Lecture7

zhike chen

October 2021

1 笔记

1.1 Pontryagin 极小值原理

状态变量 $x(t):[t_0,t_f]\to\mathbf{R}^n$ 分段连续可微，控制变量 $u(t):[t_0,t_f]\to\mathbf{R}^m$ 分段连续，且被控对象符合状态方程和初值条件

$$\dot{x}(t)=f(x(t),u(t),t),\ x(t_0)=x_0$$

符合容许控制，对任意时刻 $t\in[t_0,t_f]$，

$$u(t)\in U\in\mathbf{R}^m$$

终端时刻自由或固定，终端状态自由或固定，并且要最小化性能指标：

$$J(u)=h(x(t_f),t_f)+\int_{t_0}^{t_f}g(x(t),u(t),t)dt$$

则性能指标取得全局最小值的必要条件是最优控制 $u(t)$ 与 $x(t)$ 满足如下条件：

$$\begin{aligned}&H(x(t),u(t),p(t),t)=p(t)f(x(t),u(t),t)\\&\frac{\partial H}{\partial u}=0\\&\frac{\partial H}{\partial p}(x(t),u(t),p(t),t)=\dot{x}(t)\\&\frac{\partial H}{\partial x}(x(t),u(t),p(t),t)=-\dot{p}(t)\end{aligned}$$

边界条件：

$$[\frac{\partial h}{\partial x}(x_f,t_f)-p(t_f)]\delta x_f+[H(x(t_f),u(t_f),p(t_f),t_f)+\frac{\partial h}{\partial t}(x(t_f),t_f)]\delta t_f=0$$

1.2 稳态 Mayer 形式最优控制的极小值原理

被控对象符合稳态的状态方程和初值条件:

$$\dot{x}(t) = f(x(t), u(t)),\ x(t_0) = x_0$$

符合容许控制，对任意时刻 $t \in [t_0, t_f]$,

$$u(t) \in U \in \mathbf{R}^m$$

终端时刻自由，状态自由，并且最小化性能指标:

$$J(u) = h(x(t_f))$$

其最优控制的必要条件如下:

$$\begin{aligned}
&H(x(t), u(t), p(t), t) = g(x(t), u(t), t) + p(t)f(x(t), u(t), t)\\
&\frac{\partial H}{\partial u} = 0\\
&\frac{\partial H}{\partial p}(x(t), u(t), p(t), t) = \dot{x}(t)\\
&\frac{\partial H}{\partial x}(x(t), u(t), p(t), t) = -\dot{p}(t)
\end{aligned}$$

边界条件:

$$\begin{aligned}
&\frac{\partial h}{\partial x}(x(t_f)) - p(t_f) = 0\\
&H(x(t_f), p(t_f), u(t_f)) = 0
\end{aligned}$$

并且满足:

$$H(x(t), u(t), p(t)) = 0,\ t \in [t_0, t_f]$$

1.3 稳态 Bolza 形式最优控制的极小值原理

不同点在于 Hamilton 函数的形式:

$$H(x(t), u(t), p(t), t) = g(x(t), u(t), t) + p(t)f(x(t), u(t), t)$$

以及性能指标的形式为:

$$J(u) = h(x(t_f)) + \int_{t_0}^{t_f} g(x(t), u(t))dt$$

1.4 极小值原理求解无约束最优控制

系统的状态方程为:

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -x_2(t) + u(t)
\end{aligned}
$$

最小化性能指标:

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2}[x_1(t)^2 + u^2(t)]dt$$

终端时刻自由，状态自由

构造 Hamilton 函数:

$$H(x(t), u(t), p(t), t) = \frac{1}{2}[x_1^2(t) + u^2(t)] + p_1(t)x_2(t) + p_2(t)(u(t) - x_2(t))$$

极值条件:

$$
\begin{aligned}
0 &= \frac{\partial H}{\partial u}(x(t), u(t), p(t), t) \\
0 &= u(t) + p_2(t)
\end{aligned}
$$

状态方程和协态方程:

$$
\begin{aligned}
\dot{x}1_(t) &= \frac{\partial H}{\partial p_1}(x(t), u(t), p(t), t) = x_2(t) \\
\dot{x}_2(t) &= \frac{\partial H}{\partial p_2}(x(t), u(t), p(t), t) = -x_2(t) + u(t) \\
\dot{p}_1(t) &= -\frac{\partial H}{\partial x_1}(x(t), u(t), p(t), t) = -x_1(t) \\
\dot{p}_2(t) &= -\frac{\partial H}{\partial x_2}(x(t), u(t), p(t), t) = -p_1(t) + p_2(t)
\end{aligned}
$$

边界条件:

$$[\frac{\partial h}{\partial x}(x_f, t_f) - p(t_f)]\delta x_f + [H(x(t_f), u(t_f), p(t_f), t_f) + \frac{\partial h}{\partial t}(x(t_f), t_f)]\delta t_f = 0$$

由于终端时刻和状态均自由可变，所以 $\delta x_f, \delta t_f$ 不为零，其系数为 0，进而可以得到边界条件

$$
\begin{aligned}
&- p_1(t_f) = 0 \\
0 &= \frac{1}{2}[x_1^2(t_f) + u^2(t_f)] + p_1(t_f)x_2(t_f) + p_2(t_f)(u(t_f) - x_2(t_f))
\end{aligned}
$$

1.5 小车往返问题

小车的状态方程和初值条件:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$x_1(0) = 0$$
$$x_2(0) = 0$$

小车需要在某一 t_1 时刻经过 $x_1(t_1) = 2$ 并在 $t_f = 2$ 时停车在 $x_f = 0$

$$x_1(t_1) = 2,\ x_1(2) = 0,\ x_2(2) = 0$$

最小化性能指标:

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2} u^2(t) dt$$

引入拉格朗日算子构造增广性能指标:

$$\hat{J}(u, \lambda) = \lambda_1 x_1(t_f) + \lambda_2 x_2(t_f) + \lambda_3 (x_1(t_1) - 2) + \int_{t_0}^{t_f} \frac{1}{2} u^2(t) dt$$

该问题的 Hamilton 函数为:

$$H(x(t), u(t), p(t), t) = \frac{1}{2} u^2(t) + p_1(t) x_2(t) + p_2(t) u(t)$$

根据协态方程:

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), u(t), t)$$
$$\dot{p}_1(t) = 0$$
$$\dot{p}_2(t) = -p_1(t)$$

然后利用一般目标集最优控制问题的内殿约束以及边界条件求解

2 作业

2.1 月球软着陆问题建模

设飞船质量为 m(t)，高度为 h(t)，垂直速度为 v(t)，发动机的推力为 u(t)，月球表面的重力加速度为常数 g，推力与单位时间内燃料的消耗量成

正比，并且推力存在上限 u_{max}。初始质量为 M_0，初始高度为 h_0，初始垂直速度为 v_0。目标是在 T 时刻飞船平稳降落，问题的数学模型如下：

$$\begin{aligned}
\dot{h}(t) &= v(t) \\
\dot{v}(t) &= -g + \frac{u(t)}{m(t)} \\
\dot{m}(t) &= -ku(t)
\end{aligned}$$

边界条件：

$$0 \leq u(t) \leq u_{max},\ t \in [0, T]$$

终端约束：

$$h(T) = 0, \quad v(T) = 0$$

优化目标为

$$J(u) = -m(t_f)$$

取状态变量为 $h(t), v(t), m(t)$，其 Hamilton 函数为：

$$H(x(t), u(t), p(t)) = p_1(t)v(t) + p_2(t)(\frac{u(t)}{m(t)} - g) - p_3(t)ku(t)$$

根据极值条件可得：

$$\begin{aligned}
&\frac{\partial H}{\partial u}(x(t), u(t), p(t)) = 0 \\
&\frac{p_2(t)}{m(t)} - kp_3(t) = 0
\end{aligned}$$

列写状态方程和协态方程：

$$\begin{aligned}
&\dot{h}(t) = v(t) \\
&\dot{v}(t) = -g + \frac{u(t)}{m(t)} \\
&\dot{m}(t) = -ku(t) \\
&\frac{H}{\partial h}(x(t), u(t), p(t)) = -\dot{p}_1(t) = 0 \\
&\frac{H}{\partial v}(x(t), u(t), p(t)) = -\dot{p}_2(t) = p_1(t) \\
&\frac{H}{\partial m}(x(t), u(t), p(t)) = -\dot{p}_3(t) = -p_2(t)\frac{u(t)}{m^2(t)}
\end{aligned}$$

终端时刻和状态均固定，所以没有横截条件，带入边界条件可以用 matlab 求解。