COUNTING MANIFOLDS WITH EMBOLIC VOLUME AND SYSTOLIC VOLUME

LIZHI CHEN

ABSTRACT. We consider problem of counting manifolds with bounded embolic volume or bounded systolic volume. Yamaguchi has proved homotopy finiteness theorem with bounded embolic volume. Furthermore, Grove, Petersen and Wu have proved homeomorphism finiteness theorem with bounded embolic volume. Concerning systolic volume, there exists finiteness theorem of hyperbolic manifolds shown by Sabourau. In this paper, the goal is to provide quantitative versions of these finiteness theorems.

CONTENTS

Date: October 8, 2021.
Key words and phrases. systolic volume, hyperbolic manifolds, embolic volume.

1. Introduction

1.1. **Homotopy types of manifolds.** Mather [Mat65] and Kister [Kis68] showed that homotopy types of compact manifolds are countable.

Furthermore, Yamaguchi [Yam88] showed that the number of homotopy types of compact manifolds with bounded embolic volume is finite. More results of finiteness theorems for homotopy types with geometric constraints can be found in [GP88]. Our first result of this paper is a quantitative version of Yamaguchi's theorem.

Let M be a compact n-dimensional manifold endowed with a Riemannian metric $\mathcal{G}$, denoted by $(M, \mathcal{G})$. Denote by $\mathsf{Emb}(M)$ the embolic volume of M, which is defined to be

$$\inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Inj}(M, \mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M. According to Berger's embolic inequality, any compact manifold of dimension n has positive embolic volume. In [Yam88], Yamaguchi showed the following homotopy finiteness theorem of compact manifolds.

Theorem 1.1 (Yamaguchi [Yam88])**.** *For any positive constant L, there are only finitely many homotopy types of compact manifolds M with $\mathsf{Emb}(M) \leqslant L$.*

The first result of this paper is given in the following.

Theorem 1.2. *Let $\varphi_n(L)$ be the number of homotopy types of closed aspherical manifolds of dimension n, with $\mathsf{Emb}(M) \leqslant L$. We have*

$$a_n L \log L \leqslant \log \varphi_n(L) \leqslant b_n \tilde{L}^3 \log \tilde{L},$$

where a_n, b_n are two positive constants only depending on n, $\tilde{L}$ is a constant depending on L.

Remark 1.3. The constant $\tilde{L}$ is taken as

$$\sup\{\mathsf{CV}(M)|\,\mathsf{Emb}(M)\leqslant L\},$$

where the supremum is taken over all closed aspherical n-manifolds M with $\mathsf{Emb}(M)\leqslant L$, see the proof of this theorem in the following.

Let convexity radius of $(M,\mathcal{G})$ be denoted by $\mathsf{Conv}(M,\mathcal{G})$. Define

$$\mathsf{CV}(M)=\inf_{\mathcal{G}}\frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Conv}(M,\mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M. For a sufficiently large positive number C, denote by $\theta_n(C)$ the number of homotopy types of compact aspherical manifolds of dimension n. Theorem 1.2 is proved by the following counting estimate with $\mathsf{CV}(M)$.

Theorem 1.4. *Let C be a sufficiently large positive number. Then there exist positive constants a_n and b_n, which only depends on n, such that*

$$a_n C\log C\leqslant\log\theta_n(C)\leqslant b_n C^3\log C.$$

1.2. **Homeomorphism types of manifolds.** After Mather's theorem on homotopy type, Cheeger and Kister [CK70] showed that homeomorphism types of compact manifolds are also countable. Moreover, Cheeger's finiteness theorem implies finiteness of homeomorphism types of Riemannian manifolds with necessary geometric constraints. Grove, Petersen and Wu [GPW90, GPW91] proved finiteness of homeomorphism types of manifolds with bounded embolic volume.

The method used in the proof of Theorem 1.2 and Theorem 1.4 is based on counting in terms of fundamental groups. Therefore, for manifolds whose homeomorphism types are determined by fundamental groups, we get counting estimates of homeomorphism types.

Theorem 1.5. *Let M be a compact manifold with homeomorphism type determined by its fundamental group. Denote by $\Gamma_n(E)$ the number of homeomorphism types of M with $\mathsf{Emb}(M)\leqslant E$, and by $\nu_n(C)$ the number of homeomorphism types of M with $\mathsf{CV}(M)\leqslant C$. Then we have*

(1)
$$c_n E\log E\leqslant\log\Gamma_n(E)\leqslant d_n\tilde{E}^3\log\tilde{E},$$

where c_n and d_n are two positive constants only depending on n, and $\tilde{E}$ is a positive constant only depending on E.

(2)
$$k_n C \log C \leqslant \log \nu_n(C) \leqslant l_n C^3 \log C,$$

where k_n and l_n are two positive constants only depending on n.

In [BGS20], it is shown that number $\mathcal{P}_n(v)$ of homeomorphism types of complete Riemannian n-manifolds with normalized bounded negative curvature, and with bounded volume v, satisfies

$$\alpha v \log v \leqslant \log \mathcal{P}_n(v) \leqslant \beta v \log v,$$

where α and β are two positive constants, $n \geqslant 5$. This result is based on Farrell and Jones's work of Borel conjecture. We apply the same principle to get an estimate in terms of embolic volume, see discussion in Section 4.

1.3. **Counting hyperbolic manifolds.** For complete hyperbolic manifolds of dimension at least 4, Wang's theorem implies that there are finitely many of them if volume is bounded, see [Wan72]. In [BGLM02], Burger and Gelander et al. proved a quantitative version of Wang's theorem. Burger and Gelander et al.'s theorem directly implies the finiteness of hyperbolic n-manifolds ($n \geqslant 4$) with bounded systolic volume. Moreover, in [Sab07] Sabourau showed that the finiteness also holds for hyperbolic 3-manifolds. In this paper, we show a quantitative version of Sabourau's theorem by using the method of Burger, Gelander et al. in [BGLM02]. This result can be obtained directly from the method of [BGLM02] and Gromov's theorem in systolic geometry. We further study the problem of counting hyperbolic manifold by using embolic volume.

Let M be a complete hyperbolic n-dimensional manifold. Denote by hyp the hyperbolic metric on M, and by $\mathsf{Vol}_{\mathsf{hyp}}(M)$ the hyperbolic volume of M.

Theorem 1.6. *Let $n \geqslant 3$, and $s > 0$ be a sufficiently large positive number. Denote by $\varphi_n(s)$ the number of closed hyperbolic n-manifolds M with $\mathsf{SR}(M) \leqslant s$. Then we have*

$$\alpha_n u \log u \leqslant \log \varphi_n(s) \leqslant \begin{cases} \beta_n u \log^n u, & n \geqslant 4; \\ \beta_0 u' \log^n u', & n = 3. \end{cases}$$

where $u = s\log^n s$*,* α_n, β_n *are positive constants only depending on* n*,* $u' = c(s)\,s\log^n s$ *with* $c(s)$ *a positive constant only depending on* s*, and* β_0 *is a fixed positive constant.*

Remark 1.7. In the above theorem, the constant $c(s)$ is the lower bound of injectivity radius for any closed hyperbolic 3-manifolds M with $\mathsf{SR}(M) \leqslant s$.

Theorem 1.6 is proved by using the method in [BGLM02] and Gromov's theorem of connecting simplicial volume with systolic volume.

Let C be a sufficiently large positive constant. If we denote by $h_n(C)$ the number of compact hyperbolic manifolds with embolic volume bounded above by C, then the following result holds.

Theorem 1.8. *There exist two positive constants* f_n *and* g_n *only depending on* n*, such that*

$$f_n C \log C \leqslant \log h_n(C) \leqslant g_n C \log C.$$

1.4. **Homotopy complexity of compact manifolds.** The approach of proving counting theorems is related to the problem of uniform homotopy complexity. A (d, v)-simplicial complex is a simplicial complex with at most v vertices, and degree of each vertex is at most d. A family $\mathcal{F}$ of d-dimensional Riemannian manifolds has uniform homotopy complexity, if each $M \in \mathcal{F}$ is homotopy equivalent to some $(D, \alpha \cdot \mathsf{Vol}_{\mathcal{G}}(M))$-simplicial complex, where D, α are some fixed positive constants. Uniform homotopy complexity of locally symmetric Riemannian manifolds is studied in [Gel04]. Recent progresses can be found in [Frą21, GV21].

We prove the following curvature free version of uniform homotopy complexity for all compact manifolds.

Theorem 1.9. *Let* M *be a compact manifold of dimension* n*. Then* M *is homotopy equivalent to a* $(A_n \cdot \mathsf{CV}(M), A_n \cdot \mathsf{CV}(M))$*-simplicial complex, with* A_n *a positive constant only depending on* n*.*

Covering type of compact manifolds is defined in [KW16]. The covering type of a compact manifold M is defined to be the minimum number of contractible subsets in an open cover of any space homotopy equivalent to M. One application of Theorem 1.9 is the following result of covering type.

Corollary 1.10. *The covering type* $\mathsf{ct}(M)$ *of a compact* n*-dimensional manifold* M *has the following upper bound,*

$$\mathsf{ct}(M) \leqslant \delta_n \, \mathsf{CV}(M),$$

where δ_n *is a positive constant only depending on* n*.*

Organization. This paper is organized as follows: We introduce necessary and related backgroup knowledge in Section 2. The estimate of the number of homotopy types of compact manifolds is in Section 3. The estimate of homeomorphism types of aspherical manifolds is contained in Section 4. In section 5, we prove the counting estimate of hyperbolic manifolds in terms of systolic volume and embolic volume. In section 6, homotopy complexity of compact manifolds is studied in terms of embolic volume.

Acknowledgement. The work is supported by Youth Scientists Fund of NSCF (award No. 11901261).

2. Preliminary knowledge

2.1. **Nerve of good cover.** Given a topological space X and a covering $\mathcal{C}$ on X, the nerve $\mathcal{N}$ associated to X is a simplicial complex, such that for each subset in $\mathcal{C}$ there is a corresponding vertex in $\mathcal{N}$, and for each nonempty intersection of subsets in $\mathcal{C}$, there is a simplex spanned by the corresponding vertices.

Definition 2.1. An open covering $\mathcal{C}$ of X is called a good cover, if any nonempty intersection of subsets in $\mathcal{C}$ is contractible.

We have the following result for nerve of good covers.

Theorem 2.2 (Nerve lemma ,see Hatcher [Hat02, Corollary 4G.3.])**.** *If* $\mathcal{C}$ *is a good cover of a topological space* X*, then the nerve* Y *associated to* $\mathcal{C}$ *is homotopy equivalent to* X*.*

2.2. **Hyperbolic volume and simplicial volume.** A closed manifold M of dimension n is hyperbolic if it admits a complete Riemannian metric of constant sectional curvature -1. The simplicial volume of M, denoted $\|M\|$, is defined in terms of generator of homology group $H_n(M;\mathbb{R})$ with real coefficients. Gromov and Thurston showed that the simplicial volume is a topological description of hyperbolic volume, see [BP92] for reference.

Theorem 2.3 (Gromov and Thurston)**.** *If M is an oriented compact hyperbolic manifold then*

$$\|M\| = \frac{\mathsf{Vol}(M)}{\mathcal{V}_n},$$

where $\mathcal{V}_n$ is maximal volume of an ideal n-simplex in the hyperbolic space $\mathbb{H}^n$.

2.3. **Topology and geometric invariants.** A central theorem in systolic geometry is Gromov's result relating systolic volume of simplicial volume.

Theorem 2.4 (Gromov 1983, [Gro83])**.** *Let M be a closed essential manifold of dimension n. There exist constants C_n and C'_n only depending on n, such that*

$$\|M\| \leqslant C_n \, \mathsf{SR}(M) \log^n (C'_n \, \mathsf{SR}(M)). \tag{2.1}$$

Gromov's theorem (Theorem 2.4) indicates that systolic volume is representing that how topologically a manifold is.

3. Counting homotopy types of compact manifolds

In this section, we consider the problem of counting homotopy types of compact manifolds M with embolic volume $\mathsf{Emb}(M)$ and the invariant $\mathsf{CV}(M)$.

For a sufficiently large positive constant E, denote by $\psi_n(E)$ the number of homotopy types of compact aspherical manifolds M of dimension n with $\mathsf{Emb}(M) \leqslant E$.

Theorem 3.1. *There exist two positive constants δ_n and γ_n which only depend on n, such that*

$$\delta_n \, E \log E \leqslant \log \psi_n(E) \leqslant \gamma_n \, \tilde{E}^3 \log \tilde{E},$$

where $\tilde{E}$ is a positive constant only depending on E.

We prove Theorem 3.1 in terms of the following result on $\mathsf{CV}(M)$.

Theorem 3.2. *Let C be a sufficiently large positive number. For compact aspherical manifolds M of dimension n, deonte by $\theta_n(C)$ the number of homotopy types of M with $\mathsf{CV}(M) \leqslant C$. Then there exist positive* *constants D_n and D'_n only depending on n, such that*

$$D_n C \log C \leqslant \log \theta_n(C) \leqslant D'_n C^3 \log C.$$

Proof of Theorem 3.1:

(1) **Lower bound**
Since we have $\mathsf{SR}(M) \leqslant \mathsf{Emb}(M)$, if $\mathsf{Emb}(M) \leqslant E$ for a sufficiently large positive number E, then we must have $\mathsf{SR}(M) \leqslant E$. All hyperbolic manifolds are aspherical. Note that homotopy type and homeomorphism type of hyperbolic manifolds coincide. And the number $\varphi_n(E)$ of closed hyperbolic manifolds of dimension n has lower bound $\exp(\alpha_n E \log E)$ (see [?]), hence we have

$$\log \psi_n(E) \geqslant \log \varphi_n(E) \geqslant \alpha_n E \log E.$$

Then we take the constant δ_n in Theorem 3.1 to be α_n.

(2) **Upper bound**
Let $E > 0$ be a sufficiently large number. Set

$$\tilde{E} = \sup\{\mathsf{CV}(M) | \, \mathsf{Emb}(M) \leqslant E\},$$

then we have $\psi_n(E) \leqslant \theta_n(\tilde{E})$, so that Theorem 3.2 implies that

$$\log \psi_n(E) \leqslant D'_n \tilde{E}^3 \log \tilde{E}.$$

□

Proof of Theorem 3.2

The proof proceeds according to lower bound and upper bound respectively.

Lower bound
The lower bound holds for the similar reason as in Theorem 3.1. Since $\mathsf{Emb}(M) \leqslant \mathsf{CV}(M)$, for a sufficiently large positive number C, if $\mathsf{CV}(M) \leqslant C$, there are at least $C^{D_n C}$ number of compact hyperbolic n-manifolds according to Burger et al. 's counting result in [?].

Upper bound
Let $(M, \mathcal{G})$ be a compact manifold of dimension n. Take $r = \frac{1}{10}\,\mathsf{Conv}(M, \mathcal{G})$. Let $\mathcal{U}_1 = \{B(p_j, r)\}$ be a maximal system of metric balls with radius r in M. Then $\mathcal{U}_2 = \{B(p_j, 2r)\}$ is a covering to the manifold M. The nerve of covering $\mathcal{U}_2$ is denoted by $\mathcal{N}$. Each pair of balls in $\mathcal{U}_2$ has either empty intersection or convex intersection. Hence, according to nerve lemma (see Theorem 2.2), $\mathcal{N}$ is a simplicial complex homotopy equivalent to M. Let x_0 be the number of vertices in the nerve $\mathcal{N}$. Upper bound of x_0 can be calculated as follows,

$$\begin{aligned} x_0 &\leqslant \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\inf_j \mathsf{Vol}_{\mathcal{G}}(B(p_j, r))} \\ &\leqslant \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{c_n\, r^n} \\ &\leqslant c'_n\, \mathsf{CV}(M), \end{aligned}$$

where c'_n is a constant only depending on n. We have used Croke's local embolic inequality in the above estimate. Denote by d the upper bound of degree of vertices in the nerve $\mathcal{N}$. A crude estimate of d is x_0. In the following, we take $d = c'_n\, \mathsf{CV}(M)$. Let x_1 be the number of possible 1-skeletons of $\mathcal{N}$. We have

$$\begin{aligned} x_1 &\leqslant x_0^{d\, x_0} \\ &= e^{d\, x_0 \log x_0} \\ &\leqslant e^{d\, c''_n\, \mathsf{CV}(M) \log \mathsf{CV}(M)} \\ &\leqslant e^{c'''_n\, \mathsf{CV}(M)^2 \log \mathsf{CV}(M)}, \end{aligned}$$

where c'''_n is a positive constant only depending on n. Let x_2 be the possible number of 2-skeletons of $\mathcal{N}$. An upper bound for x_2 is estimated as follows,

$$\begin{aligned} x_2 &\leqslant 2^{d^2 x_0} x_1 \\ &\leqslant e^{\tilde{c}_n\, \mathsf{CV}(M)^3 \log \mathsf{CV}(M)}, \end{aligned}$$

where $\tilde{c}_n$ is a positive constant only depending on n.

Since the manifold M is aspherical, by Whitehead theorem we know the homotopy type of M is determined by fundamental group. Moreover, 2-skeleton determines the fundamental group. Therefore, upper bound of x_2 yields an upper bound of the number of homotopy types of M. $\square$

4. Counting homeomorphism types of manifolds

Theorem 4.1. *Assume that M is a compact n-manifold with homeomorphism types determined by fundamental group. Let C and E be two sufficiently large positive numbers. Denote by $\Gamma_n(E)$ the number of homeomorphism types of M with $\mathsf{Emb}(M) < E$, and denote by $\nu_n(C)$ the number of homeomorphism types of M with $\mathsf{CV}(M) < C$. Then we have*

(1)

$$c_n E \log E \leqslant \Gamma_n(E) \leqslant d_n \tilde{E} \log \tilde{E}, \tag{4.1}$$

where c_n and d_n are two positive constants only depending on n, and $\tilde{E}$ is a constant only depending on E.

(2)

$$k_n C \log C \leqslant \nu_n(C) \leqslant l_n C^3 \log C, \tag{4.2}$$

where k_n and l_n are positive constants only depending on n.

Proof. Since the fundamental group of compact manifold is determined by 2-skeletons, the proof of Theorem 3.2 implies the estimate (4.1) and (4.2). $\square$

Remark 4.2. We know that homeomorphism type of hyperbolic manifolds are determined by fundamental group. Moreover, in dimension 3, geometrization theorem implies that homeomorphism type of all compact 3-manifolds with infinite fundamental group are determined by fundamental group. Generally, if Borel's conjecture is true, homeomorphism type of any aspherical manifold is determined by fundamental group.

For example, if we consider complete Riemannian n-manifolds ($n \geqslant 5$) of normalized negative curvature, the following result holds.

Corollary 4.3. *Let K be a sufficiently large positive number. If M is a complete Riemannian manifold of dimension n with $n \geqslant 5$, moreover, assume that M is of normalized negative curvature, then the number $\tilde{\Gamma}_n(E)$ of homeomorphism types satisfy*

$$\tilde{c}_n \, E \log E \leqslant \log \tilde{\Gamma}_n(E) \leqslant \tilde{d}_n \, \tilde{E}^3 \log \tilde{E},$$

the number $\tilde{\nu}_n(K)$ satisfies

$$\tilde{k}_n K \log K \leqslant \nu_n(K) \leqslant \tilde{l}_n K^3 \log K,$$

where $\tilde{c}_n, \tilde{d}_n, \tilde{k}_n, \tilde{l}_n$ are positive constants only depending on n.

If we consider compact 3-manifolds with infinite fundamental group, similar counting results hold.

5. Counting hyperbolic manifolds with systolic volume and embolic volume

5.1. **Counting hyperbolic manifolds with systolic volume.** We prove Theorem 1.6 in the following. The proof is separated according to dimension $n = 3$ and $n \geqslant 4$.

5.1.1. $\mathbf{n \geqslant 4}$. Suppose that M is a closed hyperbolic n-manifold with $n \geqslant 4$. Wang's finiteness theorem [Wan72] implies that there are only finitely many M with volume bounded from above. Burger , Gelander et al. [BGLM02] proved a quantitative version of Wang's theorem. Let $\rho_n(V)$ be the number of complete hyperbolic n-manifolds with hyperbolic volume at most V.

Theorem 5.1 ([BGLM02]). *When $V > 0$ is sufficiently large, $\rho_n(V)$ satisfies*

$$a_n \, V \log V \leqslant \log \rho_n(V) \leqslant b_n \, V \log V,$$

where a_n and b_n are two positive constants only depending on n.

The lower bound in Theorem 5.1 is proved by using Gromov's construction [GPS88] of nonarithmetic groups. The proof proceeds as follows: In the isometry group $PO(n,1)$ of $\mathbb{H}^n$, there exists the nonarithmetic cocompact lattice Γ constructed by Gromov [GPS88]. By considering a finite index and torsion free subgroup $\triangle \subset \Gamma$, a hyperbolic n-manifold $M_0 = \mathbb{H}^n/\triangle$ is constructed. Then it can be shown, for some positive integer r, the number of mutually non-isometric r-sheeted cover of M_0 is at least $\exp(a_n V \log V)$ for some positive constant a_n only depending on n.

In particular, it is easy to see that the counting estimate in Theorem 5.1 still holds if we cosider closed hyperbolic n-manifolds with $n \geqslant 4$.

Let M be a closed hyperbolic manifold of dimension n, with $n \geqslant 4$. For a sufficiently large positive constant S, if $\mathsf{SR}(M) < S$, then according to Gromov's theorem of systolic volume, we have

$$\begin{aligned}\mathsf{Vol}_{\mathsf{hyp}}(M) &= \mathcal{V}_n \|M\| \\ &\leqslant \mathcal{V}_n C_n \,\mathsf{SR}(M) \log^n \left(C'_n \,\mathsf{SR}(M)\right) \\ &< \tilde{C}_n S \log^n S.\end{aligned}$$

Apply Theorem 5.1, then we have the counting estimates in Theorem 1.6.

5.1.2. $\mathbf{n=3}$. **(1) Lower bounds**

In [BGLM02], it is showed there exists a hyperbolic manifold $M_0 = \mathbb{H}^n/\triangle$, where $\triangle$ is a discrete subgroup of $PO(n,1)$. Moreover, the hyperbolic 3-manifold M_0 has at least $\frac{1}{m_1} r!$ r-sheeted mutually non-isometric covering 3-manifolds, where m_1 and r are positive integers. Note that for a r-sheeted covering manifold $(\widetilde{M}, \tilde{\mathsf{hyp}})$, the hyperbolic volume satisfies

$$\begin{aligned}\mathsf{Vol}_{\tilde{\mathsf{hyp}}}(\widetilde{M}) &= r \,\mathsf{Vol}_{\mathsf{hyp}}(M_0) \\ &\leqslant r \mathcal{V}_n C_n \,\mathsf{SR}(M_0) \log C'_n \,\mathsf{SR}(M_0),\end{aligned}$$

hence if the systolic volume σ satisfies

$$C_n \sigma \log C'_n \sigma \leqslant D_n \cdot r,$$

there will be at least $D'_n r^r$ closed hyperbolic 3-manifolds, with $r = S \log S$, and D'_n is a constant.

(2) Upper bounds

Let S be a sufficiently large positive number. When M is a closed 3-dimensional hyperbolic manifold with $\mathsf{SR}(M) \leqslant S$, then according to Sabourau's theorem, there are only finitely many of them. Hence the injectivity radius of these closed hyperbolic 3-manifolds has a lower bound. We denote this lower bound by $c(S)$, since it should be related to the number S. In the following, we apply Burger and Gelander et al.'s method in [BGLM02] to show the upper bound.

Let $r = \frac{1}{10}c(S)$, and let $\mathcal{U}_1 = \{B(p,r)\}$ be a maximal system of disjoint metric balls with radius r in M. Then $\mathcal{U}_2 = \{B(p,2r)\}$ is a covering to M. The nerve of covering $\mathcal{U}_2$ is denoted by $\mathcal{N}$. Since each metric ball with radius at most $\mathsf{Inj}_{\mathsf{hyp}}(M)$ in the hyperbolic 3-manifold M is convex, the intersection of any two balls in $\mathcal{N}$ is either empty or convex. Therefore the nerve $\mathcal{N}$ is homotopy equivalent to M. Mostow's rigidity theorem yields that M is determined by its fundamental group. Therefore, we give upper bounds of number of fundamental groups of M by considering the maximal possible number of 2-skeletons in the nerve $\mathcal{N}$. The number x_0 of vertices in $\mathcal{N}$ is at most $\frac{\mathsf{Vol}_{\mathsf{hyp}}(M)}{\mathsf{Vol}_{\mathsf{hyp}}(B(p,r))} \leqslant c_1(S)\,\mathsf{Vol}_{\mathsf{hyp}}(M)$, where $c_1(S)$ is constant only depending on S. Each vertex has a degree at most $\frac{\mathsf{Vol}_{\mathsf{hyp}}(B(p,\frac{5}{2}r))}{B(p,r)}$, which is a fixed constant, denoted by d. Then number x_1 of the 1-skeleton of $\mathcal{N}$ is at most $x_0^{dx_0}$. And number of 2-skeletons of $\mathcal{N}$ can be estimated in terms of possible number of 2-simplices at each vertex, which is at most $2^{d^2x_0}x_1$.

5.2. **Counting hyperbolic manifolds with embolic volume.**

Theorem 5.2. *Assume that C is a sufficiently large positive number. For $n \geqslant 3$, let $h_n(C)$ be the number of compact n-dimensional hyperbolic manifolds M with $\mathsf{Emb}(M) \leqslant C$. There exist two positive constants f_n and g_n which only depend on n, such that*

$$f_n C \log C \leqslant \log h_n(C) \leqslant g_n C \log C.$$

Proof. Mostow's rigidity theorem implies that compact hyperbolic manifolds are determined by fundamental groups. Hence homotopy type and homeomorphism type coincide for compact hyperbolic manifolds. Then the lower bound estimate in Theorem 3.2 remains a lower bound for $h_n(C)$. Moreover, the upper bound d of degree of vertices in the nerve $\mathcal{N}$ constructed in the proof of Theorem 3.2 has more precise estimate if the manifold M is hyperbolic,

$$\begin{aligned} d &\leqslant \frac{\mathsf{Vol}_{\mathsf{hyp}}(B(p_j, \frac{5}{2}r))}{\mathsf{Vol}_{\mathsf{hyp}}(B(p_j, r))} \\ &\leqslant const_n, \end{aligned}$$

where $const_n$ is a constant only depending on n. $\square$

6. Homotopy complexity of compact manifolds

Recall that a (d, v)-simplicial complex is a simplicial complex with at motst v vertices, and valence of each vertex is at most d. A family $\mathcal{M}$ of Riemannian manifolds has uniform homotopy complexity if any $M \in \mathcal{M}$ is homotopy equivalent to a (d, v)-simplicial complex, where v is a multiple of the Riemannian volume of M. We show that the family of all compact manifolds has the following uniform homotopy complexity in terms of $\mathsf{CV}(M)$.

Theorem 6.1. *Let M be a compact manifold of dimension n. Then M is homotopy equivalent to a $(A_n \mathsf{CV}(M), A_n \mathsf{CV}(M))$-simplicial complex, where A_n is a positive constant only depending on n.*

Proof. Let $R = \frac{1}{2}\mathsf{Conv}(M, \mathcal{G})$. Cover M by a maximal system $\mathcal{U} = \{B(p_j, R)\}_{j=1}^{\delta}$ of disjoint metric balls with common radius R, so that if any metric ball of radius R is added into $\mathcal{U}$, there will be intersections. Hence the manifold M is covered by the system $\widetilde{\mathcal{U}} = \{B(p_j, 2R)\}_{j=1}^{\delta}$ of balls with the same centers but doubled radius $2R$.

According to Croke's local embolic inequality (see [Cro80, Proposition 14]), for any point $p \in M$,

$$\mathsf{Vol}_{\mathcal{G}}(B(p, r)) \geqslant \alpha_n r^n \tag{6.1}$$

holds if $0 < r \leqslant \frac{1}{2}\mathsf{Inj}(M, \mathcal{G})$, where α_n is a positive constant only depending on n. Since $\mathsf{Inj}(M, \mathcal{G}) \geqslant 2\,\mathsf{Conv}(M, \mathcal{G})$ (see [Ber76] for a proof), the local embolic inequality (6.1) implies that similar local estimate is also true for convex radius.

Lemma 6.2. *On a compact n-dimensional Riemannian manifold $(M, \mathcal{G})$, inequality*

$$\mathsf{Vol}_{\mathcal{G}}(B(p, r)) \geqslant \alpha_n r^n \tag{6.2}$$

holds for any point $p \in M$ if $r \leqslant \mathsf{Conv}(M, \mathcal{G})$, where α_n is a constant only depending on the manifold dimension n.

Lemma 6.2 implies that

$$\begin{aligned}\mathsf{Vol}_{\mathcal{G}}(M) &\geqslant \sum_{j=1}^{\delta} \mathsf{Vol}_{\mathcal{G}}(B(p_j, R)) \\ &\geqslant \sum_{j=1}^{\delta} \alpha_n R^n \\ &= \delta\, \alpha_n R^n.\end{aligned}$$

Hence we have

$$\delta \leqslant \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\alpha_n R^n}.$$

After taking infimum over all Riemannian metrics $\mathcal{G}$ on M, we have

$$\delta \leqslant \frac{2^n}{\alpha_n} \mathsf{CV}(M).$$

Let $\mathcal{N}$ be the nerve of the covering $\widetilde{U} = \{B(p_j, 2R)\}_{j=1}^{\delta}$. According to the definition of nerve, the number δ of balls in the covering $\widetilde{\mathcal{U}}$ is equal to the number of vertices of $\mathcal{N}$. A metric ball of radius $\frac{1}{2}\mathsf{Conv}(M, \mathcal{G})$ is geodesically convex. Hence the covering $\widetilde{\mathcal{U}}$ is a good cover (see Definition 2.1). In terms of the nerve theorem (see Theorem 2.2), we know that $\mathcal{N}$ is homotopy equivalent to M.

Two balls $B(p_i, 2R)$ and $B(p_j, 2R)$ in $\widetilde{U}$ have intersections, if and only if $B(p_i, R) \subset B(p_j, 5R)$. Therefore, an upper bound of the number of intersections for a given ball $B(p_j, 2R)$ is

$$\frac{\mathsf{Vol}_{\mathcal{G}}(B(p_j, 5R))}{\inf_j \mathsf{Vol}_{\mathcal{G}}(B(p_i, R))},$$

where the infimum is taken over all balls $B(p_i, R)$ such that $B(p_i, 2R)$ having nonempty intersections with $B(p_j, 2R)$. By the local inequality (6.2), an estimate for this upper bound is $\frac{2^n}{\alpha_n}\mathsf{CV}(M)$. $\square$

The proof of Theorem 6.1 yields the following application to covering type.

Corollary 6.3. *The covering type* $\mathsf{ct}(M)$ *of a compact* n*-dimensional manifold* M *has the following upper bound,*

$$\mathsf{ct}(M) \leqslant C_n \,\mathsf{CV}(M), \tag{6.3}$$

where C_n *is a positive constant only depending on* n*.*

Proof. In the proof of Theorem 6.1, a nerve $\mathcal{N}$ homotopy equivalent to the compact manifold M is constructed. Moreover, the number δ of the vertices of $\mathcal{N}$ satisfies

$$\delta \leqslant \frac{2^n}{\alpha_n} \,\mathsf{CV}(M).$$

Let $C_n = \frac{2^n}{\alpha_n}$. Then we get the inequality (6.3). □

References

[Ber76] M. Berger, *Some relations between volume, injectivity radius, and convexity radius in Riemannian manifolds*, Differential geometry and relativity, 1976, pp. 33–42. Mathematical Phys. and Appl. Math., Vol. 3. MR 0448253

[BGLM02] M. Burger, T. Gelander, A. Lubotzky, and S. Mozes, *Counting hyperbolic manifolds*, Geom. Funct. Anal. **12** (2002), no. 6, 1161–1173. MR 1952926

[BGS20] Uri Bader, Tsachik Gelander, and Roman Sauer, *Homology and homotopy complexity in negative curvature*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 8, 2537–2571. MR 4118615

[BP92] Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992. MR 1219310

[CK70] J. Cheeger and J. M. Kister, *Counting topological manifolds*, Topology **9** (1970), 149–151. MR 256399

[Cro80] Christopher B. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 4, 419–435. MR 608287

[Frą21] Mikołaj Frączyk, *Strong limit multiplicity for arithmetic hyperbolic surfaces and 3-manifolds*, Invent. Math. **224** (2021), no. 3, 917–985. MR 4258057

[Gel04] Tsachik Gelander, *Homotopy type and volume of locally symmetric manifolds*, Duke Math. J. **124** (2004), no. 3, 459–515. MR 2084613

[GP88] Karsten Grove and Peter Petersen, V, *Bounding homotopy types by geometry*, Ann. of Math. (2) **128** (1988), no. 1, 195–206. MR 951512

[GPS88] M. Gromov and I. Piatetski-Shapiro, *Nonarithmetic groups in Lobachevsky spaces*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 66, 93–103. MR 932135

[GPW90] Karsten Grove, Peter Petersen, V, and Jyh Yang Wu, *Geometric finiteness theorems via controlled topology*, Invent. Math. **99** (1990), no. 1, 205–213. MR 1029396

[GPW91] ———, *Erratum: "Geometric finiteness theorems via controlled topology" [Invent. Math.* **99** *(1990), no. 1, 205–213; MR1029396 (90k:53075)]*, Invent. Math. **104** (1991), no. 1, 221–222. MR 1094053

[Gro83] Mikhael Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18** (1983), no. 1, 1–147. MR 697984

[GV21] Tsachik Gelander and Paul Vollrath, *Bounds on systoles and homotopy complexity*, 2021.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[Kis68] J. M. Kister, *Homotopy types of ANR's*, Proc. Amer. Math. Soc. **19** (1968), 195. MR 219031

[KW16] Max Karoubi and Charles Weibel, *On the covering type of a space*, Enseign. Math. **62** (2016), no. 3-4, 457–474. MR 3692893

[Mat65] Michael Mather, *Counting homotopy types of manifolds*, Topology **3** (1965), 93–94. MR 176470

[Sab07] Stéphane Sabourau, *Systolic volume of hyperbolic manifolds and connected sums of manifolds*, Geom. Dedicata **127** (2007), 7–18. MR 2338511

[Wan72] Hsien Chung Wang, *Topics on totally discontinuous groups*, Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970), 1972, pp. 459–487. Pure and Appl. Math., Vol. 8. MR 0414787

[Yam88] Takao Yamaguchi, *Homotopy type finiteness theorems for certain precompact families of Riemannian manifolds*, Proc. Amer. Math. Soc. **102** (1988), no. 3, 660–666. MR 928999

School of Mathematics and Statistics, Lanzhou University
Lanzhou 730000, P.R. China
E-mail address: `chenzhmath@gmail.com`