大作业

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目录

1 一阶倒立摆

1.1 问题描述

假设小车的质量为 M，倒立摆的长度为 2l, 质量为 m，且集中在摆中心，摆距离平衡状态的倾角为 θ

图 1: 一阶倒立摆

1.2 系统建模

拉格朗日方程建模:

$$T_{car} = \frac{1}{2}M\dot{x}^2$$
$$T_{ball} = \frac{1}{2}m(l^2\dot{\theta}^2 + \dot{x}^2 - 2l\cos\theta\dot{\theta}\dot{x})$$
$$V_{car} = 0$$
$$V_{ball} = mgl\cos\theta$$
$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 + \dot{x}^2 - 2l\cos\theta\dot{\theta}\dot{x}) - mgl\cos\theta$$

列写欧拉拉格朗日方程:

$$F = \frac{\partial L}{\partial x} - \frac{d}{dt}[\frac{\partial L}{\partial \dot{x}}]$$
$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}[\frac{\partial L}{\partial \dot{\theta}}]$$

化简可得:

$$F = (M+m)\ddot{x} + ml\sin\theta\dot{\theta}^2 - ml\cos\theta\ddot{\theta}$$
$$0 = mgl\sin\theta + ml^2\ddot{\theta} + ml\cos\theta\ddot{x}$$

在 $\theta = 0$ 处线性化，得到:

$$F = (M+m)\ddot{x} - ml\ddot{\theta}$$
$$0 = mgl\theta + ml^2\ddot{\theta} - ml\ddot{x}$$
$$\ddot{x} = \frac{F - mg\theta}{M}$$
$$\ddot{theta} = \frac{F - (m+M)\theta}{Ml}$$

取状态变量 $x_1=\theta, x_2=\dot{\theta}, x_3=x, x_4=\dot{x}, u=F$ 得到状态空间模型

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(m+M)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$$

1.3 能控性分析

能控性判据:

$$rank[B; AB; A^2B; ...; A^{n-1}B] = dim(A)$$

利用 matlab 可得:

$$rank(ctrb(A,B)) = dim(A)$$

所以系统能控

1.4 控制器设计

控制目标使单摆末端时刻 10s 后保持平衡即 $\theta=0$, $\dot{\theta}=0$, 并且控制能量消耗尽可能小, 由此构建优化问题:

$$\min J(u) = \frac{1}{2}x(t_f)^T Hx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[x^T(t)Qx(t) + u^T(t)Ru(t)]dt$$

$$\begin{aligned} s.t.\ \ & \dot{x}(t) = Ax(t) + Bu(t) \\ & x(0) = x_0 \\ & x_1(10) = 0 \\ & x_2(10) = 0 \\ & H = Q = diag(1000, 10, 0, 0) \\ & R = 1 \end{aligned}$$

引入协态变量 $p_1(t), p_2(t), p_3(t), p_4(t)$ 得到 Hamilton 函数

$$
\begin{aligned}
H(x(t), u(t), p(t), t) = \frac{1}{2}x^T(t)Qx(t) + \frac{1}{2}u^T(t)Ru(t) \\
+ p_1(t)x_2(t) + p_2(t)(\frac{u(t)}{Ml} - \frac{(m+M)g}{Ml}x_1(t)) \\
+ p_3(t)x_4(t) + p_4(t)(-\frac{mg}{M}x_1(t) + \frac{u(t)}{M})
\end{aligned}
$$

利用极值条件可以得到:

$$
0 = \frac{\partial H}{\partial u} = Ru(t) + \frac{p_2(t)}{Ml} + \frac{p_4(t)}{M}
$$

利用协态方程可以得到:

$$
\begin{aligned}
\dot{p}_1(t) &= -\frac{\partial H}{\partial x_1} = p_2(t)\frac{(m+M)g}{Ml} + p_4(t)\frac{mg}{M} \\
\dot{p}_2(t) &= -\frac{\partial H}{\partial x_2} = p_1(t) \\
\dot{p}_3(t) &= -\frac{\partial H}{\partial x_3} = 0 \\
\dot{p}_4(t) &= -\frac{\partial H}{\partial x_4} = -p_3(t)
\end{aligned}
$$

终端时刻固定，部分状态自由，得到边界条件:

$$
\begin{aligned}
h(x(t_f)) = \frac{1}{2}x(t_f)^T Hx(t_f) \\
\frac{\partial h(x(t_f))}{\partial x} - p(t_f) = 0
\end{aligned}
$$

1.5 Matlab 建模

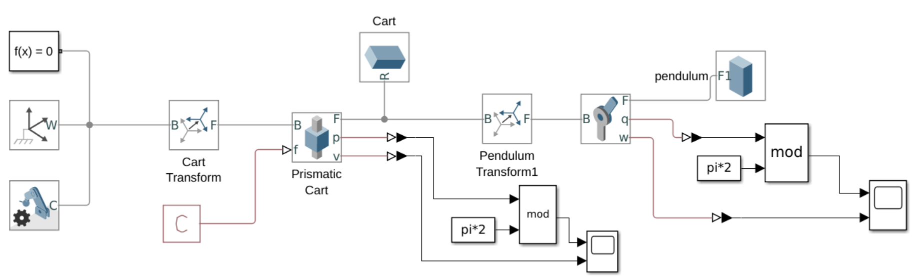

图 2: Simscape

1.6 Matlab 计算以及控制效果

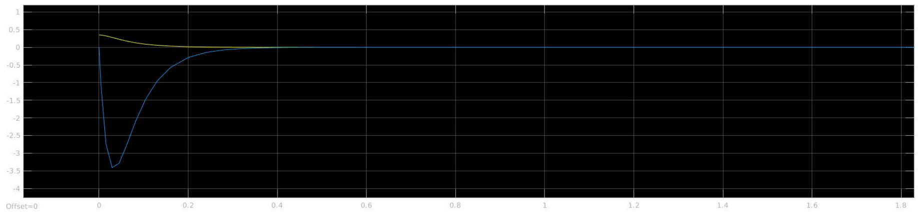

图 3: LQR control, 黄色为角度，蓝色为角速度

响应过程见附件视频

2 二阶倒立摆

2.1 问题描述

假设小车的质量为 M，两级倒立摆的长度分别为 $2l_1, 2l_2$, 质量分别为 m_1, m_2，且集中在摆重心，摆距离平衡状态的倾角为 θ

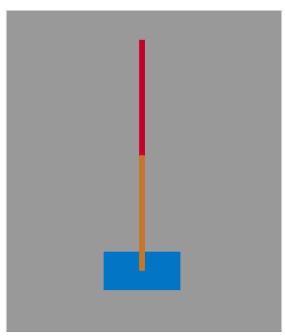

图 4: 二阶倒立摆

2.2 系统建模

拉格朗日方程建模:

$$T_{car} = \frac{1}{2}M\dot{x}^2$$

$$T_{ball_1} = \frac{1}{2}m_1(l_1^2\dot{\theta}_1^2 + \dot{x}^2 - 2l_1\cos\theta_1\dot{\theta}_1\dot{x})$$

$$T_{ball_2} = \frac{1}{2}m_2[(l_2\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) - \dot{x} + 2l_1\dot{\theta}_1\cos\theta_1)^2 + (l_2\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + 2l_1\dot{\theta}_1\sin\theta_1)^2]$$

$$V_{car} = 0$$

$$V_{ball_1} = m_1gl_1\cos\theta_1$$

$$V_{ball_2} = m_2g(2l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2))$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}[\frac{\partial L}{\partial \dot{x}}] = 0$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}[\frac{\partial L}{\partial \dot{\theta}_1}] = 0$$

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}[\frac{\partial L}{\partial \dot{\theta}_2}] = 0$$

化简可得:

$$\begin{aligned}
L =& T - V \\
=& \frac{1}{2}M\dot{x}^2 \\
&+ \frac{1}{2}m_1(l_1^2\dot{\theta}_1^2 + \dot{x}^2 - 2l_1\cos\theta_1\dot{\theta}_1\dot{x}) \\
&+ \frac{1}{2}m_2\{l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 4l_1^2\dot{\theta}_1^2 + 4l_1l_2\dot{\theta}_1\cos\theta_2 - 2\dot{x}[l_2\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + 2l_1\dot{\theta}_1\cos\theta_1] + \dot{x}^2\} \\
&- m_1gl_1\cos\theta_1 - m_2g(2l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2))
\end{aligned}$$

进而得到

$$
\begin{aligned}
u =&\frac{d}{dt}[M\dot{x} + m_1\dot{x} - m_1 l_1 \cos\theta_1\dot{\theta}_1 + m_2 l_2 \cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + m_2 l_1\dot{\theta}_1\cos\theta_1 + m_2\dot{x}]\\
0 =&m_1 l_1 \sin\theta_1\dot{\theta}_1\dot{x} - 2m_2 l_1 l_2\dot{\theta}_1\sin\theta_2 + m_2\dot{x}[l_2\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + 2l_1\dot{\theta}_1\sin\theta_1]\\
&+ m_1 g l_1\sin\theta_1 + m_2 g(2l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2))\\
&- \frac{d}{dt}[m_1 l_1^2\dot{\theta}_1 - m_1 l_1\cos\theta_1\dot{x} + m_2 l_2^2(\dot{\theta}_1+\dot{\theta}_2) + 4m_2 l_1^2\dot{\theta}_1 + 2m_2 l_1 l_2\cos\theta_2\\
&\qquad - m_2\dot{x}l_2\cos(\theta_1+\theta_2) - 2m_2\dot{x}l_1\cos\theta_1]\\
0 =&-2m_2 l_1 l_2\dot{\theta}_1\sin\theta_2 + m_2\dot{x}l_2\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + m_2 g l_2\sin(\theta_1+\theta_2)\\
&- \frac{d}{dt}[m_2 l_2^2(\dot{\theta}_1+\dot{\theta}_2) - m_2\dot{x}l_2\cos(\theta_1+\theta_2)]
\end{aligned}
$$

化简得到:

$$
\begin{aligned}
u =&(M+m_1+m_2)\ddot{x} + (m_2-m_1)l_1(\cos\theta_1\ddot{\theta}_1 - \sin\theta_1\dot{\theta}_1^2) + m_2 l_2(\cos(\theta_1+\theta_2)(\ddot{\theta}_1+\ddot{\theta}_2)\\
&- \sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)^2)\\
0 =&m_1 l_1\sin\theta_1\dot{\theta}_1\dot{x} - 2m_2 l_1 l_2\dot{\theta}_1\sin\theta_2 + m_2\dot{x}[l_2\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + 2l_1\dot{\theta}_1\sin\theta_1]\\
&+ m_1 g l_1\sin\theta_1 + m_2 g(2l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2))\\
&- [(m_1+4m_2)l_1^2\ddot{\theta}_1 - (m_1+2m_2)l_1(\cos\theta_1\ddot{x} - \sin\theta_1\dot{\theta}_1\dot{x}) + 4m_2 l_2^2\ddot{\theta}_1 - 2m_2 l_1 l_2\sin\theta_2\dot{\theta}_2\\
&\quad - m_2 l_2(\cos(\theta_1+\theta_2)\ddot{x} - \sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)\dot{x})]\\
=&\ddot{x}l_2 m_2 cos(\theta_1+\theta_2) - 4\ddot{\theta}_1(l_1^2+l_2^2)m_2 - \ddot{\theta}_1 l_1^2 m_1 + g l_2 m_2\sin(\theta_1+\theta_2) + \ddot{x}l_1(m_1+2m_2)\cos\theta_1\\
&+ g l_1(m_1 + 2*m_2)\sin(\theta_1) - 2\dot{\theta}_1 l_1 l_2 m_2\sin\theta_1 + 2\dot{\theta}_2 l_1 l_2 m_2\sin\theta_2\\
0 =&-2m_2 l_1 l_2\dot{\theta}_1\sin\theta_2 + m_2 g l_2\sin(\theta_1+\theta_2) - m_2 l_2^2(\ddot{\theta}_1+\ddot{\theta}_2)
\end{aligned}
$$

线性化可得:

$$
\begin{aligned}
\ddot{\theta}_1 = & \frac{gl_1m_2^2 + 2gl_1m_1m_2 + gl_2m_1m_2 + 3Mgl_1m_2 + Mgl_2m_2}{6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l_1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m_2}\theta_1 \\
& + \frac{-(2gl_1m_2^2 + gl_1m_1m_2 - gl_2m_1m_2 - Mgl_2m_2)}{(6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l_1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m_2)}\theta_2 \\
& \frac{(l_1m_1 + 2l_1m_2 + l_2m_2)}{(6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l_1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m_2)}u \\
\ddot{\theta}_2 = & \frac{6gl_1^2m_2^2 + 4gl_2^2m_2^2 + Mgl_1^2m_1 + 4Mgl_1^2m_2 + 3Mgl_2^2m_2 + 4gl_1^2m_1m_2 + 3gl_2^2m_1m_2 - 3Mgl_1l_2m_2 - 3gl_1}{l_2(6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l_1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m} \\
& + \frac{6gl_1^2m_2^2 + 4gl_2^2m_2^2 + Mgl_1^2m_1 + 4Mgl_1^2m_2 + 3Mgl_2^2m_2 + 3gl_1l_2m_2^2 + 4gl_1^2m_1m_2 + 3gl_2^2m_1m_2}{l_2(6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m_2)}\theta_2 \\
& + \frac{-(l_2^2m_2 + l_1l_2m_1 + 2l_1l_2m_2)}{l_2(6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l_1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m_2)}u \\
\ddot{x} = & \frac{-(7gl_1^2m_2^2 - 2gm_1l_1^2m_2 + gl_1l_2m_2^2 - gm_1l_1l_2m_2 + 4gl_2^2m_2^2)}{6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l_1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m_2)}\theta_1 \\
& + \frac{-(4gl_1^2m_2^2 + gm_1l_1^2m_2 + gl_1l_2m_2^2 - gm_1l_1l_2m_2 + 4gl_2^2m_2^2)}{6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l_1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m_2}\theta_2 \\
& + \frac{l_1^2m_1 + 4l_1^2m_2 + 4l_2^2m_2}{6l_1^2m_2^2 + 4l_2^2m_2^2 + l_1l_2m_2^2 + 4l_1^2m_1m_2 + 4l_2^2m_1m_2 + Ml_1^2m_1 + 4Ml_1^2m_2 + 4Ml_2^2m_2 - l_1l_2m_1m_2}u
\end{aligned}
$$

带入条件 $M = 1kg, m_1 = 0.2kg, m_2 = 0.1kg, l_1 = 0.4, l_2 = 0.2, g = 9.8N/kg$ 可得:

$$
\begin{aligned}
\ddot{\theta}_1 &= \frac{5831}{416}\theta_1 + \frac{735}{416}\theta_2 + \frac{875}{624}u \\
\ddot{\theta}_2 &= \frac{14553}{416}\theta_1 + \frac{19649}{416}\theta_2 - \frac{875}{624}u \\
\ddot{x} &= -\frac{49}{260}\theta_1 - \frac{49}{260}\theta_2 + \frac{25}{26}u
\end{aligned}
$$

构造状态空间模型:

$$
\begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{833}{78} & 0 & \frac{98}{39} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{539}{39} & 0 & \frac{1715}{78} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{49}{260} & 0 & -\frac{49}{260} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \\ x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{125}{117} \\ 0 \\ -\frac{125}{117} \\ 0 \\ \frac{25}{26} \end{bmatrix} u
$$

2.3 能控性分析

对带入条件之前的等式进行化简，记:

$$\ddot{\theta}_1 = a_1\theta_1 + b_1\theta_2 + c_1 u$$
$$\ddot{\theta}_2 = a_2\theta_1 + b_2\theta_2 + c_2 u$$
$$\ddot{x} = a_3\theta_1 + b_3\theta_2 + c_3 u$$

状态空间方程:

$$\begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ a_1 & 0 & b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_2 & 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_3 & 0 & b_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \\ x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ c_1 \\ 0 \\ c_2 \\ 0 \\ c_3 \end{bmatrix} u$$

利用能控性判据:

$$Q = rank([B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B])$$

$Det(Q) = -(a_2c_1^2 - b_1c_2^2 - a_1c_1c_2 + b_2c_1c_2)^2(a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1)^2$

只要行列式不为零，即系统可控，其中 $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ 需要带入
线性化之后的表达式

2.4 控制效果

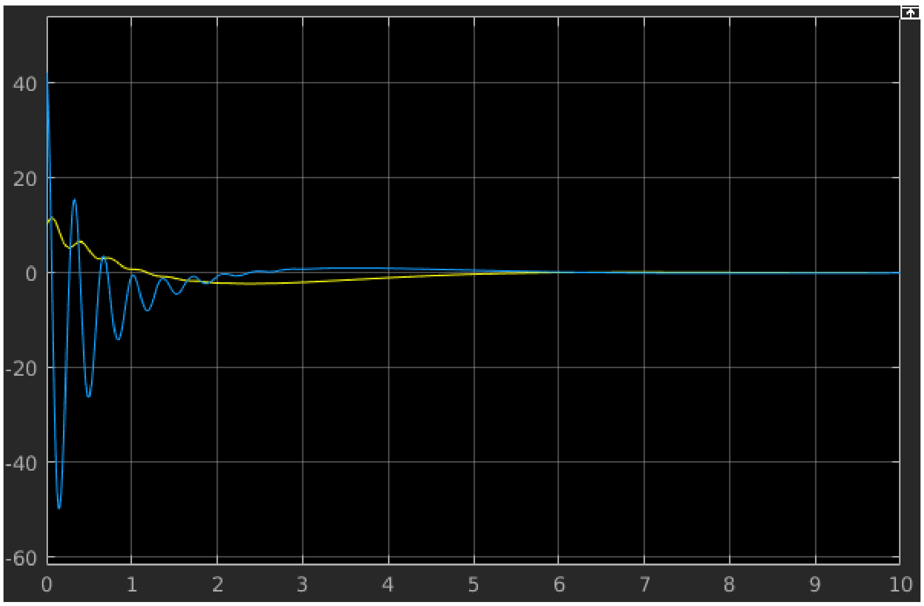

图 5: $theta_1$ 的响应曲线, 黄色为角度，蓝色为角速度

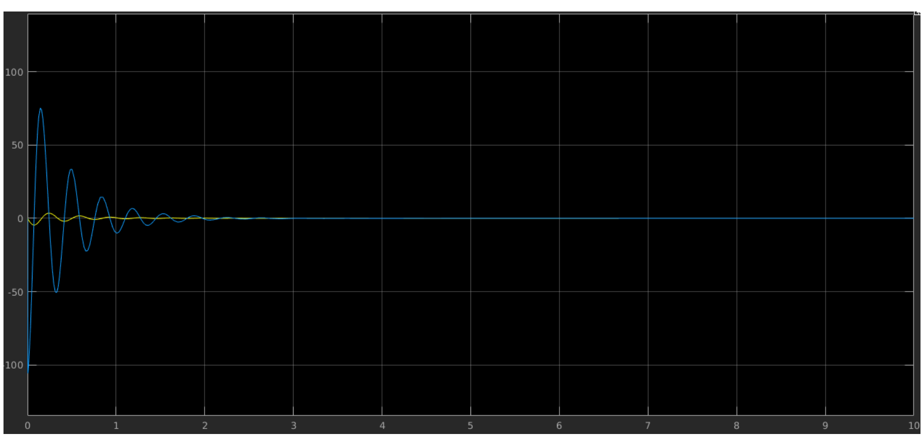

图 6: $theta_2$ 的响应曲线, 黄色为角度，蓝色为角速度

相应过程见视频附件