TRIANGULATION COMPLEXITY AND SYSTOLIC VOLUME OF HYPERBOLIC MANIFOLDS

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Abstract. Let M be a closed n-manifold with nonzero simplicial volume $\|M\|$. A central theorem in systolic geometry proved by Gromov is that systolic volume of M is related to $\|M\|$. In this paper, we generalize Gromov's theorem of relating systolic volume of hyperbolic manifolds to its triangulation complexity. The proof is based on Jørgensen and Thurston's theorem of hyperbolic volume.

Contents

1. Introduction

Gromov's systolic inequality implies that systole is bounded above in terms of Riemannian volume. The optimal constant in a systolic inequality is usually called systolic volume. Let M be a closed n-dimensional manifold with nonzero simplicial volume. Gromov proved that topological complexity of $\|M\|$ can be represented by systolic volume. In this paper, we follow Gromov's result to relate systolic volume to triangulation complexity. Triangulation complexity of manifolds is defined to be the minimum number of tetrahedra in a triangulation. Hence the triangulation complexity naturally represents how complicated a manifold is. Our result is a supplement to Gromov's theorem.

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Let M be a closed 3-manifold. If M is irreducible and not homeomorphic to $S^3, \mathbb{RP}^3$ or $L(3,1)$, the triangulation complexity coincides with the complexity defined by Matveev [11]. In the following, we use $c(M)$ to denote the triangulation complexity of a closed 3-manifold M. The triangulation complexity of 3-manifolds is studied a lot in recent years. We refer to [7, 8, 10] for recent developments of triangulation complexity of 3-manifolds. The systolic volume of M, denoted by $\mathsf{SR}(M)$, is defined to be

$$\inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Sys}\,\pi_1(M,\mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M. The systolic volume $\mathsf{SR}(M)$ is positive if M is a closed essential 3-manifold, see [4, Theorem 0.1.A.] for more details.

Let s_0 be a sufficiently large positive number.

Theorem 1.1. *Suppose that M is a closed hyperbolic 3-manifold with $\mathsf{SR}(M) \leqslant s_0$. There exists a positive constant $C(s_0)$ only depending on s_0, such that*

$$\mathsf{SR}(M) \geqslant C(s_0)\,\frac{c(M)}{\log^3 c(M)}.$$

The simplicial volume $\|M\|$ is defined to be the minimum number of tetrahedra in a cycle representing the fundamental class of M with real coefficients. For hyperbolic manifolds, there exists the proportionality principle,

$$\nu_n \|M\| = \mathsf{Vol}_{\mathsf{hyp}}(M), \tag{1.1}$$

where ν_n is a positive constant only depending on n. We refere to [1] for more about simplicial volume. A central theorem in systolic geometry is the following one of connecting systolic volume to simplicial volume.

Theorem 1.2 (Gromov 1983, see [4, Section 6.4.D.] or [5, Section 3.C.3.])**.** *Let M be a closed n-dimensional manifold with non-zero simplicial volume. Then the systolic volume $\mathsf{SR}(M)$ of M satisfies*

$$\|M\| \leqslant C_n\,\mathsf{SR}(M)\,\log^n\left(C_n'\,\mathsf{SR}(M)\right), \tag{1.2}$$

where C_n and C_n' are two positive constants only depending on n.

Remark 1.3. In [4, Section 6.4.D.], there is a typo of missing n in the logarithm part of (1.2). Sometimes the estimate (1.2) in Theorem 1.2

is also written as follows,

$$\mathsf{SR}(M) \geqslant C_n \frac{\|M\|}{\log^n \|M\|},$$

where C_n is a positive constant only depending on n. Our result in Theorem 1.1 can be seen as a version of Theorem 1.2 for integral homology.

A mail tool used to prove Theorem 1.1 is the connection between triangulation and hyperbolic volume. Jørgensen and Thurston proved that for a complete hyperbolic manifold, there exists a triangulation with the number of tetrahedra bounded from above by its volume. A detailed proof of this theorem is provided by Kobayashi and Rieck in [9].

Combined with Wang's finiteness theorem of hyperbolic manifolds, Sabourau proved that there are only finitely many hyperbolic manifolds with bounded systolic volume.

Theorem 1.4 (Sabourau 2007, [13])**.** *For a sufficiently large positive number C, there are only finitely many hyperbolic n-manifolds M with $\mathsf{SR}(M) \leqslant C$.*

The triangulation complexity of manifolds of higher dimensions is studied in [6]. Let M be a closed manifold of dimension n. The triangulation complexity of M, denoted $\sigma(M)$, is defined to be the minimum number of n-simplices in any triangulation of M. When $n = 3$, $\sigma(M)$ coincides with $c(M)$.

Theorem 1.5. *Let M be a closed hyperbolic manifold of dimension n with $n \geqslant 4$. The triangulation complexity $\sigma(M)$ and systolic volume of M is related by*

$$\mathsf{SR}(M) \geqslant D_n \frac{\sigma(M)}{\log^n \sigma(M)},$$

where D_n is a positive constant only depending on n.

Remark 1.6. It is proved in [6] that under assumptions of Theorem 1.5, $\|M\| < \sigma(M)$ holds. Hence above theorem is indeed a generalization of Gromov's theorem when the manifold M is hyperbolic and with dimension at least four.

On compact manifolds, embolic volume is defined in terms of Berger's embolic inequality.

Definition 1.7. The embolic volume of M, denoted $\mathsf{Emb}(M)$, is defined to be

$$\inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Inj}(M,\mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M.

Embolic volume is positive for all compact manifolds M, see [2]. In the following, we show that triangulation complexity $\sigma(M)$ is related to embolic volume.

Theorem 1.8. *Let M be a closed n-dimensional manifold. Then there exists a positive constant δ_n only depending on n, such that*

$$\sigma(M) \leqslant \delta_n \,\mathsf{Emb}(M). \tag{1.3}$$

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2. Systolic volume of hyperbolic 3-manifolds

Let M be a closed hyperbolic 3-manifold. For a sufficiently large positive number s_0, according to Sabourau's theorem (see Theorem 1.4), there are only finitely many closed hyperbolic 3-manifolds X with $\mathsf{SR}(X) \leqslant s_0$. Then, the injectivity radius of these hyperbolic 3-manifolds X have a common lower bound. We denote it by δ_0. It is easy to see that δ_0 is a positive constant depending on s_0.

Let M be a complete hyperbolic 3-manifold. Jørgensen and Thurston proved that the thick part of M has a triangulation such that the number of tetrahedra of this triangulation is related to volume, see [9] or [14]. We state this theorem for closed hyperbolic 3-manifold in the following.

Theorem 2.1 (Jørgensen and Thurston)**.** *Let M be a closed hyperbolic 3-manifold. Assume that $\mathsf{Inj}(M,\mathsf{hyp}) \geqslant a_0$, where a_0 is a positive constant. Then there exists a triangulation of M, with the number t of tetrahedra satisfying*

$$t \leqslant K\,\mathsf{Vol}_{\mathsf{hyp}}(M),$$

where K is a positive constant only depending on a_0.

Main steps for the proof of Theorem 2.1: Let $R = \frac{1}{2}\mathsf{Inj}(M,\mathsf{hyp})$. Assume that X is a maximal set of points in M, so that any two points in X with distance at least R. The set X is maximal under inclusion. A Voronoi cell for $x_0 \in X$ is defined to be the the following subset

$$\{y \in M | dist(y, x_0) \leqslant dist(y, x), \text{ for any } x \in X\}.$$

For all Voronoi cells corresponding to points in X, the number of these Voronoi cells will be bounded from above by a constant v depending only on a_0. After triangulating each Voronoi cell, we get a triangulation with the number t of tetrahedra bounded from above by K, where the constant K is a multiple of v.

Note that the triangulation complexity $c(M)$ of a closed hyperbolic 3-manifold M satisfies $c(M) \leqslant t$, so that we have

$$\begin{aligned} c(M) &\leqslant t \\ &\leqslant K\, \mathsf{Vol}_{\mathsf{hyp}}(M) \\ &\leqslant K\, \nu_3 \|M\| \\ &\leqslant K \nu_3 C_3\, \mathsf{SR}(M) \log^3 (C_3'\, \mathsf{SR}(M)), \end{aligned}$$

where ν_3, C_3 and C_3' are all fixed positive constants. Hence we have

$$\mathsf{SR}(M) \geqslant C(s_0) \frac{c(M)}{\log^3 c(M)},$$

where $C(s_0)$ is a positive constant only depending on s_0.

3. Systolic volume of hyperbolic n-manifolds

When $n \geqslant 4$, hyperbolic manifolds of dimension n have different properties. For example, there are only finitely many hyperbolic n-manifolds ($n \geqslant 4$) with a bounded volume, but this is not true in dimension 3. We use a generalized version of Jørgensen and Thurston's theorem as main tool to prove Theorem 1.5 in the following.

Proposition 3.1. *Let M be a closed hyperbolic manifold of dimension n, with $n \geqslant 4$. There exists a positive constant K depending only on n, such that the manifold M admits a triangulation with number t of*

n-simplices is bounded from above by its volume as follows,

$$t \leqslant K_n \,\mathsf{Vol}_{\mathsf{hyp}}(M). \tag{3.1}$$

Proof. Let M be a closed hyperbolic manifold of dimension $n \geqslant 4$. Set $R = \mathsf{Inj}(M, \mathsf{hyp})$. Suppose that $X \subset M$ is a maximal set of points with any two of them having distance at least R. We consider Voronoi cells corresponding to points in X. For any two points $p, q \in X$, $B(p, \frac{R}{2}) \cap B(q, \frac{R}{2}) = \emptyset$ since $dist(p, q) \geqslant R$. The total number of Voronoi cells is thus bounded by above by

$$\begin{aligned}\frac{\mathsf{Vol}_{\mathsf{hyp}}(M)}{\mathsf{Vol}_{\mathsf{hyp}}(B(x, \frac{R}{2}))} &= \frac{1}{\mathsf{Vol}_{\mathsf{hyp}}(B(x, \frac{R}{2}))}\,\mathsf{Vol}_{\mathsf{hyp}}(M)\\ &\leqslant \frac{1}{c_1(n)e^{(n-1)R/2}}\,\mathsf{Vol}_{\mathsf{hyp}}(M),\end{aligned}$$

where $c_1(n)$ is a positive constant depending only on n. Reznikov proved in [12] that for hyperbolic n-manifolds with $n \geqslant 4$,

$$\mathsf{Inj}(M, \mathsf{hyp}) \geqslant c_2(n)\,\mathsf{Vol}_{\mathsf{hyp}}(M)^{-(1+4/(n-3))},$$

where $c_2(n)$ is a positive constant only depending on n. Therefore, there exists a universal positive constant K'_n only depending on n, such that

$$\frac{1}{\mathsf{Vol}_{\mathsf{hyp}}(B(x, \frac{R}{2}))} \leqslant K'_n.$$

It is easy to see that $K'_n \leqslant 1$. A lower bound of K'_n can be obtained in terms of the smallest volume of complete hyperbolic n-manifolds.

Each Voronoi cell is triangulated into n-simplices, and the number of them is bounded by above by a positive constant $c_3(n)$ only depending on n. Hence the estimate (3.1) holds. □

Proof of Theorem 1.5: Theorem 1.5 holds according to above Proposition 3.1 and Gromov's theorem (Theorem 1.2). Let M be a closed hyperbolic manifold with dimension at least 4, and t be the number of n-simplices in Proposition 3.1. Then the triangulation complexity

$\sigma(M)$ is bounded above as follows,

$$\begin{aligned} c(M) &\leqslant t \\ &\leqslant Kc_3(n)\,\mathsf{Vol}_{\mathsf{hyp}}(M) \\ &= Kc_3(n)\nu_n\|M\| \\ &\leqslant Kc_3(n)\nu_n C_n\,\mathsf{SR}(M)\log^n\left(C_n'\,\mathsf{SR}(M)\right). \end{aligned}$$

Hence,

$$\mathsf{SR}(M) \geqslant D_n \frac{\sigma(M)}{\log^n \sigma(M)},$$

where D_n is a positive constant only depending on n.

4. Embolic volume and triangulation complexity

We give the proof of Theorem 1.8 in the following. The method used is analogous to the one used for hyperbolic manifolds.

For a given Riemannian metric $\mathcal{G}$ defined on M, let $\mathsf{Inj}(M,\mathcal{G})$ be injectivity radius. The distance function induced by $\mathcal{G}$ is denoted by $dist_{\mathcal{G}}(,)$. Assume that $R = \frac{1}{3}\mathsf{Inj}(M,\mathcal{G})$. Let $S \subset M$ be a maximal collection of points with distance of any two points at least R. For any $x_0 \in S$, define the Dirichlet domain as follows,

$$P_{x_0} = \{y \in S | dist(y, x_0) \leqslant dist(y, x), \forall x \in S\}.$$

We have

$$B\left(x_0, \frac{R}{2}\right) \subset P_{x_0}.$$

Hence the number of polyhedron P_{x_0} in M is bounded above by

$$\frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Vol}_{\mathcal{G}}(B(x_0, \frac{R}{2}))}.$$

We obtain a triangulation of M by triangulating each polyhedron P_x, $x \in S$. In order to let triangulations on each face of P_x match, for each P_x we choose the triangulation which induces a triangulation on each face symmetric with respect to combinatorial isomorphisms of any face. The combinatorial types of polyhedron P_x are determined by R and n. Hence there only finitely many such combinatorial types. We let T be the maximal number of simplices in all polyhedra P_x. Then we have a triangulation of the manifold M with number of simplices bounded above by

$$\frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Vol}_{\mathcal{G}}(B(x_0, \frac{R}{2}))} T.$$

Croke proved a local embolic inequality.

Lemma 4.1 (see Croke [3]). *For a Riemannian metric $\mathcal{G}$ defined on compact n-dimensional manifold M, any metric ball $B(p, r)$ with center p and radius $r \leqslant \frac{1}{2} \mathsf{Inj}(M, \mathcal{G})$ satisfies*

$$\mathsf{Vol}(B(x, r)) \geqslant \alpha_n r^n, \tag{4.1}$$

where α_n is a positive constant only depending on n.

By using Croke's local embolic inequality (4.1), we have

$$\begin{aligned} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Vol}_{\mathcal{G}}(B(x_0, \frac{R}{2}))} T &\leqslant \frac{2^n}{\alpha_n^n} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{R^n} T \\ &\leqslant \frac{6^n}{\alpha_n^n} \mathsf{Emb}(M, \mathcal{G}). \end{aligned}$$

That is, we find a triangulation on M with the number of simplices bounded by $C_n \mathsf{Emb}(M)$, where $C_n = \frac{6^n}{\alpha_n^n}$.

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