Constrained Ornstein-Uhlenbeck Process with Sum and Non-Negativity Constraints

Let $X_1, X_2, \ldots, X_n$ be n variables, each evolving as a stochastic process, with the following requirements:

- The sum of the variables is constant: $\sum_{i=1}^n X_i(t) = C$ for all t
- Each $X_i(t) \geq 0$ for all i, t
- Each X_i reverts to a (possibly different) mean μ_i, where $\mu_i \geq 0$ and $\sum_{i=1}^n \mu_i = C$
- The stationary standard deviation ("magnitude") of X_i is M_i
- The global magnitude parameter is $M = \sum_{i=1}^n M_i$
- All variables share the same cutoff frequency f_c

Parameterization

The standard Ornstein-Uhlenbeck SDE for one variable is:

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$

where θ is the mean-reversion rate, μ the mean, σ the volatility, and W_t is a standard Wiener process.

We define for all variables:

$$\theta = 2\pi f_c$$

and for each i,

$$M_i = \sqrt{\frac{\sigma_i^2}{2\theta}} \implies \sigma_i = M_i\sqrt{2\theta}$$

The weight for each variable is

$$w_i = \frac{\mu_i}{C}$$

SDE with Sum and Non-Negativity Constraints

Start with the unconstrained multi-dimensional OU process:

$$dX_i = \theta(\mu_i - X_i)dt + \sigma_i dW_i$$

To maintain the sum constraint $\sum_{i=1}^n X_i = C$, introduce a drift correction, proportional to w_i:

$$dX_i = \theta(\mu_i - X_i)dt + \sigma_i dW_i + \lambda(t)w_i dt$$

where $\lambda(t)$ is a Lagrange multiplier chosen to enforce the sum constraint.

Summing over i and using $\sum_{i=1}^{n} w_i = 1$,

$$\sum_{i=1}^{n} dX_i = \sum_{i=1}^{n} \theta(\mu_i - X_i)dt + \sum_{i=1}^{n} \sigma_i dW_i + \lambda(t)dt = 0$$

Solving for $\lambda(t)$,

$$\lambda(t) = -\sum_{j=1}^{n} \theta(\mu_j - X_j) - \sum_{j=1}^{n} \sigma_j \frac{dW_j}{dt}$$

Substituting,

$$\begin{aligned} dX_i &= \theta(\mu_i - X_i)dt + \sigma_i dW_i \\ &\quad - w_i \left[\sum_{j=1}^{n} \theta(\mu_j - X_j)dt + \sum_{j=1}^{n} \sigma_j dW_j \right] \end{aligned}$$

Or, grouping terms,

$$\boxed{dX_i = \theta \left[(\mu_i - X_i) - w_i \sum_{j=1}^{n} (\mu_j - X_j) \right] dt + \sigma_i dW_i - w_i \sum_{j=1}^{n} \sigma_j dW_j}$$

Enforcing Non-Negativity: Reflection/Projection

To ensure $X_i(t) \geq 0$, add a reflection term:

$$dX_i = \cdots + dL_i(t)$$

where $L_i(t)$ is a non-decreasing process that only increases when $X_i(t) = 0$.

Discrete-Time Simulation

Let the time step be Δt. For each time step n, let X_i^n denote the value of X_i at time n.

Define the discrete-time update:

$$\begin{aligned} X_i^{n+1,\,\text{pre-proj}} &= X_i^n + \theta(\mu_i - X_i^n)\Delta t + \sigma_i \sqrt{\Delta t}\, \xi_i^n \\ &\quad - w_i \left(\sum_{j=1}^{n} \theta(\mu_j - X_j^n)\Delta t + \sum_{j=1}^{n} \sigma_j \sqrt{\Delta t}\, \xi_j^n \right) \end{aligned}$$

where $\xi_i^n \sim \mathcal{N}(0,1)$ are independent standard normal random variables.

Projection to the Simplex

After each update, the vector $\boldsymbol{X}^{n+1,\,\text{pre-proj}}$ is projected onto the simplex:

$$\mathcal{S} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = C,\, x_i \geq 0 \ \forall i \right\}$$

The Euclidean projection $\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}\in\mathcal{S}} \|\boldsymbol{x} - \boldsymbol{y}\|^2$ for any $\boldsymbol{y}$ can be computed as follows:

1. Sort y in descending order: $y_{(1)} \geq y_{(2)} \geq \cdots \geq y_{(n)}$
2. Find $\rho = \max\left\{ j \in [n] : y_{(j)} + \frac{1}{j}(C - \sum_{k=1}^{j} y_{(k)}) > 0 \right\}$
3. Compute $\theta = \frac{1}{\rho}(C - \sum_{j=1}^{\rho} y_{(j)})$
4. The projection is $x_i^* = \max(y_i + \theta, 0)$

The final update at each time step is:

$$\boldsymbol{X}^{n+1} = \text{Proj}_{\mathcal{S}}\left(\boldsymbol{X}^{n+1,\,\text{pre-proj}}\right)$$