Lecture4

zhike chen

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1 笔记

1.1 Hamilton 方程组

针对最简最优控制问题：状态变量 $x(t) : [t_0, t_f] \to \mathbf{R}^n$ 以及控制变量 $u(t) : [t_0, t_f] \to \mathbf{R}^m$ 均为连续可微函数。其中被控对象满足状态方程和初值条件：

$$\dot{x}(t) = f(x(t), u(t), t), \quad t \in [t_0, t_f]$$

$$x(t_0) = x_0$$

控制目标为在固定终端时刻 t_f，达到固定的终端状态 $x(t_f) = x_f$ 求解最优控制 $u(t)$，最小化 Lagrange 形式的性能指标

$$J(u) = \int_{t_0}^{t_f} g(x(t), u(t), t)dt$$

求解过程：引入连续可微的 Lagrange 乘子函数 $p(t) : [t_0, t_f] \to \mathbf{R}^n$ 得到增广的性能指标：

$$\hat{g}(x(t), \dot{x}(t), u(t), t) = g(x(t), u(t), t) + p(t)[f(x(t), u(t), t) - \dot{x}(t)]$$

$$\hat{J} = \int_{t_0}^{t_f} \hat{g}(x(t), u(t), t)dt$$

利用 Euler-Lagrange 方程：

$$0 = \frac{\partial \hat{g}}{\partial x}(x(t), \dot{x}(t), u(t), p(t), t) - \frac{d}{dt}[\frac{\partial \hat{g}}{\partial \dot{x}}(x(t), \dot{x}(t), u(t), p(t), t)]$$

$$0 = \frac{\partial \hat{g}}{\partial u}(x(t), \dot{x}(t), u(t), p(t), t) - \frac{d}{dt}[\frac{\partial \hat{g}}{\partial \dot{u}}(x(t), \dot{x}(t), u(t), p(t), t)]$$

$$0 = \frac{\partial \hat{g}}{\partial p}(x(t), \dot{x}(t), u(t), p(t), t) - \frac{d}{dt}[\frac{\partial \hat{g}}{\partial \dot{p}}(x(t), \dot{x}(t), u(t), p(t), t)]$$

化简可得:

$$
\begin{aligned}
0 &= \frac{\partial g}{\partial x}(x(t),u(t),t) + p^T(t)\frac{\partial f}{\partial x}(x(t),u(t),t) - \dot{p}(t) \\
0 &= \frac{\partial g}{\partial u}(x(t),u(t),t) + p^T(t)\frac{\partial f}{\partial u}(x(t),u(t),t) \\
0 &= f(x(t),u(t),t) - \dot{x}(t)
\end{aligned}
$$

引入 Hamiltonian 函数:

$$H(x(t),u(t),p(t),t) = g(x(t),u(t),t) + p(t) * f(x(t),u(t),t)$$

结合 E-L 方程可得 Hamiltonian 方程:

$$
\begin{aligned}
\frac{\partial H}{\partial p}(x(t),u(t),p(t),t) &= \dot{x}(t) \\
\frac{\partial H}{\partial x}(x(t),u(t),p(t),t) &= \dot{p}(t) \\
\frac{\partial H}{\partial u}(x(t),u(t),p(t),t) &= 0
\end{aligned}
$$

其中第一个方程被称为状态方程、第二个方程为协态方程、第三个方程为极值条件

2 作业

2.1 女王围城问题

对于平面上位于上半平面的所有长为 l、经过 (-a,0) 和 (a,0) 的曲线, 哪一条与 [-a,a] 围城的面积最大

格林公式：平面积分与线积分之间的关系

$$
\begin{aligned}
S &= \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy \\
&= \oint_D Pdx + Qdy
\end{aligned}
$$

问题建模：对曲线 $y(x)$ 进行参数化得到参数方程：$x(t), y(t), t \in [t_0, t_f]$，则曲线的长度需要满足约束:

$$\int_{t_0}^{t_f} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} dt = L$$

曲线和 x 轴围成的面积:

$$\begin{aligned}
S &= \int\int_D dxdy \\
&= \frac{1}{2}\oint_D xdy - ydx \\
&= \frac{1}{2}\int_{t_0}^{t_f} x(t)\dot{y}(t) - y(t)\dot{x}(t)dt \quad Q = \frac{1}{2}x,\ \ P = \frac{1}{2}y,\ \ with \quad Green
\end{aligned}$$

构造优化问题:

$$\begin{aligned}
\max J(y) = \frac{1}{2}\int_{t_0}^{t_f} x(t)\dot{y}(t) - y(t)\dot{x}(t)dt \\
s.t. \quad \int_{t_0}^{t_f} \sqrt{\dot{x}^2(\tau) + \dot{y}^2(\tau)}d\tau = L \\
x(t_0) = -1,\ y(t_0) = 0 \\
x(t_f) = 1,\ y(t_f) = 0
\end{aligned}$$

引入状态变量 $z(t) = \int_{t_0}^{t_f} \sqrt{\dot{x}^2(\tau) + \dot{y}^2(\tau)}d\tau$ ，得到状态方程

$$\dot{z}(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$$

引入拉格朗日乘子 $\lambda \in \mathbf{R}$ 得到增广目标:

$$\hat{g}(x(t), y(t), \dot{x}(t), \dot{y}(t), \dot{z}(t), \lambda) = \frac{1}{2}[x(t)\dot{y}(t) - y(t)\dot{x}(t)] - \lambda[\sqrt{\dot{x}^2(t) + \dot{y}^2(t)} - \dot{z}(t)]$$

得到拉格朗日方程:

$$\begin{aligned}
0 &= \frac{\partial \hat{g}}{\partial x}(x(t), y(t), \dot{x}(t), \dot{y}(t), \dot{z}(t), \lambda) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x(t), y(t), \dot{x}(t), \dot{y}(t), \dot{z}(t), \lambda)] \\
0 &= \frac{\partial \hat{g}}{\partial y}(x(t), y(t), \dot{x}(t), \dot{y}(t), \dot{z}(t), \lambda) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{y}}(x(t), y(t), \dot{x}(t), \dot{y}(t), \dot{z}(t), \lambda)] \\
0 &= \frac{\partial \hat{g}}{\partial \lambda}(x(t), y(t), \dot{x}(t), \dot{y}(t), \dot{z}(t), \lambda)
\end{aligned}$$

化简得到:

$$\begin{aligned}
0 &= \dot{y}(t) - \frac{d}{dt}[\frac{\lambda \dot{x}(t)}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}] \\
0 &= -\dot{x}(t) - \frac{d}{dt}[\frac{\lambda \dot{y}(t)}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}] \\
0 &= \dot{z}(t) - \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}
\end{aligned}$$

积分得到:

$$y(t) - c_1 = \frac{\lambda \dot{y}(t)}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}$$
$$x(t) - c_2 = \frac{\lambda \dot{x}(t)}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}$$
$$(x(t) - c_2)^2 + (y(t) - c_1)^2 = \lambda^2$$

证得曲线为圆弧即 $x = c_2 + \lambda cost$，$y = c_1 + \lambda sint$ ，带入条件 l=$\frac{10\pi}{3}$，a=1 可得:

$$c_2 = 0$$
$$c_1 = \sqrt{3}$$
$$\lambda = 2$$