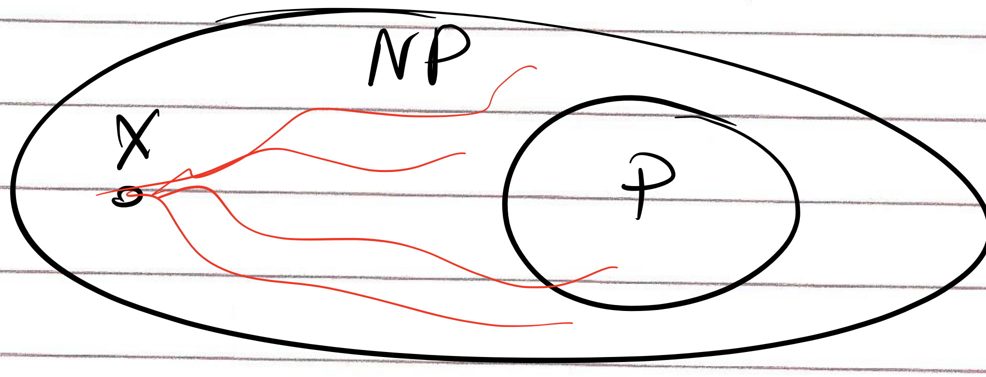
Certifier: evaluate the clauses. if all of them evaluate to 1 then it answers yes.

## Indep set

Certificate $t$ is a set of nodes of size at least $k$ in $G$.

Certifier: check each edge to make sure no edges have both ends in the set
check size of the set $\geq k$
no repeating nodes

class NP is the set of all problems for which there exists an efficient certifier




NP = P ? We don't know!

if $X \in NP$ and for all $Y \in NP$ $Y \leq_p X$, then X is the hardest problem in NP.

3-SAT has been proven to be such a problem.

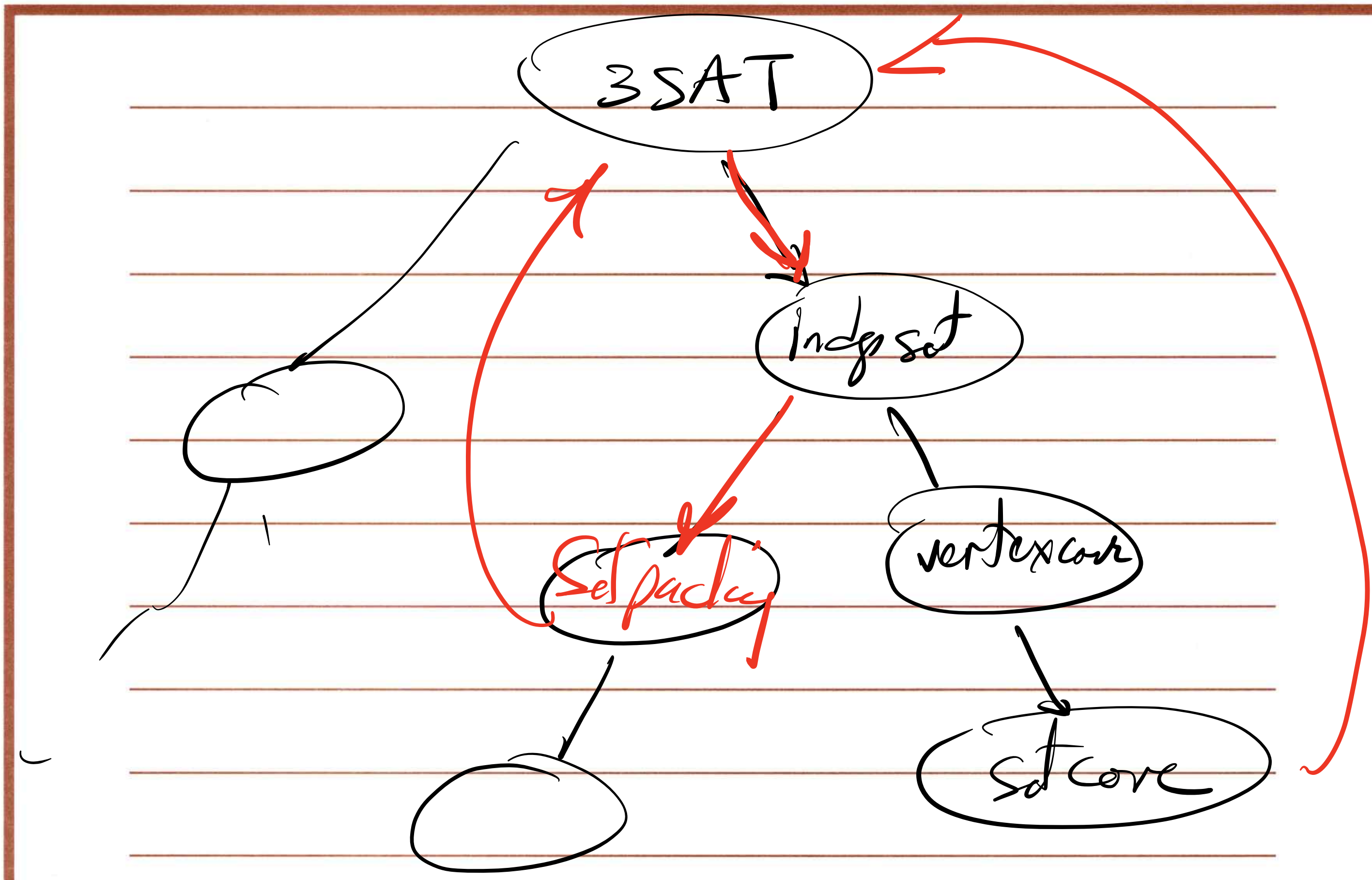
Such a problem is called NP-Complete

# Transitivity

if $Z \le_p Y$ and $Y \le_p X$

then $Z \le_p X$

3SAT $\le_p$ indepset $\le_p$ vertexcover $\le$ set cover

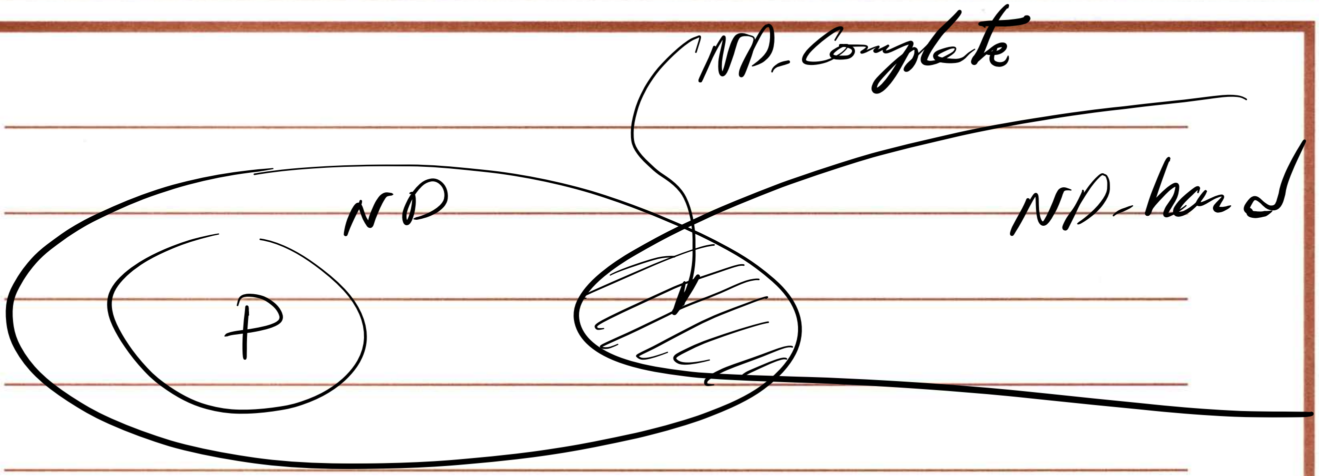


Basic Strategy to prove a problem $X$ is NP complete

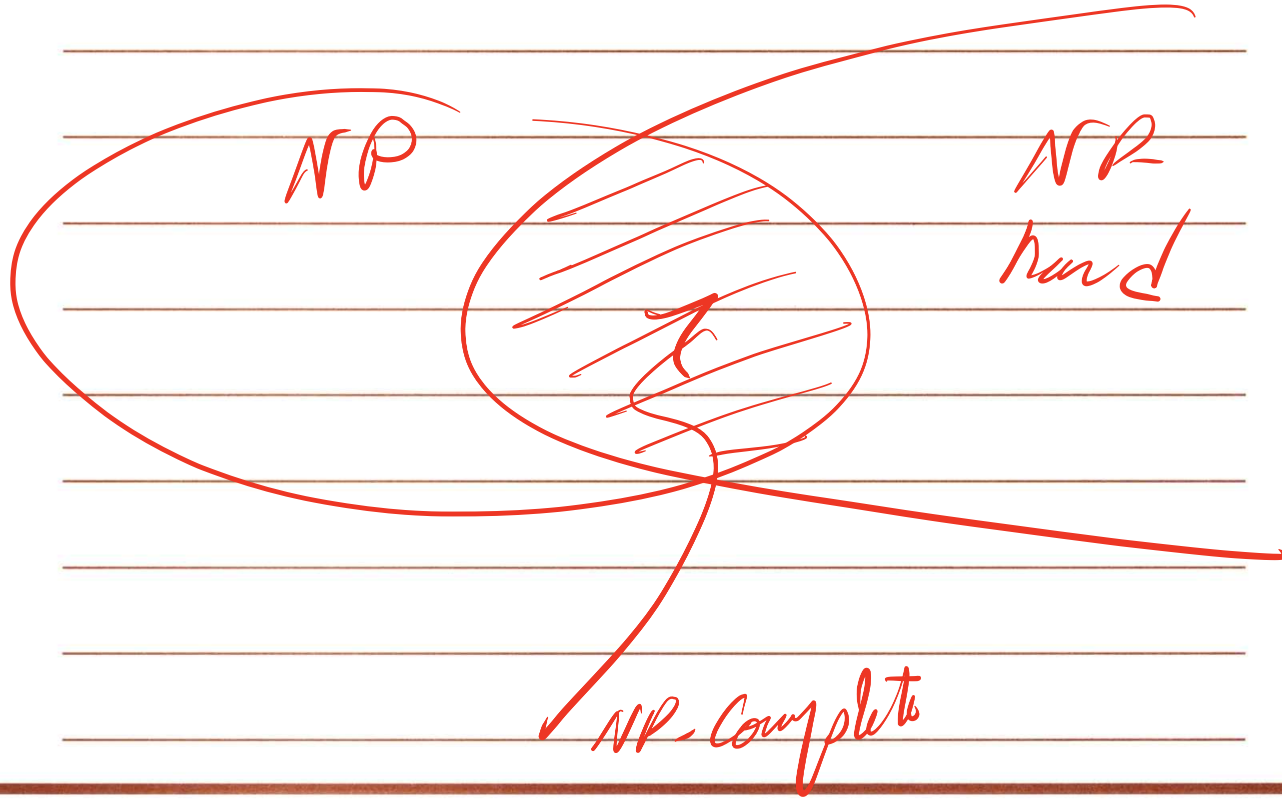
1- Prove $X \in NP$

2- Choose a problem $Y$ that is known to be NP complete

3- Prove that $Y \le_p X$




NP-hard is the class of problems that are at least as hard as NP-Complete problems


NP
NP-hard
NP-Complete

# Discussion 10

**1.** Given the SAT problem from lecture for a Boolean expression in Conjunctive Normal Form with any number of clauses and any number of literals in each clause. For example,

$$(X_1 \vee \neg X_3) \wedge (X_1 \vee \neg X_2 \vee X_4 \vee X_5) \wedge \ldots$$

Prove that SAT is polynomial time reducible to the 3-SAT problem (in which each clause contains at most 3 literals.)

**2.** The *Set Packing* problem is as follows. We are given $m$ sets $S_1, S_2, \ldots S_m$ and an integer $k$. Our goal is to select $k$ of the $m$ sets such that no selected pair have any elements in common. Prove that this problem is **NP**-complete.

**3.** The *Steiner Tree* problem is as follows. Given an undirected graph $G=(V,E)$ with nonnegative edge costs and whose vertices are partitioned into two sets, $R$ and $S$, find a tree $T \subseteq G$ such that for every $v$ in $R$, $v$ is in $T$ with total cost at most $C$. That is, the tree that contains every vertex in $R$ (and possibly some in $S$) with a total edge cost of at most $C$.
Prove that this problem is **NP**-complete.

**1.** Given the SAT problem from lecture for a Boolean expression in Conjunctive Normal Form with any number of clauses and any number of literals in each clause. For example,

$$(X_1 \vee \neg X_3) \wedge (X_1 \vee \neg X_2 \vee X_4 \vee X_5) \wedge \ldots$$

Prove that SAT is polynomial time reducible to the 3-SAT problem (in which each clause contains at most 3 literals.)

| SAT | | 3SAT |
|---|---|---|
| $(x_1)$ | $\longrightarrow$ | $(x_1 \vee x_1 \vee x_1)$ |
| $(x_1 \vee x_2)$ | $\longrightarrow$ | $(x_1 \vee x_2 \vee x_1)$ |
| $(x_1 \vee x_2 \vee x_3)$ | $\longrightarrow$ | $(x_1 \vee x_2 \vee x_3)$ |
| $(x_1 \vee x_2 \vee x_3 \vee x_4)$ | $\longrightarrow$ | $(x_1 \vee x_2 \vee S_1) \wedge (\bar{S}_1 \vee x_3 \vee x_4)$ |

$$(x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5)$$

$$\longrightarrow \quad (x_1 \vee x_2 \vee S_1) \wedge (\bar{S}_1 \vee x_3 \vee S_2) \wedge (\bar{S}_2 \vee x_4 \vee x_5)$$

**2.** The *Set Packing* problem is as follows. We are given $m$ sets $S_1, S_2, \ldots S_m$ and an integer $k$. Our goal is to select $k$ of the $m$ sets such that no selected pair have any elements in common. Prove that this problem is **NP**-complete.

1- Show Set Packing $\in$ NP

a- Certificate: a set of $\underline{k}$ sets which have no elements in common

b- form the intersection of all pairs of sets an check to see if the intersection is Null.

check the count of sets to make sure we have $\underline{k}$ sets.

2- Choose independent set

3- indep set $\leq_p$ Set packing

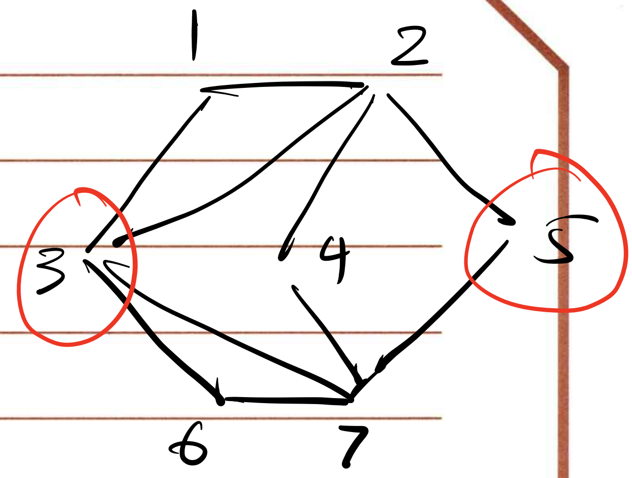
$S_1 = \{(1,2),(1,3)\}$

$S_2 = \{(1,2),(2,3),(2,4),(2,5)\}$

$S_3 =$

$S_5$

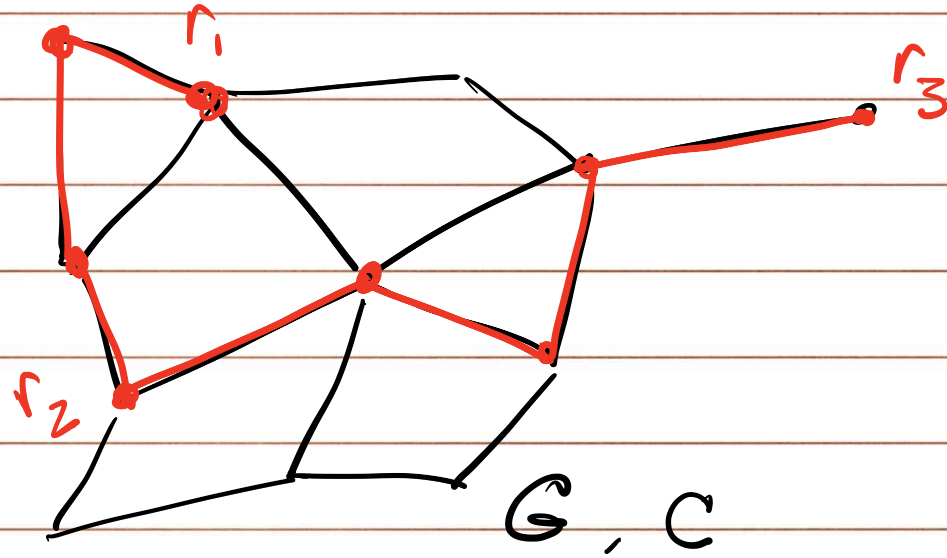
$S_9 =$




$G, k$

$k$

**3.** The *Steiner Tree* problem is as follows. Given an undirected graph $G=(V,E)$ with nonnegative edge costs and whose vertices are partitioned into two sets, $R$ and $S$, find a tree $T \subseteq G$ such that for every $v$ in $R$, $v$ is in $T$ with total cost at most $C$. That is, the tree that contains every vertex in $R$ (and possibly some in $S$) with a total edge cost of at most $C$.
Prove that this problem is **NP**-complete.



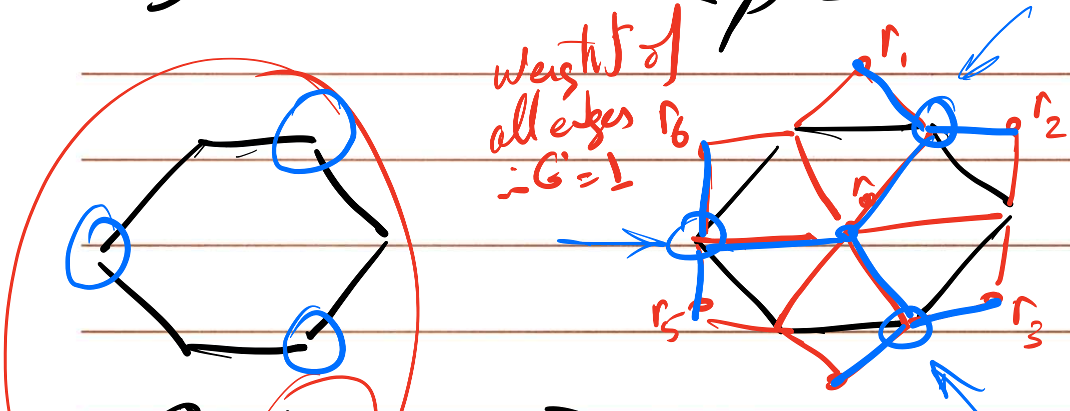

1- Show Steiner Tree $\in NP$

a- Certificate: Tree $T$
Spanning across all nodes in $R$
(and maybe some in $S$) w/ Cost $\leq C$

b- Certifier:
- Cost of $T \leq C$
- Covers all nodes in $R$
- $T$ is a tree

2- Choose Vertex Cover

3- Vertex Cover $\leq_p$ Steiner Tree




$G, k \longrightarrow G'$

$C = m + k$