

CS570
Analysis of Algorithms
Summer 2013
Exam III

Name: _____

Student ID: _____

_____ On Campus _____ DEN

	Maximum	Received
Problem 1	20	
Problem 2	20	
Problem 3	20	
Problem 4	20	
Problem 5	10	
Problem 6	10	
Total	100	

2 hr exam

Close book and notes

If a description to an algorithm is required please limit your description to within 150 words, anything beyond 150 words will not be considered.

1) 20 pts

Mark the following statements as **TRUE** or **FALSE**. No need to provide any justification.

[TRUE/FALSE] **True**

Assume $P \neq NP$. Let A and B be decision problems. If A is in NP-Complete and $A \leq_P B$, then B is not in P.

[TRUE/FALSE] **True**

There exists a decision problem X such that for all Y in NP, Y is polynomial time reducible to X.

It is in fact the Cook-Levin theorem that proves that there are problems that are NP-complete. Picking X to be 3-SAT (or any other in problem in NP-complete) ensures that ensures that every Y in NP is reducible to X.

[TRUE/FALSE] **True**

If P equals NP, then NP equals NP-complete.

True. A problem X is NP-hard iff any problem in NP can be reduced in polynomial time to X. If P equals NP, then we can reduce any problem in NP to any other problem by just solving the original problem.

[TRUE/FALSE] **False**

The running time of a dynamic programming algorithm is always $\theta(P)$ where P is the number of sub-problems.

Solution: False. The running time of a dynamic program is the number of subproblems times the time per subproblem. This would only be true if the time per subproblem is $O(1)$.

[TRUE/FALSE] **True**

A spanning tree of a given undirected, connected graph $G=(V,E)$ can be found in $O(|E|)$ time. You can just walk on the graph.

[TRUE/FALSE] **True**

To find the minimum element in a max heap of n elements, it takes $O(n)$ time

[TRUE/FALSE] **True**

Kruskal's algorithm for finding the MST works with positive and negative edge weights.

[TRUE/FALSE] **False**

If a problem is not in P, then it must be in NP.

[TRUE/FALSE] **False**

If an NP-complete problem can be solved in linear time, then all NP-complete problems can be solved in linear time.

[TRUE/FALSE] **True**

Linear programming problems can be solved in polynomial time.

2) 20 pts

Consider the following heuristic to compute a vertex cover of a connected undirected graph G . Pick an arbitrary vertex as the root and perform depth first search. Output the set of non-leaf vertices in the resulting depth first search tree.

(i) Show that the output is a vertex cover for G .

(ii) How good an approximation to a minimum vertex cover does this heuristic assure? That is, upper bound on the ratio of the number of vertices in the output to the number of vertices in a minimum vertex cover of G .

2) Let $G=(V,E)$ denote the graph, $T=(V,E')$ the resulting depth first search tree, L the set of leaf vertices in the depth first search tree and $N (=V - L)$ the set of non leaf vertices.

(i) Assume there is an edge $e=(u,v)$ in E that is not covered by N . This implies that both u and v are in L . Without loss of generality, assume that DFS explored u first. At this stage since e was available to DFS to leave u , the DFS would have left u to explore a new vertex thereby making u a non leaf. Hence our assumption is incorrect and N does indeed cover every edge in E .

(ii) If a vertex cover of G contains a vertex u in L , then u can be replaced with its parent in the DFS tree while still covering every edge in E without increasing the number of vertices. Hence there exists a min vertex cover (call A) of G that does not contain a vertex in L . In particular, A is also a vertex cover of the DFS tree T .

We next create a matching M as follows. Recall that a matching is a set of edges such that no two distinct edges in the set share a vertex.

For every vertex u in N , pick one edge that connects u to one of its descendants and call it e_u . We call the set of odd level non leaf vertices in the DFS tree ODD and the set of even level non leaf vertices as $EVEN$. If the set ODD is bigger or equal to the set $EVEN$, set $BIG:=ODD$. Else set $BIG:=EVEN$.

Since the total number of non leaf vertices in the tree T is $|N|$, BIG has size at least $|N|/2$. Now the edge set $M = \{u_e \mid u \text{ is in } BIG\}$ is a matching of size at least $|N|/2$.

Since M is a matching, to cover every edge in the matching the optimal vertex cover A has to contain at least $|M|$ vertices. (This is because in a matching no two distinct edges share a vertex). Thus A has to contain at least $|N|/2$ vertices while our solution has $|N|$ vertices. Hence our solution is at worst a 2-approximation.

It can be shown that our solution can be indeed twice as bad by considering $E = \{(a,b), (b,c)\}$ with DFS rooted at a .

3) 20 pts

There is a precious diamond that is on display in a museum at m disjoint time intervals. There are n security guards who can be deployed to protect the precious diamond. Each guard has a list of intervals for which he/she is available to be deployed. Each guard can be deployed to at most A time slots and has to be deployed to at least B time slots. Design an algorithm that decides if there is a deployment of guards to intervals such that each interval has either exactly one or exactly two guards deployed.

3.) We create a circulation network as follows. For the i th guard introduce a vertex g_i and for the j th time interval introduce a vertex t_j . If the i th guard is available for the j th interval, then introduce an edge from g_i to t_j of capacity 1. Add a source s and a sink t . To every guard vertex add an edge from s of capacity A and lower bound B . From every interval vertex add an edge to t of capacity 2 and lower bound 1. Add an edge from s to t of infinite capacity. We claim that there exists a valid deployment if and only if the above network has a valid circulation. The proof of the claim is similar to the survey design problem in the text. The algorithm proceeds by determining if the network has a circulation (by reducing it to a flow problem and then applying Ford-Fulkerson) and answers "yes" if and only if there is a circulation.

4) 20 pts

Your input is a string S which is a sequence of n characters from the English alphabet. The string S is believed to be a corrupted version of a text where the spacing between the words have been erased. You have access to a program `Dictionary()`, which takes a string w as input and returns "yes" if w is a valid word and "no" otherwise. Design an algorithm that decides if S can be partitioned (by inserting spaces) into a sequence of valid words. The running time should be polynomial in n assuming that each call to `Dictionary()` takes polynomial time.

For example, if $S = \text{"sortingiseasy"}$, then your algorithm should output "yes" , since $\text{"sorting", "is", "easy"}$ is a sequence of valid words.

Let the length of your input string of size N .

Let $b(n)$ be a boolean: true if the document can be split into words starting from position n in the string.

$b(N)$ is true (since the empty string can be split into 0 words). Given $b(N)$, $b(N - 1)$, ... $b(N - k)$, you can construct $b(N - k - 1)$ by considering all words that start at character $N - k - 1$. If there's any such word, w , with $b(N - k - 1 + \text{len}(w))$ set, then set $b(N - k - 1)$ to true. If there's no such word, then set $b(N - k - 1)$ to false.

Eventually, you compute $b(0)$ which tells you if the entire document can be split into words.

In pseudo-code:

```
def try_to_split(string):
    N = len(string)
    b = [False] * (N + 1)
    b[N] = True
    for i in range(N - 1, -1, -1):
        for word starting at position i:
            if b[i + len(word)]:
                b[i] = True
                break
    return b
```

There's some tricks you can do to get 'word starting at position i ' efficient, but you're asked for an $O(N^2)$ algorithm, so you can just look up every string starting at i in the dictionary.

To generate the words, you can either modify the above algorithm to store the good words, or just generate it like this:

```
def generate_words(doc, b, idx=0):
    length = 1
    while True:
        assert b[idx]
        if idx == len(string): return
        word = doc[idx: idx + length]
        if word in dictionary and b[idx + length]:
```

```
output(word)
idx += length
length = 1
```

Here b is the boolean array generated from the first part of the algorithm.

5) 10 pts

Show that the following problem is NP-complete:

For an undirected graph $G=(V,E)$, does G have a spanning tree with at most 2 leaf vertices?

5.) Call the decision problem in question ST2L.

Given the graph G (instance) and a set of edges that forms a spanning tree (certificate), we can verify in polynomial time if the set of edges indeed forms a spanning tree and that spanning tree has at most two leaves. Thus the ST2L problem is indeed in NP.

All that is left is to prove that ST2L is NP-Hard. Recall that a leaf vertex in a tree is a vertex of degree 1. Thus a spanning tree has to have at least 2 leaves. A spanning tree with exactly 2 leaves is in one to one correspondence with a Hamiltonian path. Hence our problem is merely a restatement of the Hamiltonian Path problem. Hence you can use the reduction in the last HW to reduce Hamiltonian cycle to Hamiltonian Path (and then Hamiltonian Path to ST2L.)

(Note: If you assumed that the spanning tree is rooted and that the root is not a leaf even if it has degree one, then we need to slightly modify the argument that reduces Hamiltonian path to ST2L. Given a graph G as an instance of Ham-Path, if G has exactly 2 vertices and one edge output "yes", Else call the blackbox that solves ST2L with input G and return the result.)

6) 10 pts

The following are a few of the design strategies we learned in class to solve problems.

1. Dynamic programming.
2. Greedy strategy.
3. Divide-and-conquer.

For each of the following problems, mention which of the above design strategies (or combinations of strategies) we used in class to solve these problems:

1. Determining if a graph has a negative cycle
Dynamic programming
2. Minimum spanning tree
Greedy strategy
3. Closest pair of points on a plane
Divide and conquer
4. Memory efficient sequence alignment
Dynamic programming + divide and conquer
5. Stable matching
Greedy Strategy

