Random Neural Network Models
Application of Dynamic Mean Field Theory

Ziao Chen

Advisor: Professor Stamatis Dostoglou

University of Missouri–Columbia

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Reference

Sompolinsky, H., Crisanti, A., & Sommers, H. J. (1988). Chaos in random neural networks. *Phys. Rev. Lett.*, *61*, 259–262. https://doi.org/10.1103/PhysRevLett.61.259

Mastrogiuseppe, F., & Ostojic, S. (2018). Linking connectivity, dynamics, and computations in low-rank recurrent neural networks. *Neuron*, *99*(3), 609–623

THE NETWORK MODEL

- A large recurrent network with N units (neurons).
- Each unit is characterized by $h_i(t) \in \mathbb{R}$, $i \in \{1, \ldots, N\}$ interpreted as membrane potential of a biological neuron.
- A related variable $S_i(t) = \phi(h_i(t))$ interpreted as its firing rate.
- $\phi(x)$ is the non-linear activation function which defines the input-output characteristic. ϕ is assumed to have a sigmoid shape $\phi(x) \to \pm 1$ as $x \to \pm\infty$, $\phi(-x) = -\phi(x)$. For concreteness, we choose

$$\phi(x) = \tanh(gx) \tag{1}$$

where $g > 0$ is a constant.

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$$\phi(x) = \tanh(gx) \tag{1}$$

where $g > 0$ is a constant.

- The dynamic of the network is given by N equations

$$\dot{h}_i = -h_i + \sum_{j=1}^{N} J_{ij} S_j = -h_i + \sum_{j=1}^{N} J_{ij} \phi(h_j) = -h_i + \hat{\eta}_i \tag{2}$$

$\hat{\eta}_i$ denotes the total input to the i-th unit.

- The connectivity matrix J is a $n \times n$ Gaussian random matrix.

Assumption (1)

J_{ij}'s are i.i.d. and $J_{ij} \sim \mathcal{N}(0, \mathrm{J}^2/N)$.

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Dynamic mean-field theory

Each $\hat{\eta}_i$ can be approximated by a **Gaussian process** $\eta_i(t)$.

Assumption (2)

The random connectivity J efficiently decorrelates single unit activity in the limit of $N \to \infty$. More specifically, for each unit j, its output S_j is uncorrelated of its outgoing weights $J_{ij}, \forall i$.

A self-consistent consequence: $\hat{\eta}_i$'s are uncorrelated with mean 0.

Calculate the first two moments of $\hat{\eta}_i$ over realizations of J.

$$\mathbb{E}[\hat{\eta}_i(t)] = \sum_{j=1}^{N} \mathbb{E}[J_{ij} S_j(t)] = \sum_{j=1}^{N} \mathbb{E}[J_{ij}]\mathbb{E}[S_j(t)] = 0 \quad \forall t \tag{3}$$

$$\begin{aligned}
\mathbb{E}[\hat{\eta}_i(t)\hat{\eta}_j(t+\tau)] &= \mathbb{E}\left[\left(\sum_{k=1}^{N} J_{ik} S_k(t)\right)\left(\sum_{l=1}^{N} J_{jl} S_l(t+\tau)\right)\right] \\
&= \sum_{k=1}^{N}\sum_{l=1}^{N} \mathbb{E}[J_{ik} J_{jl} S_k(t) S_l(t+\tau)] \\
&= \sum_{k=1}^{N}\sum_{l=1}^{N} \mathbb{E}[J_{ik} J_{jl}]\mathbb{E}[S_k(t) S_l(t+\tau)] \\
&= \delta_{ij} \frac{\mathrm{J}^2}{N} \sum_{k=1}^{N} \mathbb{E}[S_k(t) S_k(t+\tau)]
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Assume $h_i(t)$'s stationary process. Define an **autocorrelation** function:

$$C(\tau) := \mathbb{E}[N^{-1}\sum_{i=1}^{N} S_i(t)S_i(t+\tau)] \tag{5}$$

With equation (4)

$$\mathbb{E}[\hat{\eta}_i(t)\hat{\eta}_i(t+\tau)] = \mathrm{J}^2 C(\tau), \quad \forall i \tag{6}$$

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Assumption (3)

The dynamic mean field theory approximates the inputs to each neuron $\hat{\eta}_i$'s by independent Gaussian processes η_i's, with the same first and second moments as $\hat{\eta}_i$'s. This approximation becomes exact in the limit of $N \to \infty$.

Because

- $\hat{\eta}_i(t) = \sum_{j=1}^{N} J_{ij} S_j(t)$
- For any i, j, $\mathbb{E}[J_{ij} S_j(t)] = 0$
- $\mathbb{E}[(J_{ij} S_j(t))^2] = N^{-1} \mathrm{J}^2 \mathbb{E}[S_j(t)^2] = (\sigma_S \mathrm{J})^2 / N$ for some $\sigma_S < \infty$ since $|S_j| < 1$ and the network is homogeneous.
- Assuming S_j's are independent, apply central limit theorem as $N \to \infty$.

h_i is a function of each independent η_i. Then S_i's are independent. So it is self-consistent.

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There exist processes η_i's that have the same first two moments of $\hat{\eta}_i$'s.

Because for any finite set of indices $\{t_1, \ldots, t_M\}$, there exist random variables $\{\eta_i(t_j)\}_{i\in\{1,\ldots N\}, j\in\{1,\ldots,M\}}$ that form a Gaussian MN-vector with

$$\mathbb{E}[\eta_i(t_j)] = 0, \quad \mathbb{E}[\eta_i(t_j)\eta_k(t_l)] = \delta_{ik}\mathrm{J}^2 C(t_j - t_l), \quad \forall i, j, k, l$$

Then η_i's are independent Gaussian processes. In particular,

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Under assumption 3, the dynamic of each unit is reduced to a Langevin-like equation

$$\dot{h}_i = -h_i + \eta_i \tag{8}$$

Claim:

If η is stationary Gaussian process, then an h that follows $\dot{h} = -h + \eta$ is also a stationary Gaussian process.

First, solve the differential equation with initial condition $h(t_0)$

$$h(t) = \int_{t_0}^{t} e^{t'-t}\eta(t')dt' + e^{t_0-t}h(t_0)$$

Assuming $e^{t_0}h(t_0) \to 0$ as $t_0 \to -\infty$

$$h(t) = \int_{-\infty}^{t} e^{t'-t}\eta(t')dt' = \int f(t,t')\eta(t')dt' \tag{9}$$

where $f(t,t') = e^{t'-t}\chi_{(-\infty,t]}(t')$.

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The cumulant generating function of a random variable X is defined as

$$\psi(\lambda) = \ln \mathbb{E}[e^{\lambda X}]$$

If ψ can be expanded as a power series

$$\psi(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \langle X^n \rangle_c$$

then $\langle X^n \rangle_c$ is called the n-th cumulant of X.

Cumulants are related to moments of X

$$\begin{aligned} \langle X \rangle_c &= \mathbb{E}[X] \\ \langle X^2 \rangle_c &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ \langle X^3 \rangle_c &= \mathbb{E}[X^3] - 3\mathbb{E}[X^2]\mathbb{E}[X] + 2\mathbb{E}[X]^3 \end{aligned}$$

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The cumulant generating function can be extended for random vector $(X_1, \dots X_n)$.

$$\psi(\lambda) = \ln \mathbb{E}[e^{\lambda \cdot X}] = \sum_{(r_1,\dots,r_n)\in\mathbb{N}_0^n\setminus\{\mathbf{0}\}} \langle X_1^{r_1} \dots X_n^{r_n} \rangle_c \prod_{k=1}^{n} \frac{(i\lambda_k)^{r_k}}{r_k!}$$

where the coefficients $\langle X_1^{r_1} \dots X_n^{r_n} \rangle_c$ with $\sum_{k=1}^{n} r_k = m$ are cumulants of m-th order.

Fact: Each m-th cumulant can be expressed in polynomial of moments $\mathbb{E}[X_1^{r_1} \dots X_n^{r_n}]$ with $\sum_{k=1}^{n} r_k \leq m$.

Marcinkiewicz (1939)

Gaussian distribution is the only distribution whose cumulant generating function is a polynomial, i.e. the only distribution having a finite number of non-zero cumulants (in particular, first and second order).

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Any linear transform of η is a Gaussian process.

$$h(t) = \int f(t,t')\eta(t')dt'$$

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Moreover, if η is stationary and $f(t,t') = g(t-t')\chi_{(\infty,t]}(t')$ for some function g. h is stationary.

Because by change of variable $t'' = t' - t$

$$h(t) = \int_{-\infty}^{t} g(t-t')\eta(t')dt' = \int_{-\infty}^{0} g(-t'')\eta(t''+t)dt''$$

Its mean does not depend on t and its correlation depends only on $t_2 - t_1$.

$$\begin{aligned}
&\mathbb{E}[h(t)] = \int_{-\infty}^{0} g(-t')\mathbb{E}[\eta(t'+t)]dt' \\
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&= \langle h(t_1)h(t_2)\rangle_c + \mathbb{E}[h(t_1)]\mathbb{E}[h(t_2)] \\
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&= \int_{-\infty}^{0}\int_{-\infty}^{0} g(-t_1')g(-t_2')\mathbb{E}[\eta(t_1')\eta(t_2'+t_2-t_1)]dt_1'dt_2' + \mathbb{E}[h(t)]^2
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Moreover, if η is stationary and $f(t,t') = g(t-t')\chi_{(\infty,t]}(t')$ for some function g. h is stationary.

Because by change of variable $t'' = t' - t$

$$h(t) = \int_{-\infty}^{t} g(t-t')\eta(t')dt' = \int_{-\infty}^{0} g(-t'')\eta(t''+t)dt''$$

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Solving the autocorrelation function

Define the local-field autocorrelation

$$\Delta(\tau) := \mathbb{E}[h_i(t)h_i(t+\tau)] \tag{12}$$

We claim that

$$\ddot{\Delta} = \Delta - \mathrm{J}^2 C \tag{13}$$

Theorem (Wiener-Khinchin)

If $\xi(t)$ is a wide-sense stationary process, with autocorrelation function $Q(\tau) = \mathbb{E}[\xi(t)\xi(t+\tau)]$, and has power spectral density

$$I(\omega) = \lim_{T\to\infty} \frac{1}{T}|\tilde{\xi}_T(\omega)|^2$$

where $\xi_T(t) = \xi(t)\chi_{[-T/2,T/2]}(t)$ and $\tilde{\xi}_T(\omega) = \int_{-\infty}^{\infty} \xi_T(t)e^{-i\omega t}dt$ is its Fourier transform. Then $I = \tilde{Q}$.

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Apply Fourier transform to equation (8).

$$i\omega\tilde{h}(\omega) = -\tilde{h}(\omega) + \tilde{\eta}(\omega)$$

$$(1 + i\omega)\tilde{h}(\omega) = \tilde{\eta}(\omega)$$

$$(1 + \omega^2)|\tilde{h}(\omega)|^2 = |\tilde{\eta}(\omega)|^2$$

Apply Theorem 1

$$(1 + \omega^2)\tilde{\Delta}(\omega) = (1 + \omega^2)\lim_{T\to\infty}\frac{1}{T}|\tilde{h}_T(\omega)|^2 = \lim_{T\to\infty}\frac{1}{T}|\tilde{\eta}_T(\omega)|^2 = \mathrm{J}^2\tilde{C}(\omega)$$

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A **potential function** $V(\Delta)$ can be constructed

$$V(\Delta) := -\frac{1}{2}\Delta^2 + \mathrm{J}^2 \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \Phi\left(\sqrt{\Delta(0) - |\Delta|}x + \sqrt{|\Delta|}z\right) \mathcal{D}x \right\}^2 \mathcal{D}z \tag{14}$$

where

$$\mathcal{D}x := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

and in general

$$\Phi(x) := \int_0^x \phi(y) dy$$

In the particular case of equation (1)

$$\Phi(x) = g^{-1} \ln\cosh(gx)$$

We next show that V satisfies

$$\ddot{\Delta} = -\frac{\partial V}{\partial \Delta} \tag{15}$$

It suffices to show

$$\frac{\partial V}{\partial \Delta}(\Delta(\tau)) = -\Delta(\tau) + \mathrm{J}^2 C(\tau)$$

Let $\mathrm{J}^2 U(\Delta)$ be the second term in equation (14). It suffices to show $dU(\Delta(\tau))/d\Delta = C(\tau)$.

For any given t and τ, $h_i(t)$ and $h_i(t+\tau)$ are two Gaussian variables mean 0, variance $\Delta(0)$ and correlation $\Delta(\tau)$.

If $X, Y, Z \sim \mathcal{N}(0,1)$ independent, and

$$A = \sqrt{\Delta(0) - |\Delta(\tau)|}X + \sqrt{|\Delta(\tau)|}Z$$

$$B = \sqrt{\Delta(0) - |\Delta(\tau)|}Y + \mathrm{sgn}(\Delta(\tau))\sqrt{|\Delta(\tau)|}Z$$

Then (A, B) have the same joint distribution as $(h_i(t), h_i(t+\tau))$.

Check

$$\mathbb{E}[A] = \mathbb{E}[B] = 0$$

$$\mathbb{E}[A^2] = \mathbb{E}[B^2] = \Delta(0)$$

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Calculate

$$
\begin{aligned}
C(\tau) &= \mathbb{E}[N^{-1}\sum_{i=1}^{N}\phi(h_i(t))\phi(h_i(t+\tau))] = \mathbb{E}[\phi(A)\phi(B)] \\
&= \iiint \phi\left(\sqrt{\Delta(0)-|\Delta(\tau)|}x + \sqrt{|\Delta(\tau)|}z\right) \\
&\quad \phi\left(\sqrt{\Delta(0)-|\Delta(\tau)|}y + \mathrm{sgn}(\Delta(\tau))\sqrt{|\Delta(\tau)|}z\right)\mathcal{D}x\mathcal{D}y\mathcal{D}z \\
&= \int\left(\int\phi\left(\sqrt{\Delta(0)-|\Delta(\tau)|}x + \sqrt{|\Delta(\tau)|}z\right)\mathcal{D}x\right) \\
&\quad \left(\int\phi\left(\sqrt{\Delta(0)-|\Delta(\tau)|}y + \mathrm{sgn}(\Delta(\tau))\sqrt{|\Delta(\tau)|}z\right)\mathcal{D}y\right)\mathcal{D}z
\end{aligned}
$$

If $\Delta(\tau) < 0$, by change of variable $y' = -y$ and ϕ an odd function, we have the same as $\Delta(\tau) > 0$. Then

$$
C(\tau) = \int\left(\int\phi\left(\sqrt{\Delta(0)-|\Delta(\tau)|}x + \sqrt{|\Delta(\tau)|}z\right)\mathcal{D}x\right)^2\mathcal{D}z \tag{16}
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$$

On the other hand, consider $\Delta > 0$,

$$
\begin{aligned}
\frac{dU}{d\Delta} &= \frac{d}{d\Delta}\int\left(\int \Phi\left(\sqrt{\Delta(0)-|\Delta|}x+\sqrt{|\Delta|}z\right)\mathcal{D}x\right)^2\mathcal{D}z \\
&= \int\left(2\int \Phi(a)\mathcal{D}x\right)\left(\int \phi(a)\frac{\partial a}{\partial \Delta}\mathcal{D}x\right)\mathcal{D}z \\
&= \frac{1}{\sqrt{|\Delta|}}\int\left(\int \Phi\left(a\right)\mathcal{D}x\right)\left(\int \phi(a)\mathcal{D}x\right)z\mathcal{D}z \\
&\quad - \frac{1}{\sqrt{\Delta(0)-|\Delta|}}\int\left(\int \Phi(a)\mathcal{D}x\right)\left(\int \phi(a)x\mathcal{D}x\right)\mathcal{D}z
\end{aligned}
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where $a = \sqrt{\Delta(0)-|\Delta|}x+\sqrt{|\Delta|}z$, so $\dfrac{\partial a}{\partial \Delta} = \dfrac{z}{2\sqrt{|\Delta|}} - \dfrac{x}{2\sqrt{\Delta(0)-|\Delta|}}$.

Use integration by part, we have

$$
\int_{-\infty}^{\infty} f(x)x\mathcal{D}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{d}{dx}\left(-e^{-x^2/2}\right)dx = \int_{-\infty}^{\infty}\frac{df(x)}{dx}\mathcal{D}x \quad (17)
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&\quad + \frac{1}{\sqrt{|\Delta|}} \int \left(\int \Phi\left(a\right) \mathcal{D}x \right) \left(\int \phi'(a) \frac{\partial a}{\partial z} \mathcal{D}x \right) \mathcal{D}z \\
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&= \int \left(\int \phi\left(a\right) \mathcal{D}x \right) \left(\int \phi\left(a\right) \mathcal{D}x \right) \mathcal{D}z + \int \left(\int \Phi\left(a\right) \mathcal{D}x \right) \left(\int \phi'(a) \mathcal{D}x \right) \\
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By left and right continuity $\dfrac{dU(0)}{d\Delta} = \int \left(\int \phi\left(\sqrt{\Delta(0)}x \right) \mathcal{D}x \right)^2 \mathcal{D}z = 0.$

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By change of variable $\hat{h}_i(t) = h_i(t)/\mathrm{J}$, equation (2) can be rewritten as

$$\dot{\hat{h}}_i = -\hat{h}_i + \sum_{j=1}^{N} J_{ij}^{*} \phi(\mathrm{J}\hat{h}_j)$$

where $J_{ij}^{*} = J_{ij}/\mathrm{J} \sim \mathcal{N}(0, 1/N)$.

WLOG, we can reformulate the equation (14) as

$$V(\Delta) = -\frac{1}{2}\Delta^2 + \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \Phi\left(\sqrt{\Delta(0) - |\Delta|}x + \sqrt{|\Delta|}z \right) \mathcal{D}x \right\}^2 \mathcal{D}z \tag{18}$$

where

$$\phi(x) = \tanh(g\mathrm{J}x)$$

and

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Solve $\Delta(t)$

- $\ddot{\Delta} = -\partial V/\partial \Delta$ can be viewed as one-dimensional motion of $\Delta(t)$ under the potential $V(\Delta)$.
- Given that $\Delta(t)$ is an autocorrelation function, we have boundary conditions:
  1) $\Delta(t)$ is even $\Delta(t) = \Delta(-t)$, and $\dot{\Delta}(0) = 0$, implying that the orbit must have zero initial kinetic energy,
  2) $|\Delta(t)| \leq \Delta(0)$.
- Potential $V(\Delta)$ depends parametrically on $\Delta(0)$ and $g\mathrm{J}$.

The main conclusion is that there are two cases, **zero fixed point** and **chaotic phase**. We are showing only the outline.

1. When $g\mathrm{J} < 1$, there is only one solution that satisfies the boundary conditions, which is the zero fixed point, *i.e.* $\Delta(t) = 0, \forall t$.
2. When $g\mathrm{J} > 1$, there are multiple solutions depending on parameter $\Delta(0)$. By analyzing the Schrodinger equation for $\Delta(t)$, there is only one bounded state, implying only one stable solution. The system has positive Lyapunove exponent implying chaos.

Solve $\Delta(t)$

- $\ddot{\Delta} = -\partial V/\partial\Delta$ can be viewed as one-dimensional motion of $\Delta(t)$ under the potential $V(\Delta)$.
- Given that $\Delta(t)$ is an autocorrelation function, we have boundary conditions:
  1) $\Delta(t)$ is even $\Delta(t) = \Delta(-t)$, and $\dot{\Delta}(0) = 0$, implying that the orbit must have zero initial kinetic energy,
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- Potential $V(\Delta)$ depends parametrically on $\Delta(0)$ and $g\text{J}$.

The main conclusion is that there are two cases, **zero fixed point** and **chaotic phase**. We are showing only the outline.

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Compute $V(\Delta)$ numerically

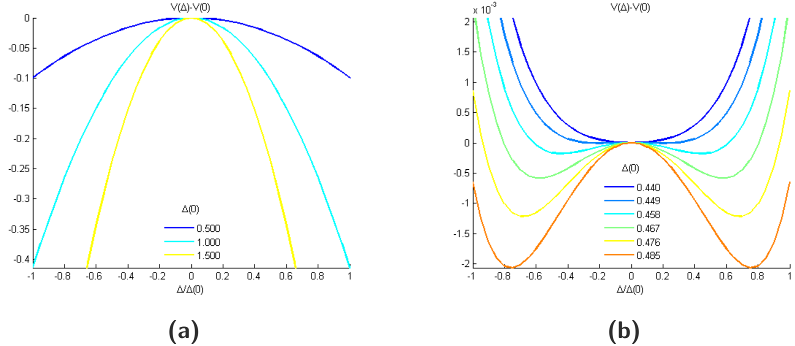
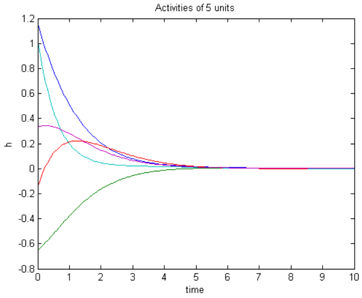


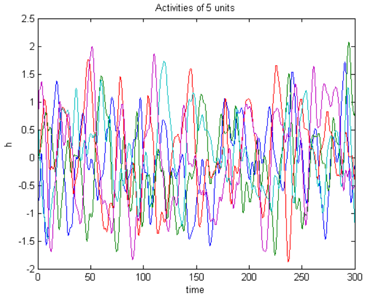
Figure (a). Computed potential function $V(\Delta)$ with $g\mathrm{J} = 0.5$, and $\Delta(0) = 0.5, 1, 1.5$, respectively.

Figure (b). Computed potential function $V(\Delta)$ with $g\mathrm{J} = 2$, and different $\Delta(0)$. For each $g\mathrm{J} > 1$, there is a critical Δ_1, such that when $0 < \Delta(0) < \Delta_1$, the potential $V(\Delta) \geq 0$ has the single-well form, and when $\Delta(0) > \Delta_1$, the potential has the double-well form.

Simulation ($N = 1000$)



(a)

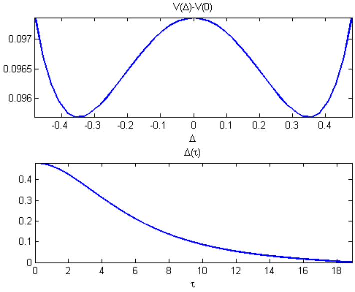


(b)

Figure (a). $g\mathrm{J} = 0.5$. There is stable zero fixed point solution. $\Delta(\tau) = 0$ is constant in this case.

Figure (b). $g\mathrm{J} = 2$. Chaotic phase.

$\Delta(\tau)$ for chaotic phase



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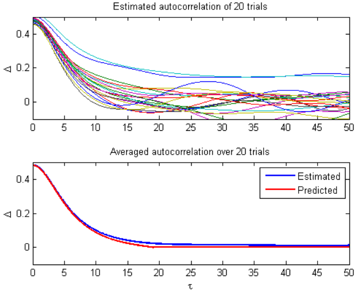

(b)

Figure (a). Stable solution is found by letting $V(\Delta(0)) = V(0)$. Then solve $\Delta(\tau)$ with such $\Delta(0)$.

Figure (b). The theoretically predicted $\Delta(\tau)$ matches the estimated autocorrelation from simulation.

Rank-one network model

- The connectivity matrix J consists of a random part and a **low-rank structured part**:

$$J = g\chi + P \tag{19}$$

where χ is a Gaussian random matrix and P is a low rank matrix.

- Consider the simplest case where P has rank one

$$P = \frac{mn^T}{N} \tag{20}$$

m and n can be interpreted as input- and output- connectivity weights.

- The network dynamic follows the same equation (2), with $\phi(x) = \tanh(x)$. Here we reduced gJ to a single parameter g.

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Assumption (4)

χ_{ij}'s are i.i.d. and $\chi_{ij} \sim \mathcal{N}(0, 1/N)$.

Assumption (5)

The structured connectivity P is independent of the random part χ. (m_i, n_i)'s are i.i.d. and $(m_i, n_i) \sim \mathcal{N}(M, \Sigma)$ where

$$M = \begin{pmatrix} M_m \\ M_n \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_m^2 & \rho\Sigma_m\Sigma_n \\ \rho\Sigma_m\Sigma_n & \Sigma_n^2 \end{pmatrix}$$

P is weak in large N limit in the sense of $P_{ij} = m_i n_j / N$.

Remark

Our analysis of the network dynamic will consider deterministic m, n, i.e. a realization of random m, n, but randomness is in χ.

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Calculate the first two moments of $\hat{\eta}_i$ over realizations of χ.

$$\begin{aligned}\mathbb{E}[\hat{\eta}_i(t)] &= \sum_{j=1}^{N} \mathbb{E}[(g\chi_{ij} + P_{ij})S_j(t)] \\ &= g\sum_{j=1}^{N} \mathbb{E}[\chi_{ij}]\mathbb{E}[S_j(t)] + \frac{m_i}{N}\sum_{j=1}^{N} n_j \mathbb{E}[S_j(t)] \\ &= \kappa m_i \end{aligned} \tag{21}$$

Here we define

$$\kappa(t) := \frac{1}{N}\sum_{i=1}^{N} n_i \mathbb{E}[S_i(t)] = \langle n_i \mathbb{E}[S_i(t)] \rangle \tag{22}$$

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\begin{aligned}
&\mathbb{E}[\hat{\eta}_i(t)\hat{\eta}_j(t+\tau)] \\
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&= \sum_{k=1}^{N}\sum_{l=1}^{N}\mathbb{E}\left[\left(g^2\chi_{ik}\chi_{jl}+\chi_{ik}P_{jl}+\chi_{jl}P_{ik}+P_{ik}P_{jl}\right)S_k(t)S_l(t+\tau)\right] \\
&= \sum_{k=1}^{N}\sum_{l=1}^{N}\left(g^2\mathbb{E}[\chi_{ik}\chi_{jl}]+\frac{m_i m_j}{N^2}n_k n_l\right)\mathbb{E}[S_k(t)S_l(t+\tau)] \\
&= \delta_{ij}\frac{g^2}{N}\sum_{k=1}^{N}\mathbb{E}[S_k(t)S_k(t+\tau)]+\frac{m_i m_j}{N^2}\sum_{k=1}^{N}n_k^2\mathbb{E}[S_k(t)S_k(t+\tau)] \\
&\quad+\frac{m_i m_j}{N}\sum_{k=1}^{N}n_k\mathbb{E}[S_k(t)]\left(\frac{1}{N}\sum_{l=1,l\neq k}^{N}n_l\mathbb{E}[S_l(t+\tau)]\right) \\
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With equation (21), we also have

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By assumption 3, we can find Gaussian processes η_i's that have the same first and second moments as $\hat{\eta}_i$'s.

Next we focus on finding the **stationary solution**. Since $\int_{-\infty}^{t} e^{t'-t} dt' = 1$, according equation (10), (11), we have the mean and variance of h_i coincide with η_i. So we can define

$$\begin{aligned} \mu_i &:= \mathbb{E}[h_i(t)] = \mathbb{E}[\eta_i(t)] = m_i \kappa \\ \Delta_0^I &:= \mathbb{E}[h_i(t)^2] - \mathbb{E}[h_i(t)]^2 = \mathbb{E}[\eta_i(t)^2] - \mathbb{E}[\eta_i(t)]^2 = g^2 \langle \mathbb{E}[S_k^2] \rangle \end{aligned} \quad (25)$$

Note: h_i's of different units are uncorrelated. μ_i and Δ_0^I are independent of time for stationary solution and Δ_0^I is the same for all units.

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Define population mean and variance by averaging equation (25)

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Want to derive expression for κ and $\langle \mathbb{E}[S_k^2] \rangle$. Start with

$$\mathbb{E}[S_i] = \mathbb{E}[\phi(h_i(t))] = \int \phi\left(\mu_i + \sqrt{\Delta_0^I} z\right) Dz \quad (27)$$

Define a density function of a bivariate distribution based on the given m, n

$$p_{mn}(m', n') := \frac{1}{N} \sum_{i=1}^{N} \delta(m' - m_i) \delta(n' - n_i)$$

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Approximate distribution p_{mn} by the Gaussian distribution $\mathcal{N}(M, \Sigma)$ in assumption 5. The mean and variance of p_{mn} equal the sample mean and variance of (m_i, n_i)'s. By law of large number, they approach M and Σ in large N limit.

Then

$$\mu \approx M_m \kappa$$
$$\Delta_0 \approx \Delta_0^l + \Sigma_m^2 \kappa^2$$

Define random variables

$$m' := M_m + \Sigma_m \sqrt{1-\rho} x_1 + \Sigma_m \sqrt{\rho} y$$
$$n' := M_n + \Sigma_n \sqrt{1-\rho} x_2 + \Sigma_n \sqrt{\rho} y \tag{30}$$

where $x_1, x_2, y \sim \mathcal{N}(0,1)$ are independent. Then $(m', n') \sim \mathcal{N}(M, \Sigma)$ and let $p^*_{mn}(m', n')$ be the density function.

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$$\begin{aligned} m' &:= M_m + \Sigma_m \sqrt{1-\rho} x_1 + \Sigma_m \sqrt{\rho} y \\ n' &:= M_n + \Sigma_n \sqrt{1-\rho} x_2 + \Sigma_n \sqrt{\rho} y \end{aligned} \tag{30}$$

where $x_1, x_2, y \sim \mathcal{N}(0,1)$ are independent. Then $(m', n') \sim \mathcal{N}(M, \Sigma)$ and let $p^*_{mn}(m', n')$ be the density function.

Approximate distribution p_{mn} by the Gaussian distribution $\mathcal{N}(M, \Sigma)$ in assumption 5. The mean and variance of p_{mn} equal the sample mean and variance of (m_i, n_i)'s. By law of large number, they approach M and Σ in large N limit.
Then

$$\begin{aligned} \mu &\approx M_m \kappa \\ \Delta_0 &\approx \Delta_0^I + \Sigma_m^2 \kappa^2 \end{aligned}$$

Define random variables

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where $x_1, x_2, y \sim \mathcal{N}(0, 1)$ are independent. Then $(m', n') \sim \mathcal{N}(M, \Sigma)$ and let $p_{mn}^*(m', n')$ be the density function.

Now substitute p_{mn} by p^*_{mn} in equation (28)

$$\kappa \approx \iint n' \left(\int \phi \left(m'\kappa + \sqrt{\Delta_0^I} z \right) Dz \right) p^*_{mn}(m', n') dm' dn'$$
$$= \iiint \left(\int \phi \left(\left(M_m + \Sigma_m \sqrt{1-\rho} x_1 + \Sigma_m \sqrt{\rho} y \right) \kappa + \sqrt{\Delta_0^I} z \right) Dz \right)$$
$$\left(M_n + \Sigma_n \sqrt{1-\rho} x_2 + \Sigma_n \sqrt{\rho} y \right) Dx_1 Dx_2 Dy \qquad (31)$$

It gives rise to three terms when expanding n'. The first term

$$M_n \int Dx_2 \iiint \phi \left(\left(M_m + \Sigma_m \sqrt{1-\rho} x_1 + \Sigma_m \sqrt{\rho} y \right) \kappa + \sqrt{\Delta_0^I} z \right) Dz Dx_1 Dy$$
$$= M_n \int \phi \left(M_m \kappa + \sqrt{\Sigma_m^2 \kappa^2 + \Delta_0^I} z \right) Dz$$
$$= M_n \int \phi \left(\mu + \sqrt{\Delta_0} z \right) Dz$$
$$= M_n \langle \mathbb{E}[S_i] \rangle$$

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$$= M_n \int \phi \left(\mu + \sqrt{\Delta_0} z \right) Dz$$

$$= M_n \langle \mathbb{E}[S_i] \rangle$$

Fact: $\left(M_m + \Sigma_m\sqrt{1-\rho}x_1 + \Sigma_m\sqrt{\rho}y\right)\kappa + \sqrt{\Delta_0^I}z$ is the linear combination of Gaussian variables, so is also Gaussian, with mean $M_n\kappa$ and variance $\Sigma_m^2\kappa^2 + \Delta_0^I$.

Also

$$
\begin{aligned}
\langle \mathbb{E}[S_i] \rangle &= \frac{1}{N}\sum_{i=1}^{N}\int \phi\left(\mu_i\kappa + \sqrt{\Delta_0^I}z\right)Dz \\
&= \iint \left(\int \phi\left(m'\kappa + \sqrt{\Delta_0^I}z\right)Dz\right)p_{mn}(m', n')dm'dn' \\
&\approx \iiint \phi\left(\left(M_m + \Sigma_m\sqrt{1-\rho}x_1 + \Sigma_m\sqrt{\rho}y\right)\kappa + \sqrt{\Delta_0^I}z\right)DzDx_1Dy \\
&= \int \phi\left(\mu + \sqrt{\Delta_0}z\right)Dz
\end{aligned}
$$

The second term

$$
\int \Sigma_n\sqrt{1-\rho}x_2Dx_2
$$

$$
\iiint \phi\left(\left(M_m + \Sigma_m\sqrt{1-\rho}x_1 + \Sigma_m\sqrt{\rho}y\right)\kappa + \sqrt{\Delta_0^I}z\right)DzDx_1Dy
$$

$$
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$$

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$$= 0$$

The third term

$$
\begin{aligned}
&\int Dx_2 \iiint \Sigma_n \sqrt{\rho} y \phi \left(\left(M_m + \Sigma_m \sqrt{1-\rho} x_1 + \Sigma_m \sqrt{\rho} y \right) \kappa + \sqrt{\Delta_0^I} z \right) DzDx \\
&= \Sigma_n \sqrt{\rho} \int y \phi \left(\left(M_m + \Sigma_m \sqrt{1-\rho} x_1 + \Sigma_m \sqrt{\rho} y \right) \kappa + \sqrt{\Delta_0^I} z \right) DzDx_1 Dy \\
&= \kappa \rho \Sigma_m \Sigma_n \int \phi' \left(\left(M_m + \Sigma_m \sqrt{1-\rho} x_1 + \Sigma_m \sqrt{\rho} y \right) \kappa + \sqrt{\Delta_0^I} z \right) DzDx_1 Dy \\
&= \kappa \rho \Sigma_m \Sigma_n \langle \mathbb{E}[S_i'] \rangle
\end{aligned}
$$

Here we used equation (17) and denote $S_i'(t) := \phi'(h_i(t))$.

Similarly, from equation (29) we have

$$
\langle \mathbb{E}[S_i^2] \rangle = \int \phi \left(\mu + \sqrt{\Delta_0} z \right)^2 Dz
$$

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$$
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&= \Sigma_n \sqrt{\rho} \int y \phi \left(\left(M_m + \Sigma_m \sqrt{1-\rho} x_1 + \Sigma_m \sqrt{\rho} y \right) \kappa + \sqrt{\Delta_0^l} z \right) DzDx_1 Dy \\
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We end up with a system of three equations.

$$\begin{aligned}
\mu &= M_m \kappa \\
\Delta_0 &= g^2 \langle \mathbb{E}[S_i^2] \rangle + \Sigma_m^2 \kappa^2 \\
\kappa &= M_n \langle \mathbb{E}[S_i] \rangle + \kappa \rho \Sigma_m \Sigma_n \langle \mathbb{E}[S_i'] \rangle
\end{aligned} \tag{32}$$

To solve the equations, we can define the following dynamic

$$\begin{aligned}
\mu &= M_m \kappa \\
\dot{\Delta}_0 &= -\Delta_0 + g^2 \langle \mathbb{E}[S_i^2] \rangle + \Sigma_m^2 \kappa^2 \\
\dot{\kappa} &= -\kappa + M_n \langle \mathbb{E}[S_i] \rangle + \kappa \rho \Sigma_m \Sigma_n \langle \mathbb{E}[S_i'] \rangle
\end{aligned} \tag{33}$$

Simulating (33) numerically until fixed point is reached. To find unstable fixed point of (33), revert the sign of time variable for the simulation.

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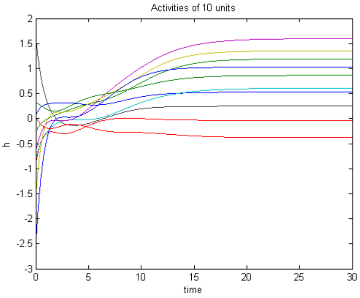
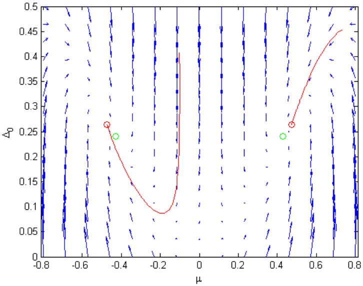
Simulating (33) numerically until fixed point is reached. To find unstable fixed point of (33), revert the sign of time variable for the simulation.

Simulation

Simulate rank-one network with $N = 1000$, $g = 0.9$, $M_m = 1$, $M_n = 1.2$, $\Sigma_m = 0.4$, $\Sigma_n = 0.8$, $\rho = 0.25$.



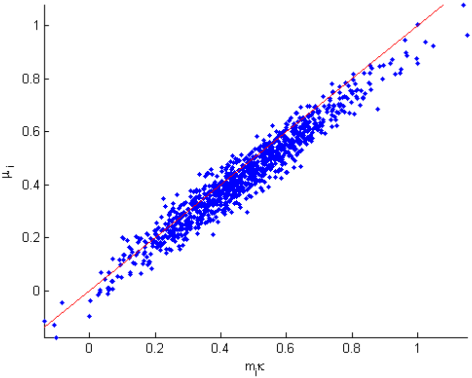
(a)



(b)

Figure (a). Stable fixed point solution of the whole network is reached.

Figure (b). The solution of equation (32) was found by simulating equation (33). Green circles are mean and variance estimated from 100 trials of simulations.



The estimated mean activities $\bar{h}_i$'s of 1000 units averaged from 100 trials and the theoretically predicted $\mu_i = m_i\kappa$. The cosine similarity between them is 0.968.