TRIANGULATION COMPLEXITY AND SYSTOLIC VOLUME OF HYPERBOLIC MANIFOLDS

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ABSTRACT. Let M be a closed n-manifold with nonzero simplicial volume $\|M\|$. A central result in systolic geometry proved by Gromov is that systolic volume of M is related to $\|M\|$. In this short note, we establish relation between systolic volume and triangulation complexity of hyperbolic manifolds. The proof is based on Jørgensen and Thurston's theorem of hyperbolic manifolds.

1. Introduction

In systolic geometry, the systole of a Riemannian manifold is defined to be the shortest length of a noncontractible loop. Gromov's systolic inequality implies that systole is bounded from above by Riemannian volume. The optimal constant in a systolic inequality is usually called systolic volume in literature. Let M be a closed n-dimensional manifold with nonzero simplicial volume. Gromov [Gro83, Section 6.4.D] proved that topological complexity of M is represented by systolic volume. In this paper, we extend Gromov's result by relating systolic volume to triangulation complexity. Triangulation complexity of a closed manifold is defined to be the minimum number of simplices in a triangulation. Hence the triangulation complexity naturally represents how complicated a manifold is. Our result is a supplement to Gromov's work of establishing relations between systolic volume and other topological invariants.

Let M be a closed 3-manifold. If M is irreducible and not homeomorphic to $S^3, \mathbb{RP}^3$ or $L(3,1)$, the triangulation complexity coincides with the complexity defined by Matveev [Mat90]. In the following, we use $c(M)$ to denote the triangulation complexity of a closed 3-manifold M. We refer to [JRT13, JRST20, LP21] for recent developments of triangulation complexity of 3-manifolds. The systolic volume of M, denoted by $\mathsf{SR}(M)$, is defined to

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be

$$\inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Sys}\,\pi_1(M,\mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M. The systolic volume $\mathsf{SR}(M)$ is positive if M is a closed essential 3-manifold, see [Gro83, Theorem 0.1.A.] for more details.

Let s_0 be a sufficiently large positive number.

Theorem 1.1. *Suppose that M is a closed hyperbolic 3-manifold with $\mathsf{SR}(M) \leqslant s_0$. There exists a positive constant $C(s_0)$ only depending on s_0, such that*

$$\mathsf{SR}(M) \geqslant C(s_0)\,\frac{c(M)}{\log^3 c(M)}.$$

The simplicial volume $\|M\|$ of a manifold of dimension n is defined to be the minimum number of simplices in a cycle representing the fundamental class of real coefficients. For hyperbolic manifolds, there exists the following proportionality principle,

$$\nu_n\,\|M\| = \mathsf{Vol}_{\mathsf{hyp}}(M), \tag{1.1}$$

where ν_n is a positive constant only depending on n. We refere to [BP92] for more details about simplicial volume. A central theorem in systolic geometry is the following one of relating systolic volume to simplicial volume.

Theorem 1.2 (Gromov 1983, see [Gro83, Section 6.4.D.] or [Gro96, Section 3.C.3.])**.** *Let M be a closed n-dimensional manifold with non-zero simplicial volume. Then the systolic volume $\mathsf{SR}(M)$ of M satisfies*

$$\|M\| \leqslant C_n\,\mathsf{SR}(M)\,\log^n\big(C'_n\,\mathsf{SR}(M)\big), \tag{1.2}$$

where C_n and C'_n are two positive constants only depending on n.

Remark 1.3. In [Gro83, Section 6.4.D.], there is a typo of missing the exponent n in the logarithm part of (1.2). In literature the estimate (1.2) in Theorem 1.2 is often written to be

$$\mathsf{SR}(M) \geqslant C_n \frac{\|M\|}{\log^n \|M\|},$$

where C_n is a positive constant only depending on n.

Theorem 1.2 builds a bridge between systolic geometry and hyperbolic geometry. We refer to [Gut10] for more explanation of this interplay.

A mail tool used to prove Theorem 1.1 is the connection between triangulation and hyperbolic volume. The work of Jørgensen and Thurston implies that any complete hyperbolic 3-manifold admits a triangulation with the number of tetrahedra bounded from above by its volume. A detailed proof of this theorem is provided by Kobayashi and Rieck in [KR11].

The triangulation complexity of manifolds in higher dimensions is studied in [FFM12]. Let M be a closed manifold of dimension n. The triangulation complexity of M, denoted $\sigma(M)$, is defined to be the minimum number of n-simplices in any triangulation of M. When $n = 3$, $\sigma(M)$ coincides with $c(M)$.

Theorem 1.4. *Let M be a closed hyperbolic manifold of dimension n with $n \geqslant 4$. The triangulation complexity $\sigma(M)$ and systolic volume of M is related by*

$$\mathsf{SR}(M) \geqslant D_n \frac{\sigma(M)}{\log^n \sigma(M)},$$

where D_n is a positive constant only depending on n.

Remark 1.5. It is proved in [FFM12] that under assumptions of Theorem 1.4, $\|M\| < \sigma(M)$ holds. Hence above theorem is indeed a generalization of Gromov's theorem (Theorem 1.2).

The embolic volume defined in terms of injectivity radius is another geometric quantitiy representing the topological complexity of manifolds. We refer to [Ber03, Section 11.2.3.] for a general description.

Definition 1.6. The embolic volume of M, denoted $\mathsf{Emb}(M)$, is defined to be

$$\inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Inj}(M, \mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M.

Embolic volume is positive for all compact n-manifolds M ($n \geqslant 2$), see [Ber03, Section 7.2.4.]. Note that on a Riemannian manifold $(M, \mathcal{G})$, $\mathsf{Sys}\,\pi_1(M, G) \geqslant 2\,\mathsf{Inj}(M, \mathcal{G})$. Hence we always have $\mathsf{Emb}(M) \geqslant \mathsf{SR}(M)$. Then for closed manifolds with nonzero simplicial volume, Theorem 1.2 implies that the embolic volume is related to simplicial volume. This result is proved by Katz and Sabourau [KS05] by a different approach. Moreover, a direct corollary of Theorem 1.4 is that any closed hyperbolic n-manifold with $n \geqslant 4$ has

$$\mathsf{Emb}(M) \geqslant D_n \frac{\sigma(M)}{\log^n (1 + \sigma(M))}, \tag{1.3}$$

where D_n is the positive constant only depending on n. In this note, we also show that embolic volume of any closed n-manifold is related to its triangulation complexity. Compared with Katz and Sabourau's above result, our theorem is more general, since it also includes all closed manifolds with zero simplicial volume.

Theorem 1.7. *Let M be a closed n-dimensional manifold. Then there exists a positive constant E_n only depending on n, such that*

$$\mathsf{Emb}(M) \geqslant E_n \sqrt{\sigma(M)}. \tag{1.4}$$

Organization. This short note is organized as follows: In Section 2, we discuss Jørgensen and Thurston's theorem for hyperbolic 3-manifolds. Proof of Theorem 1.1 is given in this section. In section 3, triangulation complexity of hyperbolic manifolds in higher dimensions is concerned. Then we prove Theorem 1.4 in this section. Section 4 concerns embolic volume and triangulation complexity. Theorem 1.7 is proved in this section.

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2. Triangulation and volume of hyperbolic 3-manifolds

We prove Theorem 1.1 in this section. The proof is based on Jørgensen and Thurston's theorem of hyperbolic 3-manifolds.

Jørgensen and Thurston's work ([Thu79], also see [KR11]) implies that triangualtion of a hyperbolic 3-manifold is related to its volume.

Theorem 2.1 (Jørgensen, Thurston)**.** *Let M be a closed hyperbolic 3-manifold, and a_0 be a positive constant. Assume that $\mathsf{Inj}(M, \mathsf{hyp}) \geqslant a_0$. Then there exists a triangulation of M, with the number t of tetrahedra satisfying*

$$t \leqslant K\ \mathsf{Vol}_{\mathsf{hyp}}(M),$$

where K is a positive constant only depending on a_0.

Proof. We briefly introduce outline of the proof. For more details, we refer to [Thu79, Chapter 5] and [KR11].

Let $R = \frac{1}{2}\,\mathsf{Inj}(M, \mathsf{hyp})$. Assume that X is a maximal set of points in M, so that any two points in X with distance at least R. The set X is maximal under inclusion. The Voronoi cell associated to $x_0 \in X$ is defined to be the the subset

$$V(x_0) = \{y \in M | dist(y, x_0) \leqslant dist(y, x), \text{ for any } x \in X \text{ and } x \neq x_0\}.$$

For all Voronoi cells corresponding to points in X, the total number is bounded from above by a constant v depending only on a_0. After triangulating each Voronoi cell, we get a triangulation of M with the number t of tetrahedra bounded from above by K, where the constant K is a multiple of v. □

Another tool we need to use in the proof of Theorem 1.1 is Sabourau's finiteness theorem. Sabourau proved that there are only finitely many hyperbolic 3-manifolds with bounded systolic volume.

Theorem 2.2 (see [Sab07, Theorem B]). *For a sufficiently large positive number s_0, there are only finitely many hyperbolic n-manifolds ($n \geqslant 2$) M with $\mathsf{SR}(M) \leqslant s_0$.*

Remark 2.3. Sabourau's theorem incudes the case of dimension $n = 3$. When $n \geqslant 4$, Theorem 2.2 is a direct implication of Wang's finiteness theorem and Theorem 1.2. The case of $n = 2$ is yielded by Theorem 1.2, since area of a closed hyperbolic surface is proportional to its simplicial volume and thus proportional to genus.

Let M be a closed hyperbolic 3-manifold. For a sufficiently large positive number s_0, according to Sabourau's theorem (see Theorem 2.2), there are only finitely many closed hyperbolic 3-manifolds X with $\mathsf{SR}(X) \leqslant s_0$. Then we know that the injectivity radii of all these hyperbolic 3-manifolds have a common lower bound. Denote this lower bound by δ_0. The constant δ_0 is only depending on s_0.

Proof of Theorem 1.1:
The triangulation complexity $c(M)$ of a closed hyperbolic 3-manifold M satisfies $c(M) \leqslant t$, so that we have

$$\begin{aligned} c(M) &\leqslant t \\ &\leqslant K\, \mathsf{Vol}_{\mathsf{hyp}}(M) \\ &\leqslant K\, \nu_3 \|M\| \\ &\leqslant K \nu_3 C_3\, \mathsf{SR}(M) \log^3\left(C_3'\, \mathsf{SR}(M)\right), \end{aligned}$$

where ν_3, C_3 and C_3' are all fixed positive constants. Hence we have

$$\mathsf{SR}(M) \geqslant C(s_0)\, \frac{c(M)}{\log^3 c(M)},$$

where $C(s_0)$ is a positive constant only depending on s_0.

□

3. Hyperbolic manifolds in higher dimensions

When $n \geqslant 4$, hyperbolic manifolds of dimension n are different than $n = 3$. For example, there are only finitely many hyperbolic n-manifolds ($n \geqslant 4$) with bounded volume (see [BGLM02]), but this is not true in $n = 3$. We generalize Jørgensen and Thurston's theorem to hyperbolic manifolds of dimension at least four. Then we prove Theorem 1.4 in this section.

Proposition 3.1. *Let M be a closed hyperbolic manifold of dimension n, with $n \geqslant 4$. There exists a positive constant K depending only on n, such that the manifold M admits a triangulation with number t of n-simplices is bounded from above by its volume as follows,*

$$t \leqslant K_n \operatorname{\mathsf{Vol}}_{\mathsf{hyp}}(M). \tag{3.1}$$

Proof. Let M be a closed hyperbolic manifold of dimension $n \geqslant 4$. Set $R = \mathsf{Inj}(M, \mathsf{hyp})$. Suppose that $X \subset M$ is a maximal set of points with any two of them having distance at least R. We consider Voronoi cells corresponding to points in X. Recall that a Voronoi cell associated to $x \in X$ is

$$V(x) = \{y \in M | dist(x, y) \leqslant dist(x', y), \text{for any } x' \in X \text{ and } x' \neq x\}.$$

For any two distinct points $p, q \in X$, $B(p, \frac{R}{2}) \cap B(q, \frac{R}{2}) = \emptyset$ since $dist(p, q) \geqslant R$. The total number of Voronoi cells is thus bounded by above by

$$\begin{aligned} \frac{\mathsf{Vol}_{\mathsf{hyp}}(M)}{\mathsf{Vol}_{\mathsf{hyp}}(B(x, \frac{R}{2}))} &= \frac{1}{\mathsf{Vol}_{\mathsf{hyp}}(B(x, \frac{R}{2}))} \mathsf{Vol}_{\mathsf{hyp}}(M) \\ &\leqslant \frac{1}{c_1(n) e^{(n-1)R/2}} \mathsf{Vol}_{\mathsf{hyp}}(M) \\ &\leqslant \frac{1}{c_1(n)} \mathsf{Vol}_{\mathsf{hyp}}(M), \end{aligned}$$

where $c_1(n)$ is a positive constant depending only on n. Note that it is proved in [BT11] there exist hyperbolic n-manifolds M with the injectivity radius R being arbitratrily small. Hence in the above estimate, $\frac{1}{\mathsf{Vol}_{\mathsf{hyp}}(B(x, \frac{R}{2}))} \leqslant \frac{1}{c_1(n)}$ cannot be improved.

A triangulation of the hyperbolic manifold M is obtained by triangulating each Voronoi cell $V(x)$ into n-simplices. Since two Voronoi cells $V(x)$ and $V(y)$ have a common face if and only if $B(y, R) \subset B(x, \frac{5}{2}R)$, the number of faces of any Voronoi cell $V(x)$ has upper bound of $\frac{\mathsf{Vol}_{\mathsf{hyp}}(B(x, \frac{5}{2}R))}{\mathsf{Vol}_{\mathsf{hyp}}(B(x, \frac{R}{2}))}$. Hence, the total number of faces for all Voronoi cells is finite and only depending on n. In the triangulation, maximal number of n-simplices in each Voronoi cell is uniformly bounded above by the number of its faces. Therefore, there exists a positive constant K_n, such that the total number of n-simplices in the triangulation obtained in the above is bounded above by $K_n \operatorname{\mathsf{Vol}}_{\mathsf{hyp}}(M)$, where K_n is a positive constant only depending on n. $\square$

Proof of Theorem 1.4: Theorem 1.4 holds according to above Proposition 3.1 and Gromov's theorem (Theorem 1.2). Let M be a closed hyperbolic manifold with dimension at least 4, and t be the number of n-simplices in Proposition 3.1. Then the triangulation complexity $\sigma(M)$ is bounded above as follows,

$$\begin{aligned} \sigma(M) &\leqslant t \\ &\leqslant K_n \operatorname{\mathsf{Vol}}_{\mathsf{hyp}}(M) \\ &= K_n \nu_n \|M\| \\ &\leqslant K_n \nu_n C_n \operatorname{\mathsf{SR}}(M) \log^n \big(C_n' \operatorname{\mathsf{SR}}(M)\big). \end{aligned}$$

Hence,

$$\operatorname{\mathsf{SR}}(M) \geqslant D_n \frac{\sigma(M)}{\log^n \sigma(M)},$$

where D_n is a positive constant only depending on n.

4. Embolic volume of compact manifolds

We prove Theorem 1.7 in this section. The proof is still based on the approach of packing balls.

Embolic volume is defined in Definition 1.6. Berger's embolic inequality ([Ber80] or [Ber03, Section 7.2.4.]) states that for any Riemannian metric $\mathcal{G}$ defined on a compact manifold of dimension n,

$$\operatorname{\mathsf{Inj}}(M, \mathcal{G})^n \leqslant C \operatorname{\mathsf{Vol}}_{\mathcal{G}}(M)$$

holds, where C is a positive constant. Hence the embolic volume of compact n-manifolds is always positive. The relation between embolic volume and other topological invariants is described in [Ber03, Section 11.2.3.]. On a closed manifold with nonzero systolic volume, we always have $\operatorname{\mathsf{Emb}}(M) \geqslant \operatorname{\mathsf{SR}}(M)$. Hence Gromov's theorem (Theorem 1.2) includes a relation between embolic volume and simplicial volume for closed manifolds with nonzero simplicial volume. Moreover, Katz and Sabourau [KS05] uses a different method to show this result. Then the results in Theorem 1.1 and 1.4 also hold if the systolic volume is replaced by embolic volume.

Let E be a sufficiently large positive constant. In [Yam88, GPW90, GPW91] the finiteness theorems are proved for compact n-manifolds M

with $\mathsf{Emb}(M) \leqslant E$. Therefore, from this point of view, the embolic volume of compact manifolds works like volume of hyperbolic manifolds. Our theorem in this section provides more evidence to this viewpoint.

The following local embolic inequality of Croke will be used in the proof of Theorem 1.7.

Lemma 4.1 (see Croke [Cro80])**.** *For a Riemannian metric $\mathcal{G}$ defined on compact n-dimensional manifold M, any metric ball $B(p, r)$ with center p and radius $r \leqslant \frac{1}{2}\,\mathsf{Inj}(M, \mathcal{G})$ satisfies*

$$\mathsf{Vol}(B(x, r)) \geqslant \alpha_n r^n, \tag{4.1}$$

where α_n is a positive constant only depending on n.

Proof of Theorem 1.7: For a Riemannian metric $\mathcal{G}$ defined on M, let $\mathsf{Inj}(M, \mathcal{G})$ be injectivity radius. The distance function induced by $\mathcal{G}$ is denoted $dist_{\mathcal{G}}(,)$. Assume that $R = \frac{1}{5}\,\mathsf{Inj}(M, \mathcal{G})$. We say a subset $A \subset M$ is R-separated if $dist_{\mathcal{G}}(x, y) \geqslant R$ for all distinct $x, y \in A$. Now let S be a maximal R-separated subset of M. Here the maximal is under inclusion relation. For $x_0 \in S$, denote by $V(x_0)$ the following Voronoi cell,

$$V(x_0) = \{y \in S | dist(y, x_0) \leqslant dist(y, x), \forall x \in S \,\text{and}\, x \neq x_0\}.$$

We have $B\left(x_0, \frac{R}{2}\right) \subset V(x_0)$. Hence the number of Voronoi cells $V(x_0)$ in M is bounded from above by

$$\frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Vol}_{\mathcal{G}}(B(x_0, \frac{R}{2}))}.$$

We obtain a triangulation of M by triangulating each Voronoi cell $V(x)$ into n-simplices. In order to let the triangulations on each face of $V(x)$ match, we choose the triangulation which induces a triangulation on each face symmetric with respect to combinatorial isomorphisms of any face. The combinatorial types of Voronoi cell $V(x)$ are determined by R and n. Hence there are only finitely many such combinatorial types. We let T be the maximal number of simplices in all Voronoi cell $V(x)$. Then the finiteness of combinatorial types of $V(x)$ implies that T is a constant only depending on R and n. In fact, two Voronoi cells $V(x)$ and $V(y)$ are adjacent, if and only if $B(x, \frac{R}{2}) \subset B(y, \frac{5R}{2})$. Therefore, the number T has upper bound $\frac{\mathsf{Vol}(B(y, \frac{5R}{2}))}{\inf_x \mathsf{Vol}(B(x, \frac{R}{2}))}$, which is bounded above by $C_n\,\mathsf{Emb}(M)$ according to Lemma 4.1, where C_n is a positive constant only depending on n, and the infimum is taken over all x such that $B(x, \frac{R}{2}) \subset B(x, \frac{5}{2}R)$ holds. The number of

n-simplices in the triangulation is equal to

$$\frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Vol}_{\mathcal{G}}(B(x_0, \frac{R}{2}))} T.$$

By using Croke's local embolic inequality (4.1) again, we have

$$\begin{aligned} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Vol}_{\mathcal{G}}(B(x_0, \frac{R}{2}))} T &\leqslant \frac{2^n}{\alpha_n^n} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{R^n} T \\ &\leqslant \beta_n \, \mathsf{Emb}(M, \mathcal{G})^2, \end{aligned}$$

where β_n is a positive constant only depending on n. Therefore we find a triangulation on M with the number of simplices bounded by $\beta_n \, \mathsf{Emb}(M)^2$.

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