最优化与最优控制 2

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1 笔记

1.1 变分法概念

泛函: 从任意定义域 Ω 到实数域或负数域的映射

泛函变分：若泛函增量可以写成函数变分的线性泛函及其高阶无穷小两项部分之和：

$$\Delta J(x, \delta x) = \delta J(x, \delta x) + g(x, \delta x)||\delta x||$$

则称 δJ 为泛函 J 对于 x 的泛函变分

泛函极值的必要条件：

$$\delta J(x, \delta x) = 0$$

引理：若连续函数 $f(x) : [t_0, t_f] \to \mathbf{R}$，对于任意满足 $h(t_0) = h(t_f) = 0$ 的连续函数 $h(t) : [t_0, t_f] \to \mathbf{R}$ 都有：

$$\int_{t_0}^{t_f} f(t)h(t) = 0$$

则有 $f(x) = 0, t \in [t_0, t_f]$

1.2 欧拉-拉格朗日方程推导

最简变分问题的欧拉-拉格朗日方程：状态变量 $x(t) : [t_0, t_f] \to \mathbf{R}^n$ 连续可微，在给定初始时刻 t_0 状态为 $x(t_0) = x_0$，在给定终端时刻 t_f 状态为 $x(t_f) = x_f$。函数 g 取值于 $\mathbf{R}$，二阶连续可微，则，状态变量 x 最小化性能指标

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt$$

的必要条件是对于任意时刻 $t \in [t_0, t_f]$,

$$\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)] = 0$$

严格证明：引入连续可微的函数变分 $\delta x(t) : [t_0, t_f] \to \mathbf{R}^n$，以保证施加扰动后 $x + \delta x$ 依然连续可微。δx 需要同时满足 $\delta x(t_0) = 0, \delta x(t_f) = 0$。首先计算 J 的泛函增量：

$$\begin{aligned}
\Delta J(x, \delta x) = J(x + \delta x) - J(x) \\
&= \int_{t_0}^{t_f} g(x(t) + \delta x(t), \dot{x}(t) + \dot{\delta} x(t), t) dt - \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt \\
&= \int_{t_0}^{t_f} (g(x(t) + \delta x(t), \dot{x}(t) + \dot{\delta} x(t), t) - g(x(t), \dot{x}(t), t)) dt \\
&= \int_{t_0}^{t_f} (\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) \delta x(t) + \frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \delta \dot{x}(t)) dt + O(||\delta x||) \\
&= \int_{t_0}^{t_f} (\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)) \delta x(t) dt + \frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)|_{t_0}^{t_f} + O(||\delta x||)
\end{aligned}$$

由初值以及终值约束以及引理可证

$$\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)] = 0$$

1.3 系统辨识与最小二乘

对于离散控制系统，输入序列 U，脉冲传递函数为 $G(z)$，输出序列为 Y，测量噪声为 V，测量值为 Z。

$$G(z) = \frac{\Sigma_{i=1}^n b_i z^{-i}}{1 + \Sigma_{i=1}^n a_i z^{-i}} = \frac{Y(z)}{U(z)}$$

$$y_k + \Sigma_{i=1}^n a_i y(k-i) = \Sigma_{i=1}^n b_i u(k-i)$$

可以得到转换为最小二乘问题求解

$$\begin{aligned}
\theta &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n & b_1 & b_2 & \cdots & b_n \end{bmatrix}^T \\
X_k &= \begin{bmatrix} -y(k-1) & -y(k-2) & \cdots & -y(k-n) & u(k-1) & u(k-2) & \cdots & u(k-n) \end{bmatrix} \\
Z_k &= \Sigma_{i=1}^n b_i u(k-i) - \Sigma_{i=1}^n a_i y(k-i) + v_k = X_k \theta + v_k
\end{aligned}$$

假设有 N 组数据

$$\Phi = \begin{bmatrix} X_1 & X_2 & \cdots & X_N \end{bmatrix}^T$$
$$Z = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_N \end{bmatrix}^T$$

取泛函 $J(\theta) = \Sigma_{i=1}^{N}(Z_i - X_i\theta) = (Z - \Phi\theta)^T(Z - \Phi\theta)$

$$\frac{\partial J}{\partial \theta} = \frac{\partial}{\partial \theta}(Z - \Phi\theta)^T(Z - \Phi\theta) = 0$$
$$\theta = (\Phi^T\Phi)^{-1}\Phi^T Z$$

系统辨识增加约束条件

2 作业

2.1 homework1

$$\begin{aligned}\frac{\partial}{\partial \theta}(Z - \Phi\theta)^T(Z - \Phi\theta) &= \frac{\partial}{\partial \theta}(Z^TZ - \theta^T\Phi^TZ - Z^T\Phi\theta + \theta^T\Phi^T\Phi\theta) \\ &= -\Phi^TZ - \Phi^TZ + 2\Phi^T\Phi\theta \\ &= -2\Phi^TZ + 2\Phi^T\Phi\theta\end{aligned}$$

2.2 homework2

KKT 条件：对于形如下式的一般优化问题

$$\begin{aligned} &\min f(x) \\ s.t.\ &g_j(x) \le 0\ (j = 1, 2, \cdots, n) \\ &h_k(x) = 0\ (k = 1, 2, \cdots, m)\end{aligned}$$

x^* 为最优解的必要条件

$$\begin{cases} \dfrac{\partial f}{\partial x_i} + \Sigma_{j=1}^{m}\mu_j\dfrac{\partial g_j}{\partial x_i} + \Sigma_{k=1}^{m}\lambda_k\dfrac{\partial h_k}{\partial x_i} = 0, (i = 1, 2, \cdots, n) \\ h_k(x) = 0,\ (k = 1, 2, \cdots, m) \\ \mu_j g_j(x) = 0,\ (j = 1, 2, \cdots, n) \end{cases} \tag{1}$$