Rationale and resources for teaching the mathematical modeling of athletic training and performance

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Clarke DC, Skiba PF. Rationale and resources for teaching the mathematical modeling of athletic training and performance. *Adv Physiol Educ* 37: 134–152, 2013; doi:10.1152/advan.00078.2011.—A number of professions rely on exercise prescription to improve health or athletic performance, including coaching, fitness/personal training, rehabilitation, and exercise physiology. It is therefore advisable that the professionals involved learn the various tools available for designing effective training programs. Mathematical modeling of athletic training and performance, which we henceforth call "performance modeling," is one such tool. Two models, the critical power (CP) model and the Banister impulse-response (IR) model, offer complementary information. The CP model describes the relationship between work rates and the durations for which an individual can sustain them during constant-work-rate or intermittent exercise. The IR model describes the dynamics by which an individual's performance capacity changes over time as a function of training. Both models elegantly abstract the underlying physiology, and both can accurately fit performance data, such that educating exercise practitioners in the science of performance modeling offers both pedagogical and practical benefits. In addition, performance modeling offers an avenue for introducing mathematical modeling skills to exercise physiology researchers. A principal limitation to the adoption of performance modeling is a lack of education. The goal of this report is therefore to encourage educators of exercise physiology practitioners and researchers to incorporate the science of performance modeling in their curricula and to serve as a resource to support this effort. The resources include a comprehensive review of the concepts associated with the development and use of the models, software to enable hands-on computer exercises, and strategies for teaching the models to different audiences.

coaching education; physical fitness; critical power model; Banister impulse-response model; exercise physiology

EXERCISE PRESCRIPTION is central to a number of professions, including those involving health and fitness training, occupational health and safety, rehabilitation from disease or injury, and sport performance. The goal of exercise prescription is to restore or improve an individual's functional capacity to a level commensurate with healthy living or with his or her fitness or athletic performance goals. In an effort to better understand and optimize athletic performance, exercise physiologists have developed mathematical models of athletic training and performance. Two models in particular, the critical power (CP) model and the Banister impulse-response (IR) model, which we henceforth denote as "performance models," are useful for training planning, analysis, and optimization. These models have been studied for decades, but they are starting to enjoy popular use owing to the commercialization of portable exercise monitoring devices such as power meters for bicycles and global positioning system wristwatches as well as easy-to-use computer software. Nowadays, the principal limiting factor in the widespread use of these models is a lack of education.

In this report, we present performance modeling as a means for addressing in part the curricular needs of exercise physiology and its applied professions and to provide a resource for its implementation. We begin by justifying the inclusion of these models in the curricula of both applied exercise physiology professionals (e.g., clinical exercise physiologists, personal trainers, coaches, etc.) and research-focused exercise physiologists. We then provide the following educational resources: *1*) lecture material in the form of comprehensive modular reviews of the models, *2*) suggested teaching strategies and example conceptual questions, and *3*) an annotated supplemental spreadsheet file featuring the computations for virtually all of the figures presented in the text, which we intend to serve as the basis for computer-based exercises.[1] We have also extensively referenced our report to facilitate supplementary reading by those interested in learning more about the models.

Rationale for Teaching Performance Modeling to Applied Exercise Physiology Practitioners

Professional health and fitness organizations, such as the American College of Sports Medicine, and the sports science community advocate evidence-based exercise prescription, in which exercise programs are based on the current best available evidence, including peer-reviewed scientific studies and professional reasoning (e.g., Refs. 2, 38, 59, 68, 82, and 93). While evidence-based exercise prescription is a noble goal, achieving it in practice is challenging. Collegiate strength and conditioning coaches, for example, rely relatively little on the scientific literature when devising their training programs (29). Resistance to evidence-based practice stems partly from many coaches and fitness professionals lacking the education to critically evaluate the scientific literature (37). Figuring importantly, however, are the impediments that exist to translating laboratory-based research, which is the principal source of scientific evidence, into real-world practice (12, 68). Such impediments include a dearth of longitudinal studies to guide long-term training program designs (12) and difficulties with comparing the efficacies of different interventions because many training studies feature only a single experimental group (68). In the study of athletes, interventions are often added on top of their "normal training," which often goes unreported

[1] Supplemental Material for this article is available online at the *Advances in Physiological Education* website.

and/or is poorly controlled during the study (68), as are each subject's training status at the beginning of the study and their states of rest or freshness, nutrition, and hydration during the performance tests (12, 68). Issues can also exist with the subjects. Training studies often feature too few subjects to be adequately statistically powered (68). Moreover, the subjects that are included in the study may not be representative of the population with which the practitioner works, in that studies examining untrained subjects may not generalize to well-trained athletes and vice versa.

Collectively, these deficiencies cause even the basic elements of training program design, such as volume, intensity, and periodization, to remain controversial in the literature (62, 68). Therefore, the scientific literature inadequately addresses critical aspects of training planning such that knowledge must be derived from other means. Such means include anecdotes and experiences of other coaches or trainers, the internet, textbooks, past experience, and trial and error (29). None of these forms of knowledge pass through the rigors of peer review such that their veracity is less assured. Even if published studies were more easily translated, their conclusions would still be based on averaged data, such that an omnipresent obstacle to evidence-based exercise prescription is the interindividual variability with which people respond to exercise (101). The individuality of the responses to exercise means that a training program deemed optimal in a published study may not be optimal for all individuals. Ultimately, therefore, exercise prescription is a single-subject experiment whose optimization requires trial and error.

We propose that performance models can address, in part, the issues cited above and can help enable bonafide evidence-based exercise prescription. Performance models elegantly integrate the principles of training into coherent frameworks that can be used as a basis for critically evaluating ideas about training from scientific studies or from other sources of knowledge. This feature would help address the lack of formal education of many exercise practitioners. They can also serve as conceptual guides for the basic elements of training program design. The CP and IR models are especially useful for guiding workout design and long-term training planning, respectively. Both models can serve to optimize training programs for individuals because their inputs are data collected from the individual and their predictions are specific to the individual. Given these benefits, we propose a renewed definition of evidence-based exercise prescription in which a general program is devised based on concepts and guidelines from peer-reviewed published studies and is optimized based on data systematically collected from the athlete or client and modeled using performance models. Performance models therefore serve as an avenue to guide professional reasoning and to systematize the trial-and-error adjustments normally featured in exercise prescription.

Rationale for Teaching Performance Modeling to Exercise Physiology Researchers

Research-focused students of exercise physiology, rehabilitation science, and biomedical engineering stand to benefit from learning performance modeling for several reasons. First, performance models can serve as an avenue to introduce students to the concepts and practice of mathematical modeling. Mathematical modeling has a long tradition in exercise physiology, and the CP model and related bioenergetic models have been used for decades to explain world records and performance as a function of time (13, 74). Mathematical models continue to inform many of the subdisciplines of exercise physiology, including the biochemistry of muscle metabolism (e.g., Refs. 27 and 60), cardiovascular regulation (e.g., Refs. 41 and 58), O$_2$ transport (e.g., Ref. 105), endocrine function (e.g., Refs. 55 and 109), and temperature regulation (e.g., Ref. 102). Mathematical models are helpful because they are an efficient and effective means to express and evaluate hypotheses about complex biological systems (61). We expect the use of mathematical models in exercise physiology to increase in the coming years because of the emergence of systems biology (4, 46, 56), the increasingly sophisticated experimental techniques that require modeling to exploit (28), and the efforts to model human physiology (49). For these same reasons, the life sciences are reconfiguring their curricula to emphasize training in the quantitative and physical sciences (81). Given these developments, we believe it is important that exercise physiology researchers be sufficiently conversant in mathematical modeling to appreciate, understand, and substantively contribute to contemporary biomedical science. Modeling depends on a core set of skills that includes the abstraction of systems into mathematical equations, parameter fitting to data (optimization), statistical evaluation of the model, simulations, and sensitivity analyses. The performance modeling literature features each of these tools (e.g., Refs. 33 and 43) such that performance models can serve as a general introduction to modeling.

Performance models also serve as tools in basic and applied research studies. For example, the IR model has been used to investigate tapering and peaking phenomena in athletes (7, 80, 86, 98–100), training transfer effects between sports in triathletes (69), and psychological effects of training (70). Performance models can also be used to address some of the shortcomings of training studies noted above. For example, quantifying a subject's training before and during the study and using this data to fit an IR model could be helpful for estimating the training and fatigue status of the subject throughout the study. In cases in which the training loads were not strictly controlled during the study, the IR model could be used to account for the variance between subjects resulting from the different training loads that might otherwise distort the effect of the experimental treatment.

Finally, we believe that performance models could facilitate closer ties between exercise physiology researchers and practitioners. On the one hand, using the models encourages the systematic collection of data by practitioners. Ideally, scientifically minded practitioners may be motivated to share their data with scientists, which could motivate formal studies. In this way, researchers would have an important source of problem finding and hypothesis generation. On the other hand, the models can be implemented in the field such that they facilitate the translation of scientific findings based on their use into everyday practice. Either scenario could foster improved evidence-based exercise prescription.

Overview of Resources

Given the justification above, we encourage educators of applied and research-focused exercise physiologists to incorporate performance modeling into their curricula. To facilitate curricular implementation, we provide the following pedagogical resources. First, we review the concepts underlying each model in a modular lecture-friendly format. As part of this review, we have cited numerous references, which we hope will inspire supplemental reading. In particular, we direct the reader to the following reviews: Refs. 5, 15, 23, 53, 75, and 97. Third, we suggest pedagogical strategies for helping students to achieve a working understanding of the models through the use of active learning modalities such as Conceptests, computer-based exercises, and laboratory sessions. Finally, we provide a supplemental spreadsheet file containing the calculations that underlie virtually all of the figures presented in the text. We intend for this file to serve as a starting point for the computer-based exercises.

Reading and Lecture Materials: Reviews of the CP and IR models

For each model, we present the following subsections: *1*) definition and history, *2*) equation derivation and assumptions, *3*) physiological basis, *4*) practical implementation, *5*) conceptual benefits and practical applications, *6*) limitations, and *7*) modifications to the model.

The CP Model

Definition and history. The CP model describes the capacity of an individual to sustain particular work rates as a function of time (t). In this way, the model summarizes the relationship between exercise intensity and duration for an individual. The historical context of the CP model has been reviewed in detail elsewhere (13, 53, 74). Briefly, a hyperbolic relationship between work rate and time was first suggested by Hill in 1925 (45), who plotted velocity versus time for world records in swimming and running over various distances. Monod and Scherrer observed a similar hyperbolic relationship in their studies of work rate versus duration in skeletal muscle, and they codified this relationship mathematically in 1965 (71). They also defined the term "critical power" (CP) as the power that can be sustained without fatigue for a very long time. Another parameter in the relationship, the "anaerobic work capacity" (AWC), nowadays called W', represents the finite amount of energy that is available for work above the critical power. In the early 1980s, Moritani et al. (72) and Whipp et al. (107) extended this concept to whole-body exercise by having human subjects exercise to exhaustion at different work rates on a cycle ergometer. Whereas Moritani et al. used the formalism of Monod and Scherrer, Whipp et al. fit a linearized two-parameter CP model to their data (107). Since those initial studies, the CP model has been applied in a variety of settings and to diverse types of subjects to evaluate muscular performance (53). In particular, the model has been applied to several sports in addition to cycling, including running (48), swimming (106), and rowing (57).

Equation derivation and assumptions. Monod and Scherrer devised the CP model by combining the equation for power (power = work/time) with the observed linear relationship between the amount of work and the duration of tests to exhaustion performed at different work rates (71). The model features two parameters, CP and work rate, which are related according to the following equation:

$$W' = (P - CP)t$$

where P is power and t is the duration for which that power was sustained (107). Note that for sports such as swimming or running, P and CP can be expressed as speed (S) and critical speed (CS), respectively, and W' can be expressed as distance (D') rather than energy. Figure 1 shows the various forms of the CP model.

Morton (74) catalogued the explicit and implicit assumptions of the CP model. The four principal assumptions are as follows: *1*) power output is a function of two energy sources, aerobic and anaerobic; *2*) aerobic energy is unlimited in capacity (i.e., one could exercise at an intensity at or below CP for infinite duration) but is limited in the rate at which it can be converted into work; *3*) anaerobic energy is unlimited in the rate of conversion (i.e., maximal power output or speed is infinite) but is limited in capacity; and *4*) exhaustion occurs when W' is depleted (74). Each of these assumptions is physiologically imprecise, but the model is nevertheless useful for modeling the power-duration relationship for maximal exercise lasting from $\sim$2 to $\sim$30 min. We discuss these and other assumptions further below in *Limitations*.

Physiological basis. W' and CP are empirical parameters in the CP model, but they both have bonafide physiological interpretations. CP is the maximal work rate that can theoretically be performed for infinite duration and corresponds to the maximal aerobic power sustainable without drawing upon W'. During exercise at power above CP, there is a clear and progressive loss of metabolic homeostasis. At the systemic level, maximal O_2 consumption ($\dot{V}O_{2\ max}$) and blood lactate concentration attain steady values in response to exercise at or below CP, whereas exercising above CP leads to the eventual attainment of $\dot{V}O_{2\ max}$ and to inexorable blood lactate accumulation (53). At the muscle level, Jones et al. (54) observed steady levels of phosphocreatine (PCr), inorganic phosphate (P_i), and pH through 20 min of leg extension exercise at a work rate $\sim$10% below CP (Fig. 2). In contrast, a work rate 10% above CP resulted in continually decreasing PCr and pH and increasing P_i until exhaustion was reached at $\sim$14.7 min (Fig. 2) (54). Interestingly, Vanhatalo et al. (104) demonstrated that different exercise intensities above CP resulted in identical PCr levels at exhaustion.

Thus, CP appears to be a true physiological "threshold" phenomenon that reflects metabolic disturbance in the working muscle mass. It corresponds to a power output that exists between those corresponding to the gas exchange threshold/lactate threshold and $\dot{V}O_{2\ max}$ (53). Of these three parameters, CP is most useful for predicting performance in endurance events, such as time trial performance in cycling (95). It is higher than the power corresponding to the maximal lactate steady state (MLSS) but is also highly correlated to MLSS (85), which is a predictor of performance for exercise lasting 30–60 min (14). However, CP is more accessibly estimated than MLSS because its measurement does not require invasive measurements.

The physiological basis of W' is less clear. Attempts to specifically characterize the biochemical nature of W' have not
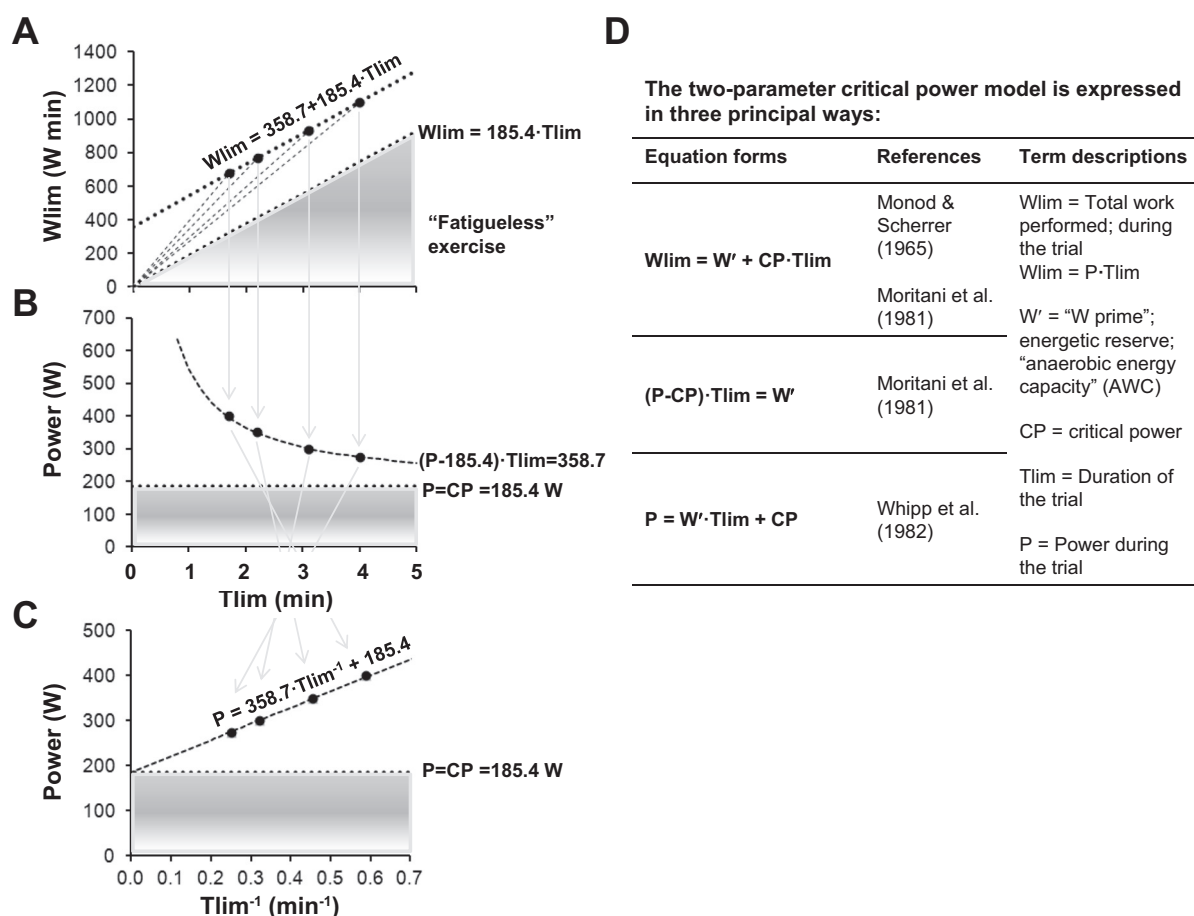
Fig. 1. Definitions and descriptions of the three principal forms of the two-parameter critical power model. The data are those of a representative subject ("M.P.") from the Moritani et al. (72) study. The units of energy are expressed as Watts·minute in keeping with the convention used by Moritani et al. (72), but W' is usually expressed in units of joules. The gray-shaded regions on each plot indicate workloads less than critical power (CP), which implies that they would not cause exhaustion ("fatigueless exercise"). A: the linear relationship between the total mechanical work done (W_{lim}) by synergistic muscle groups during constant power trials and the durations of those trials (T_{lim}). The slopes of the dashed lines between the origin and the data points are equal to the mean power of the trials. B: the hyperbolic form of the CP model, which is derived from the first equation by substituting power (P) $\times$ T_{lim} for W_{lim}. C: the linearized form of the CP model, which is derived from the hyperbolic form by solving for P.

been wholly satisfying, and it may not be possible to ascribe W' to a single physiological variable (53). Indeed, the traditional interpretation of W' as a fixed anaerobic work capacity seems dated in light of work that demonstrated decreased W' during exposure to hyperoxic gas and an inverse relationship between CP and W' (104). Other recent work correlated the discharge of W' with the progressive rise of $\dot{V}O_2$ during interval exercise (94) and the recovery of W' with the "slow" portion of the recovery of $\dot{V}O_2$ (32). In reality, W' likely represents an average or synthesis of changes in several important factors (e.g., $\dot{V}O_2$, [K^+], [P_i], [PCr], and ATP flux). Irrespective of the physiology involved, W' is useful because it represents a robust, performance-related parameter. It can be immediately discharged and is replenished with a half-time of ~3.5 min during passive (e.g., unloaded cycling) recovery (32, 94).

Practical implementation. CP and W' are traditionally estimated by having the athlete perform a series of maximal constant-power trials of varied duration and fitting these data using regression techniques (Fig. 3A and supplemental spreadsheet file). Several practical issues arise with this approach, including the choice of durations and the amount of rest between tests. With regard to the latter, if the tests are per-

formed on the same day, then sufficient recovery is needed to fully restore W', which implies a lengthy session because W' is recharged on the timescale of minutes (32, 94). These issues can be resolved by performing the tests on different days. However, doing so introduces the potential confound of training effects, and it can be cumbersome to perform the tests over multiple days. Finally, regardless of the timing of the tests, they should be performed in random order to promote statistical independence between the data points and to eliminate possible confounds introduced by the order of the tests.

To address the shortcomings of the multiple test approach, a 3-min maximal effort test has been developed to estimate CP and W' (Fig. 3B) (103). In this test, the subject exercises maximally from the start and maintains the effort throughout the test; there is no pacing. This is a stringent requirement because of the prolonged discomfort involved, such that the subject must be highly motivated and should not receive feedback during the test. The power output reaches a maximum within a few seconds and then progressively declines as W' depletes (Fig. 3B). By 2–3 min, W' depletes completely, and power output stabilizes near CP. Therefore, CP is estimated directly as the end-test power, which is calculated as the
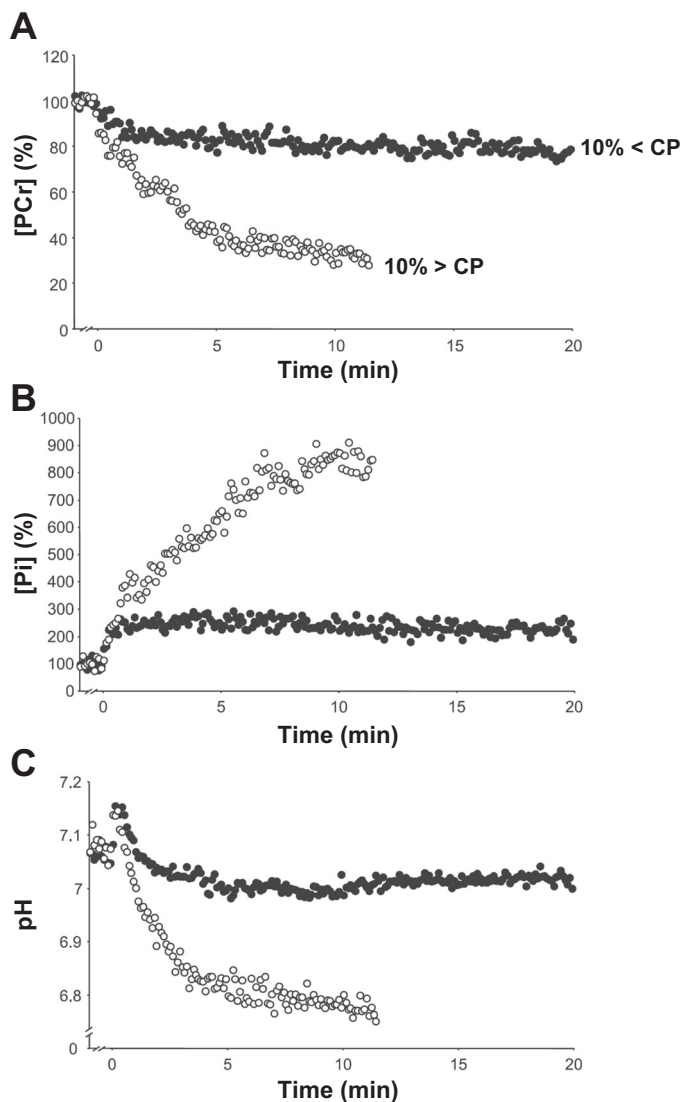
A

B

C

Fig. 2. The physiology of CP. Workloads slightly above CP lead to a loss of metabolic homeostasis, whereas workloads slightly below CP do not. Phosphocreatine (PCr; *A*), P_i (*B*), and pH (*C*) concentrations in quadriceps muscle were estimated using ^{31}P magnetic resonance spectroscopy during dynamic exercise above and below CP. Note the shorter duration of the >10% CP trial in which exhaustion was achieved. [Data reprinted from Jones et al. (54) with permission.]

average power in the final 30 s of the test, and W' is estimated by integrating the area bounded by the power profile and a horizontal line at end-test power (Fig. 3*B*). The validity of the 3-min all-out test has been supported by the high correlations of CP and W' estimates from the 3-min test with those independently estimated using the traditional protocol (103). Due to the appeal of estimating CP model parameters in a single test, the 3-min all-out test has attracted considerable interest and has recently been adapted for running (84) and rowing (25).

Conceptual benefits and practical applications. The CP model provides a physiologically sound language to express several of the qualitative sensations and observations of coaches and athletes. First, athletes often speak of "blowing up" or "dying" when exhaustion was reached. This sensation can be more accurately stated as the depletion of W'. Second,

an observation that can be explained by the CP model is the variable abilities of athletes to excel at shorter duration events or to "go all day," with the former likely exhibiting high W' relative to their CP and the latter the opposite. Finally, athletes and coaches often refer to a nebulous "threshold" to describe the dividing line between intensities that can be sustained for a long time versus those that cannot. Physiologically, this dividing line is associated with CP or MLSS. However, the term "threshold" is imprecise and is often confused with the lactate threshold or with the anaerobic threshold. The lactate threshold is defined as the intensity of exercise eliciting a 1-mM increase in blood lactate above resting levels and is less than the intensity corresponding to MLSS or the onset of blood lactate accumulation. The anaerobic threshold is a misnomer because all energy systems, whether reliant on O_2 or not, contribute to the supply of energy for exercise regardless of intensity. While the MLSS terminology is accurate, thinking in terms of MLSS encourages the erroneous notion that fatigue is caused by lactic acid when, in fact, lactate is merely a byproduct of the biochemical mechanisms responsible for energy supply during exercise. In contrast, CP is a bonafide physiological threshold, and deple-
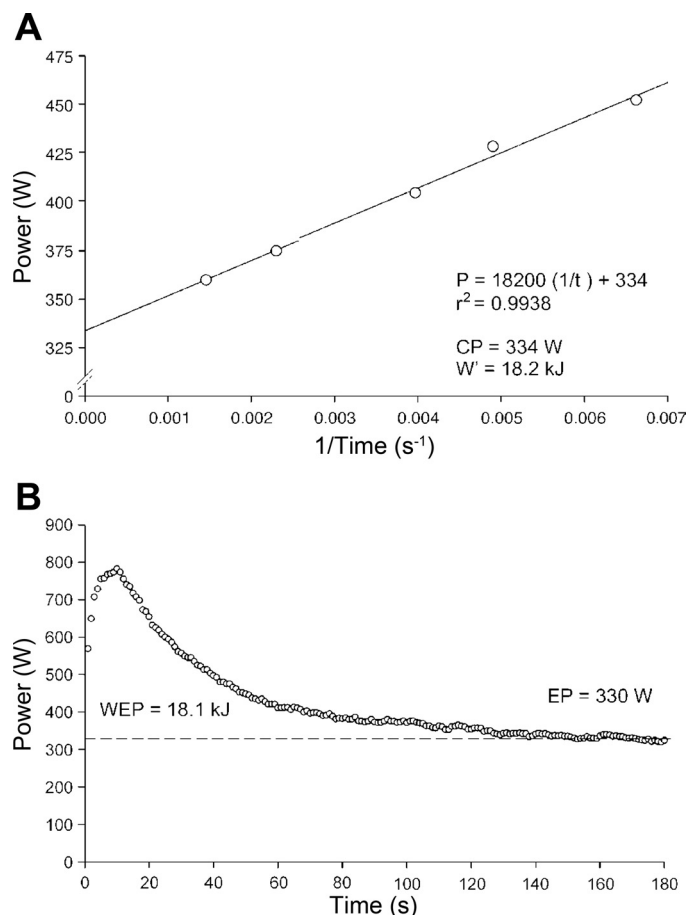
A

$$P = 18200\,(1/t) + 334$$
$$r^2 = 0.9938$$

CP = 334 W
W' = 18.2 kJ

B

WEP = 18.1 kJ   EP = 330 W

Fig. 3. Fitting the CP model. *A*: linear regression of power on duration is most commonly done using the linearized form of the CP model, in which the line of best fit is found through the method of least squares. *B*: CP parameters can be validly estimated using a 3-min all-out test. The average power over the final 30 s of the test (the "end-test" power) closely correlates with the CP estimated using the standard protocol. The area bounded by the power-time curve and the horizontal line defined by the end-test power is equal to W'. EP, end-test power; WEP, work done above EP. [Data reprinted from Vanhatalo et al. (103) with permission.]

tion of W' corresponds to exhaustion and does not invoke lactate as a causal mechanism in fatigue. Therefore, CP should be the preferred terminology.

The CP model serves as a tool for devising optimal pacing and tactical strategies in athletic competition. With regard to pacing, theoretically optimal strategies have been proposed using the CP model (53) that could inform sports such as swimming or kayaking. Running road race tactics could also be informed by the CP model. One could estimate the CS and D' values of his or her competitors from recent results and use these numbers to suggest the best tactical approach for any particular athlete. For instance, a 10K runner with a superior CS would be well advised to take the lead early, forcing his or her competitors to expend their limited D' in pursuit. Likewise, another athlete with a high D' but relatively limited CS would be advised to get to the front and attempt to dictate a slower pace, preserving his or her superior D' for a finishing sprint. The CP model could also be used to adjust tactics during competition. In cycling, for example, the decision to break away is often made in a split second based on the race scenario and on the amount of energy the athlete subjectively feels he or she has remaining (in cycling parlance, "the number of matches left to burn"), which is quantifiable as W'. Recently,

Skiba et al. (94) devised a model for the real-time monitoring of W' during dynamic exercise (Fig. 4A). An exciting possibility is to implement this model in the software of portable monitoring devices such that the athlete could be continually apprised of his or her W'.

The CP model provides a basis for prescribing individualized workout intensities during training (53). Workout intensities are commonly subdivided into discrete zones corresponding to different physiological events or states (93). For the subject whose data are shown in Fig. 3, we calculated his or her power ranges corresponding to the different intensity zones (Fig. 4B). Furthermore, a coach constructing a severe-intensity interval workout could use the CP model for intermittent exercise (which is described below in *Modifications to the CP model*) to determine the interval durations and work and rest intensities that would result in depleted W' at the end of the session, thus optimizing the quality of the workout.

Limitations. As stated above, the CP model relies on four principal assumptions that contravene known physiology. Here, we address the inaccuracies of each assumption in the same order that they were presented above:

1. Three energy-producing pathways contribute to power output, namely, high-energy phosphate compounds, glycol-
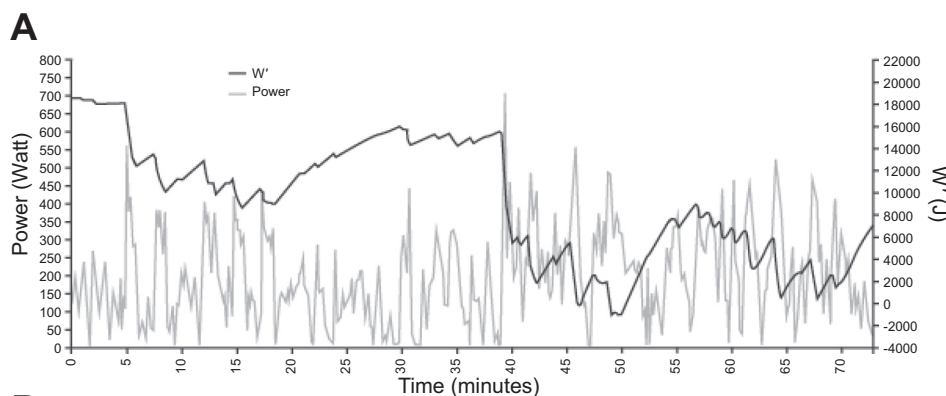
Discrete training intensity zones

Zone	Description	General ranges			Example numbers CP = 330 W HR @ CP = 170 bpm	
		% Critical Power	Heart rate (% HR @ CP)	RPE	Power (W)	Heart rate (bpm)
1	Recovery	<56	<69	<2	<185	<117
2	Endurance	56-75	69-83	2-3	185-248	118-141
3	Tempo	76-90	84-94	3-4	249-297	142-160
4	Critical power	91-105	95-105	4-5	298-347	161-179
5	$\dot{V}O_{2max}$	106-120	>106%	6-7	347-396	>179
6	Anaerobic capacity	>120	N/A	>7	>396	N/A

Fig. 4. Practical uses of the CP model. *A*: a model of W' kinetics enables continuous monitoring of energy reserves during dynamic exercise of variable intensity. Shown is the power (monitored using a power meter) and modeled W' for a cyclist participating in a brisk group ride. The subject noted the sensation of impending exhaustion as the W' balance remaining approached 0 J. In such a fashion, the model can be used to optimize pacing, race tactics, or interval workouts. *B*: discrete training intensity zones defined as the percentage of CP or pace (93). These zones facilitate precise communication between the athlete and coach with respect to workout expectations. The example numbers on the *right* were calculated from the subject's CP from Fig. 3. The heart rate [HR; in beats/min (bpm)] at CP was arbitrarily assumed. $\dot{V}O_{2max}$, maximal O_2 consumption; RPE, rating of perceived exertion; N/A, not applicable.

ysis, and oxidative phosphorylation of multiple possible substrates (74).

2. Power continues to decline below the asymptote defined by CP given enough time. The applicability of the CP model extends to exercise lasting from ~2 to 30 min in most people but up to 60 min in some individuals (47).

3. The maximum power that can be generated using W' is finite because limits exist to how fast or powerfully one can sprint (74).

4. W' need not be completely depleted at exhaustion (74). In constant-power trials, the subject ceases exercise when he or she cannot maintain the required power output. However, W' may not be depleted because if the stipulated power output was reduced to a level still above CP but less than the original power, exercise could continue. Therefore, the maximal power output is a function of the remaining W'.

The net result of these assumptions is that the two-parameter CP model tends to overestimate CP and underestimate W'.

Modifications to the CP model. To address the limitations stemming from the assumptions of the two-parameter CP model, Morton (73) created a three-parameter CP model. The three-parameter model addresses the assumptions that maximal power output is infinite and that exhaustion occurs when W' is depleted. Morton's modification was to relax the requirement of the two-parameter model that an asymptote exist at $t = 0$, which caused P to unrealistically approach infinity as t approaches zero (Fig. 5) (73). His modification is expressed mathematically as follows:

$$t = \frac{W'}{(P - CP)} + k, \, (k < 0)$$

where k is the asymptote and assumes a negative value. Because the maximal power possible (P_{max}) can only occur for instantaneous time (i.e., time to exhaustion = 0), it implies the following:

$$t = \frac{W'}{(P - CP)} + \frac{W'}{(CP - P_{max})}$$

Morton further assumed that the maximal achievable power output during a bout of exercise depends on the amount of the

Fig. 5. The three-parameter CP model. The three-parameter CP model [$CP_3(t)$] features a nonzero time asymptote. Compared with the two-parameter model [$CP_2(t)$], the three-parameter model results in lower CP estimates (compare CP_2 and CP_3) and higher W' values for the same data (*Subject A* from Ref. 73). The third parameter, P_{max}, of the three-parameter model represents the maximal instantaneous power, whereas for the two-parameter model power approaches infinity as time approaches zero.

remaining W'. Through additional reasoning and mathematics, he recovered the above equation except that the interpretation of P_{max} changed to be the "maximal instantaneous power" and was shown to be a linear function of the remaining W' (73). Therefore, with this form of the CP model, the assumption that W' is depleted at exhaustion is changed to the more realistic assumption that exhaustion occurs when P_{max} is less than the desired power output.

Morton and Billat (78) extended the two-parameter CP model to intermittent exercise, which is useful for optimizing interval workout prescription. The model can be stated mathematically as follows:

$$t = n(t_w + t_r) + \frac{W' - n\left[(P_w - CP)t_w - (CP - P_r)t_r\right]}{P_w - CP}$$

where t is total endurance time, n is the number of intervals, t_w and t_r are the durations of the work and recovery phases in each interval, respectively, and P_w and P_r are the power outputs during the work and rest phases, respectively (78). Note that proper behavior of the model requires the following constraints (78):

$$0 \le P_r < CP < P_w < CP + W'/t$$

The Banister IR Model

Definition and history. The Banister IR model quantitatively relates performance ability at a specific time to the cumulative effects of prior training loads (97). It succinctly describes an individual's exercise dose-response relationship and handles the complicating factors of nonlinear time dependence and individuality in a single framework. Banister et al. recognized the difficulty in translating the results of training studies into practice. In their original paper (6), they stated that "quantitative data relating performance to different programs of training has been obtained by several investigators but it is still difficult to predict the results of a particular training program." To address this need, they conceived the IR model for training planning and optimization. The original paper modeled the training and performance of a competitive swimmer (6). Since then, the IR model has been applied to diverse sports such as running (79, 108), swimming (43, 44, 80), cycling (16, 17, 19, 20), triathlons (7, 69), weightlifting (21, 22), and the hammer throw (18). Although its use to date has been mostly confined to laboratory studies, the model is attracting renewed interest due to the emergence of commercially available devices for real-time monitoring of exercise and software for implementing the model.

Equation derivation and assumptions. In examining a hypothesized performance time course that followed from training, Calvert et al. (24) proposed that the performance kinetics behaved like a first-order system. A system whose behavior varies over time is typically modeled using ordinary differential equations (ODEs). Calvert et al. (24) thus proposed an ODE that could recreate the qualitative form of the hypothesized performance time course. They then solved this equation using standard mathematical techniques (APPENDIX A). Calvert et al. noticed that their proposed equation did not adequately fit performance data from a competitive swimmer whose training and performance they had monitored over several months.
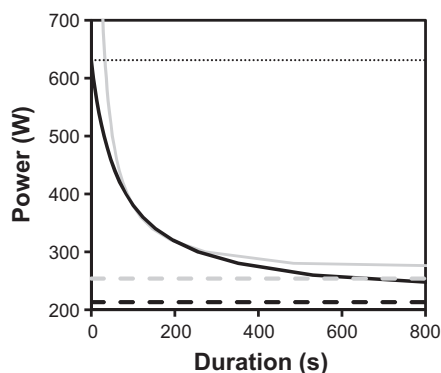
Specifically, they noticed that the swimmer's performance capacity decreased when his training load was increased (24). They therefore modified their original model to be a two-component system in which training was posited to cause both positive and negative effects, respectively, attributed to "fitness" and "fatigue." The equations for each of these two components were of the same form as the equation they had first proposed. Performance was calculated as the difference between the positive training effects (PTEs), ascribed to fitness, and the negative training effects (NTEs), ascribed to fatigue (APPENDIX A; Fig. 6, *A* and *B*). Further assumptions were specified to describe how performance changed with time. In response to a given training load, the NTE initially outweighs the PTE such that the subsequent performance capacity decreases. However, the NTE dissipates faster in time than the PTE, such that the PTE eventually outweighs the NTE and performance capacity increases (Fig. 6C). Based on these simple assumptions, the IR model can capture much of the variance in performance data collected over time ($R^2 > 0.90$ in some cases) (16, 79, 108).

Physiological basis. The IR model provides a window into the dynamics of adaptation to physical training. The PTE and NTE profiles qualitatively correlate with measurable physiological parameters related to fitness and fatigue, respectively. For example, the kinetics of iron status biomarkers in female runners generally follow that of the NTE (9), as do markers of muscle cell damage (elevated serum enzyme activities such as creatine kinase, lactate dehydrogenase, and aspartate aminotransferase) (8, 10, 11). Correlations were also found between serum hormone levels and both PTE and NTE (21, 22). Caveats with the cited studies

are that they generally included few subjects, there was considerable variability in the biochemical data, and in several of the studies, there were no statistical assessments of the correlations. More robust quantitative correlations were reported by Wood et al. (108), who observed a strong correlation between the PTE and ventilatory threshold and a moderate correlation between the NTE and scores from the Profile of Mood States questionnaire, which is sensitive to changes in perceived fatigue. Overall, these studies provide evidence demonstrating that the model parameters reflect, to some degree, the underlying physiology of training adaptations. However, as with W' in the CP model, the model parameters likely do not represent a single physiological variable. Instead, they represent the aggregate effects of multiple variables that contribute to the dynamic response of performance to training.

The physiological validity of the model is further supported by its ability to capture several well-established qualitative features of performance kinetics as a function of training. These features include the initial stagnation or decrease in performance capacity when training is increased (overreaching), the plateau effect if training load is maintained, the supercompensation effect observed with reduced training load (taper) after a period of overload, and detraining effects when training is ceased or markedly reduced (Fig. 6C). By featuring athlete-specific input data and parameter values, the model is customized to the individual and therefore reflects the principle of individuality. Hence, the IR model concisely integrates core training principles such as overload, overreaching, supercompensation, reversibility, and individuality.

A

Initial performance level

PTE (fitness)   NTE (fatigue)

$$p(t) = p(0) + k_1 \sum_{s=0}^{t-1} e^{-(t-s)/\tau_1} w(s) - k_2 \sum_{s=0}^{t-1} e^{-(t-s)/\tau_2} w(s)$$

Sum of training impulses, each modified by an exponential decay term

B

$$g(t) = g(t-s)e^{-s/\tau_1} + w(t)$$

$$h(t) = h(t-s)e^{-s/\tau_2} + w(t)$$

$$p(t) = p(0) + \underbrace{k_1 g(t)}_{PTE} - \underbrace{k_2 h(t)}_{NTE}$$

C

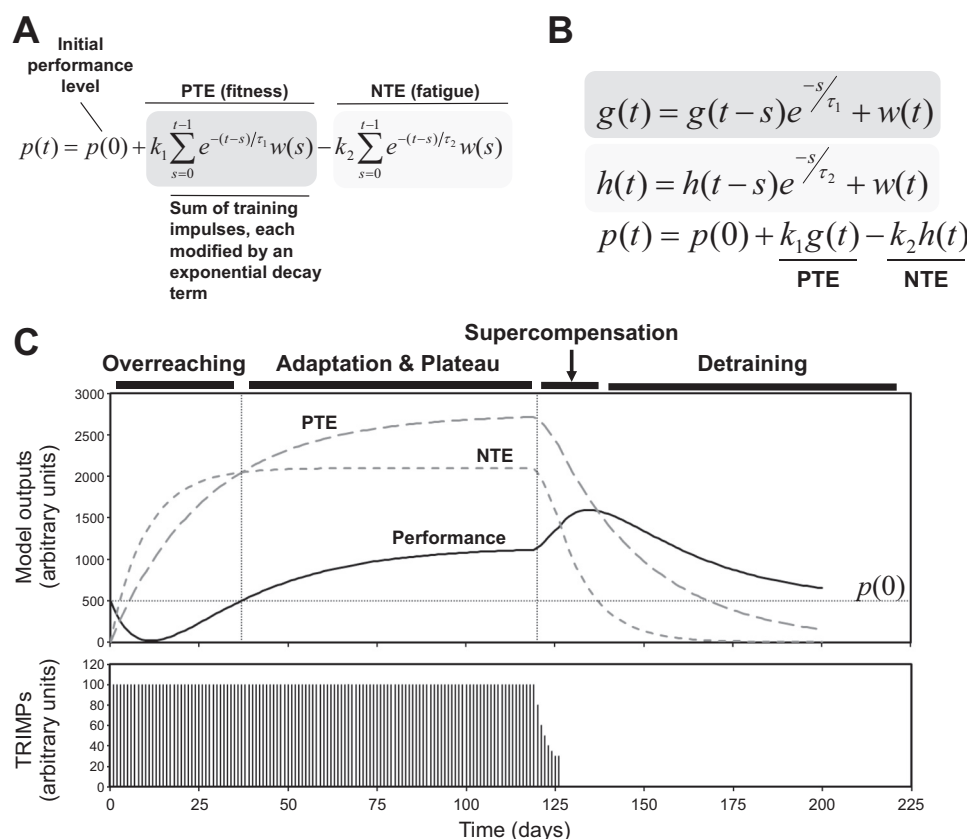Overreaching   Adaptation & Plateau   Supercompensation   Detraining

Fig. 6. Definition and description of the impulse-response (IR) model. The IR model predicts performance based on the simple premise that it is the sum of base-level performance and positive training effects (PTEs) minus negative training effects (NTEs). *A*: summation equation form of the IR model. *B*: recursion equation form of the IR model. This form is most useful for spreadsheet-based calculations. *C*: the IR model recapitulates the known qualitative features of the training response. In the *bottom* graph, simulated daily training impulses were plotted as a function of time. The athlete performed workouts of 100 training impulses (TRIMPs) per day for 120 days. The following 7 days featured a taper in which daily TRIMPs were progressively reduced to 30. Training was ceased thereafter. PTE, NTE, and performance were calculated from the simulated TRIMPs and used the following parameter values: p(0) = 500, $k_1 = 1$, $k_2 = 2$, $\tau_1 = 27$, and $\tau_2 = 10$. Arbitrary units (AU) were used for p(0), k_1, and k_2, whereas τ_1 and τ_2 were expressed in units of days.

Practical implementation. QUANTIFICATION OF DAILY TRAINING LOADS. The model takes as input daily training loads. The training load of a bout of exercise can be expressed in the following general form:

$$\text{training load} = \text{intensity} \times \text{duration}$$

Quantifying duration is simple, but quantifying intensity is difficult because work rate and the resulting metabolic stress, which chiefly determines the adaptive stimulus, are nonlinearly related. This nonlinearity is illustrated by the exponential increase of blood lactate as a function of work rate (26, 31). As such, it is a challenge to quantify and compare workouts of differing volumes and intensities in terms of their abilities to induce physiological adaptations. A number of metrics exist for estimating training load, including session rating of perceived exertion, ordinal categorization, Lucia's training impulse, summated heart rate (HR) zone score, and excess postexercise $\dot{V}O_2$ (15, 51, 97). The best-known system of training quantification, however, is Eric Banister's training impulse (TRIMP). Predicated upon HR (in beats/min) reserve as a measure of intensity, TRIMP accounts for the observation that higher workloads are more metabolically taxing (exponentially so) than workloads performed for the same duration at lower intensity (5), as follows:

$$\text{TRIMP} = t \times k \times \text{FHRR}$$

$$\text{FHRR} = \frac{\text{HR}_{\text{average}} - \text{HR}_{\text{rest}}}{\text{HR}_{\text{max}} - \text{HR}_{\text{rest}}}$$

where t is the duration of the exercise bout (in min), FHRR is the fraction of the HR reserve, and $k = 0.64e^{1.92 \times \text{FHRR}}$ or $0.86e^{1.67 \times \text{FHRR}}$ for men and women, respectively.

The reliance of TRIMP on HR can be problematic (as discussed below in *Limitations*). To address in part these shortcomings and to exploit the data from power meters, Coggan devised the training stress score (TSS) for cycling (1). A key feature of the TSS metric is the "normalized power" (NP) metric, which represents a transformed "average power" of the workout that accounts for the variability of the workout's intensity arising from changes in power output due to hills, wind, drafting, etc. In addition, physiological responses are curvilinearly related to intensity, such that large power outputs induce disproportionately higher physiological stress than lower power outputs. Therefore, average power does not adequately represent the stress incurred by a ride that features forays into higher power outputs interspersed with periods of low power output. NP is determined by calculating 30-s moving average of the raw power data from a workout, followed by raising those averages to the fourth power (thus emphasizing higher power outputs), averaging those values, and then taking the fourth root of that average (1, 51). The 30-s moving average approximates the time constant of physiological responses (such as O_2 kinetics) to changes in intensity (1).
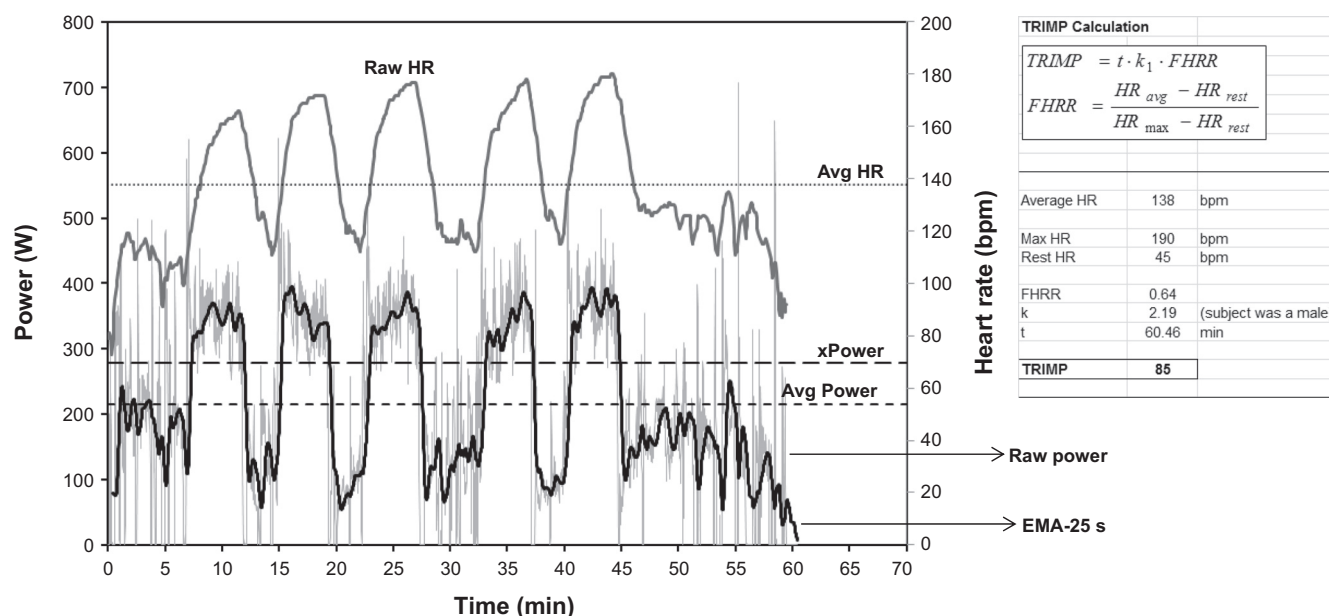
The TSS approach has been extended to runners (66), and Skiba has modified the TSS framework to create power-based metrics for swimming (SwimScore), cycling (BikeScore), and running (Gravity-Ordered Velocity Stress Score) (88–90, 92, 93). While these metrics are gaining popularity among athletes, they have yet to be rigorously validated (51), with Skiba having reported a preliminary evaluation of the TSS metric (91). Finally, a modified TRIMP based on CP has been recently

proposed by Hayes and Quinn (42), but it too remains to be validated. We describe in detail the computations for quantifying a cycling workout using both TRIMP and BikeScore in Fig. 7.

FITTING THE IR MODEL. The Banister IR model features five adjustable parameters, including the initial performance capacity, two time constants that describe the decay of the PTE and NTE, and two gain parameters that relate how the daily load determines the amplitudes of the PTE and NTE (APPENDIX A; Fig. 6). To estimate the model parameters, the model is calibrated to performance data. Performance is typically measured by performing maximal effort time trials over a prescribed distance or duration. In cycling, for example, the average power output for a 5-min maximal effort could serve as the performance estimate. In running, one might perform a 1-mi. time trial on a track and use the velocity as the performance metric.

There are several guidelines for ensuring the collection of high-quality performance data. First, the tests must be done with maximal effort and ideally with even pacing. Second, the test conditions should be kept as consistent as possible (e.g., same course, time of day, etc.) to minimize variance contributed by external factors. In ideal circumstances, the performance test would be the same distance or duration as the goal event. In practice, however, the time trials are typically kept short in duration (e.g., 3–15 min) to preserve the athlete's motivation. The shorter duration might reduce the predictive ability of the model with respect to performance in the goal event. However, the impact of this limitation is mitigated by the fact that aerobic energy production dominates the supply of energy for performances of ~75 s or longer in duration (39), such that the capacities for performances of short and longer durations correlate. Third, the tests should be done as frequently as possible. We advocate performing the tests at least weekly. Finally, the tests should be done in all stages of the training cycle, even if the athlete is tired from heavy training and not expected to perform well. The model is fit to the data using nonlinear regression in which the parameters are iteratively changed until the error between model and data is minimized (Fig. 8*A*). We provide a step-by-step procedure for fitting the IR model using the Solver function in Excel in the supplemental spreadsheet file.

For the optimal estimation of the model parameters, it is advantageous to execute a training program that leads to each parameter in the model being emphasized or, in modeling parlance, "identifiable" during the model fitting process. Such a program was performed by Morton et al. (79), in which the two subjects ran once a day for 7 days followed by running twice a day for 21 days and then ceasing the training runs for 50 days. Overall, the training program represented a step increase in training to a load that was sufficiently severe to induce negative effects on performance, thus emphasizing the fatigue gain term. When training was ceased, the fatigue and fitness time constants were emphasized, especially the latter as time wore on and fatigue was fully dissipated. Throughout the training period, at least two time trials were performed each week to collect performance data. The experimental protocol led to good model fits ($R^2 = 0.74$ and 0.90) (79). While appealing from a scientific perspective, athletes training for competition might be reluctant to adopt such a protocol. A more practical alternative would be to monitor the athlete's

TRIMP Calculation

$$TRIMP = t \cdot k_1 \cdot FHRR$$

$$FHRR = \frac{HR_{avg} - HR_{rest}}{HR_{max} - HR_{rest}}$$

Average HR	138	bpm
Max HR	190	bpm
Rest HR	45	bpm
FHRR	0.64	
k	2.19	(subject was a male)
t	60.46	min
TRIMP	**85**	

Session & athlete parameters

CP = 315 W

1 hr @ CP normalized work = NW_{CP} = CP x 3600 s = 1134 kJ

Session duration (dur) = 60.5 min

Session average power = 215 W

BikeScore calculations

Session xPower = $\{\sum[EMA(t)]^4\}^{(1/4)}$ = 278 W

Relative intensity (RI) = xPower / CP = 0.88

Session normalized work (NW_s) = xPower × dur = 1009 kJ

Raw BikeScore (RBS) = RI × NW_s = 890 kJ

BikeScore = RBS/NW_{CP} × 100 = 78

Fig. 7. Quantifying training load from a single cycling workout using TRIMP and BikeScore. The workout involved a ~5-min warm-up and 5 × ~2.7-km intervals at ~350-W average power for each interval, followed by a cool down. The athlete's CP was ~315 W, such that the intervals were *zone 5* ($\dot{V}O_{2\,max}$). The graph shows raw data from both a power meter (Saris PowerTap) and HR data from wrist-mounted global positioning system unit (Garmin Forerunner 305). Power data were overlaid with a 25-s exponential moving average (EMA-25s), which is the first step in the calculation of BikeScore (see the supplemental spreadsheet for the EMA-25s calculation), to better reflect the kinetics of the physiological stress (93). HR increased to ~180 beats/min for most of the intervals, which was well over the HR achieved at CP for this athlete (~160 beats/min). *Right*: the calculations leading to the TRIMP estimate, which in this case was a value of 85. *Bottom*: data and calculations necessary to compute BikeScore, a power-based metric of training stress that seeks to better reflect the physiological effects of training than TRIMP by expressing workout intensity as a function of CP, to account for the variability in intensity and nonlinear effects of higher intensities and to provide a single number to represent the physiological stress (88, 93). BikeScore was computed according to the following steps: *1*) the EMA at each time point was raised to the fourth power and averaged, and the fourth root of the average value was taken to calculate the "xPower," which is a metric that reflects the disproportionately higher physiological cost of high intensities; *2*) the relative intensity (RI) was computed by taking the ratio of the xPower to CP, which expresses the intensity of the ride relative to CP; *3*) the "session-normalized work" (NW_s), which reflects a scaled energy expenditure according to the intensity of the ride, was computed by multiplying xPower (in W) by the session duration (in s); *4*) a similar computation was used to calculate the "1 h at CP-normalized work" (NW_{CP}), in which CP was multiplied by 3,600 s; *5*) the "raw BikeScore" (RBS) was computed as the product of RI and NW_{CP}, and *6*) the BikeScore was computed as the ratio of RBS and NW_{CP} multiplied by 100. Note the difference between xPower (278 W) and the session average power (215 W), which reflected the high intensity of the *zone 5* intervals.

normal in-season training, which would be expected to induce substantial overload, and then during the initial portion of the off season, when the athlete is not formally training, continue the performance tests for a few weeks to monitor the decay in performance due to loss of fitness.

QUANTIFICATION OF PERFORMANCE. An underlying issue in trying to mathematically relate training and performance is that performance is a nonlinear function of training (97). Specifically, the training necessary to achieve a specific percentage improvement in performance increases approximately exponentially as a function of the stage of the athlete's career. Novice athletes rapidly improve with training, whereas experienced athletes make small improvements. Morton et al.(79) addressed this issue in quantifying running performance by modeling the progression of world records over time and devising a criterion points scale based on the model. Another study (97) compared a criterion point scale with a scale based

on the percentage of personal best time and found little difference between the two. More work is needed to define the importance of the sensitivity of the IR model predictions to the nonlinear training-performance relationship.

Conceptual benefits and practical applications. The IR model provides several conceptual benefits. First, it elegantly summarizes the principles of training into a unified and coherent framework, as noted above in *Physiological basis*. Second, by taking estimated training stress as input, the IR model reinforces the notion that exercise is a physiological stress that disturbs homeostasis, the restoration of which involves numerous adaptations across organ systems. The immune, neural, and endocrine systems coordinate these responses (34) and these systems can be compromised if excessive stress is applied, which is manifested as nonfunctional overreaching or overtraining (67, 96). Quantifying training stresses of workouts serves as a means to check that appropriate amounts of stress
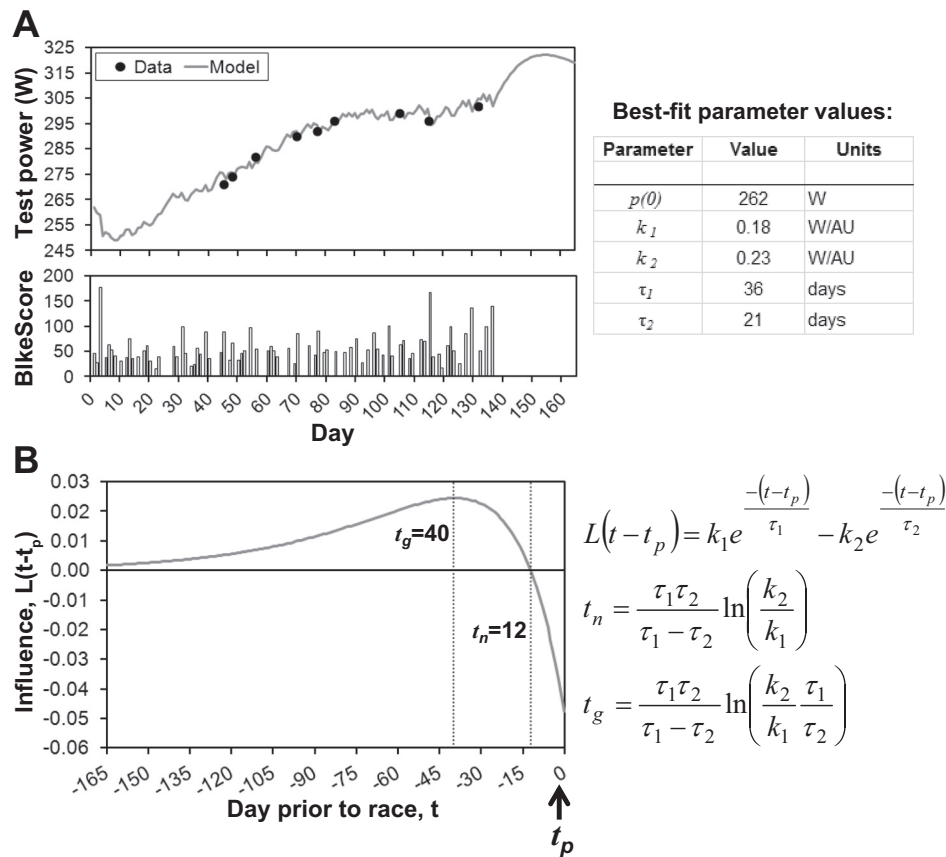
Fig. 8. Practical implementation of the IR model. *A*: training and performance data were used to fit the IR model for an individual athlete. Here, BikeScore was the metric used to estimate daily training loads (*bottom*). Performance data were determined from periodic time trials (●; *top*). The predicted performance [p(*t*)] was estimated by fitting the five IR model parameters (*right*) using nonlinear regression (Microsoft Excel Solver; see the supplemental spreadsheet for implementation). The R^2 for the model fit was 0.98. *B*: fitted model parameters were used to compute the influence curve for the athlete, which is a tool to optimize training. The curve can be interpreted in two ways in that it can be read from both left to right and from right to left. In the former case, the curve shows the net influence per unit training stress on performance on each day before the day of the goal performance (t_p). Two derived parameters, t_g and t_n (vertical dotted lines), are shown. t_g is the day before the race, during which the training performed will have the greatest positive influence on performance on t_p. t_n is the time period before t_p, in which any amount of training is predicted to have a net negative influence on performance. The literal interpretation of t_n is that training should be ceased altogether on the day corresponding to t_n. However, athletes prefer to avoid ceasing training, so from a practical standpoint t_n corresponds to the day by which the taper should be well underway. When read from right to left, the influence curve estimates the influence of a single workout over time. That is, a workout performed on *day 0* will have a negative influence up until t_n days have passed, owing to the accumulation of fatigue, but will then have a positive influence thereafter once fatigue decays.

are being applied. Furthermore, the model highlights the concept that performance adaptations are a function of aggregated training stresses, which implies that no single type of workout dominates in influence but rather consistent daily training that induces overload followed by a taper will result in peak performance.

The IR model can be used to optimize athletic performance through the tools of influence curves and simulations. Both tools require estimates of the model parameters for the individual in question. The influence curve is a plot of the effect of a unit training impulse for each day leading up to the day of a goal performance (Fig. 8*B*) (33). The curve is solely a function of the model parameters and is independent of the daily training loads. Two derived parameters, t_g and t_n, represent the day on which training will have the highest positive influence on performance and the day after which training will have a net negative influence on performance, respectively (Fig. 8*B*) (33). A few guidelines for training optimization can be gleaned from the influence curve. First, training should be concentrated during the times of high positive influence, i.e., around t_g,

while the taper should be well underway by t_n (93) Second, the influence curve implies that training done well before the day of the key event has little influence. Consequently, for typical parameter values, it implies that training blocks need only last a few weeks to ~3–4 mo, which agrees with data supporting block periodization as an optimal periodization strategy (50). Finally, influence curves provide guidance on how to train for multiple events scheduled on different days (33). A true peak performance can only be achieved on a single day, such that a compromise is required if good performances are sought in events on different days. The influence curve provides a means for optimizing this compromise.

Simulations are conducted by proposing daily training loads and inputting these into the IR model along with parameter values determined for the individual from previous model fits. The model is then solved for each hypothetical training program and performance predicted on the day of the goal event. The program that elicits the highest performance is then implemented. Simulations were used by Morton (77) to perform a theoretical study of different periodization schemes and by

other groups (7, 98–100) to study tapering schemes. We show the simulation approach in Fig. 9.

The IR model's utility extends beyond sport as it can be applied to individuals rehabilitating from disease or injury. Le Bris and coauthors (63, 65) published several studies using the models with cardiovascular rehabilitation patients to predict the loss of functional capacity once the rehabilitation program was ceased or otherwise interrupted. Furthermore, the IR model was used to predict the prolonging of fitness benefits accrued by a training regimen featuring five versus three sessions per week in phase 2 rehabilitation patients (64). Jomenez and Skiba (52) used the IR model to predict the time course of functional capacity restoration in a patient recovering from knee surgery. The IR model is therefore broadly applicable across subject populations.

Limitations. The IR model suffers from several significant practical challenges and scientific limitations as a tool for training optimization. With regard to practical challenges,

using the model requires the athlete to be highly motivated and diligent in recording all the training data and in performing frequent maximal effort performance tests. In addition, the use of power- or pace-based metrics requires the purchase of expensive specialized equipment.

The IR model has been criticized for an apparent lack of predictive ability (43, 97), which casts doubt on its usefulness for optimizing training (23). The principal criticisms stem from statistical evaluations of the model that found wide confidence intervals on the parameter estimates and the t_n and t_g parameters. Interestingly, in cases in which predictivity has actually been tested, the model performs surprisingly well. Hellard et al. (43) found that despite the variability in the parameter values, the corresponding "variability in modelled performances was quite small" and the parameter values were stable in the face of removal at random of a single data point. In our experience coaching athletes, we have found much practical utility in applying IR modeling to training planning. In partic-

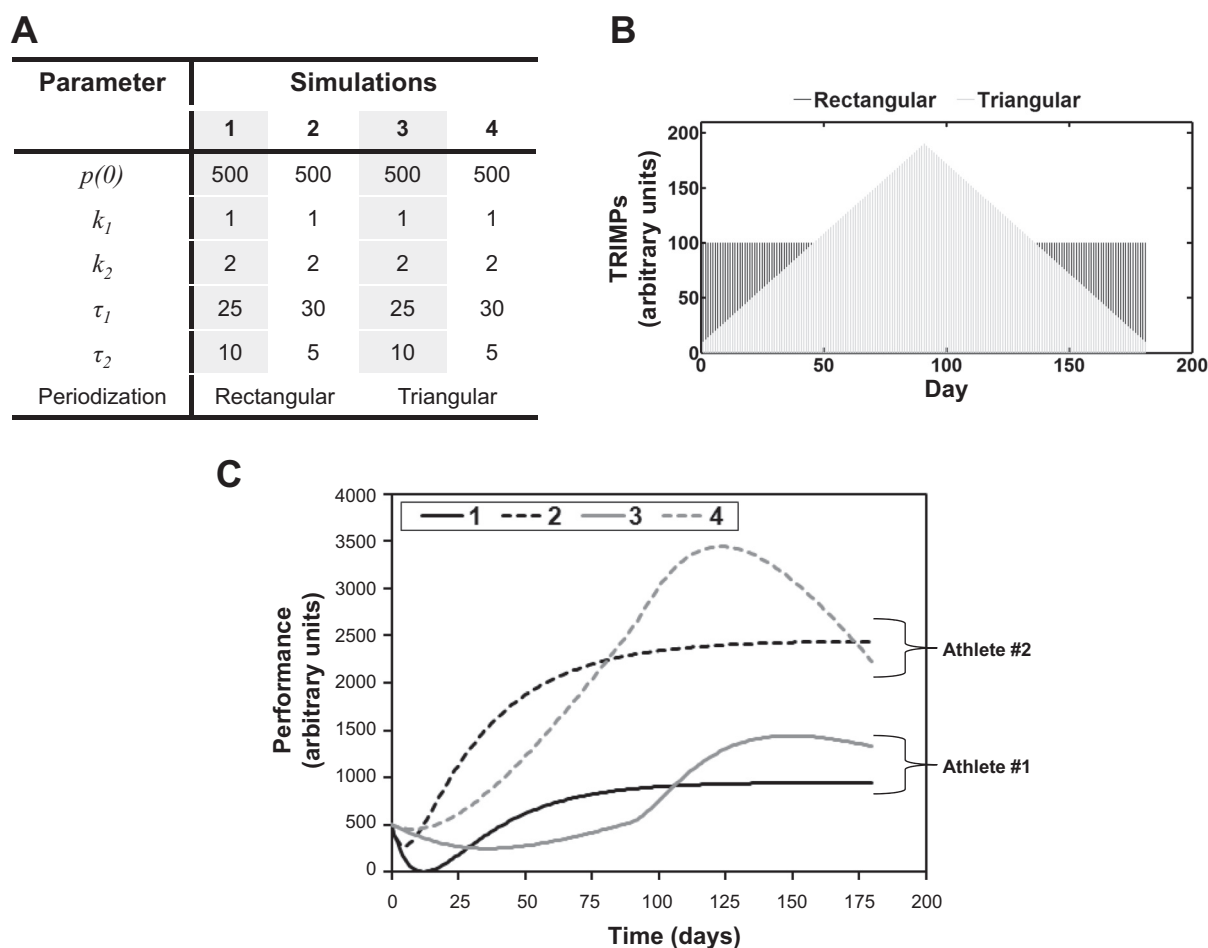| Parameter	Simulations			
	1	2	3	4
$p(0)$	500	500	500	500
k_1	1	1	1	1
k_2	2	2	2	2
τ_1	25	30	25	30
τ_2	10	5	10	5
Periodization	Rectangular		Triangular	

Fig. 9. The IR model can be simulated to predict performance based on hypothetical training programs. Simulations were performed by assigning daily training loads associated with each training program (in this case, reflecting two periodization schemes over a training period of 180 days) and using those as input to hypothetically parameterized IR models reflecting two different athletes. Four simulations were therefore performed. The day of the goal event, t_p, is defined as *day 180*. *A*: parameter values and periodization schemes for each simulation. *Simulations 1* and *3* (gray) pertain to *athlete 1* (note the same parameter values), whereas *simulations 2* and *4* pertain to *athlete 2*. *B*: illustration of the rectangular and triangular periodization schemes. The two schemes feature the same total TRIMPs accumulated over the 180 days. *C*: predicted performance curves for each simulation over the 180 days. *Athlete 1* achieved higher performance level on t_p with the triangular periodization scheme than with the rectangular periodization scheme (compare *curves 1* and *3*). Conversely, *athlete 2* achieved higher performance with the rectangular periodization scheme because the triangular periodization scheme caused the athlete to peak too early for optimal performance on *day 180* (compare *curves 2* and *4*). Therefore, the parameter values interact with the periodization schemes in determining performance on the final day of the training period. In both periodization schemes, *athlete 2* outperformed *athlete 1* owing to favorable decay constants for PTE and NTE. This simulation emphasizes the individuality of training responses, both in absolute and relative senses.

ular, one of the authors (P. F. Skiba) used the IR model to inform the training protocols of two world championship multisport performances (one of which was a world record) and several world championship podium placings as well as dozens of wins at the elite and amateur level in triathlons. We acknowledge, however, that it remains unclear how much of the model's utility was due to its ability to accurately predict the dynamics of PTE, NTE, and performance. Rather, it is possible that the model was successful because it models in a general sense the kinetics of human fatigue, adaptation, and performance. Put another way, different model parameters may have resulted in similar conclusions with respect to the absolute training prescription.

More work is therefore needed to evaluate the usefulness of the IR model in planning training. In particular, data of the highest quality must be used. We expect that power-based metrics of training quantification will be helpful in this regard, given the limitations of HR-based metrics. The principal limitations of HR are its variation with factors such as hydration, rest, illness, or cardiac drift (1) and the underestimation of stress from workloads exceeding $\dot{V}_{O_{2\,max}}$. The latter is an important deficiency given the recent interest in high-intensity

training for both health and athletic performance (40, 62). Furthermore, it is imperative that frequent (i.e., at least weekly) performance tests are carried out. Parameter fits also stand to be improved by using a training program that features sufficient variation to isolate or decouple the model parameters from each other (e.g., Ref. 79). Finally, in cases in which NTEs are not apparent, a single-component model could be used (19).

Modifications to the IR model. The IR model assumes that both fitness and fatigue respond linearly to training load, but, in reality, the body has a finite capacity to adapt to training. To reflect this physiological reality, Hellard et al. (44) proposed a modified IR model in which the daily training loads [$w(s)$] were transformed using the saturable the Hill equation (Fig. 10A). This transformation restricts the effects of high training loads on the PTE and NTE. This model outperformed the classic IR model in tracking performance of Olympic-level swimmers over 4 yr of training. The modified model also revealed that the subjects displayed varying upper thresholds for training loads, thus emphasizing the importance of individualizing training programs to ensure that the prescribed training results in positive adaptation.
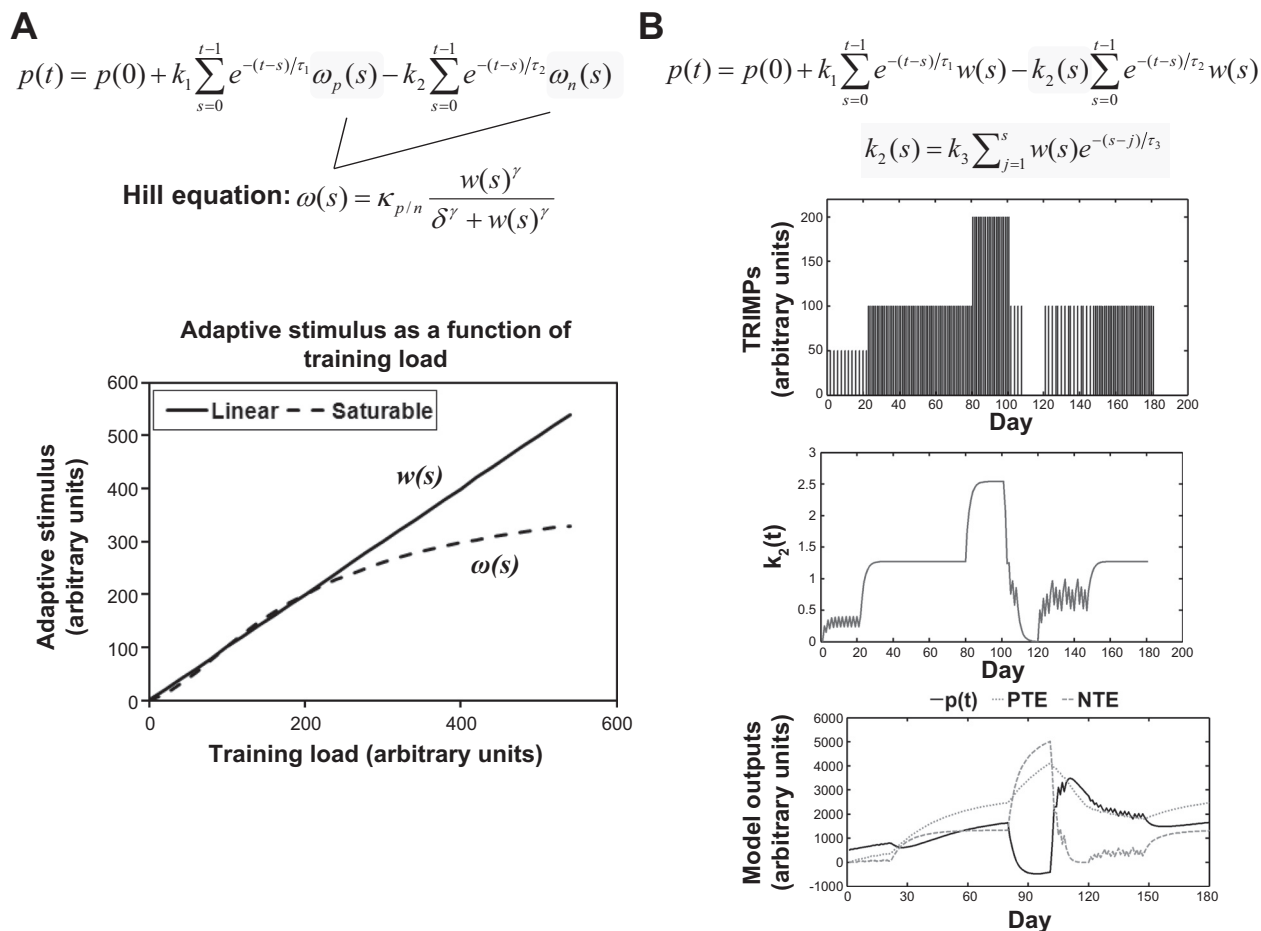
A

$$p(t) = p(0) + k_1 \sum_{s=0}^{t-1} e^{-(t-s)/\tau_1} \omega_p(s) - k_2 \sum_{s=0}^{t-1} e^{-(t-s)/\tau_2} \omega_n(s)$$

Hill equation: $\omega(s) = \kappa_{p/n} \dfrac{w(s)^{\gamma}}{\delta^{\gamma} + w(s)^{\gamma}}$

B

$$p(t) = p(0) + k_1 \sum_{s=0}^{t-1} e^{-(t-s)/\tau_1} w(s) - k_2(s) \sum_{s=0}^{t-1} e^{-(t-s)/\tau_2} w(s)$$

$$k_2(s) = k_3 \sum_{j=1}^{s} w(s) e^{-(s-j)/\tau_3}$$

Adaptive stimulus as a function of training load

Fig. 10. Modifications to the IR model. *A*: Hill equations are used to represent the notion that the body has a finite capacity to adapt to a given training load (44). In practice, this modification limits potentially unrealistically high estimates of PTE and NTE induced by high training loads. In the original IR model, the adaptive stimulus [$w(s)$] is a direct function of training load. In the modified model, the adaptive stimulus [$\omega(s)$] is a nonlinear saturable function of training load, which is represented by a Hill equation. *B*: model of the effect of training frequency on fatigue. Busso (16) proposed a modified IR model in which a first-order filter dependent on training load was applied to the k_2 parameter to represent the effect of frequency on training-induced fatigue. Simulated training data (*top*), based on a hypothetical program featuring varying training frequencies and loads, were generated to demonstrate the effect on k_2 (*middle*) and on PTE, NTE, and performance (*bottom*).

Banister (5) recognized that the model parameters likely change over time and suggested resetting the model parameters every 60–90 days. Busso and colleagues (20) also noticed that t_n values for subjects from different studies increased with training intensity. They further reasoned that different physiological processes operating at different timescales likely contribute to the NTEs, such that time-invariant parameters might be unsuitable for modeling performance from a varied training regimen (20). To explore time-varying parameters in the IR model, Busso et al. (20) used a recursive least-squares algorithm that allowed the parameters to vary over time and found that the increased flexibility of the model allowed it to better fit performance data. In a subsequent study (17), they showed that increasing training load by increasing training frequency caused the gain terms (k_1 and k_2) to change. This prompted Busso (16) to propose a modified IR model in which k_2 was redefined to vary as a function of training according to the following formula:

$$k_2^i = k_3 \sum_{j=1}^{i} w(j) e^{-(i-j)/\tau_3}$$

which modifies the NTE term in the IR model as follows:

$$p(t) = p(0) + k_1 \sum_{i=1}^{t-1} w(i) e^{-(t-i)/\tau_1} - \sum_{i=1}^{t-1} k_2^i w(i) e^{-(t-i)/\tau_2}$$

The proposed modified IR model led to more precise fits than the standard two-component IR model (16). These studies imply that training adaptations are context dependent: as overall training load is increased, the amount of performance improvement from a single unit of training varies as an inverted U-shape, such that an optimum amount of daily training exists (16).

Approaches for modeling training beyond the IR model have also been proposed. These include regression and mixed-effects models (3, 44) as well as neural networks and a dynamic meta-model (51). Each of these has unique advantages and disadvantages. To date, however, no modeling framework has emerged as superior to the IR model, but this could change as alternative frameworks are increasingly being studied.

Strategies for Teaching Performance Modeling

Various pedagogical strategies exist for teaching performance models. These include reading, didactic lectures, and active learning strategies such as thinking about conceptual questions, problem solving, computer-based exercises, critical evaluation of scientific studies, and laboratory- or field-based experiments. It is up to the instructor to choose the mix of strategies that suits the purpose, audience, and logistical constraints of the course. Our report and the references cited herein serve as content for reading and lecture materials. The rationale section of this report presents a comprehensive justification of why students ought to learn performance models, which serves to contextualize the material to be learned and to motivate the students.

Regardless of the audience, the course or module should cover at minimum the content necessary for learning how to implement the models. This minimal content includes the following subsections from the lecture material section: *1*) definition, *2*) physiological basis, *3*) practical implementation, *4*) concep-

tual benefits and practical applications, and *5*) limitations. Beyond this minimal content, additional content focusing on the practical implementation should be the focus for applied exercise physiology practitioners. Such instruction includes the steps for uploading training data from the monitoring devices that their clients typically use (e.g., HR monitors, global positioning system units, and power meters), fitting the models, interpreting their outputs, and devising practical recommendations. Monitoring devices all come with basic software to upload data, but additional software is typically required to fit the models. We have provided a comprehensive supplemental spreadsheet file (Microsoft Excel) that demonstrates the computations necessary to implement the models. Pettitt (83) has also described the use of Excel for teaching the basic aspects of the CP model. Specialized software, such as RaceDay Apollo (www.physfarm.com), is also available.

Beyond the basic use of the models, we encourage educators to use the models as conceptual guides for teaching exercise prescription. The models are instructive in this regard for two principal reasons: *1*) they succinctly integrate the core concepts of exercise prescription and *2*) they can serve as a means for experiential learning via computer exercises. Many different questions or scenarios can be investigated using the performance models. For the CP model, instructors could design computer exercises to investigate different interval workouts using the CP model for intermittent exercise (78). For the IR model, computer exercises could be designed to estimate training loads (e.g., TRIMP and BikeScore) for different types of workouts, which would demonstrate how a workout's duration and intensity interact to produce adaptive stress. Similarly, different periodization schemes and their effects on different individuals could be investigated through simulation exercises (e.g., Fig. 9) (77). In this way, performance models illuminate the principles of exercise prescription.

For both audiences, the CP and IR models provide frameworks for critically thinking about exercise physiology-related questions. The principles might not lead to the correct answer, but they can help one to achieve an educated guess. In APPENDIX B, we present questions regarding training with answers that are based on concepts underlying the CP and IR models. We encourage instructors to develop concept-based questions such as these and to use them as the basis for Conceptests (an active learning technique based on in-class clicker-based questions) (30). Inspiration for additional questions can be found in sport-related internet chat forums (e.g., www.letsrun.com and www.slowtwitch.com). Indeed, an interesting exercise is to have students go online to these forums, find questions posed by users, and answer these questions based on concepts from the performance models, perhaps supplemented with data from computer simulations and/or published data. Such exercises serve as real-life case studies that can help bridge the gap between classroom-based learning and real-world exercise prescription.

For research-focused exercise physiologists, we encourage that additional content focus on the general skills of mathematical modeling using the performance models as examples. In this regard, the subsections on model derivations and modifications can be used to explain how scientists convert observations and ideas about the physical world into mathematical equations. Linear and nonlinear regression techniques for fit-

ting the models to data can be introduced and compared, followed by statistical evaluation of the fitted model (e.g., residual diagnostics and evaluation of model predictivity through cross-validation). Papers by Gaesser et al. (36) and Hellard et al. (43) are particularly instructive for statistical evaluation of the CP and IR models, respectively. Software such as Excel, R (http://www.r-project.org/), and Matlab (The MathWorks, Natick, MA) are some of the popular options for implementation. Once working models are in hand, students should be taught various ways in which models can be analyzed. Scientific questions about the modeled system are often answered by exploring model behavior using mathematical analysis [i.e., the equations are manually manipulated (33, 35, 73)] or via simulations of the model with different parameter values or inputs (7, 77, 98–100). These methods can be taught by critical reading of the cited papers and by replicating the analyses contained within. Once the students have a working knowledge of the modeling process as it pertains to performance models, an option for further instruction is to have them critically review exercise physiology studies incorporating mathematical models to broaden their perspective on how different types of models are used to answer scientific questions.

A capstone strategy for teaching performance modeling is to have students use it in the real-world setting. Practitioners have the opportunity to apply performance models in their everyday practices. Exercise physiology students in the academic setting could be exposed to performance modeling via laboratory exercises. One option to integrate both the CP and IR models is to have students perform a longitudinal laboratory exercise. Such an exercise could proceed as follows. In the first laboratory session, the students test their CP. They then follow a training program for several weeks, record their daily training data, measure their performance weekly, and conclude by performing another CP test. Students then use their data to fit the models followed by analyzing and interpreting their results. Beyond this basic strategy, we suggest having students perform experiments that involve the models. Examples include comparing the traditional mode of CP testing (e.g., 4–6 maximal effort time trials of varying duration) versus the results of the 3-min all-out test or comparing training programs using the IR model. These exercises require minimal equipment if the training loads are computed using TRIMP (i.e., only a wrist-mounted HR monitor with stopwatch capabilities).

We conclude by discussing a perceived obstacle to teaching performance modeling, which is the need to understand mathematics at a sophisticated level. We emphasize that the mathematical sophistication can be tailored to the audience. For applied exercise physiology practitioners, the concepts and practical aspects of implementing the models using software should be emphasized. Analogously, statistics are taught to undergraduates across the spectrum of educational disciplines. In cases in which the students have limited quantitative training, the concepts of the statistical tests are taught and software, such as Minitab or SAS, is used to implement the algorithms. For exercise physiologists destined for a research career, greater emphasis can be placed on the theoretical and mathematical aspects of the models.

Conclusions

Mathematical models of training and performance are effective conceptual and practical tools for evidence-based exercise prescription. However, a lack of education contributes to limiting their widespread implementation. Our report is intended to serve as a resource for the teaching of the CP and IR models by instructors of both applied exercise physiology professionals and research-focused exercise physiologists. We emphasize that the mathematical sophistication can be tailored to the audience such that the models can be broadly accessible. We have taught these models for several years now in various formats and have distilled the course content into this report. In the years to come, we intend to build on the content, teaching approaches, and tools found in our report to continuously improve the teaching of these models and exercise prescription in general.

APPENDIX A: DERIVATION OF THE IR MODEL

Banister and colleagues posited that the response in performance to training followed first-order kinetics. Mathematically, first-order kinetics implies that the rate of change of a variable is dependent on the first-order power of its value. A model of the rate of decay of performance capacity is expressed as follows:

$$\frac{d\mathrm{p}(t)}{dt} = -\frac{1}{\tau}\mathrm{p}(t)$$

where $\mathrm{p}(t)$ is performance capacity, t is time, and τ is the decay time constant.

Similarly, when training is performed, performance is increased, and a term reflecting this increase can be added to the equation, as follows:

$$\frac{d\mathrm{p}(t)}{dt} = -\frac{1}{\tau}\mathrm{p}(t) + w(t)$$

where $w(t)$ is training stress.

The equation above is a linear ODE, which can be solved using the method of Laplace transform (87). Laplace transforms are applied to each term of the equation as follows:

$$\mathcal{L}\left[\frac{d\mathrm{p}(t)}{dt}\right] = s\mathrm{P}(s) - \mathrm{p}(0)$$

$$\mathcal{L}\left[\frac{1}{\tau}\mathrm{p}(t)\right] = \frac{\mathrm{P}(s)}{\tau}$$

$$\mathcal{L}\left[w(t)\right] = W(s)$$

and we substitute the transforms in place of their corresponding terms in the original differential equation, as follows:

$$s\mathrm{P}(s) - \mathrm{p}(0) = -\frac{P(s)}{\tau} + W(s)$$

This algebraic equation is then solved for $\mathrm{P}(s)$ to derive its "transfer function," which is the functional relationship between system input and output:

$$\mathrm{P}(s) = -\frac{W(s)}{\left(s + \frac{1}{\tau}\right)}$$

Note that $\mathrm{p}(0)$ is assumed to equal 0.

If we define $G(s) = 1/(s + 1/\tau)$, then it becomes apparent that the equation is in the form of a product, as follows:

$$P(s) = G(s)W(s)$$

This equation can be solved using the convolution theorem, which states that the inverse Laplace transform of a product is its convolution:

$$\mathcal{L}^{-1}[P(s)] = \mathcal{L}^{-1}[G(s)W(s)] = (g * w)(t)$$

A convolution is defined as follows:

$$p(t) = (g * w)(t) = \int g(\Theta)w(t - \Theta)d\Theta = \int g(t - \Theta)w(\Theta)d\Theta$$

We consult Laplace transform tables to find the inverse Laplace transforms for $g(t - \theta)$. We note that $G(s) = 1/(s + 1/\tau)$ is of the following form: $1/(s - a)$ and has the inverse Laplace transform of e^{at}. Therefore, if we let $a = -1/\tau$, $g(t - \theta) = e^{-(t - \theta)/\tau}$. We substitute this term into the above integral:

$$p(t) = \int e^{-(t - \Theta)/\tau}w(\Theta)d\Theta$$

This is the solution to the differential equation describing performance as a function of training stress and its intrinsic decay over time. To make this equation usable in a practical sense, we discretize it by expressing the continuous integral as the following sum:

$$p(t) = \sum_{i=1}^{t-1} e^{-(t-i)/\tau}w(i)\Delta t = \sum_{i=1}^{t-1} e^{-(t-i)/\tau}w(i)$$

if Δt is set equal to a constant of 1 and i is the ith day leading up to day t.

When applied to data from a swimmer, Banister and colleagues (24) found that performance capacity became negative during the initial stages of training. The original equation could not capture this behavior, forcing them to reformulate their model. They did so by modeling fitness (PTEs) and fatigue (NTEs) separately and then assuming that performance was equal to the difference of fitness and fatigue (24). PTE and NTE were modeled in an identical fashion as performance above. Two forms of the equation were eventually settled on, one in the following summation form:

$$p(t) = p(0) + k_1\sum_{i=1}^{t-1} e^{-(t-i)/\tau_1}w(i) - k_2\sum_{i=1}^{t-1} e^{-(t-i)/\tau_2}w(i)$$

where k_1 and k_2 are gain parameters, and one as the following set of computationally friendly recursion equations:

$$g(t) = g(t - i)e^{-i/\tau_1} + w(i)$$
$$h(t) = h(t - i)e^{-i/\tau_2} + w(i)$$
$$p(t) = p(0) + g(t) - h(t)$$

APPENDIX B: SAMPLE CONCEPTUAL QUESTIONS AND ANSWERS

Here we present sample conceptual questions and answers based on concepts from the CP and IR models.

Question. Explain the rationale of interval training.

Answer. The CP model tells us that depletion of W' leads to exhaustion, such that exercise performed at high intensity will be necessarily limited in duration. The IR model assumes that the metabolic adaptations to training are a function of the volume and intensity of exercise. From the TRIMP calculation, we notice that training stress is a strong function of volume. Therefore, if an athlete wishes to exercise at a power above CP, then he or she will be limited by W', which would limit the volume of work at that power. To accumulate sufficient duration at an intensity above CP without prematurely fatiguing, the athlete can insert periods of recovery between the work bouts to restore W'. Therefore, intervals provide a means for accumulating volume at high intensities.

Question. The CP model implies optimal racing strategies for different types of athletes. How might the approach of an athlete with a large W' differ from that of an athlete with a high CP?

Answer. Athletes with a high CP would likely benefit from a strategy in which they are able to set their pace in excess of the CP of their competitors but below their own CP. This would force their competitors to expend their limited W' and become exhausted sooner. In contrast, athletes with a high W' would likely benefit from attempting to control the pace, preserving their W' for a finishing sprint.

Question. Why it is important to defend W' during racing?

Answer. Recent work suggests that W' recovers quite slowly, with the recovery rate slowing as the power during the recovery period increases. Thus, during most types of races, if W' is expended, it is unlikely to be meaningfully recovered such that the athlete will position him or herself closer to exhaustion.

Question. The CP model has an asymptote, suggesting a power that can be maintained indefinitely. Is this a reasonable assumption?

Answer. It is not a valid assumption because athletes can become fatigued for a number of reasons. For example, setting a power 5% below the CP would still result in fatigue eventually, likely due to glycogen depletion and/or central mechanisms.

Question. A commonly cited justification for the taper is that athletes do not gain fitness in the final weeks before a race. Is this belief justified?

Answer. This belief is unfounded because the IR model tells us that every training session induces both fitness and fatigue. The gains in fitness are initially obscured by increased fatigue, which may lead people to believe that fitness was not gained. Accordingly, recent research using a modified form of the IR model shows potential performance benefits using tapers that follow a progressive decrease in training with an increased workload in the final days leading up to a goal event (99).

Question. Many training textbooks and coaches advocate a periodization scheme featuring 3 wk of progressive overload followed by a rest week in which training volume and intensity are reduced. Is this periodization scheme optimal? (We note that "rest weeks" are distinct from the final preevent taper.)

Answer. The IR model suggests that optimal periodization depends on the individual and that the same level of performance can be achieved by many different periodization schemes. Therefore, in some cases, rest weeks might cause too large of a decrease in fitness for some individuals, whereas others might require them to be able to sustain 3 wk of increased loads. However, if daily training loads are matched with the athlete's ability to recover from those sessions, then rest weeks should be unnecessary.

Question. A triathlete who trains with a Masters group 3–4 times/wk has plateaued at a time of 24 min for his 1,500-m swim. He wishes to swim a 1,500-m freestyle in 22 min, and, to train for this, he plans to apply the principle of specificity and do twice weekly workouts featuring 15 × 100-m intervals in which each 100 m is swum in 1:28 (= 22 min/15). The first workout will feature rest periods of 30 s between each 100 m, and subsequent workouts will feature gradually decreasing rest periods until he is able to swim 1,500 m in 22 min. In addition to these two workouts, he will train with his Masters group 2 times/wk. Will his training plan work?

Answer. His training plan is unlikely to work. The IR model predicts that improvements in performance require increasing training stress beyond what one has done previously. Training stress is a strong function of training volume, and he has not significantly modified this variable in his plan. While the workouts are specific in terms of training at goal race pace, they are unlikely to induce sufficient overload to lead to a 2-min drop in his 1,500-m swim time.

Question. What is the optimal taper duration?

Answer. The answer depends on the individual because each individual responds to training with different kinetics, which is reflected by the IR model as the values of the parameters (gain and time constants). These parameters are used to calculate the t_n parameter,

which is interpreted as the day after which training will induce net negative adaptations for performance on the day of the goal event. In practice, t_n is considered the day at which the taper should be well underway.

DISCLOSURES

P. F. Skiba is the developer of RaceDay Apollo software, which is mentioned in the text.

AUTHOR CONTRIBUTIONS

Author contributions: D.C.C. and P.F.S. conception and design of research; D.C.C. and P.F.S. analyzed data; D.C.C. and P.F.S. prepared figures; D.C.C. and P.F.S. drafted manuscript; D.C.C. and P.F.S. edited and revised manuscript; D.C.C. and P.F.S. approved final version of manuscript.

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