# Lecture5

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October 2021

# 1 笔记

## 1.1 LQ

假设有一线性系统可以表示为:

$$\begin{aligned}
&\dot{x} = Ax + Bu \\
&x(0) = x_0 \\
&0 \le t \le t_f
\end{aligned}$$

设计控制指令 $u(t)$，最小化下列指标:

$$J = \frac{1}{2}[x^T(t_f)S_f x(t_f) + \int_0^{t_f}(x^T Qx + u^T Ru)dt]$$

即

$$\begin{aligned}
&\min J = \frac{1}{2}[x^T(t_f)S_f x(t_f) + \int_0^{t_f}(x^T Qx + u^T Ru)dt] \\
&s.t \quad \dot{x} = Ax + Bu
\end{aligned}$$

利用拉格朗日乘子法得到增广目标:

$$\begin{aligned}
\hat{g}(x(t), u(t), \dot{x}(t), \lambda) &= g(x(t), u(t), t) + \lambda^T[Ax(t) + Bu(t) - \dot{x}(t)] \\
&= \frac{1}{2}(x^T(t)Qx(t) + u^T(t)Ru(t)) + \lambda^T[Ax(t) + Bu(t) - \dot{x}(t)]
\end{aligned}$$

1) 利用 E-L 方程得到:

$$\begin{aligned}
&0 = \frac{\partial \hat{g}}{\partial x}(x(t), u(t), \dot{x}(t), \lambda) - \frac{d}{dt}[\frac{\partial \hat{g}}{\partial \dot{x}}(x(t), u(t), \dot{x}(t), \lambda)] \\
&0 = \frac{\partial \hat{g}}{\partial u}(x(t), u(t), \dot{x}(t), \lambda)
\end{aligned}$$

2) 或构造 Hamiltonian

$$H(x(t),u(t),\dot{x}(t),\lambda)=\frac{1}{2}(x^T(t)Qx(t)+u^T(t)Ru(t))+\lambda(t)[Ax(t)+Bu(t)]$$

并构建 Hamilton 方程

$$\frac{\partial H}{\partial u}=0$$
$$\frac{\partial H}{\partial \lambda}=\dot{x}$$
$$\frac{\partial H}{\partial x}=\dot{\lambda}$$

化简得到:

$$0=Qx(t)+A^T\lambda(t)-\dot{\lambda}$$
$$0=Ru(t)+B^T\lambda$$

进一步求得 u:

$$u(t)=-R^{-1}B^T\lambda(t)$$
$$\dot{\lambda}(t)=Qx-A^T\lambda(t)$$

假设 $\lambda(t)=P(t)x(t)$ ，可得:

$$u(t)=-R^{-1}B^TP(t)x(t)=-K(t)x(t)$$
$$\dot{x}(t)=Ax(t)-BR^{-1}B^TP(t)x(t)$$
$$\dot{\lambda}(t)=Qx(t)-A^TP(t)x(t)=\dot{P}(t)x(t)+P(t)\dot{x}(t)$$

进一步化简得到 Riccati 方程:

$$-\dot{P}(t)=A^TP+PA-PBR^{-1}B^TP+Q$$
$$P(t_f)=P_f$$

# 2 作业

## 2.1 能量最优控制

系统状态方程为:

$$\dot{x}_1(t)=x_2(t)$$
$$\dot{x}_2(t)=u(t)$$

$t_0 = 0, t_f = 2$，要将状态从初始时刻的 $x(t_0) = [-2, 1]^T$ 到达终点时刻 $x(t_f) = [0, 0]^T$，目标为最小化控制能量:

$$
\begin{aligned}
\min J &= \int_0^2 \frac{1}{2} u^2(t) \\
s.t. \;\; \dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= u(t) \\
x_1(0) &= -2 \\
x_2(0) &= 1 \\
x_1(2) &= 0 \\
x_2(2) &= 0
\end{aligned}
$$

引入拉格朗日方程

$$H(x, u, \dot{x}, p) = \frac{1}{2} u(t)^2 + p_1[x_2(t) - \dot{x}_1(t)] + p_2[u(t) - \dot{x}_2(t)]$$

根据 E-L 方程得到:

$$
\begin{aligned}
0 &= \dot{p}_1(t) \\
0 &= p_1(t) - \dot{p}_2(t) \\
0 &= 2u(t) + p_2(t)
\end{aligned}
$$

进一步化简可得:

$$
\begin{aligned}
p_2(t) &= c_1 t + c_2 \\
u(t) &= -\frac{1}{2} p_2(2)
\end{aligned}
$$

带入边界条件可得:

$$u(t) = -\frac{3}{2} t + 1$$